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Self-adjoint higher order differential operators with eigenvalue parameter dependent boundary conditions

Manfred Möller* and Bertin Zinsou

*Correspondence:
manfred.moller@wits.ac.za
School of Mathematics, University
of the Witwatersrand, 1 Jan Smuts
Avenue, Johannesburg, 2050, South
Africa

Abstract

Eigenvalue problems for even order regular quasi-differential equations with boundary conditions which depend linearly on the eigenvalue parameter λ can be represented by an operator polynomial $L(\lambda) = \lambda^2 M - i\lambda K - A$, where M is a self-adjoint operator. Necessary and sufficient conditions are given such that also K and A are self-adjoint.

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1 Introduction

In order to solve linear partial differential equations of the form

$$\frac{\partial^2 u}{\partial t^2} + \mathcal{A}u = 0,$$

where $\mathcal{A}$ is a linear differential operator with respect to the variable x on an interval I , the separation of variables method $u(x, t) = y(x)e^{i\omega t}$ leads to

$$\omega^2 y = \mathcal{A}y.$$

For t -independent boundary conditions $Bu = 0$, setting $\lambda = \omega^2$, the operator theoretic realization leads to an eigenvalue problem for an operator A in the Lebesgue space $L^2(I)$ with domain

$$\mathcal{D}(A) = \{y \in L^2(I) : \mathcal{A}y \in L^2(I), By = 0\}.$$

Such problems are well studied, and of particular importance is the case that A is self-adjoint. Many applications in physics and engineering can be represented by such self-adjoint operators.

However, problems like the Regge problem and the vibrating beam problem have boundary conditions with partial first order derivatives with respect to t or whose mathematical

model leads to an eigenvalue problem with the eigenvalue parameter $\lambda = \omega$ occurring linearly in the boundary conditions. Such problems have an operator representation of the form

$$L(\lambda) = \lambda^2 M - i\lambda K - A \quad (1.1)$$

in a Hilbert space $H = L^2(I) \oplus \mathbb{C}^k$, where k is the number of eigenvalue dependent boundary conditions.

In general, the spectrum of L is no longer real but still has some particularly nice properties if K, M, A are self-adjoint with $M \geq 0$ and $K \geq 0$, the resolvent set of L is nonempty, and L has a compact resolvent: it is symmetric with respect to the imaginary axis and eigenvalues with negative imaginary parts must lie on the imaginary axis. In this situation, the operators M and K are quite simple bounded self-adjoint operators. However, the operator A is determined by three ingredients: the differential equation $\mathcal{A}$, the parameter independent boundary conditions as homogeneous boundary conditions for A , and the parameter dependent boundary conditions as an inhomogeneous part of A . Hence one cannot make use of the criteria for self-adjointness in the case of parameter independent boundary conditions. Rather, the parameter dependent case is a proper extension of the parameter independent case.

For parameter independent boundary conditions, *i.e.*, $k = 0$, characterizations of self-adjointness for A in the case of formally symmetric even order quasi-differential expressions are known both for the regular and the singular cases, see [1] and in particular [1], Theorem 6 for the regular case. The simplest formulation of these self-adjointness conditions makes use of quasi-derivatives, and we will henceforth mostly use quasi-derivatives $y^{[j]}$ rather than derivatives $y^{(j)}$. For the definition of the quasi-derivatives $y^{[j]}$, we refer the reader to (2.2)-(2.5), see also Remark 3.2.

Some special cases of self-adjoint boundary conditions for regular $2n$ th order differential equations with $k > 0$ are known. In [2], the second order problem related to the Regge problem was investigated, whereas the fourth order differential equation $y^{(4)} - (gy')'$ related to a vibrating beam was dealt with in [3], where the boundary conditions are of the form

$$B_j(\lambda)y = y^{[p_j]}(a_j) + \lambda\beta_j y^{[q_j]}(a_j), \quad j = 1, \dots, 4, \quad (1.2)$$

with exactly one boundary condition depending on λ . A classification of all self-adjoint boundary conditions of the form (1.2) was obtained in [4]. A corresponding result for sixth order differential equations was given in [5].

In this paper we consider $2n$ th order quasi-differential equations and derive necessary and sufficient conditions for $2n$ boundary conditions of the form (1.2) to generate self-adjoint operators K and A .

In Section 2 we give a precise definition of the boundary value problem and the quadratic operator pencil L associated with it. In Section 3 we derive necessary and sufficient conditions for K to be self-adjoint and for A to be symmetric. In Section 4 it is shown that A is self-adjoint if A is symmetric.

