Symmetry Solutions and Conservation Laws for some Partial Differential Equations in Fluid Mechanics

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ABSTRACT

In jet problems the conserved quantity plays a central role in the solution process. The conserved quantities for laminar jets have been established either from physical arguments or by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. This method of deriving conserved quantities is not entirely systematic and in problems such as the wall jet requires considerable mathematical and physical insight.

A systematic way to derive the conserved quantities for jet flows using conservation laws is presented in this dissertation. Two-dimensional, radial and axisymmetric flows are considered and conserved quantities for liquid, free and wall jets for each type of flow are derived. The jet flows are described by Prandtl's momentum boundary layer equation and the continuity equation. The stream function transforms Prandtl's momentum boundary layer equation and the continuity equation into a single third-order partial differential equation for the stream function. The multiplier approach is used to derive conserved vectors for the system as well as for the third-order partial differential equation for the stream function for each jet flow. The liquid jet, the free jet and the wall jet satisfy the same partial differential equations but the boundary conditions for each jet are different. The conserved vectors depend only on the partial differential equations. The derivation of the conserved quantity depends on the boundary conditions as well as on the differential equations. The boundary conditions therefore determine which conserved vector is associated with which jet. By integrating the corresponding conservation laws across the jet and imposing the boundary conditions, conserved quantities are derived. This approach gives a unified treatment to the derivation of conserved quantities for

jet flows and may lead to a new classification of jets through conserved vectors.

The conservation laws for second order scalar partial differential equations and systems of partial differential equations which occur in fluid mechanics are constructed using different approaches. The direct method, Noether's theorem, the characteristic method, the variational derivative method (multiplier approach) for arbitrary functions as well as on the solution space, symmetry conditions on the conserved quantities, the direct construction formula approach, the partial Noether approach and the Noether approach for the equation and its adjoint are discussed and explained with the help of an illustrative example. The conservation laws for the non-linear diffusion equation for the spreading of an axisymmetric thin liquid drop, the system of two partial differential equations governing flow in the laminar two-dimensional jet and the system of two partial differential equations governing flow in the laminar radial jet are discussed via these approaches.

The group invariant solutions for the system of equations governing flow in two-dimensional and radial free jets are derived. It is shown that the group invariant solution and similarity solution are the same.

The similarity solution to Prandtl's boundary layer equations for two-dimensional and radial flows with vanishing or constant mainstream velocity gives rise to a third-order ordinary differential equation which depends on a parameter. For specific values of the parameter the symmetry solutions for the third-order ordinary differential equation are constructed. The invariant solutions of the third-order ordinary differential equation are also derived.

DECLARATION

I declare that this is my own, unaided work. It is being submitted for the Degree of PhD to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

Rehana Naz

DEDICATION

To my parents

to my husband Imran Naeem and to my daughter Ayesha

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Contents

Chapter 1

Introduction

1.1 Introduction

The notion of a conservation law plays a vital role in the study of differential equations and their many applications. The mathematical idea of a conservation law comes from the formulation of familiar physical laws such as the balance laws for energy and momentum.

In jet problems the conserved quantity plays an important role in the solution process and is used to determine the unknown exponent in the similarity solution which cannot be obtained from the homogeneous boundary conditions. The conserved quantities for laminar jets have been established either from physical arguments or by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. By this method, Schlichting (1933,1968) established the conserved quantity for the two-dimensional free jet. The problem of the radial free jet was first studied by Squire (1955). The conserved quantity for the radial free jet was derived by integrating Prandtl's momentum boundary layer equation across the jet (see Squire 1955, Riley 1962b, Schwarz 1963). Glauert (1956) established the conserved quantities for two-dimensional and radial wall jets again by integrating the momentum boundary layer equation across the jet. The conserved quantity for the two-dimensional liquid jet (Watson 1964) and radial liquid jet (Riley 1962b, Watson 1964) was obtained from a physical argument based on conservation of volume flux in an incompressible fluid. Goldstein (1938) established the conserved quantity for an axisymmetric free jet by integrating the momentum equation across the jet. Later, Duck and Bodonyi (1986) established the conserved quantity for an axisymmetric wall jet by same procedure. This method of deriving conserved quantities is not entirely systematic and difficult to apply in problems such as the wall jets.

In this work, we have established the conserved quantities for jet flows by utilizing conservation laws. The multiplier approach is used to derive the conservation laws for the system of equations for velocity components as well as for the third-order partial differential equation for the stream function. By integrating the corresponding conservation laws across the jet and imposing the boundary conditions, conserved quantities are derived for liquid, free and wall jets for two-dimensional, radial and axisymmetric flows.

The conservation laws for fluid mechanics problems are constructed with the help of different approaches. There are nine approaches to construct the conservation laws for partial differential equations. An elegant and constructive way of finding conservation laws is by means of Noether's theorem (Noether 1918) when a system of differential equations has a Lagrangian formulation. Yet there are differential equations that do not have a Lagrangian, for example scalar evolution differential equations. There are methods to obtain conservation laws which do not rely on the knowledge of a Lagrangian function. The most elementary method is the direct method introduced by Laplace (1798). The direct method has been successfully applied to obtain conservation laws for several well-known differential equations. The other methods for obtaining conservation laws are recent or lesser known. The third approach, introduced by Steudel in 1962, involves writing a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations. In order to determine a conservation law in this approach one has to also find the related characteristics. The fourth method, which is linked to the third, involves the variational derivative. In this method, one computes the characteristics and from these the associated conservation laws (see proposition 5.49 in Olver 1993). Anco and Bluman (2002b) also used this approach. In the fifth approach, the variational derivatives are computed on the solution space of the differential equations. The characteristics obtained by this approach sometimes correspond to an adjoint symmetry rather than a conservation law. A computer algebra package has been developed by Wolf (2002) for the four methods stated which do not rely on a Lagrangian. The sixth approach introduced by Kara and Mahomed (2000) to derive a conservation law is to add a symmetry condition to the direct method. The seventh approach due to Anco and Bluman (2002a,b) provides formulae for finding conservation laws for known characteristics. The local conservation laws for the partial differential equations (single or system) expressed in a standard Cauchy-Kovalevskaya form can be computed by direct construction formulae (Anco and Bluman 2002b). The eighth and most recent approach, due to Kara and Mahomed (2006), is the partial Noether approach. It works like the Noether approach for the differential equations with or without a Lagrangian. In the ninth approach, the partial differential equation together with the adjoint equation is considered as a system. The adjoint variational principle, in general, was introduced by Atherton and Homsy (1975). Symmetry considerations for such systems were recently incorporated by Ibragimov (2007). Then the conservation laws are computed by a formula (Ibragimov 2007).

The group invariant solution for two-dimensional and radial free jets are constructed. The Lie point symmetries are calculated for both cases. A symmetry is associated with the conserved vector that is used to establish the conserved quantity for the jet and then this symmetry generates the group invariant solution for the system governing the flow in the free jet. Moreover, the symmetry and invariant solutions for the third-order ordinary differential equation, which arises from Prandtl's boundary layer equations, are derived.

1.2 Aims and objectives of thesis

There are three objectives of the thesis.

Firstly, to investigate a systematic way to derive the conserved quantities for jet flows. In the literature conserved quantities are derived either from physical arguments or by integrating the momentum balance equation across the jet and using boundary conditions and the continuity equation. This is a convenient method for simple jet flows but it is a difficult approach for more complicated jets such as the wall jet. The derivation of the conserved quantities for two-dimensional jet flows, radial jet flows and axisymmetric jet flows with the help of conservation laws is presented in Chapters 3, 4, 5 respectively.

Secondly, to review and compare the different approaches for the construction of conservation laws for partial differential equations. This is done in Chapters 6 and 7.

Thirdly, to derive the Lie point symmetries, group invariant solutions and symmetry solutions of the differential equations arising from the boundary layer approximations. In Chapter 8, a Lie symmetry analysis and the derivation of group invariant solutions for the system of partial differential equations for two-dimensional and radial jet flows are presented. Chapter 9 is concerned with the symmetry solution of a third-order ordinary differential equation which arises from Prandtl boundary layer equations.

1.3 Outline of thesis

A detail outline of the thesis is as follows.

In Chapter 2, the basic operators, definitions and the concept of a conservation law are presented.

A systematic way to derive the conserved quantities for the two-dimensional liquid jet, free jet and wall jet using conservation laws is presented in Chapter 3. The flow in the two-dimensional jet is described by Prandtl's momentum boundary layer equation and the continuity equation. The multiplier approach is first applied to construct a basis of conserved vectors for the system. The basis consists of two conserved vectors. By integrating the corresponding conservation laws across the jet and imposing the boundary conditions, conserved quantities are derived for the two-dimensional liquid and free jets. The multiplier approach is then applied to construct a basis of conserved vectors for the third-order partial differential equation for the stream function. The basis consists of two local conserved vectors one of which is a non-local conserved vector for the system. The conserved quantities for the two-dimensional free jet and the wall jet are derived from the corresponding conservation laws and boundary conditions. Corresponding results are derived for the radial liquid, free and wall jets in Chapter 4.

The results presented in Chapters 3 and 4 are published by Naz, Mason and Mahomed 2008.

In Chapter 5, a systematic way to derive the conserved quantities for the axisymmetric liquid jet, free jet and wall jet using conservation laws is presented. The multiplier approach is used to construct a basis, which consists of two conserved vectors, for the system of two partial differential equations for the two velocity components. By integrating the corresponding conservation laws across the jet and imposing the boundary conditions, conserved quantities are derived for the axisymmetric liquid and free jet. The multiplier approach applied to the third-order partial differential equation for the stream function yields two local conserved vectors one of which is a non-local conserved vector for the system. One of the conserved vectors gives the conserved quantity for the axisymmetric free jet but the conserved quantity for the wall jet cannot be obtained from the second conserved vector. The conserved quantity for the axisymmetric wall jet is derived from a non-local conserved vector of the third-order partial differential equation for the stream function which is obtained by using the stream function as a multiplier.

In Chapter 6, the different approaches to construct conservation laws for partial differential equations are discussed and are explained with the help of an illustrative example.

In Chapter 7, the conservation laws for the non-linear diffusion equation for the spreading of an axisymmetric thin liquid drop, the system of two partial differential equations governing flow in a laminar two-dimensional jet and the system of two partial differential equations governing flow in a laminar radial jet are constructed using different approaches.

The results presented in Chapters 6 and 7 are published by Naz, Mahomed and Mason 2008c.

Chapter 8 is concerned with the group invariant solution of radial and two-dimensional free jets. The Lie point symmetries are calculated for both cases and a symmetry is associated with the conserved vector that is used to establish the conserved quantity for the jet. This associated symmetry is then used to derive the group invariant solution for the system governing the flow in the free jet.

The similarity solution to Prandtl's boundary layer equations for two-dimensional and radial flows with vanishing or constant mainstream velocity gives rise to a third-order ordinary differential equation which depends on a parameter α and its symmetry solutions are considered in Chapter 9. For special values of α the third-order ordinary differential equation admits a three-dimensional symmetry Lie algebra L_3. For solvable L_3 the equation is integrated by quadrature. For non-solvable L_3 the equation reduces to the Chazy equation. The Chazy equation is reduced to a first-order differential equation in terms of differential invariants which is transformed to a Riccati equation. In general the third-order ordinary differential equation admits a two-dimensional symmetry Lie algebra L_2. For L_2 the differential equation can only be reduced to a first-order equation. The invariant solutions of the third-order ordinary differential equation are also derived.

The results presented in Chapter 9 are published by Naz, Mahomed and Mason (2008a).

Finally, the conclusions are summarized in Chapter 10.

Chapter 2

Background and definitions

The purpose of this chapter is to introduce the basic concepts of Lie theory and conservation laws for partial differential equations. The derivation of conserved quantities using conservation laws for jet flows is considered in this thesis.

We first present the notation that will be used and recall basic definitions and theorems which can be found in the literature cited. The summation convention over repeated lower and upper indices is adopted.

2.1 Boundary layer theory

Prandtl in (1904) introduced the concept of a boundary layer in large Reynolds number flows and he also showed how the Navier-Stokes equations could be simplified to yield approximate solutions. There are many books on boundary layer theory e.g. Schlichting (1968), Schlichting and Gersten (2000) and Rosenhead (1963).

Definition 2.1. A boundary layer is a thin layer in which the effect of viscosity is important no matter how high the Reynolds number may be.

The Reynold number Re is defined by

$$Re = \frac{UL}{\nu}, \tag{2.1}$$

where U is a characteristic velocity and L is a characteristic length associated with the flow and ν is the kinematic viscosity of the fluid. A boundary layer exists if

$$\sqrt{Re} >> 1. \tag{2.2}$$

A boundary layer does not necessarily need to be adjacent to a solid boundary. A thin region of sharp change can exist away from a boundary such as along the axis of a free jet. The boundary layer equations are applicable in the thin region of sharp change.

2.2 Fundamental relations

Let $x^i,\ i = 1, 2, \cdots, n$, be n independent variables and $u^\alpha,\ \alpha = 1, 2, \cdots, N$, be N dependent variables. The collection of rth-order derivatives, $r \geq 1$, is denoted by $u_{(r)}$. Subscripts denote partial derivatives. The summation convention is adopted in which there is summation over repeated upper and lower indices. As usual $\mathcal{A}$ is the vector space of differential functions. The basic operators defined in $\mathcal{A}$ are stated below.

Definition 2.2. The *Euler operator* is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}},\ \alpha = 1, 2, \cdots, N, \tag{2.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots, \quad i = 1, 2, ...n, \tag{2.4}$$

is the total derivative operator with respect to x^i. Euler operator is also denoted by E_{u^α}.

Definition 2.3. The *Lie-Bäcklund operator* is

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1\cdots i_s} \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \tag{2.5}$$

where $\zeta^\alpha_{i_1\cdots i_s}$ are defined by

$$\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j),$$

$$\zeta^\alpha_{i_1\cdots i_s} = D_{i_s}(\zeta^\alpha_{i_1\cdots i_{s-1}}) - u^\alpha_{ji_1\cdots i_{s-1}} D_{i_s}(\xi^j),\ s > 1. \tag{2.6}$$

The Lie-Bäcklund operator (2.5) in characteristic form is

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + W^\alpha \frac{\partial}{\partial u^\alpha} + D_i(W^\alpha) \frac{\partial}{\partial u^\alpha_i} + D_i D_j(W^\alpha) \frac{\partial}{\partial u^\alpha_{ij}} + \cdots, \tag{2.7}$$

where

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j,\ \alpha = 1, 2, \cdots, N, \tag{2.8}$$

are the Lie characteristic functions.

Definition 2.4. The *Noether operators* associated with a Lie-Bäcklund operator $\mathbf{X}$ are

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{ij_1\cdots j_s}},\ i = 1, 2, \cdots, n, \tag{2.9}$$

where the Euler-Lagrange operator $\delta/\delta u^\alpha_i$ is

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s\geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}$$

$$i = 1, 2, \cdots, n,\ \alpha = 1, 2, \cdots, N \tag{2.10}$$

and similarly for the other Euler-Lagrange operators with respect to higher order derivatives.

2.3 Conservation laws

Consider a k^{th}-order system of differential equations of n independent and N dependent variables

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0, \quad \alpha = 1, 2, ...N, \tag{2.11}$$

which is assumed to be of maximal rank and locally solvable.

Definition 2.5. A conserved vector of (2.11) is an n-tuple $T = (T^1, T^2, ..., T^n),\ T^i \in \mathcal{A},\ i = 1, 2...n$, such that

$$D_i T^i = 0 \tag{2.12}$$

holds for all solutions of (2.11). Equation (2.12) is called a local conservation law provided T^i are free of integral terms.

Definition 2.6. There are two kinds of trivial conservation laws for differential equations (2.11) (see Olver 1993).

The first kind of trivial conservation law is that in which the n-tuple $T = (T^1, T^2, ..., T^n)$ in (2.12) itself vanishes for all solutions of the system (2.11). For example

$$\begin{aligned} T^1 &= \sqrt{c}u\cos(\sqrt{c}v)\,[v_x - u] \\ T^2 &= c\sin(\sqrt{c}v)\,[v_x - u] + \sqrt{c}u\cos(\sqrt{c}v)\left[cxu - \frac{1}{u^3}u_x v_x - v_t\right], \end{aligned} \tag{2.13}$$

forms a trivial conservation law of the first kind for the system

$$v_x = u,\ v_t = \left(\frac{1}{u}\right)_x + cxu, \quad c > 0, \tag{2.14}$$

since T^1 and T^2 vanish for all solutions of the system.

In the second kind of triviality, the divergence expression (2.12) holds for arbitrary functions, not only for the solutions of system (2.11), or the divergence (2.12) vanishes identically. For example

$$D_t(u_x) + D_x(-u_t) = 0 \tag{2.15}$$

holds for any smooth function $u = g(t, x)$ and thus yields a trivial conservation law of the second kind.

We are interested in finding non-trivial conservation laws.

Chapter 3

Conservation laws and physical conserved quantities for laminar two-dimensional jets

3.1 Introduction

The conserved quantities are required in the solution of jet flow problems because these problems have homogeneous boundary conditions. The unknown exponent in the similarity solution cannot therefore be obtained from the boundary conditions as it can, for instance, in Blasius flow past a flat plate where the mainstream matching boundary condition is used. In jet flow problems a further condition is required and this is provided by the conserved quantity. The conserved quantity has physical significance for each jet. It is a measure of the strength of the jet.

Schlichting (1933) established the conserved quantity for the two-dimensional free jet by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. Glauert (1956) established the conserved quantity for the two-dimensional wall jet by same method. The conserved quantity for the two-dimensional liquid jet (Watson 1964) was obtained from the physical argument that volume flux is constant in an incompressible fluid.

In this chapter, we present a new approach to the construction of the conserved quantities for laminar jet flows by using the conservation laws for the partial differential equations describing the jet flows. This new method is more systematic than the existing method of integrating the momentum balance equation across the jet and for significant problems such as the wall jet it is simpler than the derivation given by Glauert (1956). Two-dimensional jets can be formulated either as a system of two partial differential equations for the two velocity components or as a single third-order partial differential equation for the stream function. Mason (2002) gave the elementary conservation law for the third-order partial differential equation for the stream function for the two-dimensional jet. We will use the multiplier approach, introduced by Steudel (1962) (see Olver 1993) and also invoked by Anco and Bluman (2002a,b), to derive a basis of two conservation laws for both the system of partial differential equations for the two velocity components and the third-order partial differential equation for the stream function. We will then show how the conservation laws can be used to construct conserved quantities for two-dimensional liquid, free and wall jets.

An outline of the chapter is as follows. In Section 3.2, a basis of conservation laws, for the system of partial differential equations for the velocity components as well as for the partial differential equation for the stream function, is derived for two-dimensional jets using the multiplier approach. In Section 3.3, the conserved quantity for the two-dimensional liquid, free and wall jet are established. Finally, the conclusions are summarized in Section 3.4.

3.2 Conservation laws for two-dimensional laminar jet flows

We will consider the two-dimensional liquid jet, free jet and wall jet.

A liquid jet is formed when a two-dimensional jet of liquid strikes a plane boundary normally and spreads over it (Watson 1964). The x-axis is along the boundary, the y-axis is perpendicular to the boundary and the boundary is $y = 0$. There is no suction or blowing of fluid at the boundary. The fluid in the jet is viscous and incompressible and the surrounding fluid is a gas. The equation of the free surface is $y = \phi(x)$.

A free jet is formed when fluid emerges from a long narrow orifice in a wall into the same viscous and incompressible fluid at rest. The x-axis is along the axis of symmetry of the jet, the y-axis is perpendicular to the jet in the plane of the jet and the origin is at the orifice.

A wall jet is formed when a sluice gate separating two sections of a canal is slightly raised (Glauert 1956). The flow of fluid into the part with lower water level is a wall jet. The fluid in the jet is viscous and incompressible and is the same as the surrounding fluid which is at rest far from the jet. The x-axis is along the wall and the y-axis is perpendicular to the wall. There is no suction or blowing of fluid at the wall.

The flow in a two-dimensional laminar jet is governed by Prandtl's momentum boundary layer equation

$$u u_x + v u_y = \nu u_{yy} \tag{3.1}$$

and the continuity equation

$$u_x + v_y = 0, \tag{3.2}$$

where $u(x, y)$ and $v(x, y)$ are the velocity components in the x and y directions, respectively and ν is the kinematic viscosity of the fluid. Subscripts denote partial derivatives.

3.2.1 Stream function

The stream function plays an important part in the subsequent analysis (Batchelor 1967). From (3.2), $udy - vdx$ is a perfect differential, denoted by $d\psi$:

$$d\psi = u(x,y)dy - v(x,y)dx. \tag{3.3}$$

The stream function $\psi(x,y)$ at any point $P(x,y)$ is obtained by integrating (3.3) from any reference point O, which we choose to be $(x_o, 0)$, to the point P along an arbitrary curve joining O and P. Thus

$$\psi(x,y) - \psi(x_o,0) = \int_O^P u(x,y)dy - v(x,y)dx. \tag{3.4}$$

Physically the integral on the right hand side of (3.4) is the flux of volume across the curve joining O and P and hence, since the fluid is incompressible, the integral is independent of the choice of curve between O and P.

From (3.3), since x and y are independent variables,

$$u(x,y) = \psi_y, \; v(x,y) = -\psi_x. \tag{3.5}$$

Now, for the free jet, since the x-axis is an axis of symmetry, $v(x,0) = 0$. For the liquid jet and the wall jet, since there is no suction or blowing at the solid boundary, $v(x,0) = 0$. Thus for all three jets,

$$\psi_x(x,0) = 0 \tag{3.6}$$

and therefore $\psi(x,0) = \psi_o$ where ψ_o is a constant. The stream function (3.4) therefore contains an additive constant which from (3.5) does not contribute to the velocity components. We specify this constant by choosing $\psi_o = 0$. Hence

$$\psi(x,0) = 0, \tag{3.7}$$

and (3.4) becomes

$$\psi(x,y) = \int_O^P u(x,y)dy - v(x,y)dx. \tag{3.8}$$

Integrating (3.8) with respect to y from $y = 0$ to $y = \infty$ along a straight line with x kept fixed gives

$$\psi(x, \infty) = \int_0^\infty u(x, y)dy. \tag{3.9}$$

We assume that $u(x, y)$ tends to zero sufficiently rapidly as y tends to infinity to ensure that the integral in (3.9) is convergent. Thus $\psi(x, \infty)$ is finite.

We substitute (3.5) into (3.1) and (3.2). Then (3.2) is identically satisfied and (3.1) yields the third-order partial differential equation

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0. \tag{3.10}$$

We will derive the conservation laws for the system (3.1)-(3.2) and also for the third-order partial differential equation (3.10), using the multiplier approach (Steudel 1962, Olver 1993, Anco and Bluman 2002a,b).

3.2.2 Conservation laws for system of equations by multiplier approach

Multipliers Λ_1 and Λ_2 for the system of equations (3.1) and (3.2) have the property that

$$\Lambda_1 (uu_x + vu_y - \nu u_{yy}) + \Lambda_2 (u_x + v_y) = D_x T^1 + D_y T^2, \tag{3.11}$$

for all functions $u(x, y)$ and $v(x, y)$ where the total derivative operators D_x and D_y from (2.4) are defined by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xy} \frac{\partial}{\partial u_y} + v_{xy} \frac{\partial}{\partial v_y} + \cdots, \tag{3.12}$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v} + u_{yy} \frac{\partial}{\partial u_y} + v_{yy} \frac{\partial}{\partial v_y} + u_{yx} \frac{\partial}{\partial u_x} + v_{yx} \frac{\partial}{\partial v_x} + \cdots. \tag{3.13}$$

The right hand side of (3.11) is a divergence expression and T^1 and T^2 are the components of the conserved vector $T = (T^1, T^2)$. We will consider multipliers of the form $\Lambda_1 = \Lambda_1(x, y, u, v)$ and $\Lambda_2 = \Lambda_2(x, y, u, v)$. The determining equations for the multipliers Λ_1 and Λ_2 are

$$E_u\left[\Lambda_1(uu_x+vu_y-\nu u_{yy})+\Lambda_2(u_x+v_y)\right]=0, \tag{3.14}$$

$$E_v\left[\Lambda_1(uu_x+vu_y-\nu u_{yy})+\Lambda_2(u_x+v_y)\right]=0, \tag{3.15}$$

where E_u and E_v are the standard Euler operators which annihilate divergence expressions:

$$E_u=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_y\frac{\partial}{\partial u_{xy}}+D_y^2\frac{\partial}{\partial u_{yy}}-\cdots, \tag{3.16}$$

and

$$E_v=\frac{\partial}{\partial v}-D_x\frac{\partial}{\partial v_x}-D_y\frac{\partial}{\partial v_y}+D_x^2\frac{\partial}{\partial v_{xx}}+D_xD_y\frac{\partial}{\partial v_{xy}}+D_y^2\frac{\partial}{\partial v_{yy}}-\cdots. \tag{3.17}$$

The expansion of (3.14) and (3.15) yields,

$$\Lambda_{1u}(uu_x+vu_y-\nu u_{yy})+\Lambda_{2u}(u_x+v_y)+\Lambda_1u_x$$
$$-D_x(\Lambda_1u+\Lambda_2)-D_y(\Lambda_1v)-\nu D_y^2(\Lambda_1)=0 \tag{3.18}$$

and

$$\Lambda_{1v}(uu_x+vu_y-\nu u_{yy})+\Lambda_{2v}(u_x+v_y)+\Lambda_1u_y-D_y(\Lambda_2)=0. \tag{3.19}$$

Equations (3.18) and (3.19) have to be satisfied for all functions $u(x,y)$ and $v(x,y)$. The equations are separated by equating the coefficients of the partial derivatives of $u(x,y)$ and $v(x,y)$. Now

$$D_y^2(\Lambda_1)=u_{yy}\Lambda_{1u}+v_{yy}\Lambda_{1v}+\text{lower derivative terms}\ . \tag{3.20}$$

Thus equating to zero the coefficients of u_{yy} and v_{yy} in (3.18) gives

$$\Lambda_{1u}=0,\ \Lambda_{1v}=0 \tag{3.21}$$

and therefore $\Lambda_1=\Lambda_1(x,y)$. When expanded, (3.18) and (3.19) reduce to

$$v_x\Lambda_{2v}+v_y(\Lambda_1-\Lambda_{2u})+u\Lambda_{1x}+v\Lambda_{1y}+\Lambda_{2x}+\nu\Lambda_{1yy}=0, \tag{3.22}$$

$$u_x\Lambda_{2v} + u_y(\Lambda_1 - \Lambda_{2u}) - \Lambda_{2y} = 0. \tag{3.23}$$

We separate (3.22) and (3.23) according to derivatives of u and v. From (3.23) it follows that $\Lambda_{2v} = 0$ and $\Lambda_{2y} = 0$ and therefore $\Lambda_2 = \Lambda_2(x, u)$ and (3.23) reduces to

$$\Lambda_1(x, y) - \Lambda_{2u} = 0. \tag{3.24}$$

Differentiating (3.24) by y gives $\Lambda_{1y} = 0$ and therefore $\Lambda_1 = A(x)$. Equation (3.24) becomes

$$\Lambda_{2u} = A(x) \tag{3.25}$$

and therefore

$$\Lambda_2 = uA(x) + B(x). \tag{3.26}$$

Equation (3.22) reduces to

$$2u\frac{dA}{dx} + \frac{dB}{dx} = 0. \tag{3.27}$$

Separating (3.27) according to powers of u gives $A(x) = c_2$ and $B(x) = c_1$ where c_1 and c_2 are constants. Thus

$$\Lambda_1 = c_2,\ \Lambda_2 = c_1 + c_2u. \tag{3.28}$$

From (3.11) and (3.28),

$$c_2(uu_x + vu_y - \nu u_{yy}) + (c_2u + c_1)(u_x + v_y) =$$

$$D_x\left[c_2u^2 + c_1u\right] + D_y\left[c_2(uv - \nu u_y) + c_1v\right], \tag{3.29}$$

for arbitrary $u(x, y)$ and $v(x, y)$ and therefore when $u(x, y)$ and $v(x, y)$ are the solutions of the system (3.1)-(3.2),

$$D_x\left[c_2u^2 + c_1u\right] + D_y\left[c_2(uv - \nu u_y) + c_1v\right] = 0. \tag{3.30}$$

Any conserved vector for the system (3.1)-(3.2) with multipliers of the form $\Lambda_1 = \Lambda_1(x, y, u, v)$ and $\Lambda_2 = \Lambda_2(x, y, u, v)$ is therefore a linear combination of the two conserved vectors

$$T^1 = u, \quad T^2 = v, \tag{3.31}$$

$$T^1 = u^2, \quad T^2 = uv - \nu u_y. \tag{3.32}$$

The conserved vectors (3.31) and (3.32) therefore form a basis of conserved vectors for the system (3.1)-(3.2) with multipliers of the form $\Lambda_1 = \Lambda_1(x, y, u, v)$ and $\Lambda_2 = \Lambda_2(x, y, u, v)$.

In this problem it was not difficult to construct the conserved vectors by elementary manipulations once the multiplier has been determined. The conserved vectors can also be derived systematically using (3.11) as the determining equation.

A more general result can be established. It can be shown that all multipliers of the form $\Lambda_1 = \Lambda_1(x, y, u, v, u_x, v_x)$ and $\Lambda_2 = \Lambda_2(x, y, u, v, u_x, v_x)$ are necessarily given by (3.28).

3.2.3 Conservation laws for two-dimensional laminar jet flow in terms of stream function

A multiplier Λ for the partial differential equation (3.10) has the property that

$$\Lambda(\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy}) = D_x T^1 + D_y T^2, \tag{3.33}$$

for all functions $\psi(x, y)$ where D_x and D_y are defined by

$$D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{xy} \frac{\partial}{\partial \psi_y} + \cdots, \tag{3.34}$$

$$D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \psi_{yx} \frac{\partial}{\partial \psi_x} + \cdots. \tag{3.35}$$

are the total derivative operators. The right hand side of (3.33) is a divergence expression and T^1 and T^2 are the components of a conserved vector

$T = (T^1, T^2)$. Consider a multiplier of the form $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$. The determining equation for the multiplier Λ is

$$E_\psi \left[\Lambda(x, y, \psi, \psi_x, \psi_y)(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy})\right] = 0, \tag{3.36}$$

where

$$E_\psi = \frac{\partial}{\partial\psi} - D_x\frac{\partial}{\partial\psi_x} - D_y\frac{\partial}{\partial\psi_y} + D_x^2\frac{\partial}{\partial\psi_{xx}} + D_xD_y\frac{\partial}{\partial\psi_{xy}} + D_y^2\frac{\partial}{\partial\psi_{yy}} - \cdots, \tag{3.37}$$

is the standard Euler operator which annihilates the divergence on the right hand side of (3.33). Expansion of (3.36) yields

$$\begin{aligned}
&(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy})\Lambda_\psi - D_x[(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy})\Lambda_{\psi_x}] \\
&-D_y[(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy})\Lambda_{\psi_y}] + D_x(\Lambda\psi_{yy}) - D_y(\Lambda\psi_{xy}) \\
&+D_xD_y(\Lambda\psi_y) - D_y^2(\Lambda\psi_x) + \nu D_y^3(\Lambda) = 0.
\end{aligned} \tag{3.38}$$

Now

$$D_y^3(\Lambda) = \psi_{xyyy}\Lambda_{\psi_x} + \psi_{yyyy}\Lambda_{\psi_y} + \text{lower derivative terms} . \tag{3.39}$$

Since (3.38) must be satisfied for all functions $\psi(x, y)$ the coefficient of the derivatives of $\psi(x, y)$ in (3.38) must vanish. The highest order derivative terms in (3.38) are $2\nu\Lambda_{\psi_x}\psi_{xyyy}$ and $2\nu\Lambda_{\psi_y}\psi_{yyyy}$. Equating their coefficients to zero yields

$$\Lambda_{\psi_x} = 0, \quad \Lambda_{\psi_y} = 0 \tag{3.40}$$

and therefore

$$\Lambda = a(x, y, \psi). \tag{3.41}$$

The substitution of (3.41) in (3.38) yields

$$\begin{aligned}
&(2a_x + 3\nu a_{\psi y})\psi_{yy} - 2a_y\psi_{xy} + 3\nu a_{\psi\psi}\psi_y\psi_{yy} + \nu a_{\psi\psi\psi}\psi_y^3 \\
&+(a_{\psi x} + 3\nu a_{\psi\psi y})\psi_y^2 - a_{\psi y}\psi_x\psi_y + (a_{xy} + 3\nu a_{\psi yy})\psi_y - \psi_x a_{yy} + \nu a_{yyy} = 0.
\end{aligned} \tag{3.42}$$

Equating to zero the coefficients of the second order derivatives ψ_{xy}, ψ_{yy} and $\psi_y\psi_{yy}$ in (3.42) yields

$$a_y = 0,\ a_x = 0,\ a_{\psi\psi} = 0. \tag{3.43}$$

We finally obtain

$$\Lambda = c_3 + c_4\psi, \tag{3.44}$$

where c_3 and c_4 are constants.

