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A Collective Field Theory Approach to the Large N Spectrum of Two Matrices

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

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Abstract

The collective field theory technique provides a method of tackling problems with two $N \times N$ matrices in the large N limit. The collective field background from one matrix is first found, then the second matrix is introduced into this background as an impurity. Within the context of the AdS/CFT correspondence, this technique can be used to describe gauge theory states in the BMN limit.

This dissertation starts by developing the collective field theory technique, firstly in general variables, then for one matrix, and subsequently for two matrices. It goes on to introduce a Yang-Mills interaction term, where two variable identifications are considered. The first is the more traditional angular momentum eigenstate model. The second is a model that directly uses two of the Higgs scalars. This model has been mentioned in the literature, but has not been considered in great depth. The exact two impurity spectrum is found, and the multi-impurity spectrum is found to first order. The resulting energy values match a spectrum that has been found for giant magnons.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

_____
Martin Cook

_____ day of _____, 2007.

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Chapter 1

Introduction

1.1 Background

't Hooft published a paper in 1974 [1] pointing out an interesting phenomena that arises when the N in $SU(N)$ gauge theories is treated as a free parameter. When the large N limit is taken, the Feynman diagrams are arranged in a similar way to that of a string. This paved the way to a proposed correspondence between gauge theories and string theory. Perhaps the best recent example of this correspondence is a duality proposed by Maldacena [2], called the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. This relates type IIB string theory involving AdS space (namely $AdS_5 \times S^5$) and $\mathcal{N} = 4$ Super Yang Mills (SYM) theory. Much work has been done on developing a "dictionary" allowing one to go from the string theory to the field theory or vice versa.

One promising area of research within the AdS/CFT correspondence has involved looking at giant gravitons in $AdS \times S$, and their SYM duals. On the field theory side, 1/2 BPS states and their interactions can be described in terms of a (single matrix) free fermion model of harmonic oscillators. This description makes use of a complex matrix in a harmonic potential ([3], [4]). Later on, Lin, Lunin and Maldacena (LLM) showed that a fermionic droplet configuration completely describes 1/2 BPS states [5]. [6] reproduced the energy and flux obtained by LLM. They extended the free fermion matrix model by considering a one dimensional Hermitian matrix in a bosonic phase space density description.

To consider more interesting cases than free fermions, it is necessary to consider models with more than one matrix. It is not known how to solve a two matrix model exactly. One way of

tackling the problem at large N is to solve one of the matrices exactly by introducing a new set of invariant collective fields. While overcomplete for finite N, the new variables tend towards independence in the large N limit. The second matrix is treated in a coherent state basis, using creation and annihilation operators. Creation operators corresponding to the second matrix are then added as impurities into the background created by the first matrix. Using this technique on a two matrix model with an harmonic potential, [6] found a sequence of eigenvalue equations that generalised the results obtained earlier for one matrix [7]. Eigenstates were obtained for a harmonic oscillator potential for any number of impurities. A 1-1 map was found that relates these eigenfunctions to a 2D subset of suitable transformed wavefunctions from Supergravity on $AdS \times S$.

By considering a different set of states, [8] obtained the full free spectrum for two Hermitian matrices. It went on to introduce the interaction term from the SYM action. This was done in two different contexts. The first used the traditional angular momentum eigenstates. First order energy results were found by considering part of the Hamiltonian. The second method involved modelling the two scalars directly.

This thesis aims firstly to review the work leading up to this point, showing the physical significance of the collective field theory approach to tackling one and two matrix models. The context of this work within the AdS/CFT correspondence is given. Secondly, it aims to extend the field by deriving the energy spectrum that results from the introduction of a Super Yang-Mills interaction term, for a particular identification of variables corresponding to a direct treatment of two Higgs.

1.2 Outline

This thesis is organised as follows.

Chapters 2 and 3 contain background material to the AdS/CFT correspondence, which will motivate the calculations of later chapters. Chapter 2 contains a summary of and motivation for the AdS/CFT correspondence. The initial motivation for this link came from 't Hooft in the context of QCD [1]. His original argument is revisited, and it is shown how the large N limit yields planar diagrams to leading order in the Feynman expansion. This large N expansion is of the form of a perturbation expansion in string theory, which provides the first indication that gauge theories and string theories are linked.

The two theories on either side of the AdS/CFT correspondence are then examined. Some of the properties and content of $\mathcal{N} = 4$ Super Yang Mills theory are found by starting with a conformal field theory, then adding supersymmetry. Anti-de Sitter space is introduced, and some fundamental properties of type IIB strings defined on $AdS_5 \times S^5$ are given. On the gauge theory side, chiral primary are introduced. These are 1/2 BPS operators which are of particular interest as their dimension does not receive coupling dependent corrections.

Maldacena's original argument [2] for the identification of $\mathcal{N} = 4$ SYM theory and IIB string on $AdS \times S$ is then explored. N parallel D3-branes are put into a ten dimensional space, and two different low energy limits are taken. In the first case, the result is free supergravity plus $\mathcal{N} = 4$ SYM, and in the other case the result is free supergravity plus strings on $AdS_5 \times S^5$. The SYM theory and the string theory are therefore identified.

Chapter 3 looks at consistency checks for the AdS/CFT, as well as exploring some of the links that have found between specific sets of states on either side of the correspondence. Parameters which are mapped to each other on both sides of the correspondence are identified. The BMN limit is explored, which relates strings in a plane wave background to single trace operators into which impurities have been introduced. Spin chains are briefly introduced, together with some motivation for their study and some basic spin chain results.

Chapters 4, 5, 6 and 7 look at applying the collective field theory technique to matrix models. Chapter 4 develops the mathematical formalism, using general notation. The change of variables to invariant collective fields is described, and the Jacobian is found by requiring Hermiticity.

Chapter 5 starts by outlining the physical motivation for examining free single matrix models using the collective field theory techniques. These models describe 1/2 BPS states – extremal states whose anomalous dimension (or energy on the string side) is protected by supersymmetry considerations. The general formalism developed in chapter 4 is then applied to an Hermitian matrix. A Fourier transform is made to an eigenvalue density description and the energy spectrum of the single matrix Hamiltonian is found.

Chapter 6 goes on to look at the identification of variables from the AdS/CFT correspondence that leads to matrix models involving two of the SYM scalars. Two possibilities arise, creating angular momentum eigenstates from the SYM scalars, or directly modelling the two SYM scalars. These two models will be examined in more detail in chapters 7 and 8. Both involve two matrix models, which are traditionally difficult to solve. The approach followed is thus to treat

the first matrix exactly, as described in chapter 5. The second matrix is then expressed in a creation/annihilation basis, and introduced in the form of impurities into the background created by the first matrix. The result of adding impurities into the zero-impurity background produced by the first matrix is then examined. The many-impurity Hamiltonian is found together with the energy spectrum, using results calculated in the appendices. The eigenfunctions are found to involve products of Chebyshev polynomials, which have a similar structure to spin chains.

The g_{YM} interaction term is added in chapters 7 and 8. Chapter 7 examines it in terms of the angular momentum eigenstate in the interaction potential of $\mathcal{N} = 4$ SYM theory, using the angular momentum variables. This is shown to generate an interaction Hamiltonian. The resulting first order energy spectrum is reviewed.

Chapter 8 presents new work on the two SYM scalar matrix model. While this model has been mentioned before in [8], it has not been developed fully. The Hamiltonian is found to generate a term of the same form as the leading contribution from chapter 7. This leads to a difference equation. The original derivation in [8] gave energy results for two impurities to first order by only considering the some of the terms in the difference equation. We take this further by introducing an ansatz which gives an analytic solution to the complete difference equation for any number of impurities. We then perform a Bogoliubov transformation. The energy spectrum resulting from the ansatz is shown to exactly determine the spectrum resulting from the g_{YM} interaction term for two impurities, and to first order in the t 'Hooft coupling for any number of impurities. The final energy is shown to be of the same form as a result found recently for giant magnons.

Chapter 9 is reserved for a discussion of the results, and future outlook.

Appendix A contain details of the calculations of the collective field theory parameters for many impurities. Appendix B has the calculation of the many-impurity energy spectrum. These results are used in chapter 6. Appendix C contains a proof for the equivalence of two operators that is used in chapter 8.

Chapter 2

AdS/CFT correspondence

The foundations of the AdS/CFT correspondence were laid within the context of understanding the strong force. In fact, string theory was first developed as an attempt to describe the observed variety of hadrons and mesons, which interact via the strong force. It was not very successful at this, and this approach was no longer pursued after the development of Quantum Chromodynamics (QCD), which describes hadrons and mesons as being made up of constituent quarks. Despite the lack of overall success, string theories retain some of the properties that make them an attractive candidate for describing the strong force.

The first useful property concerns the relationship between mass and angular momentum. Within the space of hadrons with a given spin, the hadron with the lowest mass obeys $m^2 \sim TJ^2 + \text{const}$, which is satisfied by a rotating relativistic string.

Two other useful properties are asymptotic freedom and confinement. Asymptotic freedom states that the force between quarks goes to zero at small distances. At large distances the force increases sharply, so that one is never able to observe a free quark; this is called confinement. These properties can be understood in a string-like manner. The flux lines between quarks can be thought of as stretching out in a tube, such that the quarks are always bound to each other. At small distances the force is not large, but at large distances it increases as the flux tube is extended. These flux lines are similar to strings with quarks at both ends.

While QCD on the whole has been very successful, it is not easy to do calculations at low energy scales, where QCD becomes strongly coupled. One of the problems is the lack of a good dimensionless parameter on which to perform perturbation theory. QCD is defined around $SU(3)$, in other words quarks come in three colours. t 'Hooft [1] attempted to tackle the strongly

coupled region by instead taking a gauge theory where the number of colours is N, and then taking the large N limit. If this simplified the theory, then $1/N$ corrections could be added to leading terms, which would allow one to get a perturbative expansion when $N \to 3$. The result of this approach was an expansion that matched that of free string theory. In this way, the large N limit gave the first hint that gauge theories are related to string theories, as will be described in more detail in the next section.

The kind of argument followed by 't Hooft turns out to be very general, and links can be found between many different gauge theories and string theories. A general property of dualities between gauge theories and string theories is that in the region where one is strongly coupled, the other is weakly coupled, and vice versa. This is useful, as it suggests a way to do calculations in regions where previously it was very difficult, such as in low energy QCD. It is also problematic, as it has made proving dualities between theories difficult because the regions where calculations can be done do not overlap nicely.

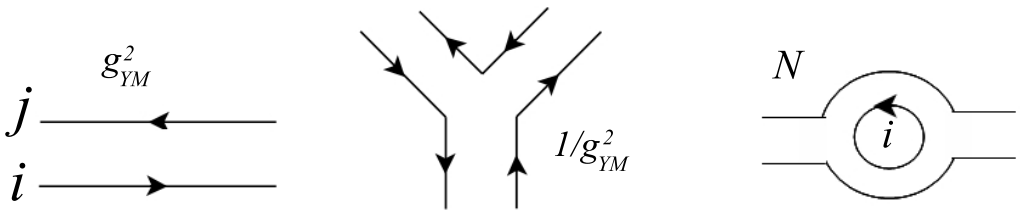
Of the dualities, the AdS/CFT correspondence in particular has received much attention in the literature, and provides the background to this thesis. It involves two highly supersymmetric theories, relating strings in a space with an Anti-de-Sitter (AdS) component to operators in Super-Yang-Mills conformal field theory (CFT).