2 The eigenvalue problem

We first summarize some basic facts about quasi-differential equations for the convenience of the reader. For a more comprehensive discussion of quasi-differential equations,

the reader is referred to [6] and to [7] in the scalar case and to [8, 9] for the general case with matrix coefficients.

Let $I = (a, b)$ be an interval with $-\infty < a < b < \infty$, and let m be a positive integer. For a given set S , $M_m(S)$ denotes the set of $m \times m$ matrices with entries from S . Let

$$Z_m(I) := \left\{ G = (g_{r,s})_{r,s=1}^m \in M_m(L^1(I)), \right. \\ \left. g_{r,r+1} \text{ invertible a.e. for } 1 \leq r \leq m-1, g_{r,s} = 0 \text{ for } 2 \leq r+1 < s \leq m \right\}, \quad (2.1)$$

where $L^1(I)$ denotes the complex-valued Lebesgue integrable functions on I .

For $G \in Z_m(I)$, define

$$Q_0 := \{y : I \rightarrow \mathbb{C}, y \text{ measurable}\} \quad (2.2)$$

and

$$y^{[0]} := y, \quad y \in Q_0. \quad (2.3)$$

Inductively, for $r = 1, \dots, m$, we define

$$Q_r = \{y \in Q_{r-1} : y^{[r-1]} \in AC(I)\}, \quad (2.4)$$

$$y^{[r]} = g_{r,r+1}^{-1} \left(y^{[r-1]'} - \sum_{s=1}^r g_{r,s} y^{[s-1]} \right), \quad y \in Q_r, \quad (2.5)$$

where $g_{m,m+1} := 1$ and where $AC(I)$ denotes the set of complex-valued functions which are absolutely continuous on I . Finally we set

$$\mathcal{A}y := i^m y^{[m]}, \quad y \in Q_m. \quad (2.6)$$

The expression $\mathcal{A} = \mathcal{A}_G$ is called the quasi-differential expression associated with G , and the function $y^{[r]}$, $0 \leq r \leq m$, is called the r th quasi-derivative of y . We also write $\mathcal{D}(\mathcal{A})$ for Q_m .

Observe that the quasi-derivatives defined in (2.5) depend on G . However, since we are only going to deal with a single quasi-differential equation, we will not indicate this dependence explicitly.

In the remainder of the paper, we assume that $m = 2n$ is an even positive integer, that $G = (g_{r,s})_{r,s=1}^{2n} \in Z_{2n}(I)$, and that $w : I \rightarrow \mathbb{R}$ is positive a.e. and satisfies $w \in L^1(I)$.

Together with (2.6) we consider the boundary conditions $B_j(\lambda)y = 0$, $j = 1, \dots, 2n$, taken at the endpoint a for $j = 1, \dots, n$ and at the endpoint b for $j = n+1, \dots, 2n$. We assume for simplicity that

$$B_j(\lambda)y = y^{[p_j]}(a_j) + i\lambda\beta_j y^{[q_j]}(a_j), \quad (2.7)$$

where $a_j = a$ for $j = 1, \dots, n$, $a_j = b$ for $j = n+1, \dots, 2n$, $\beta_j \in \mathbb{C}$ and $0 \leq p_j, q_j \leq 2n-1$. Of course, the numbers q_j are ambiguous and irrelevant in case $\beta_j = 0$.

The differential expression (2.6) and the boundary conditions (2.7) define the eigenvalue problem

$$(-1)^n y^{[2n]} = \lambda^2 w y, \quad (2.8)$$

$$B_j(\lambda)y = 0, \quad j = 1, \dots, 2n. \quad (2.9)$$

We put

$$\begin{aligned} \Theta_1 &= \{j \in \{1, \dots, 2n\} : \beta_j \neq 0\}, & \Theta_0 &= \{1, \dots, 2n\} \setminus \Theta_1, \\ \Theta_r^a &= \Theta_r \cap \{1, \dots, n\}, & \Theta_r^b &= \Theta_r \cap \{n+1, \dots, 2n\}, \quad \text{for } r = 0, 1, \end{aligned}$$

and

$$k = |\Theta_1|. \quad (2.10)$$

Assumption 2.1 We assume that the numbers $p_1, \dots, p_n, q_j$ for $j \in \Theta_1^a$ are distinct and that the numbers $p_{n+1}, \dots, p_{2n}, q_j$ for $j \in \Theta_1^b$ are distinct.

