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Closed graph theorems for locally convex topological vector spaces

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CLOSED GRAPH THEOREMS FOR LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

A Dissertation Submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg in Partial Fulfilment of the Requirements for the Degree of Master of Science

by

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ABSTRACT

Let $\mathcal{C}$ be the class of pairs of locally convex spaces (X,Y) which are such that every closed graph linear mapping from X into Y is continuous. If B is any class of locally convex Hausdorff spaces, let $\mathcal{C}(B) = \{X : (X,Y) \in \mathcal{C} \text{ for all } Y \in B\}$. In this expository dissertation, $\mathcal{C}(B)$ is investigated, firstly for arbitrary B, secondly when B is the class of D_r-complete spaces and thirdly when B is a class of locally convex webbed spaces.

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I also wish to thank Mrs. L. Berger who typed the manuscript so expertly.

DECLARATION.

I certify that

(a) this dissertation is my own unaided work ;

(b) neither the substance nor any part thereof has been nor will be submitted for a degree in any other University ;

(c) no information used has been obtained by me while employed by, or working under the aegis of, any person or organisation other than the University.

Janet Margaret Helmstedt.

Janet Margaret Helmstedt.

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CHAPTER ONE

INTRODUCTION AND NOTATION

1.1 Introduction

By "convex space" we mean "locally convex topological vector space over the field C of complex numbers". We are concerned with the class $\mathcal{C}$ of pairs (X,Y) of convex spaces with the property that every closed graph linear mapping from X into Y is continuous. One way to investigate $\mathcal{C}$ is to choose a class B of convex Hausdorff spaces and then consider $\mathcal{C}(B) = \{X : (X,Y) \in \mathcal{C}$ for all $Y \in B\}$. Banach [1] showed in 1932 that if B is the class of Frechet spaces, then $\mathcal{C}(B)$ includes B.

In Chapter two we prove some general closed graph theorems which apply to every class B of convex Hausdorff spaces. Following Iyahen in [7] we show that if B is any class of convex Hausdorff spaces, then $\mathcal{C}(B)$ is closed under the formation of inductive limits and subspaces of finite codimension. Following de Wilde in [5] we show that if I is an index set of cardinality d and $C^I \in \mathcal{C}(B)$ then $\mathcal{C}(B)$ is closed under the formation of topological products of cardinality d. In this chapter, we also prove a general open mapping theorem: if the class B of convex Hausdorff spaces is closed under the formation of Hausdorff quotient spaces and if $X \in \mathcal{C}(B)$ and $Y \in B$ then every closed graph linear mapping from Y onto X is open.

In Chapter three, we show that if B is the class of B_r-complete spaces, then $\mathcal{C}(B)$ is the class of barrelled spaces. The proofs of this chapter are adapted from the corresponding proofs for B-complete spaces in [15]. Following Robertson and Robertson [15], we show that every Frechet space is B-complete. We give an example of a barrelled space and of a B_r-complete space neither of which is a Frechet space, and so we have a generalisation of Banach's closed graph theorem.

Chapter four is concerned with the class of convex spaces with $\mathcal{C}$ webs. The theory of webbed spaces if due to de Wilde. He introduced the theory in 1967 [3] and developed it in a series of papers,

the most important being [4]. We quote Horvath ([6], p 74 96) on this theory: "His theory is undoubtedly the most important contribution to the study of locally convex spaces since the great discoveries of Dieudonné, Schwartz and Grothendieck in the late forties and early fifties". de Wilde's closed graph theorem [4] states that if $\mathcal{B}$ is the class of convex spaces with $\mathcal{C}$ webs, then $\mathcal{B}(\mathcal{B})$ includes the convex Baire spaces. The class $\mathcal{B}$ has good stability properties. It is closed under the formation of closed subspaces, continuous images, countable inductive limits, countable products and weaker convex topologies. $\mathcal{B}$ includes the Fréchet spaces and their strong duals. Proofs of theorems about spaces with $\mathcal{C}$ webs are often very technical. Robertson and Robertson [15] have simplified many of these proofs by introducing what they call "strands" of a web on a convex space. However, their proofs apply only to what they call convex spaces with "completing webs", which are special cases of convex spaces with $\mathcal{C}$ webs. We have defined what we call the "filaments" of a $\mathcal{C}$ web - a modification of the Robertson "strand". Using this definition we have obtained the more general results of de Wilde [4] with proofs similar to those of Robertson and Robertson. We have also used this method to obtain Powell's result [13], viz. if the convex Hausdorff space (X,τ) has a $\mathcal{C}$ web, so has (X,τ^u) where τ^u is the weakest ultra-bornological topology for X stronger than τ. This is a most pleasing result. Powell proves in addition that in any chain of convex Hausdorff topologies including τ^u, all those weaker than τ^u are $\mathcal{C}$ webbed and none of those strictly stronger than τ^u have a $\mathcal{C}$ web.

In Chapter five we show that neither of the closed graph theorems of Chapters two and three is a generalisation of the other. In order to do this we needed a barrelled space X and a convex space Y with a $\mathcal{C}$ web such that $(X,Y) \notin \mathcal{B}$. Valdivia [18] provided us with just such an example.

In the above chapters, we also prove some "sequentially closed graph theorems", where we derive the continuity of certain linear mappings from the assumption that their graphs are sequentially closed instead of closed.

Chapter six concludes the dissertation with a short account, without proofs, of other closed graph theorems.

As far as I am aware, Theorems 2.3 and 2.13 are new results.

1.2 Notation

We give here a list of symbols and terminology which we use frequently and which are not standard. Other terminology will be introduced as needed in particular sections of this dissertation.

Let A be a subset of a vector space X.

$\langle A \rangle$ — $\langle A \rangle$ denotes the subspace of X spanned by A.

$h(A)$ — $h(A)$ denotes the absolutely convex hull of A.

$g(A)$ — $g(A)$ denotes the gauge of A in the case when A is absolutely convex and absorbent.

(X,τ) — If we are concerned with a single vector topology τ for a vector space X, we shall refer to the topological vector space X, meaning the vector space endowed with the topology τ. We shall often be concerned with more than one topology for a vector space X. In such cases we shall use the symbol (X,τ) for the vector space X endowed with the topology τ.

Let X and Y be topological spaces.

Neighbourhood — By "neighbourhood" in X, we mean neighbourhood of the origin.

Topologies on subspace and quotient space — If M is a subspace of X, unless otherwise specified, M shall be taken to be endowed with the topology it inherits from the topology on X, and X/M with the quotient topology.

Topologically isomorphic spaces — X and Y are said to be topologically isomorphic if there exists a one-to-one, continuous, open linear mapping T from X onto Y. T is called a topological isomorphism.

Let α be a property of topological vector spaces.

α space, **α topology** — If (X,τ) has the property α, we say that (X,τ) is an α space and τ is an α topology.

Dual pair — The pair (X,Y) of vector spaces is called a dual pair if Y is a subspace of X^* which separates

the points of X. Let (X,Y) be a dual pair. If $x \in X$, $y \in Y$, we denote $y(x)$ by (x,y). Since $(x,\lambda_1 y_1+\lambda_2 y_2) = \lambda_1(x,y_1)+\lambda_2(x,y_2)$ for all $x \in X$, all $y_1,y_2 \in Y$ and all $\lambda_1,\lambda_2 \in C$, we regard each $x \in X$ as an element of Y^*. Then X separates the points of Y, and (Y,X) is a dual pair.

$W(X,Y)$ topology — The weakest topology on X which makes each $y \in Y$ continuous is called the weak topology on X determined by Y and denoted $W(X,Y)$.

Polar topology — Let (X,Y) be a dual pair and $\mathcal{D}$ a collection of $W(X,Y)$ bounded subsets of Y satisfying the conditions:

(a) if $A,B \in \mathcal{D}$, there exists $D \in \mathcal{D}$ with $A \cup B \subset D$;

(b) if $A \in \mathcal{D}$ and λ is a scalar then $\lambda A \in \mathcal{D}$;

(c) $\bigcup_{D \in \mathcal{D}} D$ spans X.

$\mathcal{D}^\circ$ / $\tau(X,Y)$ / $s(X,Y)$ — The polars in X of the elements of $\mathcal{D}$ form a base of neighbourhoods for a convex Hausdorff topology for X called the polar topology on the elements of $\mathcal{D}$. We use the symbol $\mathcal{D}^\circ$ to denote this topology. If $\mathcal{D}$ consists of all the finite subsets of Y, then $\mathcal{D}^\circ = W(X,Y)$. If $\mathcal{D}$ consists of all the absolutely convex $W(Y,X)$ compact subsets of Y, then $\mathcal{D}^\circ$ is called the Mackey topology of the dual pair (X,Y) and denoted $\tau(X,Y)$. If $\mathcal{D}$ consists of all the $W(Y,X)$ bounded subsets of Y, $\mathcal{D}^\circ$ is called the strong topology of the dual pair (X,Y) and denoted $s(X,Y)$.

Inductive limit of convex spaces — We use the definition of Robertson and Robertson ([]5]p. 78 2) for an inductive limit of a collection of convex spaces. The convex space X is said to be the inductive limit of the convex spaces X_α, $\alpha \in A$ under the linear mappings $T_\alpha : X_\alpha \to X$, if $\bigcup_{\alpha \in A} T_\alpha(X_\alpha)$ spans X and X is endowed with the strongest convex topology which makes each T_α continuous.

$\prod_{i \in I} X_i$

$\bigoplus_{i \in I} X_i$

If X_i is a topological vector space for each i in an index set I, then, unless otherwise stated, the symbols $\prod_{i \in I} X_i$, $\bigoplus_{i \in I} X_i$ represent the product and direct sum respectively of the X_i under the product and direct sum topologies. If X is either of the spaces $\prod_{i \in I} X_i$, $\bigoplus_{i \in I} X_i$, we regard each X_i as a subspace of X, by identifying X_i and $\{x \in X : p_j(x) = 0 \text{ for all } j \neq i\}$, where p_j is the jth projection from X onto X_j.

$\prod_{i \in I} C_i$

$\bigoplus_{i \in I} C_i$

When we write $\prod_{i \in I} C_i$ or $\bigoplus_{i \in I} C_i$, each C_i is always taken to be C, the complex field under the Euclidean topology.

$T|_Z$

If T is a mapping from a set X into a set Y, and Z is a subset of X, then $T|_Z$ denotes the restriction of T to Z.

τ-σ continuous

If X and Y are sets, and $f : X \to Y$ a mapping, we shall say that f is τ-σ continuous if it is continuous when X and Y are endowed with the topologies τ, σ respectively. In some cases we may simply say that f is continuous if it is clear what the topologies on X and Y are.

T', the transpose of T

Let X, Y be topological vector spaces and $T : X \to Y$ a linear mapping. Let $T^* : Y^* \to X^*$ be defined by $(x, T^*(w)) = (T(x), w)$ for all $x \in X$ and all $w \in Y^*$. Unless otherwise stated, we shall use the symbol T' for $T^*|_{Y'}$, where Y' is the continuous dual of Y.

CHAPTER TWO

SOME GENERAL CLOSED GRAPH THEOREMS

Throughout this chapter, the symbol B denotes an arbitrary class of convex Hausdorff spaces.

If f is a mapping from a set X into a set Y, the *graph*, G, of f is the subset $\{(x,f(x)) : x \in X\}$ of $X \times Y$. If X and Y are topological spaces and G is (sequentially) closed in the product topology on $X \times Y$, then f is said to be a *(sequentially) closed graph mapping.* It is easily verified that if Y is Hausdorff and f is continuous then G is closed. Our primary concern is with pairs (X,Y) of convex spaces for which every closed graph linear mapping from X into Y is continuous. However, we shall prove some results for sequentially closed graph mappings when these are easily derived from or are similar to the corresponding results for closed graph mappings. We need, then, definitions analogous to those of $\mathcal{C}$ and $\mathcal{C}(B)$ given in the introduction. Let $\mathcal{C}_1$ be the class of pairs (X,Y) of convex spaces with the property that every sequentially closed graph linear mapping from X into Y is continuous, and let $\mathcal{C}_1(B) = \{X : (X,Y) \in \mathcal{C}_1$ for all $Y \in B\}$.

If σ,τ and ν are Hausdorff topologies for the set X, with ν weaker than both σ and τ, then the identity map $I : (X,\sigma) \to (X,\tau)$ has a closed graph. If X and Y are vector spaces and $T : X \to Y$ a linear mapping, then the graph of T is a vector subspace of $X \times Y$. From this we may conclude that if $T : X \to Y$ is a closed graph linear mapping, where X and Y are convex Hausdorff spaces with continuous duals X' and Y', then the graph of T remains closed when X and Y have any other convex topologies stronger than the topologies $W(X,X')$ and $W(Y,Y')$ respectively. It is now easy to show that if (X,σ) and (Y,τ) are convex Hausdorff spaces with $((X,\sigma),(Y,\tau)) \in \mathcal{C}$ then $((X,\sigma_1),(Y,\tau_1)) \in \mathcal{C}$ where σ_1 is any topology of the dual pair (X,X') stronger than σ and τ_1 is any convex topology for Y weaker than τ. Another easily verified fact is that if $(X,Y) \in \mathcal{C}$ $(\mathcal{C}_1)$ and Z is a closed (sequentially closed) subspace of Y then $(X,Z) \in \mathcal{C}$ $(\mathcal{C}_1)$.

If $B_1 \subset B_2$, then $\mathcal{C}(B_1) \supset \mathcal{C}(B_2)$. The above remarks show that

if B_1 is a proper subset of B_2, it may happen that $\mathcal{C}(B_1) = \mathcal{C}(B_2)$.

NOTE 2.1 Closed graph theorems are a good source of counter examples of a certain kind. Let α be a property of convex spaces, and suppose that every α space is Hausdorff. If B is a class of convex spaces such that every α space is in $B \cap \mathcal{C}(B)$, then, in any chain of convex Hausdorff topologies for a vector space X there can be at most one α topology. For, if σ and τ are two α topologies of this chain, the identity mapping $I : (X,\sigma) \to (X,\tau)$ has a closed graph and so it is continuous and σ is stronger than τ. By symmetry, τ is stronger than σ. For example, if (X,τ) is a Fréchet space, then (X,σ) is not a Fréchet space if σ is any convex Hausdorff topology for X comparable with, but distinct from τ. A similar consideration of the identity mapping shows that if $(X,\tau) \in B$ and $(X,\sigma) \in \mathcal{C}(B)$, then σ is weaker than τ.

If X and Y are topological spaces and T a mapping from X into Y, then, in order to prove that the graph of T is closed (sequentially closed), it is sufficient to show that if (x_λ) is a net (sequence) in X which is such that $x_\lambda \to x$ and $T(x_\lambda) \to y$ then $y = T(x)$. The following useful lemma is easily proved using this technique.

LEMMA 2.2 *Let X, Y, Z be topological spaces. Let $f : X \to Y$ be continuous and $g : Y \to Z$ have a (sequentially) closed graph. Then the graph of $g \circ f : X \to Z$ is (sequentially) closed.*

We consider the property "Hausdorff" in connection with closed graph theorems. Let X be a topological vector space and let $L = \bigcap_{U \in \mathcal{U}} U$, where $\mathcal{U}$ is the set of all neighbourhoods in X. Then $L = \overline{\{0\}}$, and so L is a closed subspace of X (see [15] p 26 Proposition 3)*. The quotient space X/L is Hausdorff and is called the *Hausdorff topological vector space associated with* X, and we denote it $X(H)$. It is immediate that X is Hausdorff if and only if $X = X(H)$. The topology on X induces the indiscrete topology on L. Let M be an algebraic supplement of L in X. Then the projection mapping from X onto L is continuous and so X is the topological direct sum of M and L and M is topologically

*The proof of this theorem given in [15] does not require the hypothesis that the spaces concerned are convex.

isomorphic to X(H). (See [15] p 95 Proposition 29.) We shall identify M and X(H). We call L the *indiscrete summand* and M the *Hausdorff summand* of X.

Suppose X is a non-Hausdorff topological vector space. Let L and M be respectively the indiscrete and Hausdorff summands of X, then $L \neq \{0\}$. If (x_λ) is a net in X which converges to x, and if $n \in L$, then (x_λ) also converges to $x + n$. Now let Y be any other topological vector space and T a linear mapping from Y into X. For each positive integer i, let y_i be the zero element of Y. Choose $n \in L$ with $n \neq 0$. Then $y_i \to 0$ and $T(y_i) \to n$, but $T(0) \neq n$, and so the graph of T is not sequentially closed. Thus there are no closed or even sequentially closed graph linear mappings from a topological vector space into a non-Hausdorff topological vector space. For this reason, when considering $\mathcal{C}(B)$ or $\mathcal{C}_1(B)$, we shall always assume that the spaces in B are Hausdorff.

Let us now consider non-Hausdorff domain spaces. Let X and Y be as in the previous paragraph and let T be a linear mapping from X into Y. Let (x_λ) be a net in X which converges to x. Then $x_\lambda \to x + n$ for each $n \in L$. Suppose $T(x_\lambda)$ converges to y. We can only have that $y = T(x + n)$ for each $n \in L$ if $T(L) = 0$. Thus the graph of a linear mapping from a non-Hausdorff topological vector space into another topological vector space can only be closed or sequentially closed if it maps the indiscrete summand of its domain into zero. If X has the indiscrete topology then the zero mapping is the only closed graph linear mapping from X into a topological vector space Y, and so $(X,Y) = \mathcal{C}$.

THEOREM 2.3 *The closed graph theorem (sequentially closed graph theorem) holds for the pair (X,Y) of topological vector spaces if and only if it holds for the pair (X(H),Y).*

Proof. Suppose the closed graph theorem holds for the pair (X(H),Y). Let $X = M \oplus L$, where M and L are the Hausdorff and indiscrete summands of X respectively. Let T be a closed graph linear mapping from X into Y. Let $P : X \to M$ be the projection mapping, and $S : M \to Y$ be defined by $S = T\big|_M$. By the above remarks, T vanishes on L, and so $T = S \circ P$. If (m_λ) is a net in M which converges to $m \in M$ and if $S(m_\lambda)$ converges to y, since S and T agree on M and the graph of T is closed, it follows that the graph of S is closed. By hypothesis, S is

continuous, and, P being continuous, so is T.

Now suppose that the closed graph theorem holds for the pair (X,Y). Let L be as above, and T a closed graph linear mapping from X/L into Y. Let $K : X \to X/L$ be the canonical mapping and let $S = T \circ K$. By Lemma 2.2, S has a closed graph and so it is continuous. Let V be a neighbourhood in Y, then $S^{-1}(V)$ is a neighbourhood in X. Now $K(S^{-1}(V))$ is a neighbourhood in X/N, and $K(S^{-1}(V)) = K \circ K^{-1}(T^{-1}(V)) = T^{-1}(V)$. Thus T is continuous.

We note that the above proof goes through if X/L is any quotient space of X.

The corresponding results for sequentially closed graph linear mappings may be obtained in a similar way, using sequences instead of nets where necessary.

Let α be a property of convex spaces. Suppose we wish to show that all α spaces are in $\mathcal{C}(B)$ for some B. In proving a closed graph theorem, it is often convenient to assume that the domain space is Hausdorff. If the property α is inherited by associated Hausdorff convex spaces, then the above theorem allows us to make this assumption without loss of generality.

THEOREM 2.4. *$\mathcal{C}(B)$ and $\mathcal{C}_1(B)$ are closed under the formation of inductive limits.*

Proof. Let X be the inductive limit of the convex spaces X_α under the mappings $T_\alpha : X_\alpha \to X$, where $X_\alpha \in \mathcal{C}(B)\,(\mathcal{C}_1(B))$ for each α in an index set A. Let $Y \in B$, and let T be a closed (sequentially closed) graph linear mapping from X into Y. For each $\alpha \in A$, T_α is continuous and so by the previous lemma, $T \circ T_\alpha : X \to Y$ has a closed (sequentially closed) graph. By hypothesis, each $T \circ T_\alpha$ is continuous and so is T (see [15] p 79 Proposition 5).

Corollary. $\mathcal{C}(B)$ and $\mathcal{C}_1(B)$ are closed under the formation of direct sums, quotients and finite products.

If $X = \bigoplus_{i \in I} C_i$, then X is equipped with the strongest convex topology, every linear mapping from X into any topological vector space is continuous and so $X \in \mathcal{C}(B)$ for every B.

A subspace Y of a vector space X is said to be of *finite codimension* if X/Y has finite dimension. Iyahen in [7] has shown that $\mathfrak{b}(B)$ is closed under the formation of subspaces of finite codimension. Before we prove Iyahen's result we prove a lemma which can be found in Kelley [9] p.38 5.5.

LEMMA 2.5. *If T is a linear mapping from a subspace Z of a convex space X into a convex space Y, and the graph G of T is closed in $Z\times Y$, then the closure in $X\times Y$ of G is the graph of a linear extension of T. If T is continuous, so is its extension.*

Proof. We observe firstly that if V and W are vector spaces, then a subspace A of $V\times W$ is the graph of a linear mapping from a subspace of V into W if and only if $(0,y)\in A$ implies that $y=0$.

Let $\bar{G}$ be the closure of G in $X\times Y$. Let $(0,y)\in\bar{G}$, then $(0,y)\in\bar{G}\cap(Z\times Y)$ = closure of G in $Z\times Y = G$. Hence $y=0$ and so $\bar{G}$ is the graph of a linear extension, $\bar{T}$, of T.

Now suppose that T is continuous. Let $\bar{T}$ map the subspace W of X into Y. Let V be a closed neighbourhood in Y. Let U be an absolutely convex neighbourhood in X such that $T(U\cap Z)\subset V$. We shall show that $\bar{T}(\frac{1}{2}U\cap W)\subset V$ from which we can conclude that $\bar{T}$ is continuous. Let $x\in\frac{1}{2}U\cap W$, then $(x,\bar{T}(x))\in\bar{G}$ and we can choose a net $(x_\lambda, T(x_\lambda))$ in G which converges to $(x,\bar{T}(x))$. Choose μ such that $\lambda\geq\mu\Rightarrow x_\lambda\in x+\frac{1}{2}U\subset U$, then $\lambda\geq\mu\Rightarrow T(x_\lambda)\in V$. Since V is closed, $\bar{T}(x)\in V$ and the required result follows.

THEOREM 2.6. *Let X be a convex space and X_o a subspace of X of finite codimension. If $X\in\mathfrak{b}(B)$, so is X_o.*

Proof. Suppose firstly that X_o is a hyperplane in X. Let $Y\in B$ and let T be a closed graph linear mapping from X_o into Y. The closure, $\bar{G}$, in $X\times Y$ of the graph G of T is the graph of a linear extension T_1 of T by Lemma 2.5. The domain of T_1 is X_o or X. If the domain of T_1 is X, then T_1 is a closed graph linear map from X into Y and is continuous, by hypothesis. Thus T, the restriction of T_1 to X_o is also continuous.

Now suppose the domain of T_1 is X_o, then $T_1=T$ and G is closed in $X\times Y$. Let $T_2: X\to Y$ be a linear extension of T,

then the graph of $T_1 = G + \langle (a, T_2(a)) \rangle$, where $a \in X \setminus X_o$. Since $\langle (a, T_2(a)) \rangle$ is one dimensional and G is closed, the graph of T_1 is also closed by Schaefer [16] p 22, 3.3. By hypothesis, T_2 is continuous and so is T.

It is easily shown that the theorem is true when X_o is of finite codimension in X.

In Chapter 3, we shall give an example of a pair (X,Y) of convex spaces with $(X,Y) \in \mathcal{B}$, and a closed subspace Z of X such that $(Z,Y) \notin \mathcal{B}$, thus showing that some limitation is needed on the codimension of X_o in the above theorem.

We come now to a closed graph theorem for products of convex Hausdorff spaces. This theorem was proved by de Wilde in a very interesting paper [5]. We quote from the introduction to this paper. "The aim of this paper is to show that, in the study of products of topological vector spaces, two kinds of subspaces play an essential role: the factor spaces and the simple subspaces which are the products of one-dimensional subspaces contained in the factor subspaces." If X_i is a Hausdorff topological vector space for each i in an index set I, then de Wilde's "simple" subspaces of $\prod_{i \in I} X_i$ are each topologically isomorphic to $\prod_{i \in I} C_i$. We shall prove that $\mathcal{B}(B)(\mathcal{B}_i(B))$ is closed under the formation of products of cardinality d provided that $\prod_{i \in I} C_i \in \mathcal{B}(B)(\mathcal{B}_i(B))$, with cardinal $I = d$. This is a particular case of de Wilde's closed graph theorem for products. (His theorem is also valid for linear mappings whose graphs have other properties, e.g. the property of being a Borel set.)

In this dissertation we are concerned primarily with convex spaces, but as de Wilde's Theorem is no more difficult to prove for topological vector spaces, we shall give the proof for this case.

In this section on products of topological vector spaces, X shall always denote the product $\prod_{i \in I} X_i$ where X_i is a topological vector space for each i in an index set I. The following notation will be used in the next lemma and theorem.

If $f \in X$, $i \in I$ and $p_i : X \to X_i$ is the ith projection map, then we denote $p_i(f)$ by f_i.

If $f \in X$ and $A \subset I$, let $f_A \in X$ be defined by $(f_A)_i = f_i$ if

$i \in A$, $(f_A)_i = 0$ if $i \notin A$. Then

(i) $f_\emptyset = 0$,

(ii) if $f_A = 0$ then f vanishes on A,

(iii) if $f = f_A$ then f vanishes on $I \setminus A$,

(iv) if $\{A_1, A_2, \ldots, A_k\}$ is a finite partition of I, then $f_{A_1} + f_{A_2} + \ldots + f_{A_k} = f$, and in particular, $f_A + f_{I \setminus A} = f$.

We shall regard each X_i as being embedded in X, that is, we shall identify X_i and $\{f \in X : f_{I \setminus \{i\}} = 0\}$. In particular, if $t \in X_i$, then $t \in X$.

If $f \in X$, let $X_f = \prod_{i \in I} \langle f_i \rangle$, where $\langle f_i \rangle$ is the one-dimensional subspace of X_i (and hence of X) spanned by f_i. If $f_i \neq 0$ for all $i \in I$, then X_f is called a *simple subspace* of X. It is clear that each simple subspace is closed in X and if X is Hausdorff, it is topologically isomorphic to $\prod_{i \in I} C_i$.

LEMMA 2.7. *Let $(f^{(m)})$ be a sequence in X such that, for each $i \in I$, at most one of the $f_i^{(m)}$ is non zero. Then there exists a simple subspace X_f in X such that $f^{(m)} \in X_f$ for all m.*

Proof. Let $f \in X$ be defined as follows:
if $f_i^{(m)} \neq 0$ for some m, let $f_i = f_i^{(m)}$,
if $f_i^{(m)} = 0$ for all m, choose $x_i \in X_i$ with $x_i \neq 0$ and let $f_i^{(m)} = x_i$.
Then $f^{(m)} \in X_f$ for all m.

THEOREM 2.8. *Let $\mathcal{U}$ be a family of balanced subsets of X such that for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V + V \subset U$. Suppose that for each simple subspace X_f of X and for each $U \in \mathcal{U}$, there exists a finite set $A \subset I$ such that*

$$g \in X_f \text{ and } g_A = 0 \Rightarrow g \in U.$$

Then, for each $U \in \mathcal{U}$ there exists a finite set $A \subset I$ such that

$$g \in X \text{ and } g_A = 0 \Rightarrow g \in U.$$

Proof. Suppose the assertion is false for $U \in \mathcal{U}$. Then, for each finite set A in I, there exists $g \in X$ with $g_A = 0$ and $g \notin U$.

Let $V \in \mathcal{U}$ be such that $V + V \subset U$.
We construct by induction a sequence (A_m) of finite subsets of I, and a sequence $(f^{(m)})$ in X with the following properties:

(a) $A_m \cap A_n = \phi$, for all $m \neq n$;

(b) $f^{(m)} = f_{A_m}^{(m)}$ for all m ;

(c) $f^{(m)} \notin V$ for all m.

Now $U \neq X$, so choose $h \in X \setminus U$. Let $X_{h'}$ be a simple subspace containing h. There exists a finite set $B \subset I$ such that

$$f \in X_{h'} \text{ and } f_B = 0 \Rightarrow f \in V.$$

Let $A_1 = B$ and $f^{(1)} = h_B$. Now $h = f^{(1)} + h_{I \setminus B}$. Also $h \notin U$ and $h_{I \setminus B} \in V$, hence $f^{(1)} \notin V$.

Suppose that A_k , $f^{(k)}$ have been defined to satisfy (a) , (b) and (c) for $1 \leqslant k < m$.

There exists $g \in X$ such that g vanishes on $\bigcup_{k=1}^{m-1} A_k$ and $g \notin U$. Let $X_{g'}$ be a simple subspace containing g. There exists a finite set $D \subset I$ such that

$$f \in X_{g'} \text{ and } f_D = 0 \Rightarrow f \in V.$$

Let $f^{(m)} = g_D$ and $A_m = D \setminus \bigcup_{k=1}^{m-1} A_k$. Now $g = g_D + g_{I \setminus D}$ $= f^{(m)} + g_{I \setminus D}$. Also $g \notin U$ and $g_{I \setminus D} \in V$. It follows that $f^{(m)} \notin V$. Hence conditions (a) , (b) and (c) are satisfied by the A_m and the $f^{(m)}$.

The A_m are mutually disjoint, so, for each $i \in I$, there exists at most one $f_i^{(m)}$ which is non-zero. By the previous lemma, there exists a simple subspace X_f such that $f^{(m)} \in X_f$ for each m. For this f , there exists a finite subset D of I such that

$$g_D = 0 \text{ and } g \in X_f \Rightarrow g \in V \quad \ldots\ldots\ldots\ldots (1)$$

Now D is finite and the A_m are pair-wise disjoint, so D does not meet each A_m. Choose k such that $D \cap A_k = \phi$. Now $f^{(k)} = f_{A_k}^{(k)}$ and so $f^{(k)}$ vanishes on D. Also $f^{(k)} \in X_f$. By (1) , $f^{(k)} \in V$. This contradicts (c) , and the theorem is proved.

COROLLARY. *Let the product topology on X be p. If τ is any other vector topology on X which induces on each factor subsapce and on each simple subspace a topology weaker than that induced by p , then τ is weaker than p.*

Proof. Let τ induce the topology τ_i on each X_i and τ_f on each X_f , with similar notation for p.

Let $\mathcal{U}$ be a base of balanced τ-neighbourhoods in X. Let $U \in \mathcal{U}$, and let X_f be a simple subspace of X. $U \cap X_f$ is a p_f neighbourhood in X_f. Hence there exists in X_f a basic p_f neighbourhood $M = \prod_{i \in A} V_i \times \prod_{i \notin A} \langle f_i \rangle$, where A is a finite subset of I , V_i

is a neighbourhood in $\langle f_i \rangle$ for $i \in A$ and $M \subset U \cap X_f$.

If $g \in X_f$ and $g_A = 0$ then $g \in M \subset U$. Thus $\mathcal{U}$ satisfies the conditions of Theorem 2.8.

Let $V \in \mathcal{U}$ be such that $V + V \subset U$. By the theorem, there exists a finite set $B \subset I$ such that $f_B = 0$, $f \in X \Rightarrow f \in V$.

Suppose B contains m elements. Let $W \in \mathcal{U}$ be such that $W + W + \ldots + W \subset V$ (m summands). $W \cap X_i$ is a p_i-neighbourhood in X_i.

Let $Q = \{f \in X : f_i \in W \cap X_i \ \forall i \in B\}$, then Q is a p-neighbourhood in X.

If $f \in Q$, then $f = f_{I \setminus B} + \sum_{i \in B} f_{\{i\}} \in V+W+W+\ldots+W \subset U$ ($m+1$ summands).

Thus $Q \subset U$ and τ is weaker than p.

We now deduce a remarkable theorem about linear maps acting on product spaces.

THEOREM 2.9. *Let Y be a topological vector space. A linear map from X into Y is continuous if and only if its restrictions to the factor subspaces and to the simple subspaces of X are continuous.*

Proof. Let T be a linear map from X into Y whose restrictions to the factor and simple subspaces of X are continuous. Let $\mathcal{U}$ be a base of balanced neighbourhoods for Y. Let $\mathcal{U}' = \{T^{-1}(U) : U \in \mathcal{U}\}$, then $\mathcal{U}'$ is a base of neighbourhoods for a vector topology, σ, on X. Let $U \in \mathcal{U}$. If Z is a simple or factor subspace of X, then $T^{-1}(U) \cap Z = (T|_Z)^{-1}(U)$, which is a neighbourhood in Z as $T|_Z$ is continuous. It follows that σ induces on Z a topology weaker than that induced by the product topology on X. By the Corollary to Theorem 2.6, σ is weaker than the product topology on X, and so each $T^{-1}(U)$ is a neighbourhood in the product topology and T is continuous.

de Wilde's closed graph theorem for products is an immediate consequence of this theorem.

THEOREM 2.10. *Suppose that Y is a topological vector space and so is X_i for each i in an index set I. Suppose also that Y is Hausdorff and that the (sequentially) closed graph theorem holds for the pairs $(\prod_{i \in I} C_i, Y)$ and (X_i, Y) for each $i \in I$. Then*

the (sequentially) closed graph theorem holds for the pair $(\prod_{i \in I} X_i, Y)$.

Proof. Let $X = \prod_{i \in I} X_i$. We suppose firstly that each X_i is Hausdorff. Let T be a linear mapping from X into Y whose graph G is (sequentially) closed. If Z is a simple or factor subspace of X, then the graph of $T|_Z$ is $G \cap (Z \times Y)$, which is (sequentially) closed. By hypothesis, $T|_Z$ is continuous, and by Theorem 2.9, T is continuous.

In the above proof, we need the hypothesis that the X_i are Hausdorff in order that the simple subspaces of X are topologically isomorphic to $\prod_{i \in I} C_i$.