From (3.33) and (3.44)

$$(c_3 + c_4\psi)(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy}) = D_x\left[c_3\psi_y^2 + c_4\psi\psi_y^2\right]$$

$$+D_y\left[c_3(-\psi_x\psi_y - \nu\psi_{yy}) + c_4(-\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy})\right], \tag{3.45}$$

for arbitrary functions $\psi(x, y)$. When $\psi(x, y)$ is a solution of the third-order partial differential equation (3.10),

$$D_x\left[c_3\psi_y^2 + c_4\psi\psi_y^2\right]$$

$$+D_y\left[c_3(-\psi_x\psi_y - \nu\psi_{yy}) + c_4(-\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy})\right] = 0. \tag{3.46}$$

Thus any conserved vector of the third order partial differential equation (3.10) with multipliers $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$ is a linear combination of the two conserved vectors

$$T^1 = \psi_y^2,\ T^2 = -\psi_x\psi_y - \nu\psi_{yy}, \tag{3.47}$$

$$T^1 = \psi\psi_y^2,\ T^2 = -\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}. \tag{3.48}$$

The conserved vectors (3.47) and (3.48) form a basis of conserved vectors for the third order partial differential equation (3.10) for multipliers $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$. The conserved vector (3.47) is equivalent to the conserved vector (3.32) for the system (3.1)-(3.2) and it is the elementary conserved vector for (3.10) (Mason 2002). The conserved vector (3.48) yields a local conservation law for the third-order partial differential equation (3.10) but a non-local conservation law for the system (3.1)-(3.2).

The conserved vectors (3.47) and (3.48) were derived by elementary manipulations once the multiplier (3.44) had been found. They can also be derived systematically using (3.33) as the determining equation.

3.3 Conserved quantity for the two-dimensional liquid, free and wall jet

In this section we will present a new method of driving the conserved quantity for the two-dimensional liquid jet, free jet and wall jet. We will first consider the conserved vectors (3.31) and (3.32) for the system (3.1)-(3.2) and derive the conserved quantities for the two-dimensional liquid and free jets. We will then consider the stream function formulation and give an alternative derivation of the conserved quantity for the two-dimensional free jet and a new derivation of the conserved quantity for the wall jet.

The conserved vectors (T^1, T^2) which have been derived depend on $u(x,y)$, $v(x,y)$ or $\psi(x,y)$ and can therefore be expressed in terms of x and y. Thus

$$D_x T^1 + D_y T^2 = \frac{\partial T^1(x,y)}{\partial x} + \frac{\partial T^2(x,y)}{\partial y}, \tag{3.49}$$

where on the right hand side, T^1 and T^2 are regarded as functions of x and y. For a conserved vector the left hand side of (3.49) vanishes and (3.49) reduces to

$$\frac{\partial T^1(x,y)}{\partial x} + \frac{\partial T^2(x,y)}{\partial y} = 0. \tag{3.50}$$

Equation (3.50) forms the basis of the derivation of conserved quantities.

3.3.1 Conserved quantity for a two-dimensional liquid jet

The conserved quantity for the two-dimensional liquid jet is obtained from the conserved vector (3.31).

The boundary conditions for the two-dimensional liquid jet are that there is no slip or suction and blowing at the lower boundary over which the liquid is flowing and there is no shear stress along the free surface $y = \phi(x)$:

$$y = 0: \quad u(x,0) = 0, \; v(x,0) = 0, \tag{3.51}$$

$$y = \phi(x): \quad u_y(x, \phi(x)) = 0. \tag{3.52}$$

The y-component of the fluid velocity on the free surface $y = \phi(x)$ is

$$v(x, \phi(x)) = \frac{D}{Dt}[\phi(x)] = u(x, \phi(x))\frac{d\phi(x)}{dx}, \tag{3.53}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u(x,y)\frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y} \tag{3.54}$$

is the material time derivative.

Integrate (3.50) with respect to y from $y = 0$ to $y = \phi(x)$ keeping x fixed during the integration. Then for the conserved vector (3.31)

$$\int_0^{\phi(x)} \left[\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y}\right] dy = 0, \tag{3.55}$$

which, on differentiating under the integration sign (Gillespie 1959), yields

$$\frac{d}{dx}\int_0^{\phi(x)} u(x,y)dy - u\left(x, \phi(x)\right)\frac{d\phi(x)}{dx} + [v(x,y)]_0^{\phi(x)} = 0. \tag{3.56}$$

Using (3.53) for $v(x, \phi(x))$ and the boundary condition (3.51) for $v(x, 0)$ we obtain

$$\int_0^{\phi(x)} u(x,y)dy = \text{constant, independent of } \; x. \tag{3.57}$$

Equation (3.57) states that the total volume flux is constant along the jet. Thus

$$F = \int_0^{\phi(x)} u(x,y)dy \tag{3.58}$$

is the conserved quantity for the two-dimensional liquid jet (Watson 1964).

3.3.2 Conserved quantity for two-dimensional free jet

The conserved quantity for the two-dimensional free jet is obtained from the conserved vector (3.32).

The boundary conditions for the two-dimensional free jet are

$$y = 0: \quad v(x,0) = 0,\ u_y(x,0) = 0, \tag{3.59}$$

$$y = \pm\infty: \quad u(x,\pm\infty) = 0,\ u_y(x,\pm\infty) = 0. \tag{3.60}$$

Integrate (3.50) with respect to y from $y = -\infty$ to $y = \infty$ with x kept fixed during the integration. For the conserved vector (3.32) we obtain

$$\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x}\left[u^2(x,y)\right] + \frac{\partial}{\partial y}\left[u(x,y)v(x,y) - \nu u_y(x,y)\right]\right] dy = 0, \tag{3.61}$$

which yields

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2(x,y)dy + \left[u(x,y)v(x,y) - \nu u_y(x,y)\right]_{-\infty}^{\infty} = 0. \tag{3.62}$$

The second term in (3.62) vanishes due to the boundary conditions in (3.60) and we obtain

$$2\int_0^{\infty} u^2(x,y)dy = \text{constant, independent of } \ x. \tag{3.63}$$

Equation (3.63), multiplied by the density of the fluid ρ, states that the total momentum flux in the direction of the jet is constant along the jet. Thus the conserved quantity for the two-dimensional free jet is

$$F = 2\rho\int_0^{\infty} u^2(x,y)dy. \tag{3.64}$$

The definition (3.64) of F is that used by Schlichting (1933).

The conserved quantity (3.64) can also be derived using the conserved vector (3.47) for the third-order partial differential equation (3.10). We now briefly outline the derivation.

In terms of the stream function the boundary conditions (3.59) and (3.60) take the form

$$y = 0: \quad \psi_x(x,0) = 0, \ \psi_{yy}(x,0) = 0, \tag{3.65}$$

$$y = \pm\infty: \quad \psi_y(x,\pm\infty) = 0, \ \psi_{yy}(x,\pm\infty) = 0. \tag{3.66}$$

Integrate (3.50) with respect to y from $y = -\infty$ to $y = \infty$ with x kept fixed during the integration. For the conserved vector (3.47) this yields

$$\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x} \left[\psi_y^2(x,y)\right] + \frac{\partial}{\partial y} \left[-\psi_x(x,y)\psi_y(x,y) - \nu\psi_{yy}(x,y)\right] \right] dy = 0, \tag{3.67}$$

which, on differentiating under the integral sign (Gillespie 1959), gives

$$\frac{d}{dx} \int_{-\infty}^{\infty} \psi_y^2(x,y) dy + \left[-\psi_x(x,y)\psi_y(x,y) - \nu\psi_{yy}(x,y)\right]_{-\infty}^{\infty} = 0. \tag{3.68}$$

Now $\psi_x(x,\pm\infty) = -v(x,\pm\infty)$ which we assume to be finite. The second term in (3.68) is zero due to boundary conditions (3.66). Hence

$$\frac{d}{dx} \int_{-\infty}^{\infty} \psi_y^2(x,y) dy = 0, \tag{3.69}$$

which yields

$$2 \int_0^{\infty} \psi_y^2(x,y) dy = \text{constant, independent of } \ x. \tag{3.70}$$

Equation (3.70) is equivalent to (3.63) and hence we obtain again the conserved quantity F given in (3.64).

3.3.3 Conserved quantity for the two-dimensional wall jet

For the two-dimensional wall jet with no suction or blowing of fluid at the wall $y = 0$, the boundary conditions are

$$y = 0: \quad u(x, 0) = 0, \ v(x, 0) = 0, \tag{3.71}$$

$$y = \infty: \quad u(x, \infty) = 0, \ u_y(x, \infty) = 0. \tag{3.72}$$

In terms of the stream function, we obtain the boundary conditions

$$y = 0: \quad \psi_x(x, 0) = 0, \ \psi_y(x, 0) = 0, \tag{3.73}$$

$$y = \infty: \quad \psi_y(x, \infty) = 0, \ \psi_{yy}(x, \infty) = 0. \tag{3.74}$$

For the wall jet we integrate (3.50) with respect to y from $y = 0$ to $y = \infty$ keeping x fixed during the integration. Then for the conserved vector (3.48),

$$\frac{d}{dx}\int_0^\infty \psi(x,y)\psi_y^2(x,y)dy$$
$$+\left[-\psi(x,y)\psi_x(x,y)\psi_y(x,y) + \frac{\nu}{2}\psi_y^2(x,y) - \nu\psi\psi_{yy}(x,y)\right]_0^\infty = 0. \tag{3.75}$$

But from (3.7), $\psi(x, 0) = 0$ and $\psi(x, \infty)$ and $\psi_x(x, \infty)$ are assumed to be finite. Thus, imposing the boundary conditions (3.73) and (3.74), we obtain

$$\frac{d}{dx}\int_0^\infty \psi(x,y)\psi_y^2(x,y)dy = 0 \tag{3.76}$$

and therefore

$$\int_0^\infty \psi(x,y)\psi_y^2(x,y)dy = \text{constant, independent of } x. \tag{3.77}$$

In order to transform (3.77) to the form derived by Glauert (1956) integrate (3.77) by parts. Using again $\psi(x, 0) = 0$ and the assumption that $\psi(x, \infty)$ is finite, we obtain

$$\int_0^\infty \psi(x,y)\psi_y^2(x,y)dy = \int_0^\infty \psi_y(x,y)\left(\int_y^\infty \psi_{y^*}^2(x,y^*)dy^*\right)dy \tag{3.78}$$

and hence

$$\int_0^\infty \psi_y(x,y)\left(\int_y^\infty \psi_{y^*}^2(x,y^*)dy^*\right)dy = \text{constant, independent of } x. \quad (3.79)$$

We obtain the conserved quantity for the two-dimensional wall jet,

$$F = \int_0^\infty \psi_y(x,y)\left(\int_y^\infty \psi_{y^*}^2(x,y^*)dy^*\right)dy. \quad (3.80)$$

Expressed in terms of the velocity component, $u(x,y)$, (3.80) becomes

$$F = \int_0^\infty u(x,y)\left(\int_y^\infty u^2(x,y^*)dy^*\right)dy, \quad (3.81)$$

which is the form derived by Glauert (1956). The constant ρF was interpreted by Glauert as the flux of exterior momentum flux. Also (3.77) is a simple alternative form for the conserved quantity and is a local conserved quantity for the third order partial differential equation (3.10) for the stream function.

The conserved vectors (3.31) and (3.32) for the system (3.1)-(3.2) gave the conserved quantities (3.58) and (3.64) for the two-dimensional liquid jet and free jet. We cannot obtain the conserved quantity for the two-dimensional wall jet from the conservation laws of the system (3.1)-(3.2). The conserved vectors (3.47) and (3.48) for the third-order partial differential equation (3.10) gave the conserved quantities, (3.64) and (3.81), for the two-dimensional free jet and wall jet. They do not generate the conserved quantity for the two-dimensional liquid jet.

The results for two-dimensional jets are summarized in Table 3.1.

3.4 Conclusions

The conserved quantities for two-dimensional jets can be constructed with the help of conservation laws. The procedure is more systematic than integrating Prandtl's momentum boundary layer equation across the jet and using the continuity equation. The liquid jet, the free jet and the wall jet satisfy the same partial differential equations but the boundary conditions for each jet are different. The conserved vectors depend only on the partial differential equations. The derivation of the conserved quantity depends also on the boundary conditions. The boundary conditions therefore determine which conserved vector is associated with which jet. The multiplier approach gave a basis of two local conserved vectors for the system of equations for the velocity components for the two-dimensional jets. One of the conserved vectors gave the conserved quantity for the two-dimensional liquid jet and other gave the conserved quantity for the two-dimensional free jet. We cannot construct the conserved quantity for the two-dimensional wall jet from the conserved vectors for the system of equations for the velocity components. For the third-order partial differential equation for the stream function two local conservation laws were obtained, one of which is a non-local conservation law for the system of equations for the velocity components. One of the local conserved vectors was used to give an alternative derivation of the conserved quantity for the two-dimensional free jet. The second conserved vector gave the conserved quantity for the two-dimensional wall jet.

The new form, (3.77) , for the conserved quantity for the two-dimensional wall jet was derived. This form is simpler than the expression (3.80) derived by Glauert (1956) and may be easier to apply when solving the wall jet problem. The derivation of the conserved quantities require the condition $\psi(x,0) = 0$ to be satisfied. This condition is valid provided there is no suction or blowing of fluid at the solid boundary. When there is suction or blowing, (3.77) is no longer independent of x.

The results, summarized in Table 3.1, demonstrate a new way of looking at the relation between the liquid jet, the free jet and the wall jet which could lead to a new approach to the classification of jet flows through the conserved vectors or their multipliers.

Jet	Multipliers	Conserved vector	Conserved quantity
Velocity	$\Lambda_1 = c_2$		
	$\Lambda_2 = c_1 + c_2 u$		
Liquid jet	$\Lambda_1 = 0$	$T^1 = u$	$\int_0^{\phi(x)} u(x,y)dy$
	$\Lambda_2 = 1$	$T^2 = v$	
Free jet	$\Lambda_1 = 1$	$T^1 = u^2$	$\int_{-\infty}^{\infty} u^2(x,y)dy$
	$\Lambda_2 = u$	$T^2 = uv - \nu u_y$	
Stream function	$\Lambda = c_3 + c_4\psi$		
Free jet	$\Lambda = 1$	$T^1 = \psi_y^2$	$\int_{-\infty}^{\infty} \psi_y^2(x,y)dy$
		$T^2 = -\psi_x\psi_y - \nu\psi_{yy}$	
Wall jet	$\Lambda = \psi$	$T^1 = \psi\psi_y^2$	$\int_0^{\infty} \psi(x,y)\psi_y^2(x,y)dy$
		$T^2 = -\psi\psi_x\psi_y$ $+\frac{1}{2}\nu\psi_y^2 - \nu\psi\psi_{yy}$	$\int_0^{\infty} \psi_y(x,y)\Big(\int_y^{\infty} \psi_{y^*}^2(x,y^*)dy^*\Big)dy$

Table 3.1: Multipliers, conserved vectors and conserved quantities for two-dimensional jets

Chapter 4

Conservation laws and physical conserved quantities for laminar radial jets

4.1 Introduction

The problem of the radial free jet was first studied by Squire (1955) who used the spherical polar coordinates. The conserved quantity for the radial free jet was derived by integrating Prandtl's momentum boundary layer equation across the jet (Riley 1962b, Schwarz 1963). Glauert (1956) established the conserved quantity for the radial wall jet by integrating the momentum boundary layer equation across the jet. The conserved quantity for radial liquid jet (Riley 1962b, Watson 1964) was obtained from the physical argument that volume flux is constant in an incompressible fluid.

In this chapter, we will construct the conserved quantities for laminar radial jet flows by using the conservation laws for the partial differential equations describing the jet flows. As in Chapter 3, we will use the multiplier approach (Steudel 1962, Olver 1993, Anco and Bluman 2002a,b) to derive a basis of conservation laws for the system of partial differential equations for the two velocity components as well as for the third-order partial differential equation for the stream function. Then we will construct conserved quantities for the radial liquid, free and wall jets using the conservation laws.

An outline of the chapter is as follows. In Section 4.2, the multiplier approach is used to derive a basis of conservation laws for the system of partial differential equations for the velocity components and for the partial differential equation for the stream function. In Section 4.3, the conserved quantity for the radial liquid, free and wall jet are constructed. Finally, the conclusions are summarized in Section 4.4.

4.2 Conservation laws for radial laminar jet flow

We now consider the radial liquid jet, free jet and wall jet.
The radial liquid jet is formed when a circular jet of liquid strikes a plane boundary normally and spreads over it (Riley 1962b). The radial free jet is formed when fluid emerges from a pair of parallel circular plates into the surrounding fluid (Schwarz 1963). The radial wall jet is formed when a circular jet of fluid falls into a partly full container and spreads out over the base (Glauert 1956).

The fluid is viscous and incompressible and in the free jet and wall jet the surrounding fluid consist of the same fluid as the jet and is at rest far from the jet. In the liquid jet and wall jet there is no slip or suction and blowing at the solid boundary. Cylindrical polar coordinates (x, θ, y) are used. The radial coordinate is x, the axis of symmetry is $x = 0$ and all quantities are independent of θ. The y-coordinate is along the axis of symmetry. For the liquid jet and wall jet the solid boundary is $y = 0$. The free jet is symmetrical about the plane $y = 0$. For the liquid jet the fee surface is $y = \phi(x)$. The notation is chosen to make easier comparison between the solutions for the radial and two-dimensional jets.

The equations governing the flow in a radial jet are Prandtl's momentum boundary layer equation and the continuity equation:

$$uu_x + vu_y = \nu u_{yy}, \tag{4.1}$$

$$(xu)_x + (xv)_y = 0, \tag{4.2}$$

where $u(x, y)$ and $v(x, y)$ are the velocity components in the x and y directions.

4.2.1 Stream function

The stream function for a flow with an axis of symmetry is described by Batchelor (1967). From (4.2), $xudy - xvdx$ is a perfect differential, $d\psi$:

$$d\psi = xu(x, y)dy - xv(x, y)dx. \tag{4.3}$$

Consider an axial plane $\theta = \theta_o$ and choose as reference point any point $O(x_o, \theta_o, 0)$ on the line $y = 0$ in the axial plane. The stream function $\psi(x, y)$ at any point $P(x, \theta_o, y)$ in the axial plane is obtained by integrating $d\psi$ along any curve in the axial plane joining O and P:

$$\psi(x, y) - \psi(x_o, 0) = \int_O^P x\left[u(x, y)dy - v(x, y)dx\right]. \tag{4.4}$$

The integral on the right hand side of (4.4), times 2π, is the flux of fluid volume across the surface formed by rotating an arbitrary curve joining O to P in the axial plane about the axis of symmetry. The flux of fluid volume across the closed surface formed by rotating any two different curves joining O to P in the axial plane about the axis of symmetry is zero because the region inside the closed surface is completely occupied by an incompressible fluid. Thus $\psi(x, y)$ is independent of choice of path joining O to P.

Since x and y are independent variables, it follows from (4.3) that

$$u = \frac{1}{x}\psi_y, \; v = -\frac{1}{x}\psi_x. \tag{4.5}$$

But the radial free jet is symmetrical about the plane $y = 0$ and for the liquid jet and wall jet there is no suction or blowing of fluid at the solid boundary $y = 0$. Thus for all three jets, $v(x, 0) = 0$ and therefore $\psi_x(x, 0) = 0$. Thus $\psi(x, 0) = \psi_o$ where ψ_o is a constant which we choose to be zero. Thus

$$\psi(x, 0) = 0, \tag{4.6}$$

and (4.4) reduces to

$$\psi(x, y) = \int_O^P x\left[u(x, y)dy - v(x, y)dx\right]. \tag{4.7}$$

Integrating (4.7) with respect to y with x kept fixed from $y = 0$ to $y = \infty$ gives

$$\psi(x, \infty) = x\int_0^\infty u(x, y)dy. \tag{4.8}$$

We assume that $u(x, y) \to 0$ sufficiently rapidly as $y \to \infty$ to ensure that the integral in (4.8) exists. Thus $\psi(x, \infty)$ is finite.

Equation (4.2) is identically satisfied while (4.1) becomes

$$\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy} = 0. \tag{4.9}$$

We will derive a basis of conservation laws for the system (4.1)-(4.2) and also for the third-order partial differential equation (4.9), using the multiplier approach.

4.2.2 Conservation laws for system of equations by multiplier approach

Multipliers Λ_1 and Λ_2 for the system (4.1)-(4.2) satisfy

$$\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y) = D_xT^1 + D_yT^2, \tag{4.10}$$

for all functions $u(x, y)$ and $v(x, y)$ where the total derivative operators D_x and D_y are defined by (3.12) and (3.13). The determining equations for the multipliers of form $\Lambda_1 = \Lambda_1(x, y, u, v)$ and $\Lambda_2 = \Lambda_2(x, y, u, v)$ are

$$E_u \left[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y)\right] = 0, \tag{4.11}$$

$$E_v \left[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u + xu_x + xv_y)\right] = 0, \tag{4.12}$$

where E_u and E_v are standard Euler operators defined in equations (3.16) and (3.17). The expansion of (4.11) and (4.12) results in the following two equations,

$$\Lambda_{1u}(uu_x + vu_y - \nu u_{yy}) + \Lambda_{2u}(u + xu_x + xv_y) + \Lambda_1 u_x + \Lambda_2$$
$$-D_x(\Lambda_1 u + x\Lambda_2) - D_y(\Lambda_1 v) - \nu D_y^2(\Lambda_1) = 0 \tag{4.13}$$

and

$$\Lambda_{1v}(uu_x + vu_y - \nu u_{yy}) + \Lambda_{2v}(u + xu_x + xv_y) + \Lambda_1 u_y - D_y(x\Lambda_2) = 0. \tag{4.14}$$

Equations (4.13) and (4.14) have to be satisfied for all functions $u(x, y)$ and $v(x, y)$, not only for the solutions of the system (4.1)-(4.2). Using $D_y^2(\Lambda_1)$ from (3.20) and then equating to zero the coefficients of u_{yy} and v_{yy} in (4.13) yields

$$\Lambda_{1u} = 0, \ \Lambda_{1v} = 0 \tag{4.15}$$

and therefore $\Lambda_1 = \Lambda_1(x, y)$. Equations (4.13) and (4.14) become

$$xv_x\Lambda_{2v} + v_y(\Lambda_1 - x\Lambda_{2u}) + u\Lambda_{1x} + v\Lambda_{1y} + x\Lambda_{2x} - u\Lambda_{2u} + \nu\Lambda_{1yy} = 0, \tag{4.16}$$

$$xu_x\Lambda_{2v} + u_y(\Lambda_1 - x\Lambda_{2u}) - x\Lambda_{2y} + u\Lambda_{2v} = 0. \tag{4.17}$$

Equation (4.17), after separation gives

$$\Lambda_{2v} = 0, \tag{4.18}$$

$$\Lambda_1(x, y) - x\Lambda_{2u} = 0, \tag{4.19}$$

$$\Lambda_{2y} = 0. \tag{4.20}$$

It follows from (4.18) and (4.20) that $\Lambda_2 = \Lambda_2(x, u)$. Differentiating (4.19) with respect to y gives $\Lambda_{1y} = 0$ and thus

$$\Lambda_1 = A(x). \tag{4.21}$$

Equation (4.19) reduces to

$$x\Lambda_{2u} = A(x) \tag{4.22}$$

which yields

$$\Lambda_2 = \frac{u}{x}A(x) + B(x). \tag{4.23}$$

Substituting Λ_1 and Λ_2 from (4.21) and (4.23) into (4.16) results in

$$2u\frac{dA}{dx} + \frac{dB}{dx} = 0. \tag{4.24}$$

Separating (4.24) according to powers of u gives $A(x) = xc_2$ and $B(x) = c_1$ and thus

$$\Lambda_1 = c_2 x,\ \Lambda_2 = c_1 + c_2 u, \tag{4.25}$$

where c_1 and c_2 are arbitrary constants.

The conserved vectors can be derived systematically using (4.10) as the determining equation with multipliers (4.25). But it is not difficult to construct the conserved vectors by elementary manipulations for this problem. From (4.10) and (4.25), it follows that

$$c_2 x(uu_x + vu_y - \nu u_{yy}) + (c_1 + c_2 u)(u + xu_x + xv_y)$$

$$= D_x\left[c_2(xu^2) + c_1(xu)\right] + D_y\left[c_2(xuv - \nu xu_y) + c_1(xv)\right], \tag{4.26}$$

for arbitrary functions $u(x, y)$ and $v(x, y)$. Therefore

$$D_x\left[c_2(xu^2) + c_1(xu)\right] + D_y\left[c_2(xuv - \nu xu_y) + c_1(xv)\right] = 0, \tag{4.27}$$

whenever $u(x,y)$ and $v(x,y)$ are the solutions of the system (4.1)-(4.2). Any conserved vector for the system of differential equations (4.1)-(4.2) is a linear combination of the two conserved vectors

$$T^1 = xu, \quad T^2 = xv, \tag{4.28}$$

$$T^1 = xu^2, \quad T^2 = xuv - \nu x u_y. \tag{4.29}$$

The conserved vectors (4.28) and (4.29) therefore form a basis of conserved vectors for the system (4.1)-(4.2) with multipliers of the form $\Lambda_1 = \Lambda_1(x,y,u,v)$ and $\Lambda_2 = \Lambda_2(x,y,u,v)$.

It can be shown that all multipliers of the form $\Lambda_1 = \Lambda_1(x,y,u,v,u_x,v_x)$ and $\Lambda_2 = \Lambda_2(x,y,u,v,u_x,v_x)$ are given by (4.25).

4.2.3 Conservation laws for radial laminar jet flow in terms of stream function

A multiplier for (4.9) satisfies

$$\Lambda\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right) = D_xT^1 + D_yT^2, \tag{4.30}$$

for all functions $\psi(x,y)$, where D_x and D_y are defined by (3.34) and (3.35). Consider a first order multiplier of the form $\Lambda = \Lambda(x,y,\psi,\psi_x,\psi_y)$. The determining equation for the multiplier Λ is

$$E_\psi\left[\Lambda(x,y,\psi,\psi_x,\psi_y)\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right)\right] = 0, \tag{4.31}$$

where E_ψ is defined in (3.37). Expansion of (4.31) gives

$$\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right)\Lambda_\psi$$
$$-D_x\left[\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right)\Lambda_{\psi_x}\right]$$
$$-D_y\left[\left(\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \nu\psi_{yyy}\right)\Lambda_{\psi_y}\right]$$

$$+D_x\left(\frac{1}{x}\Lambda\psi_{yy}\right)+D_y\left[\left(\frac{2}{x^2}\psi_y-\frac{1}{x}\psi_{xy}\right)\Lambda\right]+D_xD_y\left(\frac{1}{x}\Lambda\psi_y\right)$$

$$-D_y^2\left(\frac{1}{x}\Lambda\psi_x\right)+\nu D_y^3(\Lambda)=0. \tag{4.32}$$

Substituting $D_y^3(\Lambda)$ from (3.39) into (4.32) and the equating the coefficients of highest order derivative terms, $2\nu\Lambda_{\psi_x}\psi_{xyyy}$ and $2\nu\Lambda_{\psi_y}\psi_{yyyy}$, to zero yields

$$\Lambda_{\psi_x}=0,\quad \Lambda_{\psi_y}=0 \tag{4.33}$$

and thus

$$\Lambda=a(x,y,\psi). \tag{4.34}$$

The substitution of (4.34) in (4.32) results in

$$\left(\frac{2}{x}a_x+3\nu a_{\psi y}\right)\psi_{yy}-\frac{2}{x}a_y\psi_{xy}+3\nu a_{\psi\psi}\psi_y\psi_{yy}-\frac{1}{x}a_{\psi y}\psi_x\psi_y$$

$$+\nu a_{\psi\psi\psi}\psi_y^3+\left(\frac{1}{x}a_{\psi x}+3\nu a_{\psi\psi y}\right)\psi_y^2+\left(\frac{1}{x^2}a_y+\frac{1}{x}a_{xy}+3\nu a_{\psi yy}\right)\psi_y$$

$$-\frac{1}{x}\psi_x a_{yy}+\nu a_{yyy}=0. \tag{4.35}$$

Equating to zero the coefficients of the second order derivatives ψ_{xy},ψ_{yy} and $\psi_y\psi_{yy}$ in (4.35) yields

$$a_y=0,\ a_x=0,\ a_{\psi\psi}=0, \tag{4.36}$$

and we finally obtain

$$\Lambda=c_3+c_4\psi, \tag{4.37}$$

where c_3 and c_4 are arbitrary constants. Thus from (4.30) and (4.37)

$$D_x\left[c_3\left(\frac{1}{x}\psi_y^2\right)+c_4\left(\frac{1}{x}\psi\psi_y^2\right)\right]$$

$$+D_y\left[c_3\left(-\frac{1}{x}\psi_x\psi_y-\nu\psi_{yy}\right)+c_4\left(-\frac{1}{x}\psi\psi_x\psi_y+\frac{\nu}{2}\psi_y^2-\nu\psi\psi_{yy}\right)\right]=0, \tag{4.38}$$

for all solutions $\psi(x, y)$ of (4.9). Hence any conserved vector for equation (4.9) is a linear combination of the two conserved vectors

$$T^1 = \frac{1}{x}\psi_y^2, \quad T^2 = -\frac{1}{x}\psi_x\psi_y - \nu\psi_{yy}, \tag{4.39}$$

$$T^1 = \frac{1}{x}\psi\psi_y^2, \quad T^2 = -\frac{1}{x}\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}. \tag{4.40}$$

The conserved vectors (4.39) and (4.40) therefore form a basis of conserved vectors for (4.9) with multiplier of the form $\Lambda = \Lambda(x, y, \psi, \psi_x, \psi_y)$. The conserved vectors (4.29) and (4.39) are equivalent.

4.3 Conserved quantity for the radial liquid, free and wall jet

In this section we will present a new method of driving the conserved quantity for the radial liquid jet, free jet and wall jet using the conservation laws. The conserved vectors (4.28) and (4.29) for the system (4.1)-(4.2) will be used to establish the conserved quantities for the radial liquid and free jets. The conserved vectors (4.39) and (4.40) for the stream function formulation will be used to give an alternative derivation of the conserved quantity for the radial free jet and a new derivation of the conserved quantity for the wall jet.

The conserved vectors (T^1, T^2) derived here depend on $u(x, y)$, $v(x, y)$ or $\psi(x, y)$ and therefore (3.50) holds for the radial case also.

4.3.1 Conserved quantity for the radial liquid jet

The conserved vector (4.28) gives the conserved quantity for the radial liquid jet.

The boundary conditions (3.51) and (3.52) and expression (3.53) for $v(x, \phi(x))$ hold for the radial liquid jet. Integration of (3.50) with respect to y from $y = 0$ to $y = \phi(x)$ keeping x constant for the conserved vector (4.28) gives

$$\int_0^{\phi(x)} \left[\frac{\partial}{\partial x}(xu(x,y)) + \frac{\partial}{\partial y}(xv(x,y)) \right] dy = 0. \tag{4.41}$$

Equation (4.41), on differentiating under integral sign Gillespie (1959), yields

$$\frac{d}{dx} \int_0^{\phi(x)} xu(x,y)dy - xu\left(x, \phi(x)\right) \frac{d\phi(x)}{dx} + \left[xv(x,y)\right]_0^{\phi(x)} = 0. \tag{4.42}$$

Using (3.53) for $v(x, \phi(x))$ and the boundary condition (3.51) for $v(x, 0)$ we obtain

$$x \int_0^{\phi(x)} u(x,y)dy = \text{constant, independent of } \; x. \tag{4.43}$$

Equation (4.43) states that the total volume flux per radian across the jet is constant along the jet. Hence

$$F = x \int_0^{\phi(x)} u(x,y)dy \tag{4.44}$$

is the conserved quantity for the radial liquid jet (Riley 1962b, Watson 1964).

4.3.2 Conserved quantity for the radial free jet

The conserved quantity for the radial free jet is derived from the conserved vector (4.29) for the system (4.1)-(4.2). The boundary conditions (3.59) and (3.60) apply for the radial free jet.

Integrating (3.50) with respect to y from $y = -\infty$ to $y = \infty$ keeping x constant gives

$$\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x}\left[xu^2(x,y)\right] + \frac{\partial}{\partial y}\left[xu(x,y)v(x,y) - \nu x u_y(x,y)\right] \right] dy = 0. \tag{4.45}$$

Integrating (4.45) and imposing the boundary conditions (3.59) and (3.60), we obtain

$$\frac{d}{dx}\left(x \int_{-\infty}^{\infty} u^2(x,y)dy \right) = 0, \tag{4.46}$$

which yields

$$2x\int_0^\infty u^2(x,y)dy = \text{constant, independent of } x. \tag{4.47}$$

Now ρ times equation (4.47) gives that the total momentum flux in the direction of the jet is constant along the jet. Thus the conserved quantity for the radial free jet is

$$F = 2\rho x\int_0^\infty u^2(x,y)dy, \tag{4.48}$$

in agreement with results established by Riley (1962b) and Schwarz (1963).

The conserved quantity (4.48) can also be constructed using the conserved vector (4.39) for the third-order partial differential equation (4.9).

The boundary conditions (3.65) and (3.66) also apply for the radial free jet. Integrate (3.50) with respect to y from $y=-\infty$ to $y=\infty$ with x kept constant. For the conserved vector (4.39) this gives

$$\int_{-\infty}^{\infty}[\frac{\partial}{\partial x}(\frac{1}{x}\psi_y^2(x,y)) + \frac{\partial}{\partial y}(-\frac{1}{x}\psi_x(x,y)\psi_y(x,y) - \nu\psi_{yy}(x,y))]dy = 0, \tag{4.49}$$

which gives

$$\frac{d}{dx}\int_{-\infty}^{\infty}\frac{1}{x}\psi_y^2(x,y)dy + \left[-\frac{1}{x}\psi_x(x,y)\psi_y(x,y) - \nu\psi_{yy}(x,y)\right]_{-\infty}^{\infty} = 0. \tag{4.50}$$

Imposing the boundary conditions (3.66) and the assumption $\psi_x(x,\pm\infty) = -xv(x,\pm\infty)$ is finite gives that the second term in (4.50) vanishes. Hence

$$\frac{2}{x}\int_0^\infty \psi_y^2(x,y)dy = \text{constant, independent of } x. \tag{4.51}$$

Equation (4.51) and (4.47) are equivalent and hence we obtain again the conserved quantity F given in (4.48).