Assumption 2.1 means that for any pair (r, a_j) the term $y^{[r]}(a_j)$ occurs at most once in the boundary conditions (2.7).

For $j \in \Theta_1$, we choose $\alpha_j \in \mathbb{R}$ and $\varepsilon_j \in \mathbb{C}$ such that $\beta_j = \alpha_j \varepsilon_j$.

For $y \in \mathcal{D}(\mathcal{A})$, we define $Y_R = \begin{pmatrix} Y^{(a)} \\ Y^{(b)} \end{pmatrix}$ with $Y = (y^{[0]}, y^{[1]}, \dots, y^{[2n-1]})^T$. We denote the collection of the $2n$ boundary conditions (2.9) by $\mathcal{U}$ and define the following matrices related to $\mathcal{U}$:

$$\begin{aligned} \mathcal{U}_r Y_R &= (y^{[p_j]}(a_j))_{j \in \Theta_r}, \quad r = 0, 1, \\ \mathcal{V}_1 Y_R &= (\varepsilon_j y^{[q_j]}(a_j))_{j \in \Theta_1}, \end{aligned} \quad \text{where } y \in \mathcal{D}(\mathcal{A}). \quad (2.11)$$

Remark 2.2 In case that $\Theta_r = \emptyset$ for $r = 0$ or $r = 1$, the corresponding matrix $\mathcal{U}_r$ will be identified with the ‘zero’ operator from $\mathbb{C}^{2n}$ into $\{0\}$.

The weighted Lebesgue space $L^2(I, w)$ is the Hilbert space of all equivalence classes of complex-valued measurable functions f such that $(f, f)_w := \int_I w(x) |f(x)|^2 dx < \infty$. For convenience we define the operator $\mathcal{A}_{\max}$ on $L^2(I, w)$ by

$$\mathcal{D}(\mathcal{A}_{\max}) = \{y \in L^2(I, w) : w^{-1} \mathcal{A}y \in L^2(I, w)\}, \quad \mathcal{A}_{\max} y = w^{-1} \mathcal{A}y.$$

We will associate the quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\lambda K - A(U) \quad (2.12)$$

in the space $L^2(I, w) \oplus \mathbb{C}^k$ with problem (2.8), (2.9), where

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ 0 & K_0 \end{pmatrix} \quad \text{with } K_0 = \text{diag}(\alpha_j : j \in \Theta_1).$$

The operator $A(U)$ in $L^2(I, w) \oplus \mathbb{C}^k$ is defined by

$$\mathcal{D}(A(U)) = \left\{ \tilde{y} = \begin{pmatrix} y \\ V_1 Y_R \end{pmatrix} : y \in \mathcal{D}(\mathcal{A}_{\max}), U_0 Y_R = 0 \right\},$$

$$(A(U))\tilde{y} = \begin{pmatrix} \mathcal{A}_{\max} y \\ U_1 Y_R \end{pmatrix}, \quad \tilde{y} \in \mathcal{D}(A(U)).$$

It is easy to see that a function $y \in \mathcal{D}(\mathcal{A}_{\max})$ satisfies $\mathcal{A}y = \lambda^2 w y$ and $B_j(\lambda)y = 0$ for $j = 1, \dots, 2n$ if and only if there is $c \in \mathbb{C}^k$ such that $(y, c)^T \in \mathcal{D}(A(U))$ such that $L(\lambda)(y, c)^T = 0$. In this case c is uniquely determined by y . Indeed, if $y \in \mathcal{D}(\mathcal{A}_{\max})$ with $\mathcal{A}y = \lambda^2 w y$ and $B_j(\lambda)y = 0$ for $j = 1, \dots, 2n$, then $U_0 Y_R = 0$ shows that $(y, V_1 Y_R)^T \in \mathcal{D}(A(U))$ and

$$L(\lambda) \begin{pmatrix} y \\ V_1 Y_R \end{pmatrix} = \begin{pmatrix} \lambda^2 y - \mathcal{A}_{\max} y \\ -i\lambda K_0 V_1 Y_R - U_1 Y_R \end{pmatrix}.$$

Clearly, the first component is 0, and so is the second component since

$$i\lambda K_0 V_1 Y_R + U_1 Y_R = i\lambda K_0 (\varepsilon_j y^{[q_j]}(a_j))_{j \in \Theta_1} + (y^{[p_j]}(a_j))_{j \in \Theta_1} = (B_j(\lambda)y)_{j \in \Theta_1}.$$

Hence the operator pencil L is an operator realization of the eigenvalue problem (2.8), (2.9).