Now suppose that the X_i are not necessarily Hausdorff. For each i, let M_i, L_i be respectively the Hausdorff and indiscrete summands of X_i. Then the (sequentially) closed graph theorem holds for the pair $(\prod_{i \in I} M_i, Y)$ by the above proof and Theorem 2.2. Now X is topologically isomorphic to $\prod_{i \in I} M_i \times \prod_{i \in I} L_i$, and it is easily proved that these factors are the Hausdorff and indiscrete summands of X. By Theorem 2.3 the (sequentially) closed graph theorem holds for the pair (X,Y).

We restate the above theorem for convex spaces.

THEOREM 2.11. *If* $\prod_{i \in I} C_i \in \mathcal{C}(B)$, *then* $\mathcal{C}(B)$ *is closed under the formation of products of cardinality* d, *where cardinal* $I = d$.

We note that $\mathcal{C}(B)$ is not always closed under the formation of products, even if countable. Let $X = \prod_{i=1}^{\infty} C_i$ and let s, p be respectively the strongest convex and product topologies on X. Then p is strictly weaker than s, for the only s-bounded subsets of X are finite dimensional (see Appendix 1), but this is clearly not true of all the p-bounded subsets. Let $B = \{(X,s)\}$, then $C_i \in \mathcal{C}(B)$ for all i, but $\prod_{i \in I} C_i \notin \mathcal{C}(B)$.

The closed graph theorems of this chapter are very pleasing, however one can obviously not expect too many of such general results. We have obtained our results by fixing a collection B of range spaces and then looking at the collection of all corresponding domain spaces. The opposite technique, of fixing a collection of domain

spaces and examining the corresponding range spaces was used by Komura [10]. Powell [13] has expanded his result, and we give a short discussion of Powell's paper in Chapter 6.

We shall now prove a lemma to be found in Robertson and Robertson [15] p 114 Lemma 5, and from it we shall derive a useful open mapping theorem.

LEMMA 2.12. *Let X and Y be convex Hausdorff spaces and T a linear mapping of X into Y. Then the graph of T is closed if and only if $(T')^{-1}(X')$ is dense in Y' under the topology $W(Y', Y)$.*

Proof.

$$\begin{array}{ccc} X & & X^* \\ T\downarrow & & \downarrow T' \\ Y & & Y' \end{array}$$

Polars will be taken in X, Y, X^* or Y'. Let G be the graph of T. Let $\mathcal{U}$ be a base of absolutely convex neighbourhoods for X. We shall first prove the following three results:

(i) $((T')^{-1}(X'))^\circ = \bigcap_{U \in \mathcal{U}} \overline{(T(U))}$;

(ii) $(T')^{-1}(X')$ is dense in Y' inder the topology $W(Y', Y)$ if and only if $((T')^{-1}(X'))^\circ = \{0\}$;

(iii) $y \in \bigcap_{U \in \mathcal{U}} \overline{T(U)}$ if and only if $(0,y) \in \bar{G}$.

(i) $X' = \bigcup_{U \in \mathcal{U}} U^\circ$, hence

$$(T')^{-1}(X') = \bigcup_{U \in \mathcal{U}} (T')^{-1}(U^\circ) = \bigcup_{U \in \mathcal{U}} (T(U))^\circ.$$

Thus $$((T')^{-1}(X'))^\circ = \bigcap_{U \in \mathcal{U}} (T(U))^{\circ\circ} = \bigcap_{U \in \mathcal{U}} \overline{T(U)}.$$

(ii) $(T')^{-1}(X')$ is dense in Y' under the topology $W(Y', Y)$ if and only if $((T')^{-1}(X'))^{\circ\circ} = Y'$ if and only if $((T')^{-1}(X'))^\circ = (Y')^\circ = \{0\}$.

(iii) Let $U \in \mathcal{U}$. $y \in \overline{T(U)}$ if and only if for every neighbourhood V in Y , there exists $x \in U$ with $T(x) \in y + V$. Thus $y \in \bigcap_{U \in \mathcal{U}} \overline{T(U)}$ if and only if for every neighbourhood V in Y and every $U \in \mathcal{U}$, the neighbourhood $(U, y + V)$ of $(0,y)$ meets G. Thus $y \in \bigcap_{U \in \mathcal{U}} \overline{T(U)}$ if and only if $(0,y) \in \bar{G}$.

Now suppose G is closed. By (iii) , $(0,y) \in G$ if and only

if $y \in \bigcap_{U \in \mathcal{U}} \overline{T(U)} = ((T')^{-1}(X'))^\circ$ by (i). But $(0,y) \in G$ if and only if $y = 0$. Hence $((T')^{-1}(X'))^\circ = \{0\}$ and so by (ii), $(T')^{-1}(X')$ is dense in Y' under the topology $W(Y', Y)$.

Now suppose that $(T')^{-1}(X')$ is dense in Y' under the $W(Y', Y)$ topology. Let $(x,y) \in \bar{G}$, then $(x, T(x)) \in \bar{G}$ and since $\bar{G}$ is a vector subspace of $X \times Y$, $(0, y - T(x)) \in \bar{G}$. By (iii) and (i), $y - T(x) \in ((T')^{-1}(X'))^\circ$. By (ii), $y = T(x)$ and the graph of G is closed.

COROLLARY. *If the graph of T is closed, then $T^{-1}(0)$ is closed.*

Proof. Let $M = (T')^{-1}(X')$. From the lemma. $M^\circ = \{0\}$. Hence $T^{-1}(0) = T^{-1}(M^\circ) = (T'(M))^\circ$. Since $(T'(M))^\circ$ is closed, so is $T^{-1}(0)$.

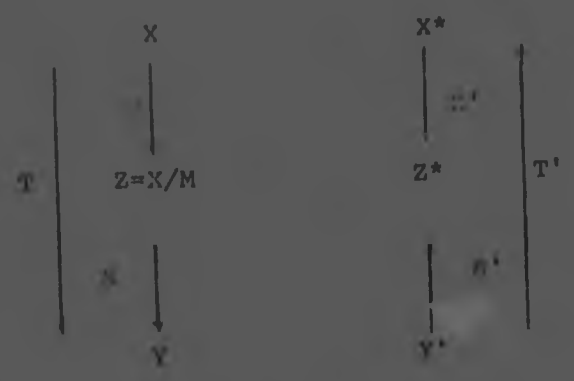
We say that the open mapping theorem holds for the pair (X,Y) of convex spaces if every closed graph linear mapping from X onto Y is open.

THEOREM 2.13. *Let X and Y be convex Hausdorff spaces. The following conditions are equivalent:*

(i) the open mapping theorem holds for the pair (X,Y) :

(ii) every closed graph one-to-one linear mapping from Y onto a Hausdorff quotient space of X is continuous.

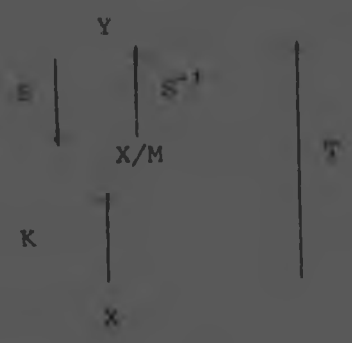
Proof. Suppose (ii) is true. Let T be a closed graph linear mapping from X onto Y. Let $M = \ker T$, then X/M is Hausdorff by the corollary to the previous lemma. Let $Z = X/M$ and $K : X \to Z$ the canonical mapping. Let $T = S \circ K$, where $S : Z \to Y$ is a one-to- one linear mapping onto Y. Let $K' : Z^* \to X^*$ be the transpose of K.



By Lemma 2.12, $(T')^{-1}(X')$ is dense in $(Y', W(Y', Y))$. Now $T' = K' \circ S'$, and so $(T')^{-1}(X') = (S')^{-1} \circ (K')^{-1}(X') = (S')^{-1}(Z')$ (see Appendix II). It follows that $(S')^{-1}(Z')$ is dense in

$(Y', W(Y', Y))$ and by Lemma 2.9, the graph of S and hence the graph of S^{-1} is closed. By hypothesis, S^{-1} is continuous, S is an open mapping and since K has the same property, T is an open mapping.

Now suppose, (i) is true and let S be a closed graph one-to-one linear mapping from Y onto X/M, where M is a closed subspace of X. Let K be the canonical mapping from X onto X/M and let $T = S^{-1} \circ K$.



Since K is continuous and S^{-1} has a closed graph, by Lemma 2.1, T has a closed graph and is open by hypothesis. Every open set in X/M is of the form $K(U)$ where U is open in X. Since $T(U)$ is open and $T(U) = S^{-1} \circ K(U)$, it follows that S^{-1} is an open mapping and so S is continuous.

In practice, we use the following corollary to derive open mapping theorems from corresponding closed graph theorems.

COROLLARY. *If B is a class of convex Hausdorff spaces which is closed under the formation of Hausdorff quotient spaces, then the open mapping theorem holds for the pair (X,Y) if X [illegible] and $Y \in \mathcal{B}(B)$.*

CHAPTER THREE

A CLOSED GRAPH THEOREM FOR B_r-COMPLETE SPACES.

A subset A of the continuous dual X' of a convex space X is said to be *nearly closed* if $A \cap U^\circ$ is $W(X', X)$ closed for every neighbourhood U in X.

The convex Hausdorff space X is said to be: (i) *B-complete* (or fully complete) if every nearly closed subspace of X' is $W(X', X)$ closed; (ii) *B_r-complete* if every nearly closed $W(X', X)$-dense subspace of X' is $W(X', X)$ closed, and hence the whole of X'.

In [15] p 107 Corollary 2, it is shown that a convex Hausdorff space is complete if and only if every nearly closed hyperplane in X' is $W(X', X)$ closed. Thus every B-complete space is B_r-complete and every B_r-complete space is complete. We shall later give an example of a complete space which is not B_r-complete. To my knowledge, it is not known if there exist B_r-complete spaces which are not B-complete.

It is easily verified that a B-complete (B_r-complete) space X remains B-complete (B_r-complete) under any stronger topology of the dual pair (X, X').

A preliminary lemma is needed before we prove that every Fréchet space is B-complete.

LEMMA 3.1. *Let U be a closed absolutely convex neighbourhood in the convex Hausdorff space X, and let A be $W(X', X)$-closed absolutely convex subset of X'. Then $(A \cap U^\circ)^\circ \subset A^\circ + 2U$.*

Proof. $A^\circ \cup U \subset A^\circ + U \subset \overline{A^\circ + U}$. Also, the closure of the absolutely convex hull of $A^\circ \cup U$ is $(A^\circ \cup U)^{\circ\circ} = (A^\circ \cap U^\circ)^\circ$, and so $(A \cap U^\circ)^\circ \subset \overline{A^\circ + U}$. Now $\overline{A^\circ + U} \subset A^\circ + 2U$. To prove this, let $x \in \overline{A^\circ + U}$ and choose $y \in A^\circ$ and $u \in U$ such that $y + u \in x + U$. Then $x \in y + u + U \subset A^\circ + 2U$. The result now follows.

THEOREM 3.2. *Every Fréchet space is B-complete.*

Proof. Let X be a Fréchet space and M a nearly closed subspace of X'. We show firstly that if U is a closed absolutely convex neighbourhood in X, then $(M \cap U^\circ)^\circ \subset M^\circ + 4U$.

Let (U_n) be a base of closed absolutely convex neighbourhoods in X with $U_1 = U$ and $U_{n+1} \subset \frac{1}{2}U_n$ for every positive integer n. Let $A_n = M \cap U_n^\circ$ for each n. Now $A_{n+1} \cap U_n^\circ = M \cap U_{n+1}^\circ \cap U_n^\circ = M \cap U_n^\circ$ since $U_{n+1}^\circ \supset U_n^\circ$, and so $A_{n+1} \cap U_n^\circ = A_n$. By Lemma 3.1, $A_n^\circ \subset A_{n+1}^\circ + 2U_n$.

Let $a \in (M \cap U^\circ)^\circ = A_1^\circ$, then $a \in A_2^\circ + 2U_1$, and so there exists a point $x_1 \in 2U_1$ such that $a - x_1 \in A_2^\circ \subset A_3^\circ + 2U_2$. Choose a point $x_2 \in 2U_2$ such that $a - x_1 - x_2 \in A_3^\circ \subset A_4^\circ + 2U_3$. Continuing in this way, we obtain a sequence (x_n) with $x_n \in 2U_n$ and

$$a - \sum_{r=1}^{n} x_r \in A_{n+1}^\circ \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (I)$$

Now $\sum_{r=n}^{n+p} x_r \in 2U_n + 2U_{n+1} + \ldots + 2U_{n+p} \subset 2U_n + U_n + \frac{1}{2}U_n + \ldots\ldots + \frac{1}{2^{p-1}} U_n \subset 4U_n$.

Hence $\sum_{r=1}^{n} x_r$ is a Cauchy sequence. Since X is complete it converges to a point x_o in X. Now $\sum_{r=1}^{n} x_r \in 4U_1$ and since U_1 is closed, $x_o \in 4U_1$.

Let $y \in M$. Then $y \in A_n = M \cap U_n^\circ$ for all sufficiently large n since $X' = \bigcup_{n=1}^{\infty} U_n^\circ$ and $U_{n+1} \subset \frac{1}{2}U_n$ for all n. It follows from this and (I) that $|(a - x_o, y)| = \lim_{n\to\infty} |(a - \sum_{r=1}^{n} x_r, y)| \leq 1$.

Hence $a - x_o \in M^\circ$, and so $a \in 4U + M^\circ$. Hence $(M \cap U^\circ)^\circ \subset M^\circ + 4U$.

Let $V = \frac{1}{4}U$, then $(M \cap V^\circ)^\circ \subset M^\circ + U$. It follows that $(M^\circ + U)^\circ \subset (M \cap V^\circ)^{\circ\circ}$, and since M is nearly closed,

$$(M^\circ + U)^\circ \subset M \cap V^\circ \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (II)$$

We now show that $(M^\circ + U)^\circ = (M^\circ \cup U)^\circ$. It is clear that $(M^\circ + U)^\circ \subset (M^\circ \cup U)^\circ$. Now M is a subspace of X' and so $M^\circ = \{x \in X : (x,y) = 0 \text{ for all } y \in M\}$. It follows that M° is a subspace of X, and so $M^{\circ\circ} = \{y \in X' : (x,y) = 0 \text{ for all } x \in M^\circ\}$. Let $y \in (M^\circ \cup U)^\circ = M^{\circ\circ} \cap U^\circ$. Now $(M^\circ + U)^\circ = \{z \in X' : |(x,z)| \leq 1$ for all $x \in M^\circ + U\}$. Let $x \in M^\circ + U$, $x = n + u$ say where $n \in M^\circ$, $n \in U$. Then $|(x,y)| \leq |(n,y)| + |(u,y)| \leq 0 + 1$. It follows that $y \in (M^\circ + U)^\circ$. Thus $(M^\circ + U)^\circ = (M^\circ \cup U)^\circ$ . . . . . (III)

From (II) and (III) it follows that $M^{\circ\circ} \cap U^\circ = (M^\circ \cup U)^\circ \subset M \cap V^\circ \subset M$.

Let $y \in M^{\circ\circ}$. Since $X' = \bigcup_{U \in \mathcal{U}} U^\circ$ where $\mathcal{U}$ is a base of closed absolutely convex neighbourhoods for X, there exists an absolutely convex neighbourhood W such that $y \in W^\circ$. By the above proof, $M^{\circ\circ} \cap W^\circ \subset M$. Thus $y \in M$, M is closed and X is fully complete.

The following definitions and proofs lead up to the closed graph theorem of this section.

Let X and Y be convex spaces and $T : X \to Y$ a linear mapping. T is said to be *nearly continuous* if $\overline{T^{-1}(V)}$ is a neighbourhood in X for every neighbourhood V in Y, and *nearly open* if $\overline{T(U)}$ is a neighbourhood in Y for every neighbourhood U in X. Clearly, if T is one-to-one and maps X onto Y, then T is nearly continuous if and only if T is nearly open. It is easily verified that if Y is barrelled and $T(X) = Y$ then T is nearly open and that if X is barrelled, T is nearly continuous. Let (X,τ) be barrelled and let σ be a convex topology for X which is strictly stronger than τ (for example, let (X,τ) be an infinite dimensional Banach space, and let $\sigma = \tau(X,X^*)$, then $\sigma \neq \tau$, as (X,σ) is not normable (see Appendix I). The identity mapping $I : (X,\tau) \to (X,\sigma)$ is continuous and nearly open. However it is not open, for it is not a topological isomorphism. The inverse of the above mapping provides an example of a mapping which is nearly continuous but not continuous.

LEMMA 3.3. *Let X and Y be convex Hausdorff spaces, T a nearly continuous linear mapping of X into Y and M a nearly closed subspace of X'. Then $(T')^{-1}(M)$ is nearly closed.*

Proof.

$$\begin{array}{ccc} X & & X^* \\ \Big| T & & \Big| T' \\ Y & & Y' \end{array}$$

Let V be a neighbourhood in Y. We must show that $(T')^{-1}(M) \cap V^\circ$ is $W(Y', Y)$ closed. $\overline{T^{-1}(V)}$ is a neighbourhood in X. $(\overline{T^{-1}(V)})^\circ = (T^{-1}(V))^\circ$ and $(T^{-1}(V))^\circ \cap M$ is $W(X', X)$ closed, since M is nearly closed. $(T^{-1}(V))^\circ \cap M$ is also $W(X^*, X)$ closed (see Appendix III) and since T' is $W(Y', Y) - W(X^*, X)$ continuous, $[T'(T^{-1}(V))^\circ \cap M]$ is $W(Y', Y)$ closed. Now $(T')^{-1}[(T^{-1}(V))^\circ \cap (T(T^{-1}(V)))^\circ \cap (T')^{-1}(M)$

$$= (V \cap T(X))^\circ \cap (T')^{-1}(M).$$

Also, $(T')^{-1}(M) \cap V^\circ = (T')^{-1}(M) \cap V^\circ \cap (V \cap T(X))^\circ$ which is an intersection of $W(Y', Y)$ closed sets. Hence $(T')^{-1}(M) \cap V^\circ$ is $W(Y', Y)$ closed.

THEOREM 3.4. *Let X and Y be convex Hausdorff spaces and let T be a nearly continuous closed graph linear mapping from X into Y.*

If Y is B_r-complete then T is continuous.

Proof. We first show that T is $W(X , X') - W(Y , Y')$ continuous by showing that $T'(Y') \subset X'$. X' is a nearly closed subspace of X' and so by Lemma 3.3, $(T')^{-1}(X')$ is a nearly closed subspace of Y'. By Lemma 2.12, $(T')^{-1}(X')$ is $W(Y' , Y)$-dense in Y'. Since Y is B_r-complete, $(T')^{-1}(X') = Y'$ and so $T'(y') \subset X'$.

Let V be a closed absolutely convex neighbourhood in Y. Then V is also $W(X , X')$ closed and so $T^{-1}(V)$ is $W(X , X')$ closed and hence closed. $\overline{T^{-1}(V)}$ is a neighbourhood in X, as T is nearly continuous, but $\overline{T^{-1}(V)} = T^{-1}(V)$ and so T is continuous.

The closed graph theorem for B_r-complete spaces is an immediate consequence of the above theorem.

THEOREM 3.5. *(Closed graph theorem for B_r-complete spaces.) If X and Y are convex spaces with X barrelled and Y B_r-complete, then $(X,Y) \in \mathcal{C}$.*

Proof. $X(H)$ is barrelled (see [15] p 81 Proposition 6), and so by Theorem 2.3, we may suppose that X is Hausdorff. Let $T : X \to Y$ be a closed graph linear mapping. Let V be an absolutely convex neighbourhood in Y. $\overline{T^{-1}(V)}$ is closed, absolutely convex and absorbent. Since X is barrelled, $\overline{T^{-1}(V)}$ is a neighbourhood in X and so T is nearly continuous. By the previous theorem, T is continuous.

COROLLARY. *A B_r-complete space cannot have a strictly weaker barrelled Hausdorff topology.*

Proof. We use Note 2.1 to prove this result.

We may now give an example of a complete convex space which is not B_r-complete. Let (X,τ) be an infinite dimensional Banach space. Under its strongest convex topology $\tau(X , X^*)$, X is barrelled, Hausdorff and complete but not normable (see Appendix I). Thus $\tau \neq \tau(X , X^*)$. By the above corollary $(X , \tau(X , X^*))$ is not B_r-complete.

We now show that Theorem 3.5 is a generalization in two senses of Banach's closed graph theorem. Every Frechet space is barrelled (see [15] p 61 Theorem 2) and B-complete (Theroem 3.?). In the

previous paragraph, we gave an example of a convex Hausdorff space which is barrelled but not normable. It remains to give an example of a B-complete space which is not a Fréchet space. Let $X = \prod_{i \in I} C_i$, with I uncountable. Then X is not metrisable and so X is not a Fréchet space. The continuous dual of X is $X' = \bigoplus_{i \in I} C_i$. On X' its strongest convex topology, $\tau(X', X'^*)$, the dual of X' is X', and every subspace of X' is closed (see Appendix I). Thus, every subspace of X' is $W(X', X)$-closed, and X is B-complete.

We proceed now to show that every Hausdorff quotient space of a B-complete space is B-complete. This will enable us to derive an open mapping theorem for B-complete spaces. Some preliminary results are needed.

Note: If A is a neighbourhood in the convex Hausdorff space X, then the polar of A in X' is the same as the polar of A in X*.

LEMMA 3.6. *Let X and Y be convex Hausdorff spaces, $T : X \to Y$ a nearly open linear mapping and M a nearly closed subspace of Y'. Then $T'(M) \cap X'$ is nearly closed in X'.*

Proof. Polars will be taken in X, X*, Y and Y'.

$$\begin{array}{ccc} X & & X^* \\ \downarrow T & & \downarrow T' \\ Y & & Y' \end{array}$$

Let U be a neighbourhood in X, then $U^\circ \cap X' = U^\circ$ by the above note, and so $T'(M) \cap X' \cap U^\circ = T'(M) \cap U^\circ$. We shall show that $T'(M) \cap U^\circ$ is $W(X', X)$ closed. It is easy to show that $T'(M) \cap U^\circ = T'(M \cap (T')^{-1}(U^\circ)) = T'(M \cap (T(U))^\circ)$. $\overline{T(U)}$ is a neighbourhood in Y, and so $(\overline{T(U)})^\circ = (T(U))^\circ$ is $W(Y', Y)$ compact. Since M is nearly closed, $M \cap T((U))^\circ$ is $W(Y', Y)$ closed and compact. T' is $W(Y', Y) - W(X^*, X)$ continuous and so $T'(M \cap (T(U))^\circ)$ is $W(X^*, X)$ compact. Since the $W(X^*, X)$ topology on X^* induces the $W(X', X)$ topology on X', (see Appendix III), and since $T'(M) \cap U^\circ \subset X'$, it follows that $T'(M) \cap U^\circ$ is $W(X', X)$ compact and closed. Thus $T'(M) \cap X'$ is nearly closed in X'.

THEOREM 3.7. *If there is a continuous nearly open mapping of a B-complete space onto a convex Hausdorff space Y, then Y is B-complete.*

Proof. Let T be a continuous nearly open linear mapping of the B-complete space X onto the convex Hausdorff space Y



Since T is continuous, $T'(Y') \subset X'$ and T' is weakly continuous. Let M be a nearly closed subspace of Y'. By Lemma 3.6, $T'(M)$ is nearly closed in X'. Since X is B-complete, $T'(M)$ is weakly closed. Also, $T(X) = Y$, and so T' is one-to-one (see Appendix IV). Thus $(T')^{-1}(T'(M)) = M$, and the weak continuity of T' implies that M is $W(Y', Y)$ closed.

COROLLARY 1. *The quotient of a B-complete space by a closed subspace is B-complete.*

Proof. Let M be a closed subspace of the B-complete space X. The canonical map $K : X \to X/M$ is open and continuous and so X/M is B-complete.

COROLLARY 2. *If a barrelled Hausdorff space is the continuous image of a B-complete space, it is B-complete.*

COROLLARY 3. (*Open mapping theorem for B-complete spaces.*) *If X is barrelled Hausdorff and Y is B-complete, the open mapping theorem holds for the pair (Y,X).*

Proof. This follows from the corollary to Theorem 2.13.

To my knowledge it is not known if every Hausdorff quotient space of a B_r-complete space is B_r complete and so we cannot prove an open mapping theorem for B_r-complete spaces. Strangely enough, this problem is related to the problem of whether every B_r-complete space is B-complete. We shall prove that the following two assertions are equivalent:

(i) every B_r-complete space is B-complete;

(ii) every Hausdorff quotient space of a B_r-complete space is B_r-complete.

This result is given as an example by Schaefer in [16] p 198 Ex. 29(c). Before proceeding we remark that we shall use results on duals of subspaces and quotient spaces given in the appendix sections II and III.

Suppose (ii) is true. Let X be a B_r-complete convex space. Let M be a nearly closed subspace of X'. Let $Y = X/M^\circ$ under the quotient topology. Then Y is B_r-complete, and its continuous dual is $M^{\circ\circ} = \bar{M}$, the $W(X', X)$ closure of M in X'. We endow X' with the topology $W(X', X)$ and Y' with the topology it inherits as a subspace of X', viz. $W(Y', Y)$. Thus M is dense in Y'. Let $K : X \to Y$ be the canonical mapping, then $K' : Y' \to X'$ is the inclusion map I.

$$\begin{array}{ccc} X & & X' \\ K \downarrow & & \downarrow K' = I \\ Y=X/M^\circ & & Y'=M^{\circ\circ} \end{array}$$

Polars taken in Y or Y' will be denoted by the symbol $^\blacktriangle$. We shall show that M is nearly closed in Y'. Let $K(U)$ be a neighbourhood in Y where U is a neighbourhood in X. $(K(U))^\blacktriangle = (K')^{-1}(U^\circ) = U^\circ \cap Y'$, and so $(K(U))^\blacktriangle \cap M = U^\circ \cap M$, which is $W(X', X)$-closed by hypothesis. Thus $(K(U))^\blacktriangle \cap M$ is $W(Y', Y)$ closed. Since Y is B_r-complete, $M = Y' = M^{\circ\circ}$. Thus M is closed in X', and X is B-complete.

The required reverse implication follows from Theorem 3.7 Corollary 1.

Let B be the class of B_r-complete spaces. We have shown that $\mathcal{C}(B)$ includes the barrelled spaces. It remains to prove the reverse inclusion, that a convex Hausdorff space is in $\mathcal{C}(B)$ only if it is barrelled.

Note. If A is an absolutely convex absorbent subset of the vector space X and p is the gauge of A then, by [15] p 13, Proposition 6 $\{x \in X : p(x) < 1\} \subset A \subset \{x \in X : p(x) \leq 1\}$.

We shall show that if X is a topological vector space, $\{x : p(x) \leq 1\} \subset \bar{A}$. We need only show that if $p(x) = 1$ then $x \in \bar{A}$. Now $\inf\{\lambda > 0 : x \in \lambda A\} = 1$. Since A is absolutely convex, $x \in \lambda A$ for all $\lambda > 1$. For each n choose $a_n \in A$ with $x = \frac{n+1}{n} a_n$. Then $a_n = \frac{n}{n+1} x \to x$ as $n \to \infty$. Hence $x \in \bar{A}$.

It follows that, if A is closed, $A = \{x \in X : p(x) \leq 1\}$. (It is interesting to note that this last result holds for the

closure of A in any vector topology for X.)

THEOREM 3.10. *If X is a convex space such that* $(X,Y) \in \mathcal{C}$ *for every Banach space* Y, *then* X *is barrelled.*

Proof. Let B be a barrel in X, then $p = g(B)$ is a semi-norm on X. Let $M = \bigcap_{\varepsilon > 0} \varepsilon B$, then clearly $M = \ker p$, and M is a closed subspace of X. Let $K : X \to X/M$ be the canonical mapping. Since K(B) is absolutely convex and absorbent in X/M, $r = g(K(B))$ is a semi-norm on X/M. Suppose $r(x + M) = 0$ for some $x + M \in X/M$. Let $\delta > 0$. We shall show that $x \in \delta B$ and then we can deduce that $x \in M$. $r(x + M) = 0 = \inf\{\lambda \geq 0 : x+M \in \lambda K(B)\}$. There exists $n \in M$, $b \in B$ and $\lambda < \frac{\delta}{2}$ such that $x = \lambda b + n$. Hence $x \in \frac{\delta}{2} B - \frac{\delta}{2} B \subset \delta B$, since B is absolutely convex. Thus r is a norm on X/M. Let q and ν denote respectively the quotient and norm topologies on X/M. We wish to show that K(B) is the closed unit ball in $(X/M, \nu)$. We must therefore prove that $K(B) = \{x + M \in X/M : r(x + M \leq 1\}$, which, by the note above is equal to K(B), if K(B) is closed in any vector topology on X/M. B is ν-closed in X, and we shall show that B is q-closed in X/M by showing that it is a union of cosets of M in X. Let $x \in B$, $n \in M$. Then $p(x + n) \leq p(x) + 0 \leq 1$ and so $x + n \in \bar{B} = B$, by the above note. Thus $x \in B$ implies that $x + M \subset B$ and so B is a union of cosets of M in X.

Let Y be the completion of $(X/M, \nu)$, then $V = \mathrm{cl}_Y(K(B))$ is the closed unit ball in the Banach space Y and $K(B) = V \cap X/M$ (see [15] p 108 remarks following Corollary 2).

We can regard K as a mapping from X into the Banach space Y. We notice that $B = K^{-1}(K(B)) = K^{-1}(V)$, since B is a union of cosets of N in X. If we can show that K is continuous as a mapping from X into Y, then we can deduce that B is a neighbourhood in X. By hypothesis, it is sufficient to show that the graph G of K is closed in $X \times Y$.

Let $(x,y) \in \bar{G}$, the closure of G in $X \times Y$. We need to show that $y = K(x)$. X/M is dense in Y. Let ε be any positive number. Choose $z \in X$ with $K(z) \in y - K(x) + \varepsilon V$ . . . . . (1) Let U be an absolutely convex neighbourhood in X. Then $(x + U, y + \varepsilon V)$ meets G. Choose $t \in x + U$ with $K(t) \in y + \varepsilon V$, then $y \in K(t) + \varepsilon V$ and $K(x) \in K(t) + K(U)$ and so $y - K(x) \in K(U) + \varepsilon V$.

From (i) , $K(z) \in K(U) + 2\varepsilon V$, and so $K(z) \in K(U) + 2\varepsilon V \cap X/M = K(U) + 2\varepsilon K(B) = 2\varepsilon K(\frac{1}{2\varepsilon} U + B)$. Now $\frac{1}{2\varepsilon} U + B$ is a union of cosets of M in X and so $z \in U + 2\varepsilon B$. From this it follows that $z \in 2\varepsilon \overline{B} = 2\varepsilon B$. From (i), $y - K(x) \in K(2\varepsilon B) + \varepsilon V \subset 3\varepsilon V$. Since this is true for every positive ε , $y = K(x)$, the graph of K is closed in $X \times Y$ and our proof is complete.

We have shown that if B is any subclass of the class of B_r-complete spaces which includes the Banach spaces, then $\mathcal{C}(B)$ is the collection of barrelled spaces. The results of Chapter two tell us nothing new in this case, as it is well known that an inductive limit and a product of barrelled spaces is barrelled and that a subspace of finite codimension of a barrelled space is barrelled. However, the results of this chapter and of Chapter two constitute proofs of the above facts. Also, if B is a proper subclass of the class of Banach spaces, $\mathcal{C}(B)$ may be larger than the class of barrelled spaces, and the theorems of Chapter two help to characterise it.

We can now give the example promised in Chapter two in the remarks following Theorem 2.6. We wish to show that if B is the class of B_r-complete spaces, then $\mathcal{C}(B)$ is not closed under the formation of closed subspaces. To do this, it is sufficient to find a barrelled space with a closed subspace which is not barrelled. Every convex Hausdorff space is topologically isomorphic to a subspace of a product of Banach spaces (see [15] p 88 Proposition 19 Corollary). A product of Banach spaces is complete, barrelled and Hausdorff. It suffices, then, to find a complete Hausdorff convex space which is not barrelled. The sequence space ℓ^∞ under the topology $\tau(\ell^\infty , \ell)$ is complete, by [15](p 104 Proposition 1 Corollary 2), but it is not barrelled, as $\tau(\ell^\infty , \ell) \neq s(\ell^\infty , \ell)$. This example is due to Köthe [11] p 368(5).

CHAPTER FOUR

A CLOSED GRAPH THEOREM FOR CONVEX SPACES WITH $\mathcal{C}$ WEBS

Let X be a vector space. Suppose that to each finite sequence $(n_1, n_2, n_3, \ldots, n_K)$ of natural numbers there corresponds a subset $A_{n_1 n_2 \ldots n_K}$ of X and that these indexed subsets are such that

$$X = \bigcup_{n_1 \in N} A_{n_1}, \quad A_{n_1} = \bigcup_{n_2 \in N} A_{n_1 n_2}, \quad \ldots \quad A_{n_1 n_2 \ldots n_{K-1}} = \bigcup_{n_K \in N} A_{n_1 n_2 \ldots n_K}.$$

Any collection of subsets of X indexed by the finite sequences of natural numbers and which satisfies the above conditions is called a *web* on X. If the sets of the web are balanced (convex, absolutely convex), we say that the web is balanced (convex, absolutely convex).

Let $\mathcal{W}$ be a web on a vector space X with elements $A_{n_1 n_2 \ldots n_K}$. The following definitions of a "strand" of a web and of a "compatible" web are due to Robertson and Robertson [15] p 155-156. If $(n_1, n_2, n_3 \ldots)$ is any infinite sequence of natural numbers, the sequence $(A_{n_1}, A_{n_1 n_2}, A_{n_1 n_2 n_3}, \ldots)$ is called a *strand* of the web $\mathcal{W}$. Each term in a strand is a subset of its predecessor. Whenever we are dealing with one strand at a time, we often denote the Kth term of the strand by a symbol such as S_K. If (X,τ) is a topological vector space, we say that $\mathcal{W}$ is *compatible* with τ if for every neighbourhood U in X, and every strand (S_K) of $\mathcal{W}$, there exists an integer n and a positive number λ such that $\lambda S_n \subset U$.