The conserved vectors (4.28) and (4.29) for the system (4.1)-(4.2) gave the conserved quantities (4.44) and (4.48) for the radial liquid jet and free jet. We cannot obtain the conserved quantity for the radial wall jet from the conservation laws for the system (4.1)-(4.2). The conserved vector (4.39) for the third-order partial differential equation (4.9) for the stream function gave the conserved quantity (4.48) for the radial free jet. The conserved quantity for the radial wall jet is obtained from the remaining conserved vector (4.40).

4.3.3 Conserved quantity for radial wall jet

The conserved vector (4.40) is used to derive the conserved quantity for the radial wall jet. The boundary conditions for the radial wall jet are (3.73) and (3.74). Integrate (3.50) with respect to y from $y = 0$ to $y = \infty$ in an axial plane keeping x fixed during the integration. This gives, for the conserved vector (4.40),

$$\frac{d}{dx}\int_0^\infty [\frac{1}{x}\psi(x,y)\psi_y^2(x,y)]dy$$
$$+\left[-\frac{1}{x}\psi(x,y)\psi_x(x,y)\psi_y(x,y) + \frac{\nu}{2}\psi_y^2(x,y) - \nu\psi(x,y)\psi_{yy}(x,y)\right]_0^\infty = 0. \quad (4.52)$$

It follows from (4.52), by imposing the boundary conditions (3.73) and (3.74) and the condition $\psi(x, 0) = 0$ from (4.6), that

$$\frac{1}{x}\int_0^\infty \psi(x,y)\psi_y^2(x,y)dy = \text{constant, independent of } x. \quad (4.53)$$

Integrating (4.53) by parts, we obtain

$$F = \frac{1}{x}\int_0^\infty \psi_y(x,y)\left(\int_y^\infty \psi_{y^*}^2(x,y^*)dy^*\right)dy, \quad (4.54)$$

or expressed in terms of $u(x, y)$,

$$F = x^2\int_0^\infty u(x,y)\left(\int_y^\infty u^2(x,y^*)dy^*\right)dy, \quad (4.55)$$

which is the conserved quantity derived by Glauert (1956). The results for radial jets are summarized in Table 4.1.

4.4 Conclusions

A basis of two local conserved vectors for the system of equations for the velocity components for radial jets was obtained by the multiplier approach.

The conserved quantities for the radial liquid jet and radial free jet were derived from these two conserved vectors. The conserved quantity for the wall jet cannot be obtained from the conserved vectors of the system of equations for the velocity components. A basis of two local conservation laws was obtained for the third-order partial differential equation for the stream function. One of conservation laws for stream function formulation is a non-local conservation law for the system of equations for the velocity components. These conserved vectors were used to give an alternative derivation of the conserved quantity for the free jet and a new form for the conserved quantity for the wall jet.

The new form (4.53) for the conserved quantity for the radial wall jet is simpler than the expression (4.54) derived by Glauert (1956). The derivation of the conserved quantities require the condition $\psi(x, 0) = 0$ to be satisfied which is valid provided there is no suction or blowing of fluid at the solid boundary. When there is suction or blowing, (4.53) is no longer independent of x.

Jet	Multipliers	Conserved vector	Conserved quantity
Velocity	$\Lambda_1 = c_2 x$		
	$\Lambda_2 = c_1 + c_2 u$		
Liquid jet	$\Lambda_1 = 0$	$T^1 = xu$	$x\int_0^{\phi(x)} u(x,y)dy$
	$\Lambda_2 = 1$	$T^2 = xv$	
Free jet	$\Lambda_1 = x$	$T^1 = xu^2$	$x\int_{-\infty}^{\infty} u^2(x,y)dy$
	$\Lambda_2 = u$	$T^2 = x(uv - \nu u_y)$	
Stream function	$\Lambda = c_3 + c_4\psi$		
Free jet	$\Lambda = 1$	$T^1 = \frac{1}{x}\psi_y^2$	$\frac{1}{x}\int_{-\infty}^{\infty} \psi_y^2(x,y)dy$
		$T^2 = -\frac{1}{x}\psi_x\psi_y$ $-\nu\psi_{yy}$	
Wall jet	$\Lambda = \psi$	$T^1 = \frac{1}{x}\psi\psi_y^2$	$\frac{1}{x}\int_0^{\infty} \psi(x,y)\psi_y^2(x,y)dy$
		$T^2 = -\frac{1}{x}\psi\psi_x\psi_y$ $+\frac{1}{2}\nu\psi_y^2 - \nu\psi\psi_{yy}$	$\frac{1}{x}\int_0^{\infty} \psi_y(x,y)\left(\int_y^{\infty} \psi_{y^*}^2(x,y^*)dy^*\right)dy$

Table 4.1: Multipliers, conserved vectors and conserved quantities for radial jets

Chapter 5

Physical conserved quantities for the axisymmetric liquid, free and wall jets

5.1 Introduction

Goldstein (1938) established the conserved quantity for an axisymmetric free jet by integrating the momentum equation across the jet. Later, Duck and Bodoyni (1986) established the conserved quantity for the axisymmetric wall jet by the same procedure. The conserved quantity for the axisymmetric liquid jet may be obtained from a physical argument based on conservation of volume flux in an incompressible fluid.

The conserved quantity for an axisymmetric liquid jet is the volume flux. Goldstein (1938) showed that the conserved quantity for the axisymmetric free jet is the total momentum flux in the downstream direction. This flux is constant along the jet. For the wall jet on an axisymmetric body, the conservation integral involves a balance between the viscous and inertial forces of the jet (see Duck and Bodoyni 1986). There is no physical interpretation of this conservation integral.

In this Chapter, we construct the conserved quantities for the axisymmetric liquid, free and wall jets using the approach introduced in Naz, Mason and Mahomed (2008) as described in Chapters 3 and 4. The conserved quantities for the axisymmetric liquid, free and wall jets are derived using conservation laws. Mason and Ruscic (2004) discussed the elementary conservation law for the third-order partial differential equation for the stream function for an axisymmetric free jet. These authors obtained that conservation law from inspection of the momentum equation. To the best of our knowledge the conservation laws for the system of equations for the velocity components and the third-order partial differential equation for the stream function have not been computed. We will construct the conservation laws for the system of two partial differential equations for the velocity components as well as for the third-order partial differential equation for the stream function for the axisymmetric jets using the multiplier approach (see for example Steudel 1962, Olver 1993, Anco and Bluman 2002a,b, Naz, Mahomed and Mason 2008c).

5.2 Conservation laws for system of equations for axisymmetric jet flows

We consider the axisymmetric liquid jet, free jet and wall jet. An axisymmetric free jet is formed when fluid emerges from a small circular orifice in a wall into the surrounding fluid which is the same fluid as the jet (Goldstein 1938). An axisymmetric wall jet and liquid jet on an axisymmetric body is formed when an axisymmetric jet is incident on the surface of the body along the axis of symmetry and spreads out over the surface. For the wall jet surrounding fluid is the same as the jet (Duck and Bodoyni 1986) but for the liquid jet the surrounding fluid is a gas. There are several physical situations which could provide an axisymmetric wall jet on the surface of an axisymmetric body (Duck and Bodoyni 1986). The collision between an axisymmetric radially directed inward jet and a circular cylinder lying along the axis could produce an axisymmetric wall jet. Also an axisymmetric jet directed towards an axisymmetric body would produce an axisymmetric wall jet on the surface. We can expect that sufficiently far downstream the conditions will be approximately independent of the way the jet was formed.

The fluid of the jet is viscous and incompressible. There is no slip or suction or blowing at the solid boundary for the axisymmetric liquid and wall jets. Cylindrical polar coordinates (z, r, θ) are used. The $z-$axis is along the axis of the jet and all the fluid variables are independent of θ. For the free jet the origin of the coordinate system is at the orifice. For the wall and liquid jet on a circular cylinder the origin of the coordinate system is on the central axis of the cylinder at the point where the jet has zero initial thickness.

Prandtl's boundary layer equations governing the steady flow in the axisymmetric liquid, free and wall jets, in the absence of a pressure gradient, are

$$uu_z + vu_r = \nu(u_{rr} + \frac{u_r}{r}), \tag{5.1}$$

$$(ru)_z + (rv)_r = 0, \tag{5.2}$$

where $u(z, r)$ and $v(z, r)$ are the velocity components in the z and r directions, respectively and ν is the kinematic viscosity of the fluid.

5.2.1 Stream function

The stream function for a flow with an axis of symmetry is described by Batchelor (1967). From (5.2), $rudr - rvdz$ is a perfect differential, $d\psi$:

$$d\psi = ru(z, r)dr - rv(z, r)dz. \tag{5.3}$$

Choose any point $O(z_o, \theta_o, r_o)$ as the reference point on the line $r = r_o$ in the axial plane $\theta = \theta_o$. For the free jet $r_o = 0$ and the line is the axis of symmetry. For the wall and liquid jet $r_o = a$ where a is the radius of the cylinder and the line is a generator of the cylinder on the surface. The stream function $\psi(z, r)$ at any point $P(z, \theta_o, r)$ in the axial plane is obtained by integrating $d\psi$ along any curve in the axial plane joining O and P:

$$\psi(z, r) - \psi(z_o, r_o) = \int_O^P r\left(u(z, r)dr - v(z, r)dz\right). \tag{5.4}$$

Equation (5.3) yields

$$u = \frac{1}{r}\psi_r, \ v = -\frac{1}{r}\psi_z. \tag{5.5}$$

The axisymmetric free jet is symmetrical about the axis $r = 0$ and for the liquid jet and wall jet there is no suction or blowing of fluid at the solid boundary of the cylinder $r = r_o = a$. Thus for all three jets, $v(z, r_o) = 0$ and therefore $\psi_z(z, r_o) = 0$. Thus $\psi(z, r_o) = \psi_o$ where ψ_o is a constant which we choose to be zero and (5.4) reduces to

$$\psi(z, r) = \int_O^P r\left(u(z, r)dr - v(z, r)dz\right), \tag{5.6}$$

where $r_o = 0$ for free jet and $r_o = a$ for liquid and wall jets. Integrating (5.6) with respect to r with z kept fixed from $r = r_o$ to $r = \infty$ gives

$$\psi(z, \infty) = \int_{r_o}^{\infty} ru(z, r)dr. \tag{5.7}$$

We assume that $u(z, r) \to 0$ sufficiently rapidly as $r \to \infty$ to ensure that the integral in (5.7) exists. Thus $\psi(z, \infty)$ is finite.

We substitute (5.5) into (5.1)-(5.2). Equation (5.2) is identically satisfied while (5.1) gives rise to a third-order partial differential equation for the stream function ψ:

$$\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu(\psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r) = 0. \tag{5.8}$$

We derive a basis of conservation laws for the system (5.1)-(5.2) as well as for the third-order partial differential equation (5.8), using the multiplier approach (Steudel 1962, Olver 1993, Anco and Bluman 2002a,b, Naz, Mahomed and Mason 2008c).

5.2.2 Conservation laws for system of equations for velocity components

Multipliers Λ_1 and Λ_2, for all functions $u(z,r)$ and $v(z,r)$, not only for the solutions of (5.1)-(5.2), satisfy

$$\Lambda_1\left[uu_z + vu_r - \nu(u_{rr} + \frac{u_r}{r})\right] + \Lambda_2\left[ru_z + rv_r + v\right] = D_zT^1 + D_rT^2, \quad (5.9)$$

where D_z and D_r, defined by

$$D_z = \frac{\partial}{\partial z} + u_z\frac{\partial}{\partial u} + v_z\frac{\partial}{\partial v} + u_{zz}\frac{\partial}{\partial u_z} + v_{zz}\frac{\partial}{\partial v_z} + u_{zr}\frac{\partial}{\partial u_r} + v_{zr}\frac{\partial}{\partial v_r} + \cdots, \quad (5.10)$$

$$D_r = \frac{\partial}{\partial r} + u_r\frac{\partial}{\partial u} + v_r\frac{\partial}{\partial v} + u_{rr}\frac{\partial}{\partial u_r} + v_{rr}\frac{\partial}{\partial v_r} + u_{rz}\frac{\partial}{\partial u_z} + v_{rz}\frac{\partial}{\partial v_z} + \cdots, \quad (5.11)$$

are the total derivative operators. The two determining equations for the multipliers are obtained by applying the standard Euler operators E_u and E_v to (5.9). The Euler operators E_u and E_v are defined by

$$E_u = \frac{\partial}{\partial u} - D_z\frac{\partial}{\partial u_z} - D_r\frac{\partial}{\partial u_r} + D_z^2\frac{\partial}{\partial u_{zz}} + D_zD_r\frac{\partial}{\partial u_{zr}} + D_r^2\frac{\partial}{\partial u_{rr}} - \cdots, \quad (5.12)$$

and

$$E_v = \frac{\partial}{\partial v} - D_z\frac{\partial}{\partial v_z} - D_r\frac{\partial}{\partial v_r} + D_z^2\frac{\partial}{\partial v_{zz}} + D_zD_r\frac{\partial}{\partial v_{zr}} + D_r^2\frac{\partial}{\partial v_{rr}} - \cdots. \quad (5.13)$$

Thus the determining equations for multipliers of the form $\Lambda_1 = \Lambda_1(z,r,u,v)$ and $\Lambda_2 = \Lambda_2(z,r,u,v)$ are

$$E_u\left[\Lambda_1(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}) + \Lambda_2(ru_z + rv_r + v)\right] = 0, \quad (5.14)$$

$$E_v\left[\Lambda_1(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}) + \Lambda_2(ru_z + rv_r + v)\right] = 0. \quad (5.15)$$

After expansion, (5.14) and (5.15) become

$$\Lambda_{1u}(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}) + \Lambda_{2u}(ru_z + rv_r + v) - u(\Lambda_{1z} + u_z\Lambda_{1u} + v_z\Lambda_{1v})$$

$$-r(\Lambda_{2z} + u_z\Lambda_{2u} + v_z\Lambda_{2v}) - \Lambda_1 v_r - \frac{\nu}{r^2}\Lambda_1 - (v - \frac{\nu}{r})(\Lambda_{1r} + u_r\Lambda_{1u} + v_r\Lambda_{1v})$$

$$-\nu[u_{rr}\Lambda_{1u} + v_{rr}\Lambda_{1v} + D_r(\Lambda_{1r}) + u_r D_r(\Lambda_{1u}) + v_r D_r(\Lambda_{1v})] = 0, \qquad (5.16)$$

$$\Lambda_{1v}(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}) + \Lambda_{2v}(ru_z + rv_r + v)$$

$$-r(\Lambda_{2r} + u_r\Lambda_{2u} + v_r\Lambda_{2v}) + \Lambda_1 u_r = 0, \qquad (5.17)$$

which must be satisfied for all functions $u(z,r)$ and $v(z,r)$. Equating the coefficients of u_{rr} and v_{rr} in (5.16) to zero yields $\Lambda_{1u} = 0,\ \Lambda_{1v} = 0$ and thus $\Lambda_1 = \Lambda_1(z,r)$. Equations (5.16) and (5.17) reduce to

$$(\Lambda_1 - r\Lambda_{2u})\, v_r + r\Lambda_{2v}v_z - v\Lambda_{2u} + r\Lambda_{2z} + u\Lambda_{1z} + \frac{\nu\Lambda_1}{r^2}$$

$$+(v - \frac{\nu}{r})\Lambda_{1r} + \nu\Lambda_{1rr} = 0, \qquad (5.18)$$

$$r\Lambda_{2v}u_z + (\Lambda_1 - r\Lambda_{2u})u_r + v\Lambda_{2v} - r\Lambda_{2r} = 0. \qquad (5.19)$$

Equation (5.19) separates into the equations

$$\Lambda_{2v} = 0,\ \Lambda_{2r} = 0,\ \Lambda_1(z,r) - r\Lambda_{2u} = 0, \qquad (5.20)$$

which yield

$$\Lambda_1 = rA(z),\ \Lambda_2 = uA(z) + B(z). \qquad (5.21)$$

Now, (5.21) reduces (5.18) to

$$2u\frac{dA}{dz} + \frac{dB}{dz} = 0. \qquad (5.22)$$

Separating (5.22) according to powers of u, we obtain the multipliers

$$\Lambda_1 = c_2 r,\ \Lambda_2 = c_1 + c_2 u, \qquad (5.23)$$

where c_1 and c_2 are constants.

Equation (5.9) together with (5.23) gives

$$c_2 r\Big(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}\Big) + (c_1 + c_2 u)(ru_z + rv_r + v)$$

$$= D_z\left[c_1 ru + c_2 ru^2\right] + D_r\left[c_1 rv + c_2 r(uv - \nu u_r)\right], \tag{5.24}$$

for arbitrary $u(z,r)$ and $v(z,r)$. When $u(z,r)$ and $v(z,r)$ are the solutions of the system of differential equations (5.1)-(5.2), then

$$D_z[c_1 ru + c_2 ru^2] + D_r[c_1 rv + c_2 r(uv - \nu u_r)] = 0. \tag{5.25}$$

Thus

$$T^1 = ru, \quad T^2 = rv, \tag{5.26}$$

$$T^1 = ru^2, \quad T^2 = r(uv - \nu u_r), \tag{5.27}$$

form a basis of conserved vectors for the system (5.1)-(5.2) with multipliers of the form $\Lambda_1(z,r,u,v)$ and $\Lambda_2(z,r,u,v)$.

5.2.3 Conservation laws for axisymmetric laminar jet flow in terms of stream function

A multiplier for the third-order partial differential equation (5.8) is a function Λ satisfying

$$\Lambda\left[\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu(\psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r)\right]$$

$$= D_z T^1 + D_r T^2, \tag{5.28}$$

for all functions $\psi(z,r)$. In (5.28), D_z and D_r are total derivative operators defined by

$$D_z = \frac{\partial}{\partial z} + \psi_z\frac{\partial}{\partial \psi} + \psi_{zz}\frac{\partial}{\partial \psi_z} + \psi_{zr}\frac{\partial}{\partial \psi_r} + \cdots, \tag{5.29}$$

$$D_r = \frac{\partial}{\partial r} + \psi_r\frac{\partial}{\partial \psi} + \psi_{rr}\frac{\partial}{\partial \psi_r} + \psi_{rz}\frac{\partial}{\partial \psi_z} + \cdots. \tag{5.30}$$

The determining equation for a multiplier of the form $\Lambda = \Lambda(z, r, \psi, \psi_z, \psi_r)$ is

$$E_\psi\left[\Lambda\left(\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu\psi_{rrr} + \frac{\nu}{r}\psi_{rr} - \frac{\nu}{r^2}\psi_r\right)\right] = 0, \quad (5.31)$$

where

$$E_\psi = \frac{\partial}{\partial\psi} - D_z\frac{\partial}{\partial\psi_z} - D_r\frac{\partial}{\partial\psi_r} + D_z^2\frac{\partial}{\partial\psi_{zz}} + D_zD_r\frac{\partial}{\partial\psi_{zr}} + D_r^2\frac{\partial}{\partial\psi_{rr}} - \cdots, \quad (5.32)$$

is standard Euler operator. Expanding equation (5.31), we have

$$\begin{aligned}
&\Lambda_\psi[\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu\psi_{rrr} + \frac{\nu}{r}\psi_{rr} - \frac{\nu}{r^2}\psi_r] \\
&-D_z[\Lambda_{\psi_z}(\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu\psi_{rrr} + \frac{\nu}{r}\psi_{rr} - \frac{\nu}{r^2}\psi_r)] \\
&-D_r[\Lambda_{\psi_r}(\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu\psi_{rrr} + \frac{\nu}{r}\psi_{rr} - \frac{\nu}{r^2}\psi_r)] \\
&-D_z[\Lambda(\frac{1}{r^2}\psi_r - \frac{1}{r}\psi_{rr})] - D_r[\Lambda(\frac{1}{r^2}\psi_z + \frac{1}{r}\psi_{rz} - \frac{\nu}{r^2}\Lambda)] \\
&+D_rD_z[\frac{1}{r}\Lambda\psi_r] + D_r^2[\Lambda(\frac{\nu}{r} - \frac{1}{r}\psi_z)] + \nu D_r^3[\Lambda] = 0. \qquad (5.33)
\end{aligned}$$

Now

$$D_r^3(\Lambda) = \psi_{zrrr}\Lambda_{\psi_z} + \psi_{rrrr}\Lambda_{\psi_r} + \text{lower derivative terms} . \quad (5.34)$$

In equation (5.33) the highest order derivative terms are $2\nu\Lambda_{\psi_z}\psi_{zrrr}$ and $2\nu\Lambda_{\psi_r}\psi_{rrrr}$ which yield

$$\Lambda_{\psi_z} = 0, \quad \Lambda_{\psi_r} = 0, \quad (5.35)$$

and (5.33) reduces to

$$\begin{aligned}
&-\frac{2}{r}\Lambda_r\psi_{zr} + \left(\frac{2\nu}{r}\Lambda_\psi + \frac{2}{r}\Lambda_z + 3\nu\Lambda_{\psi r}\right)\psi_{rr} + 3\nu\Lambda_{\psi\psi}\psi_r\psi_{rr} - \frac{1}{r}\Lambda_{\psi r}\psi_z\psi_r \\
&+\nu\Lambda_{\psi\psi\psi}\psi_r^3 + \left(\frac{\nu}{r}\Lambda_{\psi\psi} + \frac{1}{r}\Lambda_{\psi z} + 3\nu\Lambda_{\psi\psi r}\right)\psi_r^2 + \left(\frac{1}{r^2}\Lambda_r - \frac{1}{r}\Lambda_{rr}\right)\psi_z \\
&+\left(\frac{2\nu}{r}\Lambda_{\psi r} - \frac{2\nu}{r^2}\Lambda_\psi - \frac{2}{r^2}\Lambda_z + \frac{1}{r}\Lambda_{zr} + 3\nu\Lambda_{\psi rr}\right)\psi_r
\end{aligned}$$

$$-\frac{\nu}{r^2}\Lambda_r + \frac{\nu}{r}\Lambda_{rr} + \nu\Lambda_{rrr} = 0. \tag{5.36}$$

Equating to zero the coefficients of the second order derivatives $\psi_{zr}, \psi_r\psi_{rr}$ and ψ_{rr} in (5.36) yields

$$\Lambda_r = 0,\ \Lambda_{\psi\psi} = 0,\ \frac{2\nu}{r}\Lambda_\psi + \frac{2}{r}\Lambda_z + 3\nu\Lambda_{\psi r} = 0. \tag{5.37}$$

Solving the system (5.37), we finally obtain

$$\Lambda = c_3 + c_4\left(\psi - \nu z\right), \tag{5.38}$$

where c_3 and c_4 are constants. The remainder of (5.36) is identically zero..

The substitution of multiplier Λ from (5.38) in (5.28) gives

$$\begin{aligned}
&\left[c_3 + c_4\left(\psi - \nu z\right)\right]\left[\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu\left(\psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r\right)\right] \\
&= D_z\left[c_3\left(\frac{1}{r}\psi_r^2\right) + c_4\frac{1}{r}\psi_r^2\left(\psi - \nu z\right)\right] + D_r\left[c_3\left(-\frac{1}{r}\psi_z\psi_r - \nu\psi_{rr} + \frac{\nu}{r}\psi_r\right)\right. \\
&\left.+c_4\left(\frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)\left(\psi - \nu z\right)\right)\right],
\end{aligned} \tag{5.39}$$

for arbitrary $\psi(z, r)$. Thus

$$\begin{aligned}
&D_z\left[c_3\left(\frac{1}{r}\psi_r^2\right) + c_4\frac{1}{r}\psi_r^2\left(\psi - \nu z\right)\right] + D_r\left[c_3\left(-\frac{1}{r}\psi_z\psi_r - \nu\psi_{rr} + \frac{\nu}{r}\psi_r\right)\right. \\
&\left.+c_4\left(\frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)\left(\psi - \nu z\right)\right)\right] = 0,
\end{aligned} \tag{5.40}$$

holds when $\psi(z, r)$ is a solution of third-order partial differential equation (5.8). For each arbitrary constant in (5.40) we obtain a conserved vector. Thus

$$T^1 = \frac{1}{r}\psi_r^2,\ T^2 = -\frac{1}{r}\psi_z\psi_r - \nu\psi_{rr} + \frac{\nu}{r}\psi_r, \tag{5.41}$$

$$T^1 = \frac{1}{r}\psi_r^2(\psi - \nu z),\ T^2 = \frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)(\psi - \nu z), \tag{5.42}$$

are the conserved vectors for the third-order partial differential equation (5.8) with multipliers of the form $\Lambda = \Lambda(z, r, \psi, \psi_z, \psi_r)$. We obtained two local conservation laws for the third-order partial differential equation (5.8). The conserved vector (5.42) is a local conserved vector for the third-order partial differential equation (5.8) but it is a non-local conserved vector for the system (5.1)-(5.2).

The multiplier approach gives only those multipliers which yield local conservation laws. We cannot obtain the multipliers for the non-local conservation laws using the multiplier approach. It is of interest to observe that we can derive a non-local conservation law for the third-order partial differential equation (5.8) by multiplying with ψ. If we multiply equation (5.8) by ψ, we obtain

$$\psi\left[\frac{1}{r}\psi_r\psi_{rz}+\frac{1}{r^2}\psi_z\psi_r-\frac{1}{r}\psi_z\psi_{rr}-\nu\left(\psi_{rrr}-\frac{1}{r}\psi_{rr}+\frac{1}{r^2}\psi_r\right)\right]$$

$$=D_z\left[\frac{1}{r}\psi\psi_r^2-\frac{\nu}{r}\int_0^z\psi_r^2dz\right]+D_r\left[\frac{\nu}{2}\psi_r^2+\left(\frac{\nu}{r}\psi_r-\nu\psi_{rr}-\frac{1}{r}\psi_z\psi_r\right)\psi\right], \quad (5.43)$$

for arbitrary functions $\psi(z,r)$. We chose the lower limit of the integral as $z=0$ which is the origin of the $z-$coordinate. The choice specifies the arbitrary function of r in the conserved vector. When $\psi(z,r)$ is a solution of partial differential equation (5.8), we have

$$D_z[\frac{1}{r}\psi\psi_r^2-\frac{\nu}{r}\int_0^z\psi_r^2dz]+D_r[\frac{\nu}{2}\psi_r^2+(\frac{\nu}{r}\psi_r-\nu\psi_{rr}-\frac{1}{r}\psi_z\psi_r)\psi]=0, \quad (5.44)$$

which yields

$$T^1=\frac{1}{r}\psi\psi_r^2-\frac{\nu}{r}\int_0^z\psi_r^2dz,\ T^2=\frac{\nu}{2}\psi_r^2+\left(\frac{\nu}{r}\psi_r-\nu\psi_{rr}-\frac{1}{r}\psi_z\psi_r\right)\psi. \quad (5.45)$$

Thus we have obtained two local conserved vectors (5.41), (5.42) and one non-local conserved vector (5.45) for the third-order partial differential equation (5.8).

5.3 Conserved quantities for axisymmetric liquid, free and wall jets

In this section we derive the conserved quantities for the axisymmetric liquid jet, free jet and wall jet by a new method. The conserved vectors (5.26) and (5.27) for the system (5.1)-(5.2) give the conserved quantities for the axisymmetric liquid and free jets. The conserved vectors for stream function equation (5.8) are (5.41), (5.42) and (5.45). The conserved vector (5.41) is used to give an alternative derivation of the conserved quantity for the axisymmetric free jet and the conserved vector (5.45) gives a new derivation of the conserved quantity for the wall jet. The conserved vector (5.42) may give the conserved quantity for some other flow.

The conserved vectors (T^1, T^2) depend on $u(z,r)$, $v(z,r)$ or $\psi(z,r)$ and can therefore be expressed in terms of z and r as follows:

$$D_z T^1 + D_r T^2 = \frac{\partial T^1(z,r)}{\partial z} + \frac{\partial T^2(z,r)}{\partial r}. \tag{5.46}$$

But for a conserved vector, $D_z T^1 + D_r T^2 = 0$ and (5.46) becomes

$$\frac{\partial T^1(z,r)}{\partial z} + \frac{\partial T^2(z,r)}{\partial r} = 0. \tag{5.47}$$

Expression (5.47) is the basis of the derivation of conserved quantities for the axisymmetric liquid jet, free jet and wall jet.

5.3.1 Conserved quantity for an axisymmetric liquid jet

The surface of the cylinder is $r = a$. There is no slip or suction and blowing on the surface $r = a$. The boundary conditions for the axisymmetric liquid jet therefore are

$$r = a: \quad u(z,a) = 0, \; v(z,a) = 0, \tag{5.48}$$

$$r = \phi(z): \quad u_r(z, \phi(z)) = 0. \tag{5.49}$$

The r-component of the fluid velocity on the free surface $r = \phi(z)$ is

$$v(z, \phi(z)) = \frac{D}{Dt}\left[\phi(z)\right] = u(z, \phi(z))\frac{d\phi(z)}{dz}, \tag{5.50}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u(z, r)\frac{\partial}{\partial z} + v(z, r)\frac{\partial}{\partial r} \tag{5.51}$$

is the material time derivative. The conserved vector (5.26) gives the conserved quantity for the axisymmetric liquid jet. Integrating (5.47) with respect to r from $r = a$ to $r = \phi(z)$, keeping z fixed during the integration, gives for the conserved vector (5.26)

$$\int_a^{\phi(z)} \left[\frac{\partial}{\partial z}\left(ru(z, r)\right) + \frac{\partial}{\partial r}(rv(z, r))\right] dr = 0. \tag{5.52}$$

Applying the result for differentiating under an integral sign (Gillespie 1959) to the first term in (5.52), we obtain

$$\frac{d}{dz}\int_a^{\phi(z)} ru(z, r)dr - \phi(z)u(z, \phi(z))\frac{d\phi(z)}{dz} + \left[rv(z, r)\right]_a^{\phi(z)} = 0. \tag{5.53}$$

Using (5.50) and the boundary condition (5.48) for $v(z, a)$, we have

$$\frac{d}{dz}\int_a^{\phi(z)} ru(z, r)dr = 0, \tag{5.54}$$

which gives

$$\int_a^{\phi(z)} ru(z, r)dr = \text{constant, independent of } z. \tag{5.55}$$

Therefore

$$F = \int_a^{\phi(z)} ru(z, r)dr \tag{5.56}$$

is the conserved quantity for the axisymmetric liquid jet which gives the total volume flux as constant along the jet.

5.3.2 Conserved quantity for axisymmetric free jet

The boundary conditions for the axisymmetric free jet are

$$r = 0 : v(z, 0) = 0,\ u_r(z, 0) = 0, \tag{5.57}$$

$$r \to \infty : ru(z, r) = 0,\ ru_r(z, r) = 0. \tag{5.58}$$

The conserved vector (5.27) gives the conserved quantity for the axisymmetric free jet. Integrating (5.47) with respect to r from $r = 0$ to $r = \infty$ keeping z fixed, we have for the conserved vector (5.27),

$$\int_0^\infty \left[\frac{\partial}{\partial z}[ru^2(z, r)] + \frac{\partial}{\partial r}[r(u(z, r)v(z, r) - \nu u_r(z, r))]\right] dr = 0,$$

which yields

$$\frac{d}{dz}\int_0^\infty ru^2(z, r)dr + [r(u(z, r)v(z, r) - \nu u_r(z, r))]_0^\infty = 0. \tag{5.59}$$

The boundary conditions (5.57) and (5.58) and the fact that $v(z, \infty)$ is finite yield

$$\int_0^\infty ru^2(z, r)dr = \text{constant, independent of } z. \tag{5.60}$$

Thus the conserved quantity is

$$F = \int_0^\infty ru^2(z, r)dr. \tag{5.61}$$

Goldstein (1938) used $2\pi\rho F$ as the conserved quantity for the axisymmetric free jet.

The conserved quantity (5.61) can also be constructed for the stream function formulation using the conserved vector (5.41). In terms of the stream function the boundary conditions (5.57) and (5.58) take the following form:

$$r = 0 : \frac{1}{r}\psi_z(z, r) = 0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\psi_r(z, r)\right) = 0, \tag{5.62}$$

$$r = \infty : \psi_r(z, r) = 0,\ r\frac{\partial}{\partial r}\left(\frac{1}{r}\psi_r(z, r)\right) = 0. \tag{5.63}$$

The conserved quantity for the axisymmetric free jet is obtained by integrating (5.47) with respect to r from $r = 0$ to $r = \infty$ keeping z fixed. Thus

$$\frac{d}{dz}\int_0^\infty \frac{1}{r}\psi_r^2(z,r)dr$$
$$+\left[-\frac{1}{r}\psi_z(z,r)\psi_r(z,r) - \nu\psi_{rr}(z,r) + \frac{\nu}{r}\psi_r(z,r)\right]_0^\infty = 0. \tag{5.64}$$

We assume that $v(r,\infty) = -\frac{1}{r}\psi_z(z,\infty)$ is finite. The boundary conditions (5.62) and (5.63) finally yield

$$\int_0^\infty \frac{1}{r}\psi_r^2(z,r)dr = \text{constant, independent of } \ z, \tag{5.65}$$

which is equivalent to (5.60) and hence we obtain the conserved quantity (5.61). Thus the conserved quantity is the same if we use the stream function formulation or the system of equations for the velocity components.