It is clear that M and K are bounded self-adjoint operators and that M is non-negative. The operator $A(U)$ is not self-adjoint, in general, and we will give necessary and sufficient conditions for the operator $A(U)$ to be self-adjoint.

3 Symmetry conditions for $A(U)$

We will denote the canonical inner product in $L^2(I, w) \oplus \mathbb{C}^k$ by $\langle \cdot, \cdot \rangle$.

The Lagrange form of $A(U)$ is defined by

$$F_U(\tilde{y}, \tilde{z}) = \langle A(U)\tilde{y}, \tilde{z} \rangle - \langle \tilde{y}, A(U)\tilde{z} \rangle, \quad \tilde{y}, \tilde{z} \in \mathcal{D}(A(U)).$$

The operator $A(U)$ is symmetric if and only if its Lagrange form is identically zero. For this it is necessary that $\mathcal{A}$ is formally symmetric, and for the remainder of this paper we make therefore the following assumption.

Assumption 3.1 We assume that

$$G = -CG^*C,$$

where

$$C = \left((-1)^r \delta_{r, 2n+1-s} \right)_{r,s=1}^{2n} \quad (3.1)$$

and δ is the Kronecker delta.

It is easy to verify that Assumption 3.1 holds if and only if

$$g_{r,s} = (-1)^{r+s+1} \bar{g}_{2n+1-s, 2n+1-r}, \quad r, s = 1, \dots, 2n. \quad (3.2)$$

Remark 3.2 Classical formally self-adjoint differential expressions are of the form

$$(-1)^n \sum_{j=0}^n (g_j y^{(j)})^{(j)}$$

with $g_j \in C^j[0, a]$ for $j = 0, \dots, n$ and invertible g_n . It is easy to verify that this is a quasi-differential equation with quasi-derivatives

$$\begin{aligned} y^{[r]} &= y^{(r)}, \quad r = 0, \dots, n-1, \\ y^{[n]} &= g_n y^{(n)}, \\ y^{[r]} &= y^{[r-1]'} + g_{2n-r} y^{[2n-r]}, \quad r = n+1, \dots, 2n. \end{aligned}$$

The corresponding matrix $G = (g_{r,s})_{r,s=1}^{2n}$ has the entries $g_{r,r+1} = 1$ for $r = 1, \dots, n-1$ and $r = n+1, \dots, 2n-1$, $g_{n,n+1} = g_n^{-1}$, $g_{r,2n-r+1} = -g_{2n-r}$ for $r = n+1, \dots, 2n$, while all other entries are zero. It is easy to see that Assumption 3.1 holds in this case if and only if $g_j = \bar{g}_j$ for $j = 0, \dots, n$, so that the formal self-adjointness condition reduces to the well-known condition that all g_j , $j = 0, \dots, n$, are real-valued functions.

From [10], Lemma 3.3 we know that the Lagrange identity

$$(w^{-1} \mathcal{A}y, z)_w - (y, w^{-1} \mathcal{A}z)_w = Z_R^* D Y_R, \quad y, z \in \mathcal{D}(\mathcal{A}_{\max}) \quad (3.3)$$

holds, where

$$D = (-1)^n \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}. \quad (3.4)$$

Proposition 3.3 *The Lagrange form F_U of $A(U)$ has the representation*

$$F_U(\tilde{y}, \tilde{z}) = Z_R^* W Y_R, \quad \tilde{y}, \tilde{z} \in \mathcal{D}(A(U)),$$

where

$$W = D + (V_1^* U_1 - U_1^* V_1). \quad (3.5)$$

Proof Let $\tilde{y}, \tilde{z} \in \mathcal{D}(A(U))$. Then

$$F_U(\tilde{y}, \tilde{z}) = (w^{-1} \mathcal{A}y, z)_w + (V_1 Z_R)^* U_1 Y_R - (y, w^{-1} \mathcal{A}z)_w - (U_1 Z_R)^* V_1 Y_R,$$

and an application of the Lagrange identity (3.3) completes the proof of the lemma. $\square$

By definition, an operator in a Hilbert space is symmetric if and only if its Lagrange form is identically zero. Hence we have the following.

Corollary 3.4 *The differential operator $A(U)$ is symmetric if and only if $Z_R^* W Y_R = 0$ for all $\tilde{y}, \tilde{z} \in \mathcal{D}(A(U))$.*

The nullspace and range of a matrix M are denoted by $N(M)$ and $R(M)$, respectively.