Using the above definition of a strand we now give the following definition of a $\mathcal{C}$ web which is equivalent to de Wilde's definition in [4]. If the topological vector space (X,τ) has a web $\mathcal{W}$, we say that $\mathcal{W}$ is a $\mathcal{C}$ web if to every strand (S_K) of $\mathcal{W}$ there corresponds a sequence (λ_K) of strictly positive numbers such that if $|\rho_K| \leq \lambda_K$ and $x_K \in S_K$ for each $K \in N$, then the sequence $\sum_{K=1}^{n} \rho_K x_K$ is convergent. The sequence (λ_K) is clearly not unique.

In particular, if (μ_K) is any other sequence of positive numbers such that $0 < \mu_K \leq \lambda_K$ for each $K \in N$, then

$\sum_{K=1}^{n} \rho_K x_K$ is convergent if $|\rho_K| \leq \mu_K$ and $x_K \in S_K$ for each K.

From this, we see that, if (S_K) is a strand of $\mathcal{W}$ we can choose a decreasing sequence (λ_K) is positive numbers such that if $x_K \in S_K$ and $|\rho_K| \leq \lambda_K$ for each K, the sequence

$\sum_{K=1}^{n} \rho_K x_K$ is convergent. This leads to our definition of a "filament" of a $\mathcal{C}$ web. If (λ_K) is such a decreasing sequence of positive numbers, we say that the sequence $(F_K) = (\lambda_K S_K)$ of subsets of X is a *filament* of the web $\mathcal{W}$ corresponding to the strand (S_K). We have then, that, if (F_K) is any filament of the web $\mathcal{W}$, the series $\sum_{K=1}^{n} \rho_K x_K$ is convergent if $x_K \in F_K$ and $|\rho_K| \leq 1$ for each K.

We now show that if the topological vector space has a web $\mathcal{W}$, we can construct a balanced web $\mathcal{V}$ on X such that, if $\mathcal{W}$ is compatible, so is $\mathcal{V}$ and if $\mathcal{W}$ is a $\mathcal{C}$ web, so is $\mathcal{V}$. For each set $A_{n_1 n_2 \ldots n_K}$ of $\mathcal{W}$, let $B_{n_1 n_2 \ldots n_K} = \{\lambda x : x \in A_{n_1 n_2 \ldots n_K} \text{ and } |\lambda| \leq 1\}$.

It is easily verified that the sets $B_{n_1 n_2 \ldots n_K}$ constitute a balanced web on X. If $\mathcal{W}$ is compatible, so is $\mathcal{V}$, since X has a base of balanced neighbourhoods. It is clear that if $\mathcal{W}$ is a $\mathcal{C}$ web so is $\mathcal{V}$. From now on, we shall assume that all webs are balanced.

We now have, that if X is a topological vector space with a $\mathcal{C}$ web $\mathcal{W}$ and (F_K) is any filament of $\mathcal{W}$, then

$\sum_{K=1}^{n} x_K$ is convergent if $x_K \in F_K$ for all $K \in N$.

Conversely, if X is a topological vector space with a web $\mathcal{W}$, and to every strand (S_K) of $\mathcal{W}$ there corresponds a filament $(F_K) = (\lambda_K S_K)$, with $\lambda_K > 0$ each K, such that

$\sum_{K=1}^{n} x_K$ is convergent when $x_K \in F_K$, then $\mathcal{W}$ is a $\mathcal{C}$ web.

This idea of a filament of a web enables us to give neater proofs of most of the theorems concerning $\mathcal{C}$-webbed spaces.

If $\mathcal{W}$ is a $\mathcal{C}$ web for (X,τ) and (S_K) is a strand with

corresponding filament $(F_K) = (\lambda_K S_K)$, since the sets S_K are balanced and (λ_K) is a decreasing sequence of positive numbers, and (S_K) is a decreasing sequence of subsets of X, it follows that the filament (F_K) is a decreasing sequence of subsets of X.

If the topological vector space (X,τ) has a $\mathcal{C}$ web, then for every neighbourhood U and every filament (F_K), $F_K \subset U$ for some K. Suppose this is not the case. Choose a neighbourhood U and a filament (F_K) such that $F_K \not\subset U$ for each $K \in N$. Choose $x_K \in F_K \setminus U$, each K. Then $\sum_{K=1}^{n} x_K$ is convergent, but $x_K \not\to 0$, which is a contradiction. From this, we see that a $\mathcal{C}$ web for (X,τ) is compatible with τ.

Let us recapitulate before proceeding further. A topological vector space X has a web $\mathcal{W}$ if and only if to each finite sequence $(n_1 n_2, \ldots, n_K)$ of natural numbers, there corresponds a balanced subset $A_{n_1 n_2 \ldots n_K}$ of X such that $X = \bigcup_{n_1 \in N} A_{n_1}$, ...

$A_{n_1 n_2 \ldots n_{K-1}} = \bigcup_{n_K \in N} A_{n_1 n_2 \ldots n_K}$. The sets $A_{n_1 n_2 \ldots n_K}$ constitute the web $\mathcal{W}$. If $(n_1, n_2, n_3, \ldots)$ is any infinite sequence of natural numbers, the sequence $(A_{n_1}, A_{n_1 n_2}, A_{n_1 n_2 n_3}, \ldots)$ is called a strand of $\mathcal{W}$. $\mathcal{W}$ is a $\mathcal{C}$ web if and only if to every strand $(S_1, S_2, S_3, \ldots)$ there corresponds a filament $(F_1, F_2, F_3, \ldots)$ of $\mathcal{W}$, where $F_K = \lambda_K S_K$ for each $K \in N$ and (λ_K) is a decreasing sequence of strictly positive real numbers, and if $x_K \in F_K$ for each K, the sequence $\sum_{K=1}^{n} x_K$ converges.

We have shown that every filament of a web $\mathcal{W}$ is a decreasing sequence of balanced subsets of X, and if U is any neighbourhood in X and (F_K) any filament, then there exists $r \in N$ such that $F_r \subset U$.

For our next theorem, we need the following easily proved result: if A is a subset of a vector space and $\lambda, \mu \in C$, then $\lambda A + \mu A \subset (|\lambda| + |\mu|)h(A)$.

THEOREM 4.1. *If X is a sequentially complete convex space with a compatible web $\mathcal{W}$ then $\mathcal{W}$ is a $\mathcal{C}$ web.*

Proof. Let (S_K) be a strand of $\mathcal{W}$ and let $F_K = 2^{-k} S_K$ for each K. Choose $x_K \in F_K$ for each K, and let

$T_n = \sum_{K=1}^{n} x_K$. Let U be an absolutely convex neighbourhood in X and choose $\lambda > 0$ and $r \in N$ such that $\lambda S_r \subset U$. Let $m > n \geq r$ and $2^{-n} < \lambda$. Then $T_m - T_n = x_{n+1} + \ldots + x_m$

$$\in 2^{-n-1}S_{n+1} + \ldots + 2^{-m}S_m$$

$$\subset 2^{-n-1}S_{n+1} + \ldots + 2^{-m}S_{n+1}$$

$$\subset (2^{-n-1} + \ldots + 2^{-m})h(S_{n+1}) \quad \text{by the note above}$$

$$\subset 2^{-n}h(S_{n+1}) \subset \lambda h(S_{n+1}) \subset U.$$

Thus (T_r) is a Cauchy sequence, and since X is sequentially complete, (S_n) converges and $\mathcal{W}$ is a $\mathcal{C}$ web.

It is clear that if $\mathcal{W}$ is a $\mathcal{C}$ web for (X,τ) and σ is any weaker vector topology for X, then $\mathcal{W}$ is a $\mathcal{C}$ web for (X,σ). The question as to what vector topologies on X stronger than τ have $\mathcal{C}$ webs has been answered completely by Powell [13] in the case when τ is Hausdorff. We shall prove Powell's result later in this chapter.

We now give two examples of convex spaces with $\mathcal{C}$ webs.

Every Fréchet space has a $\mathcal{C}$ web. We construct a web $\mathcal{W}$ for the Fréchet space X as follows. Let $\{U_n\}$ be a countable base of closed absolutely convex neighbourhoods for X with $U_{n+1} \subset U_n$ for each n. For each finite sequence $(n_1, n_2, \ldots, n_K)$ in N, let $A_{n_1 n_2 \ldots n_K} = \bigcap_{j=1}^{K} n_j U_j$. Then $X = \bigcup_{n_1=1}^{\infty} n_1 U_1$, since U_1 is absolutely convex and absorbent, and

$$\bigcup_{n_K=1}^{\infty} A_{n_1 n_2 \ldots n_K} = \bigcup_{n_K=1}^{\infty} \bigcap_{j=1}^{K} n_j U_j = \bigcap_{j=1}^{K-1} n_j U_j \cap \left(\bigcup_{n_K=1}^{\infty} n_K U_K \right) = A_{n_1 \ldots n_{K-1}} \cap X.$$

Thus the sets $A_{n_1 n_2 \ldots n_K}$ constitute a web $\mathcal{W}$ on X. A strand of $\mathcal{W}$ is of the form $(n_1 U_1, n_1 U_1 \cap n_2 U_2, n_1 U_1 \cap n_2 U_2 \cap n_3 U_3, \ldots)$ and so $\mathcal{W}$ is a compatible web. Since X is complete, $\mathcal{W}$ is a $\mathcal{C}$ web.

The strong dual, X', of a metrisable convex space X has a $\mathcal{C}$ web. Let $\{U_n : n \in N\}$ be a base of closed absolutely convex neighbourhoods for X. For each finite sequence $(n_1 n_2 \ldots n_K)$ of

natural numbers, let $A_{n_1 n_2 \dots n_K} = U^*_n$. The sets $A_{n_1 n_2 \dots n_K}$ form a web $\mathcal{W}$ in X', since $X' = \bigcup_{m=1} U^*_m$. Every strand of $\mathcal{W}$ is of the form $(U^*_m, U^\circ_m, U^\circ_m, \dots)$ for some $m \in N$. Each U°_m is strongly bounded, (this result is easily deduced from [15] p 71 Lemma 2) and so $\mathcal{W}$ is compatible with the topology $s(X', X)$. Since X' is complete under this topology, (see [15] p 104 Proposition 1 Corollary 1) $\mathcal{W}$ is a $\mathcal{C}$ web.

The following two lemmas are needed for the proof of the closed graph theorem for spaces with $\mathcal{C}$ webs.

LEMMA 4.2. *If X is a Baire topological vector space with a web $\mathcal{W}$, then $\mathcal{W}$ has strand (S_n) such that $\bar{S}_n$ has interior for each n.*

Proof. Let the sets of $\mathcal{W}$ be $A_{n_1 n_2 \dots n_K}$, where $n_1, n_2, \dots, n_K \in N$. Now $X = \bigcup_{n \in N} A_n$, hence for some $n_1 \in N$, A_{n_1} is not meagre. $A_{n_1} = \bigcup_{n \in N} A_{n_1 n}$, and so for some $n_2 \in N$, $A_{n_1 n_2}$ is not meagre. Proceeding in this way, we obtain the required result.

LEMMA 4.3. *Let X and Y be topological vector spaces and T a closed graph linear mapping from X [illegible]. If $y \in W + T(U)$ for every neighbourhood W in Y and every neighbourhood U in X, then $y = 0$.*

Proof. Let U, W be balanced neighbourhoods in X, Y respectively. $(U, y + W)$ is a neighbourhood of $(0,y)$ in $X \times Y$. Let $y = w + T(u)$, where $w \in W$, $u \in U$. Then $(u, T(u)) = (u, y-w) \in (U, y+W)$. Hence $(0,y) \in \bar{G} = G$, where G is the graph of T, and so $y = 0$.

We remark that if X and Y are vector spaces, $T : X \to Y$ a linear mapping and if $\mathcal{W}$ is a web in Y, then the inverse images under T of the elements of $\mathcal{W}$ form a web in X.

A further observation needed in the following theorem is that if X is a topological space and A a subset of X whose closure $\bar{A}$ has interior, then there exists an interior point of $\bar{A}$ with $a \in A$.

The proof of the following theorem has been adapted from de Wilde [3] p 376 Theorème.

THEOREM 4.4. (*Closed graph theorem for convex spaces with* $\mathcal{C}$ *webs.*) *Let* X *be a convex Baire space,* Y *a convex space with a* $\mathcal{C}$ *web and* T *a linear mapping from* X *into* Y. *Then* T *is continuous if*

(i) T *has a closed graph,*

or (ii) T *has a sequentially closed graph and* X *is metrisable.*

Proof. Let $\mathcal{W}$ be a $\mathcal{C}$ web in Y. The inverse images under T of the sets of $\mathcal{W}$ form a web in X. By Lemma 4.2, there is a strand (S_n) of $\mathcal{W}$ such that $\overline{T^{-1}(S_n)}$ has interior for each n. Let (F_n) be a filament corresponding to (S_n), then $\overline{T^{-1}(F_n)}$ has an interior point y_n with $y_n \in T^{-1}(F_n)$ for each n . . . . . (a)

Since F_n is balanced, $y_n + \overline{T^{-1}(F_n)}$ is a neighbourhood in X for each n.

For each neighbourhood U in X and each $n \in N$, $\overline{T^{-1}(F_n)} \subset T^{-1}(F_n) + U$, and so $T(\overline{T^{-1}(F_n)}) \subset F_n + T(U)$ . . . . . (b)

Let V be a closed absolutely convex neighbourhood in Y. Choose m so that $F_m \subset V$ . . . . . (c)

(i) Suppose that the graph of T is closed. We shall show that $T(y_m + \overline{T^{-1}(F_m)}) \subset 4V$, which will prove T continuous. From (b), $T(\overline{T^{-1}(F_n)}) \subset F_n + T(\tfrac{1}{2}y_{n+1} + \overline{T^{-1}(\tfrac{1}{2}F_{n+1})})$ for each n. Let $x_o \in \overline{T^{-1}(F_m)}$. Choose $x_1 \in \overline{T^{-1}(\tfrac{1}{2}F_{m+1})}$ such that $T(x_o) - T(\tfrac{1}{2}y_{m+1}) - T(x_1) \in F_m$. Now choose $x_2 \in \overline{T^{-1}(\tfrac{1}{4}F_{m+2})}$ such that $T(x_1) - T(\tfrac{1}{2}y_{m+2}) - T(x_2) \in \tfrac{1}{2}F_{m+1}$. Proceeding in this way, choose a sequence (x_r) such that $x_r \in \overline{T^{-1}(2^{-r}F_{m+r})}$ and $T(x_{r-1}) - T(2^{-r}y_{m+r}) - T(x_r) \in 2^{-r+1}F_{m+r-1}$ for each r . . . . . (d)

Now, for each $n \in N$,

$$T(x_o) - T(x_n) = \sum_{r=1}^{n} (T(x_{r-1}) - T(2^{-r}y_{m+r}) - T(x_r)) + \sum_{r=1}^{n} T(2^{-r}y_{m+r}) \quad . \quad . \quad (e).$$

(F_n) is a filament of $\mathcal{W}$ and so, using (a) and (d) we see that the right hand side converges to a point y in Y. We shall use Lemma 4.3 to show that $y = T(x_o)$. Let U and W be absolutely convex neighbourhoods in X, Y respectively. For each r, $x_r \in \overline{T^{-1}(2^{-r}F_{m+r})}$, and so $x_r \in 2^{-r}T^{-1}(F_{m+r}) + U$. Hence $T(x_r) \in 2^{-r}F_{m+1} + T(U) \subset \tfrac{1}{2}W + T(U)$ for all r sufficiently large, since (F_K) is a filament of the $\mathcal{C}$ web $\mathcal{W}$. Also, $T(x_o) - T(x_r) - y \in \tfrac{1}{2}W$ for all r sufficiently large. It follows that

$T(x_o) - y \in \frac{1}{2}W + \frac{1}{2}W + T(U) \subset W + T(U)$. By Lemma 4.3, $T(x_o) = y$.

Now, for every n, by (e), (d) and (a),

$$T(x_o) - T(x_n) \in F_m + 2^{-1}F_{m+1} + \ldots + 2^{-n+1}F_{m+n-1} + 2^{-1}F_{m+1} + \ldots + 2^{-n}F_{m+n}$$

$$\subset F_m + 2^{-1}F_m + \ldots + 2^{-n+1}F_m + 2^{-1}F_m + \ldots + 2^{-n}F_m$$

$$\subset 3h(F_m) \quad \text{by the note preceding Theorem 4.1}$$

$$\subset 3V \text{ , since } V \text{ is absolutely convex.}$$

Now $T(x_n) \to 0$, and so $T(x_o) \in 3\bar{V} = 3V$.

By (a) and (c), $T(y_m) \in V$. Since x_o is an arbitrary element of $\overline{T^{-1}(F_m)}$, it follows that $T(y_m - \overline{T^{-1}(F_m)}) \subset 4V$. Thus T is continuous.

(ii) Suppose now that T has a sequentially closed graph and X is metrisable. Choose a base (U_n) of absolutely convex neighbourhoods in X such that $U_n \subset y_n + \overline{T^{-1}(F_n)}$ for each n. We shall prove that $T(U_m) \subset 3V$. From (b), choosing $\frac{1}{2}U_{n+1}$ as U, $T(U_n) \subset T(y_n) + F_n + T(\frac{1}{2}U_{n+1})$ for each n.

Let $x_o \in U_m$. Using a similar procedure to that of part (i) of the proof, we obtain a sequence (x_r) such that $x_r \in 2^{-r}U_{m+r}$ and $T(x_{r-1}) - T(2^{-r-1}y_{m+r-1}) - T(x_r) \in 2^{-r-1}F_{m+r-1}$ for each r. For each $n \in N$,

$$T(x_o) - T(x_n) = \sum_{r=1}^{n} (T(x_{r-1}) - T(2^{-r-1}y_{m+r-1}) - T(x_r)) + \sum_{r=1}^{n} T(2^{-r-1}y_{m+r-1}).$$

The right hand side converges to a point $y \in Y$ and $x_n \to 0$, since (U_n) is a base of neighbourhoods for X. Since the graph of T is sequentially closed, $T(x_n) \to 0$ and so $T(x_o) = y$. The proof that $T(x_o) \in 3V$ proceeds in a similar fashion to that of the corresponding proof in part (i).

COROLLARY. *A convex Hausdorff space with a $\mathcal{C}$ web is not a Baire space under any strictly weaker convex Hausdorff topology.*

Proof. See note 2.1.

Let B be the class of convex spaces with $\mathcal{C}$ webs. The above theorem tells us that $\mathcal{B}(B)$ includes the convex Baire spaces and $\mathcal{B}_1(B)$ the metrisable convex Baire spaces. Let us apply the results of Chapter two to this case. By Theorem 2.4, we have that $\mathcal{C}(B)$

includes the inductive limits of Baire convex spaces and $\mathcal{b}_1(B)$ the inductive limits of metrisable Baire convex spaces, among these the ultra-bornological spaces. Theorem 2.4 has some substance in this case, for $X = \bigoplus_{i=1}^{\infty} C_i$ is an example of an inductive limit of metrisable Baire convex spaces which is not a Baire space, for, if $X_n = \bigoplus_{i=1}^{n} C_n$, then $X = \bigcup_{n=1}^{\infty} X_n$, and each X_n is nowhere dense in X. (We identify X_n and $\{x \in X : x_i = 0 \text{ for } i > n\}$.)

By Theorem 2.6, $\mathcal{b}(B)$ is closed under the formation of subspaces of finite codimension. I do not know if a subspace of finite codimension of a convex Baire space is also a Baire space.

By Theorem 2.11, $\mathcal{b}(B)$ ($\mathcal{b}_1(B)$) is closed under the formation of products of cardinality d if $\prod_{i \in I} C_i \in \mathcal{b}(B)$ ($\mathcal{b}_1(B)$) with cardinal $I = d$.

A cardinal d is said to be *strongly inaccessible* if (a) $d > \aleph_0$, (b) if $c = \sum_{\gamma} d_\gamma$ where each $d_\gamma < d$ and there are less than d summands in $\sum_{\gamma} d_\gamma$ then $c < d$, (c) if $f < d$ then $2^f < d$.

It is not known if any strongly inaccessible cardinals exist. The Mackey-Ulam theorem states that the product of d bornological spaces is bornological if d is smaller than the smallest strongly inaccessible cardinal. (A proof of this theorem may be found in [11] p 392.)

Let d be smaller than the least strongly inaccessible cardinal and let I be an index set with cardinal $I = d$. By the Mackey-Ulam theorem, $\prod_{i \in I} C_i$ is bornological and being complete and Hausdorff is ultra-bornological (see [15] p 83 Theorem 1, noting that Robertson and Robertson call a bornological space a Mackey space). Thus $\mathcal{b}_1(B)$ is closed under the formation of products of cardinality d.

We now ask if $\mathcal{b}(B)$ is closed ubder the formation of products. It is if $\prod_{i \in I} C_i \in \mathcal{b}(B)$ for every index set I. We give now a proof that every product of complete metric spaces is a Baire space, giving us an affirmative answer to the above question. This theorem is given as an exercise by Bourbaki in [2] p 254 Ex 17(a).

THEOREM 4.5. *Any product of complete metric spaces is a Baire space.*

Proof. Let X be the topological product $\prod_{i \in I} X_i$ where X_i is a complete metric space for each i in the index set I. A base of open sets for the topology on X is the collection of sets of the form $\prod_{i \in J} B_i(r) \times \prod_{i \notin J} X_i$, where J is a finite subset of I and for each $i \in J$, $B_i(r)$ is an open ball in X_i with radius r. We call such a set an open hyperball of radius r. The closure of an open hyperball is called a closed hyperball. We note that every open hyperball includes a closed hyperball.

Suppose there exists an open subset A of X such that $A = \bigcup_{n=1}^{\infty} A_n$ and A_n is nowhere dense for each n. $A \notin \bar{A}_1$, hence we can choose a closed hyperball H_1 of radius r_1 such that $H_1 \subset A$ and $H_1 \cap A_1 = \phi$. $H_1 \notin \bar{A}_2$, hence we can choose a closed hyperball H_2 of radius $r_2 < \frac{1}{2} r_1$ such that $H_2 \subset H_1$ and $H_2 \cap A_2 = \phi$. Proceeding thus, we obtain for each n a closed hyperball H_n with radius $r_n < \frac{1}{n} r_1$ such that $H_n \cap A_n = \phi$ and $H_{n+1} \subset H_n$. Each H_n is of the form $\prod_{i \in I} D_{(i,n)}$, where $D_{(i,n)}$ is a closed ball in X_i of radius $r_n < \frac{1}{n} r_1$ or $D_{(i,n)} = X_i$.

Now $\bigcap_{n=1}^{\infty} H_n = \prod_{i \in I} \bigcap_{n=1}^{\infty} D_{(i,n)}$. For each i, $\bigcap_{n=1}^{\infty} (D_{i,n}) = X_i$, or $\bigcap_{n=1}^{\infty} (D_{i,n})$ is the intersection of a decreasing sequence of closed balls in X_i, whose radii converge to zero. In this case, since each X_i is complete, $\bigcap_{n=1}^{\infty} D_{(i,n)}$ is a point x_i in X_i. Thus $\bigcap_{n=1}^{\infty} H_n$ is not zero. Let $f \in \bigcap_{n=1}^{\infty} H_n$, then $f \notin A_n$ for each n, hence $f \notin A$. But $f \in H_1 \subset A$, which is a contradiction. It follows that X is a Baire space.

We consider now stability properties of the class B of convex spaces with $\mathcal{C}$ webs. It is easily verified that if the convex space X has a $\mathcal{C}$ web, so does any sequentially closed subspace Z of X. The sets $Z \cap W$, with $W \in \mathcal{W}$ form a web $\mathcal{V}$ in Z. If (F_K) is a filament of $\mathcal{W}$ then $(F_K \cap Z)$ is a filament of $\mathcal{V}$. A continuous linear image $T(X)$ of a convex space X with a $\mathcal{C}$ web $\mathcal{W}$ also has

a $\mathcal{C}$ web. The sets $T(W)$ with $W \in \mathcal{W}$ form a web $\mathcal{V}$ in $T(X)$. If (F_K) is a filament of $\mathcal{W}$, then $(T(F_K))$ is a filament of $\mathcal{V}$.

Thus every quotient space of a convex space with a $\mathcal{C}$ web also has a $\mathcal{C}$ web. From this, and from Theorem 2.10 Corollary we may conclude that the open mapping theorem holds for the pair (X,Y) of convex Hausdorff spaces if X has a $\mathcal{C}$ web and Y is an inductive limit or a Hausdorff product of Baire convex spaces.

Note (i) If X is the projective (inductive) limit of the finite sequence $(X_1, X_2, \dots, X_n)$ of convex spaces under the linear mappings $T_r : X \to X_r$ $(T_r : X_r \to X)$, then X is also the projective (inductive) limit of the countable infinite sequence $(X_1, X_2, \dots)$ of convex spaces under the linear mappings $T_r : X \to X_r$ $(T_r : X_r \to X)$, where $X_r = X_n$ and $T_r = T_n$ for all $r \geq n$.

Before proceeding to consider countable projective and inductive limits of convex spaces with $\mathcal{C}$ webs, we introduce further notation. Let X be a convex space with a $\mathcal{C}$ web $\mathcal{W}$ and let $A_{n_1 n_2 \dots n_k}$ be the element of $\mathcal{W}$ corresponding to the finite sequence $(n_1, n_2 \dots n_k)$. Let $n = (n_1, n_2, n_3, \dots)$ be an infinite sequence of natural numbers. We call the strand $(A_{n_1}, A_{n_1 n_2}, A_{n_1 n_2 n_3}, \dots)$ the strand corresponding to n. If (F_K) is a filament corresponding to the above strand, we call (F_K) a filament corresponding to n. If $x_K \in F_K$ for each K, we say that (x_K) is a *point sequence* of the filament (F_K).

Note (ii) If A and B are balanced subsets of a vector space, and if $0 \leq \lambda \leq 1$, $0 \leq \mu \leq 1$, then $\lambda\mu(A \cap B) \subset \lambda A \cap \mu B$.

THEOREM 4.6. *Let X be the projective limit of the finite or countable sequence (X_r) of convex spaces under the mappings $T_r : X \to X_r$. Suppose that, whenever (x_λ) is a sequence in X such that $(T_r(x_\lambda))$ converges in X_r for each r, then (x_λ) converges in X. Then, if each X_r has a $\mathcal{C}$ web, so has X.*

Proof. Without loss of generality, we may suppose that the sequence (X_r) is infinite. For each r, let $\mathcal{V}^{(r)}$ be a $\mathcal{C}$ web for X_r, with element $A^{(r)}_{n_1 n_2 \dots n_k}$ corresponding to the finite sequence $(n_1, n_2, \dots, n_k)$ of natural numbers. We construct a

web $\mathcal{W}$ for X as follows. For each $n_1 \in N$, let $B_{n_1} = T_1^{-1}(A_{n_1}^{(1)})$.

Then $\bigcup_{n_1 \in N} B_{n_1} = T_1^{-1} \bigcup_{n_1 \in N} A_{n_1}^{(1)} = T_1^{-1}(X_1) = X$. Let

$B_{n_1 n_2} = T_1^{-1}(A_{n_1 n_2}^{(1)}) \cap T_2^{-1}(A_{n_2}^{(2)})$. Then

$$\bigcup_{n_1 \in N} B_{n_1 n_2} = T_1^{-1} \bigcup_{n_1 \in N} (A_{n_1 n_2}^{(1)}) \cap T_2^{-1} \bigcup_{n_2 \in N} A_{n_2}^{(2)}$$

$$= T_1^{-1}(A_{n_1}) \cap T_2^{-1}(X_2) = B_{n_1}.$$

In general, if $K > 1$, let

$$B_{n_1 n_2 \ldots n_K} = T_1^{-1}(A_{n_1 n_2 \ldots n_K}^{(1)}) \cap T_2^{-1}(A_{n_2 n_3 \ldots n_K}^{(2)}) \cap \ldots \cap T_K^{-1}(A_{n_K}^{(K)}) \quad \ldots \quad (a)$$

Then

$$\bigcup_{n_K \in N} B_{n_1 n_2 \ldots n_K} = T_1^{-1}(A_{n_1 \ldots n_{K-1}}^{(1)}) \cap T_2^{-1}(A_{n_2 \ldots n_{K-1}}^{(2)}) \cap \ldots \cap T_K^{-1}(X_K)$$

$$= B_{n_1 \ldots n_{K-1}}$$

Thus the sets $B_{n_1 n_2 \ldots n_K}$ constitute a web for X. Fix a sequence $n = (n_1, n_2, n_3, \ldots)$ of natural numbers. To show that $\mathcal{W}$ is a $\mathcal{C}$ web for X, we must find a suitable filament (F_K) of $\mathcal{W}$ corresponding to the above sequence. We shall choose (F_K) in such a way that if (x_K) is a point sequence of (F_K) then for each $r \in N$, $(T_r(x_K) : K = r, r+1, r+2, \ldots)$ is a point sequence of a filament of the web $\mathcal{W}^{(r)}$ of X_r. For each $r \in N$ let $(S_1^{(r)}, S_2^{(r)}, \ldots)$ be the strand of $\mathcal{W}^{(r)}$ corresponding to the sequence $(n_r, n_{r+1}, \ldots)$, and let $(F_1^{(r)}, F_2^{(r)}, \ldots)$ be a corresponding filament, with $F_K^{(r)} = \lambda_K^{(r)} S_K^{(r)}$, each K, and $0 < \lambda_K^{(r)} \leq 1$. Notice that, from (a), if (S_K) is the strand of $\mathcal{W}$ corresponding to the sequence n, then

$$S_K = T_1^{-1}(S_K^{(1)}) \cap T_2^{-1}(S_{K-1}^{(2)}) \cap \ldots \cap T_K^{-1}(S_1^{(K)}).$$

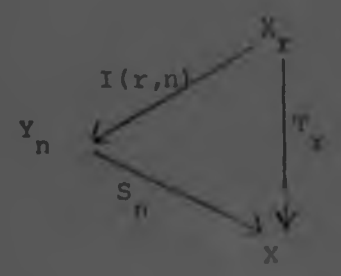
Now let $F_K = \lambda_K^{(1)} \lambda_{K-1}^{(2)} \ldots \lambda_1^{(K)} S_K$ for each $K \in N$. Then

$F_K \subset T_1^{-1}(F_K^{(1)}) \cap T_2^{-1}(F_{K-1}^{(2)}) \cap \ldots \cap T_K^{-1}(F_1^{(K)})$ by note (ii) above. It is clear now, that if (x_K) is a point sequence of the filament (F_K), then for each r, $(T_r(x_r), T_r(x_{r+1}), T_r(x_{r+2}), \ldots)$ is a point sequence of a filament of the web $\mathcal{W}^{(r)}$. Now $\sum_{i=1}^{n} T_r(x_{r+i})$ is

convergent and so is $\sum_{i=1}^{n} T_r(x_i)$ for each r. Using the hypothesis of the theorem, we see that $\mathcal{W}$ is a $\mathcal{C}$ web for X.

COROLLARY. *A finite or countable product of convex spaces with $\mathcal{C}$ webs has a $\mathcal{C}$ web, and so does a finite direct sum.*

Let (X,τ) be the inductive limit of the sequence (X_n) of convex spaces under the mappings $T_n : X_n \to X$, then X is spanned by $\bigcup_{n=1}^{\infty} T_n(X_n)$. We shall show that (X,τ) is also the inductive limit of a sequence (Y_n) of convex spaces under mappings $S_n : Y_n \to X$, with $X = \bigcup_{n=1}^{\infty} S_n(Y_n)$. For each n, let Y_n be the direct sum $\sum_{i=1}^{n} X_i$. Let $S_n : Y_n \to X$ be defined by $S_n = T_1 + T_2 + \ldots + T_n$. Then $X = \bigcup_{n=1}^{\infty} S_n(Y_n)$. Let (X,σ) be the inductive limit of the Y_n under the mappings S_n. Let $I(r,n) : X_r \to Y_n$ be the injection mapping for each n and $1 \leq r \leq n$. Then $T_r = S_n \circ I(r,n)$ in each case.



Each T_r is continuous when X has the topology τ, hence so is each S_n (see [15] p 79 Proposition 5). Thus τ is weaker than σ. Each T_r is clearly continuous when X has the topology σ and so σ is weaker than τ.

THEOREM 4.7. *An inductive limit of a sequence of convex spaces with $\mathcal{C}$ webs also has a $\mathcal{C}$ web.*

Proof. Let X be the inductive limit of the sequence (X_r) of convex spaces with $\mathcal{C}$ webs under the mappings $T_r : X_r \to X$. By the above remarks and by the Corollary to Theorem 4.6, we may assume that

$X = \bigcup_{r=1}^{\infty} T_r(X_r)$. For each r, let $\mathcal{W}^{(r)}$ be a $\mathcal{C}$ web for X_r, and let $A^{(r)}_{n_1 \dots n_K}$ be the element of $\mathcal{W}^{(r)}$ corresponding to the sequence $(n_1, \dots, n_K)$. We construct a web $\mathcal{W}$ for X as follows. For each $n_1 \in N$, let $A_{n_1} = T_{n_1}(X_{n_1})$. If $K > 1$, let $A_{n_1 n_2 \dots n_K} = T_{n_1}(A^{(n_1)}_{n_1 n_2 \dots n_K})$. It is easy to verify that the sets $A_{n_1 \dots n_K}$ constitute a web for X. Let S be a strand of $\mathcal{W}$, then for some r, $S = (T_r(X_r), T_r(S_1), T_r(S_2), \dots)$, where $(S_1, S_2, \dots)$ is a strand of $\mathcal{W}^{(r)}$. Let $(F_1, F_2, \dots)$ be a filament of $\mathcal{W}^{(r)}$ corresponding to the above strand. As a filament corresponding to S, we choose, $(T_r(X_r), T_r(F_1), T_r(F_2), \dots)$. With this choice of filaments for the strands of $\mathcal{W}$, it is clear that $\mathcal{W}$ is a $\mathcal{C}$ web, since each T_r is continuous, and $\mathcal{W}^{(r)}$ is a $\mathcal{C}$ web for X_r.