We observe that the conserved vector (5.41) which establishes the conserved quantity for the stream function formulation is equivalent to the conserved vector (5.27) for the system of equations.

5.3.3 Conserved quantity for axisymmetric wall jet

For an axisymmetric wall jet on a cylinder of radius a, the boundary conditions are

$$r = a : u(z,a) = 0, \ v(z,a) = 0, \tag{5.66}$$

$$r \to \infty : u = O(\frac{1}{r^2}). \tag{5.67}$$

In terms of the stream function, we obtain

$$r = a : \psi_z(z,a) = 0, \ \psi_r(z,a) = 0, \tag{5.68}$$

$$r \to \infty : \psi_r = O(\frac{1}{r}) \tag{5.69}$$

and the stream function is zero at $r = a$, that is $\psi(z,a) = 0$. The local conserved vector (5.42) obtained by the multiplier approach cannot give a conserved quantity for the axisymmetric wall jet because it is not compatible with the boundary conditions. The non-local conserved vector (5.45) gives the conserved quantity for the axisymmetric wall jet. Integrating (5.47) with respect to r from $r = a$ to $r = \infty$ keeping z as fixed and by considering the conserved vector (5.45), we have

$$\int_a^\infty \frac{\partial}{\partial z}\left[\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \frac{\nu}{r}\int_0^z \psi_r^2(z^*,r)dz^*\right]dr + \left[\frac{\nu}{2}\psi_r^2(z,r) + \left(\frac{\nu}{r}\psi_r(z,r)\right.\right.$$

$$\left.\left. -\nu\psi_{rr}(z,r) - \frac{1}{r}\psi_z(z,r)\psi_r(z,r)\right)\psi(z,r)\right]_a^\infty = 0. \tag{5.70}$$

However $\psi(z, a) = 0$ and $\psi(z, \infty)$ and $\psi_z(z, \infty)$ are assumed to be finite. The second term in (5.70) vanishes due the to the boundary conditions (5.68) and (5.69) and we obtain

$$\frac{d}{dz}\int_a^\infty \left[\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \frac{\nu}{r}\int_0^z \psi_r^2(z^*,r)dz^*\right]dr = 0, \tag{5.71}$$

which yields

$$\int_a^\infty \left[\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \frac{\nu}{r}\int_0^z \psi_r^2(z^*,r)dz^*\right]dr = \text{constant, independent of } z. \tag{5.72}$$

We obtain the conserved quantity for the axisymmetric wall jet

$$F = \int_a^\infty \left[\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \frac{\nu}{r}\int_0^z \psi_r^2(z^*,r)dz^*\right]dr. \tag{5.73}$$

Duck and Bodoyni (1986) established the conserved quantity (5.73) by integrating the momentum equation and using the continuity equation and boundary conditions.

The conserved vectors (5.26) and (5.27) for the system (5.1)-(5.2) give the conserved quantities for the axisymmetric liquid jet and axisymmetric free jet respectively. The conserved quantity for the axisymmetric wall jet cannot be obtained even if we consider higher order multipliers because it is a nonlocal conserved quantity for the system. It may be that some non-local conserved vector for the system (5.1)-(5.2) will give the conserved quantity for the axisymmetric wall jet. Since the multiplier approach yields only local conserved vectors an alternative method had to be used. We therefore considered the stream function formulation. The conserved vector (5.41) for the third-order partial differential equation (5.8) gives the conserved quantity for the free jet and the conserved vector (5.45) gives the conserved quantity for the wall jet.

The results for the axisymmetric liquid, free and wall jets are summarized in Table 5.1.

5.4 Conclusions

The multiplier approach gave two local conservation laws for the system of equations for the velocity components. One of the conserved vectors gave the conserved quantity for the axisymmetric liquid jet and the second conserved vector gave the conserved quantity for the axisymmetric free jet.

For the third-order partial differential equation for the stream function two local conserved vectors were obtained, one of which was the non-local conserved vector for the system of equations for the velocity components. One of the local conserved vectors for the third-order partial differential equation for the stream function was used to give an alternative derivation of the conserved quantity for the axisymmetric free jet but the other local conserved vector cannot be used to derive the conserved quantity for the axisymmetric wall jet. The conserved quantity for the axisymmetric wall jet was established with the help of a non-local conserved vector for the third-order partial differential equation for the stream function. That non-local conservation law for the third-order partial differential equation for the stream function was not obtained by the multiplier approach. The reason is that the multiplier approach only gives multipliers for local conservation laws.

Table 5.1: Multipliers, conserved vectors and conserved quantities for axisymmetric jets

Jet	Multipliers	Conserved vector	Conserved quantity
Velocity	$\Lambda_1 = c_2 r$ $\Lambda_2 = c_1 + c_2 u$		
Liquid jet	$\Lambda_1 = 0$ $\Lambda_2 = 1$	$T^1 = ru$ $T^2 = rv$	$\int_0^{\phi(z)} ru(z,r)dr$
Free jet	$\Lambda_1 = r$ $\Lambda_2 = u$	$T^1 = ru^2$ $T^2 = r(uv - \nu u_r)$	$\int_0^{\infty} ru^2(z,r)dr$
Stream function	$\Lambda = c_3$ $+c_4(\psi - \nu z)$		
Free jet	$\Lambda = 1$	$T^1 = \frac{1}{r}\psi_r^2$ $T^2 = -\frac{1}{r}\psi_z\psi_r - \nu\psi_{rr} + \frac{1}{r}\nu\psi_r$	$\int_0^{\infty} \frac{1}{r}\psi_r^2(z,r)dr$
	$\Lambda = \psi - \nu z$	$T^1 = \frac{1}{r}\psi_r^2(\psi - \nu z)$ $T^2 = \frac{1}{2}\nu\psi_r^2$ $+(\frac{1}{r}\nu\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r)$ $\times(\psi - \nu z)$	
Wall jet	$\Lambda = \psi$	$T^1 = \frac{1}{r}\psi\psi_r^2 - \frac{\nu}{r}\int_0^z \psi_r^2 dz^*$ $T^2 = \frac{1}{2}\nu\psi_r^2$ $+\left(\frac{1}{r}\nu\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)\psi$	$\int_1^{\infty}[\frac{1}{r}\psi(z,r)\psi_r^2(z,r)$ $-\frac{\nu}{r}\int_0^z \psi_r^2(z^*,r)dz^*]dr$

Chapter 6

Comparison of different approaches for deriving conservation

6.1 Introduction

The concept of a conservation law plays a vital role in the study of differential equations and in many applications. The mathematical idea of a conservation law comes from the formulation of familiar physical laws such as conservation of energy and momentum. In jet problems the conserved quantity plays an important role in the solution process and is used to determine the unknown exponent in the similarity solution which cannot be obtained from the homogeneous boundary conditions. Previously, the conserved quantities for jets were established either from physical arguments or by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. This method of deriving conserved quantities is not entirely systematic and difficult to apply in problems such as the wall jet. We have shown in Chapters 3, 4 and 5 that the conserved quantities for jet flows can be established by utilizing the conservation laws.

An elegant and constructive way of finding conservation laws is by means of Noether's theorem (1918). This theorem roughly states that for Euler-Lagrange differential equations, to each Noether symmetry associated with the Lagrangian there corresponds a conservation law which can be determined explicitly by a formula. The application of Noether's theorem depends upon the knowledge of a suitable Lagrangian. There have been several contributions to the inverse problem in variational calculus, that is, to determine when a system of differential equations has a Lagrangian formulation. Yet there are differential equations that do not have a Lagrangian, for example, scalar evolution differential equations. There are methods to obtain conservation laws which do not rely on the knowledge of a Lagrangian function. The direct method, characteristic method, variational derivative method (multiplier approach) for arbitrary functions as well as on the solution space, symmetry conditions on the conserved quantities, direct construction formula approach, the partial Noether approach and Noether approach for the equation and its adjoint are reviewed and explained with the help of an illustrative example on the relaxation to a Maxwellian distribution.

The outline of this Chapter is as follows. In Section 6.2, a review of all the approaches is given. In Section 6.3, each approach is used to construct the conservation laws for a nonlinear field equation describing the relaxation to a Maxwell distribution. Finally, concluding remarks are given in Section 6.4.

6.2 Approaches to construct conservation laws

In this section various approaches to construct local conservation laws are discussed. We present nine approaches taken from the literature.

6.2.1 Direct Method

The Direct Method uses (2.12), subject to the system of differential equations (2.11) being satisfied, as the determining equation for the conserved vectors, i.e. we solve

$$D_i T^i \mid_{E_\alpha = 0} = 0 \tag{6.1}$$

for the components $T^1, T^2, ..., T^n$. This approach was first used by Laplace (1798) and gives all local conservation laws.

6.2.2 Noether's approach

An elegant and constructive way of finding conservation laws is by means of Noether's theorem (Noether 1918).

Definition 1.7. If there exists a function $L(x, u, u_{(1)}, u_{(2)}, ..., u_{(s)}) \in \mathcal{A},\ s \leq k$, such that (2.11) are equivalent to

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, ..., N, \tag{6.2}$$

then L is called a Lagrangian of (2.11) and (6.2) are the corresponding Euler-Lagrange differential equations.

Definition 1.8. A Lie-Backlund operator X is a Noether symmetry generator associated with a Lagrangian L of (6.2) if there exists a vector $B =$

$(B^1, B^2, ..., B^n)$ such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{6.3}$$

Theorem 1.1. For each Noether symmetry generator X associated with a given Lagrangian L corresponding to the Euler-Lagrange differential equations, there corresponds a vector $T = (T^1, T^2, ..., T^n)$ with T^i defined by

$$\begin{aligned} T^i &= B^i - N^i L \\ &= B^i - \xi^i L - W^\alpha \frac{\delta L}{\delta u_i^\alpha} - \sum_{s \geq 1} D_{i_1 \cdots i_s}(W^\alpha) \frac{\delta L}{\delta u^\alpha_{i_1 \cdots i_s}}, \end{aligned} \tag{6.4}$$

which is a conserved vector for the Euler-Lagrange differential equations (6.2). In the Noether approach we find $L(x, u, ..., u_{(k-1)})$ and then (6.3) is used for the determination of the Noether symmetries. Finally (6.4) yields the corresponding Noether conserved vectors. The characteristics W^α of the Noether symmetry generator are the characteristics of the conservation law.

6.2.3 Characteristic method

The conservation law is written in characteristic form (Steudel 1962 and see also Olver 1993) as

$$D_i T^i = \Lambda^\alpha E_\alpha \tag{6.5}$$

where Λ^α are the characteristics. The characteristics are the multipliers which make the equation exact.

6.2.4 Variational derivative method

This approach, presented by Olver (1993), involves the variational derivative of (6.5):

$$\frac{\delta}{\delta u^\beta}(\Lambda^\alpha E_\alpha) = 0. \tag{6.6}$$

Conditions (6.6) hold for arbitrary functions $u(x^1, x^2, \cdots, x^n)$. All the multipliers can be calculated with the help of (6.6) for which the equation can be expressed as a local conservation law.

6.2.5 Variational derivative method on space of solutions of the differential equation

In this approach the variational derivative of (6.5) is computed on the space of solutions of the differential equation, i.e.

$$\frac{\delta}{\delta u^\beta}(\Lambda^\alpha E_\alpha) \mid_{E_\alpha=0} = 0. \tag{6.7}$$

Conditions (6.7) are less overdetermined than conditions (6.6) and the characteristics computed by (6.7) may not correspond to a conservation law but may correspond to adjoint symmetries (see e.g. Wolf 2002).

6.2.6 Symmetry and conservation law relation

The fundamental relation between the Lie-Backlund symmetry generator X and the conserved vector T for a differential equation is governed by (Kara and Mahomed 2006)

$$X(T^i) + D_k(\xi^k)T^i - D_k(\xi^i)T^k = 0. \tag{6.8}$$

The joint conditions (6.8) together with (6.1) are used to find conserved vectors T^i. Here a symmetry condition is added to the direct method.

6.2.7 Direct construction method for conservation laws

Anco and Bluman (2002a), by utilizing the multipliers given in Olver (1993), gave an algorithmic method for finding the local conservation laws for partial differential equations expressed in a standard Cauchy-Kovalevskaya form. The method does not require the use or existence of a variational principle. The following definitions and results are adopted from Anco and Bluman (2002a,b).

Definition 1.9. If system (2.11) is in solved form for a pure derivative of the dependent variables u^α with respect to an independent variable say t and all other derivatives (mixed derivatives) of u^α are of lower order with respect to t then the system (2.11) is in Cauchy-Kovalevskaya form.

Burgers' equation

$$u_t - u_{xx} - uu_x = 0, \tag{6.9}$$

admits two Cauchy-Kovalevskaya forms, $u_t = u_{xx} + uu_x$ with respect to t and $u_{xx} = u_t + uu_x$ with respect to x.

A system of partial differential equations with $n+1$ independent variables $(t, x) = (t, x^1, \cdots, x^n)$ and N dependent variables $u = (u^1, \cdots, u^N)$ expressed in first-order Cauchy-Kovalevskaya form with respect to t has the following form:

$$\frac{\partial u^\alpha}{\partial t} + g^\alpha\left(t, x, u, u_{(1)}, \cdots, u_{(m)}\right),\ \alpha = 1, 2, \cdots, N, \tag{6.10}$$

where $u_{(j)}$ denotes the j^{th} partial derivatives of u. We can replace $u_t^\alpha = -g^\alpha$ in the conserved densities T^t and T^i, $i = 1, 2 \cdots n$, on the solution space of system (6.10). Therefore without loss of generality we can assume that the T^t and T^i depend only on t, x, u and x derivatives of u. This is referred to as the normal form of the conservation law:

$$D_t T^t(t, x, u, u_{(1)}, \cdots, u_{(k)}) + D_i T^i(t, x, u, u_{(1)}, \cdots, u_{(k)}) = 0. \tag{6.11}$$

It is shown by Anco and Bluman (2002b) that every nontrivial conservation law in normal form (6.11) is uniquely characterized by a set of multipliers $\Lambda(t, x, u, u_{(1)}, \cdots, u_{(p)})$ with no dependence on u_t and differential consequences.

Conservation law construction formula: The conserved densities of any nontrivial conservation law in normal form for the Cauchy-Kovalevskaya system (6.10) for given multipliers Λ_α are

$$T^t = \int_0^1 (u^\alpha - \tilde{u}^\alpha)\Lambda_\alpha[u_{(\lambda)}]d\lambda + \int_0^1 K(\lambda t, \lambda x)d\lambda, \tag{6.12}$$

$$T^i = x^i \int_0^1 \lambda^n K(\lambda t, \lambda x)d\lambda + \int_0^1 \Bigg(S^i\left[u - \tilde{u}, \Lambda[u_{(\lambda)}]; g[u_{(\lambda)}]\right]$$

$$+ S^i\left[u - \tilde{u}, g[u_{(\lambda)}] - \lambda g[u] + (1-\lambda)\tilde{u}_t; \Lambda[u_{(\lambda)}]\right] \Bigg) d\lambda, \tag{6.13}$$

where

$$u^\alpha_{(\lambda)} = \lambda u^\alpha + (1-\lambda)\tilde{u}^\alpha, \text{where } u^\alpha_{(1)} = u^\alpha,\ u^\alpha_{(0)} = \tilde{u}^\alpha, \tag{6.14}$$

$$\Lambda_\rho[u_{(\lambda)}] = \Lambda_\rho\left(t, x, u_{(\lambda)}, \partial_x u_{(\lambda)}, \cdots, \partial_x^p u_{(\lambda)}\right),\ \alpha, \rho = 1, 2, \cdots, N, \tag{6.15}$$

$$g^\rho[u_{(\lambda)}] = \Lambda_\rho\left(t, x, u_{(\lambda)}, \partial_x u_{(\lambda)}, \cdots, \partial_x^m u_{(\lambda)}\right), \tag{6.16}$$

$$K(t,x) = \left[\left(u^\rho_{(\lambda)t} + g^\rho[u_{(\lambda)}]\right)\Lambda_\rho[u_{(\lambda)}]\right]_{\lambda=0} \tag{6.17}$$

$$S^i\left[V, W; g\right] = \sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1} (-1)^k \left(D_{i_1}\cdots D_{i_l} V^\rho\right)$$

$$D_{j_1}\cdots D_{j_k}\left(W_\alpha \frac{\partial g^\alpha}{\partial u^\rho_{ii_1\cdots i_k j_1 \cdots j_l}}\right), \tag{6.18}$$

$$S^i\left[V, W; \Lambda\right] = \sum_{l=0}^{p-1}\sum_{k=0}^{p-l-1} (-1)^k \left(D_{i_1}\cdots D_{i_l} V^\rho\right)$$

$$D_{j_1}\cdots D_{j_k}\left(W^\alpha \frac{\partial \Lambda_\alpha}{\partial u^\rho_{ii_1\cdots i_k j_1 \cdots j_l}}\right). \tag{6.19}$$

If Λ_α and g^α are nonsingular at $u^\alpha = 0$ then choose $\tilde{u}^\alpha = 0$. If the system (6.10) satisfies $u^\alpha = \tilde{u}^\alpha = 0$, then the K integral vanishes.

6.2.8 Partial Noether approach

If the standard Lagrangian does not exist or is difficult to find, then we write its partial Lagrangian and derive the conservation laws by the partial Noether approach introduced by Kara and Mahomed (2006).

Definition 1.10. Suppose that the k^{th}-order differential system (2.11) can be written as

$$E_\alpha = E_\alpha^0 + E_\alpha^1 = 0. \tag{6.20}$$

A function $L = L(x, u, u_{(1)}, u_{(2)}, ..., u_{(l)})$, $l \leq k$ is called a partial Lagrangian of system (2.11) if system (2.11) can be expressed as $\delta L/\delta u^\alpha = f_\alpha^\beta E_\beta^1$ provided $E_\beta^1 \neq 0$ for some β. Here $\left(f_\alpha^\beta\right)$ is an invertible matrix.

Definition 1.11. The operator X defined in (2.7) satisfying

$$X(L) + LD_i(\xi^i) = D_i(B^i) + (\eta^\alpha - \xi^j u_j^\alpha)\frac{\delta L}{\delta u^\alpha},$$

$$i = 1, 2, \cdots, n; \quad \alpha = 1, 2, \cdots, N, \tag{6.21}$$

is a partial Noether operator corresponding to the partial Lagrangian L.

Theorem 1.2. The conserved vector of the system (2.11) associated with a partial Noether operator X corresponding to the partial Lagrangian L is determined from (6.4). Here also W^α are the characteristics of the conservation law.

We can also use the partial Noether approach for equations that have Lagrangian formulations.

6.2.9 Noether approach for a system and its adjoint

Definition 1.12. The system of adjoint equations to the system of k^{th} order differential equations (2.11) are defined by Atherton and Homsy (1975)

$$E_\alpha^*(x, u, v, ..., u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, 2, ...N, \tag{6.22}$$

where

$$E^*_\alpha(x,u,v,...,u_{(k)},v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1,2,...N,\ v = v(x) \tag{6.23}$$

and $v = (v^1, v^2, \cdots, v^N)$ are new dependent variables.

Definition 1.13. Suppose system (2.11) admits the generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{6.24}$$

Then the adjoint system (6.22) admits the operator (Ibragimov 2007)

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha},\ \eta^\alpha_* = -(\lambda^\alpha_\beta v^\beta + v^\alpha D_i(\xi^i)), \tag{6.25}$$

which is an extension of (6.24) to the variable v^α and λ^α_β are obtained from

$$X(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{6.26}$$

Theorem 1.3. Every Lie point, Lie Bäcklund and non-local symmetry of the system of k^{th} order differential equations (2.11) yields a conservation law for the system consisting of equations (2.11) and the adjoint equations (6.22). The conserved vector components are

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u^\alpha_i} + \sum_{s\geq 1} D_{i_1\cdots i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i_1\cdots i_s}}, \tag{6.27}$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x,u,...,u_{(k)}) \tag{6.28}$$

and ξ^i, η^α are the coefficient functions of the generator (6.24). The conserved vectors obtained from (6.27) involve the arbitrary solutions v of the system of adjoint equations (6.22) and hence one obtains an infinite number of conservation laws for (2.11) by specifying v.

6.3 Illustrative Example

To illustrate and compare all the approaches we derive using each approach the conservation laws for a nonlinear field equation describing the relaxation to a Maxwell distribution (Euler, Leach, Mahomed and Steeb 1988)

$$u_{tx} + u^2 = 0. \tag{6.29}$$

Direct method

Equation (6.1) with $T^1(t, x, u, u_t, u_x)$ and $T^2(t, x, u, u_t, u_x)$ becomes

$$D_t T^1 + D_x T^2 \mid_{u_{tx}+u^2=0} = 0,$$

or

$$T^1_t + T^1_u u_t + T^1_{u_t} u_{tt} + T^1_{u_x} u_{tx} + T^2_x + T^2_u u_x + T^2_{u_x} u_{xx} + T^2_{u_t} u_{xt} \mid_{u_{tx}+u^2=0} = 0. \tag{6.30}$$

Substitute $u_{tx} = -u^2$ in (6.30). Splitting with respect to u_{tt} and u_{xx} gives that T^1 and T^2 are independent of u_t and u_x, respectively. The remaining terms of equation (6.30) are

$$T^1_t + T^1_u u_t + T^2_x + T^2_u u_x - u^2(T^1_{u_x} + T^2_{u_t}) = 0. \tag{6.31}$$

If we further restrict T^1 and T^2 to be

$$T^1 = a(t, x, u)\frac{u_x^2}{2} + b(t, x, u),\ T^2 = c(t, x, u)\frac{u_t^2}{2} + d(t, x, u), \tag{6.32}$$

then (6.31) becomes

$$\frac{1}{2}c_u u_t^2 u_x + \frac{1}{2}a_u u_t u_x^2 + \frac{1}{2}c_x u_t^2 + \frac{1}{2}a_t u_x^2 + (b_u - cu^2)u_t$$

$$+(d_u - au^2)u_x + b_t + d_x = 0. \tag{6.33}$$

Splitting (6.33) according to the derivatives of u, we obtain

$$u_x u_t^2 : \quad c_u = 0, \tag{6.34}$$

$$u_t u_x^2: \quad a_u = 0, \tag{6.35}$$

$$u_t^2: \quad c_x = 0, \tag{6.36}$$

$$u_x^2: \quad a_t = 0, \tag{6.37}$$

$$u_t: \quad b_u - cu^2 = 0, \tag{6.38}$$

$$u_x: \quad d_u - au^2 = 0, \tag{6.39}$$

$$remainder: \quad b_t + d_x = 0. \tag{6.40}$$

The solution of equations (6.34)- (6.37) yields $a = a(x)$ and $c = c(t)$. Thus (6.38)- (6.39) give

$$b = \frac{1}{3}c(t)u^3 + e(t,x), \; d = \frac{1}{3}a(x)u^3 + f(t,x). \tag{6.41}$$

Equation (6.40) together with (6.41) gives

$$a = -c_1 x + c_3, \; c = c_1 t + c_2, \; e_t + f_x = 0, \tag{6.42}$$

and therefore from equation (6.41)

$$b = \frac{1}{3}(c_1 t + c_2)u^3 + e(t,x), \; d = \frac{1}{3}(-c_1 x + c_3)u^3 + f(t,x). \tag{6.43}$$

We can set $e = f = 0$ since they contribute to the trivial part of the conservation law. Substituting (6.42) and (6.43) into equation (6.32), we obtain three conserved vectors

$$\begin{aligned} T^1 &= -\frac{1}{2}xu_x^2 + \frac{1}{3}tu^3, \; T^2 = \frac{1}{2}tu_t^2 - \frac{1}{3}xu^3, \\ T^1 &= \frac{1}{3}u^3, \; T^2 = \frac{1}{2}u_t^2, \\ T^1 &= \frac{1}{2}u_x^2, \; T^2 = \frac{1}{3}u^3. \end{aligned} \tag{6.44}$$

Noether's approach

A Lagrangian for equation (6.29) satisfying the Euler-Lagrange equation $\delta L/\delta u = 0$, is

$$L = -\frac{u_t u_x}{2} + \frac{u^3}{3}. \tag{6.45}$$

We now show how the Noether point symmetries for the Lagrangian (6.45) can be constructed. From (2.5) upto first-order derivatives and (2.6)

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

$$+\zeta_t(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t} + \zeta_x(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}, \tag{6.46}$$

where

$$\zeta_t = D_t\eta - u_t D_t\tau - u_x D_t\xi,\ \zeta_x = D_x\eta - u_t D_x\tau - u_x D_x\xi,$$

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots, \tag{6.47}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots .$$

The Noether symmetry determining equation is, by (6.3),

$$XL + L(D_t\tau + D_x\xi) = D_tB^1 + D_xB^2, \tag{6.48}$$

where $B^1 = B^1(t,x,u)$ and $B^2 = B^2(t,x,u)$ are the gauge terms.

The Noether operators corresponding to L, given by (6.45), are determined using (6.48) which becomes

$$\eta u^2 - \frac{u_x}{2}[\eta_t + \eta_u u_t - \tau_t u_t - \tau_u u_t^2 - \xi_t u_x - \xi_u u_x u_t]$$

$$-\frac{u_t}{2}[\eta_x + \eta_u u_x - \tau_x u_t - \tau_u u_t u_x - \xi_x u_x - \xi_u u_x^2]$$

$$+(-\frac{u_t u_x}{2} + \frac{u^3}{3})[\tau_t + \tau_u u_t + \xi_x + \xi_u u_x] = B^1_t + B^1_u u_t + B^2_x + B^2_u u_x. \tag{6.49}$$

We split equation (6.49) with respect to derivatives of u and after simplification, we obtain

$$u_x u_t^2 : \quad \tau_u = 0, \tag{6.50}$$

$$u_t u_x^2 : \quad \xi_u = 0, \tag{6.51}$$

$$u_x u_t : \quad \eta_u = 0, \tag{6.52}$$

$$u_t^2 : \quad \tau_x = 0, \tag{6.53}$$

$$u_x^2 : \quad \xi_t = 0, \tag{6.54}$$

$$u_t : \quad \frac{u^3}{3}\tau_u - \frac{\eta_x}{2} = B_u^1, \tag{6.55}$$

$$u_x : \quad \frac{u^3}{3}\xi_u - \frac{\eta_t}{2} = B_u^2, \tag{6.56}$$

$$remainder : \quad \eta u^2 + \frac{u^3}{3}\xi_x + \frac{u^3}{3}\tau_t = B_t^1 + B_x^2. \tag{6.57}$$

The solution of equations (6.50)- (6.54) yields

$$\tau = A(t),\ \xi = B(x),\ \eta = C(x,t). \tag{6.58}$$

Equations (6.55) and (6.56), with the help of equation (6.58), give

$$B^1 = -\frac{1}{2}uc_x + D(t,x),\ B^2 = -\frac{1}{2}uc_t + E(t,x). \tag{6.59}$$

Equation (6.57) becomes

$$\frac{1}{3}u^3\left(\frac{dA}{dt} + \frac{dB}{dx}\right) + u^2C + uC_{xt} - D_t - E_x = 0, \tag{6.60}$$

which results in

$$A = c_1t + c_2,\ B = -c_1x + c_3,\ C = 0,\ D_t + E_x = 0, \tag{6.61}$$

where c_1, c_2 and c_3 are constants. From equations (6.58) and (6.61), we conclude that

$$\tau = c_1t + c_2,\ \xi = -c_1x + c_3,\ \eta = 0,\ B^1 = D(x,t),\ B^2 = E(x,t). \tag{6.62}$$

We can set $D = E = 0$ as they contribute to the trivial part of the conservation law. The three Noether operators, in extended form, are

$$X_1 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - u_t\frac{\partial}{\partial u_t} + u_x\frac{\partial}{\partial u_x},\ (c_1 = 1)$$

$$X_2 = \frac{\partial}{\partial t},\ (c_2 = 1) \tag{6.63}$$

$$X_3 = \frac{\partial}{\partial x},\ (c_3 = 1).$$

The first-order Noether conserved vector is $T = (T^1, T^2)$, where, by (6.4),

$$T^1 = B^1 - \tau L - (\eta - \tau u_t - \xi u_x)\frac{\partial L}{\partial u_t}, \tag{6.64}$$

$$T^2 = B^2 - \xi L - (\eta - \tau u_t - \xi u_x)\frac{\partial L}{\partial u_x}, \tag{6.65}$$

which together with (6.62) yield

$$\begin{aligned} T^1 &= -(c_1 t + c_2)\frac{u^3}{3} - (-c_1 x + c_3)\frac{u_x^2}{2}, \\ T^2 &= -(-c_1 x + c_3)\frac{u^3}{3} - (c_1 t + c_2)\frac{u_t^2}{2}. \end{aligned} \tag{6.66}$$

The three conserved vectors obtained from (6.66) are equivalent to (6.44).

Characteristic method

If we use this approach on (6.29) with Λ and T^i having at most first order derivatives, we find

$$T^1_t + T^1_u u_t + T^1_{u_t} u_{tt} + T^1_{u_x} u_{tx} + T^2_x + T^2_u u_x + T^2_{u_x} u_{xx} + T^2_{u_t} u_{xt} = \Lambda(u_{tx} + u^2). \tag{6.67}$$

Equating the coefficients of u_{tx}, we obtain

$$\Lambda = T^2_{u_t} + T^1_{u_x}. \tag{6.68}$$

Splitting with respect to u_{tt} and u_{xx} yields T^1 and T^2 independent of u_t and u_x, respectively, and equation (6.67) together with (6.68) reduces to (6.31). If we assume the quadratic solution ansatz as we did for (6.31), then we obtain precisely the conserved vectors (6.44). Moreover, (6.68) furnishes us with the multipliers so that (6.29) has conserved form.

Variational derivative method

If we use this approach on (6.29) with $\Lambda = \Lambda(t, x, u, u_x, u_t)$, we have

$$E_u[\Lambda(t, x, u, u_x, u_t)(u_{tx} + u^2)] = 0, \tag{6.69}$$

where from (2.3)

$$E_u = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} - \cdots, \quad (6.70)$$

is the Euler operator with respect to u. Equation (6.69) has to be satisfied for all functions $u(t,x)$ and not only the solutions of equation (6.29). The expansion of equation (6.69) results in

$$\Lambda_u(u_{tx} + u^2) - D_t[\Lambda_{u_t}(u_{tx} + u^2)]$$

$$-D_x[\Lambda_{u_x}(u_{tx} + u^2)] + D_t D_x(\Lambda) + 2u\Lambda = 0, \quad (6.71)$$

which yields

$$u_{xx}u_{tt}\Lambda_{u_t u_x} + u_{xx}[u_t\Lambda_{uu_x} - u^2\Lambda_{u_x u_x} + \Lambda_{tu_x}] + u_{tt}[u_x\Lambda_{uu_t} - u^2\Lambda_{u_t u_t} + \Lambda_{xu_t}]$$

$$-u_{xt}^2\Lambda_{u_t u_x} - 2u_{xt}(u^2\Lambda_{u_t u_x} - \Lambda_u) + u_x u_t \Lambda_{uu} + u_t(\Lambda_{ux} - u^2\Lambda_{uu_t} - 2u\Lambda_{u_t})$$

$$+u_x(\Lambda_{ut} - u^2\Lambda_{uu_x} - 2u\Lambda_{u_x}) + u^2(\Lambda_u - \Lambda_{tu_t} - \Lambda_{xu_x}) + 2u\Lambda + \Lambda_{tx} = 0. \quad (6.72)$$

Equation (6.72) is separated according to the second-order derivatives of u which yields the following system of determining equations for the multipliers:

$$u_{xx}u_{tt}: \quad \Lambda_{u_t u_x} = 0, \quad (6.73)$$

$$u_{xx}: \quad u_t\Lambda_{uu_x} - u^2\Lambda_{u_x u_x} + \Lambda_{tu_x} = 0, \quad (6.74)$$

$$u_{tt}: \quad u_x\Lambda_{uu_t} - u^2\Lambda_{u_t u_t} + \Lambda_{xu_t} = 0, \quad (6.75)$$

$$u_{xt}: \quad u^2\Lambda_{u_t u_x} - \Lambda_u = 0, \quad (6.76)$$

$$\mathrm{r}emainder: \quad u_x u_t \Lambda_{uu} + u_t(\Lambda_{ux} - u^2\Lambda_{uu_t} - 2u\Lambda_{u_t})$$

$$+u_x(\Lambda_{ut} - u^2\Lambda_{uu_x} - 2u\Lambda_{u_x}) + u^2(\Lambda_u - \Lambda_{tu_t} - \Lambda_{xu_x}) + 2u\Lambda + \Lambda_{tx} = 0. \quad (6.77)$$

Equations (6.73) and (6.76) give

$$\Lambda_{u_t} = \Lambda(t, x, u_t), \quad (6.78)$$

which together with (6.75) results in

$$\Lambda = u_t A(t) + B(t, x, u_x). \tag{6.79}$$

Substitution of (6.79) into (6.74) yields

$$B_{tu_x} - u^2 B_{u_x u_x} = 0, \tag{6.80}$$

which gives $B = u_x C(x) + D(t, x)$ and therefore

$$\Lambda = u_t A(t) + u_x C(x) + D(t, x). \tag{6.81}$$

Equation (6.77), after using (6.81), takes the following form:

$$u^2 \left(\frac{dA}{dt} + \frac{dC}{dx} \right) - 2uD - D_{tx} = 0. \tag{6.82}$$

The solution of (6.82) is

$$A = d_1 t + d_2, \ C = -d_1 x + d_3, \ D = 0, \tag{6.83}$$

and thus

$$\Lambda = (d_1 t + d_2) u_t + (-d_1 x + d_3) u_x. \tag{6.84}$$

A multiplier Λ for the partial differential equation (6.29) has the property that

$$\Lambda(u_{tx} + u^2) = D_t T^1 + D_x T^2, \tag{6.85}$$

for all functions $u(t, x)$. It is not difficult to construct the conserved vectors by elementary manipulations for the multipliers (6.84) and thus we have

$$[(d_1 t + d_2) u_t + (-d_1 x + d_3) u_x][u_{tx} + u^2] = D_t[\frac{1}{2}(-d_1 x + d_3) u_x^2 + \frac{1}{3}(d_1 t + d_2) u^3]$$

$$+ D_x[\frac{1}{2}(d_1 t + d_2) u_t^2 + \frac{1}{3}(-d_1 x + d_3) u^3], \tag{6.86}$$

which holds for arbitrary functions $u(t, x)$. When $u(t, x)$ is a solution of equation (6.29), then the left hand side of (6.86) vanishes and we obtain the conserved vectors (6.44). The conserved vectors can also be derived systematically using (6.85) as the determining equation with Λ given by (6.84). Assuming that T^1 and T^2 are of first order, we have

$$[(d_1t+d_2)u_t+(-d_1x+d_3)u_x][u_{tx}+u^2]$$

$$=T^1_t+T^1_uu_t+T^1_{u_t}u_{tt}+T^1_{u_x}u_{tx}+T^2_x+T^2_uu_x+T^1_{u_x}u_{xx}+T^2_{u_t}u_{tx}, \qquad (6.87)$$

which yields finally the conserved vectors (6.44). The variational derivative approach is also sometimes known as the multiplier approach.