2.1 Large N limit

Following the review in [9], we consider a general theory based on $SU(N)$ with fields Φ_a^i. For a less specific version of this argument, involving a general Hermitian matrix M, see [10]. In Φ_a^i, a is an index in the adjoint representation of SU(N), and i is another index, such as flavour or colour. Three point vertices will contribute g_{YM}, and four point vertices g_{YM}^2. After rescaling by taking $\phi \to \phi/g_{YM}$, the Lagrangian is of the form

$$\mathcal{L} = \frac{1}{g_{YM}^2}\left[\mathrm{Tr}(d\Phi_i d\Phi_i) + c^{ijk}\mathrm{Tr}(\Phi_i\Phi_j\Phi_k) + d^{ijkl}\mathrm{Tr}(\Phi_i\Phi_j\Phi_k\Phi_l)\right] \tag{2.1}$$

The original argument followed 't Hooft in [1] dealt with the gauge field theory that describes the strong interaction. The Lagrangian was therefore more complex, involving quark fields and ghost fields (in the Feynman gauge) in addition to the gauge fields $A_{\mu j}{}^i$. The general principles of the argument are the same, however.



(a) Propagator (b) Three point vertex (c) Closed loop

Figure 2.1: The contributions to g^2_{YM} and N that arise from various diagrams

We can represent each adjoint field as a direct product of a fundamental and an anti-fundamental field, so the theory will be built out of objects with two indices, Φ^i_j. The propagators (neglecting terms sub-leading in N for some groups) are of the form

$$\langle \Phi^i_j \Phi^k_l \rangle \propto \delta^i_l \delta^j_k$$

To keep track of the two indices, we introduce a double line notation into the Feynman diagrams, where the upper indices are denoted by incoming arrows and lower indices by outgoing arrows. Where arrows join, such as at vertices, the indices are summed. The resulting Feynman diagram will be a network of directed double lines. This can be viewed as a simplical decomposition of a surface, where the lines (arrows) are edges or faces in the decomposition.

To see what happens at large N, the scaling of g_{YM} as $N \to \infty$ must be determined. Requiring that the leading terms of the beta function for $SU(N)$ YM theory scale consistently at large N (see [9] for details) means that N limit should be taken in such a way as to keep $\lambda \equiv g^2_{YM} N$ finite. This is called the 't Hooft limit.

We determine the g_{YM} and N dependence from the geometrical properties of the surface. From equation 2.1, vertices will contribute a factor of $1/g^2_{YM}$. Propagators (or edges) are inversely proportional to the quadratic part, and thus contribute a factor of g^2_{YM}. Closed loops are equivalent to δ^i_i, so each closed loop, or face, contributes a factor of N. These cases are shown in figure 2.1.

A diagram with E edges, V vertices and F faces will be proportional to

$$N^{V-E+F} \lambda^{E-V} = N^{2-2g} \lambda^{E-V}$$

where Euler's formula has been used to introduce the genus, or number of handles, g. (We are assuming connected diagrams – the extension to disconnected diagrams is easily made by

adding the terms for the different components.) A full perturbative expansion will be a sum of terms of the above form, and can be written as

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{2.2}$$

where the functions f_g capture the λ dependence. The leading terms are the ones with $g = 0$, which are diagrams that can be drawn on a plane. The large N limit is thus often referred to as the "planar limit". For each handle that is added, the diagram will be suppressed by a factor of $1/N^2$.

Equation (2.2) has the same form as the perturbative expansion for closed oriented strings, if we identify $1/N$ as the string coupling constant. We therefore see that in the large N limit, gauge fields are linked to string theories. It should be noted that this connection is based on a perturbation expansion that does not converge, and can thus not be called a rigorous derivation. It is thought, however, that it is indicative of a true equivalence between gauge field theories and string theories.

2.2 Properties of $\mathcal{N} = 4$ SYM and IIB strings on $AdS \times S$

Having given the general motivation for links between gauge theories and string theories, we now turn to the example of interest, namely the AdS/CFT correspondence. One the string side, the correspondence involves type IIB string theory on a ten dimensional space that consists of a 5-sphere and a five dimension Anti-de-Sitter space, i.e. $AdS_5 \times S^5$. This is identified with the maximally supersymmetric four dimensional conformal field theory, which is $\mathcal{N} = 4$ Super Yang Mills (SYM) with the gauge group SU(N) [2, 11]. Before giving an outline of the justification for this link, let us briefly explore the theories on either side of the correspondence, starting with the SYM field theory.

A conformal field theory (CFT) is a quantum field theory that is invariant at quantum level under the group of conformal transformations. These transformations preserve the metric up to an overall (in general x dependant) scaling factor, $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$, thus preserving angles. The conformal group is the smallest group that contains both the Poincaré group as well as the inversion symmetry $x^\mu \to x^\mu/x^2$. In Minkowski space, the conformal group is generated by Poincaré transformations, scale transformations and a special conformal transformation given

by

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2} \tag{2.3}$$

For details on the full set of commutators obeyed by the generators of these transformations, see for example the review in [9]. For now, there are two that are of interest. Denote the generator of the conformal transformation (2.3) by K_μ, the generator of translations by P_μ and the scaling operator (also called the dilatation operator), which takes $x^\mu \to \lambda x^\mu$, by D. Then,

$$[D, K_\mu] = iK_\mu \qquad \text{and} \qquad [D, P_\mu] = -iP_\mu \tag{2.4}$$

Consider now representations that consist of operators that are eigenfunctions of D with eigenvalue $-i\Delta$ where Δ is referred to as the scaling dimension (or just dimension) of the field. From the commutators in (2.4), the action of P_μ is to raise the scaling dimension of the field by one, while K_μ will lower it by one. For example,

$$\begin{aligned} D(P_\mu(\Phi)) &= P_\mu(D(\Phi)) - iP_\mu(\Phi) \\ &= -i(\Delta + 1)P_\mu(\Phi) \end{aligned}$$

There is a lower bound on the dimension of the field for unitary field theories. The operators with lowest scaling dimension are called *primary operators*, and will be annihilated by K_μ.

We now add supersymmetry to the conformal field theory, to get superconformal field theory. Supersymmetry relates bosons to fermions, and contains fermionic operators, or supercharges. We look for the maximally supersymmetric ($\mathcal{N} = 4$) four dimensional Yang-Mills algebra that includes both the conformal group and the Poincaré group. These constraints are very restrictive, essentially uniquely determining the field content and the Lagrangian of the theory, up to the gauge group and the coupling constant. We will take the gauge group to be $SU(N)$, as was the case in 't Hooft's original exploration of the planar limit in the context of QCD. We will briefly look at the content of this theory, for a more detailed summary see for example the reviews in [12], [13] and [9].

If no particles with spin greater than one are included, in other words the theory does not include gravity, the maximum number of supercharges for a free theory is 16. Adding conformal invariance doubles this number to make 32 supersymmetries. This is because Poincaré supersymmetries and the transformations K_μ do not commute, yet both are symmetries. The

result of their commutation must therefore be another conformal symmetry, which is generated by an additional four supercharges S and their complex conjugates, adding an additional 16 supercharges.

$\mathcal{N} = 4$ supersymmetry in four dimensions has a unique vector multiplet. This contains complex Weyl fermions, a vector field and six real scalars, ϕ^I, $I = 1...6$. See [9] for a summary of the action of supersymmetry generators on these fields. The group that rotates the six scalars into each other is $SO(6)$ (or $SU(4)$) which is called the R-symmetry. Including the R-symmetry, $\mathcal{N} = 4$ SYM obeys a global supersymmetry corresponding to the supergroup $SU(2,2|4)$.

We now turn briefly to the string theory side. $AdS_5 \times S^5$ consists of a 5-sphere and five dimensional Anti-de-Sitter (AdS). AdS is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. It can be defined by embedding the six dimensional hyperboloid

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = R^2 \tag{2.5}$$

into a space similar to Minkowski space, but with two time-like coordinates:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2$$

We now change coordinates such that

$$(X_{-1}, X_0) \to R \cosh\rho \; \hat{k}_2$$
$$(X_1, X_2, X_3, X_4) \to R \sinh\rho \; \hat{k}_4$$

where $\hat{k}_2$ and $\hat{k}_4$ are the two and four dimensional unit vectors respectively that give the direction of the two vectors. This can be seen to satisfy the hyperboloid constraint (2.5). The AdS_5 metric is then

$$ds^2 = R^2 \left[-dt^2 \cosh^2\rho + d\rho^2 + d\rho^2 \sinh\rho \; d\Omega_3^2\right]$$

Once a 5-sphere of radius (R) is added, the full metric for $AdS_5 \times S^5$ is

$$ds^2 = R^2 \left[-dt^2 \cosh^2\rho + d\rho^2 + d\rho^2 \sinh^2\rho \; d\Omega_3^2 + d\psi^2 \cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2\right] \tag{2.6}$$

$AdS_5 \times S^5$ can be shown to be a solution to the type IIB supergravity equations of motion [10].

These arise from considering an action with a term involving the metric and a term involving the fiveform field strength,

$$S = \int \sqrt{g}R + S_5^2$$

and constraining the field strength to be self-dual. In this solution, the radius of the AdS space is equal to that of the 5-sphere, and is given by

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s \tag{2.7}$$

where g_s is the string coupling constant and l_s is the string length. The symmetry group of AdS_5 is $SO(2,4)$, and the symmetry group of S^5 is $SO(6)$, thus $AdS_5 \times S^5$ has an overall symmetry group of $SO(2,4) \times SO(6)$. In complex terms this is $SU(2,2|4)$, which is the same as that of the SYM theory.

2.3 Chiral Primary Operators

We now introduce a particular set of operators within the SYM theory, called chiral primaries, which are of particular interest because their dimension does not receive coupling dependent corrections. This makes it easy to compare them to similarly protected states on the string theory side.

In the previous section, we found that additional supercharges S arise from the introduction of conformal symmetry. The commutator between the generators S and the generator of the scaling transformation, D, is $[D, S] = \frac{i}{2}S$. This means that the generators S act in a similar way to K_μ, in that they reduce the dimension of operators. Primary operators in superconformal field theory are therefore operators that are annihilated by both K_μ and S. Because the conformal algebra is a subalgebra of the conformal algebra, in general each primary of the superconformal algebra will include several primaries of the conformal algebra. These can be found by acting on the superconformal primary operator with the supercharges Q.

Chiral Primary operators are defined as operators that are annihilated by some combination of the supercharges Q. Their dimension is uniquely determined by their R-symmetry representations and thus cannot receive any quantum corrections. In any given representation, the chiral primary will always have the lowest dimension; other operators will have a strictly higher dimension. Representations of the superconformal algebra can be formed by starting with a

chiral primary, and then generating descendants by acting on it with the operators Q and P_μ. For Abelian R-symmetry groups, $\Delta \geq a|R|$ for some constant a, and this bound is saturated by chiral primary operators. They are 1/2 BPS states, preserving 16 of the 32 supersymmetries.

It is possible to find chiral primaries by looking for components of a primary multiplet that cannot be written as a supercharge Q acting on another operator, as these will be of lowest dimension. Any operators built from the fermions or vector fields can be expressed as Q acting on other fields, and are thus not chiral primary. We therefore look for operators that are built out of scalar fields, and it turns out that chiral primary operators correspond to symmetric, traceless combinations of the ϕ^I.

2.4 Initial motivation for the AdS/CFT correspondence

Having provided a general explanation of why one expects field theories to be linked to string theories, we now provide some justification for why in particular we expect $\mathcal{N} = 4$ $SU(N)$ to be dual to a string theory on $AdS_5 \times S^5$.

The starting point is type IIB string theory, which is a supersymmetric ten dimensional string theory. We now introduce D-branes. These are extended objects that can be defined by the fact that open strings can end on them [14]. They arose from the study of Dirichlet boundary conditions, where it was realised (in the context of compactification of open string theory) that they define a hyperplane that is a dynamical object. They are a necessary part of the theory; attempts to include only Neumann boundary conditions result in Dirichlet boundary conditions being introduced via the T-duality which links type IIA and IIB string theories.