COROLLARY 1. *If each X_r has a $\mathcal{C}$ web, so does $\bigoplus_{r=1}^{\infty} X_r$.*

COROLLARY 2. *If (X,τ) is a convex space of countable dimension, then (X,τ) has a $\mathcal{C}$ web.*

Proof. Let σ be the strongest convex topology for X. Then $(X,\sigma) = \bigoplus_{i=1}^{\infty} C_i$, which has a $\mathcal{C}$ web by Corollary 1. (X,τ) also has a $\mathcal{C}$ web since τ is weaker than σ.

This last corollary provides us with an example of an incomplete convex space with a $\mathcal{C}$ web. Let $p > 1$, and let Z be the subspace of ℓ^p consisting of those sequences of complex numbers with only a finite number of non-zero terms. Then Z, under the topology inherited from ℓ^p, is not complete, but, being of countable dimension, does have a $\mathcal{C}$ web.

We continue our study of stability properties of the class of convex spaces with $\mathcal{C}$ webs. Let (X,τ) be a convex space. We shall show that there is a weakest ultra-bornological topology, τ^u, for X stronger than τ. We shall then prove an interesting result of Powell [13]. If the convex Hausdorff space (X,τ) has a $\mathcal{C}$ web, so does

(X,T^u) , and (X,σ) does not have a G web if σ is a convex topology for X strictly stronger than T^u.

Let α be a property of convex spaces. We say that (X,T) is an $\bar{\alpha}$ space if (X,T) is an inductive limit of α-spaces. If (X,T) is an α-space, it is an $\bar{\alpha}$ space. If (X,T) is an inductive limit of $\bar{\alpha}$ spaces, it is an $\bar{\alpha}$ space. To prove this last statement, let us suppose that (X,T) is the inductive limit of the $\bar{\alpha}$ spaces X_i under the mapping $T_i : X_i \to X$. Let each X_i be the inductive limit of the α spaces X_{ij} under the mappings $T_{ij} : X_{ij} \to X_i$. The maps $T_i \circ T_{ij} : X_{ij} \to X$ are continuous. Let σ be any other convex topology on X under which the maps $T_i \circ T_{ij}$ are continuous, then the maps T_i are also continuous (see [15] p 79 Proposition 5). Hence σ is weaker than T and so (X,T) is the inductive limit of the α spaces X_{ij}.

Let (X,T) be a convex space and let $\{T_i : i \in I\}$ be the family of all $\bar{\alpha}$ topologies on X which are stronger than T. Suppose this set is not empty. The intersection of the T_i is a convex topology σ on X which is stronger than T. If T is Hausdorff, so is σ. (X,σ) is the inductive limit of the (X,T_i) under the inclusion maps, hence σ is an $\bar{\alpha}$ topology. σ is the weakest $\bar{\alpha}$ topology on X which is stronger than T. We denote this topology by $T^{\bar{\alpha}}$.

If X is a vector space, then, under its strongest convex topology $\tau(X,X')$, X is barrelled, bornological and complete, hence ultra-bornological (see Appendix I). Thus, if α is the property "Banach", then $T^{\bar{\alpha}}$ exists for every convex topology T on X. We denote $T^{\bar{\alpha}}$ by T^u.

An inductive limit of bornological or barrelled spaces has the same property. From this and the above remarks we see that if (X,T) is any convex space, there exists for X a strongest bornological topology, T^b, which is stronger than T , and a weakest barrelled topology, T^t, stronger than T. It is clear that $s(X,X')$, T^X and T^t are each weaker than T^u.

If A is an absolutely convex bounded subset of the convex space (X,T) , and if X_A is the subspace of X spanned by A , then it is easy to show that $g(A)$, the gauge of A, is a norm on X_A inducing on X_A a topology n_A stronger than the topology T_A induced by T. If, in addition, A is closed and sequentially complete, then X_A is a Banach space under the norm $g(A)$. (See [11] p 252 (2).)

Let $\{A_\gamma : \gamma \in \Gamma\}$ be a collection of absolutely convex bounded subsets of (X,T) whose union spans X. Endow each X_{A_γ} with the

topology induced by the norm $g(A_\gamma)$ and let $I_\alpha : X_{A_\gamma} \to X$ be the inclusion mapping for each α. Then X is spanned by $\bigcup_{\gamma \in \Gamma} T_\alpha(X_{A_\gamma})$. It is easy to verify that if (X,σ) is the inductive limit of the spaces X_{A_γ} under the maps I_γ, then τ is weaker than σ.

THEOREM 4.8. *Let the convex Hausdorff space (X,τ) have a $\mathcal{C}$ web $\mathcal{W}$. Then $\mathcal{W}$ is a $\mathcal{C}$ web for (X,τ^u), and if σ is any convex topology for X strictly stronger than τ^u, then (X,σ) does not have a $\mathcal{C}$ web.*

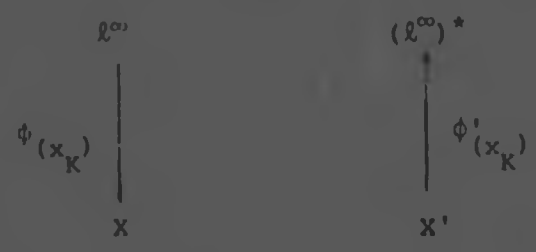
Proof. Let $\mathcal{S} = \{(x_\nu) : x_K \in F_K$ for each positive integer K, where (F_K) is a filament of $\mathcal{W}\}$. Thus $\mathcal{S}$ is the collection of all point sequences of all filaments of $\mathcal{W}$. If $(a_K) \in \ell^\infty$ and $(x_K) \in \mathcal{S}$, then the sequence $(\sum_{K=1}^{n} a_K x_K)$ converges in X, for suppose $|a_K| < m$, all K, then $(\frac{a_\nu}{m} x_\nu) \in \mathcal{S}$, since the sets in $\mathcal{W}$ are balanced. For each $(x_K) \in \mathcal{S}$, we define a linear function $\phi_{(x_K)} : \ell^\infty \to X$ by $\phi_{(x_K)}(a) = \sum_{K=1}^{\infty} a_K x_K$, where $a = (a_K) \in \ell^\infty$. Let U be the closed unit ball in ℓ^∞ and let

$$B_{(x_K)} = \phi_{(x_K)}(U) \quad \{\sum_{K=1}^{\infty} a_K x_K : |a_K| \leq 1 \text{ for all } K \in N\}.$$

Then each $B_{(x_K)}$ is absolutely convex and X is spanned by $\bigcup_{(x_K) \in \mathcal{S}} B_{(x_K)}$. For suppose $y \in X$, then there exists a filament (F_K) of $\mathcal{W}$ and a positive number λ such that $\lambda y \in F_1$. Let $(x_K) = (\lambda y, 0, 0, \ldots)$ and $(a_K) = (1, 0, 0, \ldots)$ then $y = \frac{1}{\lambda} \sum_{K=1}^{\infty} a_K x_K \in \frac{1}{\lambda} B_{(x_K)}$.

We shall show that each $B_{(x_K)}$ is bounded and complete.

Let $(x_K) \in \mathcal{S}$ and consider $\phi'_{(x_K)} : X \to (\ell^\infty)^*$.



Let $y \in X'$, $a = (a_K) \in \ell^\infty$. Then $(a, \phi'_{(x_K)}(y)) = (\phi_{(x_K)}(a), y)$ $= (\sum_{K=1}^{\infty} a_K x_K, y) = \sum_{K=1}^{\infty} a_K (x_K, y)$, since $y : X \to C$ is continuous.

The series $\sum_{K=1}^{\infty} a_K(x_K , y)$ is convergent for each $(a_K) \in \ell^\infty$, hence $((x_K , y)) \in \ell'$ and so $\phi'_{(x_K)}(y) \in \ell'$. Thus $\phi_{(x_K)}$ is $W(\ell^\infty,\ell') - W(X,X')$ continuous. Now U is $W(\ell^\infty,\ell)$-compact (see [15] p 62, Theorem 6, Corollary 2) hence $B_{(x_K)}$ is $W(X,X')$- compact and complete. τ is a stronger polar topology than $W(X,X')$ and so $B_{(x_K)}$ is τ-complete (see [15] p 105 Proposition 3, Corollary). Let (X,σ) be the inductive limit of the Banach spaces $X_{B_{(x_K)}}$ under the inclusion maps, then (X,σ) is ultra-bornological and τ is weaker than σ.

By constructing suitable filaments from the strands of $\mathcal{W}$, we shall show that it is a $\mathcal{C}$ web for (X,σ). For each filament (F_K) of $\mathcal{W}$ for the topology τ, let $G_K = 2^{-K}F_K$. Let $y_K \in G_K$ for each K, then $(2^K y_K) \in \mathcal{S}$. $(\sum_{K=1}^{n} y_K)$ is τ convergent to an element y of X. Now, $y - \sum_{K=1}^{m} y_K = \frac{1}{2^m}\sum_{K>m} 2^{m-K}(2^K y_K) = \frac{1}{2^m}\Phi_{(2^K y_K)}(a) \in \frac{1}{2^m} P_{(2^K y_K)}$, where $a = (a_K) \in \ell_\infty$ is defined by

$a_K = 0$, $K \leq m$; $a_K = 2^{m-K}$, $K > m$.

A base of neighbourhoods for the topology σ on X is the collection of absolutely convex hulls of sets of the form $\bigcup_{(x_K)\in\mathcal{S}} \lambda_{(x_K)} B_{(x_K)}$, where $\lambda_{(x_K)}$ is a positive real number for each $(x_K) \in \mathcal{S}$.

Thus $\sum_{K=1}^{m} y_K$ is σ-convergent to y, and so if we choose the K σ-filaments for $\mathcal{W}$, then $\mathcal{W}$ is a $\mathcal{C}$ web for (X,σ).

An application of the relevant closed graph theorem proves that $\sigma = \tau^u$. Suppose u is an ultrabornological topology for X stronger than τ. The identity map $I : (X,u) \to (X,\sigma)$ has a closed graph and so it is continuous, and so u is stronger than σ. A similar argument proves the final statement of the theorem.

COROLLARY 1. *If $\mathcal{W}$ is a $\mathcal{C}$ web for the convex Hausdorff space (X,τ) and τ' is a convex Hausdorff topology for X with $\tau' \subset \tau''$ then $\mathcal{W}$ is a $\mathcal{C}$ web for (X,τ'). (τ' may not be comparable with τ.) In particular, $\mathcal{W}$ is a $\mathcal{C}$ web for $s(X,X')$, (X,τ^t) and (X,τ^{X_1}), where*

X' is the τ dual of X.

COROLLARY 2. *If (X,τ) is Hausdorff and has a $\mathcal{C}$ web, then τ^u is minimal among the ultra-bornological topologies for X.*

We can now give an example of a complete convex space which does not have a $\mathcal{C}$ web. Let (X,τ) be an infinite dimensional Banach space. Then (X,τ) has a $\mathcal{C}$ web. $(X,\tau(X,X^*))$ is complete and $\tau(X,X^*)$ is strictly stronger than τ (see Appendix I). By the above theorem $(X,\tau(X,X^*))$ does not have a $\mathcal{C}$ web.

From this example we can conclude that not every inductive limit of convex $\mathcal{C}$ -webbed spaces has a $\mathcal{C}$ web, for $(X,\tau(X,X^*))$ is topologically isomorphic to $\bigoplus_{i=I} C_i$ with cardinal I = dimension X. (We have shown that there is no Banach space of countably infinite dimension, for we have seen that $\bigoplus_{i=1}^{\infty} C_i$ does have a $\mathcal{C}$ web and, not being normable, is not a Banach space (see Appendix I). There is no weaker ultra-bornological topology for $\bigoplus_{i=1}^{\infty} C_i$).

CHAPTER FIVE

A COUNTER-EXAMPLE

Let X and Y be convex spaces. In Chapter three we showed that if X is barrelled and Y is B_r-complete, then $(X,Y) \in \mathcal{B}$. In Chapter four we showed that if X is an inductive limit or a topological product of convex Baire spaces and Y has a $\mathcal{C}$ web, then $(X,Y) \in \mathcal{B}$. In this chapter, we show that neither of these two theorems is a generalization of the other. We shall find a barrelled Hausdorff space X, a convex space Y with a $\mathcal{C}$ web and a closed graph linear mapping from X onto Y which is not continuous. The example we give is a particular case of an example of Valdivia in [18]. We first need a lemma from another paper of Valdivia [17].

LEMMA 5.1. *Let X be a separable convex space and let (X_m) be a strictly increasing sequence of subspaces of X with* $X = \bigcup_{m=1}^{\infty} X_m$. *If there is a bounded set A in X such that $A \not\subset X_m$ for each $m \in N$, then there is a dense subspace F of X, $F \neq X$ such that $F \cap X_m$ is finite dimensional for all $m \in N$.*

Proof. Let $\{x_1, x_2, \dots\}$ be a countable dense subset of X. We choose a subsequence (F_n) of (X_m), and a sequence (y_n) in A to satisfy $\{x_1, x_2, \dots x_n\} \subset F_n$ and $y_n \in F_{n+1} \setminus F_n$ for each n. To do this, we choose X_{m_1} such that $x_1 \in X_{m_1}$, choose $y_1 \in A \setminus X_{m_1}$. Let $F_1 = X_{m_1}$. Now choose X_{m_2} so that $\{x_2, y_1\} \cup F_1 \subset X_{m_2}$, let $F_2 = X_{m_2}$ and choose $y_2 \in A \setminus F_2$. Proceeding in this way, we obtain the required sequences.

Let $H = \langle \{x_1 + y_1, x_2 + \frac{1}{2}y_2, x_3 + \frac{1}{3}y_3, \dots\} \rangle$. We shall show that $H \cap X_m$ has dimension at most n, for each n. Let

$x \in H \cap X_n$ where $x = \sum_{p=1}^{K} \alpha_p (x_p + \frac{1}{p} y_p)$, with $\alpha_K \neq 0$. We shall show that $K \leq n$. Suppose $K > n$. Then

$x - \alpha_K x_K - \sum_{p=1}^{K-1} \alpha_p (x_p + \frac{1}{p} y_p) = \frac{1}{K} \alpha_K y_K \in F_n + \Gamma_K + V_K \subset \Gamma_K$. This is a contradiction, and so $H \cap X_n \subset \langle [x_1 + y_1, \ldots, x_n + \frac{1}{n} y_n] \rangle$.

We now show that H is dense in X. Let $z \in X$ and let U be an absolutely convex neighbourhood in X. Since A is bounded, the sequence $(\frac{1}{n} y_n)$ converges to the origin in X. Choose n_1 so that $n > n_1 \Rightarrow \frac{1}{n} y_n \in \frac{1}{2}U$. $\{x_{n_1+1}, x_{n_1+2}, \ldots\}$ is dense in X, so we can choose $n_0 > n_1$ so that $z - x_{n_0} \in \frac{1}{2}U$. Then $z - (x_{n_0} + \frac{1}{n_0} y_{n_0}) \in U$.

Hence H is dense in X.

If $H \neq X$, let $F = H$.

Suppose $H = X$. The set $B = \{y_1, y_2, \ldots\}$ is linearly independent. We extend B to a Hamel base $\mathcal{H}$ of X, $\mathcal{H} = B \cup \{z_i\}_{i \in I}$. Define a linear functional U on X by $(y_n, U) = n$, all $n \in N$ and $(z_i, U) = 0$ for each $i \in I$. U is not bounded on A and so U is not continuous. $U^{-1}(0)$ is a hyperplane in X which is not closed and so $U^{-1}(0)$ is dense in X. Let $F = U^{-1}(0) \cap H$. Then F is dense in X, $F \neq X$ and $F \cap X_n$ is finite dimensional.

We now prove a less general form of a theorem of Valdivia in [18]. We shall need the following result: if, for each i in an index set I, the convex Hausdorff space X_i is endowed with the topology $W(X_i, X_i')$, then the product topology, p, on $X = \prod_{i \in I} X_i$ is $W(X, X')$, where X' is the dual of X under p. To prove this, we note firstly that $W(X, X')$ is weaker than p. Next, for each $i \in I$, let $p_i : X \to X_i$ be the projection mapping. Then each p_i is continuous when X has the topology p. By [15] p 39 Proposition 13, p_i is also continuous when X has the topology $W(X, X')$. Thus p is weaker than $W(X, X')$.

THEOREM 5.2. *Let (X_n) and (Y_n) be two sequences of convex Hausdorff spaces, with X_i separable, such that, for every positive integer n, there exists a one-to-one continuous linear mapping*

$U_n : Y_n \to X_n$ *such that* $U_n(Y_n)$ *is separable, dense in* X_n *and* $U_n(Y_n) \neq X_n$. *Then there is in* $L = \overset{\infty}{\underset{n=1}{\oplus}} X_n \times \overset{\infty}{\underset{n=1}{\prod}} Y_n$ *a dense subspace* G *different from* L *which meets every bounded and closed subset of* L *in a closed subset of* L.

Proof. We shall write $\oplus$ and $\prod$ for $\overset{\infty}{\underset{n=1}{\oplus}}$ and $\overset{\infty}{\underset{n=1}{\prod}}$ respectively. Unless otherwise stated, the following dual spaces will be regarded as being equipped with the weak topologies indicated in the table below.

Vector space	Topology
X'_n	$W(X'_n, X_n)$
Y'_n	$W(Y'_n, Y_n)$
$\oplus X'_n$	$W(\oplus X'_n, \prod X_n)$
L'	$W(L', L)$

Throughout the proof of this theorem, x, y, z, t shall be regarded as arbitrary elements of the spaces indicated:

$x = (x^1, x^2, x^3, \dots) \in \oplus X_n$

$y = (y^1, y^2, y^3, \dots) \in \prod Y_n$

$z = (z^1, z^2, z^3, \dots) \in \oplus X'_n$

$t = (t^1, t^2, t^3, \dots) \in \prod X'_n$.

Let $U : \prod Y_n \to \prod X_n$ be defined by $U(y) = (U_1(y^1), U_2(y^2), U_3(y^3), \dots)$, and $U^* : \prod X'_n \to \prod Y'_n$ by $U^*(t) = (U'_1(t^1), U'_2(t^2), U'_3(t^3), \dots)$.

Consider U', the transpose of U.

$$\begin{array}{ccc} \prod Y_n & & (\prod (Y_n)^* \\ \Big\downarrow U & & \Big\uparrow U' \\ \prod X_n & & \oplus X'_n \end{array}$$

$(y, U'(z)) = (U(y), z) = \sum (U_n(y^n), z^n) = \sum (y^n, U'_n(z^n))$. Thus $U'(z) = (U'_1(z^1), U'_2(z^2), U'_3(z^3), \dots) \in \oplus Y'_n$. It follows that U is weakly continuous and that $U' = U^*\big|_{\oplus X_n}$.

Now let $f : L \to \prod X_n$ be defined by $f(x,y) = x + U(y)$. f is linear and continuous and hence f' is a continuous map from $\oplus X'_n$ into L' under the topologies assigned to these spaces in the table.

$$L = \oplus X_n \times \Pi Y_n \qquad\qquad L' = \Pi X'_n \times \oplus Y'_n$$

$$f \downarrow \qquad\qquad f' \uparrow$$

$$\Pi X_n \qquad\qquad \oplus X'_n$$

We wish to prove that f is open as a map from L under the topology $W(L , L')$ onto $f(L)$ under the topology τ induced on it by $W(\Pi X_n , \oplus X'_n)$. In order to prove this, it is sufficient to show that $f'(\oplus X'_n)$ is closed in L' (see Appendix V).

Let us evaluate $f'(z)$. Suppose that $f'(z) = (p,q)$ where $p \in \Pi X'_n$, $q \in \oplus Y'_n$, then

$((x,y),f'(z)) = (x+U(y),z) = (x,z)+(U(y),z) = (x,z)+(y,U'(z)) =$

$((x,y),(p,q)) = (x,p) + (y,q)$.

Letting $y = 0$, we see that $p = z$. From $x = 0$, we obtain $q = U'(z)$, and so $f'(z) = (z,U'(z))$.

Let $M = \{(t,U^*(t)) : t \in \Pi X'_n\}$ and $Z = M \cap (\Pi X'_n \times \oplus Y'_n)$. We show that $Z = f'(\oplus X'_n)$. Let $(t , U^*(t)) \in Z$, then $U^*(t) \in \oplus Y'_n$ and so $U'_n(t^n) = 0$ for all but finitely many values of n. Now $U_n(Y_n)$ is dense in X_n , by hypothesis, and so U'_n is one-to-one for each n (see Appendix IV). Thus t^n is zero for all but finitely may values of n. Hence $t \in \oplus X'_n$ and $(t , U^*(t)) \in f'(\oplus X'_n)$. Using the fact that U' and U^* agree on $\oplus X'_n$, we obtain that $f'(\oplus X'_n) = Z$. It remains to show that Z is closed in L'.

$M = \{((t^1, t^2, t^3, \dots) , (U'_1(t^1) , U'_2(t^2) , U'_3(t^3) , \dots)) :$

$t = (t^n) \in \Pi X'_n\} \subset \Pi X'_n \times \Pi Y'_n$.

Now $\Pi X'_n \times \Pi Y'_n$ is topologically isomorphic to $\Pi(X'_n \times Y'_n)$. For each n , the graph of $U'_n : X'_n \to Y'_n$ is $\{(t^n,U'_n(t^n)) : t^n \in X'_n\}$ which is closed in $X'_n \times Y'_n$ since U'_n is continuous. From this we see that M is closed in $\Pi X'_n \times \Pi Y'_n$. Now the product topology on $\Pi X'_n \times \Pi Y'_n$ is the topology $W(\Pi X'_n \times \Pi Y'_n , \oplus X_n \times \oplus Y_n)$ (see the remarks preceeding this theorem). Thus Z is closed in $L' = \Pi X'_n \times \oplus Y'_n$ in the topology ν that L' inherits as a subspace of $\Pi X'_n \times \Pi Y'_n$. Now $\nu = W(L' , L)$ (see Appendix III), and so Z is closed in L' under $W(L' , L)$.

We can now conclude that f is an open mapping in the sense required.

Let $H = f(L)$ under the topology it inherits as a subspace of ΠX_n. We shall show that H satisfies the conditions of Lemma 5.1. Let $L_p = \oplus_{n=1}^{p} X_n \times \Pi_{n=1}^{\infty} Y_n$, and let $H_p = f(L_p)$. Then

$H_1 = f(X_1 \times \Pi Y_n) = X_1 + U(\Pi Y_n) = X_1 + \Pi U_n(Y_n)$. (Again we regard X_1 as a subspace of ΠX_n.) Now each $U_n(Y_n)$ is dense in X_n and so $cl_H H_1 = (cl_{\Pi X_n} H_1) \cap H \supset \Pi X_n \cap H = H$. (If A and B are subsets of a convex space then $\overline{A + B} \supset \bar{A} + \bar{B}$.) Thus H_1 is dense in H. H_1 is also separable and so it follows that H is separable. For each n, choose $W_n \in X_n \setminus U_n(Y_n)$, and for each p, let $W^{(p)} = (W_1, W_2, \ldots W_p, 0, 0, \ldots)$. Let $A = \{W^{(1)}, W^{(2)}, W^{(3)} \ldots\}$, then $A \subset H$, since $X_n \subset H$. A is bounded in ΠX_n, and so A is bounded in H. f being continuous. Now

$H_p = f(\bigoplus_{n=1}^{p} X_n \times \prod_{n=1}^{\infty} Y_n) = \bigoplus_{n=1}^{p} X_n + \prod_{n=1}^{\infty} U_n(Y_n)$, and so if $W^{(p+1)} \in H_p$ then $W_{p+1} \in U_{p+1}(Y_{p+1})$ which is a contradiction. Thus $W^{(p+1)} \notin H_p$. By Lemma 5.1 there exists a dense subspace D of H, $D \neq H$, such that $D \cap \Pi_p$ is finite dimensional for each p.

Let $G = f^{-1}(D)$. We shall show that G is the subspace of L required by the theorem. Clearly $G \neq L$. Let U be a neighbourhood in L. f(U) is a neighbourhood in H in the topology induced on it by $W(\Pi X_n, \bigoplus X'_n)$. Choose $x \in U$ such that $f(x) \in f(U) \cap D$. Then $x \in U \cap f^{-1}(D)$, and G is dense in L. Now suppose there exists a bounded closed subset B of L with $B \cap G$ not closed. Let $y \in \overline{G \cap B} \setminus G \cap B$. Since $\overline{G \cap B} \subset \bar{G} \cap B$, $y \in B$, and so $y \notin G$. We may regard L as the topological direct sum $\bigoplus_{n=0}^{\infty} X_n$, with $X_0 = \Pi Y_n$. Then there exists a positive integer q such that $B \subset L_q$ (see [15] p 92 Proposition 24). Now $f(y) \notin D$, and $f(y) \in f(\overline{G \cap B}) \subset \overline{f(G \cap B)} \subset \overline{f(G) \cap f(B)} \subset \overline{D \cap H_q} = D \cap H_q$, since $D \cap H_q$ is a finite dimensional subspace of H. Hence $f(y) \in D$, which is a contradiction. The proof of the theorem is now complete.

Let us consider a particular case of the above theorem. We let $X_n = Y_n = \ell^2$ for each n, and let $U_n : Y_n \to X_n$ be defined by $U_n((y_m)) = (\frac{1}{m} y_m)$, where the sequence (y_m) is an element of Y_n. U_n is clearly continuous, one-to-one and linear and $U_n(Y_n) \neq X_n$. Let $x = (x_m) \in X_n$ and let ε be any positive number. Choose K so that $\sum_{m=K}^{\infty} |x_m|^2 < \varepsilon^2$. Let $(y_m) \in Y_n$ be defined by $y_m = mx_m$, $m \leq K$, $y_m = 0$, $m > K$. Then $\|U_n(y_m) - x\| = (\sum_{m=K}^{\infty} |x_m|^2)^{\frac{1}{2}} < \varepsilon$. It

follows that $U_n(Y_n)$ is dense in X_n for each n.

The conditions of Theorem 5.2 are satisfied by $L = \bigoplus_{n=1}^{\infty} X_n = \prod_{n=1}^{\infty} Y_n$ with $X_n = Y_n = \ell^2$. The continuous dual of L is $L' = \prod_{n=1}^{\infty} X_n' \times \bigoplus_{n=1}^{\infty} Y_n'$. If we endow X_n' and Y_n' with the usual norm topology, then of course $X_n' = Y_n' = \ell^2$. We also let L' be the topological product of $\prod_{n=1}^{\infty} X_n'$ with $\bigoplus_{n=1}^{\infty} Y_n'$. Then $L = L'$, and we can choose a dense subspace G' of L' different from L' which meets every bounded and closed subset of L' in a closed subset of L'.

Let the topology on L be σ. We shall find a convex topology τ for L, strictly weaker than σ such that (L,τ) is barrelled and Hausdorff. (L,G') is a dual pair, since, under the topology G' inherits from L', the dual of G' is L. Let $\tau = \tau(L,G')$, then τ is strictly weaker than σ. Let A be a τ barrel in L. Then A is also a σ barrel and so A is a σ neighbourhood. Let A° be the polar of A in L', then $A^\circ \cap G'$ is the polar of A in G'. A° is $W(L', L)$ compact, $A^\circ \cap G'$ is $W(L', L)$ closed since A° is $W(L', L)$ bounded and closed. Hence $A^\circ \cap G'$ is $W(G', L)$ compact (see Appendix III). Thus $(A^\circ \cap G')^\circ = A$ is a $\tau(L,G)$ neighbourhood and so (L,τ) is barrelled.

We now have the example we need. (L,σ) has a $\mathfrak{C}$ web and (L,τ) is barrelled and Hausdorff. The identity map $I : (L,\tau) \to (L,\sigma)$ has a closed graph but is not continuous.

(L,σ) provides us with an example of a convex space with a $\mathfrak{C}$ web which is not B_r-complete under any stronger convex topology. (L,τ) is an example of a barrelled Hausdorff space which is not an inductive limit nor a product of convex Baire spaces.

CHAPTER SIX

SOME OTHER CLOSED GRAPH THEOREMS - A SUMMARY

In this chapter, we give a short account without proofs of other closed graph theorems.

Many results have been obtained by studying $\mathcal{C}(B)$ where B is a particular class of convex spaces. We discuss some of these first.

In 1956, Robertson and Robertson [14] showed that if B is the class of inductive limits of sequences of B-complete spaces, then $\mathcal{C}(B)$ includes the convex Baire spaces.

A convex Hausdorff space X is barrelled if and only if every $W(X',X)$ bounded subset of X' is equicontinuous (see [15] p 66 Proposition 1 Corollary 1). Using this result various generalisations of the idea of barrelledness have been made. Two of these are used in the following two papers we discuss.

Let B_1 be the class of separable B_r-complete spaces, B_2 be the class of separable Banach spaces, B_3 the convex space $C[0,1]$ under the topology of uniform convergence, and A the class $\{X : X$ is a convex space and every $W(X',X)$-bounded metrisable absolutely convex subset of X' is equi-continuous$\}$. In 1971 Kalton [8] proved that $\mathcal{C}(B_1) = \mathcal{C}(B_2) = \mathcal{C}(B_3) = A$. That $\mathcal{C}(B_2) = \mathcal{C}(B_3)$ follows easily from the fact that every separable Banach space is norm isomorphic to a closed subspace of $C[0,1]$. Kalton also gives an example of an element of $\mathcal{C}(B_1)$ which is not barrelled, showing that $\mathcal{C}\{B_r$-complete spaces$\} \neq \mathcal{C}\{$separable B_r-complete spaces$\}$.

In 1976, Popoola and Tweddle [12] characterised the class $\mathcal{C}(B)$, where B is the class of Banach spaces of dimension at most c. They define a subset A of the algebraic dual X^* of a vector space X to be *essentially separable* if it is included in a $W(X^*,X)$-separable set. They call a convex Hausdorff space X *δ-barrelled* if every essentially separable $W(X',X)$-bounded subset of X' is equi-continuous. Their closed graph theorem states that $\mathcal{C}(B)$ is the class of δ-barrelled spaces. By an example they show that $\mathcal{C}$(Banach spaces) $\neq \mathcal{C}$(Banach spaces of dimension at most c). They also show that $\mathcal{C}(B)$ is closed

under the formation of completions and subspaces of countable co-dimension.

Other techniques have been used to investigate $\mathcal{b}$:

Iyahen [7] investigates what he calls the *closed graph spaces* or *CG spaces*. X is said to be a CG space if $(X,X) \in \mathcal{b}$. Clearly a barrelled Hausdorff, B_r-complete space is a CG space. The author shows that $(X,W(X,Y))$ is a CG space if and only if $(X,\tau(X,Y))$ is a CG space, and that if X is a CG space then it is the topological direct sum of two subspaces if it is their algebraic direct sum.

Powell [13] has applied a general closed graph theorem of Komura [10] to many special cases. Let $\{X_i : i \in I\}$ be a family of convex spaces, and for each $i \in I$, let T_i be a linear mapping from X_i into a vector space X , then the *final topology* on X determined by the linear mappings T_i is the strongest convex topology on X which makes each T_i continuous This definition is not the same as the one we use for an inductive limit topology, as it omits the condition that X is spanned by $\bigcup_{i \in I} T_i(X_i)$. Let α be a property of convex spaces. Powell calls a convex space (Y,σ) an α space if σ is the final topology determined by a set of mappings $T_i : X_i \to Y$, where each X_i is an α space. For each convex space (X,τ) , and for each α , he shows that there is a weakest $\bar{\alpha}$ topology for X which is stronger than τ , and denotes this topology by $\tau^{\bar{\alpha}}$. In Chapter four, we modified Powell's definitions in the case when α is the property "Banach". According to Komura's closed graph theorem, the following statements about a convex Hausdorff space (Y,τ) are equivalent:

(i) for every α space X , $(X,Y) \in \mathcal{b}$;

(ii) for every convex Hausdorff topology τ_o on Y weaker than τ , we have $\tau \subset \tau_o^{\bar{\alpha}}$

(iii) for every convex Hausdorff topology τ_o on Y weaker than τ , we have $\tau_o^{\bar{\alpha}} = \tau^{\bar{\alpha}}$.

If a convex Hausdorff space Y satisfies (i) Powell calls Y an $(\alpha\mathcal{C})$ space. Powell considers four special cases for the property α : α = universal, α = barrelled, α = normable, α = Banach. The technique used in this paper is to fix a collection of domain spaces for the closed graph theorem and examine the corresponding collection of range spaces. We mention a few of the interesting results Powell obtains which characterise $(\alpha\mathcal{C})$ spaces for the

above four values of α.

(a) $(X,\tau$ is a (universal $\mathcal{C}$) space if and only if τ is minimal among the convex Hausdorff topologies for τ.

(b) (X,τ) is a (barrelled $\mathcal{C}$) space if and only if for every $W(X',X)$-dense subspace H of X', we have that $X' \subset \overline{H}$, where $\overline{H}$ is the quasi-completion of H. (For a definition of the quasi-completion of a convex space, see [11] p 297.) This is a generalisation of Theorem 3.5.

(c) (X,τ) is a (normable $\mathcal{C}$) space if and only if for every convex Hausdorff topology τ_0 for X weaker than τ every τ_0-bounded set is τ-bounded.

(d) (X,τ) is a (Banach $\mathcal{C}$) space if and only if for every convex topology τ_0 for X which is weaker than τ, every τ_0-compact absolutely convex subset of X is τ-bounded.