Proposition 3.5 *The differential operator $A(U)$ is symmetric if and only if $W(N(U_0)) \subset (N(U_0))^\perp$.*

Proof From [10], Corollary 5.5 we know that

$$\{Y_R : y \in \mathcal{D}(\mathcal{A}_{\max})\} = \mathbb{C}^{4n}. \quad (3.6)$$

Hence $\{Y_R : \tilde{y} \in \mathcal{D}(A(U))\} = N(U_0)$. An application of Proposition 3.4 completes the proof. $\square$

Corollary 3.6 *If $A(U)$ is symmetric, then $\text{rank } W = 2(2n - k)$ and $W(N(U_0)) = (N(U_0))^\perp$.*

Proof Since $\dim(N(U_0))^\perp = \text{rank } U_0 = 2n - k$, we have

$$2n - k \geq \dim W(N(U_0)) \geq \dim N(U_0) - (4n - \text{rank } W) = -2n + k + \text{rank } W. \quad (3.7)$$

Hence $\text{rank } W \leq 2(2n - k)$. Since $V_1^* U_1 - U_1^* V_1$ has $2k$ non-zero entries and D is invertible, $\text{rank } W \geq 2(2n - k)$ and $\text{rank } W = 2(2n - k)$ follows. In this case, all the inequalities of (3.7) are equalities and $\dim W(N(U_0)) = \dim(N(U_0))^\perp$ holds. Thus it follows from Proposition 3.5 that $W(N(U_0)) = (N(U_0))^\perp$. $\square$

In view of Corollary 3.6, we may assume that $\text{rank } W = 2(2n - k)$ when investigating the symmetry of $A(U)$. Since $(N(U_0))^\perp = R(U_0^*)$, see [11], Theorem IV.5.13, Proposition 3.5 and Corollary 3.6 lead to the following.

Corollary 3.7 *Let $\text{rank } W = 2(2n - k)$. Then the differential operator $A(U)$ is symmetric if and only if $W(N(U_0)) = R(U_0^*)$.*

We now give an explicit description for the condition $\text{rank } W = 2(2n - k)$.

Proposition 3.8 *$\text{rank } W = 2(2n - k)$ if and only if the following conditions hold:*

1. For $s \in \Theta_1$, $p_s + q_s = 2n - 1$;
2. For $s \in \Theta_1^{(a)}$, $\varepsilon_s = (-1)^{q_s + n}$;
3. For $s \in \Theta_1^{(b)}$, $\varepsilon_s = (-1)^{q_s + n + 1}$.

Proof Note that

$$V_1^* U_1 - U_1^* V_1 = \begin{pmatrix} V_2 & 0 \\ 0 & V_3 \end{pmatrix}, \quad (3.8)$$

where

$$V_2 = \sum_{s \in \Theta_1^{(a)}} (\overline{\varepsilon_s} \delta_{i, q_s + 1} \delta_{j, p_s + 1} - \varepsilon_s \delta_{i, p_s + 1} \delta_{j, q_s + 1})_{i, j=1}^{2n},$$

$$V_3 = \sum_{s \in \Theta_1^{(b)}} (\bar{\varepsilon}_s \delta_{i,q_s+1} \delta_{j,p_s+1} - \varepsilon_s \delta_{i,p_s+1} \delta_{j,q_s+1})_{i,j=1}^{2n}.$$

Since D has exactly one non-zero entry in each row and column and $V_1^* V_0 - V_0^* V_1$ has exactly $2k$ non-zero entries, it follows that $\text{rank } W = 2(2n - k)$ if and only if each non-zero entry of V_2 cancels a non-zero entry of $(-1)^{n-1} C$ and each non-zero entry of V_3 cancels a non-zero entry of $(-1)^n C$. Since the non-zero entries of C are in rows i and columns j such that $i + j = 2n + 1$, we obtain that $\text{rank } W = 2(2n - k)$ if and only if conditions 1, 2, and 3 are satisfied. $\square$

Corollary 3.9 *The boundary eigenvalue problem (2.8), (2.9) has an operator pencil representation (2.12) with self-adjoint operator K and symmetric operator $A(U)$ if and only if*

1. $\beta_j \in \mathbb{R}$ and $p_j + q_j = 2n - 1$ for all $j \in \Theta_1$;
2. $W(N(U_0)) = R(U_0^*)$.