Specifically, consider IIB string theory on ten dimensional Minkowski space, with N parallel D3 branes that very close to each other (or even co-incident). There are two different ways of looking at the low energy limit of this system, and the identification of the results from each will lead us to the link between strings on AdS and SYM, originally postulated by Maldacena [2].

In the first point of view, perturbative excitations corresponding to both closed and open strings are examined. Closed strings are not connected to D-branes, and are therefore excitations of open space. Open strings end on D-branes, and can be viewed as excitations of the D-branes. At low energies, only massless string states can be excited, and the effective Lagrangian of these massless modes is of the form

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \tag{2.8}$$

where S_{bulk} is the Lagrangian in the bulk due to closed string modes, S_{brane} is the Lagrangian on the D-branes to due open string modes and S_{int} describes the interaction between the bulk and brane modes.

S_{bulk} consists of a quadratic part plus higher order interaction terms that disappear at low energies. We are thus left with only the free part, which includes the graviton and describes free supergravity in the bulk. It is then useful to take low energy limit by keeping the energy fixed, and taking l_s and α' to zero. The coupling, given by $g_s\alpha'^2$, goes to zero in this limit, and S_{int} falls away. S_{brane} is the Lagrangian of strings that start and end on one of N different branes, and therefore describes objects with two indices running from 1 to N, so one might expect a description in terms of field theory. It turns out to consist of a piece that is the same as (3+1) dimensional $\mathcal{N} = 4$ $U(N)$ SYM theory, plus higher order terms that also disappear in the low energy limit. We are thus left with two pieces, free gravity in the bulk and $\mathcal{N} = 4$ SYM which lives on the world-volume of the D3-branes.

The second point of view looks at D-branes as being a source for supergravity fields, and looks at a D3 brane solution (see [9]) of the N parrallel D3 branes described above. From the viewpoint of an observer at infinity, the energy of an object would get smaller and smaller as it gets closer in to the D-brane. As $r \to 0$, there are two types of low energy excitations that the observer could see, massless low energy (i.e. very high wavelength) particles in the bulk, or excitations that are brought close to $r = 0$. At low energy, the wavelength of bulk particles is much larger than the brane size, and thus the cross-section goes to zero. From the other side, near horizon (close to $r = 0$) excitations find it hard to escape the gravitational potential of the D-brane at low energy. The two different excitations therefore become decoupled, leaving two separate pieces, namely free bulk supergravity and the near horizon region, which turns out to have the same geometry as $AdS_5 \times S^5$.

We thus have two different low energy solutions, each of which yield free supergravity plus something else. It is natural to identify the second part in each system, which leads to an identification of (3+1) dimensional $U(N)$[1] SYM theory with IIB strings on $AdS_5 \times S^5$. Note that the claim is not just that there is a similarity between the theories; rather the two theories are actually identified with each other.

[1]There is some little flexibility when looking at the near horizon region, where there are some zero modes that live in the region connecting the near horizon with the bulk. Depending on whether or not these are included, the AdS correspondence is either with $SU(N)$ or $U(N)$.

Chapter 3

Testing the AdS/CFT correspondence – developing a dictionary

Having found a good reason to want to identify the two theories, we should take a moment to examine whether this is, in fact, a sensible thing to do. Everything that goes into the conformal theory should have a matching description on the string side, and vice versa. It turns out that this is a highly non-trivial problem, and much work over the last several years has gone into developing this "dictionary", describing what we should be matching on either side of the correspondence. [9] contains a good review of the early work on this.

The first obvious objection is that it seems strange to link a four dimensional field theory to a ten dimensional string theory – it would be more intuitively appealing to link the field theory to something with the same number of dimensions. This is not possible however, because string theory is not consistent in four dimensions. The first problem is the Weyl symmetry ($g_{ab} \rightarrow \Omega g_{ab}$). Quantum mechanically, taking $g_{ab} \rightarrow e^{\phi}\hat{g}_{ab}$ generates an effective action (the Liouville action) for ϕ, and integrating over ϕ is equivalent to adding a dimension. So we immediately see that the string theory we are looking for cannot be four dimensional, and we need a theory that has at least five dimensions. After adding one dimension, it is not so unreasonable to add another five to get to ten, the critical number of supersymmetric strings. The correspondence between a ten dimensional theory and a four dimensional theory is an example of the holographic principle, which states that all the information contained in a volume

of space can be represented by another theory which lives on the boundary of that volume. From the "derivation" of the AdS/CFT correspondence given in section 2.4, the SYM theory can be thought of as living on the four dimensional boundary of the five dimensional AdS space.

We can also see that it is reasonable for the theory to contain an Anti-de Sitter component by looking at scale invariance [10]. Consider a string theory with 4d Poincaré symmetry, of the form

$$ds^2 = w(z)^2(dx_\mu dx^\mu + dz^2), \quad \mu = 0, \ldots, 3 \tag{3.1}$$

At first glance it looks contradictory for it to have scale invariance, because string theories have a length scale, set by the string tension. The only way for this to make sense is if the scale transformation is an isometry of the (3.1), which leads to

$$ds^2 = R^2 \frac{dx_\mu dx^\mu + dz^2}{z^2}$$

which is five dimensional Anti-de Sitter space. It is therefore not at all strange that the final string theory that we have arrived at has a five dimensional Anti-de Sitter component.

The next easy area for comparison is the global symmetry of the two theories. From section 2.2 we see that both the SYM theory and the string theory have the same supergroup, $SU(2,2|4)$. Matching the $SU(2)$ section on both sides of the correspondence implies that the anomalous dimension of a state (or operator) in the gauge theory corresponds to the energy of the a string state.

We now turn to the perturbative behaviour of the two theories. From the section on the planar limit (section 2.1), we don't expect this to be too pretty. This is indeed the case – the regimes where we can trust perturbation theory are completely incompatible. The following parameters are mapped to each other:

Field theory		**String theory**
g_{YM}^2	$\leftrightarrow$	g_s
$g_{YM}^2 N$	$\leftrightarrow$	R^4/α'^2

(3.2)

Field theory perturbation is reliable when $g_{YM}^2 N \ll 1$. The string theory perturbation is reliable in the low energy limit when the radius becomes large compared to the string length, therefore $R^4/\alpha'^2 \sim g_{YM}^2 N \gg 1$. This mismatch allows the identification of two theories which look so

different without immediate contradiction. It also makes proving the correspondence difficult.

What we would like to do next is find an exact matching between all the states in the two different theories. This is unfortunately a problem that is still far from being solved. There are limits in which certain subsets of the states on either side have been matched however. While it would be beyond the scope of this thesis to provide a comprehensive review of all the work done on this, there are some examples of particular interest which are discussed next.

3.1 The plane wave solution and the BMN limit

It is not yet known how to obtain the full perturbative string spectrum in $AdS_5 \times S^5$, in order to match it to the spectrum of single trace operators in SYM theory. At first, it was only known how to match strings in the large 't Hooft limit, in which case the space becomes flat. These string states correspond to chiral primary operators in the gauge theory. Later on, a more interesting maximally supersymmetric solution to IIB string theory was found, called the plane wave background [15, 16], or pp-wave solution. This solution can be found by taking a Penrose limit of $AdS_5 \times S^5$. It can be thought of as a space which lies between flat space and $AdS_5 \times S^5$, because it looks like flat space, but with the addition of first order corrections towards the AdS space. The pp-wave background is interesting because in this background the free string theory spectrum can be solved exactly in light cone coordinates. Berenstein, Maldacena and Nastase (BMN)[17] showed how to reproduce this spectrum on the gauge theory side. A good pedagogical account of this approach can be found in [18].

The approach followed by [17] is to examine the geometry seen by a particle travelling very fast along the S^5. The first step is to examine the $AdS_5 \times S^5$ metric (2.6), which is reproduced below.

$$ds^2 = R^2 \left[-dt^2 \cosh^2\rho + d\rho^2 + d\rho^2 \sinh^2\rho \; d\Omega_3^2 + d\psi^2 \cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2 \right]$$

The parameters can be understood as follows:

θ	Latitude on S^5
ψ	Coordinate along the equator of S^5
$\rho = 0$	Centre of AdS_5
$\rho = \infty$	Boundary of AdS_5

We can consider a particle in the centre of AdS_5 moving along the equator of S^5 by going to (rescaled) light-cone coordinates

$$x^+ = \frac{1}{2}(t+\psi), \qquad x^- = \frac{R^2}{2}(t-\psi), \qquad \theta = \frac{y}{R}, \qquad \rho = \frac{r}{R}$$

and then enforcing $\rho \to 0$ and $\theta \to 0$ by taking the limit $R \to \infty$. For finite x^-, the particle goes along a path where $t \sim \psi$. Under these transformations, the metric becomes that of a plane wave:

$$ds^2 = -4dx^+dx^- - \mu^2\vec{z}^2dx^{+2} + d\vec{z}^2$$

where $\vec{z}$ parametrises a point in R^8 and μ is a mass parameter that has been introduced.

The free IIB string on a plane wave background is exactly solvable. The momentum can be decomposed into Fourier modes, where $n < 0$ indicates right movers and $n > 0$ indicates left movers. In terms of these modes, the Hamiltonian can be written as [19], [17]

$$2p^- = -p^+ = \sum_{n=-\infty}^{n=\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \tag{3.3}$$

N_n counts the occupation number of the nth mode, and is the eigenvalue of a number operator made from creation and annihilation operators, $(a_n^I)^\dagger(a_n^I)$, where $I = 1, \ldots, 8$ labels bosonic excitations. The spectrum of string states can be generated by these creation operators acting on a ground state with momenta $p^- = 0$ and p^+, denoted $|0; p^+\rangle$. In order for a string to be physical, its total momentum must vanish, so the sum of the products $(n \times N_n)$ for all modes must be zero.

We would like to see what states on the gauge theory side match this solution. The S^5 originally had $SO(6)$ symmetry; choosing an equator broke this to $SO(4) \times U(1)$. $U(1)$ is the angle in the 1 2 plane of R^6, generated by the angular momentum $J \equiv -i\partial_\psi$. The $SO(6)$ R-symmetry on the gauge side that rotates the six scalar fields ϕ^I must similarly be broken. This is done by defining a $U(1)$ subgroup corresponding to rotations of the ϕ^1-ϕ^2 plane[1]. The charge of a state under this $U(1)$ rotation is referred to as its R-charge.

We now look at what the light cone coordinate momenta on the string side (p^+ and p^-) correspond to. Because energy is mapped to conformal dimension, the energy operator ($i\partial_t$)

[1]There are different conventions over which two SYM scalars are chosen. For the sake of consistency, the plane defined by ϕ_1 and ϕ_2 will be used throughout this dissertation. BMN originally used ϕ_5 and ϕ_6.

gets mapped to Δ. Tracing the momenta through to the gauge theory:

$$\begin{aligned} 2p^- = -p_- = i(\partial_t + \partial_\psi) &\iff \Delta - J \\ 2p^+ = -p_- = \frac{i}{R^2}(\partial_t - \partial_\psi) &\iff \frac{\Delta + J}{R^2} \end{aligned} \tag{3.4}$$

For finite momenta, we are therefore looking for states where $\Delta - J$ is finite and $\Delta + J$ is of the same order as R^2. Noting that $R^2 \propto \sqrt{N}$ (equation 2.7) and $g_{YM}^2 N \propto R^4/\alpha'^2$ (3.2), the following quantities are useful:

$$\lambda' = \frac{g_{\mathrm{YM}}^2 N}{J^2} \qquad \text{and} \qquad g_2 = \frac{J^2}{N}$$

The limit where these are kept finite is referred to as the BMN limit, and operators with finite g_2 and $\Delta - J$ are referred to as BMN operators.