(e) If (X,τ) is a (Banach $\mathcal{C}$) space with dimension smaller than the least strongly inaccessible cardinal, and τ is not a minimal convex Hausdorff topology for X, then there is no minimal convex Hausdorff topology for X weaker than τ.

APPENDIX

I. *The strongest convex topology for a vector space.*

Let X be a vector space with algebraic dual X^*. (X,X^*) is a dual pair. $\tau(X,X^*)$ is the strongest convex topology of this dual pair, and, since any other convex topology for X must have its continuous dual a subspace of X^*, it follows that $\tau(X,X^*)$ is the strongest convex topology for X. Let $s = \tau(X,X^*)$.

Let d be the dimension of X. Then X is algebraically isomorphic to $\bigoplus_{\alpha \in A} C_\alpha$, where cardinal $A = d$. If τ is any convex Hausdorff topology for X, τ induces the Euclidean topology on each C_α and so each injection map $I_\alpha : C_\alpha \to X$ is continuous. Now the strongest convex topology on X which makes the injection maps continuous is the inductive limit topology, and so the direct sum topology on $\bigoplus_{\alpha \in A} C_\alpha$ is the strongest convex topology on X, and (X,s) is topologically isomorphic to $\bigoplus_{\alpha \in A} C_\alpha$. (X,s) is complete (see [15] p 92 Proposition 23). Clearly $\tau(X,X^*) \leq s(X,X^*)$ and so (X,s) is barrelled.

The collection of all absolutely convex absorbent subsets of X form a base of neighbourhoods for s. It follows that every linear mapping T from X into a convex space Y is continuous, for the inverse image under T of every absolutely convex neighbourhood in Y is a neighbourhood in X. Thus $(X,\tau(X,X^*))$ is bornological.

Every subspace of (X,s) is closed. To prove this, let L be a subspace of X, and M an algebraic supplement of L. The projection mapping $P : X \to M$ is continuous and its kernel is L. Hence L is closed in (X,s).

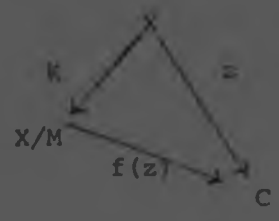
Only finite dimensional subsets of X can be s-bounded. For, let $A = \{x_1, x_2, x_3, \dots\}$ be an infinite countable set of linearly independent elements of X. Extend A to a Hamel base B for X. Let $z \in X^*$ be defined by $(x_n, z) = n$, $(x,z) = 0$ if $x \in B \setminus A$, and z is defined by linearity otherwise. Then z is continuous, but z is not bounded on A, and so A is not s-bounded.

(X,θ) can only be normable if X is finite dimensional, for (X,s) is normable if and only if it has a bounded neighbourhood. A neighbourhood, being absorbent, cannot be finite dimensional unless X is finite dimensional.

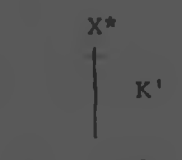
(X,θ) is complete (see [15] p 92 Proposition 23).

II *A note on Quotient Spaces of Convex Spaces.*

Let M be a subspace of the vector space X and let M° be the polar of M in X^*. Let $z \in M^\circ$, then $(M,z) = 0$. If $x_1 + M = x_2 + M$ where $x_1, x_2 \in X$, then $(x_1,z) = (x_2,z)$. Thus, we may define a linear mapping $f : M^\circ \to (X/M)^*$ by $(x+M, f(z)) = (x,z)$ for all $x \in X$ and all $z \in M^\circ$. It is easy to show that f is one-to-one and maps M° onto $(X/M)^*$. We identify $(X/M)^*$ and M°. If $z \in M^\circ$ and $x + M \in X/M$, then $(x + M, z) = (x,z)$. Let $K : X \to X/M$ be the canonical mapping, and let $z \in M^\circ$.



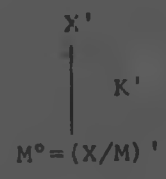




Then $z = f(z) \circ K$, and $K' : M^\circ \to X^*$ is the inclusion mapping. If now X is a topological vector space and X/M has the quotient topology, then z is continuous if and only if $f(z)$ is continuous. From this, it follows that $(X/M)' = M^\circ \cap X'$, the polar of M in X'. It is also clear that $(K')^{-1}(X') = M^\circ \cap X'$.

Now suppose M is a closed subspace of the convex Hausdorff space (X,τ). Then K' maps $(X/M)'$ into X'.





Polars taken in X and X' will be denoted by the symbol $^\circ$, and those taken in X/M and $(X/M)'$ by $^\blacktriangle$. Let $\tau = W(X,X')$. Let $K(F^\circ)$ be a basic neighbourhood in the quotient topology, q, on X/M, where F is a finite subset of X'. Then $K(F^\circ) = ((K')^{-1}(F))^\blacktriangle = (F \cap M^\circ)^\blacktriangle$. $F \cap M^\circ$ is a finite subset of M° and so $K(F^\circ)$ is a $W(X/M, (X/M)')$ neighbourhood in X/M. Since $W(X/M, (X/M)')$ is weaker than q, the two topologies are the same.

11: *A note on Subspaces of Convex Hausdorff Spaces*

Let (A,τ_A) be a subspace of the convex Hausdorff space (X,τ), where τ_A is the topology induced on A by τ. Let A° be the polar of A in X'. If $z_1 + A^\circ = z_2 + A^\circ$, where $z_1, z_2 \in X'$, then $(a,z_1) = (a,z_2)$ for all $a \in A$. Thus, we may define a linear mapping $f : X'/A^\circ \to A'$ by $f(z+A^\circ) = z|_A$, where $z \in X'$. It is easy to show that f is one-to-one and maps X'/A° onto A'. We identify X'/A° and A'. If $z + A^\circ \in X'/A^\circ$, and $a \in A$, then $(a, z + A^\circ) = (a,z)$. Let $I : A \to X$ be the inclusion mapping, then $I' : X' \to X'/A^\circ$ is the canonical mapping.



We consider now the conditions under which the continuous duals of X and A coincide. Clearly $X' = A'$ if and only if $A^\circ = \{0\}$ if and only if A is $W(X,X')$ dense in X if and only if (A,X') is a dual pair if and only if A separates the points of X'.

Now suppose that $\tau = W(X,X')$. Let $F^\circ \cap A$ be a basic τ_A neighbourhood in A, where F is a finite subset of X'. Polars taken in A or A' will be denited by the symbol $^\wedge$. $(I'(F))^\wedge = I'^{-1}(F^\circ) = F^\circ \cap A$. Now $I'(F)$ is a finite subset of A', and so $F^\circ \cap A$ is a $W(A,A')$ neighbourhood. Since $W(A,A')$ is clearly weaker than τ_A, we have that $W(A,A') = \tau_A$.

We thus have that if (A,X') is a dual pair, then $\tau_A = W(A,X')$.

IV *A necessary and sufficient condition for the transpose of a linear mapping to be one-to-one.*

THEOREM. *Let (X,Z) and (Y,W) be dual pairs and $T : X \to Y$ a linear mapping. Let $T' : W \to Z$ be the transpose of T. Then T' is a one-to-one if and only if $T(X)$ is $W(Y,W)$ dense in Y.*

Proof.



Endow Y with the $W(Y,W)$ topology.

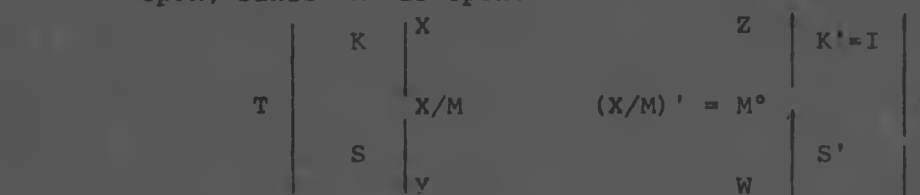
Suppose $T(X)$ is dense in Y, and that $T'(w) = 0$, where $w \in W$. Then $(x, T'(w)) = (T(x),w) = 0$ for all $x \in X$. Since W is the continuous dual of Y and since $T(X)$ is dense in Y, it follows that $(y,w) = 0$ for all $y \in Y$. Thus $w = 0$ since Y separates the points of W and so T' is one-to-one.

Suppose T' is one-to-one. Then $(T(X))^\circ = (T')^{-1}(X^\circ) = (T')^{-1}(\{0_1\}) = (\{0_2\})$, since T' is one-to-one, where 0_1 and 0_2 are the zero elements of Z and W respectively. It follows that $\overline{T(X)} = (T(X))^{\circ\circ} = \{0_1\}^\circ = Y$. Thus $T(X)$ is dense in Y.

V *An open mapping theorem.*

THEOREM. *Let (X,Z) , (Y,W) be dual pairs, T a weakly continuous linear mapping from X into Y and suppose that $T'(W)$ is a $W(Z,X)$-closed subspace of Z. Then T is open as a mapping from $(X,W(X,Z))$ onto $f(X)$ under the topology induced on it by $W(Y,W)$.*

Proof. Let $M = \operatorname{Ker} T$, then M is $W(X,Z)$ closed. Let M^c be the polar of M in Z. We are concerned with three dual pairs: (X,Z) , $(X/M,M^\circ)$ and $Y,W)$. We shall assume that each of the six vector spaces involved is equipped with the relevant weak topology, and that $T(X)$ is equipped with the topology induced on it by $W(Y,W)$. By Appendix II, the weak and quotient topologies on X/M coincide. Let $T = S \circ K$, where K is the canonical mapping from X onto X/M and $S : X/M \to Y$ is one-to-one and continuous. It is sufficient to prove that $S : X/M \to T(X)$ is open, since K is open.



$T'(W) = (I \circ S')(W) = S'(W)$ where I is the inclusion map from M° into Z , and so by hypothesis, $S'(W)$ is closed in Z. Now M° is closed in Z and $S'(W) \subset M^\circ$, hence $S'(W)$ is closed in M°. But $S'(W)$ is dense in M° (see Appendix IV), hence $S'(W) = M^\circ$.

Suppose $S(t_\alpha)$ is a net in $S(X/M)$ which converges to $S(t) \in S(X/M)$. Then, for each $w \in W$, $(S(t_\alpha),w) \to (S(t),w)$ and so $(t_\alpha,S'(w)) \to (t,S'(w))$, for each $w \in W$. Hence $(t_\alpha,u) \to (t,u)$ for each $u \in M^\circ$ and so $t_\alpha \to t$, and $S : X/M \to T(X)$ is open as required.

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$\bar{A}$	[15]	6
A°	[15]	34
α space		3
α topology		3
$\bar{\alpha}$ space		41
$\aleph_0$	cardinal of the set of natural numbers	
$\mathcal{L}$, $\mathcal{L}^{(B)}$		(i)
$\mathcal{L}_1$, $\mathcal{L}_1^{(B)}$		6
C	the complex number field	
$\mathcal{C}$ web		28
$cl_X E$	[19]	25
$\mathcal{L}$ [0,1]	[15]	17
CG space		52
δ-barrelled		51
g(A)		1
h(a)		1
$\widehat{H}$		53
ℓ^p	[15]	21
ℓ^∞	[15]	21
N	the set of natural numbers	
$\prod_{i \in I} X_i$		5
$\prod_{I \in I} C_j$		5
$\bigoplus_{i \in I} X_i$		5
$\bigoplus_{i \in I} C_i$		5

$\mathcal{B}(X,Y)$	4
T'	5
$T\|_{\mathcal{Z}}$	5
τ^{U}	41
τ^{N}	41
τ^{c}	41
τ-σ continuous	5
$\tau(X,Y)$	4
$w(X,Y)$	4
X'	the continuous dual of a topological vector space
X^{*}	the algebraic dual of a topological vector space
$X(H)$	7
X_{A}	41
X/M	3
(X,T)	3

TOPOLOGICAL VECTOR SPACES

FIRST PROJECT:

COMPLETIONS OF LOCALLY CONVEX HAUSDORFF TOPOLOGICAL VECTOR SPACES.

Janet M. Helmstead.

1. INTRODUCTION

By *convex space* we mean locally convex topological vector space over the field C of complex numbers.

Two metric spaces M and N with metrics ρ and σ respectively are said to be *isometric* if and only if there exists a one-to-one mapping f from X onto Y which is such that $\rho(x,y) = \sigma(f(x),f(y))$ for all x and y in M. The mapping f is called an *isometry*, and is clearly a homeomorphism. Every metric space M is isometric to a dense subspace of a complete metric space $\hat{M}$ called a *completion* of M. The completion $\hat{M}$ is unique up to an isometry which leaves M pointwise fixed. Explicitly, if f_1 and f_2 are isometries from M onto dense subspaces of the complete metric spaces $\hat{M}_1$ and $\hat{M}_2$, then there exists an isometry g from $\hat{M}_1$ onto $\hat{M}_2$ such that $g(f_1(x)) = f_2(x)$ for all $x \in M$. These results can be found in [6] p.176 Th.24.4 . The proof of this theorem can easily be adapted to the case where M is a convex metric space. The operations of addition and scalar multiplication can be extended to $\hat{M}$ in a natural way so that $\hat{M}$ becomes a convex metric space and the isometry defined in the theorem from M to a dense subspace of $\hat{M}$ becomes a linear mapping .

The idea of completeness in metric spaces can be extended to convex Hausdorff spaces by the use of filters. Let X be a convex Hausdorff space. We shall call a neighbourhood of the origin in X simply a *"neighbourhood"* . Let U be a neighbourhood in X and A a subset of X. Then A is said to be *small of order* U if $a_1, a_2 \in A \Rightarrow a_1 - a_2 \in U$. A filter $\mathcal{F}$ on A is said to be a *Cauchy filter* on A if, for every neighbourhood U in X , there exists $F \in \mathcal{F}$ with F small of order U . The subset A of X is said to be *complete* if every Cauchy filter on A converges to a point of A. It is easy to show that A is complete if and only if every Cauchy filter on X which contains A converges to a point of A. Every convergent filter on A is a Cauchy filter on A. If X is a complete convex space, then the subset A of X is complete if and only if it is closed. If X is a convex metric space, then a subset A of X is complete according to the above definition if and only if every Cauchy sequence in A converges to a point of A.

If X,Y are convex spaces and $T : X \to Y$ is a linear homeomorphism from X onto Y, we say that X is *topologically isomorphic* to Y and that T is a *topological isomorphism* from X onto Y. Also, the complete convex Hausdorff space $\hat{X}$ is a *completion* of the convex Hausdorff space X if X is topologically isomorphic to a dense subspace of $\hat{X}$. Now X is topologically isomorphic to a subspace of a product of Banach spaces (see [4] p.88 prop.19 Cor.). Let Y denote such a product and let $T : X \to Y$ be a topological isomorphism from X onto the subspace $T(X)$ of Y. Since Y is complete, so is $\overline{T(X)}$, and so it is a completion of X. In this project, we shall show that, as for metric spaces, a completion of X is unique up to a topological isomorphism which leaves X pointwise fixed, and so we may refer to the completion of X and denote it $\hat{X}$.

The above construction of the completion of the convex Hausdorff space has not proved useful in practice. In this project we develop another construction of $\hat{X}$ due to Grothendieck [2]. Using this construction, we shall derive many results about completeness of convex spaces. When X and Y are convex Hausdorff spaces, some of the results we shall prove are :

(i) if X has the weak, Mackey or strong topology with respect to X', then $\hat{X}$ has the weak, Mackey or strong topology with respect to X' ;

(ii) completeness of subsets of X is stable under the formation of stronger polar topologies with respect to the same subspace of X^*.

(iii) if $T : X \to Y$ is a continuous linear mapping, then T has a unique continuous linear extension $\hat{T} : \hat{X} \to \hat{Y}$;

(iv) if X is complete then the closure of every weakly sequentially compact subset of X is weakly compact.

The principal source for the material of this project is [4], Chapter VI. A few ideas have been obtained from [5].

2. NOTATION

We shall often be concerned in the same proof with several dual pairs. In order to avoid obscurity or repetitive explanations, we have devised the following special notation.

If $\mathcal{U}$ is a base of neighbourhoods for a convex topology on a vector space X we shall sometimes say that the topology on X is $\mathcal{U}$.

$\mathcal{D}^\circ$ If (X,Y) is a dual pair, and $\mathcal{D}$ is a collection of subsets of Y, then, taking polars in X and then in Y, $\mathcal{D}^\circ$ denotes $\{D^\circ : D \in \mathcal{D}\}$ and

$\mathcal{D}^{\circ\circ}$ $\mathcal{D}^{\circ\circ}$ denotes $\{D^{\circ\circ} : D \in \mathcal{D}\}$. If A is a subset

$\mathcal{D}^\circ \cap A$ of X then $\mathcal{D}^\circ \cap A$ denotes $\{D^\circ \cap A : D \in \mathcal{D}\}$.

w(X,Y) Let (X,Y) be a dual pair. The weakest convex topology on X which makes each y in Y continuous is called the *weak topology on X determined by Y*, and is denoted w(X,Y).

The polar topology $\mathcal{D}$. Let (X,Y) be a dual pair and let $\mathcal{D}$ be a collection of subsets of Y which satisfy the following conditions:

C_1 : each D in $\mathcal{D}$ is w(Y,X) bounded;

C_2 : if $A, B \in \mathcal{D}$, there exists $D \in \mathcal{D}$ with $A \cup B \subset D$;

C_3 : if $D \in \mathcal{D}$ and $\alpha \in C$ with $\alpha \neq 0$, then $\alpha D \in \mathcal{D}$;

C_4 : $\bigcup_{D \in \mathcal{D}} D$ spans Y.

The polars in X of the elements of $\mathcal{D}$ form neighbourhood base for a convex Hausdorff topology for X, called *a polar topology for X with respect to Y*. Combining previous definitions, we call this polar topology $\mathcal{D}^\circ$.

$\mathcal{U}^{\circ\circ}$ Let (X,T) be a convex Hausdorff space with a base $\mathcal{U}$ of neighbourhoods closed under non-zero scalar multiplication. Then the collection $\mathcal{C}$ of polars in X' of the elements of $\mathcal{U}$ satisfies the conditions C_1 - C_4 for a polar topology on X. This topology is denoted $\mathcal{U}^{\circ\circ}$ and is equal to T.

The polar topology $\mathcal{B}^{\circ} \cap X$

Let (X,Y) and (Y,Z) be dual pairs, with $X \subset Z$. Let $\mathcal{B}$ be a collection of subsets of Y. Taking polars in Z, according to the above notation, $\mathcal{B}^{\circ} \cap X$ is a collection of subsets of X, indeed, it is the collection of polars in X of the elements of $\mathcal{B}$. If $\mathcal{B}^{\circ} \cap X$ is a neighborhood base for a convex topology on X, this topology is denoted $\mathcal{B}^{\circ} \cap X$.

Further special notation and also more standard notation may be found in the candidate's dissertation.

3. DENSE SUBSPACES OF CONVEX HAUSDORFF SPACES.

Let (Y,τ) be a convex Hausdorff space and (X,τ_X) a dense subspace of Y. Then $X' = Y'$ and if $\tau = w(Y,Y')$, then $\tau_X = w(X,X')$ (see Appendix III of the candidate's dissertation). Thus a convex Hausdorff space X and its completion have the same dual, and if the completion has the weak topology with respect to this dual, so does X.

The following lemma and theorem show how the neighbourhoods of a convex Hausdorff space are related to the neighbourhoods of its completion.

LEMMA 3·1. *If Y is a topological space, O and open and X a dense subset of Y, then $O \subset$ interior $(\overline{O \cap X})$.*

Proof. Let $a \in O$ and suppose a is not an interior point of $\overline{O \cap X}$. Then O contains a point b not in $\overline{O \cap X}$. There exists an open neighbourhood U of b such that $U \cap (O \cap X)$ is empty. Now $U \cap O$ is not empty and so we have a contradiction, as X meets every non-empty open subset of Y.

COROLLARY *If X is dense in the topological space Y, then the closure in Y of any relatively open subset of X has interior in Y.*

If A is an absolutely convex subset of the convex space X and A has interior, then zero is an interior point of A, and so A is a neighbourhood.

THEOREM 3·2 *Let X be a dense subspace of the convex Hausdorff space Y, and $\mathcal{U}$ a base of absolutely convex neighbourhoods for X. Then the closures in Y of the elements of $\mathcal{U}$ form a base of neighbourhoods for Y.*

Proof Let $U \in \mathcal{U}$ then $\overline{U}$ is a neighbourhood in Y by Lemma 3·1 Corollary and the above remarks. Let V be a closed neighbourhood in Y. Then $V \cap X$ is a neighbourhood in X, and so there exists $U \in \mathcal{U}$ with $U \subset V \cap X$. Then $\overline{U} \subset \overline{V} = V$. Since Y has a base of closed neighbourhoods, the theorem has been proved.

COROLLARY If X is barrelled, so is Y.

Proof Let B be a barrel in Y. Then $B \cap X$ is a barrel in X, and so it is a neighbourhood in X. Taking closure in Y, $\overline{B \cap X}$ is a neighbourhood in Y. But $\overline{B \cap X} = B$.

4. THE COMPLETION OF A WEAK CONVEX SPACE.

We consider the completion of $(X, w(X,Y)$ were (X,Y) is a dual pair. Now (Y,X) is a dual pair and X is $w(Y^*,Y)$ dense in Y^*. To show this, consider the dual pair (Y^*,Y). Taking polars in Y or Y^*, $X^{oo} = w(Y^*,Y)$ closure of X. But $X^{oo} = \{0\}^o = Y^*$. Now $(Y^*,W(Y^*,Y)$ is complete (see [4] p.61 Prop. 13) and so $(Y^*,w(y^*,Y))$ is the completion of $(X;w(X,Y))$. If we assume that the completion of a convex Hausdorff space is unique up to topological isomorphism, it follows that $(X,w(X,Y))$ is only complete if $X = Y^*$. For example, if $p > 1$, ℓ^p under the topology $w(\ell^p,\ell^q)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) is not complete although it is sequentially complete (see [1] p.69 Cor. 29, noting that Dunford and Schwarz call a weakly sequentially complete convex space "weakly complete").

Now let (X,Y) be a dual pair and endow X with a polar topology with respect to Y. In the following paragraph we shall show that the completion of X is a subspace of Y^* under a polar topology with respect to Y.

5. A CHARACTERISATION OF THE COMPLETION OF A CONVEX HAUSDORFF SPACE UNDER A POLAR TOPOLOGY.

THEOREM 5·1. *Let (X,Y) be a dual pair. Let $\mathscr{D}$ be a collection of subsets of Y satisfying the conditions for a polar topology on X. Let $\hat{X} = \bigcap_{D\in\mathscr{D}} (X + D^{\circ})$, where polars are taken in Y^*.*

Then $\hat{X}$ is a subspace of Y^*, $(\hat{X},Y)$ is a dual pair, $\mathscr{D}$ satisfies the conditions for a polar topology on $\hat{X}$, and $\hat{X}$ under the topology $\mathscr{D}^{\circ}\cap\hat{X}$ is the completion of X under the topology $\mathscr{D}^{\circ}\cap X$.

Proof. In this theorem, polars are taken in Y or Y^*. It is easy to show that $\hat{X}$ is a subspace of Y^*. Now $(Y,\hat{X})$ is a dual pair and $X \subset \hat{X} \subset Y^*$. Thus $(Y,\hat{X})$ is a dual pair and so is $(\hat{X},Y)$.

We show that the elements of $\mathscr{D}$ are $w(Y,\hat{X})$ bounded. Let $D\in\mathscr{D}$. A sub-basic $w(Y,\hat{X})$ neighbourhood in Y is $\{z\}^{\circ}$ where $z\in\hat{X}$. Now $z\in X + D^{\circ}$ and so there exists $x\in X$ with $z\in x + D^{\circ}$. Since $D^{\circ}\cap X$ absorbs the points of X, so does D°. Thus $z\in\lambda D^{\circ}$ for some λ, and

$\{z\}^{\circ} \supset \frac{1}{\lambda} D^{\circ\circ} \supset \frac{1}{\lambda} D$. Since D is absorbed by a sub-basic $w(Y,\hat{X})$ neighbourhood in Y it is also absorbed by a finite intersection of such neighbourhoods and so D is $w(Y,\hat{X})$ bounded. Thus $\mathscr{D}$ satisfies the conditions for a polar topology, $\mathscr{D}^{\circ}\cap\hat{X}$, on $\hat{X}$.

We now show that $(X,\mathscr{D}^{\circ}\cap X)$ is dense in $(\hat{X},\mathscr{D}^{\circ}\cap\hat{X})$. Firstly, it is clear the the topology $\mathscr{D}^{\circ}\cap\hat{X}$ induces the topology $\mathscr{D}^{\circ}\cap X$ on X, since, for each $D\in\mathscr{D}$, $(D^{\circ}\cap\hat{X})\cap X = D^{\circ}\cap X$. Let $z\in\hat{X}$ and $D\in\mathscr{D}$, then $z + D^{\circ}\cap\hat{X}$ is a basic neighbourhood of z in $\hat{X}$. Now $z\in X + D^{\circ}$. Let $z = x + d$ where $x\in X$ and $d\in D^{\circ}$. Then $d = z - x \in \hat{X}$ and so $x\in z + D^{\circ}\cap\hat{X}$. Hence the closure of X is $\hat{X}$.

It remains to show that $\hat{X}$ is complete. Let $\mathcal{H}$ be a $\mathcal{B}^\circ \cap \hat{X}$ Cauchy filter on $\hat{X}$. The $\mathcal{B}^* \cap \hat{X}$ topology on $\hat{X}$ is stronger than the $w(\hat{X},Y)$ topology induced on $\hat{X}$ by the $w(Y^*,Y)$ topology on Y^*. Thus $\mathcal{H}$ is the base of a $w(Y^*,Y)$ Cauchy filter $\mathcal{G}$ on Y^*. Now Y^* is $w(Y^*,Y)$ complete (see [4] p.61 Prop.13) and so $\mathcal{G}$ converges to a point $y^* \in Y^*$ in the $w(Y^*,Y)$ topology. Let $D \in \mathcal{B}$. We shall show that $y^* \in X + D^\circ$ and that $\mathcal{H}$ contains an element B such that $B \subset y^* + D^\circ \cap \hat{X}$. It will then follow that $y^* \in \hat{X}$, $\mathcal{H} \to y^*$ and $\hat{X}$ is complete.

$\mathcal{H}$ contains an element B small of order $\frac{1}{2} D^\circ \cap \hat{X}$. Let $b \in B$, then $B \subset b + \frac{1}{2} D^\circ \in \mathcal{G}$. Now $b + \frac{1}{2}D^\circ$ is $w(Y^*,Y)$ closed and so $y^* \in b + \frac{1}{2} D^\circ$ . . . . . (1)
Also, $b \in \hat{X}$ and so $b \in X + \frac{1}{2} D^\circ$, hence $y^* \in X + \frac{1}{2} D^\circ + \frac{1}{2} D^\circ = X + D^\circ$. Thus $y^* \in \hat{X}$. From (i), $y^* = b + a$, where $a \in \frac{1}{2} D^\circ$. Since y^* and b are in $\hat{X}$, so is a, and $y^* \in b + D^\circ \cap \hat{X}$. Hence $B \subset y^* + D^\circ \cap \hat{X}$, as required.

COROLLARY 1. *Each $D \in \mathcal{B}$ is $w(Y,\hat{X})$ bounded.*

Proof This was proved in the course of the theorem.

COROLLARY 2. *If X has the toplogy $s(X,Y)$ then $\hat{X}$ has the topology $s(\hat{X},Y)$.*

Proof. Suppose X has the topology $s(X,Y)$. Then $\mathcal{B}$ is the collection of all $w(Y,X)$ bounded subsets of Y, each of which is $w(Y,\hat{X})$ bounded. Also, every $w(Y,\hat{X})$ bounded subset of Y is $w(Y,X)$ bounded. Thus $\mathcal{B}^\circ \cap \hat{X} = s(\hat{X},Y)$.

COROLLARY 3. *Let X be a convex Hausdorff space with a base $\mathcal{U}$ of neighbourhoods closed under non-zero scalar multiplication. Taking polars firstly in X' and then in X'^*, the completion of X is $\hat{X} = \bigcap_{U \in \mathcal{U}} (X + \mathcal{U}^{\circ\circ})$ under the topology $\mathcal{U}^{\circ\circ} \cap \hat{X}$.*

Proof. This result follows from Theorem 5·1 with Y replaced by X' and $\mathcal{B}$ by $\mathcal{U}^\circ$.

Suppose that the convex Hausdorff topology τ on the vector space X is a polar topology, $\mathscr{P}^{\circ}$, with respect to a subspace Y of X^*, but that τ is not a topology of the dual pair (X,Y). Then we have two forms for the completion of (X,τ). One is a subspace of Y^* under a polar topology with respect to Y and the other a subspace of X'^* under a polar topology with respect to X'.

The completion $\hat{X}$ of the convex Hausdorff space $(X,\mathscr{P}^{\circ}\cap Y)$ is a subspace of Y^*, where $\mathscr{P}^{\circ}\cap Y$ is a polar topology with respect to a subspace Y of X^*. When is $\hat{X}$ the whole of Y^* under a polar topology with respect to Y? We have seen that this is the case when $\mathscr{P}^{\circ}\cap Y$ is $w(X,Y)$. Now suppose $\hat{X} = Y^*$ under a polar topology with respect to Y. All such polar topologies coincide (see [4] p.64 (2). Since the $w(Y^*,Y)$ topology on Y^* induces the $w(X,Y)$ topology on X, it follows that the topology on X is $w(X,Y)$.

6. A SECOND CHARACTERISATION OF $\hat{X}$.

If X and Y are topological spaces, X with topology τ and $A \subset X$ and T is a mapping from X into Y, we shall say that T is *relatively continuous* on A if T is continuous on A in the topology induced on A by τ.

THEOREM 6·1. *Let A be an absolutely convex subset of the convex space X and let T be any linear mapping of X into another convex space Y. Then T is relatively continuous on A if (and only if) T is continuous at the origin in the relative topology on A.*

Proof. Suppose T is continuous at the origin in the relative topology on A. Let $a \in A$. Let V be any neighbourhood in Y. There exists an absolutely convex neighbourhood U in X such that $T(U \cap A) \subset \frac{1}{2} V$. We show that $T((a+U) \cap A) \subset T(a) + V$. Let $x \in (a+U) \cap A$. Then $x - a \in U \cap (2A) \subset 2(U \cap A)$. Hence $T(x) \in T(a) + V$. Thus T is relatively continuous on A.

Before proving the next theorem, we remark that if U is a neighbourhood in a convex Hausdorff space Y and if U° is the polar of U in Y^*, then $U^\circ = U^\circ \cap Y'$, the polar of U in Y', since a linear functional on Y is continuous if and only if it is bounded on a neighbourhood.

THEOREM 6·2. *Let (X, Y) be a dual pair. Endow Y with any topology of the dual pair (Y, X) and let $\mathcal{D}$ be a collection of closed absolutely convex subsets of Y satisfying the conditions for a polar topology on X. Let $\hat{X}$ be the completion of X under this topology. Let $z \in Y^*$. Then $z \in \hat{X}$ if and only if z is relatively continuous on each $D \in \mathcal{D}$.*

Proof. In this theorem, polars will be taken in Y or Y^*. Endow X with the topology $\mathcal{D}^\circ \cap X$ and let $\hat{X}$ be the completion of X. Let $\mathcal{U}$ be a base of closed, absolutely convex neighbourhoods for Y. Then $Y' = X = \bigcup_{U \in \mathcal{U}} U^\circ$ . . . . . (i)

Let $z \in Y^*$. We first show that $z \in \hat{X}$ if and only if for each $D \in \mathcal{D}$, there exists $U \in \mathcal{U}$ with $z \in U^\circ + D^\circ$ . . . . . (ii)

If $z \in \hat{X} = \bigcap_{D \in \mathcal{D}} (X + \mathcal{D}^\circ)$, then for each $D \in \mathcal{D}$ there exists $U \in \mathcal{U}$ with $z \in U^\circ + D^\circ$ by (i). Now suppose that for each $D \in \mathcal{D}$ there exists $U \in \mathcal{U}$ with $z \in U^\circ + D^\circ$. Since U is a neighbourhood in Y, $U^\circ \in Y'$, by the remark preceding this theorm. Thus $z \in X + D^\circ$ for each $D \in \mathcal{D}$, and so $z \in \hat{X}$.

We show next that, if $U \in \mathcal{U}$ and $D \in \mathcal{D}$ then

$$(U \cap D)^\circ \subset U^\circ + D^\circ \subset 2(U \cap D)^\circ \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (iii)$$

Now $U \cap D \subset U$ and $U \cap D \subset D$, hence $(U \cap D)^\circ \supset U^\circ$ and $(U \cap D)^\circ \supset D^\circ$ and so $U^\circ + D^\circ \subset 2(U \cap D)^\circ$.

By the Banach-Alaoglu theorem ([4] p.61 Th. 6), U° is $w(Y',Y)$ compact and so it is also $w(Y^*,Y)$ compact, since the topology $w(Y^*,Y)$ on Y^* induces the topology $w(Y',Y)$ on Y'. As D° is $w(Y^*,Y)$ closed, so is $D^\circ + U^\circ$ ([4] p.53 Lemma 7). Now $U^\circ \cup D^\circ \subset U^\circ + D^\circ$, and $U^\circ + D^\circ$ is absolutely convex. Hence $U^\circ + D^\circ$ includes the $w(Y^*,Y)$ closure of the absolutely convex hull of $U^\circ \cup D^\circ$ which is $(U^\circ \cup D^\circ)^{\circ\circ}$. Now $(U^\circ \cup D^\circ)^{\circ\circ} = (U^{\circ\circ} \cap D^{\circ\circ})^\circ = (U \cap D)^\circ$, and we have proved result (i).