Proof We have seen in Proposition 3.8 that three sets of conditions have to be satisfied in order that the necessary condition $\text{rank } W = 2(2n - k)$ for symmetry of $A(U)$ holds. Conditions 2 and 3 can always be satisfied if we put $\alpha_j = \beta_j (-1)^{q_s+n}$ for $j \in \Theta_1^a$ and $\alpha_j = \beta_j (-1)^{q_s+n+1}$ for $j \in \Theta_1^b$, and for K to be self-adjoint it is therefore necessary and sufficient that β_j are real. The remaining conditions now follow easily from Proposition 3.8 and Corollary 3.7. $\square$

We could now give explicit conditions for symmetry of $A(U)$ in terms of the boundary conditions (2.7). However, we will see in the next section that $A(U)$ is self-adjoint if and only if it is symmetric. In order to avoid duplication we will therefore postpone deriving these explicit conditions to the next section.

4 Self-adjointness conditions for $A(U)$

From Corollary 3.9 we know that for self-adjointness of K and $A(U)$ the condition $\beta_j \in \mathbb{R}$ for all $j \in \Theta_1$ is necessary. Hence we require without loss of generality that the numbers ε_s for $s \in \Theta_1$ are chosen as in Proposition 3.8, conditions 2 and 3.

Assumption 4.1 For $s \in \Theta_1^{(a)}$, let $\varepsilon_s = (-1)^{q_s+n}$, and for $s \in \Theta_1^{(b)}$, let $\varepsilon_s = (-1)^{q_s+n+1}$.

For convenience, we set

$$\begin{aligned} \tilde{p}_j &= p_j + 1, \tilde{q}_j = q_j + 1 \quad \text{for } j = 1, \dots, n, \\ \tilde{p}_j &= p_j + 2n + 1, \tilde{q}_j = q_j + 2n + 1 \quad \text{for } j = n + 1, \dots, 2n. \end{aligned}$$

The range $R(U_r^*)$ of U_r^* for $r = 0, 1$ is the span of all standard unit vectors $e_{\tilde{p}_j}$ in $\mathbb{C}^{4n}$ with $j \in \Theta_r$, and $R(V_1^*)$ is the span of all standard unit vectors $e_{\tilde{q}_j}$ in $\mathbb{C}^{4n}$ with $j \in \Theta_1$. Hence it follows from Assumptions 2.1 and 4.1 that

$$U_0 U_0^* = \text{id}_{\mathbb{C}^{2n-k}}, \quad U_1 U_1^* = \text{id}_{\mathbb{C}^k}, \quad V_1 V_1^* = \text{id}_{\mathbb{C}^k}, \quad (4.1)$$

$$U_1 U_0^* = 0, \quad V_1 U_0^* = 0, \quad U_1 V_1^* = 0. \quad (4.2)$$

Theorem 4.2 *The operator $A(U)$ is densely defined, the domain $\mathcal{D}((A(U))^*)$ of its adjoint $(A(U))^*$ is the set of all $\tilde{z} = \begin{pmatrix} z \\ d \end{pmatrix}$ in $L^2(I, w) \oplus \mathbb{C}^k$ such that there is $c \in \mathbb{C}^k$ such that $z \in \mathcal{D}(\mathcal{A}_{\max})$ and*

$$D^*Z_R + U_1^*d - V_1^*c \in R(U_0^*). \quad (4.3)$$

For $\tilde{z} = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathcal{D}((A(U))^)$, the vectors d and c are uniquely determined by z , namely, $d = -U_1D^*Z_R$ and $c = V_1D^*Z_R$, and*

$$(A(U))^*\tilde{z} = \begin{pmatrix} \mathcal{A}_{\max}z \\ V_1D^*Z_R \end{pmatrix}. \quad (4.4)$$

Proof By definition of the adjoint (possibly as a linear relation), $\tilde{z} = \begin{pmatrix} z \\ d \end{pmatrix} \in L^2(I, w) \oplus \mathbb{C}^k$ belongs to $\mathcal{D}((A(U))^*)$ if and only if there is $\tilde{u} = \begin{pmatrix} u \\ c \end{pmatrix} \in L^2(I, w) \oplus \mathbb{C}^k$ such that for all $\tilde{y} = \begin{pmatrix} y \\ V_1Y_R \end{pmatrix} \in \mathcal{D}(A(U))$ the identity

$$\langle A(U)\tilde{y}, \tilde{z} \rangle = \langle \tilde{y}, \tilde{u} \rangle \quad (4.5)$$

holds.