The ground state $|0; p^+\rangle$ must (by (3.4)) correspond to an operator with $\Delta - J = 0$. Defining $Z = (\phi^1 + i\phi^2)/\sqrt{2}$, the gauge operator which satisfies this is $\mathrm{Tr}(Z^J)$. Next, we can look at operators whose dimension does not depend on coupling parameters. These states are BPS states. From (3.3), recalling that $2p^- = \Delta - J$, we can see that on the string side, these are states where $n = 0$. On their own, these will generate the flat space spectrum. On the gauge side, states dual to $(a_0^I)^\dagger|0; p^+\rangle$ are created by adding impurities and then summing over all possible placements within $\mathrm{Tr}(Z^J)$. The impurities consist of fields with $\Delta - J = 1$, for example one of the other scalars (ϕ^I where $I = 3, \ldots, 6$), or derivatives with respect to Z. Therefore,

$$\frac{1}{\sqrt{J}} \sum \mathrm{Tr}[\ldots Z\phi_{i_1} Z \ldots Z\phi_{i_k} Z \ldots] \iff a_0^{i_1} a_0^{i_2} \ldots a_0^{i_k} |0; p^+\rangle \tag{3.5}$$

Berenstein, Maldacena and Nastase's breakthrough in [17] was to extend this to states which are not BPS, but are still close to BPS in that they consist of a set number of excitations away from a BPS state. The proposal is that modes with $n \neq 0$ should correspond to operators where the impurities are introduced with a position (and n) dependent phase given by $\exp[2\pi i n l/J]$, where l is the position of the impurity within the trace. So, for instance,

$$(a_n^8)^\dagger|0; p^+\rangle \iff \frac{1}{\sqrt{J}} \sum_{l=1}^{J} \frac{1}{\sqrt{J} N^{\frac{J+1}{2}}} \mathrm{Tr}\left[Z^l \phi^4 Z^{J-l}\right] e^{\frac{2\pi i n l}{J}}$$

All states with only one impurity disappear, because the trace can be used to move the impurity

to the front, and the exponentials sum to zero. This is good, because $(a_n^8)^\dagger|0;p^+\rangle$ does not have total momentum equal to zero, and is thus not a physical state. The simplest states will therefore need at least two impurities, corresponding to excitations with momenta that cancel. For many excitations, the general form is

$$(a_{n_1})^\dagger \dots (a_{n_m})^\dagger|0;p^+\rangle \iff \frac{1}{\sqrt{J}} \sum_{l_l,\dots,l_m=1}^{J} \frac{1}{\sqrt{J}N^{\frac{J+m}{2}}} \mathrm{Tr}\left[\phi \dots Z\phi Z \dots Z\phi Z \dots\right] \times \\ \times\, e^{[2\pi i(n_1 l_1 + \dots + n_m l_m)]/J} \tag{3.6}$$

where the freedom of the trace has been used to put one of the impurities in the front. The assumption is always made that J is large, so that the trace behaves like a "dilute gas", with many Zs in between each impurity. BMN states of this form, with two impurities, will be of interest later in this thesis. The condition of zero total momentum means that the values of n for all the impurities must sum to zero. If this were not the case on the gauge side, cyclicity of the trace would make the operator zero. From equation (3.6), in the BMN limit where $N, J \to \infty$ with $J \sim \sqrt{N}$, the contribution of an impurity with phase n is

$$(\Delta - J)_n = \sqrt{1 + \lambda' n^2}$$

which matches equation (3.3). The AdS/CFT correspondence is thus shown to hold for a class of states that go beyond the BPS condition.

3.2 Spin Chains

In the BMN limit, the AdS/CFT correspondence was examined by looking at a particular background in which the string theory could be solved, and finding the corresponding gauge theory operators. In contrast, Minahan and Zarembo [20] followed the opposite approach. They proposed starting with a perturbative examination of operators with two or more impurities on the gauge theory side. The corresponding string spectrum can then be constructed by solving for the anomalous dimensions of these operators. This lead to a study of spin chain models, which has received much attention.

[20] started by looking at operators constructed from the six scalars that do not contain

derivates. They are thus of the form

$$\mathcal{O}[\psi] = \psi^{i_1,\ldots,i_L}\mathrm{Tr}(\Phi_{i_1},\ldots,\Phi_{i_L})$$

The renormalisation properties of these operators defines a matrix of anomalous dimensions, Γ. If an operator $\mathcal{O}_n$ is an eigenvalue of this matrix with eigenvalue γ_n, then its correlator is of the form

$$\langle \mathcal{O}_n(x)\mathcal{O}_n(y)\rangle = \frac{\text{const}}{|x-y|^{2(L+\gamma_n)}}$$

The (finite-dimensional) Hilbert space of operators with bare dimension L is of the form $\mathcal{H} = V_1 \otimes \ldots \otimes V_L$ where each of the V_l is a six dimensional vector that corresponds to an index from $SO(6)$, the group that rotates the SYM scalars. This Hilbert space is isomorphic to the Hilbert space of a one dimensional lattice with six lattice sites where the ends are identified. Each lattice site is home to a six dimensional vector. This describes an (integrable) spin chain, which are well known from the study of magnetism. The matrix of anomalous dimensions is an operator in $\mathcal{H}$, and it corresponds to the Hamiltonian of the spin chain. Thus we can find the anomalous dimensions of operators by finding the energy eigenvalues of spin chains. In [20], Minahan and Zarembo used this approach to exact results for BMN operators with two impurities, and $1/J$ corrections for BMN operators with many impurities, up to first order.

The relationship between the CFT and an integrable system raises the interesting possibility that planar $\mathcal{N} = 4$ SYM theory is integrable. There is also increasing evidence that free IIB string theory on $AdS_5 \times S^5$ could be integrable ([21] contains a short summary of developments in solving the relevant sigma model). If both theories are integrable, then the spectra on both sides could be calculated exactly. This would make the AdS/CFT correspondence falsifiable – if the spectra do not match, the conjecture is false, and if they do match then it is supported.

Working on the assumption that both models are completely integrable, Staudacher [21] reasoned that the S-matrix would be the most appropriate object to consider, as it determines the spectrum of a quantum system. This is potentially far simpler than calculating the spectra perturbatively, because integrable systems do not necessarily have simple Hamiltonians. The proposed technique is based on an application of the Bethe Ansatz [22].

The simplest case to consider is one loop scattering in the $SU(2)$ bosonic sector of planar $\mathcal{N} = 4$ SYM theory. In this sector, operators consist of sums of all possible ordering of states of

the form $\mathrm{Tr}\phi^M Z^J$. These states are re-expressed as spin chains of total length $L = M + J$. For instance, the two body states are of the form

$$|\Psi\rangle = \sum_{1 \le x_1 < x_2 < L} \Psi(x_1, x_2) |Z^{x_1 - 1} \phi Z^{x_2 - x_1 - 1} \phi Z^{L - x_2}\rangle$$

so that the first impurity is at the x_1th lattice site and the second is at the x_2th within the spin chain ket. The Hamiltonian is of the form

$$H = \sum_{x=1}^{L} (1 - \mathcal{P}_{x,x+1}) = \sum_{x=1}^{L} \frac{1}{2} (1 - \vec{\sigma}_x \cdot \vec{\sigma}_{x+1})$$

where $\mathcal{P}_{x,x+1}$ is the permutation operator which swops the xth and $(x+1)$th lattice sites, and σ_x is a three dimensional vector of the Pauli matrices at the xth site.

The analysis now follows the standard procedure for the application of the Bethe Ansatz to the spin 1/2 Heisenberg spin chain model ([23] contains a good introduction to this approach in a different context). The Hamiltonian leads to two sets of difference equations, depending on whether the impurities are adjacent or not. For adjacent impurities,

$$\begin{aligned} E_0 \Psi(x_1, x_2) &= 2\Psi(x_1, x_2) - \Psi(x_1 - 1, x_2) - \Psi(x_1, x_2 + 1) \\ (x_2 = x_1 + 1) & \end{aligned} \tag{3.7}$$

and for non-adjacent impurities,

$$\begin{aligned} E_0 \Psi(x_1, x_2) &= 2\Psi(x_1, x_2) - \Psi(x_1 - 1, x_2) - \Psi(x_1 + 1, x_2) \\ (x_1 > x_1 + 1) \qquad & + 2\Psi(x_1, x_2) - \Psi(x_1, x_2 - 1) - \Psi(x_1, x_2 + 1) \end{aligned} \tag{3.8}$$

Bethe's ansatz is inspired by one dimensional scattering. The two impurities can be thought of as excitations which circle around the spin chain with momenta k_1 and k_2. When they meet[2] they can either scatter off each other or pass through each other unaffected. The scattering probability is governed by a scattering amplitude, $S(x_1, x_2)$, and will exchange the momenta of

[2] provided that the system is integrable

the excitations. The ansatz is thus of the form

$$\Psi(x_1, x_2) = \underset{\substack{\uparrow \\ \text{(preserve momenta)}}}{e^{i(k_1 x_1 + k_2 x_2)}} + \underset{\substack{\uparrow \\ \text{(swap momenta)}}}{S(x_1, x_2) e^{i(k_1 x_2 + k_2 x_1)}}$$

The energy, E_0, can be found by substituting this into the difference equation for non-adjacent impurities (3.8)

$$E_0 = 4\left[\sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right)\right] \tag{3.9}$$

which generalises easily to more impurities. To find the scattering amplitude, one must then use the special case for adjacent impurities (3.7). This makes physical sense, because in the absence of the special case when $x_2 = x_1 + 1$, the two impurities are unaffected by each other, and hence no scattering will occur. The solution for $S(x_1, x_2)$ is

$$S(x_1, x_2) = -\frac{1 + e^{i(k_1+k_2)} - 2e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2e^{ik_2}}$$

The spin chain is periodic, so we must identify the $(L+1)$th lattice site with the first site. This, together with the restriction that the second impurity must appear later than the first leads to the periodic boundary conditions

$$\Psi(x_1, x_2) = \Psi(x_2, x_1 + L) \tag{3.10}$$

Spin chains are of interest here, because the states that are derived later in this dissertation resemble spin chains, and much of the "spin chain thinking" will be applicable there.

Chapter 4

Collective field theory

4.1 Variable change

This chapter contains a general outline of the collective field theory technique; this is applied to the specific matrix models that are of interest in later chapters. The idea behind the collective field theory technique [24] is to change variables from the original degrees of freedom to an infinite new set. These are typically invariant under the symmetries of the system. Because there are infinitely many, the new variables will be overcomplete when the original theory has finitely many variables. When N tends to infinity, however, the new variables become independent, and thus the collective field theory approaches the original theory. The approach was originally based on well-known techniques used in N-body statistical mechanics problems.

Let the original variables be denoted by q_i, and the new ones by ϕ_α. The notation is intentionally left as general as possible. The values that i and α can take on depends on the particular case under consideration. For example, if we are dealing with a single matrix M_{ij}, then $i \to (i,j),\ i,j = 1,\ldots,N$. If there are d matrices, $M^a_{ij},\ a = 1,\ldots d$, then $i \to (a,i,j),\ a = 1,\ldots d,\ i,j = 1,\ldots N$. α can also take on a variety of values, and could even be a continuous index in the infinite limit.