Variational derivative method on space of solutions of the differential equation

For equation (6.29), condition (6.7) results in

$$E_u[\Lambda(t,x,u,u_x,u_t)(u_{tx}+u^2)]\mid_{u_{tx}+u^2=0}=0. \qquad (6.88)$$

Expansion of equation (6.88) and replacing u_{tx} by $-u^2$ gives

$$u_{xx}u_{tt}\Lambda_{u_tu_x}+u_{xx}[u_t\Lambda_{uu_x}-u^2\Lambda_{u_xu_x}+\Lambda_{tu_x}]+u_{tt}[u_x\Lambda_{uu_t}-u^2\Lambda_{u_tu_t}+\Lambda_{xu_t}]$$

$$+u_xu_t\Lambda_{uu}+u_t(\Lambda_{ux}-u^2\Lambda_{uu_t}-2u\Lambda_{u_t})+u_x(\Lambda_{ut}-u^2\Lambda_{uu_x}-2u\Lambda_{u_x})$$

$$+u^4\Lambda_{u_tu_x}-u^2(\Lambda_u+\Lambda_{tu_t}+\Lambda_{xu_x})+2u\Lambda+\Lambda_{tx}=0. \qquad (6.89)$$

Splitting equation (6.89) with respect to second order derivatives of u and solving the resulting system, we obtain the following system of determining equations for the multipliers:

$$u_{xx}u_{tt}: \quad \Lambda_{u_tu_x}=0, \qquad (6.90)$$

$$u_{xx}: \quad u_t\Lambda_{uu_x}-u^2\Lambda_{u_xu_x}+\Lambda_{tu_x}=0, \qquad (6.91)$$

$$u_{tt}: \quad u_x\Lambda_{uu_t}-u^2\Lambda_{u_tu_t}+\Lambda_{xu_t}=0, \qquad (6.92)$$

$$\text{r}emainder: \quad u_xu_t\Lambda_{uu}+u_t(\Lambda_{ux}-u^2\Lambda_{uu_t}-2u\Lambda_{u_t})+u_x(\Lambda_{ut}-u^2\Lambda_{uu_x}-2u\Lambda_{u_x})$$

$$+u^4\Lambda_{u_tu_x}-u^2(\Lambda_u+\Lambda_{tu_t}+\Lambda_{xu_x})+2u\Lambda+\Lambda_{tx}=0. \qquad (6.93)$$

Equations (6.90) and (6.92) give

$$\Lambda = u_t A(t) + B(t, x, u, u_x). \tag{6.94}$$

Substitution of (6.94) into (6.91) yields

$$u_t B_{uu_x} - u^2 B_{u_x u_x} + B_{tu_x} = 0, \tag{6.95}$$

which yields

$$B = u_x C(x) + D(t, x, u). \tag{6.96}$$

Now, using $\Lambda = u_t A(t) + u_x C(x) + D(t, x, u)$, equation (6.93) reduces to

$$u_t u_x D_{uu} + u_t D_{ux} + u_x D_{ut} - u^2 \left(D_u + \frac{dA}{dt} + \frac{dC}{dx} \right) + 2uD + D_{tx} = 0, \tag{6.97}$$

which yields

$$A = d_1 t + d_2,\ C = (d_4 - d_1)x + d_3,\ D = d_4 u, \tag{6.98}$$

where d_1, d_2, d_3 and d_4 are constants. Therefore

$$\Lambda = (d_1 t + d_2)u_t + [(d_4 - d_1)x + d_3]u_x + ud_4. \tag{6.99}$$

The multipliers corresponding to the constants d_1, d_2, d_3 are the same as obtained in the previous case and give the conserved vectors (6.44). The multiplier corresponding to d_4 is $xu_x + u$ and there exists no T^1 and T^2 such that

$$(xu_x + u)(u_{tx} + u^2) = D_t T^1 + D_x T^2. \tag{6.100}$$

This means that the multiplier $xu_x + u$ does not correspond to a conserved vector; it may correspond to an adjoint symmetry.

Symmetry and conservation law relation

The Lie-point symmetry determining equations by the computer package YaLie by Diaz are

$$\tau_u = 0,\ \tau_x = 0, \tag{6.101}$$

$$\xi_u = 0,\ \xi_t = 0, \tag{6.102}$$

$$\eta_{uu} = 0,\ \eta_{tu} = 0,\ \eta_{xu} = 0, \tag{6.103}$$

$$u^2(\tau_t + \xi_x - \eta_u) + 2u\eta + \eta_{tx} = 0. \tag{6.104}$$

Equations (6.101)-(6.104) result in

$$\tau = k_1 t + k_2,\ \xi = k_3 - (k_1 + k_4)x,\ \eta = k_4 u, \tag{6.105}$$

where $k_1,\ k_2,\ k_3$ and k_4 are constants. The Lie point symmetry generators of equation (6.29) in prolonged form are

$$X = [k_1 t + k_2]\frac{\partial}{\partial t} + [k_3 - (k_1 + k_4)x]\frac{\partial}{\partial x} + k_4 u\frac{\partial}{\partial u}$$

$$+[k_4 - k_1]u_t\frac{\partial}{\partial u_t} + [2k_4 + k_1]u_x\frac{\partial}{\partial u_x}. \tag{6.106}$$

We construct a conserved vector $T = (T^1, T^2)$ which has associated with it the following symmetry generator (in extended form) obtained from (6.106) with $k_1 \neq 0$ and $k_2 = k_3 = k_4 = 0$,

$$X = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - u_t\frac{\partial}{\partial u_t} + u_x\frac{\partial}{\partial u_x}. \tag{6.107}$$

Expansion of (6.1) results in equation (6.31) and

$$T^1 = T^1(t, x, u, u_x),\ T^2 = T^2(t, x, u, u_t). \tag{6.108}$$

Instead of restricting T^1 and T^2 to satisfy (6.32), we use condition (6.8) with equation (6.31). The symmetry condition (6.8) results in the following two equations

$$X(T^1) + T^1 D_x(\xi) - T^2 D_x(\tau) = 0, \tag{6.109}$$

$$X(T^2) + T^2 D_t(\tau) - T^1 D_t(\xi) = 0, \tag{6.110}$$

which yield

$$XT^1 - T^1 = 0,\ XT^2 + T^2 = 0, \tag{6.111}$$

where X is given in (6.107). The solution of equation (6.111) yields

$$T^1 = tf(a, \alpha, \beta),\ T^2 = xg(a, \alpha, \gamma), \tag{6.112}$$

where $a = xt,\ \alpha = u,\ \beta = xu_x$, and $\gamma = tu_t$. The substitution of (6.112) into (6.31) results in

$$f + a\frac{\partial f}{\partial a} + \gamma\frac{\partial f}{\partial \alpha} - a\alpha^2\frac{\partial f}{\partial \beta} + g + a\frac{\partial g}{\partial a} - a\alpha^2\frac{\partial g}{\partial \gamma} + \beta\frac{\partial g}{\partial \alpha} = 0, \tag{6.113}$$

which gives

$$f = -\frac{\beta^2}{2a} + \frac{\alpha^3}{3},\ g = \frac{\gamma^2}{2a} - \frac{\alpha^3}{3}. \tag{6.114}$$

Hence we obtain

$$T^1 = -\frac{xu_x^2}{2} + \frac{tu^3}{3},\ T^2 = \frac{tu_t^2}{2} - \frac{xu^3}{3}. \tag{6.115}$$

The other two conserved vectors can be obtained in the same manner.

The second aspect of this approach is that we can associate a symmetry with a known conserved vector. For the conserved vector $T^1 = \frac{1}{3}u^3,\ T^2 = \frac{1}{2}u_t^2$, the symmetry condition (6.8) with X given in (6.106) gives

$$[2k_4 - k_1]T^1 = 0,\ [2k_4 - k_1]T^2 = 0, \tag{6.116}$$

which is satisfied only if $k_4/k_1 = 1/2$. Thus, taking $k_2 = k_3 = 0$ and $k_4/k_1 = 1/2$, we have

$$X = t\frac{\partial}{\partial t} + \frac{3}{2}x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u}, \tag{6.117}$$

as the symmetry associated with the conserved vector $T^1 = \frac{1}{3}u^3,\ T^2 = \frac{1}{2}u_t^2$.

Direct construction method for conservation laws

Anco and Bluman (2002a) considered the Klein-Gordon equation $u_{tx} - g(u) = 0$ in one of the examples and derived the expression for the conserved density $T^1(x, u, u_{(1)}, \cdots, u_{(q)})$ for given multipliers of the form $\Lambda(x, u, u_{(1)}, \cdots, u_{(p)}),\ p = 2q - 1$, as

$$T^1 = \int_0^1 (u_x - \tilde{u}_x)\Lambda[\lambda u + (1 - \lambda)\tilde{u}]d\lambda, \tag{6.118}$$

where $\Lambda[u] = \Lambda(x, u, u_{(1)}, \cdots, u_{(p)})$ and $\tilde{u} = \tilde{u}(x)$ is any function chosen such that $\Lambda[\tilde{u}]$ and $g(\tilde{u})$ are non-singular. In particular, $\tilde{u} = 0$ if $\Lambda[0]$ and $g(0)$ are non-singular (see Anco and Bluman 2002a).

For our case $g(u) = -u^2$. The only first order multiplier of the form $\Lambda(x, u, u_x)$ for (6.29) calculated with the help of (6.6) is $\Lambda(x, u, u_x) = u_x$ and then the conserved vector $T^1(x, u, u_x)$ can be found using the formula (6.118). Now $\Lambda[0]$ and $g(0)$ are non-singular and therefore we can choose $\tilde{u} = 0$. Thus from (6.118) we have

$$T^1 = u_x^2 \int_0^1 \lambda d\lambda = \frac{u_x^2}{2}. \tag{6.119}$$

Equation (6.5), with T^1 given in (6.119) and $T^2(x, u, u_x)$, gives

$$D_t\left(\frac{u_x^2}{2}\right) + D_x T^2 = u_x(u_{tx} + u^2), \tag{6.120}$$

which yields

$$T^2 = \frac{u^3}{3}. \tag{6.121}$$

If we replace x with t, (6.118) yields the conserved vector

$$T^1 = \frac{1}{3}u^3. \tag{6.122}$$

The third conservation law cannot be obtained by this approach.

Partial Noether approach

Consider the partial Lagrangian

$$L = -\frac{1}{2}u_t u_x \tag{6.123}$$

so that equation (6.29) becomes

$$\frac{\delta L}{\delta u} = -u^2. \tag{6.124}$$

The partial Noether symmetry determining equation is, by (6.21),

$$X^{[1]}L + L(D_t\tau + D_x\xi) = D_t B^1 + D_x B^2 + (\eta - \tau u_t - \xi u_x)\frac{\delta L}{\delta u}. \tag{6.125}$$

Equation (6.125) for $L = -\frac{1}{2}u_t u_x$ gives

$$
\begin{aligned}
&-\frac{u_x}{2}[\eta_t + \eta_u u_t - \tau_t u_t - \tau_u u_t^2 - \xi_t u_x - \xi_u u_x u_t] \\
&-\frac{u_t}{2}[\eta_x + \eta_u u_x - \tau_x u_t - \tau_u u_t u_x - \xi_x u_x - \xi_u u_x^2] \\
-\frac{u_t u_x}{2}&[\tau_t + \tau_u u_t + \xi_x + \xi_u u_x] = B_t^1 + B_u^1 u_t + B_x^2 + B_u^2 u_x - u^2[\eta - \tau u_t - \xi u_x], \quad (6.126)
\end{aligned}
$$

where $B^1 = B^1(t,x,u)$ and $B^2 = B^2(t,x,u)$ are the gauge terms. Separating equation (6.126) with respect to derivatives of u, we obtain

$$u_x u_t^2 : \quad \tau_u = 0, \qquad (6.127)$$

$$u_t u_x^2 : \quad \xi_u = 0, \qquad (6.128)$$

$$u_x u_t : \quad \eta_u = 0, \qquad (6.129)$$

$$u_t^2 : \quad \tau_x = 0, \qquad (6.130)$$

$$u_x^2 : \quad \xi_t = 0, \qquad (6.131)$$

$$u_t : \quad -\frac{\eta_x}{2} = B_u^1 + u^2\tau, \qquad (6.132)$$

$$u_x : \quad -\frac{\eta_t}{2} = B_u^2 + u^2\xi, \qquad (6.133)$$

$$\text{r}emainder : \quad B_t^1 + B_x^2 - \eta u^2 = 0. \qquad (6.134)$$

The solution of equations (6.127)- (6.131) yields

$$\tau = A(t),\ \xi = B(x),\ \eta = C(x,t). \qquad (6.135)$$

Equations (6.132) and (6.133) give

$$B^1 = -\frac{1}{3}u^3 A(t) - \frac{1}{2}uc_x + D(t,x),\ B^2 = -\frac{1}{3}u^3 B(x) - \frac{1}{2}uc_t + E(t,x). \quad (6.136)$$

Equation (6.134), after substitution of (6.136), takes the following form

$$\frac{1}{3}u^3\left(\frac{dA}{dt} + \frac{dB}{dx}\right) + u^2 C - D_t - E_x = 0, \qquad (6.137)$$

which results in

$$A = c_1 t + c_2,\ B = -c_1 x + c_3,\ C = 0,\ D_t + E_x = 0, \qquad (6.138)$$

where c_1, c_2 and c_3 are constants. Therefore

$$\tau = c_1 t + c_2,\ \xi = -c_1 x + c_3,\ \eta = 0,$$

$$B^1 = -\frac{u^3}{3}(c_1 t + c_2) + D(t,x),\ B^2 = -\frac{u^3}{3}(-c_1 x + c_3) + E(t,x),$$

$$D_t + E_x = 0. \tag{6.139}$$

We choose $D = E = 0$ since they contribute to the trivial part of the conserved vector. We obtain three partial Noether operators which are the same as the Noether operators (6.63) since the partial Euler-Lagrange equations (6.124) are independent of derivatives. The gauge terms for the Noether and partial Noether operators are different.

The first order partial Noether conserved vector $T = (T^1, T^2)$ can be derived from (6.64) and (6.65) together with (6.139). We obtain the conserved vectors (6.44).

Conservation theorem

Consider now the conservation theorem given by Ibragimov (2007). The adjoint equation for (6.29) is

$$E^*(t,x,u,v,\cdots,v_{tx}) = E_u\left[v(u_{tx} + u^2)\right] = 0,\ v = v(t,x), \tag{6.140}$$

which yields

$$v_{tx} + 2uv = 0. \tag{6.141}$$

Now, consider equation (6.29) and the adjoint equation (6.141) as a system. The Lagrangian for the system is, from (6.28),

$$L = v(u_{tx} + u^2), \tag{6.142}$$

that is

$$E_v = u_{tx} + u^2 = 0,\ E_u = v_{tx} + 2uv = 0. \tag{6.143}$$

The conserved vectors of the system of equations (6.29) and (6.141), associated with a symmetry, can be obtained from (6.27) as follows:

$$T^1 = \tau L + W\left[\frac{\partial L}{\partial u_t} - D_t\left(\frac{\partial L}{\partial u_{tt}}\right) - D_x\left(\frac{\partial L}{\partial u_{tx}}\right)\right]$$

$$+D_t(W)\frac{\partial L}{\partial u_{tt}} + D_x(W)\frac{\partial L}{\partial u_{tx}}, \tag{6.144}$$

$$T^2 = \xi L + W\left[\frac{\partial L}{\partial u_x} - D_t\left(\frac{\partial L}{\partial u_{tx}}\right) - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right]$$

$$+D_t(W)\frac{\partial L}{\partial u_{tx}} + D_x(W)\frac{\partial L}{\partial u_{xx}}, \tag{6.145}$$

where

$$W = \eta - \tau u_t - \xi u_x. \tag{6.146}$$

The Lie point symmetries for (6.29) are given by (6.106). We consider the symmetry $X_1 = \partial/\partial t$. Equation (6.25) reveals that Y_1 coincides with X_1. The Lie characteristic function is $W = -u_t$ and the conserved vector from (6.144) and (6.145) is

$$T^1 = u^2 v + u_t v_x,\ T^2 = u_t v_t - u_{tt} v. \tag{6.147}$$

The conserved vector (6.147) involves solutions v of the adjoint equation (6.141) and hence gives an infinite number of conservation laws. Next, we consider the symmetry $X_2 = \partial/\partial x$. Equation (6.25) reveals that Y_2 also coincides with X_2. For this case, we have, $\tau = 0$, $\xi = 1$ and the Lie characteristic function is $W = -u_x$. The conserved vector from (6.144) and (6.145) is

$$T^1 = u_x v_x - u_{xx} v,\ T^2 = u^2 v + u_x v_t. \tag{6.148}$$

The conserved vector (6.148) involves solutions v of the adjoint equation (6.141) and hence yields an infinite number of conservation laws. Similarly one can use the other symmetries contained in (6.106). However, one requires the solution of the adjoint equation to work out the conservation law.

6.4 Concluding remarks

The conservation laws for single partial differential equations or systems of partial differential equations that arise in fluid mechanics were computed using different approaches. Firstly, we explained all the approaches with the help of an example. The Noether approach is simple and is a systematic way to construct conservation laws for partial differential equations that posses a standard Lagrangian and corresponding Noether symmetries. We constructed the Noether point symmetries and the Noether conserved vectors for a partial differential equation. The partial Noether approach is as effective as the Noether approach for differential equations with or without standard Lagrangians. In particular, for second order partial differential equations (single or system) which arise in fluid mechanics and for which the standard Lagrangian does not exist conservation laws can be constructed with the partial Noether approach. We also commentated on the direct method as well as its use with a symmetry condition for systems without a standard Lagrangian. Furthermore, we looked at some other approaches which do not rely upon the knowledge of a Lagrangian, for example, the characteristic method, variational derivative method for arbitrary functions as well as for the solution space, direct construction formulae and the Noether approach for the equation and its adjoint. The simplest and most effective way to compute characteristics (multipliers) is by taking the variational derivative of $D_i T^i = \Lambda^\alpha E_\alpha$ for arbitrary functions, not only for solutions.

Chapter 7

Application of different approaches for deriving conservation laws in fluid mechanics problems

7.1 Introduction

In this Chapter the different approaches to construct conservation laws are applied to selected equations in fluid mechanics. The approaches are compared. The equations considered are, the non-linear diffusion equation for the radial spreading of an axisymmetric thin liquid drop, the system of partial differential equations for the two-dimensional jet flows and the system of partial differential equations for the radial laminar jet flows.

The outline of this Chapter is as follows. In Section 7.2, the conservation laws for the non-linear diffusion equation describing the spreading of an axisymmetric thin liquid drop are derived using the different approaches. The conservation laws for the system of partial differential equations for the two-dimensional and radial laminar jet flows are constructed using the different approaches in Sections 7.3 and 7.4 respectively. A discussion and the concluding remarks are given in Section 7.5.

7.2 Non-linear diffusion equation for an axisymmetric thin liquid drop

The non-linear diffusion equation governing the gravity-driven spreading of an axisymmetric liquid drop on a fixed horizontal plane (Momoniat, Mason and Mahomed 2001) is

$$h_t - \frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3} = 0. \tag{7.1}$$

Direct method:

We apply the direct method to (7.1). Note that we can use (7.1) to replace h_t by h and derivatives of h with respect to r. We assume T^i to be first order in the r derivative: $T^i = T^i(t, r, h, h_r)$. Now, from (6.1),

$$D_t T^1 + D_r T^2 \mid_{(7.1)} = 0, \tag{7.2}$$

and replacement of h_{rr} using (7.1) gives

$$\begin{aligned} &T^1_t + T^1_h h_t + T^1_{h_r} h_{tr} + T^2_r + T^2_h h_r \\ &+ T^2_{h_r} \frac{3}{h^3}[h_t - \frac{h^3 h_r}{3r} - h^2 h_r^2] = 0. \end{aligned} \tag{7.3}$$

Separation by h_{tr} and h_t in (7.3) yields

$$T^1_{h_r} = 0, \; T^1_h + \frac{3}{h^3} T^2_{h_r} = 0, \tag{7.4}$$

and (7.3) reduces to

$$T^1_t + T^2_r + T^2_h h_r + T^2_{h_r} \frac{3}{h^3}[-\frac{h^3 h_r}{3r} - h^2 h_r^2] = 0. \tag{7.5}$$

The solution of equations (7.4) is

$$T^1 = A(t,r,h),\ T^2 = -\frac{h^3}{3}h_r A_h + B(t,r,h), \tag{7.6}$$

and equation (7.5) becomes

$$\frac{1}{3}h_r^2 h^3 A_{hh} + h_r[\frac{h^3}{3}A_{hr} - \frac{h^3}{3r}A_h - B_h] - A_t - B_r = 0. \tag{7.7}$$

Separating equation (7.7) with respect to derivatives of h, we have

$$h_r^2: \quad A_{hh} = 0, \tag{7.8}$$

$$h_r: \quad \frac{h^3}{3}A_{hr} - \frac{h^3}{3r}A_h - B_h = 0, \tag{7.9}$$

$$\text{r}emainder: \quad A_t + B_r = 0. \tag{7.10}$$

Equation (7.8) yields

$$A(t,r,h) = hC(t,r) + D(t,r). \tag{7.11}$$

Substitution of (7.11) into (7.9) results in

$$B_h - \frac{h^3}{3}(C_r - \frac{C}{r}) = 0, \tag{7.12}$$

and thus we have

$$B(t,r,h) = \frac{h^4}{12}(C_r - \frac{C}{r}) + E(t,r). \tag{7.13}$$

Substituting (7.11) and (7.13) into (7.10) gives rise to

$$\frac{h^4}{12}\left(C_{rr} - \frac{C_r}{r} + \frac{C}{r^2}\right) + hC_t + D_t + E_r = 0. \tag{7.14}$$

Splitting equation (7.14) with respect to powers of h gives

$$h^4: \quad C_{rr} - \frac{C_r}{r} + \frac{C}{r^2} = 0,$$

$$h: C_t = 0, \tag{7.15}$$

$$\text{remainder}: \quad D_t + E_r = 0.$$

Solving the system of equations (7.15), we obtain

$$C = c_1 r + c_2 r \ln r,\ D_t + E_r = 0, \tag{7.16}$$

and we set $D = E = 0$ as they contribute to the trivial part of the conservation law. Equations (7.11) and (7.13), together with (7.16), give

$$A(t,r,h) = (c_1 r + c_2 r \ln r)h,\ B(t,r,h) = \frac{c_2}{12}h^4, \tag{7.17}$$

and then using in equation (7.6), we obtain two conserved vectors

$$T^1 = (c_1 r + c_2 r \ln r)h,\ T^2 = -\frac{1}{3}(c_1 r + c_2 r \ln r)h_r h^3 + \frac{c_2}{12}h^4. \tag{7.18}$$

Characteristic approach:

If we employ the characteristic approach on (7.1) with T^1, T^2 and Λ functions of t, r, h and h_r, we obtain (by (6.5))

$$\begin{aligned} &T^1_t + T^1_h h_t + T^1_{h_r} h_{tr} + T^2_r + T^2_h h_r + T^2_{h_r} h_{rr} \\ &= \Lambda[h_t - \frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3}]. \end{aligned} \tag{7.19}$$

Equating the coefficients of h_{rr} in equation (7.19) yields the characteristic

$$\Lambda = -\frac{3}{h^3}T^2_{h_r}. \tag{7.20}$$

The substitution of the characteristic from equation (7.20) into the remaining part of equation (7.19) gives rise to equation (7.3). The solution of equation (7.3) was discussed before. We obtain the conserved vectors (7.18).

Variational derivative method:

If we use this approach on (7.1) with $\Lambda(t, r, h, h_r)$, we have

$$E_h\left[\Lambda(t,r,h,h_r)\left(h_t - \frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3}\right)\right] = 0, \tag{7.21}$$

where E_h is the Euler operator with respect to h and can be computed from (2.3). Equation (7.21) has to be satisfied for all functions $h(t, r)$ and not only the solutions of equation (7.1). The expansion of equation (7.21) results in

$$\Lambda_h\left(h_t - \frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3}\right) - D_r\left[\Lambda_{h_r}\left(h_t - \frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3}\right)\right]$$

$$-\Lambda\left(\frac{h^2h_r}{r}+2hh_r^2+h^2h_{rr}\right)-D_t(\Lambda)$$
$$+D_r\left[\Lambda\left(\frac{h^3}{3r}+2h^2h_r\right)\right]-D_r^2\left(\frac{1}{3}\Lambda h^3\right)=0. \quad (7.22)$$

After expansion, the coefficient of h_{rt} in (7.22) gives $\Lambda_{h_r}=0$ and (7.22) reduces to

$$\Lambda_h\left(h_t-\frac{h^3h_r}{3r}-h^2h_r^2-\frac{h^3h_{rr}}{3}\right)-\Lambda\left(\frac{h^2h_r}{r}+2hh_r^2+h^2h_{rr}\right)-\Lambda_t-h_t\Lambda_h$$
$$+(\Lambda_r+h_r\Lambda_h)\left(\frac{h^3}{3r}+2h^2h_r\right)+\Lambda\left(-\frac{h^3}{3r^2}+\frac{h^2h_r}{r}+4hh_r^2+2h^2h_{rr}\right)-2hh_r^2\Lambda$$
$$-h^2h_{rr}\Lambda-2h^2h_r(\Lambda_r+h_r\Lambda_h)-\frac{h^3}{3}\left[\Lambda_{rr}+2h_r\Lambda_{rh}+h_{rr}\Lambda_h+h_r^2\Lambda_{hh}\right]=0. \quad (7.23)$$

The coefficient of h_{rr} in (7.23) gives $\Lambda_h=0$ and thus

$$\Lambda=A(r,t). \quad (7.24)$$

Equation (7.23) becomes

$$\frac{h^3}{3}\left[A_{rr}-\frac{1}{r}A_r+\frac{1}{r^2}A\right]+A_t=0 \quad (7.25)$$

which yields $A=c_1r+c_2r\ln r$ and therefore

$$\Lambda=c_1r+c_2r\ln r. \quad (7.26)$$

A multiplier Λ for the partial differential equation (7.1) satisfies

$$\Lambda\left(h_t-\frac{h^3h_r}{3r}-h^2h_r^2-\frac{h^3h_{rr}}{3}\right)=D_tT^1+D_rT^2, \quad (7.27)$$

for all functions $h(t,r)$. For the multipliers (7.26), we have

$$[c_1r+c_2r\ln r]\left(h_t-\frac{h^3h_r}{3r}-h^2h_r^2-\frac{h^3h_{rr}}{3}\right)=D_t[(c_1r+c_2r\ln r)h]$$
$$+D_r[-\frac{1}{3}(c_1r+c_2r\ln r)h_rh^3+\frac{c_2}{12}h^4], \quad (7.28)$$

which holds for arbitrary functions $h(t,r)$. When $h(t,r)$ is a solution of equation (7.1), then the left hand side of (7.28) vanishes and we obtain the conserved vectors (7.18).

Partial Lagrangian approach:

A Lagrangian for (7.1) does not exist. However, we can write a partial Lagrangian. A partial Lagrangian $L = h^3 h_r^2/6$ reduces equation (7.1) to

$$\frac{\delta L}{\delta h} = -h_t + \frac{h^3 h_r}{3r} + \frac{h^2 h_r^2}{2}. \tag{7.29}$$

The partial Noether symmetry determining equation is, by (6.21),

$$XL + L(D_t\tau + D_r\xi) = D_t B^1(t,r,h) + D_r B^2(t,r,h)$$
$$+(\eta - \tau h_t - \xi h_r)\frac{\delta L}{\delta h}, \tag{7.30}$$

which yields

$$\frac{h^2 h_r^2}{2}\eta + \frac{h^3 h_r}{3}[\eta_r + \eta_h h_r - \tau_r h_t - \tau_h h_r h_t - \xi_r h_r - \xi_h h_r^2)]$$
$$+\frac{h^3 h_r^2}{6}[\tau_t + \tau_h h_t + \xi_r + \xi_h h_r] = B_t^1 + B_h^1 h_t + B_r^2 + B_h^2 h_r$$
$$+[\eta - \tau h_t - \xi h_r][-h_t + \frac{h^3 h_r}{3r} + \frac{h^2 h_r^2}{2}]. \tag{7.31}$$

Separation by h_t^2 and $h_t h_r$ in (7.31) yields

$$\tau = 0,\ \xi = 0, \tag{7.32}$$

and equation (7.31) reduces to

$$\frac{h^2 h_r^2}{2}\eta + \frac{h^3 h_r}{3}[\eta_r + \eta_h h_r] = B_t^1 + B_h^1 h_t + B_r^2 + B_h^2 h_r$$
$$+\eta[-h_t + \frac{h^3 h_r}{3r} + \frac{h^2 h_r^2}{2}]. \tag{7.33}$$

Separating equation (7.33) according to derivatives of h, we obtain the following determining equations:

$$h_r^2: \quad \eta_h = 0, \tag{7.34}$$

$$h_t: \quad B_h^1 - \eta = 0, \tag{7.35}$$

$$h_t: \quad B_h^2 - \frac{h^3}{3}\left(\eta_r - \frac{1}{r}\eta\right) = 0, \tag{7.36}$$

$$\text{remainder}: \quad B^1_t + B^2_r = 0. \tag{7.37}$$

Equation (7.34) gives

$$\eta = A(t, r). \tag{7.38}$$

Equation (7.35) and (7.36) with $\eta = A(t, r)$ yield

$$B^1 = hA(t, r) + f(t, r),\ B^2 = \frac{h^4}{12}\left(A_r - \frac{1}{r}A\right) + g(t, r). \tag{7.39}$$

Substituting (7.39) in (7.37) results in

$$\frac{h^4}{12}\left(A_{rr} - \frac{1}{r}A_r + \frac{1}{r^2}A\right) + hA_t + f_t + g_r = 0, \tag{7.40}$$

which gives

$$A = c_1 r + c_2 r \ln r,\ f_t + g_r = 0, \tag{7.41}$$

where c_1 and c_2 are constants. We set $f = g = 0$ as they contribute to the trivial part of the conservation law. We obtain the following partial Noether operators and gauge terms:

$$X = (c_1 r + c_2 r \ln r)\frac{\partial}{\partial h},$$

$$B^1 = h(c_1 r + c_2 r \ln r),\ B^2 = \frac{c_2}{12}h^4. \tag{7.42}$$

The partial Noether conserved vectors by (6.4) are

$$T^1 = B^1 - \tau L - (\eta - \tau h_t - \xi h_r)\frac{\partial L}{\partial h_t}, \tag{7.43}$$

$$T^2 = B^2 - \xi L - (\eta - \tau h_t - \xi h_r)\frac{\partial L}{\partial h_r}, \tag{7.44}$$

which together with (7.42) yields the conserved vectors (7.18).