Hence let $\tilde{z}, \tilde{u} \in L^2(I, w) \oplus \mathbb{C}^k$ such that (4.5) holds for all $\tilde{y} \in \mathcal{D}(A(U))$. If y has compact support in I , then (4.5) reduces to

$$(\mathcal{A}_{\max}y, z)_w = (y, u)_w.$$

This, the formal symmetry Assumption 3.1 and [10], Theorem 4.2 show that $z \in \mathcal{D}(\mathcal{A}_{\max})$ and $\mathcal{A}_{\max}z = u$. We can now conclude that (4.5) holds if and only if

$$(\mathcal{A}_{\max}y, z)_w + d^*U_1Y_R = (y, \mathcal{A}_{\max}z)_w + c^*V_1Y_R.$$

In view of the Lagrange identity (3.3), the above is equivalent to

$$Z_R^*DY_R + d^*U_1Y_R = c^*V_1Y_R.$$

Since the range of all Y_R with $y \in \mathcal{D}(A(U))$ is $N(U_0)$, it follows that (4.5) is equivalent to $z \in \mathcal{D}(\mathcal{A}_{\max})$, $u = \mathcal{A}_{\max}z$ and

$$D^*Z_R + U_1^*d - V_1^*c \in N(U_0)^\perp = R(U_0^*). \quad (4.6)$$

Applying U_1 and V_1 , respectively, to (4.6) and observing (4.1) and (4.2) it follows that d and c are uniquely given by $d = -U_1D^*Z_R$ and $c = V_1D^*Z_R$. From the uniqueness of u and c we see that $(A(U))^*$ is not only a linear relation but a linear operator, so that $A(U)$ is densely defined. $\square$

Remark 4.3 The matrix D is invertible and

$$D^{-1} = -D = D^*, \quad (4.7)$$

see [8], (2.7).

Proposition 4.4 Assume that $\text{rank } W = 2(2n - k)$. Then $U_1 D = V_1$ and $V_1 D = -U_1$.

Proof By definition of U_1 and D we can write

$$U_1 D = (-1)^n \begin{pmatrix} U_1^a C & 0 \\ 0 & -U_1^b C \end{pmatrix},$$

where $U_1^\alpha = (\delta_{j,p_i+1})_{i \in \Theta_1^\alpha, j=1, \dots, 2n}$ for $\alpha = a, b$. In view of Proposition 3.8 we conclude that

$$\begin{aligned} U_1^\alpha C &= (\delta_{2n+1-j, p_i+1} (-1)^{p_i+1})_{i \in \Theta_1^\alpha, j=1, \dots, 2n} \\ &= (\delta_{j, q_i+1} (-1)^{q_i})_{i \in \Theta_1^\alpha, j=1, \dots, 2n}. \end{aligned}$$

Hence $U_1 D = V_1$, and (4.7) gives $V_1 D = U_1 D^2 = -U_1$. $\square$

Proposition 4.5 If $A(U)$ is symmetric, then $A(U)$ is self-adjoint.

Proof We have to show that $\mathcal{D}((A(U))^*) \subset \mathcal{D}(A(U))$. By Theorem 4.2, $\mathcal{D}((A(U))^*)$ is the set of all $\begin{pmatrix} z \\ V_1 Z_R \end{pmatrix}$ such that $z \in \mathcal{D}(\mathcal{A}_{\max})$ and $D^* Z_R + U_1^* d - V_1 c \in R(U_0^*)$. But Theorem 4.2, Proposition 4.4 and (4.7) imply

$$\begin{aligned} D^* Z_R - V_1^* c + U_1^* d &= D^* Z_R - V_1^* V_1 D^* Z_R - U_1^* U_1 D^* Z_R \\ &= -D Z_R - V_1^* U_1 Z_R + U_1^* V_1 Z_R \\ &= -W Z_R, \end{aligned}$$

so that $\mathcal{D}((A(U))^*) \subset \mathcal{D}(A(U))$ if and only if $W^{-1}(R(U_0^*)) \subset N(U_0)$.

We know that $\text{rank } U_0 = 2n - k$ and $\dim N(U_0) = 4n - \text{rank } U_0 = 2n + k$, whereas $\dim N(W) = 4n - \text{rank } W = 2k$ by Corollary 3.6. Altogether, we conclude

$$\dim W^{-1}(R(U_0^*)) \leq \dim N(W) + \text{rank } U_0 = 2n + k = \dim N(U_0).$$

But from Corollary 3.7 we conclude that $N(U_0) \subset W^{-1}(R(U_0^*))$, and it follows that $N(U_0) = W^{-1}(R(U_0^*))$. $\square$

Proposition 4.6 Assume $\text{rank } W = 2(2n - k)$. Then $W(N(U_0)) = R(U_0^*)$ if and only if

- (i) $p_s + p_r \neq 2n - 1$ for all $r, s \in \Theta_0^a$,
- (ii) $p_s + p_r \neq 2n - 1$ for all $r, s \in \Theta_0^b$.