Let the conjugate momentum for each coordinate q_i be P_i. A basic Hamiltonian consisting with an interaction potential V is then

$$\begin{aligned} H &= \frac{1}{2}\sum_{i,j}\Big[P_iP_j + V\big(\{q_i\}\big)\Big] \\ &= \frac{1}{2}\sum_{i,j}\Big[-\frac{\partial^2}{\partial q_i \partial q_j} + V\big(\{q_i\}\big)\Big] \end{aligned} \tag{4.1}$$

the goal is to express V and the Hamiltonian as a (possibly infinite) combination of functions, $\phi\big(\{q_i\}\big)$, taking into account the Jacobian that arises from the change in variables. Applying the chain rule to the first term yields

$$\begin{aligned} -\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial q_i \partial q_j} &= -\frac{1}{2}\sum_{i,j}\sum_{\alpha}\frac{\partial}{\partial q_i}\left(\frac{\partial \phi_\alpha}{\partial q_j}\frac{\partial}{\partial \phi_\alpha}\right) \\ &= -\frac{1}{2}\sum_{i,j}\sum_{\alpha,\beta}\left(\frac{\partial^2 \phi_\alpha}{\partial q_i \partial q_j}\frac{\partial}{\partial \phi_\alpha} + \frac{\partial \phi_\alpha}{\partial q_i}\frac{\partial \phi_\beta}{\partial q_j}\frac{\partial^2}{\partial \phi_\alpha \partial \phi_\beta}\right) \\ &= -\frac{1}{2}\sum_{\alpha,\beta}\left(\omega_\alpha \frac{\partial}{\partial \phi_\alpha} + \Omega_{\alpha,\beta}\frac{\partial^2}{\partial \phi_\alpha \partial \phi_\beta}\right) \end{aligned} \tag{4.2}$$

where ω and Ω have been defined such that

$$\omega_\alpha = \sum_{i,j}\frac{\partial^2 \phi_\alpha}{\partial q_i \partial q_j} \tag{4.3}$$

$$\Omega_{\alpha,\beta} = \sum_{i,j}\frac{\partial \phi_\alpha}{\partial q_i}\frac{\partial \phi_\beta}{\partial q_j} \tag{4.4}$$

4.2 Jacobian

The functions upon which the Hamiltonian operates were functions of the q_i, but in the new formalism we would like them to be functionals of the invariant states that we have introduced. A change of variables implies the introduction of a Jacobian. Inner products should be conserved if the coordinate change is valid. Therefore, if $\psi(q)$ are functions of the original variables, and $\Psi(\phi)$ are functionals of the new variables, then we want

$$\int [dq]\psi^\star(q)\psi(q) = \int [d\phi] J \Psi^\star(\phi)\Psi(\phi) \tag{4.5}$$

where J is the Jacobian that results from the change in variables.

We would like to absorb the Jacobian by defining $\Psi \to J^{1/2}\Psi$. This however has implications for operators in the theory, as can be seen by looking at the expectation value of the derivative $\partial/\partial\phi_\alpha$.

$$\left\langle \Psi_1 \left| \frac{\partial}{\partial \phi_\alpha} \right| \Psi_2 \right\rangle = \int [d\phi] J^{1/2}\Psi_1^\star J^{1/2}\frac{\partial}{\partial \phi_\alpha} J^{-1/2} J^{1/2}\Psi_2 \tag{4.6}$$

This means that (making use of the chain rule)

$$\begin{aligned}\frac{\partial}{\partial\phi_\alpha} &\rightarrow J^{1/2}\frac{\partial}{\partial\phi_\alpha}J^{-1/2}\\ &= \frac{\partial}{\partial\phi_\alpha}-\frac{1}{2}\frac{1}{J}\frac{\partial J}{\partial\phi_\alpha}\\ &= \frac{\partial}{\partial\phi_\alpha}-\frac{1}{2}\frac{\partial\ln J}{\partial\phi_\alpha}\end{aligned} \tag{4.7}$$

In what follows, any repeated indices are regarded as being summed according to Einstein's summation convention. The new expression for the derivative (4.7) is substituted into the kinetic term (4.2) (ignoring the overall factor of $-1/2$ for now), and ∂_α is used as a shorthand for $\partial/\partial\phi_\alpha$ to produce[1]

$$\begin{aligned}&\omega_\alpha\partial_\alpha+\Omega_{\alpha,\beta}\partial_\alpha\partial_\beta\\ &=\omega_\alpha\left(\partial_\alpha-\frac{1}{2}(\partial_\alpha\ln J)\right)+\Omega_{\alpha,\beta}\left(\partial_\alpha-\frac{1}{2}(\partial_\alpha\ln J)\right)\left(\partial_\beta-\frac{1}{2}(\partial_\beta\ln J)\right)\\ &=w_\alpha\partial_\alpha-\frac{1}{2}\omega_\alpha(\partial_\alpha\ln J)+\Omega_{\alpha,\beta}\partial_\alpha\partial_\beta-\Omega_{\alpha,\beta}\partial_\alpha(\partial_\beta\ln J)+\frac{1}{4}\Omega_{\alpha,\beta}(\partial_\alpha\ln J)(\partial_\beta\ln J)\\ &=w_\alpha\partial_\alpha-\frac{1}{2}\omega_\alpha(\partial_\alpha\ln J)+[\partial_\alpha\Omega_{\alpha,\beta}\partial_\beta-(\partial_\alpha\Omega_{\alpha,\beta})\partial_\beta]\\ &\quad-[\Omega_{\alpha,\beta}(\partial_\alpha\partial_\beta\ln J)+\Omega_{\alpha,\beta}(\partial_\beta\ln J)\partial_\alpha]+\frac{1}{4}\Omega_{\alpha,\beta}(\partial_\alpha\ln J)(\partial_\beta\ln J)\end{aligned} \tag{4.8}$$

Because these terms are part of an Hamiltonian, they must be Hermitian, which means that any non-Hermitian terms must sum to zero. Ω, ω and derivatives of $\ln J$ are not operators, so they are automatically Hermitian. Also, $\partial_\alpha\Omega_{\alpha,\beta}\partial_\beta$ is Hermitian. This leaves only the other terms in (4.8) which end in a derivative:

$$\begin{aligned}0 &= w_\alpha\partial_\alpha-(\partial_\alpha\Omega_{\alpha,\beta})\partial_\beta-\Omega_{\alpha,\beta}(\partial_\beta\ln J)\partial_\alpha\\ \partial_\beta\ln J &= \Omega^{-1}_{\beta,\alpha}\omega_\alpha-\Omega^{-1}_{\beta,\alpha}\partial_\gamma\Omega_{\gamma\alpha}\end{aligned} \tag{4.9}$$

Ω^{-1} has been introduced as the inverse of Ω, i.e. the function that will satisfy

[1]The scope of derivatives will be indicated by the presence or absence of brackets. For example, given $\partial_\alpha\ln J$, the derivative should be read as acting on $\ln J$ as well as everything that follows, whereas in $(\partial\alpha\ln J)$, the derivative acts only on $\ln J$.

$$\Omega^{-1}_{\alpha\beta}\Omega_{\beta\gamma} = \delta_{\alpha\gamma} \tag{4.10}$$

Equation (4.9) gives a (differential) expression for the Jacobian J that would otherwise have been far more difficult to calculate. Substituting this into the remaining Hermitian terms in (4.8) gives

$$\begin{aligned}
&-\frac{1}{2}\omega_\alpha(\partial_\alpha \ln J) + \partial_\alpha\Omega_{\alpha,\beta}\partial_\beta - \Omega_{\alpha,\beta}(\partial_\alpha\partial_\beta \ln J) + \frac{1}{4}\Omega_{\alpha,\beta}(\partial_\alpha \ln J)(\partial_\beta \ln J) \\
&= -\frac{1}{2}\omega_\alpha(\Omega^{-1}_{\alpha,\beta}\omega_\beta - \Omega^{-1}_{\alpha,\beta}\partial_\gamma\Omega_{\gamma\beta}) + \partial_\alpha\Omega_{\alpha,\beta}\partial_\beta \\
&\quad - \Omega_{\alpha,\beta}(\partial_\beta(\Omega^{-1}_{\alpha,\lambda}\omega_\lambda - \Omega^{-1}_{\alpha,\lambda}\partial_\gamma\Omega_{\gamma\lambda})) \\
&\quad + \frac{1}{4}\Omega_{\alpha,\beta}(\Omega^{-1}_{\alpha,\lambda}\omega_\lambda - \Omega^{-1}_{\alpha,\lambda}\partial_\gamma\Omega_{\gamma\lambda})(\Omega^{-1}_{\beta,\mu}\omega_\mu - \Omega^{-1}_{\beta,\mu}\partial_\gamma\Omega_{\gamma\mu})
\end{aligned} \tag{4.11}$$

In the finite case, all of the above terms would need to be taken into account. We, however, will only be interested in what happens in the large N limit. To make the N dependence explicit, the following are rescaled:

$$\begin{aligned}
\phi_\alpha &\rightarrow \sqrt{N}\phi_\alpha \\
\partial_\alpha &\rightarrow \frac{1}{N}\partial_\alpha \\
\omega_\alpha &\rightarrow \sqrt{N}\omega_\alpha \\
\Omega_{\alpha,\beta} &\rightarrow \frac{1}{N}\Omega_{\alpha,\beta}
\end{aligned} \tag{4.12}$$

Looking at (4.9), only the first term in the expression for the Jacobian will contribute as $N \rightarrow \infty$. In (4.11), all the terms are interaction potential terms except for the derivative term $\partial_\alpha\Omega_{\alpha,\beta}\partial_\beta$. To generate the spectrum plus fluctuations around the spectrum in the large N limit, we need only keep the interaction terms with highest N dependence, plus the derivative term, namely:

$$\begin{aligned}
&-\frac{1}{2}\omega_\alpha\Omega^{-1}_{\alpha,\beta}\omega_\beta + \frac{1}{4}\Omega_{\alpha\beta}\Omega^{-1}_{\alpha\lambda}\omega_\lambda\Omega^{-1}_{\beta\mu}\omega_\mu + \partial_\alpha\Omega_{\alpha,\beta}\partial_\beta \\
= \quad &-\frac{1}{4}\omega_\alpha\Omega^{-1}_{\alpha,\beta}\omega_\beta + \partial_\alpha\Omega_{\alpha,\beta}\partial_\beta
\end{aligned}$$

Once the overall factor of $-1/2$ in equation (4.2) is reintroduced, the final kinetic term is

$$K = -\frac{1}{2}\partial_\alpha \Omega_{\alpha,\beta}\partial_\beta + \frac{1}{8}\omega_\alpha \Omega^{-1}_{\alpha,\beta}\omega_\beta \tag{4.13}$$

We have derived the necessary ingredients to construct a collective field theory Hamiltonian in general variables. The form of the potential V will depend on the particular system under consideration. The next chapter goes on to apply the above formalism to the case of a single matrix.

Chapter 5

Free single matrix model

5.1 Physical justification

Having developed the general mathematical formalism of the collective field theory technique, we now apply it to a model with a single Hermitian matrix. For the purpose of this thesis, this will be viewed as merely a stepping stone to the more complicated model where another matrix is introduced as an impurity into the single matrix background. It is worth noting however, that the single matrix model has been studied as an end in itself, for example when Jevicki and Sakita [24] developed the collective field theory technique in the form that we will use. They considered the Langrangian introduced by Bezin, Itzykson, Parisi and Zuber ([25])

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}(\dot{M}^2) - \frac{1}{2}\mathrm{Tr}(M^2) - \frac{g}{N}\mathrm{Tr}(M^4) \tag{5.1}$$

In the Hamiltonian formulation, this is

$$H = -\frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2}{\partial M_{ij}M_{ji}} + \frac{1}{2}\mathrm{Tr}(M^2) + \frac{g}{N}\mathrm{Tr}(M^4) \tag{5.2}$$

Using the collective field theory techniques described in this chapter, together with some standard complex analysis, [24] reproduced the BIPZ energies from [25].

In the context of the AdS/CFT correspondence, it can be shown (see chapter 6) that a single Hermitian matrix model with an harmonic potential can be used to describe 1/2 BPS states on the SYM side.

5.2 Collective field theory

We will be interested in Hamiltonians that have an harmonic potential. The general Hamiltonian in (4.1) is expressed in terms of an Hermitian matrix M,

$$H_0 = -\frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial}{\partial M_{ij}M_{ji}} + \frac{w^2}{2}\mathrm{Tr}(M^2) \tag{5.3}$$

The subscript 0 denotes that this is the zero-impurity Hamiltonian, in contradistinction to later Hamiltonians which will contain added impurities.