Symmetry condition:

The Lie point symmetries for (7.1) are (Momoniat, Mason and Mahomed 2001)

$$X_1 = \frac{\partial}{\partial t},\ X_2 = t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r},\ X_3 = \frac{3r}{2}\frac{\partial}{\partial r} + h\frac{\partial}{\partial h}. \tag{7.45}$$

The symmetry condition (6.8) yields following two equations

$$X(T^1) + T^1 D_r(\xi) - T^2 D_r(\tau) = 0, \tag{7.46}$$

$$X(T^2) + T^2 D_t(\tau) - T^1 D_t(\xi) = 0. \tag{7.47}$$

We now construct conserved vectors for equation (7.1) by the joint conditions of the direct method and symmetry condition. The direct method on equation (7.1) gives rise to equations (7.4) and (7.5). Instead of solving these equations directly we add the symmetry conditions. From equations (7.46) and (7.47) the conserved vectors $T^1(t, r, h, h_r)$ and $T^2(t, r, h, h_r)$ associated with the symmetry generator X_1 will satisfy $X_1(T^1) = X_1(T^2) = 0$ and we deduce that

$$T^1_t = 0,\ T^2_t = 0. \tag{7.48}$$

Equations (7.4) and (7.48) give

$$T^1 = A(r, h),\ T^2 = -\frac{h^3}{3} h_r A_h + B(r, h), \tag{7.49}$$

and equation (7.5) reduces to

$$\frac{1}{3} h_r^2 h^3 A_{hh} + h_r[\frac{h^3}{3} A_{hr} - \frac{h^3}{3r} A_h - B_h] - B_r = 0. \tag{7.50}$$

Separating equation (7.50) with respect to derivatives of h, we have

$$h_r^2 : \quad A_{hh} = 0, \tag{7.51}$$

$$h_r : \quad \frac{h^3}{3} A_{hr} - \frac{h^3}{3r} A_h - B_h = 0, \tag{7.52}$$

$$remainder : \quad B_r = 0. \tag{7.53}$$

Equations (7.51) and (7.53) give

$$A = hC(r) + D(r),\ B = B(h). \tag{7.54}$$

Equation (7.52), after using (7.54), becomes

$$\frac{dB}{dh} - \frac{h^3}{3}\left(\frac{dC}{dr} - \frac{1}{r} C\right) = 0. \tag{7.55}$$

Differentiating (7.55) with respect to r, we have

$$\frac{d^2C}{dr^2} - \frac{1}{r}\frac{dC}{dr} + \frac{1}{r^2}C = 0, \tag{7.56}$$

which yields

$$C(r) = c_1 r + c_2 r \ln r \tag{7.57}$$

and thus $A = (c_1 r + c_2 r \ln r)h + D(r)$. Equation (7.55), with C in (7.57), gives

$$B(h) = \frac{c_2}{12}h^4 + c_3. \tag{7.58}$$

We can choose $c_3 = D(r) = 0$. Thus from (7.49), we obtain the conserved vectors (7.18).

Direct construction formula:

We will use formulae (6.12) and (6.13) together with equations (6.14)-(6.19) to derive conserved densities for (7.1). Equation (7.1) can be rewritten as

$$h_t + g(r, h, h_r, h_{rr}) = 0, \tag{7.59}$$

where

$$g(r, h, h_r, h_{rr}) = -\frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3}, \tag{7.60}$$

which is in Cauchy-Kovalevskaya form with respect to t. We have two independent variables t and x and one dependent variable h. The multiplier of the form $\Lambda(t, r, h, h_r)$, calculated by the characteristic approach, is given in (7.26). Now at $h = 0$, the multiplier $\Lambda = c_1 r + c_2 r \ln r$ and g are non-singular. Therefore we can choose $\tilde{h} = 0$ and hence $h_{[\lambda]} = \lambda h$. Also $\tilde{h} = 0$, $h = 0$ satisfies (7.59) and therefore the K integrals in (6.12) and (6.13) vanish. Hence integrals (6.12) and (6.13) for our case reduce to

$$T^1 = \int_0^1 h\Lambda[h_{(\lambda)}]d\lambda = \int_0^1 h\Lambda(t, r, \lambda h, \lambda h_r)d\lambda, \tag{7.61}$$

and

$$T^2 = \int_0^1 \left(S^r\left[h, \Lambda[h_{[\lambda]}]; g[h_{[\lambda]}]\right] + S^r\left[h, g[h_{[\lambda]}] - \lambda g[h]; \Lambda[h_{[\lambda]}]\right]\right) d\lambda. \tag{7.62}$$

Since g involves upto second-order derivatives with respect to r it follows that $m = 2$ and the multiplier is of first-order, $p = 1$. Equations (6.18) and (6.19) reduce to

$$S^r\left[V, W; g\right] = \sum_{l=0}^{1} \sum_{k=0}^{1-l} (-1)^k \left(D_{i_1} \cdots D_{i_l} V\right) D_{j_1} \cdots D_{j_k} \left(W \frac{\partial g}{\partial h_{r i_1 \cdots i_k j_i \cdots j_l}} \right)$$

$$= VW \frac{\partial g}{\partial h_r} - V D_r \left(W \frac{\partial g}{\partial h_{rr}} \right) + D_r(V) \left(W \frac{\partial g}{\partial h_{rr}} \right), \tag{7.63}$$

$$S^r\left[V, W; \Lambda\right] = VW \frac{\partial \Lambda}{\partial h_r}. \tag{7.64}$$

Equation (7.61), with $\Lambda = c_1 r + c_2 r \ln r$ from (7.26), yields

$$T^1 = (c_1 r + c_2 r \ln r) h \int_0^1 d\lambda = (c_1 r + c_2 r \ln r) h, \tag{7.65}$$

which agrees with (7.18) for T^1 . From (7.63), we obtain

$$S^r\left[h, \Lambda[h_{[\lambda]}]; g[h_{[\lambda]}]\right] = S^r\left[h, \Lambda[\lambda h]; g[\lambda h]\right]$$

$$= h\Lambda[\lambda h] \frac{\partial g}{\partial(\lambda h_r)} - h D_r \left(\Lambda[\lambda h] \frac{\partial g}{\partial(\lambda h_{rr})} \right) + h_r \left(\Lambda[\lambda h] \frac{\partial g}{\partial(\lambda h_{rr})} \right). \tag{7.66}$$

Equation (7.66) gives

$$S^r\left[h, \Lambda[h_{[\lambda]}]; g[h_{[\lambda]}]\right] = -\frac{4}{3} \lambda^3 h^3 h_r [c_1 r + c_2 r \ln r] + \frac{c_2}{3} \lambda^3 h^4, \tag{7.67}$$

where we have used

$$\Lambda[h_{[\lambda]}] = \Lambda[\lambda h] = c_1 r + c_2 r \ln r,$$

$$g[h_{[\lambda]}] = g[\lambda h] = \lambda^4 \left[-\frac{h^3 h_r}{3r} - h^2 h_r^2 - \frac{h^3 h_{rr}}{3} \right]. \tag{7.68}$$

Equation (7.64) yields

$$S^r\left[h, g[h_{[\lambda]}] - \lambda g[h]; \Lambda[h_{[\lambda]}]\right]$$

$$= h \left(g[h_{[\lambda]}] - \lambda g[h] \right) \frac{\partial}{\partial \lambda h_r} (c_1 r + c_2 r \ln r) = 0. \tag{7.69}$$

Equation (7.62) with (7.67) and (7.69) gives

$$T^2 = \left(-\frac{4}{3} h^3 h_r [c_1 r + c_2 r \ln r] + \frac{c_2}{3} h^4 \right) \int_0^1 \lambda^3 d\lambda, \tag{7.70}$$

which yields T^2 in (7.18).

7.3 System of equations for the two-dimensional laminar jet

The boundary layer equations governing the flow in the two-dimensional laminar jet are (3.1) and (3.2).

Partial Lagrangian approach:

We will derive the conserved vectors for the system of equations (3.1) and (3.2) by the partial Lagrangian approach. The partial Lagrangian

$$L = \frac{1}{2}w_y^2 - \frac{1}{2}\nu u_y^2, \quad \text{where } v = w_y(x,y) \tag{7.71}$$

for the system of equations (3.1) and (3.2) reduces the system to

$$\frac{\delta L}{\delta u} = uu_x + w_y u_y, \quad \frac{\delta L}{\delta w} = u_x. \tag{7.72}$$

In equation (7.72), $\delta/\delta w$ from (2.3) is given by

$$\frac{\delta}{\delta w} = \frac{\partial}{\partial w} - D_x\frac{\partial}{\partial w_x} - D_y\frac{\partial}{\partial w_y} + D_x^2\frac{\partial}{\partial w_{xx}} + D_xD_y\frac{\partial}{\partial w_{xy}} + D_y^2\frac{\partial}{\partial w_{yy}} - \cdots, \tag{7.73}$$

and $\delta/\delta u$ is defined in (3.16). The total derivative operators D_x and D_y are defined by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + w_x\frac{\partial}{\partial w} + u_{xx}\frac{\partial}{\partial u_x} + w_{xx}\frac{\partial}{\partial w_x} + u_{xy}\frac{\partial}{\partial u_y} + w_{xy}\frac{\partial}{\partial w_y} + \cdots, \tag{7.74}$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + w_y\frac{\partial}{\partial w} + u_{yy}\frac{\partial}{\partial u_y} + w_{yy}\frac{\partial}{\partial w_y} + u_{yx}\frac{\partial}{\partial u_x} + w_{yx}\frac{\partial}{\partial w_x} + \cdots. \tag{7.75}$$

The partial Noether symmetry determining equation is (by (6.21))

$$X^{[1]}L + L(D_x\xi^1 + D_y\xi^2) = D_xB^1(x,y,u,w) + D_yB^2(x,y,u,w)$$
$$+(\eta^1 - \xi^1 u_x - \xi^2 u_y)\frac{\delta L}{\delta u} + (\eta^2 - \xi^1 w_x - \xi^2 w_y)\frac{\delta L}{\delta w}, \tag{7.76}$$

where $\xi^i = \xi^i(x,y,u,w)$, $\eta^\alpha = \eta^\alpha(x,y,u,w)$ and $B^i = B^i(x,y,u,w)$. The operator $X^{[1]}$ is given by (invoke (2.5) upto first-order derivatives together with (2.6))

$$X^{[1]} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial w} + \zeta^1_x \frac{\partial}{\partial u_x} + \zeta^2_x \frac{\partial}{\partial w_x} + \zeta^1_y \frac{\partial}{\partial u_y} + \zeta^2_y \frac{\partial}{\partial w_y}, \quad (7.77)$$

where

$$\zeta^1_x = D_x\eta^1 - u_x D_x \xi^1 - u_y D_x \xi^2,\ \zeta^1_y = D_y \eta^1 - u_x D_y \xi^1 - u_y D_y \xi^2,$$

$$\zeta^2_x = D_x\eta^2 - w_x D_x \xi^1 - w_y D_x \xi^2,\ \zeta^2_y = D_y \eta^2 - w_x D_y \xi^1 - w_y D_y \xi^2. \quad (7.78)$$

Equation (7.76) with L given by (7.71) yields

$$\begin{aligned}
&w_y[\eta^2_y + u_y\eta^2_u + w_y\eta^2_w - w_x(\xi^1_y + u_y\xi^1_u + w_y\xi^1_w) - w_y(\xi^2_y + u_y\xi^2_u + w_y\xi^2_w)] \\
&-\nu u_y[\eta^1_y + u_y\eta^1_u + w_y\eta^1_w - u_x(\xi^1_y + u_y\xi^1_u + w_y\xi^1_w) - u_y(\xi^2_y + u_y\xi^2_u + w_y\xi^2_w)] \\
&+(\frac{{w_y}^2}{2} - \frac{\nu}{2}{u_y}^2)[\xi^1_x + u_x\xi^1_u + w_x\xi^1_w + \xi^2_x + u_x\xi^2_u + w_x\xi^2_w] \\
&= B^1_x + u_x B^1_u + w_x B^1_w + B^2_y + u_y B^2_u + w_y B^2_w \\
&+(\eta^1 - u_x\xi^1 - u_y\xi^2)(uu_x + w_y u_y) + (\eta^2 - w_x\xi^1 - w_y\xi^2)u_x. \qquad (7.79)
\end{aligned}$$

Separation by u_x^2 and $u_x w_y$ gives

$$\xi^1 = 0,\ \xi^2 = 0, \quad (7.80)$$

and equation (7.79) reduces to

$$w_y[\eta^2_y + u_y\eta^2_u + w_y\eta^2_w] - \nu u_y[\eta^1_y + u_y\eta^1_u + w_y\eta^1_w]$$

$$= B^1_x + u_x B^1_u + w_x B^1_w + B^2_y + u_y B^2_u + w_y B^2_w + \eta^1(uu_x + w_y u_y) + \eta^2 u_x. \quad (7.81)$$

Separation of (7.81) by derivatives of u and w gives rise to the following overdetermined system of equations:

$$u_y^2: \quad \eta^1_u = 0, \quad (7.82)$$

$$w_y^2: \quad \eta^2_w = 0, \quad (7.83)$$

$$w_y u_y: \quad \eta^2_u - \nu\eta^1_w = \eta^1, \quad (7.84)$$

$$u_x: \quad u\eta^1 + \eta^2 + B^1_u = 0, \tag{7.85}$$

$$w_x: \quad B^1_w = 0, \tag{7.86}$$

$$u_y: \quad -\nu\eta^1_y = B^2_u, \tag{7.87}$$

$$w_y: \quad \eta^2_y = B^2_w, \tag{7.88}$$

$$1: \quad B^1_x + B^2_y = 0. \tag{7.89}$$

Equations (7.82), (7.83) and (7.86) yield

$$\eta^1 = \eta^1(x, y, w),\ \eta^2 = \eta^2(x, y, u),\ B^1 = B^1(x, y, u). \tag{7.90}$$

Differentiation of (7.85) with respect to w yields $\eta^1_w = 0$ and then from equations (7.84)-(7.85), we obtain

$$\eta^1 = A(x, y),\ \eta^2 = uA(x, y) + B(x, y),\ B^1 = -u^2A - uB + C(x, y). \tag{7.91}$$

Substitution of (7.91) into equations (7.87)-(7.88) gives

$$A_y = 0,\ B^2 = wB_y + D(x, y). \tag{7.92}$$

Using equations (7.91) and (7.92) in (7.89) results in

$$u^2A_x + uB_x - wB_{yy} - C_x - D_y = 0, \tag{7.93}$$

which gives

$$A = c_1,\ B = c_2y + c_3,\ C_x + D_y = 0. \tag{7.94}$$

Therefore

$$\xi^1 = 0,\ \xi^2 = 0,\ \eta^1 = c_1,\ \eta^2 = c_1u + c_2y + c_3,$$

$$B^1 = -c_1u^2 - c_2yu - c_3u + C(x, y),\ B^2 = c_2w + D(x, y), C_x + D_y = 0. \tag{7.95}$$

We can set $C = D = 0$ as they contribute to the trivial part of the conservation law. The components of the first-order partial Noether conserved vector are, by (6.4),

$$T^1 = B^1 - \xi^1L - \left(\eta^1 - \xi^1u_x - \xi^2u_y\right)\frac{\partial L}{\partial u_x} - \left(\eta^2 - \xi^1w_x - \xi^2w_y\right)\frac{\partial L}{\partial w_x}, \tag{7.96}$$

$$T^2 = B^2 - \xi^2 L - \left(\eta^1 - \xi^1 u_x - \xi^2 u_y\right) \frac{\partial L}{\partial u_y} - \left(\eta^2 - \xi^1 w_x - \xi^2 w_y\right) \frac{\partial L}{\partial w_y}, \quad (7.97)$$

which together with (7.95) and $v = w_y$ give the conserved vectors (3.31) and (3.32) and

$$T^1 = yu, \quad T^2 = yv - \int v dy. \quad (7.98)$$

The non-local conserved vector (7.98) is obtained due to $v = w_y$ and the reason for defining $v = w_y$ is to make the system of equations (3.1) and (3.2) a second-order system so that we can apply the partial Noether approach. It is of interest to observe that the conserved vector (7.98) was not obtained in Chapter 2 by the multiplier approach. It is a non-local conserved vector and the variational derivative approach gives multipliers only for local conserved vectors. The conserved vector (7.98) can be obtained by the variational derivative approach by defining $v = w_y$.

Variational derivative approach on solution space:

The conserved vectors for the system of equations (3.1) and (3.2) are derived by the variational derivative method on solution space. The determining equations for multipliers of the form $\Lambda_1 = \Lambda_1(x, y, u, v)$ and $\Lambda_2 = \Lambda_2(x, y, u, v)$ are

$$E_u[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u_x + v_y)] \mid_{(3.1),\,(3.2)} = 0, \quad (7.99)$$

$$E_v[\Lambda_1(uu_x + vu_y - \nu u_{yy}) + \Lambda_2(u_x + v_y)] \mid_{(3.1),\,(3.2)} = 0, \quad (7.100)$$

where E_u and E_v are the standard Euler operators defined in (3.16) and (3.17). The expansion of equations (7.99) and (7.100) gives rise to the following two equations,

$$\begin{aligned} &(2uu_x + 2vu_y)\Lambda_{1u} + u\Lambda_{1x} + uv_x\Lambda_{1v} + \Lambda_{2x} + u_x\Lambda_{2u} + v_x\Lambda_{2v} \\ &-u_x\Lambda_1 + v\Lambda_{1y} - vu_x\Lambda_{1v} + \nu\Big[\Lambda_{1yy} + 2u_y\Lambda_{1uy} - 2u_x\Lambda_{1vy} \\ &+u_y^2\Lambda_{1uu} - 2u_xu_y\Lambda_{1uv} - u_{xy}\Lambda_{1v} + u_x^2\Lambda_{1vy}\Big] = 0 \end{aligned} \quad (7.101)$$

and

$$u_x\Lambda_{2v} + u_y(\Lambda_1 - \Lambda_{2u}) - \Lambda_{2y} = 0. \tag{7.102}$$

Equations (7.101) and (7.102) have to be satisfied for all functions $u(x, y)$ and $v(x, y)$. Separating equation (7.102) with respect to derivatives of u, we obtain

$$\Lambda_1 = A_u(x, u), \quad \Lambda_2 = A(x, u), \tag{7.103}$$

and equation (7.101) reduces to

$$\nu u_y^2 A_{uuu} + 2v u_y A_{uu} + 2u u_x A_{uu} + u A_{ux} + A_x = 0. \tag{7.104}$$

Separation of equation (7.104) results in

$$\begin{aligned} u_y^2 : &\quad A_{uuu} = 0, \\ u_y : &\quad A_{uu} = 0, \\ remainder : &\quad u A_{ux} + A_x = 0, \end{aligned} \tag{7.105}$$

which finally yields

$$\Lambda_1 = A_u = d_2, \ \Lambda_2 = A = d_1 + d_2 u. \tag{7.106}$$

Notice that the multipliers in (7.106) are the same as the multipliers in (3.28) obtained by the variational derivative approach for arbitrary functions.

7.4 System of equations for the radial laminar jet

The boundary layer equations governing the flow in the radial laminar jet are (4.1) and (4.2).

Partial Lagrangian approach:

We derive the conservation laws for the system of two partial differential equations (4.1) and (4.2) by the partial Lagrangian method of Kara and Mahomed (2006). A partial Lagrangian for the system of equations (4.1) and (4.2) is

$$L = \frac{1}{2}xw_y^2 - \frac{1}{2}\nu u_y^2, \quad v = w_y(x,y). \tag{7.107}$$

The reason for defining $v = w_y(x,y)$ is to make the system of equations (4.1) and (4.2) a second-order system so that we can apply the partial Noether approach. The system of equations (4.1) and (4.2) can therefore be written as

$$\frac{\delta L}{\delta u} = uu_x + w_y u_y, \quad \frac{\delta L}{\delta w} = u + xu_x. \tag{7.108}$$

The partial Noether symmetry determining equation is, by (6.21),

$$X^{[1]}L + L(D_x\xi^1 + D_y\xi^2)$$
$$= D_xB^1 + D_yB^2 + (\eta^1 - \xi^1 u_x - \xi^2 u_y)\frac{\delta L}{\delta u} + (\eta^2 - \xi^1 w_x - \xi^2 w_y)\frac{\delta L}{\delta w}, \tag{7.109}$$

where $\xi^i = \xi^i(x,y,u,w)$, $\eta^\alpha = \eta^\alpha(x,y,u,w)$ and $B^i = B^i(x,y,u,w)$. The operator $X^{[1]}$ is given by (7.76).

Equation (7.109) with L given by (7.107) yields

$$xw_y\left[\eta^2_y + u_y\eta^2_u + w_y\eta^2_w - w_x\left(\xi^1_y + u_y\xi^1_u + w_y\xi^1_w\right) - w_y\left(\xi^2_y + u_y\xi^2_u + w_y\xi^2_w\right)\right]$$
$$-\nu u_y\left[\eta^1_y + u_y\eta^1_u + w_y\eta^1_w - u_x\left(\xi^1_y + u_y\xi^1_u + w_y\xi^1_w\right) - u_y\left(\xi^2_y + u_y\xi^2_u + w_y\xi^2_w\right)\right]$$
$$+\frac{1}{2}\xi^1 w_y^2 + \frac{1}{2}\left(xw_y^2 - \nu u_y^2\right)\left[\xi^1_x + u_x\xi^1_u + w_x\xi^1_w + \xi^2_y + u_y\xi^2_u + w_y\xi^2_w\right]$$
$$= B^1_x + u_xB^1_u + w_xB^1_w + B^2_y + u_yB^2_u + w_yB^2_w$$
$$+\left(\eta^1 - u_x\xi^1 - u_y\xi^2\right)(uu_x + w_yu_y) + \left(\eta^2 - w_x\xi^1 - w_y\xi^2\right)(u + xu_x). \tag{7.110}$$

Separation of (7.110) by u_x^2 and u_xw_y gives

$$\xi^1 = 0,\ \xi^2 = 0, \tag{7.111}$$

and equation (7.110) becomes

$$xw_y\left[\eta^2_y+u_y\eta^2_u+w_y\eta^2_w\right]-\nu u_y\left[\eta^1_y+u_y\eta^1_u+w_y\eta^1_w\right]=B^1_x+u_xB^1_u+w_xB^1_w$$

$$+B^2_y+u_yB^2_u+w_yB^2_w+\eta^1\left(uu_x+w_yu_y\right)+\eta^2\left(u+xu_x\right). \tag{7.112}$$

Separation by derivatives of u and w results in the following overdetermined system of equations:

$$u^2_y: \quad \eta^1_u=0, \tag{7.113}$$

$$w^2_y: \quad \eta^2_w=0, \tag{7.114}$$

$$w_yu_y: \quad x\eta^2_u-\nu\eta^1_w=\eta^1, \tag{7.115}$$

$$u_x: \quad u\eta^1+x\eta^2+B^1_u=0, \tag{7.116}$$

$$w_x: \quad B^1_w=0, \tag{7.117}$$

$$u_y: \quad -\nu\eta^1_y=B^2_u, \tag{7.118}$$

$$w_y: \quad x\eta^2_y=B^2_w, \tag{7.119}$$

$$1: \quad B^1_x+B^2_y+u\eta^2=0. \tag{7.120}$$

From equations (7.113), (7.114) and (7.117), we conclude that

$$\eta^1=\eta^1(x,y,w),\ \eta^2=\eta^2(x,y,u),\ B^1=B^1(x,y,u). \tag{7.121}$$

Equations (7.115) and (7.116) yield

$$\eta^1=A(x,y),\ \eta^2=\frac{u}{x}A(x,y)+B(x,y),\ B^1=-u^2A-uxB+C(x,y). \tag{7.122}$$

Making use of (7.122) in equations (7.118)-(7.119), we obtain

$$A_y=0,\ B^2=wxB_y+D(x,y). \tag{7.123}$$

Substitution of (7.122) and (7.123) into (7.120) gives

$$u^2\left(A_x-\frac{A}{x}\right)+uxB_x-wB_{yy}-C_x-D_y=0. \tag{7.124}$$

Splitting equation (7.124) according to powers of u and w, we obtain

$$A = xc_1,\ B = c_2y + c_3,\ C_x + D_y = 0, \tag{7.125}$$

and thus

$$\xi^1 = 0,\ \xi^2 = 0,\ \eta^1 = xc_1,\ \eta^2 = c_1u + c_2y + c_3,$$

$$B^1 = -c_1xu^2 - c_2xyu - c_3xu + C(x, y),$$

$$B^2 = c_2xw + D(x, y),\ C_x + D_y = 0. \tag{7.126}$$

We can set $C = D = 0$ as they only contribute to the trivial part of the conservation law.

The components of the first-order partial Noether conserved vector are given by (7.96) and (7.97). Equation (7.126) and $v = w_y$ yield the conserved vectors (4.28), (4.29) and

$$T_2^1 = xyu,\ T_2^2 = xyv - x\int vdy. \tag{7.127}$$

In Chapter 3, the conservation laws for the system of equations (4.1) and (4.2) were constructed by the multiplier approach. The conserved vector (7.127) was not obtained by the multiplier approach because it is a non-local conserved vector and the multiplier approach gives multipliers for local conserved vectors. The conserved vector (7.127) can be obtained by the multiplier approach by defining $v = w_y(x, y)$ and then computing the conservation laws for the system of equations for the velocity components.

7.5 Discussion and concluding remarks

In this section we highlight the complications and functionality of each approach and make some conclusions.

The most elementary method is the direct method which was first used in 1798 to derive conservation laws. This method involves a single partial differential equation possessing derivatives of T^i with respect to variables (x^i, u^k_m) which is the determining equation for the conserved vectors. Sometimes it is quite straightforward to solve this determining equation. But for some problems the determining equation for conserved vectors is difficult to solve and certain assumptions are made in order to obtain solutions as we did in our illustrative example. Noether in 1918 gave an elegant approach for the construction of conserved vectors for partial differential equations possessing standard Lagrangians. The limitation of this approach is that we can only construct conservation laws for differential equations having standard Lagrangians and also the corresponding Noether symmetries.

In 1962 Stuedel introduced the idea of writing the conservation law in characteristic form. The characteristics are obtained by solving $D_iT^i = \Lambda^\alpha E_\alpha$ and substitution of these characteristics which involve derivatives of T^i yields the same determining equations as are obtained in the direct method. Later, Olver discovered that the characteristics can be obtained by computing the variational derivative of $D_iT^i = \Lambda^\alpha E_\alpha$ for arbitrary $u(x_1, \cdots, x_i)$, not only for the solutions of $E_\alpha = 0$. The characteristics obtained here are explicitly in terms of (x^i, u^k_m) and each characteristic corresponds to a conserved vector. The conserved vectors can either be obtained by elementary manipulations or by considering $D_iT^i = \Lambda^\alpha E_\alpha$ as determining equation for the conserved vectors. Furthermore, it is simpler and more systematic to derive conservation laws with the help of characteristics written explicitly in terms of (x^i, u^k_m) than with characteristics involving derivatives of T^i as computed in the first case. The final form of the characteristics is the same for both methods. The characteristics are also obtained by taking the variational derivative of $D_iT^i = \Lambda^\alpha E_\alpha$ on the solution space. The system of determining equations for the characteristics is less than the previous case. The number of characteristics obtained here may be less than, equal to or more than for the variational derivative approach for arbitrary functions. The characteristic obtained sometimes may not be associated with a conservation law but to an adjoint symmetry. These points are illustrated in the examples which were considered. In the example on a nonlinear field equation describing the relaxation to a Maxwellian distribution more characteristics were obtained but the extra characteristic belong to an adjoint symmetry. For the laminar two-dimensional jet flow the number of characteristics calculated for arbitrary functions and on the solution space was the same.

There is a fundamental relation between symmetries and conservation laws, both for the Euler-Lagrange equations and general equations without Lagrangians. The conservation laws can be computed with the joint conditions of the direct method and symmetry. The difficulties of solving the determining equations obtained by the direct method are resolved by imposing the symmetries conditions. The second aspect of the symmetry conditions is that we can associate a symmetry with a known conserved vector. This symmetry is important in the derivation of group invariant solutions, for example, for jet flows and thin fluid films.

The time component of a conserved vector, for non-variational evolution equations, non-linear evolution partial differential equations that possesses extra conservation laws and for integrable partial differential equations, can be computed directly by Bluman's direct construction formulae for given characteristics (multipliers). However, it is still necessary to solve $D_iT^i = \Lambda^\alpha E_\alpha$ for the spatial component of the conserved vector. Also all local conservation laws for partial differential equations (single or system) which can be expressed in a standard Cauchy-Kovalevskaya form can be computed by direct construction formulae. The formulae which correspond to the spatial coordinate are complicated and difficult to apply. These formulae are restricted to partial differential equations which can be expressed in a standard Cauchy-Kovalevskaya form.

The Noether approach for the system and its adjoint gives the conservation laws for the system and its adjoint. To obtain the conserved vectors for the system, one has to compute the solution of the adjoint equation. It is often quite difficult to derive the solution of the adjoint equation.

The failure of the Noether approach for non-variational problems motivated the idea of the partial Noether approach. This approach considers the symmetries of the partial Lagrangian. Their corresponding conserved vectors are determined by a formula. The partial Noether approach works in the same way as the Noether approach for differential equations with or without standard Lagrangians. This method is also applicable to scalar evolution differential equations.

Chapter 8

Group invariants solution for two-dimensional and radial free jets

8.1 Introduction

Schlichting (1933) was the first to apply laminar boundary layer theory to the two-dimensional free jet. He solved the boundary value problem by transforming the system of two partial differential equations in terms of velocity components to a single third-order partial differential equation in terms of the stream function. A similarity solution was derived. The third-order partial differential equation was transformed to a third-order ordinary differential equation in the similarity variable which was solved numerically. Later, Bickley (1937) solved the third-order ordinary differential equation analytically. The standard procedure for solving the boundary value problems of laminar jets is discussed in the texts by Rosenhead (1963), Schlichting (1968), Schlichting and Gersten (2000). A similarity solution is derived assuming a certain form for the stream function. Later, Mason (2002) solved the third-order partial differential equation for the stream function for the two-dimensional free jet, using a linear combination of Lie point symmetries and showed that the similarity solution derived by Schlichting (1933) and Bickley (1937) is a group invariant solution. The problem of the radial jet was first introduced by Squire (1955). The similarity solution for the radial free jet was derived by Riley 1962a and Schwarz (1963). The group invariant solution for the system of equations in terms of the velocity components for both the two-dimensional and radial free jet has not been derived.

The Lie point symmetries are calculated for two-dimensional and radial free jets and then using the approach introduced by Kara and Mahomed (2000), we find the symmetry associated with the conserved vector which is used to establish the conserved quantity. This symmetry yields the group invariant solution for the system of equations for the free jets.

An outline of the Chapter is as follows. In Section 8.2, the Lie point symmetries and the group invariant solution for the system of equations for the two-dimensional free jet are derived. In Section 8.3, we derive the Lie point symmetries and the group invariant solution for the radial free jet. In Section 8.4, the group invariant solution for the radial and two-dimensional free jets are compared. Finally, in Section 8.5 the conclusions are summarized.

8.2 Two-dimensional free jet

The flow in a two-dimensional free jet is governed by equations (3.1) and (3.2). The boundary conditions for the two-dimensional free jet are (3.59) and (3.60) and conserved quantity is given in (3.64).

8.2.1 Lie point symmetries for two-dimensional jet

The system of equations (3.1) and (3.2) is of the form

$$F(u, v, u_x, u_y, u_{yy}) = 0,$$

$$G(x, u, u_x, v_y) = 0, \tag{8.1}$$

where

$$F = uu_x + vu_y - \nu u_{yy},$$

$$G = u_x + v_y. \tag{8.2}$$

The Lie point symmetry generator

$$X = \xi^1(x, y, u, v)\frac{\partial}{\partial x} + \xi^2(x, y, u, v)\frac{\partial}{\partial y} + \eta^1(x, y, u, v)\frac{\partial}{\partial u} + \eta^2(x, y, u, v)\frac{\partial}{\partial v}, \tag{8.3}$$

of the system (3.1)-(3.2) is derived by solving

$$X^{[2]}F \mid_{(F=0,\ G=0)} = 0,$$

$$X^{[2]}G \mid_{(F=0,\ G=0)} = 0, \tag{8.4}$$

where

$$X^{[2]} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \zeta^1_x\frac{\partial}{\partial u_x} + \zeta^2_x\frac{\partial}{\partial v_x} + \zeta^1_y\frac{\partial}{\partial u_y} + \zeta^2_y\frac{\partial}{\partial v_y}$$

$$+\zeta^1_{xx}\frac{\partial}{\partial u_{xx}} + \zeta^2_{xx}\frac{\partial}{\partial v_{xx}} + \zeta^1_{xy}\frac{\partial}{\partial u_{xy}} + \zeta^2_{xy}\frac{\partial}{\partial v_{xy}} + \zeta^1_{yy}\frac{\partial}{\partial u_{yy}} + \zeta^2_{yy}\frac{\partial}{\partial v_{yy}}, \tag{8.5}$$

is the second prolongation of the operator X with

$$\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_s D_i(\xi^s), \tag{8.6}$$

$$\zeta^\alpha_{ij} = D_j(\zeta^\alpha_i) - u^\alpha_{is} D_j(\xi^s), \tag{8.7}$$

and the total derivative operators $D_1 = D_x$ and $D_2 = D_y$ are given in equation (3.12) and (3.13). The system (8.4) is separated according to the derivatives of u and v and an overdetermined system of partial differential equations for the unknown coefficients ξ^1, ξ^2, η^1 and η^2 is obtained. Alternatively this system can also be obtained from the computer package YaLie by Diaz. The YaLie package yields the following determining equations:

$$\xi^1_y = 0,\ \xi^1_u = 0,\ \xi^1_v = 0, \tag{8.8}$$

$$\xi^2_u = 0,\ \xi^2_v = 0, \tag{8.9}$$

$$\eta^1_v = 0,\ \eta^1_{uu} = 0, \tag{8.10}$$

$$v\eta^1_y - \nu\eta^1_{yy} + u\eta^1_x = 0, \tag{8.11}$$

$$\eta^1 + 2u\xi^2_y - u\xi^1_x = 0, \tag{8.12}$$

$$\eta^2 + v\xi^2_y - 2\nu\eta^1_{uy} + \nu\xi^2_{yy} - u\xi^2_x = 0, \tag{8.13}$$

$$\eta^1_x + \eta^2_y = 0, \tag{8.14}$$

$$\eta^1_u - \eta^2_v + \xi^2_y - \xi^1_x = 0, \tag{8.15}$$

$$\eta^2_u - \xi^2_x = 0. \tag{8.16}$$

Equations (8.8)- (8.10) gives

$$\xi^1 = A(x),\ \xi^2 = B(x,y),\ \eta^1 = uC(x,y) + D(x,y). \tag{8.17}$$

Substitution of (8.17) into (8.12)- (8.13) yields

$$C(x,y) = A' - 2B_y,\ D(x,y) = 0,\ \eta^2 = -vB_y - 5\nu B_{yy} + uB_x. \tag{8.18}$$

From (8.14), we have

$$B(x,y) = yE(x) + k(x),\ A'(x) = E(x) + c_2. \tag{8.19}$$

Equations (8.15) and (8.16) are satisfied. Equation (8.11) gives

$$E(x) = c_1. \tag{8.20}$$

Therefore

$$A(x) = (c_1 + c_2)x + c_3,\ B(x,y) = c_1y + k(x),\ C(x,y) = c_2 - c_1, \tag{8.21}$$

where c_1, c_2 and c_3 are constants and $k(x)$ is an arbitrary function. Hence

$$\xi^1 = (c_1+c_2)x+c_3,\ \xi^2 = c_1 y+k(x),\ \eta^1 = (c_2-c_1)u,\ \eta^2 = -c_1 v+uk'(x). \quad (8.22)$$

The Lie point symmetry generator for system (3.1)-(3.2) is

$$X = [(c_1 + c_2)\, x + c_3]\, \frac{\partial}{\partial x} + [c_1 y + k(x)]\, \frac{\partial}{\partial y} + (c_2 - c_1)\, u \frac{\partial}{\partial u}$$
$$+ [-c_1 v + uk'(x)]\, \frac{\partial}{\partial v}. \quad (8.23)$$

The conserved vector (3.32) was used to derive the conserved quantity for the two-dimensional free jet. The Lie point symmetry associated with the conserved vector (3.32) can be deduced using the condition (6.8) and that symmetry will give the group invariant solution for the two-dimensional free jet. It is found that (8.23) is associated with (3.32) provided $c_2 = \frac{1}{2}c_1$, that is, provided

$$X = \left[\frac{3}{2}c_1 x + c_3\right] \frac{\partial}{\partial x} + [c_1 y + k(x)]\, \frac{\partial}{\partial y} - \frac{c_1 u}{2} \frac{\partial}{\partial u} + [-c_1 v + uk'(x)]\, \frac{\partial}{\partial v}. \quad (8.24)$$

The Lie point symmetry (8.24) generates the group invariant solution for the two-dimensional free jet.