Proof Defining for $c = a, b$,

$$\begin{aligned} M_c &= \text{span}\{e_{p_j+1} : j \in \Theta_0^c\} \subset \mathbb{C}^{2n}, \quad c = a, b, \\ N_c &= \mathbb{C}^{2n} \ominus M_c = \text{span}\{e_j : j \in \{1, \dots, 2n\} \setminus \{p_s + 1 : s \in \Theta_0^c\}\} \subset \mathbb{C}^{2n}, \\ W_a &= (-1)^n C + V_2, \quad W_b = (-1)^{n+1} C + V_3, \end{aligned}$$

where V_2 and V_3 are as in (3.7), it follows that

$$R(U_0^*) = \left\{ \begin{pmatrix} u_a \\ u_b \end{pmatrix} : u_a \in M_a, u_b \in M_b \right\}, \quad N(U_0) = \left\{ \begin{pmatrix} u_a \\ u_b \end{pmatrix} : u_a \in N_a, u_b \in N_b \right\},$$

and

$$W = D + V_1^* U_1 - U_1^* V_1 = \begin{pmatrix} W_a & 0 \\ 0 & W_b \end{pmatrix}$$

in view of (3.5) and (3.8). Therefore $W(N(U_0)) = R(U_0^*)$ if and only if $W_c(N_c) = M_c$ for $c = a, b$. Now let $c \in \{a, b\}$. From Proposition 3.8 and its proof we find for $j \in \{1, \dots, 2n\}$ that

$$W_c(e_j) = \begin{cases} \pm e_{2n+1-j} & \text{if } j \in \{1, \dots, 2n\} \setminus \{p_s + 1, q_s + 1 : s \in \Theta_1^c\}, \\ 0 & \text{if } j \in \{p_s + 1, q_s + 1 : s \in \Theta_1^c\}. \end{cases}$$

Observing condition 1 in Proposition 3.8 it follows that

$$\begin{aligned} W_c(N_c) &= \text{span}\{e_{2n+1-j} : j \in \{1, \dots, 2n\} \\ &\quad \setminus (\{p_s + 1, q_s + 1 : s \in \Theta_1^c\} \cup \{p_s + 1 : s \in \Theta_0^c\})\} \\ &= \text{span}\{e_j : j \in \{1, \dots, 2n\} \\ &\quad \setminus (\{p_s + 1, q_s + 1 : s \in \Theta_1^c\} \cup \{2n - p_s : s \in \Theta_0^c\})\}. \end{aligned}$$

Hence $W_c(N_c) = M_c$ holds if and only if the sets

$$\Psi_1^c := \{p_s + 1, q_s + 1 : s \in \Theta_1^c\} \cup \{2n - p_s : s \in \Theta_0^c\} \quad \text{and} \quad \Psi_0^c := \{p_s + 1 : s \in \Theta_0^c\}$$

are complementary subsets of $\{1, \dots, 2n\}$. But by Assumption 2.1 and condition 1 in Proposition 3.8 the listed elements in Ψ_0^c as well as in Ψ_1^c are mutually distinct, so that the sets Ψ_0^c and Ψ_1^c are complementary if and only if they are disjoint. It is clear that this latter property holds if and only if $2n - p_j \notin \Psi_0^c$ for all $j \in \Theta_0^c$. This completes the proof of the proposition. $\square$

Theorem 4.7 *The boundary eigenvalue problem (2.8), (2.9) has an operator pencil representation (2.12) with self-adjoint operators K and $A(U)$ if and only if*

1. $\beta_j \in \mathbb{R}$ and $p_j + q_j = 2n - 1$ for all $j \in \Theta_1$;
2. $p_s + p_r \neq 2n - 1$ for all $r, s \in \Theta_0^a$,
3. $p_s + p_r \neq 2n - 1$ for all $r, s \in \Theta_0^b$.

Proof This theorem is an immediate consequence of Corollary 3.9 and Propositions 4.5 and 4.6. $\square$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to the manuscript and read and approved the final submitted version.

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