Let us change variables to the invariant set

$$\mathrm{Tr}(M), \mathrm{Tr}(M^2), \ldots, \mathrm{Tr}(M^N) \tag{5.4}$$

which is invariant under the unitary transformation $M \to U^\dagger MU$. At most N of these are independent for finite N, but as $N \to \infty$, the $\mathrm{Tr}(M^n)$ become independent variables. V can then be expressed as a combination of infinite sums of the $\mathrm{Tr}(M^n)$, of the form

$$\begin{aligned}\psi_k &= \mathrm{Tr}(e^{ikM}) \\ &= N + ik\mathrm{Tr}(M) - \frac{k^2}{2}\mathrm{Tr}(M^2) + \ldots\end{aligned} \tag{5.5}$$

where k is a real number, which is a specific case of the generic loop index α used in chapter 4.

We now need to re-express the Hamiltonian (5.3) in terms of the new variables ψ_k. Starting with the kinetic part, and applying the results of chapter 4 (specifically (4.2) and (4.13)) to the first term yields

$$\begin{aligned}K_0 \equiv -\frac{1}{2}\frac{\partial}{\partial M_{ij}\partial M_{ji}} &= -\frac{1}{2}\left(\omega_k\frac{\partial}{\partial\psi_k} + \Omega_{k,k'}\frac{\partial^2}{\partial\psi_k\partial\psi_{k'}}\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial\psi_k}\Omega_{k,k'}\frac{\partial}{\partial\psi_{k'}} + \frac{1}{8}\omega_k\Omega^{-1}_{k,k'}\omega_{k'}\end{aligned} \tag{5.6}$$

where ω and Ω are now

$$\omega_k = \sum_{i,j}\frac{\partial^2\psi_k}{\partial M_{ij}\partial M_{ji}} \tag{5.7}$$

$$\Omega_{k,k'} = \sum_{i,j}\frac{\partial\psi_k}{\partial M_{ij}}\frac{\partial\psi_{k'}}{\partial M_{ji}} \tag{5.8}$$

and Ω^{-1} is the inverse of Ω.

Using the formula

$$\frac{\partial}{\partial M_{ij}}\left(e^{ikM}\right)_{ab} = \int_0^1 d\alpha \left(e^{i\alpha M}\right)_{ai} (ik) \left(e^{i(1-\alpha)kM}\right)_{jb} \tag{5.9}$$

ω and Ω become

$$\begin{aligned}
\omega_k &= ik\frac{\partial}{\partial M_{ij}}\left(e^{ikm}\right)_{ij} \\
&= -k^2 \int_0^1 d\alpha \left(\text{Tr } e^{i\alpha kM}\right)\left(\text{Tr } e^{i(1-\alpha)kM}\right) \\
&= -k\int_0^k dk' \psi_{k'}\psi_{k-k'} \qquad (5.10) \\
\Omega_{k,k'} &= ik\left(e^{ikM}\right)_{ji} ik'\left(e^{ik'M}\right)_{ij} \\
&= -kk'\psi_{k+k'} \qquad (5.11)
\end{aligned}$$

5.3 Density description

It is easier to work with the Fourier transforms of the above operators, rather than using them in their current form. We thus move from k space to x space, where the invariant operators become density functions of the eigenvalues of the matrix M. Representing these eigenvalues as λ_i, we define $\psi(x)$ as

$$\psi(x) = \int \frac{dk}{2\pi}\, e^{-ikx}\psi_k = \text{Tr}(\delta(x-M)) = \sum_{i=1}^{N} \delta(x-\lambda_i) \tag{5.12}$$

Applying Fourier transforms to (5.10) and (5.11) produces

$$\begin{aligned}
\omega(x) &= \int \frac{dk}{2\pi} e^{-ikx} \omega_k \\
&= \int \frac{dk}{2\pi} e^{-ikx} (-k) \int_0^k dk' \psi_{k'} \psi_{k-k'} \\
&= \int \frac{dk}{2\pi} e^{-ikx} (ik) \sum_{i \neq j} \frac{e^{ik\lambda_i} - e^{ik\lambda_j}}{\lambda_i - \lambda_j} \\
&= \int \frac{dk}{2\pi} e^{-ikx} (ik) \int dy \oint dz \psi(y) \psi(z) \frac{e^{iky} - e^{ikz}}{y - z} \\
&= -2 \frac{\partial}{\partial x} \left[\psi(x) \oint dz \frac{\psi(z)}{x - z} \right] \qquad (5.13) \\
\Omega(x,y) &= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ikx} e^{-ik'y} (-kk') \mathrm{Tr} \left(e^{i(k+k')M} \right) \\
\overset{(k+k' \to l)}{} &= \frac{\partial}{\partial x} \frac{\partial}{\partial y} \int \frac{dl}{2\pi} \int \frac{dk'}{2\pi} e^{-i(l-k')x} e^{-ik'y} (-kk') \mathrm{Tr} \left(e^{ilM} \right) \\
&= \frac{\partial}{\partial x} \frac{\partial}{\partial y} \left(\psi(x) \delta(x-y) \right) \qquad (5.14)
\end{aligned}$$

When expressing the kinetic term (5.6) in the density description, the sums over the indices become integrals:

$$K_0 = \int dx \int dy \ \left[-\frac{1}{2} \frac{\partial}{\partial \psi(x)} \Omega(x,y) \frac{\partial}{\partial \psi(y)} + \frac{1}{8} \omega(x) \Omega^{-1}(x,y) \omega(y) \right] \qquad (5.15)$$

Substituting in equations (5.13) and (5.14) and integrating by parts gives

$$\begin{aligned}
K_0 &= \int dx \int dy \left[-\frac{1}{2} \partial_x \frac{\partial}{\partial \psi(x)} \psi(x) \delta(x-y) \partial_y \frac{\partial}{\partial \psi(y)} \right. \\
&\quad \left. + \frac{1}{2} \left(\psi(x) \oint dz \frac{\psi(z)}{x - z} \right) \partial_x \partial_y \Omega^{-1}(x,y) \left(\psi(y) \oint dv \frac{\psi(v)}{y - v} \right) \right]
\end{aligned}$$

To simplify the second term, we examine the double derivative of the inverse of Ω. In the density description, (4.10) becomes

$$\begin{aligned}
\int dy \ \Omega^{-1}(x,y) \Omega(y,z) &= \delta(x-z) \\
\int dy \ \Omega^{-1}(x,y) \partial_y \partial_z \psi(y) \delta(y-z) &= \delta(x-z) \\
-\partial_z \left(\partial_x \partial_z \Omega^{-1}(x,y) \psi(z) \right) &= \partial_x \delta(x-y) \\
\partial_x \partial_z \Omega^{-1}(x,z) &= \frac{\delta(x-z)}{\psi(x)} \qquad (5.16)
\end{aligned}$$

We will then need to deal with a term that looks like

$$\int dx\ \psi(x) \left(\fint dy \frac{\psi(y)}{x-y} \right)^2$$
$$= \int dk_1 \int dk_2 \int dk_3\ e^{-i(k_1 x + k_2 y + k_3 z)} \int dx \fint dy \fint dz\ \psi_{k_1} \left(\frac{\psi_{k_2}}{x-y} \right) \left(\frac{\psi_{k_3}}{x-z} \right)$$

We note that

$$\fint dy \frac{e^{-iky}}{x-y} = i\pi\epsilon(k) e^{-ikx}$$

where $\epsilon(k) = 1$ for positive k and -1 for negative k. Adding contour integral contributions depending on whether each k is positive or negative leads to the following identity:

$$\int dx\ \psi(x) \left(\fint dy \frac{\psi(y)}{x-y} \right)^2 = \frac{\pi^2}{3} \int dx\ \psi^3(x) \tag{5.17}$$

Putting it all together, the kinetic term is

$$K_0 = -\frac{1}{2} \int dx\ \partial_x \frac{\partial}{\partial\psi(x)} \psi(x) \partial_x \frac{\partial}{\partial\psi(x)} + \frac{\pi^2}{6} \int dx\ \psi^3(x) \tag{5.18}$$

Finally, we express the Jacobian, equation (4.9) in the density description. Only the first term is of leading order in N, which becomes

$$\partial_x \frac{\partial \ln J}{\partial\psi(x)} = \partial_x \int dy\ \Omega^{-1}(x,y)\omega(y)$$

Using equations (5.13) and (5.16), this is

$$\partial_x \frac{\partial \ln J}{\partial\psi(x)} = 2 \fint dy \frac{\psi(y)}{x-y} \tag{5.19}$$

We use equation (5.12) to express the potential in terms of x:

$$\begin{aligned} \frac{w^2}{2} \mathrm{Tr}(M^2) &= \frac{w^2}{2} \int dx\, \mathrm{Tr}(\delta(x-M)) x^2 \\ &= \frac{w^2}{2} \int dx\, \psi(x) x^2 \end{aligned} \tag{5.20}$$

Adding the kinetic and potential terms yields

$$\begin{aligned} H_0 &= -\frac{1}{2}\int dx\,\partial_x \frac{\partial}{\partial\psi(x)}\psi(x)\partial_x\frac{\partial}{\partial\psi(x)} + \frac{\pi^2}{6}\int dx\,\psi^3(x) \\ &\quad + \frac{w^2}{2}\int dx\,\psi(x)x^2 \end{aligned} \tag{5.21}$$

The matrices that went into creating equation (5.21) were $N \times N$ matrices, but this N dependence is not immediately apparent. The necessary condition on ψ, which can be seen from (5.12), is

$$\int dx\,\psi(x) = N \tag{5.22}$$

To enforce this constraint, we introduce a Lagrange multiplier μ, and the Hamiltonian becomes

$$-\frac{1}{2}\int dx\,\partial_x\frac{\partial}{\partial\psi(x)}\psi(x)\partial_x\frac{\partial}{\partial\psi(x)} + \int dx\left(\frac{\pi^2}{6}\psi^3(x) + \psi(x)\left(\frac{w^2x^2}{2} - \mu\right)\right) + \mu N \tag{5.23}$$

To make the N dependence explicit, we rescale according to (4.12), with

$$x \to \sqrt{N}x\ , \quad \psi \to \sqrt{N}\psi, \quad \mu \to N\mu \quad \text{and} \quad -i\frac{\partial}{\partial\psi} \equiv \Pi \to \frac{1}{N}\Pi$$

to get

$$H_0 = \frac{1}{2N^2}\int dx\ \partial_x\Pi(x)\psi(x)\partial_x\Pi(x) + N^2\left(\int dx\,\frac{\pi^2}{6}\psi^3(x) + \psi(x)\left(\frac{w^2x^2}{2} - \mu\right)\right) + \mu N^2 \tag{5.24}$$

5.4 Energy spectrum

Equation (5.24) has a standard solution (see for example [24], [26], [27] and [6]), obtained by minimising the effective potential as $N \to \infty$. The second term of equation (5.24) is of leading order in N. This will therefore generate the background, and the first term will generate the fluctuations around this background. We will examine these fluctuations by defining the background ϕ_0 as the result of extremising the second term of (5.24) with respect to $\psi(x)$,

$$\pi\phi_0 = \sqrt{2\mu - w^2x^2} \tag{5.25}$$

We then look at small shifts away from this background

$$\psi(x) = \phi_0 + \frac{1}{\sqrt{\pi}N}\partial_x\eta; \qquad \partial_x\Pi(x) = -\sqrt{\pi}NP(x) \tag{5.26}$$

η and P are cannonical conjugates, so

$$[P(x), \eta(y)] = -i\delta(x-y) \tag{5.27}$$

Because we are in effect expanding around the extremum, the leading contribution to the small fluctuations will come from the terms that are quadratic in $\partial_x\eta$ and P. The quadratic terms from equation (5.24) are

$$H_0^{(2)} = \frac{1}{2}\int dx\, \pi\phi_0 P^2(x) + \frac{1}{2}\int dx\, \pi\phi_0(\partial_x\eta)^2$$

This can be further simplified by introducing a coordinate q, called the "time of flight", such that

$$\frac{dx}{dq} = \pi\phi_0 \tag{5.28}$$

Looking at equation (5.25), and using the formula for the derivative of arccos, one arrives at

$$\begin{aligned} & \arccos\left(\sqrt{\frac{w}{2}}x\right) = -wq \\ \text{and therefore} \qquad & x(q) = -\sqrt{\frac{2}{w}}\cos(wq) \\ \text{and} \qquad & \pi\phi_0 = \sqrt{2w}\sin(wq) \end{aligned} \tag{5.29}$$

for $0 \le q \le \pi/w$. In terms of q, equation (5.27) becomes

$$[P(x), \eta(y)] = -i\delta(x-y) = -i\frac{\delta(q-q')}{\left|\frac{dx}{dq}\right|} = -i\frac{\delta(q-q')}{\pi\phi_0}$$

If we therefore define $\bar{P}(q) \equiv \pi\phi_0 P(x(q))$ and $\bar{\eta}(q) \equiv \eta(x(q))$, then $\bar{P}(q)$ and $\bar{\eta}(q)$ will also be cannonical conjugates.