Suppose $z \in Y^*$ is relatively continuous on each $D \in \mathcal{D}$. For each $D \in \mathcal{D}$ there exists a neighbourhood $U \in \mathcal{U}$ such that $y \in D \cap U \Rightarrow |(y,z)| \leq 1$. Hence $z \in (D \cap U)^\circ \subset U^\circ + D^\circ$ by (iii) and so $z \in \hat{X}$ by (ii).

Conversely, suppose $z \in \hat{X}$ and $D \in \mathcal{D}$. Let $\varepsilon > 0$, then $\frac{2}{\varepsilon} D \in \mathcal{D}$, and so there exists $U \in \mathcal{U}$ with $z \in \left(\frac{2}{\varepsilon} D\right)^\circ + \left(\frac{2}{\varepsilon} U\right)^\circ$ by (ii). Thus $z \in \tfrac{1}{2} \varepsilon (U^\circ + D^\circ) \subset \varepsilon (U \cap D)^\circ$ by (iii), and $y \in U \cap D \Rightarrow \left|(y, \frac{z}{\varepsilon})\right| \leq 1 \Rightarrow |(y,z)| \leq \varepsilon$. It follows that z is continuous at 0 in the relative topology on D and so z is relatively continuous on D.

We do not lose generality by supposing that the elements of D are absolutely convex and closed. The theorem does give us a characterisation of $\hat{X}$ when X has a polar topology on any suitable collection $\mathcal{C}$ of subsets of Y. If $\mathcal{D}$ is the collection of closures of the absolutely convex hulls of the elements of $\mathcal{C}$, then the polar topologies $\mathcal{D}^\circ$ and $\mathcal{C}^\circ$ on X are the same.

COROLLARY 1. *If every linear functional on Y which is relatively continuous on each $D \in \mathfrak{D}$ is also continuous on Y, then X is complete under the topology $\mathfrak{D}^\circ \cap X$.*

COROLLARY 2. *The same linear functionals on Y are relatively continuous on each $D \in \mathfrak{D}$ for all topologies of the dual pair (Y,X).*

Proof This follows because the same absolutely convex sets are bounded and closed in Y for every topology of the dual dual pair (Y,X).

COROLLARY 3. *The topologies induced on each $D \in \mathfrak{D}$ by the topologies $w(Y,X)$ and $w(Y,\hat{X})$ coincide.*

Proof. In Theorem 6·2, let Y have the topology $w(Y,X)$. Let $D \in \mathfrak{D}$ and $a \in D$. Let $z_1, z_2, \ldots, z_n \in \hat{X}$. Then $a + \{z_1, z_2, \ldots, z_n\}^\circ$ is a basic $w(Y,\hat{X})$ neighbourhood of a in Y.

Since $z_1, z_2, \ldots, z_n$ are relatively $w(Y,X)$ continuous on D, there exists a $w(Y,X)$ neighbourhood U such that

$$y \in (a+U) \cap D \Rightarrow |(y,z_i) - (a,z_i)| < 1, \quad 1 \leq i \leq n,$$

$$\Rightarrow |(y-a, z_i)| < 1, \quad 1 \leq i \leq n,$$

$$\Rightarrow y - a \in \{z_1, z_2, \ldots, z_n\}^\circ.$$

Hence $(a+U) \cap D \subset a + \{z_1, z_2, \ldots, z_n\}^\circ$, and so the topology induced on D by $w(Y,\hat{X})$ is weaker than that induced on it by $w(Y,X)$. However, the $w(Y,X)$ topology is finer than the $w(Y,\hat{X})$ topology, and so the two topologies coincide on D.

COROLLARY 4. *If the convex Hausdorff space has the Mackey topology $\tau(X,Y)$ then its completion, $\hat{X}$, has the Mackey topology $\tau(\hat{X},Y)$.*

Proof. In the theorem, let $\mathcal{D}$ be the collection of all absolutely convex $w(Y,X)$-compact subsets of Y. By Corollary 3, each D in $\mathcal{D}$ is also $w(Y,\hat{X})$ compact. Taking polars in Y^*, by Theorem 5·1, the topology on $\hat{X}$ is $\mathcal{D}^\circ \cap \hat{X}$ which is $\tau(\hat{X},Y)$, since every $w(Y,\hat{X})$ compact subset of Y is also $w(Y,X)$ compact.

If X and Y are convex spaces and $T : X \to Y$ a linear mapping which maps bounded subsets of X into bounded subsets of Y, then T is called a bounded linear mapping. A convex space is said to be *barrelogical* if every bounded linear mapping from X into any other convex space is continuous. In the following section we shall prove some results about completeness of duals of bornological spaces under certain polar topologies. For this reason, we restate Theorem 6·2 for duals of convex spaces.

THEOREM 6·3. *Let X be a convex Hausdorff space and let X' have the polar topology on a collection $\mathcal{D}$ of closed absolutely convex subsets of X. Then the completion, M, of X' under this topology is the set of all linear functionals on X which are relatively continuous on each D in $\mathcal{D}$.*

COROLLARY. *If every linear functional on X which is relatively continuous on each D in $\mathcal{D}$ is also continuous on X, then X' is complete under the polar topology $\mathcal{D}^\circ$.*

7. THE CONTINUOUS DUAL OF A HAUSDORFF BORNOLOGICAL SPACE.

THEOREM 7·1. *Let X be a Hausdorff bornological space and $\mathcal{D}$ a collection of closed absolutely convex subsets of X satisfying the conditions for a polar topology on X'. If every compact subset of X is included in some $D \in \mathcal{D}$ then X' is complete under the topology $\mathcal{D}^\circ$.*

Proof. Let $z \in M$, the completion of X'. Then z is relatively continuous on each $D \in \mathcal{D}$. We shall show that z maps bounded sets in X into bounded sets in C. It will follow that z is continuous on X and hence that $z \in X'$.

Suppose that z is not a bounded functional on X. Then there exists a bounded sequence (x_n) of X with $|(x_n, z)| > n^2$ for each n. Now $\frac{1}{n} x_n \to 0$ as $n \to \infty$,

since, if U is any absolutely convex neighbourhood in X, there exists an integer N such that $x_n \in NU$ for all n. Hence $n > N \Rightarrow \frac{1}{n} x_n \in U$.

Let $A = \{0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\}$. Then A is compact, and, by hypothesis, $A \subset D$ for some $D \in \mathcal{D}$. Now z is relatively continuous on D and so maps bounded subsets of D into bounded subsets of C. This contradicts the fact that $|(\frac{1}{n} x_n, z)| > n$ for all n.

COROLLARY 1. *If X is a Hausdorff bornological space then X' is complete under the topology $s(X',X)$. In particular if X is a metrisable convex space then X' is complete under $s(X',X)$ and if X is a normed linear space, X' is a Banach space under its strong topology.*

Proof. Let $\mathcal{D}$ in the theorem be the collection of all absolutely convex, closed, bounded subsets of X. Since every compact subset of X is bounded, the conditions of the theorem are satisfied, and so X' is complete under $s(X',X)$.

COROLLARY 2. *If X is a Hausdorff bornological space, X' is complete under the topology $\tau(X',X)$.*

Proof. Let $\mathcal{B}$ in the theorem be the collection of all absolutely convex $w(X,X')$ compact subsets of X. Then $\mathcal{B}^{\circ}$ is the topology $\tau(X',X)$. Every compact subset of X is also $w(X,X')$ compact and so is its absolutely convex hull. By the theorem, X' is complete.

COROLLARY 2. *If X is a Hausdorff bornological space, X' is complete under the topology* $t(X',X)$.

Proof. Let $\mathcal{B}$ in the theorem be the collection of all absolutely convex $w(X,X')$ compact subsets of X. Then $\mathcal{B}^\circ$ is the topology $t(X',X)$. Every compact subset of X is also $w(X,X')$ compact and so is its absolutely convex hull. By the theorem, X' is complete.

8. A THIRD CHARACTERISATION OF THE COMPLETION OF A CONVEX HAUSDORFF SPACE

In this section we obtain a characterisation of the completion $\hat{X}$ of a convex Hausdorff space X which will enable us to show that completeness of subsets of X is stable under the formation of stronger polar topologies on X with respect to the same subspace of X^* .

THEOREM 8·1. *Let (X,Y) be a dual pair and let $\mathcal{D}$ be a collection of $w(Y,X)$ closed absolutely convex subsets of Y satisfying the conditions for a polar topology on X. Let $\hat{X}$ be the completion of X under this topology. Let $z \in Y^*$. Then $z \in \hat{X}$ if and only if $(z^{-1}(0)) \cap D$ is $w(Y,X)$ closed for each $D \in \mathcal{D}$.*

Proof. Endow Y with the topology $w(Y,X)$. Let $z \in \hat{X}$, then z is relatively continuous on each D in $\mathcal{D}$ by Theorem 6·2, and so $(z^{-1}(0)) \cap D$ is closed in D . Since D is closed, this set is also closed in X.

Conversely, suppose $H = z^{-1}(0)$ intersects each D in $\mathcal{D}$ in a closed set. Let $D \in \mathcal{D}$. If z vanishes on D , then z is relatively continuous on D , so suppose z does not vanish on D. Let ε be any positive number, then we can choose a point $a \in D$ with $(a,z) = \alpha$, $0 < \alpha \leq \varepsilon$. For, let $b \in D$ be such that $(b,z) = \beta \neq 0$. If $|\beta| \leq \varepsilon$, choose a to be $\frac{|\beta|}{\beta} b$, which is also in D . If $|\beta| > \varepsilon$, let $a = \frac{\varepsilon}{\beta} b \in D$.

Now $a \notin H$ and so $a \notin H \cap 2D$, which is closed by hypothesis. Hence there exists an absolutely convex neighbourhood U in Y with $a+U$ not meeting $H \cap 2D$.

Now $|(y,z)| < \alpha \leq \varepsilon$ for all $y \in U \cap D$. Suppose this is not true. Since $U \cap D$ is absolutely convex, there must exist $y_1 \in U \cap D$ with $(y_1,z) = -\alpha$. Then $(y_1 + a, z) = 0$, $y_1 + a \in 2D$ and $y_1 + a \in H$, and so $y_1 + a \in H \cap 2D \cap (a + U)$, which is a contradiction. Hence z is relatively continuous on at 0, and so $z \in \hat{X}$ by Theorem 6·2 .

COROLLARY 1. *Let X be a convex Hausdorff space with completion $\hat{X}$. Let $z \in X'^*$, then $z \in \hat{X}$ if and only if $z^{-1}(0) \cap U^\circ$ is $w(X',X)$ closed in X' for every neighbourhood U in X.*

Proof. In the theorem, we let Y be X' and $\mathcal{S}$ be the collection of polars in X' of the class of all neighbourhoods in X.

If Y is a vector space, and if $z \in Y^*$ and $z \neq 0$, then $z^{-1}(0)$ is a hyperplane in Y. Conversely, every hyperplane in Y is the kernel of a non-zero linear functional on Y.

A subset A of the continuous dual X' of a convex Hausdorff space X is said to be *nearly closed* if $A \cap U^\circ$ is $w(X',X)$ closed for every neighbourhood U in X.

We give now a useful characterisation of complete convex Hausdorff spaces.

COROLLARY 2. *The convex Hausdorff space X is complete if and only if every nearly closed hyperplane in X' is $w(X',X)$ closed.*

Proof. Endow X' with the topology $w(X',X)$. Suppose X is complete. Let $z^{-1}(0)$ be a nearly closed hyperplane in X' where $z \in X'^*$. By Corollary 1, $z \in \hat{X} = X = X''$. Thus $z : X' \to \mathbb{C}$ is continuous and $z^{-1}(0)$ is closed.

Now suppose every nearly closed hyperplane in X' is $w(X',X)$ closed. Let $z \in \hat{X}$. Then by Corollary 1, $z^{-1}(0)$ is closed and $z : X' \to \mathbb{C}$ is continuous and so $z \in X = X''$.

(The above characterisation of complete convex Hausdorff spaces leads to the definitions of B-complete and B_r-complete spaces. The convex Hausdorff space X is said to be

(i) B-complete if every nearly closed subspace of X' is $w(X',X)$ closed;

(ii) B_r-complete if every nearly closed, $w(X',X)$ dense subspace of X' is $w(X',X)$ closed.)

Our next theorem compares the completions of a vector space X under two comparable polar topologies.

If $\mathcal{F}$ is a filter on a vector space X and A is a subset of X then $\mathcal{F} \cap A$ is defined to be $\{A \cap F : F \in \mathcal{F}\}$.

If A is a subset of the subspace X of the convex Hausdorff space Y, then A is complete in X if and only if A is complete in Y. Suppose A is complete in X. Let $\mathcal{F}$ be a Cauchy filter on Y which contains A. Then $\mathcal{F} \cap X$ is a Cauchy filter on X which contains A. Let $\mathcal{F} \cap X \to a \in A$, then $\mathcal{F} \to a$, and A is complete in Y. Conversely, suppose A is complete in Y. Let $\mathcal{F}$ be a Cauchy filter on X which contains A. Then $\mathcal{F}$ is a base for a Cauchy filter $\mathcal{G}$ on Y which contains A. If $\mathcal{G} \to a \in A$, then $\mathcal{F} \to a$, and A is complete in X.

THEOREM 8·2. *Let (X,Y) be a dual pair and let $\mathcal{D}$ and $\mathcal{E}$ be collections of absolutely convex $w(Y,X)$ closed subsets of Y satisfying the conditions for a polar topology on X. Taking polars in Y, let M, N be the completions of X under the topologies $\mathcal{D}^\circ \cap X$, $\mathcal{E}^\circ \cap X$ respectively. If $\mathcal{D} \subset \mathcal{E}$, then $N \subset M$. In particular, if X is complete under the topology $\mathcal{D}^\circ \cap X$, it is also complete under the topology $\mathcal{E}^\circ \cap X$.*

Proof. If $z \in N$, then $z^{-1}(0) \cap D$ is $w(Y,X)$ closed for each $D \in \mathcal{D}$, by Theorem 8·1 and the fact that $\mathcal{D} \subset \mathcal{E}$. By a second application of Theorem 8·1, $z \in M$.

COROLLARY 1. *If also, K is a subset of X which is $\mathcal{D}^\circ \cap X$-complete in X, it is also $\mathcal{E}^\circ \cap X$-complete.*

Proof By the note preceding this theorem, K is complete and hence closed in M, and so K is closed in N under the topology induced on it by the $\mathcal{D}^\circ \cap M$ topology on M. This topology is weaker than the topology $\mathcal{E}^\circ \cap N$, and so K is closed and hence complete in N. Thus K is complete in X under the topology $\mathcal{E}^\circ \cap X$.

COROLLARY 2 *If the vector space X is complete under one polar topology with respect to a subspace Y of X^*, then X is complete under any stronger polar topology with respect to Y.*

9. A GENERALISATION OF THE BANACH-ALAOGLU THEOREM.

The Banch-Alaoglu theorem states that if U is a neighbourhood in a convex Hausdorff space X, then U° is $w(X',X)$ compact. We shall find the strongest convex topology on X' under which U° is compact for every neighbourhood U in X.

Let (X,ξ) be a convex Hausdorff space and let $\mathcal{B}$ be the the collection of all closed absolutely convex pre-compact subsets of X. Then the topology, $\mathcal{B}^\circ$, on X' is denoted $\Pi(\xi)$.

THEOREM 9·1. *Let X be a convex Hausdorff space with topology ξ. For every neighbourhood U in X, U° is $\Pi(\xi)$ compact. $\Pi(\xi)$ is the strongest polar topology on X' under which the sets U° are compact or even pre-compact.*

Proof. Let $\mathcal{A}$ be the collection of absolutely convex, closed, pre-compact subsets of X, and $\mathcal{B}$ be the collection of polars in X' of neighbourhoods in X. If $U^\circ \in \mathcal{B}$, then U° is $w(X',X)$ complete, since it is $w(X',X)$ compact. By Theorem 8·2 Corollary 1, U° is also $\Pi(\xi)$ complete. We now show that U° is $\Pi(\xi)$ pre-compact.

Now $\mathcal{B}^\circ = \xi$ and $\mathcal{A}^\circ = \Pi(\xi)$. We use Th. 3 Cor. p.51 of [4]. Since each $A \in \mathcal{A}$ is $\mathcal{B}^\circ$ pre-compact, each $U^\circ \in \mathcal{B}$ is $\mathcal{A}^\circ$ pre-compact.

Let $\mathcal{C}^\circ$ be any other polar topology on X' such that each $U^\circ \in \mathcal{B}$ is $\mathcal{C}^\circ$ pre-compact. Then each $C \in \mathcal{C}$ is $\mathcal{B}^\circ$ pre-compact, and so $\mathcal{C}^\circ$ is weaker than $\mathcal{A}^\circ$.

10. AN EXTENSION OF A CONTINUOUS LINEAR MAPPING.

In this section we show that a continuous linear mapping from one convex Hausdorff space into another can be uniquely extended to a continuous linear mapping from the completion of the first space into the completion of the second. Using this result, we shall show that the completion of a convex Hausdorff space is unique up to topological isomorphism.

LEMMA 10·1. *Let (X,Y), (Y,Z), (U,V), (V,W) be dual pairs and $T : X \to U$ a linear mapping whose transpose, T', maps V into Y. Let the transpose, T'', of T' map Z into W. Then, if $A \subset X$, $T''(A^{oo}) \subset (T(A))^{oo}$, where polars are taken in Y,Z,V or W.*

Proof.

$$\begin{array}{ccccc} X & & Y & & Z \\ T \downarrow & & \uparrow T' & & \downarrow T'' \\ U & & V & & W \end{array}$$

Since $T'(V) \subset Y$, T is $w(X,Y) - w(U,V)$ continuous (see [4] p.38 Prop.12), and so $(T(A))^{o} = (T')^{-1}(A^{o})$ (see [4] p.39 Lemma 6). Thus $T'(T(A))^{o} \subset A^{o}$ and $T'(T(A))^{o})^{o} \supset A^{oo}$. Now $T''(W) \subset Z$ and so T' is $w(V,W) - w(Y,Z)$ continuous. Hence $(T'(T(A))^{o})^{o} = (T'')^{-1}(T(A))^{oo}$ using the same reference as above. It now follows that $T''(A^{oo}) \subset (T(A))^{oo}$.

THEOREM 10·2. *Let X and Y be convex Hausdorff spaces and let T be a continuous linear mapping of X into Y. Then T has a unique extension $\hat{T}$ which is a continuous linear mapping of $\hat{X}$ into $\hat{Y}$.*

Proof. Let $\mathcal{U}$, $\mathcal{V}$ be bases of closed absolutely convex neighbourhoods in X and Y respectively. Let $T'^{*} : X'^{*} \to Y'^{*}$ be the transpose of T'. Since T is continuous, T' maps Y' into X'.

$$\begin{array}{ccccc} X & & X' & & X'^* \\ T\downarrow & & \uparrow T' & & \downarrow T'^* \\ Y & & Y' & & Y'^* \end{array}$$

All polars will be taken in X', X'^*, Y' or Y'^*.

For each $U \in \mathcal{U}$, by Lemma 10·1,

$$T'^*(U^{oo}) \subset (T(U))^{oo} \quad \ldots\ldots\ldots\ldots \quad (a)$$

Now $T'^*(X) \subset \bigcap_{U\in\mathcal{U}} T'^*(X + U^{oo})$ (by Theorem 5·1 Corollary 3)

$$\subset \bigcap_{U\in\mathcal{U}} (Y + (T(U))^{oo}) \quad \text{by (a) and the fact that } T'^*\big|_X = T.$$

Let $z \in \bigcap_{U\in\mathcal{U}} (Y + (T(U))^{oo})$. Let $V \in \mathcal{V}$. There exists $U \in \mathcal{U}$ such that $T(U) \subset V$, hence $z \in Y + V^{oo}$, and so

$$T'^*(\hat{X}) \subset \bigcap_{V\in\mathcal{V}} (Y + V^{oo}) = \hat{Y} \quad \ldots\ldots\ldots\ldots \quad (b)$$

Let $V^{oo} \cap Y$ be a basic neighbourhood in $\hat{Y}$, where $V \in \mathcal{V}$. Let $U \in \mathcal{U}$ be such that $T(U) \subset V$. Then $T'^*(U^{oo} \cap \hat{X}) \subset V^{oo} \cap \hat{Y}$ by (a) and (b). It follows that $\hat{T} = T'^*\big|_{\hat{X}}$ is a continuous linear map from $\hat{X}$ into $\hat{Y}$ and it is an extension of T. Since X is dense in $\hat{X}$, $\hat{T}$ is unique.

We now consider $\hat{T}$ in the case when T is one-to-one and open as a mapping onto $\hat{T}(\hat{X})$

THEOREM 10·3. *With the hypothesis of Theorem 10·2, if also, T is a topological isomorphism of X onto $T(X)$, then $\hat{T}$ is a topological isomorphism of $\hat{X}$ onto $\hat{T}(\hat{X})$.*

Proof. We use the notation of Theorem 10.2. All closures will be taken in $\hat{X}$ or $\hat{T}(\hat{X})$.

The sets $T(U)$, $U \in \mathcal{U}$, form a base of neighbourhoods for $T(X)$. Also, $\overline{T(X)} = \overline{T(X)} \supset T(\overline{X}) = \hat{T}(\hat{X})$, and so $T(X)$ is dense in $\hat{T}(\hat{X})$. By Theorem 4·2, the sets $\overline{T(U)}$, $U \in \mathcal{U}$, form a base of neighbourhoods for $\hat{T}(\hat{X})$.

Let $U \in \mathcal{U}$ and $x \in \hat{X}$ be such that $\hat{T}(x) \in \overline{T(U)}$. We shall show that $x \in 2\bar{U}$. For, if this is not the case, we may choose $W \in \mathcal{U}$ with $W \subset U$ such that $x + \bar{W}$ does not meet $2U$. X is dense in $\hat{X}$, so we may choose $y \in X$ with $y \in x + \bar{W}$. Then $y \notin 2U$, and $\hat{T}(y) \in \hat{T}(x) + \hat{T}(\bar{U}) \subset 2\overline{T(U)}$. Now $\hat{T}(y) = T(y) \in 2\overline{T(U)} \cap T(X) = 2T(U)$, U being closed in X and T a topological isomorphism onto $T(X)$. This is a contradiction and so $x \in 2\bar{U}$.

It now follows that $\hat{T}$ is one-to-one and its inverse is continuous. Suppose $\hat{T}(x) = 0$, then $x \in 2\bar{U}$ for every $U \in \mathcal{U}$ and so $x = 0$. Also, for each $U \in \mathcal{U}$, $\hat{T}^{-1}(\overline{T(U)}) \subset 2\bar{U}$, and so $\hat{T}^{-1} : \hat{T}(\hat{X}) \to \hat{X}$ is continuous.

COROLLARY. *Let X be a convex Hausdorff space and let $\hat{X}$ be the completion of X as defined in Theorem 5·1 Corollary 3. Let Y be any other completion of X. Then there is a topological isomorphism from $\hat{X}$ onto Y which leaves X pointwise fixed.*

Proof Let T be a topological isomorphism from X onto the dense subspace $T(X)$ of the complete convex Hausdorff space Y. Then $\hat{T}$, as defined in Theorem 10·2 maps $\hat{X}$ into Y. By Theorem 10·3, $\hat{T}$ is a topological isomorphism from $\hat{X}$ onto $\hat{T}(\hat{X})$. Since $\hat{X}$ is complete, so is $\hat{T}(\hat{X})$. Thus $\hat{T}(\hat{X})$ is both closed and dense in Y, and so $\hat{T}(\hat{X}) = Y$. Since $\hat{T}$ is an extension of T, $\hat{T}$ leaves X pointwise fixed.

This corollary justifies our use of the phrase "*the completion of the convex Hausdorff space* X".

11. COMPLETIONS OF METRISABLE AND NORMED CONVEX SPACES.

In this section, we show that the completion of a convex metric space is a Fréchet space and the completion of a normed linear space is a Banach space.

We show firstly that a continuous seminorm on a convex Hausdorff space can be extended to a continuous seminorm on its completion.

If p is a continuous seminorm on the convex Hausdorff space X, then p is the gauge of an absolutely convex neighbourhood U in X. Taking polars firstly in X', then in X, let $\hat{p}$ be the gauge in $\hat{X}$ of U^{oo}. Now U^{oo} is the $w(\hat{X},X')$ closure of U. Since the dual of $\hat{X}$ is X', U^{oo} is the closure of U in $\hat{X}$. By Theorem 3.2, U^{oo} is a neighbourhood in X. Let $z \in \hat{X}$, then

$$\hat{p}(z) = \inf\{\lambda > 0 : z \in \lambda U^{oo}\}$$

$$= \inf\{\lambda > 0 : |(z,y)| \leq \lambda \text{ for all } y \in U^{o}\}$$

$$= \inf \bigcap_{y \in U^{o}} \{\lambda > 0 : |(z,y)| \leq \lambda\}$$

$$= \inf \bigcap_{y \in U^{o}} [\,|(z,y)|, \infty)$$

$$= \sup \{\,|(z,y)| : y \in U^{o}\} \qquad \ldots\ldots\ldots\ldots \quad (i)$$

By the same argument, if $x \in X$,

$$p(x) = \inf\{\lambda > 0 : x \in \lambda(U^{oo} \cap X)\} = \sup\{\,|(x,y)| : y \in U^{o}\}$$

Thus $\hat{p}$ is an extension of p. Also, $\hat{p}$ is continuous since U^{oo} is a neighbourhood in $\hat{X}$ and $\hat{p}$ is unique, since X is dense in $\hat{X}$.

If X is a normed linear space with norm p, let $p = g(B)$ where B is the closed unit ball in X. We show that $\hat{p}$ is a norm on $\hat{X}$. Suppose $\hat{p}(z) = 0$, where $z \in \hat{X}$. Then by (i), $(z,y) = 0$ for all $y \in B^{o}$ which is an absorbent subset of X'. Hence $(z,y) = 0$ for all $y \in X'$ and so $z = 0$. The sets λB, $\lambda \in C$, form a neighbourhood base for the topology on X and so the sets λB^{oo}, $\lambda \in C$, form a neighbourhood base for the topology on $\hat{X}$. Thus $\hat{p}$ determines the topology on $\hat{X}$, and $\hat{X}$ is a Banach space.

If X is a metrisable convex space, then the closures in $\hat{X}$ of a countable neighbourhood base for $\hat{X}$ form a neighbourhood base for $\hat{X}$, and so $\hat{X}$ is a Frechet space.

We have shown that if the convex Hausdorff space X has the weak, Mackey or strong topologies with respect to a subspace Y of X^*, so does $\hat{X}$. If X is metrisable, normable or barrelled, so is $\hat{X}$ (see Theorem 3.7 Corollary). According to McKennon and Robertson [3], it is not known if the properties of being reflexive, semi-reflexive, bornological or ultra-bornological are inherited by completions of convex Hausdorff spaces.

12. EBERLEIN'S THEOREM.

We shall now prove a result about weakly compact sets in complete convex Hausdorff spaces. We need the following lemma.

LEMMA 12·1. *If A is a subset of a convex space such that every sequence in A has a cluster point, then A is pre-compact.*

Proof. Suppose A is not pre-compact. There exists an absolutely convex neighbourhood U such that for every finite sequence $(x_1, x_2, \ldots, x_n)$, $A \not\subset \bigcup_{i=1}^{n} (x_i + U)$. Let $x_1 \in A$. Choose $x_2 \in A$ such that $x_2 - x_1 \notin U$. Then $A \not\subset (x_1 + U) \cup (x_2 + U)$. Choose $x_3 \in A$ such that $x_3 - x_1 \notin U$ and $x_3 - x_2 \notin U$. Proceeding thus we obtain a sequence (x_n) of points of A such that $x_n - x_m \notin U$ for all $n \neq m$. This sequence has no cluster point which is a contradiction.

COROLLARY. *If A is a subset of a complete convex space then $\bar{A}$ is compact if every sequence in A has a cluster point.*

Proof. This result follows from the above lemma, the fact that the closure of a pre-compact set is pre-compact and the fact that a complete pre-compact set is compact.

We note that if A is a subset of the subspace X of the convex space Y, and if A is pre-compact in Y. it is also pre-compact in X.

THEOREM 12.2. (Eberlein's Theorem). *Let X be a complete convex Hausdorff space. If every sequence of points in the subset A of X has a weak cluster point in X then the weak closure of A is weakly compact.*

Proof. In this theorem, polars are taken in X' or X'^*. By lemma 13·1 A is $w(X,X')$ pre-compact. It follows that A is $w(X'^*,X')$ pre-compact, as the topology $w(X,X')$ on X is induced on it by the $w(X'^*,X')$ topology on X'^*. Let $B = \mathrm{cl}_{X'^*} A$ when X'^* has the topology $w(X'^*,X')$. Then B is $w(X'^*,X')$ pre-compact (see [4] p.49 Lemma 3) and since X'^* is $w(X'^*,X')$ complete, B is $w(X'^*,X')$ compact. It suffices to show that $B \subset X$, for then $B = \mathrm{cl}_X A$.

Let $z \in B$. Let $\mathcal{U}$ be a base of closed absolutely convex neighbourhoods for X. We shall show that z is relatively $w(X,X')$ continuous on U° for each $U \in \mathcal{U}$. Since X is complete it will follow by Theorem 6·2 that $z \in X$.

Let $U \in \mathcal{U}$. z is relatively $w(X',X)$ continuous on U° if and only if for every $\varepsilon > 0$, there exists a $w(X',X)$ neighbourhood V such that

$$y \in V \cap U^\circ \Rightarrow |(z,y)| \leq \varepsilon \Rightarrow |(z/\varepsilon, y| \leq 1 .$$

So z is relatively $w(X',X)$ continuous on U° if and only if $z \in \varepsilon(V \cap U^\circ)^\circ$ for some $w(X',X)$ neighbourhood V in X'.

Suppose z is not relatively $w(X',X)$ continuous on some U°. There exists $\varepsilon > 0$ such that for every weak neighbourhood V in X, $z \notin \varepsilon(V \cap U^\circ)^\circ$. i.e. For every weak neighbourhood V, there exists $y \in V \cap U^\circ$ such that $|(z,y)| > \varepsilon$. Thus, for every finite subset $\{x_1, x_2, \ldots, x_r\}$ of X, there exists $y \in U^\circ \cap \{x_1, x_2, \ldots, x_r\}^\circ$ such that $|(z,y)| > \varepsilon$. For every $\{x_1, \ldots, x_r\} \subset X$, there exists $y \in U^\circ$ such that

$$|(z,y)| > \varepsilon \quad \text{and} \quad |(x_i,y)| < \varepsilon/3 , \quad 1 \leq i \leq r. \qquad \ldots\ldots\ldots \quad (i)$$

Now $z \in B = c\ell_{X',*}A$, hence for every finite subset $\{y_1, y_2, \ldots, y_n\}$ of X', there exists $x \in A$ with

$$x \in z + \varepsilon/3 \{y_1, y_2, \ldots, y_n\}^\circ .$$

Thus, for every finite subset $\{y_1, y_2, \ldots, y_n\}$ of X', there exists $x \in A$ with $|(x-z, y_i)| \leq \varepsilon/3$, $1 \leq i \leq n$ . . . . . (ii)

Using (i) and (ii),

choose $y_1 \in U^\circ$ such that $|(z, y_1)| > \varepsilon$;

choose $x_1 \in A$ such that $|(x_1 - z, y_1)| < \varepsilon_3$;

choose $y_2 \in U^\circ$ such that $|(z, y_2)| > \varepsilon$ and $|(x_1, y_2)| < \varepsilon/3$;

choose $x_2 \in A$ such that $|(x_2 - z, y_i)| < \varepsilon/3$, $i = 1$ or 2 ;

choose $y_3 \in U^\circ$ such that $|(z, y_3)| > \varepsilon$ and $|(x_i, y_3)| < \varepsilon/3$, $i = 1$ or 2.

Proceeding thus, we obtain sequences (x_n) in A and (y_n) in U° such that, for all n,

$|(x_n - z, y_i)| < \varepsilon/3$, $1 \leq i \leq n$; $|(z, y_n)| > \varepsilon$ and $|(x_i, y_n)| \quad \varepsilon/3$, $1 \leq i \leq n$,

. . . . . (iii)

By hypothesis, (x_n) has a weak cluster point t. Now U° is $w(X', X)$ compact and so (y_n) has a weak cluster point $d \in U^\circ$.

We may not assert that these sequences have convergent subsequences. However, for each $x \in X$ and each $y \in X'$, the sequences of complex numbers (x_n, y) and (x, y_n) have cluster points (t, y) and (x, d) respectively, and so we may claim that for each $m, n \in N$, there exist sequences (m_μ) and (n_λ) of positive integers such that

$\lim_{\lambda \to \infty} (x_n, y_{n_\lambda}) = (x_n, d)$ and $\lim_{\mu \to \infty} (x_{m_\mu}, y_m) = (t, y_m)$. [At the end of this section, we give an example of a sequence in a convex space which has a cluster point, but no convergent subsequence] .

For each fixed λ, if $m_\mu > n_\lambda$, then by (iii), $|(x_{m_\mu} - z, y_{n_\lambda})| < \varepsilon/3$ and so $\lim_{\mu \to \infty} |(x_{m_\mu} - z, y_{n_\lambda})| = |(t - z, y_{n_\lambda})| \leq \varepsilon/3$ . . . . . (iv) .

Also, for each fixed λ, using (iii), $|(t, y_{n\lambda})| \geq |(z, y_{n\lambda})| - |(t-z, y_{n_\lambda})| > \varepsilon - \varepsilon/3$. Thus $\lim_{\lambda \to \infty} |(t, y_{n\lambda})| = |(t, d)| \geq 2\varepsilon/3$.

This is a contradiction of (iv) and the theorem is proved.