8.2.2 Group invariant solution for two-dimensional free jet

Schlichting (1933) derived the numerical solution, Bickley (1937) found the analytical solution and Mason (2002) established the group invariant solution for the two-dimensional free jet. In these papers the third-order partial differential equation (3.10) for the stream function was solved. We will construct the group invariant solution for the two-dimensional free jet by directly solving the system (3.1)-(3.2).

Now, $u = U(x, y)$ and $v = V(x, y)$ is the group invariant solution of the system of equations (3.1) and (3.2) if

$$X(u - U(x, y)) \mid_{u=U} = 0, \quad (8.25)$$

$$X(v - V(x,y))\mid_{v=V}= 0, \tag{8.26}$$

where X is given by (8.24). Equation (8.25) can be rewritten as

$$\left(\frac{3}{2}c_1 x + c_3\right) U_x + (c_1 y + k(x))\, U_y = -\frac{c_1 U}{2}, \tag{8.27}$$

Equation (8.27) is quasi-linear first order partial differential equation for $U(x,y)$. Two independent solutions of the differential equations of the characteristic curves are

$$\frac{y}{(x+\frac{2c_3}{3c_1})^{2/3}} - K(x) = a_1, \tag{8.28}$$

$$U(x+\frac{2c_3}{3c_1})^{1/3} = a_2, \tag{8.29}$$

where

$$K(x) = \frac{2}{3c_1}\int^x \frac{k(x)}{(x+\frac{2c_3}{3c_1})^{5/3}}dx \tag{8.30}$$

and a_1 and a_2 are constants. The general solution of equation (8.27) is $a_2 = g(a_1)$ and for $u = U(x,y)$ is of the form

$$u = (x+\frac{2c_3}{3c_1})^{-1/3}g(\chi), \quad \chi = \frac{y}{(x+\frac{2c_3}{3c_1})^{2/3}} - K(x). \tag{8.31}$$

The conserved quantity (3.64) becomes

$$J = 2\rho\int_{-K(x)}^{\infty} g^2(\chi)d\chi, \tag{8.32}$$

and is independent of x provided $K(x)$ is a constant. We choose the constant to be zero and therefore $\chi = 0$ corresponds to $y = 0$. To make $K = 0$ we choose $k(x) = 0$. We choose $c_3 = 0$ to ensure that the singularity $u(x,0) = \infty$ occurs at the orifice $x = 0$. Therefore equation (8.31) reduces to

$$u = x^{-\frac{1}{3}}g(\chi),\ \chi = \frac{y}{x^{2/3}}. \tag{8.33}$$

Now, equation (8.26) yields

$$\frac{3}{2}xV_x + yV_y = -V, \tag{8.34}$$

and the solution of (8.34) for $v = V(x, y)$ is

$$v = x^{-\frac{2}{3}}h(\chi). \tag{8.35}$$

The boundary conditions (3.59) and (3.60) on $u(x, y)$ and $v(x, y)$ give the following boundary conditions on $g(\chi)$ and $h(\chi)$:

$$h(0) = 0, \quad \frac{dg}{d\chi}(0) = 0, \quad g(\pm\infty) = 0. \tag{8.36}$$

The conserved quantity (8.32) becomes

$$J = 2\rho \int_0^\infty g^2 d\chi. \tag{8.37}$$

Substitution of (8.33) and (8.35) in equations (3.1) and (3.2) yields

$$\nu\frac{d^2g}{d\chi^2} + \left(\frac{2}{3}\chi g - h\right)\frac{dg}{d\chi} + \frac{1}{3}g^2 = 0 \tag{8.38}$$

and

$$\frac{2}{3}\chi\frac{dg}{d\chi} + \frac{1}{3}g - \frac{dh}{d\chi} = 0. \tag{8.39}$$

Notice that equation (8.39) can be rewritten as

$$\frac{dh}{d\chi} = \frac{2}{3}\chi^{1/2}\frac{d}{d\chi}(g\chi^{1/2}), \tag{8.40}$$

which on integrating from 0 to χ and using the boundary conditions (8.36) yields

$$h(\chi) = \frac{2}{3}g\chi - \frac{1}{3}\int_0^\chi g d\chi. \tag{8.41}$$

Substitution of (8.41) in (8.38) gives rise to the third order ordinary differential equation

$$\frac{d^3f}{d\chi^3} + \frac{1}{3\nu}\frac{d}{d\chi}\left(f\frac{df}{d\chi}\right) = 0, \tag{8.42}$$

where

$$f(\chi) = \int_0^\chi g d\chi. \tag{8.43}$$

The boundary conditions (8.36) and conserved quantity (8.37), in terms of $f(\chi)$ becomes

$$f(0) = 0, \quad \frac{d^2 f}{d\chi^2}(0) = 0, \quad \frac{df}{d\chi}(\pm\infty) = 0, \tag{8.44}$$

and

$$J = 2\rho \int_0^\infty \left(\frac{df}{d\chi}\right)^2 d\chi. \tag{8.45}$$

For completeness we outline briefly the solution of the boundary value problem (8.42)-(8.44). Integrating (8.42) with respect to χ, we obtain

$$\frac{d^2 f}{d\chi^2} + \frac{1}{3\nu} f \frac{df}{d\chi} = c_1, \tag{8.46}$$

where c_1 is a constant. The boundary conditions (8.44) at $\chi = 0$ yield $c_1 = 0$ only if

$$f(0)\frac{df(0)}{d\chi} = 0. \tag{8.47}$$

We choose $c_1 = 0$ and check at the end that the solution obtained satisfies (8.47). Integration of (8.46) results in a variable separable differential equation which yields

$$f(\chi) = \alpha \tanh\left(\frac{\alpha}{6\nu}\chi\right), \tag{8.48}$$

where α is a constant. Substitution of (8.48) in (8.45) yields α in terms of J as

$$\alpha = \left(\frac{9\nu J}{2\rho}\right)^{1/3}. \tag{8.49}$$

Substitution of equation (8.48) in (8.41) and (8.43) gives

$$g(\chi) = \frac{\alpha^2}{6\nu}\mathrm{sech}^2\left(\frac{\alpha}{6\nu}\chi\right), \tag{8.50}$$

$$h(\chi) = \frac{\alpha^2\chi}{9\nu}\mathrm{sech}^2\left(\frac{\alpha}{6\nu}\chi\right) - \frac{\alpha}{3}\tanh\left(\frac{\alpha}{6\nu}\chi\right). \tag{8.51}$$

Substitution of (8.50) and (8.51) in (8.33) and (8.35) yields

$$u = x^{-1/3}\frac{\alpha^2}{6\nu}\text{sech}^2\left(\frac{\alpha}{6\nu}\chi\right), \tag{8.52}$$

$$v = \alpha x^{-2/3}\left[\frac{2}{3}\left(\frac{\alpha\chi}{6\nu}\right)\text{sech}^2\left(\frac{\alpha}{6\nu}\chi\right) - \frac{1}{3}\tanh\left(\frac{\alpha}{6\nu}\chi\right)\right], \tag{8.53}$$

where

$$\chi = \frac{y}{x^{2/3}} \quad \text{and} \quad \alpha = \left(\frac{9\nu J}{2\rho}\right)^{1/3}, \tag{8.54}$$

where J is the given constant which describes strength of the jet. These results are in agreement with those of Mason (2002) and Bickley (1937). The Lie point symmetry that generates the group invariant solution is

$$X = \frac{3x}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{u}{2}\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{8.55}$$

Finally, we observe that (8.48) satisfies the condition (8.47).

8.3 Radial free jet

The flow in a radial free jet is governed by equations (4.1) and (4.2). The boundary conditions for the radial free jet are (3.59) and (3.60). The conserved quantity for the radial free jet is given in equation (4.48).

8.3.1 Lie point symmetries for radial jet

The YaLie package by Diaz yields the following determining equations for the system (4.1)-(4.2):

$$\xi^1_y = 0,\ \xi^1_u = 0,\ \xi^1_v = 0, \tag{8.56}$$

$$\xi^2_u = 0,\ \xi^2_v = 0, \tag{8.57}$$

$$\eta^1_v = 0,\ \eta^1_{uu} = 0, \tag{8.58}$$

$$v\eta^1_y - \nu\eta^1_{yy} + u\eta^1_x = 0, \tag{8.59}$$

$$\eta^1 + 2u\xi^2_y - u\xi^1_x = 0, \tag{8.60}$$

$$\eta^2 + v\xi^2_y - 2\nu\eta^1_{uy} + \nu\xi^2_{yy} - u\xi^2_x = 0, \tag{8.61}$$

$$\eta^1 + x\eta^1_x + x\eta^2_y - u\eta^2_v - \frac{u}{x}\xi^1 + u\xi^2_y = 0, \tag{8.62}$$

$$x\eta^1_u - x\eta^2_v + x\xi^2_y - x\xi^1_x = 0, \tag{8.63}$$

$$x(\eta^2_u - \xi^2_x) = 0. \tag{8.64}$$

Equations (8.56)- (8.58) and (8.60)- (8.61) result in

$$\xi^1 = A(x),\ \xi^2 = B(x,y),\ \eta^1 = u(A'-2B_y),\ \eta^2 = -vB_y - 5\nu B_{yy} + uB_x. \tag{8.65}$$

Equations (8.63) and (8.64) are identically satisfied by (8.65). Equations (8.59) and (8.62) finally yield

$$A(x) = c_1 x + \frac{c_2}{x^2},\ B(x,y) = \left(\frac{c_1}{2} - \frac{c_2}{x^3} + c_3\right) y + k(x), \tag{8.66}$$

where c_1, c_2 and c_3 are constants and $k(x)$ is an arbitrary function. Thus

$$\xi^1 = c_1 x + \frac{c_2}{x^2},\ \xi^2 = \left(\frac{c_1}{2} - \frac{c_2}{x^3} + c_3\right) y + k(x),$$

$$\eta^1 = -2c_3 u,\ \eta^2 = -\left(\frac{c_1}{2} - \frac{c_2}{x^3} + c_3\right) v + \frac{3c_2 uy}{x^4} + uk'(x). \tag{8.67}$$

The Lie point symmetry generator for the system (4.1)-(4.2) is

$$X = \left(c_1 x + \frac{c_2}{x^2}\right)\frac{\partial}{\partial x} + \left[\left(\frac{c_1}{2} - \frac{c_2}{x^3} + c_3\right) y + k(x)\right]\frac{\partial}{\partial y} - 2c_3 u\frac{\partial}{\partial u}$$
$$+\left[-\left(\frac{c_1}{2} - \frac{c_2}{x^3} + c_3\right) v + \frac{3c_2 uy}{x^4} + uk'(x)\right]\frac{\partial}{\partial v}. \tag{8.68}$$

We want to determine the symmetry associated with the conserved vector (4.29) which gave the conserved quantity for radial free jet. Kara and Mahomed (2000) showed that the symmetries associated with a known conserved vector can be determined by using (6.8). Equation (6.8) after substitution of (4.29) and (8.68) yields

$$T^1(c_1 - 2c_3) = 0,\ T^2(c_1 - 2c_3) = 0, \tag{8.69}$$

which is satisfied if and only if $c_1 = 2c_3$. Thus

$$X = (c_1 x + \frac{c_2}{x^2})\frac{\partial}{\partial x} + [(c_1 - \frac{c_2}{x^3})y + k(x)]\frac{\partial}{\partial y} - c_1 u\frac{\partial}{\partial u}$$

$$+[-(c_1 - \frac{c_2}{x^3})v + \frac{3c_2 uy}{x^4} + uk'(x)]\frac{\partial}{\partial v} \tag{8.70}$$

is the Lie point symmetry generator associated with the conserved vector given in (4.29).

The Lie point symmetry associated with the conserved vector which gives the conserved quantity for the radial free jet gives the group invariant solution.

8.3.2 Group invariant solution for system of equations for radial free jet

By introducing a stream function of the form (4.5) the system of equations (4.1) and (4.2) is transformed to the third-order partial differential equation (4.9). Riley (1962a) and Schwarz (1963) derived the similarity solution for the third-order partial differential equation (4.9) for the stream function ψ. We will derive the group invariant solution directly for the system of equations (4.1) and (4.2) governing the flow in a radial free jet.

Now, $u = U(x,y)$ and $v = V(x,y)$ are group invariant solutions of the system of equations (4.1) and (4.2) if

$$X(u - U(x,y)) \mid_{u=U} = 0, \tag{8.71}$$

and

$$X(v - V(x,y)) \mid_{v=V} = 0, \tag{8.72}$$

where the operator X is given in equation (8.70). Equation (8.71) yields

$$\left(c_1 x + \frac{c_2}{x^2}\right) U_x + \left[\left(c_1 - \frac{c_2}{x^3}\right) y + k(x)\right] U_y = -c_1 U. \tag{8.73}$$

Two independent solutions of the differential equations of the characteristic curves for quasi-linear first order partial differential equation (8.73), are

$$\frac{xy}{\left(x^3+\frac{c_2}{c_1}\right)^{2/3}} - K(x) = a_1, \tag{8.74}$$

$$U\left(x^3+\frac{c_2}{c_1}\right)^{1/3} = a_2, \tag{8.75}$$

where

$$K(x) = \frac{1}{c_1}\int^x \frac{x^3 k(x)}{\left(x^3+\frac{c_2}{c_1}\right)^{5/3}}dx \tag{8.76}$$

and a_1 and a_2 are constants. The general solution of equation (8.73) for $u = U(x,y)$ is of the form

$$u = \left(x^3+\frac{c_2}{c_1}\right)^{-1/3} g(\chi), \quad \chi = \frac{xy}{\left(x^3+\frac{c_2}{c_1}\right)^{2/3}} - K(x). \tag{8.77}$$

The conserved quantity, given by equation (4.48), becomes

$$J = 2\rho\int_{-K(x)}^{\infty} g^2(\chi)d\chi, \tag{8.78}$$

and is independent of x provided $K(x)$ is a constant. We choose the constant to be zero and to make $K = 0$ we choose $k(x) = 0$. Now, since $K = 0$, $\chi = 0$ corresponds to $y = 0$. The radial free jet has infinite fluid velocity at the orifice. We therefore choose $c_2 = 0$ to ensure that $u(x,0) = \infty$ at the orifice $x = 0$. Therefore equation (8.77) reduces to

$$u = \frac{1}{x}g(\chi), \quad \chi = \frac{y}{x}. \tag{8.79}$$

Now equation (8.72) gives

$$xV_x + yV_y = -V. \tag{8.80}$$

The general solution of equation (8.80) for $v = V(x,y)$ is

$$v = \frac{1}{x}h(\chi). \tag{8.81}$$

The boundary conditions (3.59), (3.60) yield the boundary conditions (8.36) on $g(\chi)$ and $h(\chi)$. Also, since $K = 0$, (8.78) reduces to (8.37). Substitution of equations (8.79) and (8.81) in the system of equations (4.1) and (4.2) gives rise to

$$\nu \frac{d^2 g}{d\chi^2} + (\chi g - h) \frac{dg}{d\chi} + g^2 = 0 \tag{8.82}$$

and

$$\chi \frac{dg}{d\chi} - \frac{dh}{d\chi} = 0. \tag{8.83}$$

Integrating (8.83) from 0 to χ and using the boundary condition $h(0) = 0$ yields

$$h(\chi) = f'\chi - f, \tag{8.84}$$

where

$$f(\chi) = \int_0^\chi g(\chi) d\chi. \tag{8.85}$$

Substitution of (8.84) and (8.85) in (8.82) yields the third-order ordinary differential equation

$$\frac{d^3 f}{d\chi^3} + \frac{1}{\nu} \frac{d}{d\chi} \left(f \frac{df}{d\chi} \right) = 0, \tag{8.86}$$

and we obtain boundary conditions (8.44) and conserved quantity (8.45). The solution of equation (8.86) subject to (8.44) is

$$f(\chi) = \alpha \tanh \left(\frac{\alpha}{2\nu} \chi \right), \tag{8.87}$$

where α is a constant and is determined in terms of J as

$$\alpha = \left(\frac{3\nu J}{2\rho} \right)^{1/3}. \tag{8.88}$$

Substitution of equation (8.87) in (8.84) and (8.85) gives

$$g(\chi) = \frac{\alpha^2}{2\nu} \operatorname{sech}^2 \left(\frac{\alpha}{2\nu} \chi \right), \tag{8.89}$$

$$h(\chi) = \frac{\alpha^2\chi}{2\nu}\text{sech}^2\Big(\frac{\alpha}{2\nu}\chi\Big) - \alpha\tanh\Big(\frac{\alpha}{2\nu}\chi\Big), \tag{8.90}$$

which together with (8.79) and (8.81) yields

$$u = \frac{1}{x}\frac{\alpha^2}{2\nu}\text{sech}^2\Big(\frac{\alpha}{2\nu}\chi\Big), \tag{8.91}$$

$$v = \frac{\alpha}{x}\Big[\frac{\alpha\chi}{2\nu}\text{sech}^2\Big(\frac{\alpha}{2\nu}\chi\Big) - \tanh\Big(\frac{\alpha}{2\nu}\chi\Big)\Big], \tag{8.92}$$

where

$$\chi = \frac{y}{x} \quad \text{and} \quad \alpha = \left(\frac{3\nu J}{2\rho}\right)^{1/3}, \tag{8.93}$$

where J is the given constant which describes strength of the jet. These results agree with those of Riley (1962a) and Schwarz (1963). The Lie point symmetry which generates the solution is from (8.70) with $c_2 = 0$ and $k(x) = 0$,

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{8.94}$$

8.4 Comparison between two-dimensional and radial free jets

The comparison between two-dimensional and radial free jets is presented in Table 8.1.

Table 8.1 shows that Prandtl's momentum boundary layer equation for the two free jets is the same. However, the continuity equation is different. For both cases the Lie point symmetry that generates the group invariant solution is a scaling symmetry. The symmetries can be related through a similarity parameter b. The value of the similarity parameter for the two-dimensional free jet given by Schlichting (1933) is $b = 2/3$ and Schwarz (1963) gives $b = 1$ for the radial free jet.

The Lie point symmetry that generates the group invariant solution can be expressed as

$$X = x\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} - cu\frac{\partial}{\partial u} - bv\frac{\partial}{\partial v}, \tag{8.95}$$

where $c = 1 - b$ for the two-dimensional free jet and $c = 2 - b$ for the radial free jet. For $b = 2/3$, $c = 1/3$ we obtain the Lie point symmetry for the two-dimensional free jet and $b = 1$, $c = 1$ gives the Lie point symmetry for the radial free jet. The x and y components of the velocity for the free jets can be written as

$$u = \frac{1}{x^c}\frac{\alpha^2}{2\nu}\mathrm{sech}^2\left(\frac{\alpha}{2\nu}\chi\right),$$

$$v = \frac{\alpha}{x^b}\left[\frac{b\alpha\chi}{2\nu}\mathrm{sech}^2\left(\frac{1}{2\nu}\chi\right) - c\tanh\left(\frac{\alpha}{2\nu}\chi\right)\right],$$

where $b = 2/3$, $c = 1/3$, $\nu \to 3\nu$ gives the velocity components for the two-dimensional free jet and $b = 1$, $c = 1$ gives the velocity components for the two-dimensional free jet. The same similarity parameters which connect the two-dimensional and radial jet flows in the similarity solution connect the two flows in the group invariant solution.

8.5 Conclusions

The Lie point symmetry generator for the system of two partial differential equations governing flow in the two-dimensional jet was derived. The Lie point symmetry associated with the conserved vector which gave the conserved quantity for the two-dimensional free jet generated the group invariant solution. For the radial free jet, we computed the Lie point symmetry generator for the system of two partial differential equations for the velocity components. Then the group invariant solution was constructed for the radial free jet using the symmetry associated with the conserved vector that was used to derive the conserved quantity for the radial free jet.

	Radial free jet	Two-dimensional free jet
System of partial differential equations	$uu_x + vu_y = \nu u_{yy}$ $(xu)_x + (xv)_y = 0$	$uu_x + vu_y = \nu u_{yy}$ $u_x + v_y = 0$
Associated symmetry	$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$	$X = \frac{3}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$
Velocity components	$u = \frac{1}{x}\frac{\alpha^2}{2\nu}\text{sech}^2\left(\frac{\alpha}{2\nu}\chi\right)$	$u = \frac{1}{x^{1/3}}\frac{\alpha^2}{6\nu}\text{sech}^2\left(\frac{\alpha}{6\nu}\chi\right)$
	$v = \frac{\alpha}{x}\left[\frac{\alpha\chi}{2\nu}\text{sech}^2\left(\frac{\alpha}{2\nu}\chi\right) - \tanh\left(\frac{\alpha}{2\nu}\chi\right)\right]$	$v = \frac{\alpha}{x^{2/3}}\left[\frac{2}{3}\left(\frac{\alpha\chi}{6\nu}\right)\text{sech}^2\left(\frac{\alpha}{6\nu}\chi\right) - \frac{1}{3}\tanh\left(\frac{\alpha}{6\nu}\chi\right)\right]$
	$\chi = \frac{y}{x}, \quad \alpha = \left(\frac{3\nu J}{2\rho}\right)^{1/3}$	$\chi = \frac{y}{x^{2/3}}, \quad \alpha = \left(\frac{9\nu J}{2\rho}\right)^{1/3}$

Table 8.1: Comparison of two-dimensional with radial free jets

The similarity parameter introduced by Schlichting (1933) and Schwarz (1963) to connect the two-dimensional and radial similarity solutions for the free jet also connects the Lie point symmetries which generate the group invariant solutions for the velocity components for the two-dimensional and radial free jet.

Chapter 9

Symmetry solutions of a third-order ordinary differential equation which arises from Prandtl boundary layer equations

9.1 Introduction

Prandtl (1904) introduced the concept of a boundary layer in large Reynolds number flows and he also showed how the Navier-Stokes equation could be simplified to yield approximate solutions. The similarity solution of Prandtl's boundary layer equation for the stream function for steady two-dimensional and radial flows with vanishing or constant mainstream velocity yields the third-order ordinary differential equation

$$\frac{d^3y}{dx^3} + By\frac{d^2y}{dx^2} + C\left(\frac{dy}{dx}\right)^2 = 0, \qquad (9.1)$$

where

$$B = \begin{cases} 1-\alpha & \text{two-dimensional} \\ 2-\alpha & \text{radial} \end{cases}, \; C = 2\alpha - 1, \tag{9.2}$$

and α is a constant determined from further conditions. Equation (9.1) arises in the study of steady flows produced by free jets, wall jets and liquid jets (two-dimensional or radial), the flow past a stretching plate and Blasius flow. The numerical solution for a free two-dimensional jet for which $\alpha = 2/3$ was obtained by Schlichting (1933) and later an analytic solution was derived by Bickley (1937). Schwarz in (1963) obtained the solution for the free radial jet for which $\alpha = 1$. The solutions for two-dimensional and radial wall jets for which $\alpha = 3/4$ (two-dimensional) and $\alpha = 5/4$ (radial) were obtained in parametric form by Glauert (1956). Riley (1962a,b) derived the solution for a radial liquid jet for which $\alpha = 2$. Later, two-dimensional flow past a stretching plate with $\alpha = 0$ was discussed by Crane (1970).

The purpose of this work is to obtain reductions and solutions of the third-order differential equations which arise from Prandtl boundary layer equations for two-dimensional and radial flows with vanishing or constant mainstream velocity using Lie symmetry methods of reduction. The invariant solutions are also derived.

The Lie point symmetry generators of (9.1) for general values of α are

$$X_1 = \partial/\partial x, \quad X_2 = x\partial/\partial x - y\partial/\partial y. \tag{9.3}$$

For special values of α three Lie point symmetry generators exist and the third-order ordinary differential equation is solved by the Lie approach as described, for example, by Ibragimov and Nucci (1994), Mahomed (2008) and Olver (1993). For $B = 0$ the third-order ordinary differential equation (9.1) describes radial and two-dimensional liquid jets and admits a solvable Lie algebra. We solve the equation by the Lie approach (Ibragimov and Nucci 1994, Mahomed 2008, Olver 1993). For $\alpha = -1$ (two-dimensional) and $\alpha = -4$ (radial), the third-order ordinary differential equation (9.1) admits a non-solvable Lie algebra and can be reduced to the Chazy equation (Chazy 1909, 1910, 1911, Clarkson and Olver 1996). Clarkson and Olver (1996) expressed the general solution of the Chazy equation as the ratio of two solutions of a hypergeometric equation. We reduce the Chazy equation by the Lie approach using the semi-canonical variables of Ibragimov and Nucci (1994). Another approach is given by Adam and Mahomed (2002).

9.2 Mathematical formulation

Prandtl's boundary layer equation for the stream function for an incompressible, steady two-dimensional flow with uniform or vanishing mainstream velocity is (Rosenhead 1963)

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \psi_{yyy}, \tag{9.4}$$

where ν is the kinematic viscosity. Using the classical Lie method of infinitesimal transformations (Olver 1993) the similarity solution for (9.4) is found to be

$$\psi(x, y) = x^{1-\alpha} F(\chi),\ \chi = \frac{y}{x^{\alpha}}. \tag{9.5}$$

The substitution of (9.5) in (9.4) yields a third-order ordinary differential equation in $F(\chi)$:

$$\nu \frac{d^3F}{d\chi^3} + (1-\alpha) F \frac{d^2F}{d\chi^2} + (2\alpha - 1) \left(\frac{dF}{d\chi} \right)^2 = 0. \tag{9.6}$$

The variable $\chi = y/x^{\alpha}$ is the similarity variable.

For radial flow, Prandtl's boundary layer equation for uniform or vanishing mainstream velocity is (Glauert 1956)

$$\frac{1}{r}\psi_z \psi_{rz} - \frac{1}{r^2}\psi_z^2 - \frac{1}{r}\psi_r \psi_{zz} = \nu \psi_{zzz}. \tag{9.7}$$

The similarity solution of (9.7) derived using the Lie method is

$$\psi(r,z) = r^{2-\alpha}F(\chi),\ \chi = \frac{z}{r^{\alpha}}, \tag{9.8}$$

which reduces (9.7) to

$$\nu\frac{d^3F}{d\chi^3} + (2-\alpha)F\frac{d^2F}{d\chi^2} + (2\alpha-1)\left(\frac{dF}{d\chi}\right)^2 = 0. \tag{9.9}$$

Equations (9.6) and (9.9) can be combined to give the following third-order ordinary differential equation:

$$\nu\frac{d^3F}{d\chi^3} + BF\frac{d^2F}{d\chi^2} + C\left(\frac{dF}{d\chi}\right)^2 = 0, \tag{9.10}$$

where B and C are defined in terms of α by (9.2) . The transformation $(\chi, F) \mapsto (x, \nu y)$ reduces (9.10) to (9.1).

9.3 Lie point symmetry generators

Equation (9.1) can be written as

$$E(y, y', y'', y''') = 0, \tag{9.11}$$

where

$$E = \frac{d^3y}{dx^3} + By\frac{d^2y}{dx^2} + C\left(\frac{dy}{dx}\right)^2. \tag{9.12}$$

The Lie point symmetry generators $X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$, are obtained from the determining equation (Bluman and Kumei 1989)

$$X^{[3]}E\mid_{E=0} = 0, \tag{9.13}$$

where $X^{[3]}$ is the third-order prolongation given by

$$X^{[3]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''} + \zeta_3\frac{\partial}{\partial y'''}, \tag{9.14}$$

with

$$\zeta_1 = D(\eta) - y'D(\xi), \quad \zeta_2 = D(\zeta_1) - y''D(\xi), \quad \zeta_3 = D(\zeta_2) - y'''D(\xi), \tag{9.15}$$

and

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''}. \tag{9.16}$$

Equation (9.13) is separated according to derivatives of y and thus the determining equations for Lie point symmetry generators are obtained. The determining equations for the Lie point symmetry generator can also be derived directly by the computer package YaLie by Diaz. The solution depends whether $B = 0$ or $B \neq 0$. The determining equations for Lie point symmetry generators $X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$, for third-order ordinary differential equation (9.1) obtained by the YaLie package by Diaz are

$$\xi_y = 0, \tag{9.17}$$

$$\eta_{yy} = 0, \tag{9.18}$$

$$B\eta + By\xi_x + 3\eta_{xy} - 3\xi_{xx} = 0, \tag{9.19}$$

$$C\eta_y + C\xi_x = 0, \tag{9.20}$$

$$2C\eta_x + By(2\eta_{xy} - \xi_{xx}) + 3\eta_{xxy} - \xi_{xxx} = 0, \tag{9.21}$$

$$By\eta_{xx} + \eta_{xxx} = 0. \tag{9.22}$$

Equations (9.17) and (9.18) imply

$$\xi = f(x), \eta = yg(x) + h(x). \tag{9.23}$$

The solution of equations (9.19)-(9.22) depends whether $B = 0$ or $B \neq 0$.

Case I: $B = 0$

For this case, $\alpha = 1$, $C = 1$ for two-dimensional flow and $\alpha = 2$, $C = 3$ for radial flow. Since $C \neq 0$ for both flows, (9.20) gives $g(x) = -f'(x)$ and then (9.19) yields

$$f''(x) = 0. \tag{9.24}$$

Hence, we obtain

$$f(x) = c_1 + c_2 x, \; g(x) = -c_2, \tag{9.25}$$

and thus

$$\xi = c_1 + c_2 x, \; \eta = -c_2 y + h(x). \tag{9.26}$$

Equation (9.21) together with (9.26) gives $h'(x) = 0$ and therefore

$$h(x) = c_3. \tag{9.27}$$

Equation (9.22) is identically satisfied. We obtain

$$\xi = c_1 + c_2 x, \; \eta = -c_2 y + c_3. \tag{9.28}$$

Thus

$$X = (c_1 + c_2 x)\frac{\partial}{\partial x} + (-c_2 y + c_3)\frac{\partial}{\partial y}, \tag{9.29}$$

where c_1, c_2 and c_3 are arbitrary constants.

Case I: $B \neq 0$

Substitution of (9.23) in (9.19) yields

$$By[g(x) + f'(x)] + Bh(x) + 3g'(x) - 3f''(x) = 0. \tag{9.30}$$

Equation (9.30) after separation gives

$$g(x) = -f'(x), \; h(x) = \frac{6}{B}f''(x) \tag{9.31}$$

and hence

$$\xi = f(x), \; \eta = -yf' + \frac{6}{B}f''(x). \tag{9.32}$$

Equation (9.20) is identically satisfied. From (9.21) and (9.22) we conclude

$$(2C + 3B)f''(x) = 0, \tag{9.33}$$

$$f'''(x) = 0. \tag{9.34}$$

The solution of equations (9.33) and (9.34) depends whether $2C + 3B = 0$ or $2C+3B \neq 0$. But $2C+3B = \alpha+1$ for two-dimensional flow and $2C+3B = \alpha+4$ for the radial flow. When $2C + 3B \neq 0$, then $\alpha \neq -1,\ B \neq 2$ for two-dimensional flow and $\alpha \neq -4,\ B \neq 6$ for the radial flow. Thus equations (9.33) and (9.34) give

$$f(x) = c_1 + c_2 x \tag{9.35}$$

and therefore

$$\xi = c_1 + c_2 x,\ \eta = -c_2 y. \tag{9.36}$$

For the case $2C + 3B = 0$, we have $\alpha = -1,\ B = 2,\ C = -3$ for two-dimensional flow and $\alpha = -4,\ B = 6,\ C = -9$ for radial flow. Equations (9.33) and (9.34) for this case yield

$$f(x) = c_1 + c_2 x + c_3 x^2, \tag{9.37}$$

and thus

$$\xi = c_1 + c_2 x + c_3 x^2,\ \eta = -[2c_3 x + c_2]y + \frac{12}{B}c_3. \tag{9.38}$$

The Lie point symmetries and Lie algebra for both flows are summarized in Table 9.1.