Changing variables from x to q, the quadratic Hamiltonian is now

$$\begin{aligned} H_0^{(2)} &= \frac{1}{2}\int (dq\pi\phi_0)\pi\phi_0 P^2(x) + \frac{1}{2}\int (dq\pi\phi_0)\pi\phi_0 \left(\frac{\partial_q \eta}{\pi\phi_0}\right)^2 \\ &= \frac{1}{2}\int dq\bar{P}^2(q) + \frac{1}{2}\int dq(\partial_q\bar{\eta})^2 \end{aligned} \tag{5.30}$$

This is now a standard Klein-Gordon Hamiltonian, and can thus easily be solved. To ensure that the constraint (5.22) is satisfied consistently at all times, we find the classical turning points and impose Dirichlet boundary conditions. The spectrum is then

$$w_j = j \quad ; \qquad \phi_j = \sin(jq). \tag{5.31}$$

Chapter 6

Free two matrix model – adding impurities

We now go in greater detail into the physical reasons for studying the matrix models that we are considering. This will lead to a two matrix model, rather than the single matrix model considered in chapter 5.

In the BMN limit, one looks at states that are close to the plane defined by two of the SYM scalars. The Hamiltonians that one needs to consider will be functions of two of the SYM scalars, and will typically have an harmonic potential. For example, in the case of SYM defined on $S^3 \times R$, the positive curvature of S^3 leads to an action of the form [28]

$$R^3 \frac{\Omega_3}{2g_{YM}^2} \int dt \, \mathrm{Tr} \left(\dot{\phi_1}^2 + \dot{\phi_2}^2 - \frac{1}{R^2}(\phi_1^2 + \phi_2^2) \right) \tag{6.1}$$

where Ω_3 is the volume of S_3 at the boundary of AdS_5. The g_{YM} interaction term of the SYM theory has been neglected for the time being – this will be added in later.

The action in (6.1) can be put in a more general form by dropping the overall factor and denoting the constant that multiplies the potential by w^2. A momentum conjugate to each of the scalars is introduced, in order to move to the Hamiltonian formulation of (6.1).

$$H_{\text{free}} = \frac{1}{2} \mathrm{Tr} \big(P_1^2 + P_2^2 + w^2(\phi_1^2 + \phi_2^2) \big) \tag{6.2}$$

The conjugate momenta and the SYM scalars satisfy cannonical commutation relations. The

angular momentum J is given by

$$J = \frac{1}{2}\mathrm{Tr}(\phi_2 P_1 - \phi_1 P_2)$$

6.1 Angular momentum eigenstates

There is more than one way to define the matrices that go into the model. The first is the more standard approach, which involves creating angular momentum eigenstates. This approach is followed in [3], [29] and [6]. The initial motivation in [3] was to understand which AdS states correspond to the chiral primaries in the SYM theory. It was known that 1/2 BPS Kaluza-Klein modes of the supergravity fields on AdS_5 correspond to the chiral primary operators in the SYM theory. In this identification, the energy of the excited Kaluza Klein modes gets mapped to the dimension of the chiral primary operator. The picture was complicated by [30] however, which showed that the same set of quantum numbers is shared by both Kaluza-Klein modes carrying momentum in S_5, and a stable configuration of spherical branes in S_5. This raised the question of which of the two AdS states should correspond to the chiral primaries in SYM, and what the other state would correspond to. [28] then went on to compound the problem by showing that there is yet another (1/2 BPS) supergravity state that has the same quantum numbers, namely a stable configuration of spherical 3-branes in AdS_5. Matrix models similar to the ones that we will look at were therefore used in an attempt to understand the SYM 1/2 BPS states more fully. While this is not our final goal, the method that was followed will still be relevant.

The approach is to introduce complex matrices by defining (together with their complex conjugates)

$$\begin{aligned} Z &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \Pi &= \frac{1}{\sqrt{2}}(P_1 + iP_2) \end{aligned}$$

These are cannonical conjugates, allowing us to introduce creation and annihilation operators,

following [6, 8, 31]:

$$Z = \frac{1}{\sqrt{2w}}(A + B^\dagger) \ , \quad \Pi = -i\sqrt{\frac{w}{2}}(A^\dagger - B)$$
$$\left(\quad \text{or} \quad A = \frac{1}{\sqrt{2}}\left(\sqrt{w}Z - \frac{i}{\sqrt{w}}\Pi^\dagger\right), \quad B = \frac{1}{\sqrt{2}}\left(\sqrt{w}Z^\dagger - \frac{i}{\sqrt{w}}\Pi\right) \quad \right)$$

In terms of these the Hamiltonian and angular momentum operators are

$$H_{\text{free}} = w\left(\text{Tr}(A^\dagger A) + \text{Tr}(B^\dagger B)\right) \quad \text{and} \qquad J = \text{Tr}(A^\dagger A) - \text{Tr}(B^\dagger B) \tag{6.3}$$

1/2 BPS states are now states without B excitations. To see this, consider the eigenstates associated with the Hamiltonian in equation (6.3), which are of the form

$$\text{Tr}\left[(A^\dagger)^m (B^\dagger)^n\right]|0\rangle$$

The energy and momentum of such a state will be $w(m+n)$ and $(m-n)$ respectively. 1/2 BPS states will saturate the bound that the quantum number corresponding to energy must be greater or equal to that corresponding to momentum, i.e. $E > |J|$ (taking $w = 1$ for the time being). We must therefore have $m + n = \pm(m - n)$, so either n or m must be zero. We arbitrarily choose $n = 0$, which implies that 1/2 BPS states are states without B impurities. These are the chiral primaries in the SYM theory.

In the absence of B operators one can define a single $N \times N$ matrix M as the Hermitian matrix associated with the A, $A^\dagger$ part[1] [6, 8]

$$M = \frac{1}{\sqrt{2w}}(A + A^\dagger) \ , \quad P_M = -i\sqrt{\frac{w}{2}}(A - A^\dagger) \tag{6.4}$$

in terms of which the Hamiltonian is

$$H = \frac{1}{2}\text{Tr}(P_M^2) + \frac{w^2}{2}\text{Tr}(M^2) \tag{6.5}$$

[1] In terms of the original SYM scalars,

$$\begin{aligned} M &= \frac{1}{\sqrt{2w}}\left(\sqrt{w}\phi_1 - \frac{P_2}{\sqrt{w}}\right) \\ P_M &= \sqrt{\frac{w}{2}}\left(\sqrt{w}\phi_2 + \frac{P_1}{\sqrt{2}}\right) \end{aligned}$$

On the SYM side, 1/2 BPS states thus correspond to the single matrix model that has been derived in chapter 5.

The link to 1/2 BPS states in AdS is now understood in terms of the free fermionic droplet picture introduced by Lin, Lunin and Maldacena [5]. They showed that a classical ansatz for the AdS space could have an equivalent energy and flux content to a general fermionic droplet configuration. The geometry of the AdS solutions is completely determined by the configuration of fermionic droplets on a (suitably defined) two dimensional sub-plane of the full ten dimensional space. [6] contains a summary of the link between the fermionic picture produced by the single matrix model (as derived in chapter 5) and the LLM Sugra ansatz.

In order to study states beyond the 1/2 BPS restriction, it is necessary to solve the full two matrix problem corresponding to both A and B excitations. This is in general a very difficult problem. The approach followed by [6] is thus to apply collective field theory techniques to the first part of the Hamiltonian involving A excitations to generate a background. B excitations are then added as "impurities" to perturb this background. We thus add B impurities back into (6.5). Going to a coherent state basis, $B^\dagger \to B$, $B \to \partial/\partial B$, produces

$$H_{\text{free}} = \frac{1}{2}\text{Tr}(P_M^2) + \frac{w^2}{2}\text{Tr}(M^2) + w\text{Tr}\left(B\frac{\partial}{\partial B}\right) \tag{6.6}$$

6.2 Two Higgs scalar model

Another possible physical application is to directly treat the two SYM scalars, ϕ_1 and ϕ_2, as the matrices that go into the two matrix model.

Introducing creation and annihilation operators for the second scalar, ϕ_2,

$$\phi_2 = \frac{1}{\sqrt{2w}}(C + C^\dagger), \qquad P_2 = -i\sqrt{\frac{w}{2}}(C - C^\dagger)$$

the Hamiltonian becomes

$$H = \frac{1}{2}\text{Tr}(P_1^2) + \frac{1}{2}w^2\text{Tr}(\phi_1^2) + w\text{Tr}(C^\dagger C) \tag{6.7}$$

Going to a coherent state basis makes this identical to (6.6).

This approach was first mentioned in [8], but it was not developed fully. Chapter 8 will explore this association of variables in depth, in the presence of a g_{YM} interaction term.

6.3 Collective field theory Hamiltonian

We wish to arrive at the collective field theory Hamiltonian that results from equation (6.6) (or (6.7)), taking into account the presence of B (or C) impurities. We review below the method used in [8]. The approach is similar to that followed in section 5.4, namely looking at small fluctuations away from the (zero-impurity) background. Firstly, a more general set of invariant states that take into account B impurities is introduced:

$$\begin{aligned} \psi_0(k) &= \mathrm{Tr}(e^{ikM}) \\ \psi_1(k_1) &= \mathrm{Tr}(Be^{ik_1M}) \\ &\cdots \\ \psi_s(k_1,k_2,\ldots,k_s) &= \mathrm{Tr}\left(\prod_{i=1}^{s} Be^{ik_iM}\right) \end{aligned} \tag{6.8}$$

Analogous states to these, without any interaction terms, are of interest in and of themselves. For instance, [6] applied a transformation which lead to a 1-1 mapping of the spectrum of the above states to a class of Supergravity wave functions on the $AdS \times S$ background. The wave functions are described in terms of hypergeometric functions. This kind of mapping will not be the focus of this thesis, however. Rather, we view the many-impurity spectrum as an intermediate step in the introduction of the g_{YM} interaction term, explored in chapters 7 and 8.

To see what new fluctuations arise around the zero-impurity background, we introduce a number (s) of impurities. The expectation value of the zero impurity states will be the background, $\pi\phi_0$, introduced in equation (5.25). The s-impurity states take the form of small fluctations around the zero-impurity background, and will thus not pick up an expectation value. Because the expectation value is zero for $s > 0$, we cannot apply an argument analogous to that in (4.6) and (4.7). Derivatives with respect to states with $s > 0$ therefore do not pick up factors involving the Jacobian. The multi-impurity Jacobian is thus the same as the zero impurity Jacobian, namely equation (5.19).