COROLLARY. *Let (X,τ) be a convex Hausdorff space and A a subset of X with the property that every sequence of points of A has a cluster point. If X is complete under $\tau(X,X')$ then $\bar{A}$ is τ-compact.*

Proof. Every sequence of points in A has a weak cluster point. Let B be the weak closure of A, then, by the theorem, B is $w(X,X')$ compact and so $w(X,X')$ complete. By Theorem 8·2 Corollary 1, B is τ-complete. $\bar{A}$, being a τ-closed subset of B is also τ-complete. By lemma 13·1, $\bar{A}$ is τ pre-compact and so $\bar{A}$ is τ-compact.

A subset A of a topological space X is said to be sequentiallv compact if every sequence in A has a convergent subsequenc . Thus, if A is a sequentially compact subset of X, then every sequence in A has a cluster point. We may conclude from the above corollary that if A is a sequentially compact subset of a convex Hausdorff space (X,τ) which is complete under some stronger topology of the dual pair (X,X') then the τ-closure of A is τ-compact.

We conclude with two examples which show that compactness does not imply sequential compactness and vice versa. These examples are given as an exercise in [5] p.200 Ex. 37.

Let $\mathbb{R}$ be the set of real numbers. For each $\alpha \in \mathbb{R}$, let $C_\alpha = C$, the field of complex numbers, and let D_α be the closed unit disc in C . Let X be the topological product $\prod_{\alpha \in \mathbb{R}} C_\alpha$ and B be $\prod_{\alpha \in \mathbb{R}} D_\alpha$, a subset of X under the topology induced on it by X. Let F be the collection of elements of B which have at most countably many non-zero co-ordinates. Now B is a compact subset of X by Tychonoff's theorem. We shall show that B is not sequentially compact. F is not closed in X, so it is not compact. We shall show that it is sequentially compact.

Define the sequence $(x^{(n)})$ in B by $x^{(n)}_\alpha = e^{in\Pi\alpha}$ for each natural number n and each $\alpha \in \mathbb{R}$. Suppose that $(x^{(n)})$ has a convergent subsequence. Choose a strictly increasing sequence (t_n) of positive integers such that $(e^{i\pi t_n \alpha})$ converges for each real α. For each $q \in N$, the set $\{e^{i\pi t_n \frac{1}{q}} : n \in N\}$ has at most q values, and so there must exist an integer N_q such that $n > N_q \Rightarrow e^{i\pi t_n \frac{1}{q}}$ is a constant. Thus $n > N_q \Rightarrow t_{n+1} - t_n$ is divisible by q.

We choose a subsequence (U_n) of (t_n) to satisfy

(i) $2V_1 V_2 \ldots V_{n-1}$ divides V_n where $V_n = U_{n+1} - U_n$ for each $n \in N$

(ii) $V_{n+1} > 2^n V_n$ for each $n \in N$.

From (ii) we have that $V_{n+1} > 2^n \cdot 2^{n-1} V_{n-1} > \ldots > 2^n \cdot 2^{n-1} \ldots 2^1 V_1 = 2^{\frac{1}{2}n(n+1)} V_1$ and that $V_{n+r} > 2^{\frac{1}{2}r(2n+r-1)} V_n$ for each n and r in N. It follows that each of the series $\sum_{r=1}^{\infty} \frac{1}{V_r}$ and $\sum_{r=1}^{\infty} \frac{V_n}{V_{n+r}}$ is convergent for each $n \in N$, and $\lim_{n\to\infty} \sum_{r=1}^{\infty} \frac{V_n}{V_{n+r}} = 0$. Now let $\alpha_o = \frac{1}{V_1} + \frac{2}{V_2} + \frac{1}{V_3} + \frac{2}{V_4} + \ldots$, which is of course a convergent series.

Also, for each $n \in N$ and each $\alpha \in \mathbb{R}$, $e^{i\pi V_n \alpha} = e^{i\pi u_{n+1}\alpha} \cdot e^{-i\pi u_n \alpha}$, and so $e^{i\pi V_n \alpha}$ converges for each real α. Now, if n is even,

$$V_n \alpha_o = V_n\left(\frac{1}{V_1} + \frac{2}{V_2} + \ldots + \frac{1}{V_{n-1}}\right) + 2 + V_n\left(\frac{1}{V_{n+1}} + \frac{2}{V_{n+2}} + \ldots\right)$$

Thus, by (i), if n is even $V_n \alpha_o = k_n + 2 + S_n$, where K_n is an even integer and $S_n \to 0$ as $n \to \infty$. Similarly, if n is odd, $V_n \alpha_o = \ell_n + 1 + S_n$, where ℓ_n is even. From this it

follows that $e^{i\pi y_n^{\alpha_o}}$ does not converge which is a contradiction. [Since $(x^{(n)})$ is a sequence in the compact set B, it must have a cluster point, and so we have an example of a sequence which has a cluster point but no convergent subsequence].

Now let $(y^{(n)})$ be any sequence in F. For each n, $y_\alpha^{(n)}$ is non-zero for at most countably many values of α. Thus there are at most countably many values of α for which the set $A_\alpha = \{y_\alpha^{(n)} : n = 1,2,3, \ldots\}$ is not $\{0\}$. Denote these values of α by $\alpha_1, \alpha_2, \alpha_3, \ldots$. Choose a subsequence $(n'_k)_{k=1,2,\ldots}$ of the natural numbers, such that the sequence $(y_{\alpha_1}^{(n'_k)})$ converges to a point z_1 in C_{α_1}. Choose a subsequence $(n^2_k)_{k=1,2,3\ldots}$ of $(n'_k)_{k=1,2,\ldots}$ such that the sequence $(y_{\alpha_2}^{(n^2_k)})$ converges to a point z_2 in C. Then, of course, $y_{\alpha_1}^{(n^2_k)} \to z_1$ as $k \to \infty$. Proceeding in this way, for each natural number r, we obtain a subsequence $(n^r_k)_{k=1,2,\ldots}$ of the sequence of natural numbers such that, as $k \to \infty$, $y_{\alpha_r}^{(n^r_k)} \to z_r \in C_{\alpha_r}$.

Now let $y \in F$ be defined by $y_{\alpha_r} = z_r$, $r = 1,2,3, \ldots$

$y_\alpha = 0$ if $\alpha \neq \alpha_r$ for any r.

We shall construct a subsequence of $(y^{(n)})$ which converges to y.

Consider the subsequence $(n^1_1, n^2_2, n^3_3, \ldots) = (n^k_k)$ of the sequence of natural numbers. To form this sequence, we have taken the first term of the first subsequence constructed above and then the second term of the second subsequence and so on. For each r, the sequence $(n^r_r, n^{r+1}_{r+1}, \ldots)$ is a subsequence of $(n^r_k)_{k=1,2,\ldots}$, and so, for each r, $y_{\alpha_r}^{(n^k_k)} \to z_r$ as $k \to \infty$. If $\alpha \neq \alpha_r$ for any r, then $y_\alpha^{(n^k_k)} = 0$ for all k. Thus $y^{(n^k_k)} \to y$ as $k \to \infty$.

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M.SC. PROJECT NO. 2

TOPOLOGICAL VECTOR SPACES : COMPACT LINEAR MAPPINGS

by

Janet M. Helmstedt

A notation list and an index of definitions

for this project can be found in the candidate's dissertation.

§ 1. INTRODUCTION

By "convex space" we mean "locally convex topological vector space over the field C of complex numbers.

If X and Y are convex spaces and T is a linear mapping from X into Y, then T is said to be a *compact linear mapping* if there exists a neighbourhood U in X and a compact set K in Y such that $T(U) \subset K$.

If X and Y are convex spaces and Y is Hausdorff, any continuous linear map T from X into Y whose image is a finite dimensional subspace N, is compact. For N has the Euclidean topology, and its closed unit ball B is both a neighbourhood and compact. T, being continuous, maps a neighbourhood into B.

For example, let $z_o \in Y$, $y_o \in X'$ and let $T : X \to Y$ be defined by $T(x) = (x,y_o)z_o$. Then T is continuous and its image is a one-dimensional subspace of Y. Any linear combination of such maps will map X into a finite dimensional subspace of Y.

We shall be concerned with the following problem : If $\lambda \in C$ and if T is a compact linear operator on the convex space X, is there a subspace Y of X on which $V = \lambda I - T$ is invertible ? We need V to be 1-1 and onto Y. If $V(X) \neq X$, the obvious space to consider is $V(X)$, and, if this fails, $V^2(X)$ and so on. If T is not compact, it may happen that $V^n(X)$ is distinct from $V^{n+1}(X)$ for all n. This happens, for example, if $X = \ell^1$ and T is defined by

$$T(x_1, x_2, x_3, \dots) = (x_1, x_2 - x_1, x_3 - x_2, \dots)$$

Then $I-T$ is the "shift" operator :

$$(I-T)(x_1, x_2, x_3, x_4, \dots) = (0, x_1, x_2, x_3, \dots)$$

It may also happen, if T is not compact, and X is infinite dimensional, that V is nilpotent i.e. $V^n(X) = \{0\}$ for some n.

We shall show, that if T is a compact operator on X and $\lambda \neq 0$, then there exists a non-negative integer n such that

(i) $V^n(X) = V^{n+1}(X)$;

(ii) X is the topological direct sum of $V^n(X)$ and $(V^n)^{-1}(0)$;

(iii) V restricted to $V^n(X)$ is a topological isomorphism and so has a continuous inverse ;

(iv) $(V^n)^{-1}(0)$ is finite dimensional. [Thus the subspace Y on which we can invert V is "large"in a sense] .

We shall also show that $V = V_1 + V_2$ where V_1 is a topological isomorphism on X and V_2 maps X into a finite dimensional subspace.

If X is infinite dimensional, (ii) and (iv) preclude $V^r(X) = 0$ for any integer r

We follow [1] p.142, in defining the spectrum of a continuous linear operator T as the set of all complex numbers λ such that $\lambda I - T$ does not have a continuous inverse. This definition can be justified by regarding the set of all continuous linear operators on X as a non-commutative algebra under the operations of $+$ and composition . It is of course not consistent with normal usage in Hilbert space.

We shall consider the spectrum of the compact linear operator T on the convex Hausdorff space X and show that, except possibly for the number 0, all elements of the spectrum are eigenvalues, and that the spectrum is finite or a sequence converging to zero.

In the last section of this project we shall consider the transpose, T', of the compact linear map $T : X \to Y$. We shall show, that except possibly for 0, T and T' have the same spectrum . We shall also prove Schauder's theorem :

If X and Y are Banach spaces whose continuous duals have their norm topologies, and if T is a linear map from X into Y, then T is compact if and only if T' is compact .

The main source we have used for the above material is [1] , Chapter VIII .

As far as I am aware, Theorem 5.8 is a new result .

§ 2. THE ASCENT AND DESCENT OF A LINEAR OPERATOR ON A VECTOR SPACE .

Let X be a vector space and V a linear operator on X, and let $V^{-0}(0) = \{0\}$; $V^{-r}(0) = (V^r)^{-1}(0)$.

Then, if n is a non-negative integer ,

$V^{-n}(0) \supset V^{-n-1}(0)$ and $V^{-1}(V^{-n}(0)) = V^{-n-1}(0)$.

If there exists an integer $n \geq 0$ such that

$V^{-n}(0) = V^{-n-1}(0)$, then

$V^{-n-2}(0) = V^{-1}(V^{-n-1}(0)) = V^{-1}(V^{-n}(0)) = V^{-n-1}(0) = V^{-n}(0)$;

and, inductively, $V^{-r}(0) = V^{-n}(0)$, all $r \geq n$. Hence, either the sequence $(V^{-r}(0))$ is strictly increasing or there is a least integer $n \geq 0$ such that the $V^{-r}(0)$ are all distinct for $0 \leq r \leq n$ and subsequent subspaces are identical with $V^{-n}(0)$. If this latter is the case, we say that V has *finite ascent* n .

Similarly, the vector subspaces $X = V^0(V)$, $V(X)$, $V^2(X)$, ... , form a sequence which either decreases strictly or has a least integer $m \geq 0$ such that the $V^r(X)$ are all distinct for $0 \leq r \leq m$ and subsequent subspaces coincide with $V^m(X)$. In this latter case we say that V has *finite descent* m .

We shall show that if the ascent and descent of V are finite, then they are equal.

LEMMA 2.1 *If V is any linear operator on the vector space X and r,s are non-negative integers, then*

(i) $V^{-r}(0) = V^{-r-s}(0)$ *if and only if* $V^r(X) \cap V^{-s}(0) = \{0\}$;

(ii) $V^s(X) = V^{s+r}(X)$ *if and only if* $V^r(X) + V^{-s}(0) = X$.

Proof. (i) Suppose $V^{-r}(0) = V^{-r-s}(0)$.
Let $x \in V^r(X) \cap V^{-s}(0)$. Then $V^s(x) = 0$ and there exists y such that $V^r(y) = x$. Hence $V^{r+s}(y) = 0$ and so $y \in V^{-r-s}(0) = V^{-r}(0)$. Thus $V^r(y) = 0 = x$. Now suppose $V^r(X) \cap V^{-s}(0) = \{0\}$. We know that $V^{-r-s}(0) \supset V^{-r}(0)$. Let $x \in V^{-r-s}(0)$, then $V^{r+s}(x) = 0$ and so $V^r(x) \in V^{-s}(0)$. But $V^r(x) \in V^r(X)$ and so $V^r(x) = 0$ and $x \in V^{-r}(0)$.

(ii) Suppose $V^r(X) + V^{-s}(0) = X$. We know that $V^{s+r}(x) \subset V^s(x)$.
Let $V^s(y) \in V^s(X)$. Let $y = y_1 + y_2$ where $y_1 \in V^r(X)$ and
$y_2 \in V^{-s}(0)$. Then $V^s(y) = V^s(y_1) = V^{r+s}(y_3)$ say , since $y_1 \in V^r(X)$.
Hence $V^s(X) \subset V^{s+r}(X)$.
Now suppose that $V^s(X) = V^{s+r}(X)$. Let $x \in X$. There exists
$y \in X$ such that $V^s(x) = V^{s+r}(y)$. Now $x = x - V^r(y) + V^r(y)$ and
$V^s(x - V^r(y)) = 0$, hence $x \in V^{-s}(0) + V^r(X)$.

Theorem 2.2 *If the linear operator V on the vector space X has finite ascent n and finite descent m, then $m = n$, and $X = V^n(X) \oplus V^{-n}(0)$.*

Proof (i) Suppose $m \leq n$, then $V^{-n}(0) = V^{-n-1}(0)$ and so by the previous lemma, $V^n(X) \cap V^{-1}(0) = \{0\}$.
Now $V^m(X) = V^n(X)$, hence $V^m(X) \cap V^{-1}(0) = \{0\}$. By the lemma,
$V^{-m}(0) = V^{-m-1}(0)$, hence $m = n$.

(ii) Suppose $n \leq m$. $V^m(X) = V^{m+1}(X)$ and so, by the lemma
$V^1(X) + V^{-m}(0) = X$.
But $V^{-n}(0) = V^{-m}(0)$, hence $V^1(X) + V^{-n}(0) = X$. By the lemma, $V^n(X) = V^{n+1}(X)$.
Hence $m = n$.

(iii) $V^{-n}(0) = V^{-n-n}(0)$. Hence $V^n(X) \cap V^{-n}(0) = \{0\}$ by the lemma.
Also, $V^n(X) = V^{n+n}(X)$. Hence $V^n(X) + V^{-n}(0) = X$ bythe lemma.

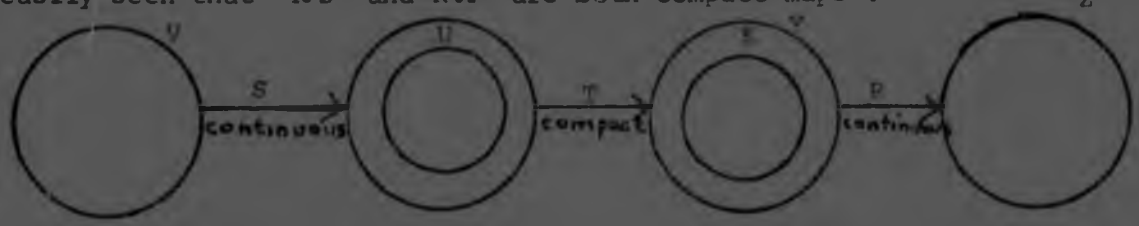
Thus $X = V^n(X) \oplus V^{-n}(0)$.

Suppose V is a linear operator on X with finite ascent and descent.
If V is 1-1, its ascent is 0, so by the theorem it maps X onto X .
If, conversely, V maps X onto X , its descent is 0 and so by the theorem it is 1-1. We shall show in the next section that if T is a *compact* operator on a convex space, then $\lambda I - T$ has finite ascent and descent, where $\lambda \neq 0$.

§ 3. COMPACT LINEAR MAPPINGS.

A compact linear map is continuous. For suppose $T(U) \subset K$ where U is a neighbourhood in the convex space X and K is a compact subset of the convex space Y, and T is a linear map. Let V be any neighbourhood in Y. Since K is bounded, there exists a complex number $\lambda \neq 0$ such that $\lambda K \subset V$ and so $T(\lambda U) \subset V$.

Let V, X, Y, Z be convex spaces and $S : V \to X$; $R : Y \to Z$ be continuous linear maps and $T : X \to Y$ a compact linear map. It is easily seen that $T \circ S$ and $R \circ T$ are both compact maps .



If T, S are both compact maps from X into Y, then so are $T+S$ and λT , $\lambda \in C$. For if T, S map the neighbourhoods U and V into the compact sets K and C , then $T+S$ maps $U \cap V$ into the compact set $K+C$, and T maps λU into λK .

From the above remarks, we see that if T is a compact linear operator on the convex space X, then so is any polynomial in T which has no constant term.

We shall make this study a little more general by considering "potentially compact" linear maps. A linear map S is said to be *potentially compact* if and only if S^k is compact for some integer $k \geq 1$. This generalisation does not complicate the work, for :

Let $V = \mu I - S$ and $W = \mu^k I - S^k$ where S^k is a compact linear map. Then V is a factor of W. Let the other factor be U. Then $U \circ V = V \circ U = \mu^k I - S^k = \lambda I - T$ say. So, instead of studying the linear operator $V = \lambda I - T$, where T is compact, we study continuous linear operators U and V which are such that

$$U \circ V = V \circ U = \lambda I - T \text{ , where } T \text{ is compact .}$$

NOTE: If $\mathcal{F}$ and $\mathcal{G}$ are filters on a convex space X and λ is a non-zero scalar, then we define $\mathcal{F}+\mathcal{G}$ to be $\{F+G : F\in\mathcal{F} \text{ and } G\in\mathcal{G}\}$ and $\lambda\mathcal{F}$ to be $\{\lambda F : F\in\mathcal{F}\}$. Then $\mathcal{F}+\mathcal{G}$ is a filter base and $\lambda\mathcal{F}$ is a filter on X. If $\mathcal{F}\to a$ and $\mathcal{G}\to b$, then $\mathcal{F}+\mathcal{G}\to a+b$ and $\lambda\mathcal{F}\to\lambda a$.

Theorem 3.1 *If T is a compact linear operator on the convex Hausdorff space X, $\lambda\neq 0$, and U,V are two continuous linear operators on X satisfying $U\circ V$ $\lambda I-T$, then*

(i) $V^{-1}(0)$ *is finite dimensional;*

(ii) V *is open as a map from X onto $V(X)$;*

(iii) $V(X)$ *is closed in X.*

Proof Let T map the neighbourhood U into the compact set K; and let $N = V^{-1}(0)$.

(i) Let $x\in\overline{U}\cap N$. Then $\lambda x = T(x)$. Hence $\lambda x\in T(\overline{U})\subset\overline{T(U)}\subset K$, and so $\overline{U}\cap N$ is compact. Hence N, having a compact neighbourhood, is finite dimensional.

(ii) Let $\mathcal{W}$ be a base of absolutely convex neighbourhoods in X. Suppose V is not open as a map from X onto $V(X)$. Then there exists $W\in\mathcal{W}$ such that $W\subset U$ and $V(W)$ is not a neighbourhood in $V(X)$.

Let $B = 2W\cap {}^{c}(W+N)$. We show firstly that the sets $V^{-1}(A)\cap B$, where $A\in\mathcal{W}$ form a base for a filter on X:

Let $A\in\mathcal{W}$. If $A\cap V(X)$ does not meet ${}^{c}(V(W))\cap V(X)$ then $A\cap V(X)\subset V(W)$ and $V(W)$ is a neighbourhood in $V(X)$, which is a contradiction.

Let $x\in A\cap V(X)\cap {}^{c}(V(W))$ and let $x = V(y)$, then $y\notin W$.

We show that there exists μ, $0<\mu\leq 1$, such that $\mu y\in 2W\cap {}^{c}W\cap V^{-1}(A)$. If $y\in 2W$, let $\mu = 1$. Suppose $y\notin 2W$. Since W is absolutely convex and absorbent, the set $\{\lambda : \lambda y\in W\}$ is of the form $[0,a)$ or $[0,a]$ with $0<a\leq\frac{1}{2}$. Hence the set $\{\lambda : \lambda y\in 2W\}$

is of the form $[0,2a)$ or $[0,2a]$. Choose $\mu \in (a,2a)$, then $\mu y \in 2W \cap {}^{c}W \cap V^{-1}(A)$.

We have thus shown that the sets $V^{-1}(A) \cap B$ are non-empty for all $A \in \mathcal{W}$.

Also, if $A_1, A_2 \in \mathcal{W}$ there exists $A_3 \in \mathcal{W}$ with $A_3 \subset A_1 \cap A_2$ and $(V^{-1}(A_1) \cap B) \cap (V^{-1}(A_2) \cap B) \supset V^{-1}(A_3) \cap B$.

So the sets $V^{-1}(A) \cap B$ with $A \in \mathcal{W}$ form the base for a filter $\mathcal{F}$ on X which contains B. Also, $V(\omega') \to 0$, hence $U \circ V(\mathcal{F}) \to 0$. Now $B \in \mathcal{F}$, hence $T(B) \in T(\mathcal{F})$ and $T(B) \subset T(2W) \subset 2K$. So $T(\mathcal{F})$ clusters at some point $z \in 2K$. Now $\lambda(\mathcal{F}) = U \circ V(\mathcal{F}) + T(\mathcal{F})$. Hence $\lambda\mathcal{F}$ clusters at z, and so $z \in \overline{\lambda B} = \lambda\overline{B}$. Also, $V(\lambda\mathcal{F})$ clusters at $V(z)$, and $V(\lambda\mathcal{F}) = \lambda(V(\mathcal{F})) \to 0$. Hence $V(z) = 0$ and $z \in N$. Thus $\frac{1}{\lambda} z \in N \cap \overline{B}$. Now $W+N$ is a neighbourhood of $\frac{1}{\lambda} z$, but, by definition of B, $W+N$ does not meet B, which is a contradiction. It follows that V is open as a map from X onto $V(X)$.

(iii) Let $a \in \overline{V(X)}$. By (ii), $V(U)$ is a neighbourhood in $V(X)$. Choose $W_1 \in \mathcal{W}$ such that $W_1 \cap V(X) \subset V(U)$. Since $a \in \overline{V(X)}$, the set $(a + \frac{1}{2}W_1) \cap V(X)$ is non-empty. It is also small of order $W_1 \cap V(X)$. Let $V(b) \in (a + \frac{1}{2} W_1) \cap V(X)$, then $(a+\frac{1}{2}W) \cap V(X) \subset V(b) + W_1 \cap V(X) \subset V(b) + V(U) = V(b+U)$. Now, since $a \in \overline{V(X)}$, the sets $(a+W) \cap V(X)$, with $W \in \mathcal{W}$ are non-empty, and each one meets every other. Since $(a + \frac{1}{2}W_1) \cap V(X) \subset V(b + U)$, it follows that the sets $(a + W) \cap V(X) \cap V(b+U) = (a + W) \cap V(b + U)$ are also non-empty, and clearly each one meets every other.

The above statement also applies to the sets $V^{-1}(a + W) \cap (b + U)$, for $W \in \mathcal{W}$. Hence these sets form the base of a filter $\mathcal{F}$ on X. Now, $V(\mathcal{F}) \to a$. Also, $T((b + U) \cap V^{-1}(a + W_1)) \in T(\mathcal{F})$, hence $T(b + U) \in T(\mathcal{F})$. Now $T(b + U) = T(b) + T(U) \subset T(b) + K$, which is compact. Hence $T(\mathcal{F})$ clusters at a point $y \in T(b) + K$. Also, $\lambda(\mathcal{F}) = T(\mathcal{F}) + U \circ V(\mathcal{F})$ clusters at $y + U(a)$. Thus $V(\lambda\mathcal{F})$ clusters at $V(y + U(a)) \in V(X)$, and $V(\lambda\mathcal{F}) = \lambda(V\mathcal{F}) \to \lambda a$. It follows that $\lambda a \in V(X)$ and so $a \in V(X)$ and $V(X)$ is closed.

COROLLARY. *If, in addition to the hypotheses of the theorem,* $U \circ V = V \circ U$, *then for any integer* $r \geq 1$, $V^{-r}(0)$ *is finite dimensional and* V^r *is an open map of* X *onto the closed subspace* $V^r(X)$.

Proof. $U^r \circ V^r = (U \circ V)^r = \lambda^r I - S$ where S is a polynomial in T without a constant term and is thus compact. The corollary now follows from the theorem .

LEMMA 3.2 *Let* T *be a linear operation on the convex space* X *which maps the neighbourhood* U *into a compact set* K . *Let* $W = \lambda I - T$ *where* $\lambda \neq 0$, *and let* F *and* G *be distinct subspaces of* X *with* G *closed,* $G \subset F$ *and* $W(F) \subset G$. *Then there exists a point* x *in* $F \cap (2U)$ *with* $x \notin G + U$.

Proof. We first show that $F \not\subset G + U$. For suppose $F \subset G + U$.

Let $\varepsilon > 0$.

$$\begin{aligned} F = \varepsilon \lambda F &\subset G + \varepsilon \lambda U \subset G + \varepsilon \lambda (U \cap F) \quad [\text{since } G \subset F] \\ &\subset G + \varepsilon [W(U \cap F) + T(U \cap F)] \\ &\subset G + \varepsilon [G + T(U)] \\ &\subset G + \varepsilon T(U) \\ &\subset G + \varepsilon K . \end{aligned}$$

Let V be any neighbourhood. There exists $\mu > 0$ such that $K \subset \mu V$, since K is a bounded set. For every positive ε , $F \subset G + \varepsilon \mu V$. In particular , $F \subset G + V$. From this it follows that $F \subset \overline{G} = G$. This contradicts the hypotheses of the theorem, and so $F \not\subset G + U$. .... (i)

We now show that $F \cap 2U \not\subset G + U$. Suppose that $F \cap 2U \subset G + U$. We shall deduce that $F \cap 2^n U \subset G + F \cap 2^{n-1} U$ where n is any integer > 1 . Let $2^n u \in F \cap 2^n U$ where $u \in U$. $2^n u = 2^{n-1}(2u) \in 2^{n-1}(G + U) = G + 2^{n-1} U$. Let $2^n u = g + 2^{n-1} u_1$. Then $2^{n-1} u_1 \in F$, as $G \subset F$. Hence $F \cap 2^n U \subset G + 2^{n-1} U \cap F$, for every integer $n > 1$. It follows that $F \cap 2^n U \subset G + F \cap 2^{n-1} U \subset G + G + F \cap 2^{n-2} U = G + F \cap 2^{n-2} U \subset \ldots \subset G + U$.

As U is absorbent, each $f \in F$ is contained in $2^n U$ for some positive integer n. It follows that $F \subset G + U$, which contradicts (i) above. We may now conclude that $F \cap 2U \not\subset G + U$, and so there exists $x \in F \cap 2U$, $x \notin G + U$.

Theorem 3.3 *Let* T *be a compact linear operator on the convex Hausdorff space* X , *let* $\lambda \neq 0$ *and let* U, V *be two continuous linear operators such that* $U \circ V = V \circ U = \lambda I - T$.

Then U *and* V *have finite ascent and descent* .

Proof Let T map the neighbourhood U into the compact set K.
Let $W = U \circ V = \lambda I - T$.
Suppose V does not have finite ascent. For each integer $r \geq 1$, let $N_r = V^{-r}(0)$. Then N_r is a closed subspace of X and $r \neq s \Rightarrow N_r \neq N_s$. Let $x \in N_{r+1}$ then $V^{r+1}(x) = 0$.

Hence $V^r(W(x)) = U \circ V^{r+1}(x) = 0$, and so $W(x) \in N_r$. Thus $W(N_{r+1}) \subset N_r$.

We now apply Lemma 3.2 with $F = N_{r+1}$, $G = N_r$. For each $r \geq 1$, there exists a point $x_r \in N_{r+1} \cap (2U)$, with $x_r \notin N_r + U$. . . . (i)

If $r > s$,

$$T(x_r) - T(x_s) = \lambda x_r - W(x_r) - \lambda x_s + W(x_s)$$

$$\in \lambda x_r - N_r - N_r + N_r = \lambda x_r + N_r,$$

as $N_s \subset N_{r+1}$ and $W(N_{r+1}) \subset N_r$.

Suppose that for some r and s with $r > s$, $T(x_r) - T(x_s) \in \lambda U$. Since $T(x_r) - T(x_s) \in \lambda x_r + N_r$, $\lambda x_r \in T(x_r) - T(x_s) - N_r \subset \lambda U + N_r$. This contradicts (i), so that for every r and s with $r > s$,

$$T(x_r) - T(x_s) \notin \lambda U. \quad . . . \quad (ii)$$

We shall show that this contradicts the fact that $2K$ is pre-compact. There exist $y_1, y_2, \ldots, y_n \in 2K$ such that $2K \subset \bigcup_{r=1}^{n} (y_r + \lambda U)$.

Now, by (ii), $T(x_r) \neq T(x_s)$ if $r \neq s$, so the points $T(x_r)$ are all distinct. Also, $x_r \in N_{r+1} \cap (2U)$ and so $T(x_r) \subset 2K$ for each r. Hence there exist r,s,i with $r > s$ such that $T(x_r) \in y_i + \lambda U$ and $T(x_s) \in y_i + \lambda U$ and so $T(x_r) - T(x_s) \in \lambda U$. This is a contradiction, and so V must have finite ascent.

Suppose that V does not have finite descent. Then the subspaces $M_r = V^r(X)$ are closed (by 3.1 Corollary) and distinct.

Let $x = V^r(y) \in M_r$. Then $W(x) = U \circ V^{r+1}(y) = V^{r+1}(U(y)) \in M_{r+1}$

and so $W(M_r) \subset M_{r+1}$.

By Lemma 3.2 , for each r , there exists a point $z_r \in M_r \cap 2U$ with $z_r \in M_r \cap 2U$ with $z_r \notin M_{r+1} + U$ and $T(z_r) \in 2K$.

If $r < s$,

$T(z_r) - T(z_s) = \lambda z_r - W(z_r) - \lambda z_s + W(z_s) \in \lambda z_r + M_{r+1}$.

This leads to a similar contradiction and so V has finite descent .

COROLLARY 1 . *If n is the ascent of V , then X is the topological direct sum of* $V^{-n}(0)$ *and* $V^n(X)$.

Proof By Theorem 2.2 , the ascent and descent of V are both n and X is the algebraic direct sum of $V^n(X)$ and $V^{-n}(0)$. By the corollary to theorem 2.1 $V^{-n}(0)$ is finite dimensional and $V^n(X)$ is closed. Thus X is the topological direct sum of $V^n(X)$ and $V^{-n}(0)$, using [1 | page 96, Prop. 29 Cor. .

COROLLARY 2 . *If T is a compact linear mapping of the convex Hausdorff space into itself and* $W = \lambda I - T$, $\lambda \neq 0$, *the following are equivalent :*

(i) λ *is not an eigenvalue of T ;*

(ii) W is 1-1 ;

(iii) the ascent of W is 0 ;

(iv) the descent of W is 0 ;

(v) W maps X onto itself .

Use theorem 3.3 with U = W , V = I .

Theorem 3.4 *Let T, U, V be linear operators on the convex Hausdorff space X with T compact and U, V continuous, satisfying* $U \circ V = V U = \lambda I - T$, *where* $\lambda \neq 0$. *Then*

(i) X is the topological direct sum of the closed subspaces $V^n(X) = M$ *and* $V^{-n}(0) = N$ *where n is the ascent of V ;*

(ii) $V(M) = M$ *and* $V(N) = N$;

(iii) V is a topological isomorhphism on M ;

(iv) N is finite dimensional and $V^n(N) = 0$;

(v) For each integer $r > 0$, $V^{-r}(0)$ *and* $X / V^r(X)$ *have the same dimension ;*

(vi) $V = V_1 + V_2$ *where V is a topological isomorphism of X onto itself and* $V_2(X)$ *is finite dimensional.*

Proof. (i) This follows from 3.1 Corollary and 3.3 Corollary (i) .

(ii) $V(M) = V^{n+1}(X) = M$; and

$V(N) = V(V^{-n}(O)) = V(V^{-n-1}(O)) = V \circ V^{-1}(N) \subset N$.

(iii) $U(M) = U(V^n(X)) = V^n(U(X)) \subset V^n(X) = M$. Since $\lambda M = M$, it follows that $T(M) \subset M$. Clearly T is a compact linear operator on M . We may thus apply 3.1 with $X=M$ and U,V,T restricted to M . Since $V(M) = M$, it follows from 3.1 that V is an open mapping . By 3.3 Corollary 2 , V is 1-1 on M . Hence V is a topological isomorphism on M .

(iv) Let r be a positive integer, then

$X/V_r(X) \cong (M+N)/(V^r(M) + V^r(N)) \cong (M+N)/(M+V^r(N))$.

We show that

$$(M+N)/(M+V^r(N)) \cong N/V^r(N) .$$

Let $m+n + M + V^r(N) = n + M + V^r(N) \in (M+N)/(M+V^r(N))$.