Two-dimensional	Radial flow	Lie point symmetries for both flows	Lie algebra
$B = 0$	$B = 0$	$X_1 = \frac{\partial}{\partial x}$	$[X_1, X_2] = 0$
$\alpha = 1$	$\alpha = 2$	$X_2 = \frac{\partial}{\partial y}$	$[X_1, X_3] = X_1$
		$X_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$	$[X_2, X_3] = -X_2$
$B = 2$	$B = 6$	$X_1 = \frac{\partial}{\partial x}$	$[X_1, X_2] = X_1$
$\alpha = -1$	$\alpha = -4$	$X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$	$[X_1, X_3] = 2X_2$
		$X_3 = x^2\frac{\partial}{\partial x} + (\frac{12}{B} - 2xy)\frac{\partial}{\partial y}$	$[X_2, X_3] = X_3$
$B \neq 0$, $\alpha \neq 1$	$B \neq 0$, $\alpha \neq 2$	$X_1 = \frac{\partial}{\partial x}$	$[X_1, X_2] = X_1$
$B \neq 2$, $\alpha \neq -1$	$B \neq 6$, $\alpha \neq -4$	$X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$	

Table 9.1: Lie point symmetries for $B = 0$ and $B \neq 0$

9.4 Symmetry solutions

9.4.1 *Case I:* $B = 0$ *(two-dimensional and radial)*

For this case, $\alpha = 1$, $C = 1$ for two-dimensional flow and $\alpha = 2$, $C = 3$ for radial flow. Using the transformation $y \rightarrow \frac{3}{C}Y$ in (9.1), we obtain

$$\frac{d^3Y}{dx^3} + 3\left(\frac{dY}{dx}\right)^2 = 0. \tag{9.39}$$

Equation (9.39) applies for both radial and two-dimensional liquid jets (Riley 1962a,b). Using $y \rightarrow \frac{3}{C}Y$ in the Lie point symmetries for this case in Table 9.1, we obtain

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{C}{3}\frac{\partial}{\partial Y}, \quad X_3 = x\frac{\partial}{\partial x} - Y\frac{\partial}{\partial Y}. \tag{9.40}$$

The commutators of the Lie point symmetry generators in (9.40) are

$$[X_1, X_2] = 0, \; [X_1, X_3] = X_1, \; [X_2, X_3] = -X_2. \tag{9.41}$$

Thus (9.39) admits a solvable Lie algebra L_3 and can be solved by the Lie approach as outlined, for example, by Ibragimov and Nucci (1994), Mahomed (2008) and Olver (1993).

Consider the subalgebra $L_2 =< X_1, X_2 >$ which spans an Abelian ideal. Invariants of X_2 are obtained as follows:

$$\frac{dx}{0} = \frac{3dY}{C} = \frac{dY'}{0} = \frac{dY''}{0}, \tag{9.42}$$

which gives

$$u = x, \; v = Y', \; \frac{dv}{du} = Y'' = w. \tag{9.43}$$

Equation (9.39) in the new variables u and v becomes

$$\frac{d^2v}{du^2} + 3v^2 = 0. \tag{9.44}$$

Now, X_1 in the new variables transforms to

$$\tilde{X}_1 = X_1^{[2]} u \frac{\partial}{\partial u} + X_1^{[2]} v \frac{\partial}{\partial v} + X_1^{[2]} w \frac{\partial}{\partial w}, \tag{9.45}$$

and thus we have

$$\tilde{X}_1 = \frac{\partial}{\partial u}. \tag{9.46}$$

The generator $\tilde{X}_1$ is the symmetry of equation (9.44). The invariants of $\tilde{X}_1$ are

$$\frac{du}{1} = \frac{dv}{0} = \frac{dw}{0}, \tag{9.47}$$

which yields

$$s = v = Y', \; t = w = Y'', \tag{9.48}$$

and forms a basis of differential invariants of the subalgebra $L_2 =< X_1, X_2 >$. Equation (9.44) reduces to the following first-order ordinary differential equation:

$$\frac{dt}{ds} + \frac{3s^2}{t} = 0. \tag{9.49}$$

Now, $X_3 \to \tilde{X}_3$ and

$$\tilde{X}_3 = X_3^{[2]} s \frac{\partial}{\partial s} + X_3^{[2]} t \frac{\partial}{\partial t}, \tag{9.50}$$

where

$$X_3^{[2]} = x\frac{\partial}{\partial x} - Y\frac{\partial}{\partial Y} - 2Y'\frac{\partial}{\partial Y'} - 3Y''\frac{\partial}{\partial Y''}. \tag{9.51}$$

From (9.50), we obtain

$$\tilde{X}_3 = -2s\frac{\partial}{\partial s} - 3t\frac{\partial}{\partial t} \tag{9.52}$$

which is the symmetry of reduced equation (9.49). The Invariant of $\tilde{X}_3$ is the solution of characteristic equation

$$\frac{ds}{2s} = \frac{dt}{3t}, \tag{9.53}$$

which gives

$$T = \frac{t}{s^{3/2}}. \tag{9.54}$$

Equation (9.49) is in variables separable form, we can either solve it directly or solve by using the invariant T of $\tilde{X}_3$. The solution of (9.49) is

$$t = \left[2(c_1 - s^3)\right]^{\frac{1}{2}}, \tag{9.55}$$

which can be expressed in the original variables as

$$Y'' = \left[2(c_1 - Y'^3)\right]^{\frac{1}{2}}. \tag{9.56}$$

Equation (9.56) gives

$$\int \frac{ds}{\sqrt{c_1 - s^3}} = \sqrt{2}x + c_2,\ Y' = s. \tag{9.57}$$

To solve the left hand side of equation (9.57), introduce $c_1 - s^3 = z$. Then

$$\int \frac{ds}{\sqrt{c_1 - s^3}} = -\frac{1}{3}\int \frac{dz}{\sqrt{z}(c_1 - z)^{2/3}}$$

$$= -\frac{2}{3c_1^{\frac{2}{3}}}\sqrt{z} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; \frac{z}{c_1}],$$

$$= -\frac{2}{3c_1^{\frac{2}{3}}}(c_1 - s^3)^{\frac{1}{2}} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1 - \frac{s^3}{c_1}]. \tag{9.58}$$

Equation (9.57) together with (9.58) gives

$$-\frac{2}{3c_1^{\frac{2}{3}}}(c_1 - s^3)^{\frac{1}{2}} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1 - \frac{s^3}{c_1}] = \sqrt{2}x + c_2, \tag{9.59}$$

where ${}_2F_1$ is the Hypergeometric function of first kind and c_1, c_2 are arbitrary constants.

For both the radial and two-dimensional liquid jets the boundary conditions are

$$Y(0) = 0, \quad Y'(0) = 0, \quad Y''(1) = 0 \tag{9.60}$$

and it is also required that $Y'(1) = 1$. The conditions $Y'(1) = 1$ and $Y''(1) = 0$ on (9.56) give $c_1 = 1$. Using $Y(0) = 0$ and $Y'(0) = s = 0$ in (9.59), we obtain

$$c_2 = -\frac{2}{3}\,{}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1]. \tag{9.61}$$

Equation (9.59), after substitution of c_1 and c_2, finally yields

$$x = \frac{\sqrt{2}}{3}\left({}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1] - (1 - s^3)^{\frac{1}{2}} \times {}_2F_1[\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; 1 - s^3]\right), \tag{9.62}$$

which can be used to tabulate the values of x for given values of the parameter $s = Y'$. The scaled velocity profiles for radial and two-dimensional liquid jets are the same and are shown in Figure 7.1. Figure 7.1 agrees with the velocity profile of a radial jet given by Riley (1962a,b).

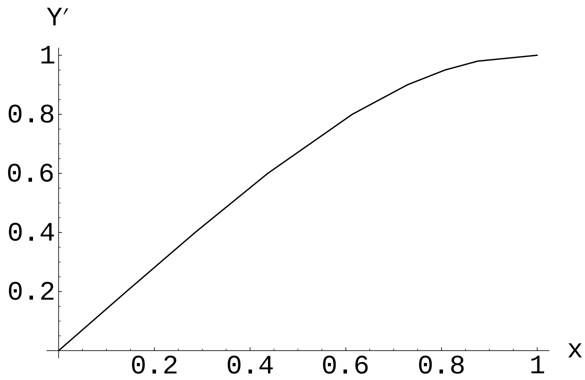

Figure 9.1: The velocity function $Y'(x)$ for two-dimensional and radial liquid jets

9.4.2 *Case II: $B = 2, \alpha = -1$ (two-dimensional), $B = 6, \alpha = -4$ (radial)*

Then $C = -3B/2$ and with this value of C the transformation $y \to -2Y/B$ reduces (9.1) to

$$\frac{d^3Y}{dx^3} - 2Y\frac{d^2Y}{dx^2} + 3\left(\frac{dY}{dx}\right)^2 = 0, \tag{9.63}$$

which is the Chazy equation (Chazy 1909, 1910, 1911, Clarkson and Olver 1996). Using $y \to -2Y/B$ in the Lie point symmetries for this case in Table 9.1, we obtain

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - Y\frac{\partial}{\partial Y}, \quad X_3 = x^2\frac{\partial}{\partial x} - 2(3 + xY)\frac{\partial}{\partial Y}, \tag{9.64}$$

which are the Lie point symmetries of the Chazy equation (Chazy 1909, 1910, 1911, Clarkson and Olver 1996). The commutators of the Lie point symmetry generators in (9.64) are

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{9.65}$$

Thus (9.63) possesses a non-solvable Lie symmetry algebra L_3 and, therefore, cannot be solved by the Lie approach (Olver 1993, Ibragimov and Nucci 1994, Mahomed 2008).

Consider the subalgebra $L_2 =< X_1, X_2 >$. The invariants of X_1 are

$$u = Y, \; v = Y', \; p = Y'' = v\frac{dv}{du}. \tag{9.66}$$

Equation (9.63) in (u, v) space takes the following form:

$$v\frac{d^2v}{du^2} + \left(\frac{dv}{du}\right)^2 - 2u\frac{dv}{du} + 3v = 0. \tag{9.67}$$

The generator $X_2 \to \tilde{X}_2$ in variables u and v becomes

$$\tilde{X}_2 = -u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 3p\frac{\partial}{\partial p}, \tag{9.68}$$

and is the symmetry generator of the reduced equation (9.67). The invariants of $\tilde{X}_2$ are

$$s = vu^{-2} = Y'Y^{-2}, \quad t = v'u^{-3} = Y''Y^{-3}, \tag{9.69}$$

and form a basis of differential invariants of the subalgebra $L_2 =< X_1, X_2 >$. Equation (9.67) transforms to the following first-order ordinary differential equation in terms of (s, t):

$$\frac{dt}{ds} = \frac{(3s - 2)t + 3s^2}{2s^2 - t}. \tag{9.70}$$

The non-local generator X_3 admitted by equation (9.70) in the space (s,t) is (Ibragimov and Nucci 1994)

$$X_3 = Y^{-1}[-2(1-6s)\frac{\partial}{\partial s} + 6(3t-s)\frac{\partial}{\partial t}]. \tag{9.71}$$

By solving the first-order linear partial differential equation

$$(1-6s)\frac{\partial w}{\partial s} - 3(3t-s)\frac{\partial w}{\partial t} = 0, \tag{9.72}$$

we obtain a new variable

$$w = \frac{1+9(t-s)}{(1-6s)^{\frac{3}{2}}}, \tag{9.73}$$

which transforms the generator (9.71) to its semi-canonical form

$$X_3 = Y^{-1}(1-6s)\frac{\partial}{\partial s}. \tag{9.74}$$

We want to express equation (9.70) in variables s and w. Solve equation (9.73) for t. This gives

$$t = \frac{1}{9}\left[w(1-6s)^{\frac{3}{2}} + 9s - 1\right], \tag{9.75}$$

which on differentiation with respect to s yields

$$\frac{dw}{ds} = \frac{9}{(1-6s)^{\frac{3}{2}}}\left[\frac{dt}{ds} + w(1-6s)^{\frac{1}{2}} - 1\right]. \tag{9.76}$$

But from (9.70), with t given by (9.75), we have

$$\frac{dt}{ds} = \frac{27s^2 + (3s-2)(9s-1) + w(3s-2)(1-6s)^{\frac{3}{2}}}{18s^2 - 9s + 1 - w(1-6s)^{\frac{3}{2}}}, \tag{9.77}$$

which simplifies to

$$\frac{dt}{ds} = \frac{2 - 9s + w(3s-2)(1-6s)^{\frac{1}{2}}}{1 - 3s - w(1-6s)^{\frac{1}{2}}}. \tag{9.78}$$

Substitution of (9.78) in (9.76) gives

$$\frac{dw}{ds} = \frac{9}{(1-6s)^{\frac{3}{2}}}\left[\frac{2 - 9s + w(3s-2)(1-6s)^{\frac{1}{2}}}{1 - 3s - w(1-6s)^{\frac{1}{2}}} + w(1-6s)^{\frac{1}{2}} - 1\right],$$

$$= \frac{9(w^2-1)}{(1-6s)^{\frac{1}{2}}(-1+3s)+(1-6s)w}. \tag{9.79}$$

Thus, we have expressed equation (9.70) in (s, w) space and therefore

$$\frac{ds}{dw} = \frac{(1-6s)^{\frac{1}{2}}(-1+3s)+(1-6s)w}{9(w^2-1)}. \tag{9.80}$$

By the Vessiot-Guldberg-Lie theorem (Ibragimov and Nucci 1992, 1994), the generators

$$\Lambda_1 = s(1-6s)^{\frac{1}{2}}\frac{\partial}{\partial s}, \quad \Lambda_2 = (1-6s)^{\frac{1}{2}}\frac{\partial}{\partial s}, \quad \Lambda_3 = (1-6s)\frac{\partial}{\partial s}, \tag{9.81}$$

form a three-dimensional Lie algebra. To convert equation (9.80) to a Riccati equation, consider

$$\bar{\Lambda}_1 = -\frac{1}{3}\Lambda_2, \quad \bar{\Lambda}_2 = -\frac{1}{3}\Lambda_3, \quad \bar{\Lambda}_3 = 2\Lambda_1 - \frac{1}{3}\Lambda_2, \tag{9.82}$$

and define ϕ

$$\phi = (1-6s)^{\frac{1}{2}}, \tag{9.83}$$

where ϕ satisfies

$$\frac{1}{3}(1-6s)\frac{d\phi}{dt} = -\phi. \tag{9.84}$$

From (9.83)

$$s = \frac{1}{6}(1-\phi^2). \tag{9.85}$$

Differentiating (9.85) with respect to w, we obtain

$$\frac{ds}{dw} = -\frac{1}{3}\phi\frac{d\phi}{dw}, \tag{9.86}$$

which together with (9.80) yields

$$\frac{d\phi}{dw} = \frac{1}{6(w^2-1)}\phi^2 - \frac{w}{3(w^2-1)}\phi + \frac{1}{6(w^2-1)}. \tag{9.87}$$

Thus, we have transformed equation (9.80) to a Riccati equation (9.87) in the variables (w, ϕ). The substitution

$$u = \exp[-\frac{1}{6}(\int \frac{\phi}{w^2 - 1} dw)], \tag{9.88}$$

reduces the Riccati equation (9.87) to a second order linear differential equation:

$$\frac{d^2u}{dw^2} + \frac{7w}{3(w^2-1)}\frac{du}{dw} - \frac{1}{36(w^2-1)^2}u = 0. \tag{9.89}$$

The solution of (9.89), using the computer program Mathematica, is

$$u = \frac{c_1 P[\frac{1}{6}, w] + c_2 Q[\frac{1}{6}, w]}{(w^2-1)^{\frac{1}{12}}}, \tag{9.90}$$

where $P[\frac{1}{6}, w]$ and $Q[\frac{1}{6}, w]$ are Legendre functions of first and second kind. Thus, the reduction of the Chazy equation, from (9.85) with $s = Y'Y^{-2}$, is

$$Y^2 = \frac{6Y'}{1 - \phi^2(w)}, \tag{9.91}$$

where w in the original variables is

$$w = \frac{Y^3 - 9YY' + 9Y''}{(Y^2 - 6Y')^{\frac{3}{2}}}, \tag{9.92}$$

and ϕ is given by

$$\phi = \frac{[8w(c_1 P[\frac{1}{6}, w] + c_2 Q[\frac{1}{6}, w]) - 7(c_1 P[\frac{7}{6}, w] + c_2 Q[\frac{7}{6}, w])}{c_1 P[\frac{1}{6}, w] + c_2 Q[\frac{1}{6}, w]}. \tag{9.93}$$

To obtain the reduction of (9.1) in (x, y) variables with $B = 2, C = -3$ (two-dimensional) $B = 6, C = -9$ (radial) replace $Y \to \frac{-By}{2}$. For $(x, Y) \to (\chi, \frac{-BF}{2\nu})$ the reduction of (9.10) can be recovered.

The approach used here is due to Ibragimov and Nucci (1994) and is different from those of Clarkson and Olver (1996) and Adam and Mahomed (2002).

9.4.3 *Case III:* $B \neq 0,\ B \neq 2$ *(two-dimensional),* $B \neq 0,\ B \neq 6$ *(radial)*

From Table 9.1, for this case we have only two generators. Using the transformation $y \to \frac{1}{B}Y$ in (9.1), we obtain

$$\frac{d^3Y}{dx^3} + Y\frac{d^2Y}{dx^2} + \frac{C}{B}\left(\frac{dY}{dx}\right)^2 = 0. \tag{9.94}$$

The Lie point symmetry generators transform to

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - Y\frac{\partial}{\partial Y}. \tag{9.95}$$

The invariants of X_1 are

$$u = Y, \quad v = Y', \quad \frac{dv}{du} = \frac{Y''}{Y'} = w, \tag{9.96}$$

which reduce (9.94) to the second-order ordinary differential equation

$$v\frac{d^2v}{du^2} + \left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} + \frac{C}{B}v = 0. \tag{9.97}$$

The generator

$$X_2 = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, \tag{9.98}$$

in (u, v) coordinates is a Lie point symmetry of the reduced equation (9.97). The invariants of the generator X_2 are

$$s = vu^{-2}, \quad t = u^{-1}w = u^{-1}\frac{dv}{du}, \tag{9.99}$$

which reduce the second-order ordinary differential equation (9.97) to the first-order differential equation

$$\frac{dt}{ds} = \frac{t^2 + t + st + \frac{C}{B}s}{s(2s - t)}. \tag{9.100}$$

There are several subcases depending on the value of the ratio C/B.

For $C/B = 0$, $\alpha = 1/2$ with $B = 1/2$ (two-dimensional) and $B = 3/2$ (radial), equation (9.94) reduces to

$$\frac{d^3Y}{dx^3} + Y\frac{d^2Y}{dx^2} = 0, \tag{9.101}$$

which is the Blasius equation (Bluman and Kumei 1989). Equation (9.101) in terms of invariants reduces to the following first-order ordinary differential equation:

$$\frac{dt}{ds} = \frac{t^2 + t + st}{s(2s - t)}. \tag{9.102}$$

For $C/B = -1$, $\alpha = 0$ for two-dimensional flows and $\alpha = -1$ for radial flows. Using appropriate boundary conditions, Crane (1970) derived an exact solution for a two-dimensional stretching plate with $\alpha = 0$. For $C/B = -1$ (9.94) in terms of differential invariants becomes

$$\frac{dt}{ds} = \frac{t^2 + t + st - s}{s(2s - t)}. \tag{9.103}$$

It is of interest to observe that for $C/B = 1$ and $C/B = 2$, the second-order equation (9.97) obtained by using the invariants of X_1 becomes exact. The second reduction is not needed to obtain a solution.

For $C/B = 1$, $\alpha = 2/3$ for two-dimensional flows and $\alpha = 1$ for radial flows. Equation (9.97) can be integrated immediately with respect to u to give

$$\frac{dv}{du} = -u + \frac{c_1}{v}, \tag{9.104}$$

where c_1 is a constant. Integrating (9.104) again with respect to u and expressing the result in terms of the original variables gives the Riccati equation

$$Y' = -\frac{Y^2}{2} + c_1 x + c_2, \tag{9.105}$$

where c_2 is a constant. For a free two-dimensional jet (Schlichting 1933 and Bickley 1937) the conserved quantity gives $\alpha = 2/3$ and for the free radial jet

(Squire 1955) the conserved quantity gives $\alpha = 1$. In both cases the boundary conditions are

$$Y(0) = 0,\ Y''(0) = 0,\ Y'(\pm\infty) = 0. \tag{9.106}$$

Imposing the boundary condition $Y'(\pm\infty) = 0$ gives

$$c_1 = 0, \quad c_2 = \frac{1}{2}Y^2(\infty), \tag{9.107}$$

and (9.105) reduces to

$$Y' = \frac{1}{2}\left(Y^2(\infty) - Y^2\right). \tag{9.108}$$

The solution may be completed for the free two-dimensional jet as described by Bickley (1937) and for the free radial jet as described by Squire (1955).

For $C/B = 2$, $\alpha = 3/4$ for two-dimensional flows and $\alpha = 5/4$ for radial flows. By first multiplying (9.97) by u, (9.97) can be integrated with respect to u to give

$$\frac{dv}{du} - \frac{1}{2u}v = -u + \frac{c_3}{uv}, \tag{9.109}$$

where c_3 is a constant. Multiplying (9.109) by the integrating factor $u^{-1/2}$, integrating again with respect to u and expressing the result in the original variables gives

$$Y' = -\frac{2Y^2}{3} + \left(c_3 \int^x Y^{-3/2}dx + c_4\right) Y^{1/2}, \tag{9.110}$$

where c_4 is a constant. Using a conserved quantity, Glauert (1956) showed that for a two-dimensional wall jet $\alpha = 3/4$ and for a radial wall jet $\alpha = 5/4$. The boundary conditions in both cases are

$$Y(0) = 0,\ Y'(0) = 0,\ Y'(\infty) = 0. \tag{9.111}$$

Now for a wall jet the stress at the wall is non-zero and finite and therefore $Y''(0)$ is non-zero and finite. Since $Y(0) = 0$ and $Y'(0) = 0$ it follows that

$$Y(x) \sim \frac{1}{2}Y''(0)x^2 \quad \text{as } x \to 0. \tag{9.112}$$

Hence

$$c_3 Y^{1/2} \int^x \frac{1}{Y^{3/2}} dx \sim -\frac{c_3}{Y''(0)} \frac{1}{x}, \quad \text{as } x \to 0. \tag{9.113}$$

Imposing on (9.110) the boundary condition at $x = 0$ therefore gives $c_3 = 0$ and imposing $Y'(\infty) = 0$ yields

$$c_4 = \frac{2}{3} Y^{3/2}(\infty). \tag{9.114}$$

Equation (9.110) reduces to

$$Y' = \frac{2}{3} Y^{1/2} (Y^{3/2}(\infty) - Y^{3/2}). \tag{9.115}$$

The solution for two-dimensional and radial wall jets may be completed as described by Glauert (1956).

9.5 Invariant solutions

9.5.1 *Case I: $B = 0$ (two-dimensional and radial)*

From Table 9.1 the generator X of the invariant solution, for $y \to 3Y/C$, is

$$X = (c_1 + c_3 x) \frac{\partial}{\partial x} + (\frac{C c_2}{3} - Y c_3) \frac{\partial}{\partial Y}, \tag{9.116}$$

where c_1, c_2 and c_3 are constants. The invariant solution will depend only on the ratio of the constants. The invariant solution is obtained by solving the following characteristic equation

$$\frac{dx}{c_1 + c_3 x} = \frac{dY}{\frac{C c_2}{3} - Y c_3}, \tag{9.117}$$

which yields

$$Y = \frac{1}{c_3} [\frac{C}{3} c_2 + \frac{k}{(c_1 + c_3 x)}], \tag{9.118}$$

provided $c_3 \neq 0$, where k is the constant of integration. To obtain k, (9.118) is substituted into (9.39). This gives $k = 0$ and $k = 2c_3^2$. For $k = 0$, we obtain the constant solution $Y = c_2 C/3c_3$. The invariant solution for $k = 2c_3^2$ is

$$Y = \frac{C}{3}\frac{c_2}{c_3} + \frac{2c_3}{(c_1 + c_3 x)}. \tag{9.119}$$

After using $y \to 3Y/C$, the invariant solutions for two-dimensional and radial flows can be obtained from (9.119) by taking $C = 1$ and $C = 3$, respectively. The trivial constant solution is obtained for $c_3 = 0$.

9.5.2 *Case II: $B = 2, \alpha = -1$ (two-dimensional), $B = 6, \alpha = -4$ (radial)*

In Section 9.4, we saw that (9.1) reduces to the Chazy equation (9.63). From (9.64), the generator X of the invariant solution is

$$X = (c_1 + c_2 x + c_3 x^2)\frac{\partial}{\partial x} - (c_2 Y + 2c_3(3 + xY))\frac{\partial}{\partial Y}, \tag{9.120}$$

where c_1, c_2 and c_3 are constants. The characteristic equation is

$$\frac{dx}{c_1 + c_2 x + c_3 x^2} = -\frac{dY}{c_2 Y + 2c_3(3 + xY)}, \tag{9.121}$$

which yields

$$Y = \frac{k - 6c_3 x}{c_1 + c_2 x + c_3 x^2}. \tag{9.122}$$

By substituting (9.122) into (9.63) it is found that the constant of integration k satisfies

$$(c_2^2 - 4c_1 c_3)(k^2 + 6kc_2 + 36c_1 c_3) = 0. \tag{9.123}$$

Thus either $c_2^2 - 4c_1 c_3 = 0$ and k is arbitrary or $c_2^2 - 4c_1 c_3 \neq 0$ and

$$k = 3[-c_2 \pm (c_2^2 - 4c_1 c_3)^{1/2}]. \tag{9.124}$$

If $c_2^2 - 4c_1 c_3 < 0$, k is complex and the invariant solution does not exist.

9.5.3 *Case III:* $B \neq 0,\ B \neq 2$ *(two-dimensional),* $B \neq 0,\ B \neq 6$ *(radial)*

From Table 9.1, for all real values of α, except $\alpha = 1$ and $\alpha = -1$ for two-dimensional flow and $\alpha = 2$ and $\alpha = -4$ for radial flow, the generator X of the invariant solution of (9.1) is

$$X = (c_1 + c_2 x)\frac{\partial}{\partial x} - c_2 y \frac{\partial}{\partial y}, \tag{9.125}$$

where c_1 and c_2 are constants. The non-trivial invariant solution is

$$y = \frac{6c_2}{(2B + C)(c_1 + c_2 x)}, \tag{9.126}$$

where $2B + C$ does not depend on α and is 1 for two-dimensional flow and 3 for radial flow. Thus for Blasius flow we obtain the invariant solution

$$y = \frac{6c_2}{(c_1 + c_2 x)}. \tag{9.127}$$

9.6 Conclusions

For $\alpha = 1$ (two-dimensional) $\alpha = 2$ (radial), equation (9.1) admits three independent Lie point symmetries generating a solvable Lie algebra. We therefore solved (9.1) by the Lie approach and obtained the scaled velocity profile for two-dimensional and radial jets. For $\alpha = -1$ (two-dimensional) $\alpha = -4$ (radial), equation (9.1) has three independent Lie point symmetries generating a non-solvable Lie algebra. The Chazy equation was recovered and its reduction was obtained using the semi-canonical variables of Ibragimov and Nucci (1994).

For the values of α which correspond to two-dimensional and radial (free or wall) jets, we have given an alternative method of solution which is more systematic. Equation (9.1) has two independent Lie point symmetries. The equation can be integrated because the second-order differential equations obtained using the invariants of X_1 are exact and the boundary conditions give the constants of integration special values. For all other real values of α, the differential invariants of X_1 and X_2 can be used to reduce the third-order ordinary differential equation to a first-order ordinary differential equation.

We have also derived the invariant solutions of equation (9.1) which give singular solutions of the third-order ordinary differential equation. Particularly, for the Chazy equation and Blasius equation the invariant solutions have been obtained.

Chapter 10

Conclusions

10.1 New method for derivation of conserved quantities for jet flows

A systematic way to derive the conserved quantities for jet flows using conservation laws is presented in this thesis. In jet problems the conserved quantity plays an important part in the derivation of the solution. The conserved quantities for laminar jets were established either from physical arguments or by integrating Prandtl's momentum boundary layer equation across the jet and using the boundary conditions and the continuity equation. We discussed three types of flows, namely, two-dimensional, radial and axisymmetric. The conserved quantities for liquid, free and wall jets, for two-dimensional, radial and axisymmetric flows, were derived. The flow in jets was governed by a system of two partial differential equations for the velocity components or by a third-order partial differential equation for the stream function. The conserved vectors for the system as well as for the third-order partial differential equation for the stream function for each jet flow were constructed by utilizing the multiplier approach. The liquid jet, the free jet and the wall jet satisfy the same partial differential equations but the boundary conditions for each jet are different. The conserved vectors depend only on the partial differential equations whereas the derivation of the conserved quantity depends also on the boundary conditions. By integrating the corresponding conservation laws across the jet and imposing the boundary conditions, conserved quantities were derived for liquid, free and wall jets for each type of flow. This approach gives a unified treatment to the derivation of conserved quantities for jet flows and may lead to a new classification of jets through conserved vectors or multipliers.

For the two-dimensional and radial flows, the multiplier approach gave two conserved vectors for the system of equations for the velocity components. One was used to derive the conserved quantity for the liquid jet and other for the free jet for both flows. The conserved quantity for the wall jet was not obtained from the conserved vectors for the system. The multiplier approach applied to the third-order partial differential equation for the stream function yielded two conserved vectors. The first conserved vector was used to give an alternative derivation of the conserved quantity for the free jet and the second conserved vector gave the conserved quantity for the wall jet, for both two-dimensional and radial jets. A new form of the conserved quantity for the wall jet was obtained which was simpler than the one obtained by Glauert (1956).

For the axisymmetric flow, the multiplier approach gave two local conservation laws for the system of equations for the velocity components. One of the conserved vectors gave the conserved quantity for an axisymmetric liquid jet and second conserved vector gave the conserved quantity for the axisymmetric free jet. For the third-order partial differential equation for the stream function two local conserved vectors were obtained, one of which was a non-local conserved vector for the system of equations for the velocity components. One of local conserved vectors for the third-order partial differential equation for the stream function was used to give an alternative derivation of the conserved quantity for an axisymmetric free jet but the other local conserved vector cannot be used to derive the conserved quantity for the axisymmetric wall jet. The conserved quantity for an axisymmetric wall jet was established with the help of a non-local conserved vector for the third-order partial differential equation for the stream function. That non-local conservation law for the third-order partial differential equation for stream function was not obtained by the multiplier approach. The reason is that the multiplier approach only gives multipliers for local conservation laws.

The conservation laws for two-dimensional, radial and axisymmetric jet flows are new. The conserved quantities for liquid, free and wall jets for all three type of flows are the same as in the literature but now they are derived by a new method using conservation laws. Moreover, new simple forms for the conserved quantities for the two-dimensional and radial wall jets are obtained.

10.2 Comparison of approaches for derivation of conservation laws

The conservation laws for second order scalar partial differential equations and systems of partial differential equations which occur in fluid mechanics were constructed using different approaches. There are nine different approaches to construct conservation laws for partial differential equations. We have discussed and explained each approach with the help of an illustrative example which describes the relaxation to a Maxwellian distribution. The conservation laws for the non-linear diffusion equation for the spreading of an axisymmetric thin liquid drop, for the system of two partial differential equations governing the flow in the laminar two-dimensional jet and for the system of two partial differential equations governing the flow in the laminar radial jet were also discussed to highlight the advantages and disadvantages of each approach.

The Noether approach is simple and is a systematic way to construct conservation laws for partial differential equations that have a standard Lagrangian and corresponding Noether symmetries. The partial Noether approach works in the same way as the Noether approach for differential equations with or without standard Lagrangians. We also commented on the direct method as well as its use with a symmetry condition for systems without a standard Lagrangian. Furthermore, we looked at some other approaches which do not rely upon the knowledge of a Lagrangian. In characteristic method conserved vector can be expressed as $D_iT^i = \Lambda^\alpha E_\alpha$. There are three ways to construct the characteristics (multipliers). Firstly, by expanding directly $D_iT^i = \Lambda^\alpha E_\alpha$. Secondly, taking variational derivative of $D_iT^i = \Lambda^\alpha E_\alpha$ on solution space. The simplest and most effective way is the third way to compute characteristics (multipliers) which is by taking the variational derivative of $D_iT^i = \Lambda^\alpha E_\alpha$ for arbitrary functions, not only for solutions.

10.3 Symmetry and group invariant solutions

We have derived the group invariant solutions for the system of equations governing flow in two-dimensional and radial free jets. It is shown that the group invariant solution and the similarity solution derived by other authors are the same.

The similarity solution to Prandtl's boundary layer equations for two-dimensional and radial flows with vanishing or constant mainstream velocity gives rise to a third-order ordinary differential equation which depends on one parameter. For specific values of the parameter the symmetry solutions for the third-order ordinary differential equation were constructed. These results are new and may lead to further studies. The invariant solutions of the third-order ordinary differential equation were also derived.

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