Let $f(n + M + V^r(N)) = n + V^r(N)$, then f is well defined, for if $n_1 + M + V^r(N) = n_2 + M + V^r(N)$, then $n_1 - n_2 \in M + V^r(N)$, and since $M \cap N = \{O\}$, $n_1 - n_2 \in V^r(N)$ and so $n_1 + V^r(N) = n_2 + V^r(N)$.

It is easily seen that f is linear, 1-1 and maps $X/V^r(X)$ onto $N/V^r(N)$, ...(*)

Now $V^{-r}(O) \subset N$ hence $\dim N = \dim V^r(N) + \dim (N/V^r(N))$ and $\dim N = \dim V^{-r}(O) + \dim (N/V^{-r}(O))$. But $N/(V^r)^{-1}(O) \cong V^r(N)$, hence $\dim (N/V^r(N)) = \dim V^{-r}(O) = \dim X/V^r(X)$ by (*) .

(vi) Let P and Q be the projections of X onto M and N respectively.

Let $V_1 = V \circ P + Q$, then V_1 is a topological isomorphism of X onto itself ; for

(a) V_1 is clearly linear ;

Let $x = m + n \in X$ where $m \in M$, $n \in N$. Then

(b) suppose $V_1(x) = O = V(m) + n$. Now $V(m) \in M$ and $M \cap N = \{O\}$ hence $V(m) = n = O$. Since V is 1-1 on M , $m = O$. Hence V_1 is 1-1 .

(c) Since V is an isomorphism on M, there exists $m_1 \in M$ such that $V(m_1) = m$. Then $V_1(m_1 + n) = m + n$ and V_1 maps X onto X.

(d) The continuity of V_1 follows from that of V, P and Q.

(e) $V_1^{-1}(x) = (V|M)^{-1}(P(x)) + Q(x)$, from (c).

The continuity of V_1^{-1} follows from that of P, Q and $(V|M)^{-1}$.

Now let $V_2 = V \circ Q - Q$, then $V_2(x) = V(n) - n \in N$. and $V_1 + V_2 = V \circ (P+Q) = V$.

COROLLARY 1. *Let S be a continuous linear operator on the convex Hausdorff space X. Let ϕ be a polynomial with $\phi(S)$ compact. Then, if $\phi(\mu) \neq 0$, the map $\mu I - S$ has finite ascent and descent and all the properties of V in the above theorem.*

Proof Let $\psi(\xi)$ be the quotient when $\phi(\xi)$ is divided by $(\xi - \mu)$. By the remainder theorem, $\psi(\xi)(\mu - \xi) = \phi(\mu) - \phi(\xi)$ and so, $\psi(S) \circ (\mu I - S) = \phi(\mu) I - \phi(S)$.

We now apply theorem 3.4 with $V = \mu I - S$, $U = \psi(S)$, $\lambda = \phi(\mu)$, $T = \phi(S)$.

COROLLARY 2. *If T is a compact linear operator on the convex Hausdorff space X and $W = \lambda I - T$, $\lambda \neq 0$, then W has all the properties of V in the above theorem. In addition on $M = W^n(X)$ on which W is a topological isomorphism, W^{-1} is of the form $\lambda^{-2}(\lambda I - S)$ where S is a compact linear operator on M and S commutes with T. If $\lambda \neq 0$ and λ is not an eigenvalue of T, W^{-1} has the above form on the whole of X.*

Proof. We restrict all mappings to M. $W^{-1} = \lambda^{-2}(\lambda I - (\lambda I - \lambda^2 W^{-1}))$. Let $S = \lambda I - \lambda^2 W^{-1}$. Then S is continuous on M.

$I = W \circ W^{-1} = (\lambda I - T) \circ \lambda^{-2}(\lambda I - S) = I - \lambda^{-1}T - \lambda^{-1}S + \lambda^{-2} T \circ S = W^{-1} \circ W = \lambda^{-2}(\lambda I - S) \circ (\lambda I - T) = I - \lambda^{-1}T - \lambda^{-1}S + \lambda^{-2} S \circ T$ and so $T \circ S = S \circ T$.

Also $T \circ (\lambda I - S) = (\lambda I - W) \circ \lambda^2 W^{-1} = \lambda^2(\lambda W^{-1} - I) = -\lambda S$. $T \circ (\lambda I - S)$ is compact, hence S is compact.

Now suppose $\lambda \neq 0$ and λ is not an eigenvalue of T. Then $\lambda I - T$ maps X onto X and $M = X$.

COROLLARY 3. *If T is a compact linear operator on the convex Hausdorff space X, then apart possibly from 0, the spectrum of T consists of eigenvalues only. Each eigenvalue has a finite dimensional subspace of eigenvectors.*

Proof. Let $\lambda \neq 0$, $\lambda \in$ spectrum of T . Then either

(i) $\lambda I - T$ is not 1-1 ;

or (ii) $(\lambda I - T)(X) \neq X$

or (iii) $\lambda I - T$ is 1-1 and maps X onto itself, but its inverse is not continuous .

Now (iii) is impossible by Corollary 2 and (i) and (ii) are equivalent by 3.3 Corollary 2 .

Hence λ is in the spectrum of T .

The last part of the corollary follows from the fact that $V^{-1}(0)$ is finite dimensional, where $V = \lambda I - T$.

COROLLARY 4. *(The Fredholm Alternative) If T is a compact linear operator on the convex Hausdorff space X and $\lambda \neq 0$ [or, more generally, if T is continuous, and $\phi(T)$ is compact where ϕ is a polynomial and $\phi(\lambda) \neq 0$] then the equation in x :*

$$(\lambda I - T)x = y$$

either has a unique solution for each $y \in X$ or there exists $x \in X$ with $(\lambda I - T)x = 0$ and $x \neq 0$. If this latter is the case, the set of all x such that $(\lambda I - T)x = 0$ is finite dimensional .

§ 4. THE SPECTRUM OF A COMPACT LINEAR OPERATOR

Theorem 4.1 *If X is a convex Hausdorff space and T a compact linear operator on X , then the spectrum of T consists either of a finite set or of a sequence convergent to zero .*

Proof. We have seen that, apart possibly from 0, the points of the spectrum of T are all eigenvalues.

Let $\varepsilon > 0$, and suppose there exists an infinite sequence (λ_r) of distinct eigenvalues of T with $|\lambda_r| \geq \varepsilon$, all $r \geq 1$. Let (x_r) be a sequence of corresponding eigenvectors, and H_n the space spanned by the first n of these. Then, the H_n, being finite dimensional are closed, and are all distinct, as eigenvectors corresponding to distinct eigenvalues are linearly independent .

Let $W_r = \lambda_r I - T$; $r \geq 1$. Then

$$W_r(H_r) = W_r \langle x_1, x_2, \ldots x_r \rangle$$

$$= \langle W_r(x_1), W_r(x_2), \ldots, W_r(x_{r-1}) \rangle, \text{ since } W_r(x_r) = 0 .$$

$$= \langle \lambda_r x_1 - T(x_1), \lambda_r x_2 - T(x_2), \ldots, \lambda_r x_{r-1} - T(x_{r-1}) \rangle$$

$$= \langle \lambda_r x_1 - \lambda_1 x_1, \lambda_r x_2 - \lambda_2 x_2, \ldots, \lambda_r x_{r-1} - \lambda_{r-1} x_{r-1} \rangle$$

$$= H_{r-1} , \text{ since none of the above elements are zero .}$$

Let T map the absolutely convex neighbourhood U into the compact set K. We apply Lemma 3.2 with $W = W_r$, $F = H_r$, $G = H_{r-1}$. For each $r \geq 2$, there exists a point y_r such that $y_r \in H_r \cap 2U$ and $y_r \notin H_{r-1} + U$. ... (i)

Then $T(y_r) \in 2K$.

If $r > s$,

$$T(y_r) - T(y_s) = \lambda_r y_r - W_r(y_r) - \lambda_s y_s + W_s(y_s) \in \lambda_r y_r + H_{r-1} \quad \ldots \text{(ii)}$$

Suppose that $T(y_r) - T(y_s) \in \varepsilon U$. Then from (ii),

$$\lambda_r y_r \in T(y_r) - T(y_s) + H_{r-1} \subset \varepsilon U + H_{r-1}.$$

Hence $y_r \in \frac{\varepsilon}{\lambda_r} U + H_{r-1} \subset U + H_{r-1}$, since U is absolutely convex and $\left|\frac{\varepsilon}{\lambda_r}\right| < 1$.

This contradicts (i); hence $T(y_r) - T(y_s) \notin \varepsilon U$, for all r,s with $r > s$. By the same argument as that used in theorem 3.3, we obtain a contradiction of the fact that $2K$ is pre-compact.

Thus, for each positive integer n, there exists at most a finite set of eigenvalues λ such that $|\lambda| \geq \frac{1}{n}$. Hence the set of eigenvalues of T is either finite or countable. Clearly if the set of eigenvalues is infinite, it is a sequence converging to zero.

COROLLARY *Let S be a continuous linear operator on the convex Hausdorff space X and let ϕ be a polynomial (not a constant) with $\phi(S)$ compact. Then the spectrum of S is finite or countable and its only limit points are zeros of ϕ.*

Proof We first show : if λ is in the spectrum of S, where λ and $\phi(\lambda)$ are non-zero, then $\phi(\lambda)$ is an eigenvalue of $\phi(S)$. Let $\lambda \in$ spectrum of S, $\lambda \neq 0$, $\phi(\lambda) \neq 0$ and let the polynomial ψ satisfy $\psi(S) \cdot (\lambda I - S) = \phi(\lambda) I - \phi(S)$. If $\lambda I - S$ or $\psi(S)$ do not map X onto X, nor does $\phi(\lambda) I - \phi(S)$. If $\lambda I - S$ or $\psi(S)$ is not 1-1, nor is $\phi(\lambda) I - \phi(S)$. In these cases, since $\phi(S)$ is a compact operator, $\phi(\lambda)$ is an eigenvalue of $\phi(S)$.

If $\lambda I - S$ and $\psi(S)$ are both 1-1 and both map X onto X, suppose that the inverse of $I - S$ is not continuous. But $(\lambda I - S)^{-1} = \psi(S) \cdot (\phi(\lambda) I - \phi(S))^{-1}$, and $\phi(S)$ being compact, the right-hand side is continuous, which is a contradiction.

Thus, in all cases, $\phi(\lambda)$ is an eigenvalue of $\phi(S)$.

If the spectrum of S were uncountable, since a polynomial in a complex variable has only a finite number of zeroes, it would follow that the set of eigenvalues of $\phi(S)$ would be uncountable. Hence the spectrum of S is countable .

Let λ_o be a limit point of the spectrum of S. There exists a sequence (λ_n) of distinct points of the spectrum such that $\lambda_n \to \lambda_o$. Now $\phi(\lambda_n) \to \phi(\lambda_o)$, and, since ϕ is a polynomial $(\phi(\lambda_n))$ has infinitely many distinct values. Hence $\phi(\lambda_n) \to 0 = \phi(\lambda_o)$, and so λ_o is a zero of ϕ .

The above corollary is also true if $S \neq 0$ and ϕ is a constant , for then the identity map would be compact, X would have a compact neighbourhood and so be finite dimensional and the spectrum of S a finite set.

§ 5. DUALITY THEORY

LEMMA 5.1 *If T is a compact linear operator on the convex Hausdorff space X , its transpose, T' is compact when X' has the topology $\mathcal{S}^o$ where $\mathcal{S}$ is the set of absolutely convex compact subsets of X .*

Proof. Let T map the absolutely convex neighbourhood U into the compact set K .

Let $V = (T(U))^o = (\overline{T(U)})^o$. Then V is a neighbourhood in X' . Also, using [1] p.39, lemma 6 , $V = (T')^{-1}(U^o)$

and hence $T'(V) \subset U^o$.

We shall show that U^o is $\mathcal{S}^o$ compact . By Project 1 , theorem 9.1 , if $\mathcal{S}_1$ is the collection of all closed absolutely convex pre-compact subsets of X , then U^o is $\mathcal{S}_1^o$ compact .

Now $\mathcal{S}^o \subset \mathcal{S}_1^o$ hence U^o is also $\mathcal{S}^o$ compact, and T' is a compact operator on X' .

NOTE It is easy to show that if S,T are continuous linear operators on X and $\lambda \in \mathbb{C}$ then $(S+T)' = S' + T'$; $(\lambda S)' = \lambda S'$; $(S \circ T)' = T' \circ S'$ $I' = I$; and we assume these results.

Theorem 5.2 *Let T be a compact linear operator on the convex Hausdorff space X , $\lambda \neq 0$, and U and V continuous linear operators on X such that $U \circ V = V \circ U = \lambda I - T$. Then*

(i) V and V' have the same (finite) ascent and descent n ;

(ii) for each positive integer r , $V^{-r}(0)$ and $(V')^{-r}(0)$ have the same dimension ;

(iii) if X' has any topology for which T' is compact, then X' is the topological direct sum of $(V')^n(X')$ and $(V')^{-n}(0)$. On the finite dimensional space $(V')^{-n}(0)$, V' is nilpotent, and on $(V')^n(X)$, V' is a topological isomorphism.

Proof (i) If r is any non-negative integer, then by [1] page 39, lemma 6 $(V^r(X))^\circ = ((V^r)')^{-1}(X^\circ) = (V')^{-r}(0)$.

If $V^r(X) = V^{r+1}(X)$ then $(V')^{-r}(0) = (V')^{-r-1}(0)$, and so ascent of $V' \leq$ descent of V. Now $V^r(X)$ is closed and absolutely convex, hence $(V^r(X))^{\circ\circ} = V^r(X) = ((V')^{-r}(0))^\circ$. Thus, if $(V')^{-r}(0) = (V')^{-r-1}(0)$ then $V^r(X) = V^{r+1}(X)$ and so descent of $V \leq$ ascent of V'.

Now let X' have the topology $\mathscr{B}^\circ$ where $\mathscr{B}$ is the set of absolutely convex compact subsets of X. By Lemma 5.1, T' is compact, and by the Mackey-Arens theorem, the dual of X' is X, and so the transpose of T' is T.

We may thus reverse the roles of T and T', V and V', X and X', to obtain the fact that ascent of V = descent of V'. The result (i) now follows.

(ii) We have that $V^{-r}(0) = X/V^r(X)$. Also, $X/V^r(X) = (V^r(X))^\circ$ where the polar is taken in X' and we have used [1] p.78, proposition 3. But $(V^r(X))^\circ = (V')^{-r}(0)$, hence $V^{-r}(0)$ and $(V')^{-r}(0)$ have the same dimension.

Now (iii) follows from theorem 3.4.

COROLLARY *Apart possibly from $\lambda = 0$, T and T' have the same eigenvalues. Let λ be a non-zero eigenvalue of T and T' and let $W = \lambda I - T$. Then the equation in x, $W(x) = y$ has a solution if and only if y vanishes on $(W')^{-1}(0)$.*

Also, the equation in x', $W'(x') = y'$ has a solution if and only if y' vanishes on $W^{-1}(0)$.

Proof Let $\lambda \neq 0$. If λ is not an eigenvalue of T, then the ascent of $V = 0 =$ ascent of V', and λ is not an eigenvalue of T'. Now exchange the roles of T and T'.

$(W(X))^{\circ} = (W')^{-1}(0)$, and since $W(X$, is closed and absolutely convex, $W(X) = ((W')^{-1}(0))^{\circ} = \{z \in X : (z,t) = 0$ for all $t \in W'^{-1}(0)\}$ since $(W')^{-1}(0)$ is a subspace of X' .

We thus have :

the equation $W(x) = y$ has a solution $\Leftrightarrow y \in W(X) \Leftrightarrow y \in ((W')^{-1}(0))^{\circ} \Leftrightarrow$ y vanishes on $(W')^{-1}(0)$.

The last result may be obtained by giving X' the topology $\mathcal{B}^{\circ}$ as in the theorem .

We shall obtain a few results about compact operators on normed linear spaces. We note that, if X and Y are convex spaces and T is a compact linear operator from X into Y then T maps bounded sets into compact sets, for every bounded subset of X is absorbed by the neighbourhood which T maps into a compact subset of Y .

Conversely, if X is a normed linear space, and T maps bounded subsets of X into compact subsets of Y, then since X has a bounded neighbourhood, T is a compact operator.

In [2] section 55 , Taylor gives the following definition of a compact linear operator T from the normed space X to the normed space Y : "T is compact if, for each bounded sequence (x_n) in X , the sequence $(T(x_n))$ contains a subsequence converging to some limit in Y."

We shall show that the above definition is equivalent to our definition in the case when X is a normed linear space and Y is a convex metric space.

Suppose X is a normed linear space and Y a convex metric space with metric d. Let T be a linear operator from X into Y such that, for each bounded sequence (x_n) in X, (Tx_n) contains a subsequence converging to zero .

Let B be the closed unit ball in X . We shall show that $\overline{T(B)}$ is compact .

Let (y_n) be a sequence in $\overline{T(B)}$. For each n, let $(T(x_{nr}))$ be a sequence which converges to y_n.

Consider the matrix :

$$A = \begin{pmatrix} T(x_{11}) & T(x_{12}) & \cdots \\ T(x_{21}) & T(x_{22}) & \cdots \\ \vdots & \vdots & \\ T(x_{m1}) & T(x_{m2}) & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

We form a new matrix from A as follows :

In the mth row of A, for each n, choose an element whose distance from y_m is less than $\frac{1}{n}$. Call this element $T(U_{mn})$, thus obtaining the matrix B, where

$$B = \begin{pmatrix} T(u_{11}) & T(u_{12}) & \cdot & \cdot \\ T(u_{21}) & T(u_{22}) & \cdot & \\ \vdots & & & \\ T(um_1) & T(um_2) & \cdot & \cdot \\ \vdots & \vdots & & \end{pmatrix}$$

For each n , $d(T(u_{nn}) , y_n)) < \frac{1}{n}$. Now the sequence $T(u_{nn})$ on the diagonal of B has a convergent subsequence. Call this subsequence $T(V_{nn})$ and call each corresponding y_n in the rows in which the subsequence appears z_n . Let $T(V_{nn}) \to y \in Y$.

Let $\varepsilon > 0$ be given . Let M be an integer such that $\frac{1}{2M} < \varepsilon$.

There exists an integer N such that $n > N \Rightarrow d(T(V_{nn}) , y) < \frac{1}{M}$.

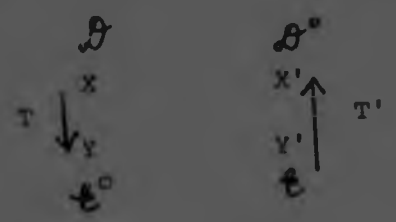
Let $R = \max(N,M)$, then $r > R \Rightarrow d(z_r,y) < d(z_r, T(V_{rr})) + d(T(V_{rr}) , y)$

$$< \frac{1}{r} + \frac{1}{M} < \varepsilon .$$

Hence every sequence in $\overline{T(B)}$ has a convergent subsequence and so $\overline{T(B)}$ is compact .

Theorem 5.3 *Let X and Y be convex Hausdorff spaces and T a weakly continuous linear map from X into Y . Let X' have the polar topology $\mathcal{B}^\circ$, the elements of $\mathcal{B}$ being subsets of X . Then T maps the sets of $\mathcal{B}$ into pre-compact sets if and only if T' maps equi-continuous sets into compact sets .*

Proof.



Let τ be the given topology on Y. Let $\mathcal{C}$ be the collection of equi-continuous subsets of Y' . Then $\tau = \mathcal{C}^\circ$.

Note that T maps the sets of $\mathcal{B}$ into pre-compact sets $\Leftrightarrow$ T maps the sets of $\mathcal{B}$ onto pre-compact sets, as a subset of a pre-compact set is pre-compact.

By [1] page 51 theorem 3 , T(D) is pre-compact for each $D \in \mathcal{B}$ if and only if $T'(E)$ is pre-compact for each $E \in \mathcal{C}$. . . . (i)

Suppose T maps the sets of $\mathcal{B}$ into pre-compact sets. Let $E \in \mathcal{C}$. Then $E \subset E^{oo}$, and E^{o} is a neighbourhood in Y and hence E^{oo} is $W(Y',Y)$ compact by the Banach-Alaoglu Theorem . T' is weakly continuous and so $T'(E^{oo})$ is $W(X',X)$ compact and complete . By [1] , p.105 , proposition 3 , Corollary , $T'(E^{oo})$ is also $\mathcal{B}^{c}$ complete . Now $T'(E)$ is $\mathcal{B}^{o}$ pre-compact, hence so is $\overline{h(T'(E)} = \overline{(T'(h(E))} \subset T'(\overline{h(E)}) = T'(E^{oo})$. Thus $\overline{h(T'(E))}$ is $\mathcal{B}^{o}$ pre-compact and complete, and hence it is $\mathcal{B}^{o}$ compact . Thus T' maps E into a compact set .

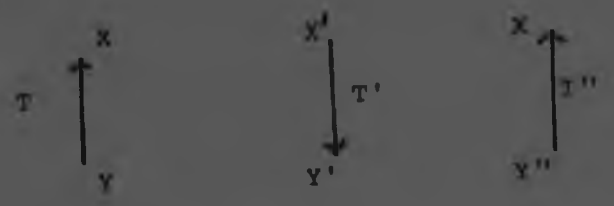
The converse is clear from (i) .

COROLLARY *1* *Let X and Y be normed linear spaces and T a compact linear operator from X into Y . Then T' is also a compact operator when X', Y' have their norm topologies.*

Proof This follows from the above lemma by choosing $\mathcal{B}$ to be the collection of balls in X, by noting that balls in Y' are equi-continuous sets and that T compact $\Rightarrow$ T continuous $\Rightarrow$ T weakly continuous .

COROLLARY *2* *Let X and Y be normed linear spaces, X reflexive and T a linear operator from Y into X . Let X' and Y' have their norm topologies . Then T' compact $\Rightarrow$ T is compact .*

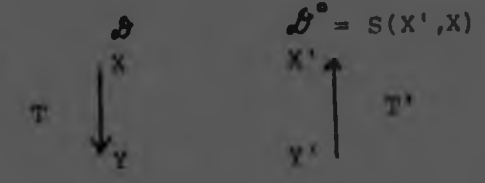
Proof In the lemma, replace X, Y, T by X',Y',T' and let $\mathcal{B}$ be the collection of balls in X' .



Suppose T' is a compact operator. Then T' is continuous and so T' is $W(X',X) - W(Y',Y'')$ continuous and so $T''(Y'') \subset X$. By the lemma, T'' maps balls in Y'' into compact sets in X. Hence $T''|_{Y}$ maps balls in Y into compact sets in X , but $T''|_{Y} = T$.

Theorem 5.4 *Let X and Y be convex Hausdorff spaces whose duals X' and Y' have the strong topologies, $S(X',X)$, $S(Y',Y)$, and let T be a weakly continuous linear mapping from X into Y. Let Y be complete and barrelled. Then T maps bounded sets into compact sets if and only if T' has the same property.*

Proof



Suppose T maps bounded sets into compact sets. Y is barrelled, hence Y has the topology $S(Y,Y')$. Let A be an $S(Y',Y)$ bounded subset of Y'. Then A is $W(Y',Y)$ bounded and A° is a neighbourhood in Y. Hence $A^{\circ\circ}$ is an equi-continuous set in Y', and so is A.

By the previous Lemma, T' maps A into a compact set.

Conversely, suppose T' maps bounded sets into compact sets. By [1] p.71, lemma 2, Corollary, every equi-continuous subset of Y' is $S(Y',Y)$ bounded, hence T' maps equi-continuous sets into compact sets. By the previous Lemma with $\mathcal{B}$ the collection of bounded subsets of X, T maps bounded subsets of X, onto pre-compact sets.

Let A be a bounded subset of X, then $T(A)$ is pre-compact, $\overline{h(T(A))}$ is pre-compact, and since y is complete, it is also complete. Hence $\overline{h(T(A))}$ is compact and so T maps bounded sets into compact sets.

Theorem 5.5 *(Schauder) Let X and Y be Banach spaces and let X' and Y' have their norm topologies. Let T be a linear mapping from X into Y. Then T is compact if and only if T' is compact.*

Proof We note firstly, that if T is a compact operator it is continuous and hence weakly continuous. If T' is compact, it is weakly continuous and hence T is weakly continuous.

We see that this theorem follows from the previous theorem by noting that the balls in X and Y' are bounded sets and neighbourhoods.

NOTE: If X is a convex Hausdorff space with topology ξ, and $\mathcal{B}$ is the collection of equi-continuous subsets of X', then $\mathcal{B}^\circ = \xi$.

If A is an equi-continuous subset of X' then the $W(X',X)$ closure of $h(A)$ is also equi-continuous, for there exists a neighbourhood U in X such that $A \subset U^\circ$, which is $W(X',X)$ closed and absolutely convex.

It follows that, if $\mathcal{E}$ is the collection of closed absolutely convex equi-continuous subsets of X' then $\mathcal{B}^\circ = \mathcal{E}^\circ$.

We shall use this result in the following lemma.

LEMMA 5.6 *Suppose that X and Y are convex Hausdorff spaces, that X' has the polar topology $\mathcal{B}^\circ$, where $\mathcal{B}$ is a collection of absolutely convex subsets of X and that T is a weakly continuous mapping from X into Y. Then*

(i) T maps the sets of $\mathcal{B}$ into $W(Y,Y')$ compact sets if and only if $T''(X'') \subset Y$;

(ii) T' maps equi-continuous sets into $W(X',X'')$-compact sets if and only if $T''(X'') \subset Y$.

Proof

$$\begin{array}{ccccc} X & & X' & & X'' \\ T\downarrow & & \uparrow T' & & \downarrow T'' \\ Y & & Y' & & Y'^* \end{array}$$

In this theorem, polars of subsets of X or Y are taken in X',Y' respectively. Polars of subsets of X',Y' are taken in X'', Y'' respectively.

Since T is weakly continuous, $T'(Y') \subset X'$. The conditions of Lemma 10.1, project 1, are satisfied and so, if $D \in \mathcal{B}$, then $T''(D^{\circ\circ}) \; T''(D^{\circ\circ}) \subset (T(D))^{\circ\circ}$.

Now $X'' = \bigcup_{D \in \mathcal{B}} D^{\circ\circ}$, and so

$$T''(X'') = \bigcup_{D \in \mathcal{B}} T''(D^{\circ\circ}) \subset \bigcup_{D \in \mathcal{B}} (T(D))^{\circ\circ} \quad . \quad . \quad . \quad . \qquad (a)$$

(i) Let $D \in \mathcal{B}$. Suppose T maps D into a W(Y,Y')-compact set $B \subset Y \subset Y'^*$. The W(Y'*, Y') topology on Y'* induces the W(Y,Y') topology on Y , and so B is also W(Y'*, Y')-compact. Now $(T(D))^{oo}$ is the W(Y'*, Y') closure of T(D) , since D is absolutely convex and so $(T(D))^{oo} \subset B \subset Y$. By (a), $T''(X'') \subset Y$.

Conversely, suppose $T''(X'') \subset Y$. Then T'' is W(X'', X)-W(Y,Y') continuous. Let $D \in \mathcal{B}$, then D^{o} is a $\mathcal{B}^{\circ}$ neighbourhood in X' and so D^{oo} is W(X'', X') compact. Hence $T''(D^{oo})$ is W(Y,Y') compact. We have that $D \subset D^{oo}$ hence $T(D) = T''(D) \subset T''(D^{oo})$, and so T maps D into a W(Y,Y') compact set.

(ii) Suppose $T''(X'') \subset \hat{Y}$. Then T' is $W(Y',\hat{Y})$ - W(X', X'') continuous. . . . . (b)

We apply Theorem 6.2 Corollary 3 of Project 1 . Replace the X of the theorem by Y under the given topology, Y by Y' under the topology W(Y',Y) and $\mathcal{B}$ by the collection $\mathcal{A}$ of the closed absolutely convex equi-continuous subsets of Y' . Then $\mathcal{A}^{\circ}$ (polars taken in Y) is the given topology on Y. By Corollary 3 of the theorem, the topologies W(Y',Y) and $W(Y',\hat{Y})$ coincide on each $A \in \mathcal{A}$.

Let C be an equi-continuous subset of Y'. Then $C \subset A$ for some $A \in \mathcal{A}$. There exists a neighbourhood B in Y such that $A \subset B^{\circ}$. B° is W(Y', Y) compact by the Banach-Alaoglu Theorem , and $B^{\circ} \in \mathcal{A}$, so B° is also $W(Y',\hat{Y})$ compact. By (b), $T'(B^{\circ})$ is W(X', X'') compact, and so T' maps C into a W(X',X'') compact set .

Conversely, suppose T' maps equi-continuous sets into W(X',X'')-compact sets. Let $\mathcal{A}$ be as above. We apply Theorem 5.1 of Project 1 with the dual pair (X,Y) replaced by the dual pair (Y,Y') and $\mathcal{B}$ replaced by $\mathcal{A}$. Then $\hat{Y} = \bigcap_{A \in \mathcal{A}} (Y + A^{\square})$.

Let $z \in X''$ and $A \in \mathcal{A}$. We need only show that $T''(z) \in Y + A^{\circ}$. Now $X'' = \bigcup_{D \in \mathcal{B}} D^{oo}$, and so $z \in D^{oo}$ for some $D \in \mathcal{B}$.

D^{oo} = w(X'', X') closure of D (since D is absolutely convex)

= $\tau(X'', X')$ closure of D (since $\tau(X'', X')$ is a topology of the dual pair (X'', X')) . . . (c)

Now T'(A) is absolutely convex and , by hypothesis , is included in a W(X', X'') compact set . Also, $(T'(A))^{\circ}$ = (W(X',X'') closure of $T'(A))^{\circ}$. Hence $(T'(A))^{\circ}$ is a $\tau(X'', X')$ neighbourhood in X'' . Now $z - (T'(A))^{\circ}$ is a $\tau(X'', X')$ neighbourhood of z in X'' , and z is the $\tau(X'', X')$ closure of D by (c) . Hence there exists $x \in D$, $x \in z - (T'(A))^{\circ}$

and so $z \in x + (T'(A))^{\circ} = x + (T'')^{-1}(A^{\circ})$. Hence $T''(z) \in Y + A^{\circ}$, since T'' maps the elements of X into Y . It now follows that $T''(X'') \subset \hat{Y}$.

Theorem 5.7 *Let X and Y be Banach spaces whose duals X' and Y' have their norm topologies, and T a weakly continuous linear map from X into Y . Let T'' be the transpose of T' with respect to the system (Y', Y'^*) , (X', X'') . Then the following are equivalent :*

(i) T maps bounded sets into $W(Y,Y')$ compact sets ;
(ii) T' maps bounded sets into $W(X', X'')$ compact sets ;
(iii) $T''(X'') \subset Y$.

Proof In the lemma, let $\mathcal{B}$ be the collection of balls in X . Then : T maps bounded sets into $W(Y,Y')$ compact sets $\Leftrightarrow$ T maps the elements of $\mathcal{B}$ into $W(Y,Y')$ compact sets .

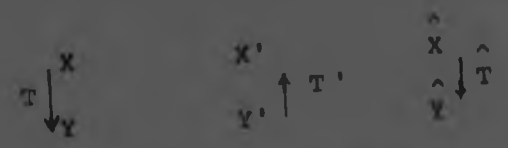
Also, a set A in Y' is equi-continuous if and only if it is bounded, and since Y is complete $Y = \hat{Y}$.

The theorem then follows.

In the first project we showed that every convex Hausdorff space X can be embedded as a dense subspace of a complete convex Hausdorff space which we denoted $\hat{X}$. We also showed that if X and Y are convex Hausdorff spaces and T is a continuous linear mapping of X into Y then T has a unique continuous linear extension from $\hat{X}$ into $\hat{Y}$. We now show that if T is compact, so is $\hat{T}$. We recall that $\hat{T}$ was defined as follows : Let T'^* be the transpose of T' with respect to the system (X', X'^*) , (X', X'^*) then $\hat{T} = T'^* | \hat{X}$.

Theorem 5.8 *If X and Y are convex Hausdorff spaces with completions $\hat{X}$ and $\hat{Y}$, and if T is a compact linear mapping from X into Y, then $\hat{T}$ the continuous linear extension of T from $\hat{X}$ into $\hat{Y}$ is also a compact mapping .*

Proof



In this theorem, polars are taken in X' , Y' , $\hat{X}$ or $\hat{Y}$. Let $\mathcal{U}$ and $\mathcal{V}$ be bases of closed, absolutely convex neighbourhoods for X , Y respectively. Then $\hat{X}$, $\hat{Y}$ have the topologies $\mathcal{U}^{oo}$ and $\mathcal{V}^{oo}$ respectively. If $B \subset \hat{Y}$, let $\overline{B}$ denote the closure of B in $\hat{Y}$ under the topology $\mathcal{V}^{oo}$. Since Y is dense in $\hat{Y}$, they have the same continuous dual. It follows that if B is absolutely convex, then $\overline{B}$ is the $W(\hat{Y},Y')$ closure of B .

Let T map the neighbourhood U into the compact set K, where $U \in \mathcal{U}$. From project 1 lemma 5.6, $\hat{T}(U^{oo}) \subset (T(U))^{oo} \subset K^{oo}$.

Let $h(K) = J$, then J is pre-compact in X and $J^{oo} = \bar{J}$ in $\hat{Y}$ and and hence complete. In order to prove the theorem we need only show that J^{oo} is pre-compact in Y.

Let V^{oo} be a neighbourhood in Y. There exist $A_1, A_2, \ldots, A_n$, subsets of X which are small of order V such that $J \subset \bigcup_{r=1}^{n} A_r$. Then

$$J^{oo} = \bar{J} \subset \bigcup_{r=1}^{n} \bar{A}_r .$$

We now show that each $\bar{A}_r$ is small of order $\bar{V}$. Let $a,b \in \bar{A}_r$ and let (a_λ), (b_λ) be nets in A_r which converge to a and b respectively. Then $a_\lambda - b_\lambda \in V$ and so $a - b \in \bar{V}$. It follows that J^{oo} is pre-compact and the theorem is proved.

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