The kinetic term (corresponding to (4.13)) of the many impurity Hamiltonian is [6, 8]

$$\begin{aligned} K_s = & - \frac{1}{2}\sum_s \int dx_1 \dots dx_s \omega_s(x_1,\dots,x_s) \frac{\partial}{\partial\psi_s(x_1,\dots,x_s)} \\ & - \frac{1}{2}\sum_{s,s'} \int dx_1 \dots dx_s \int dy_1 \dots dy'_s \Omega_{s,s'}(x_1,\dots,x_s,y_1,\dots,y_s) \times \\ & \times \frac{\partial^2}{\partial\psi_s(x_1,\dots,x_s)\partial\psi_{s'}(y_1,\dots,y_s)} \end{aligned} \tag{6.9}$$

where ω_s and $\Omega_{s,s'}$ are generalisations of the ω and Ω that we have considered in the case with no impurities, defined in k space as

$$\begin{aligned} \omega_s(k_1,\dots,k_s) &= \frac{\partial^2\psi_s(k_1,\dots,k_s)}{\partial M_{ij}\partial M_{ji}} \\ \Omega_{s;s'}(k_1,\dots,k_s;k'_1,\dots,k'_{s'}) &= \frac{\partial\psi_s(k_1,\dots,k_s)}{\partial M_{ij}} \frac{\partial\psi_{s'}(k'_1,\dots,k'_{s'})}{\partial M_{ji}} \end{aligned} \tag{6.10}$$

To determine the quadratic Hamiltonian, we look for terms within (6.9) that are quadratic in fluctuations away from the background. Consider the first term of (6.9). From equation (5.13) (or equivalently (5.10)), we see that ω splits into two loops when no impurities are present. We expect similar behaviour in the case with impurities, which is confirmed by the calculations in appendix A.1. In general ω_s will split into two loops, with the total number of impurities summing to s. Most of the terms will therefore be cubic (or higher) in fluctuations away from the background, with one contribution from each the loops introduced by ω_s, and one from the derivative. The exception is the case where ω_s splits into a loop with zero impurities and one with s impurities. The zero-impurity loop will include a ϕ_0 background term (see equation (5.26)) and this term will be only quadratic in fluctuations. It is therefore useful to define $\bar{\omega}$ as the piece of ω that contains one zero-impurity loop and one s-impurity loop, as only this piece of the Hamiltonian will be present in the quadratic part.

Consider now the second term in equation (6.9). The action of Ω is to combine two loops, so $\Omega_{s,s'}$ will generate a single loop with $s+s'$ impurities. It is in general again the case that terms will be cubic in fluctuations, with one loop from Ω and two derivatives. The exception in this case is where one of the derivatives is with respect to a zero-impurity loop. Zero impurity derivatives contain a piece that does not contribute towards fluctuations. This can be seen by

noting that zero impurity derivatives transform according to equation (4.7):

$$\frac{\partial}{\partial\psi_0(x)} \to \frac{\partial}{\partial\psi_0(x)} - \frac{1}{2}\frac{\partial \ln J}{\partial\psi_0(x)} \tag{6.11}$$

The Jacobian term contains a piece that will depend only on the background ϕ_0. For the derivatives term therefore, we will keep only the background dependent part of (5.19), i.e.

$$\begin{aligned} \partial_x \frac{\partial \ln J}{\partial\psi_0(x)} &\to 2 \fint dy \frac{\phi_0(y)}{x-y} \\ \text{or} \quad \frac{\partial \ln J}{\partial\psi_0(x)} &\to 2\int_0^x dq \fint dy \frac{\phi_0(y)}{q-y} \end{aligned} \tag{6.12}$$

To ensure that zero impurity loop is present, we set $s' = 0$ in equation (6.9). The quadratic Hamiltonian is thus

$$\begin{aligned} H_s^{(2)} &= -\frac{1}{2}\int dx_1, \ldots, dx_s \bar{\omega}_s(\{x_i\}) \frac{\partial}{\partial\psi_s(\{x_i\})} \\ &\quad + \frac{1}{2}\int dx \int dy_1 \ldots dy_s \Omega_{0,x}(x;\{y_i\}) \frac{\partial \ln J}{\partial\psi_0(x)} \frac{\partial}{\partial\psi_s(\{y_i\})} \end{aligned} \tag{6.13}$$

To find the spectrum of this Hamiltonian, we now need to find $\bar{\omega}$ and Ω as well as the Jacobian J in the density description. Some calculation (see appendices A.1 and A.2 for details) yields:

$$\begin{aligned} \bar{\omega}(k_1, \ldots, k_s) &= -2\sum_{i=1}^{s} \int_0^{k_i} dk' k' \phi_0(k_i - k') \psi_s(k_1, \ldots, k_{i-1}, k', k_{i+1}, \ldots, k_s) \\ \Omega_{0;s}(k_0; k_1, \ldots, k_s) &= -k_0 \left(\sum_{i=1}^{s} k_i \psi_s(k_1, \ldots, k_i + k_0, \ldots, k_s) \right) \end{aligned} \tag{6.14}$$

where we have taken $\psi_0 \to \phi_0$ in order to keep only the background term.

We now take the Fourier transform of all the above operators, starting with the set of gauge invariant states:

$$\begin{aligned} \psi_s(x_1, \ldots, x_s) &= \int \frac{dk_1}{2\pi} e^{-ik_1 x_1} \ldots \int \frac{dk_s}{2\pi} e^{-ik_s x_s} \psi_s(k_1, \ldots, k_s) \\ &= Tr\left(B\delta(x_1 - M) \ldots B\delta(x_s - M)\right) \end{aligned} \tag{6.15}$$

Further calculation (see appendices A.3 and A.4 for details) gives the final result for ω and Ω as

$$\begin{aligned}\bar{\omega}(\{x_i\}) &= -2\sum_{i=1}^{s} \oint dz\ \psi_0(z) \left[\frac{\partial}{\partial x_i}\left(\frac{\psi_s(\{x_i\})}{x_i - z}\right)\right. \\ &\qquad \left. + \frac{\psi_s(\{x_i\})}{(x_i - z)^2} - \delta(z - x_i) \oint dy_i \frac{\psi(x_1, \ldots, y_i, \ldots, x_s; s)}{(y_i - z)^2}\right] \end{aligned} \tag{6.16}$$

$$\Omega_{0;s}(z; x_1, \ldots, x_s) = \sum_{i=1}^{s} \frac{\partial}{\partial z}\frac{\partial}{\partial x_i}\left(\delta(z - x_i)\psi_s(x_1, \ldots, x_i, \ldots, x_s)\right) \tag{6.17}$$

We now look at the two terms in equation (6.13) in turn, starting with the second term. (6.12) states how to deal with the Jacobian term. This, together with (6.17) leads to

$$\begin{aligned}&\frac{1}{2}\int dz \int dx_1 \ldots dx_s\ \Omega_{0;s}(z; x_1, \ldots, x_s)\frac{\partial \ln J}{\partial \psi_0(z)}\frac{\partial}{\partial \psi_s(x_1, \ldots, x_s)} \\ &= \frac{1}{2}\int dz \int dx_1 \ldots dx_s\ \sum_i \frac{\partial}{\partial z}\frac{\partial}{\partial x_i}\left[\delta(z - x_i)\psi_s(\{x_i\})\right]\ 2\int_0^z dq \oint dy \frac{\phi_0(y)}{q - y}\frac{\partial}{\partial \psi_s(\{x_i\})}\end{aligned}$$

The derivatives can be performed on the section in square brackets producing

$$-\delta''(z - x_i)\psi_s(\{x_i\}) + \delta'(z - x_i)\partial_{x_i}\psi_s(\{x_i\})$$

and the second term of (6.13) becomes

$$\int dx_1 \ldots dx_s \oint dy \sum_i \left[\frac{\psi_s(\{x_i\})}{(x_i - y)^2} - \frac{\partial_{x_i}\psi_s(\{x_i\})}{(x_i - y)}\right]\phi_0(y)\frac{\partial}{\partial \psi_s(\{x_i\})} \tag{6.18}$$

Putting this aside for the moment, we turn to the first term of (6.13). When we substitute the results from (6.16) for ω, we keep only the background pieces of the zero impurity loop by taking $\psi_0 \to \phi_0$. Consider the first term of (6.16), which is proportional to

$$\frac{\partial}{\partial x_i}\left(\frac{\psi_s(\{x_i\})}{x_i - z}\right)$$

when this term is substituted into (6.13), the result is

$$\int dx_1 \ldots dx_s\ \sum_{i=1}^{s} \oint dz\ \phi_0(z)\left[\frac{\partial_{x_i}\psi_s(\{x_i\})}{(x_i - z)} - \frac{\psi_s(\{x_i\})}{(x_i - z)^2}\right]\frac{\partial}{\partial \psi_s(\{x_i\})} \tag{6.19}$$

This neatly cancels with (6.18). What remains of the Hamiltonian is therefore the result of

substituting the other terms in (6.16) into (6.13). The quadratic Hamiltonian is therefore

$$H_s^{(2)} = \int dx_1 \ldots \int dx_s \oint dz \sum_{i=1}^{s} \frac{\phi_0(z)\psi_s(\{x_i\}) - \phi_0(x_i)\psi_s(x_1, \ldots, z, \ldots, x_s)}{(x_i - z)^2} \frac{\partial}{\partial \psi_s(\{x_i\})} \tag{6.20}$$

The background grows as N^2, so one would expect the fluctuations generated be a quadratic term to be of order $N^0 = 1$. This can be shown to be the case by extending the rescaling (4.12) so that

$$\begin{aligned} \phi_s(\{x_i\}) &\rightarrow N^{s/2}\phi_s(\{x_i\}) \\ \frac{\partial}{\psi_s(\{x_i\})} &\rightarrow \frac{1}{N^s}\frac{\partial}{\psi_s(\{x_i\})} \end{aligned}$$

6.4 Many impurity energy spectrum and states

Following [6] and [8], the spectrum that results from the quadratic Hamiltonian can be solved by expressing it in turns of a s-impurity kernel. Swopping the x_ith integral with z in the second term, (6.20) becomes

$$\begin{aligned} H_s^{(2)} &= \int dx_1 \ldots \int dx_s \psi_s(\{x_i\}) \oint dz \frac{\phi_0(z)}{(x_i - z)^2} \times \\ &\times \sum_{i=1}^{s} \frac{\partial}{\partial \psi_s(\{x_i\})} - \frac{\partial}{\partial \psi_s(x_1, \ldots, z, \ldots, x_s)} \\ &= \int dx_1 \ldots dx_s \int dy_1 \ldots dy_s \psi_s(\{x_i\}) K(\{x_i\}, \{y_i\}; s) \frac{\partial}{\partial \psi_s(\{y_i\})} \end{aligned} \tag{6.21}$$

It follows from this that the s-impurity kernel is

$$K_s(\{x_i\}, \{y_i\}) = \sum_{i=1}^{s} \oint dz \frac{\phi_0(z)}{(x_i - z)^2} \left(\prod_{j \neq i} \delta(x_j - y_j) \right) (\delta(x_i - y_i) - \delta(z - y_i)) \tag{6.22}$$

The form of a general eigenfunctional in ψ space is

$$\int \ldots \int dw_1 \ldots dw_s f(w_1, \ldots, w_s) \psi_s(w_1, \ldots, w_s). \tag{6.23}$$

The functions f can be found by solving the eigenvalue problem created by acting the kernel on

this eigenfunctional. The details of the calculation are given in appendix B. The result is

$$\begin{aligned} \epsilon_{\{n_i\}} &= w\left(\sum_{i=1}^{s} n_i - s\right) \\ f^s_{n_1,\ldots,n_s}(\{x_i\}) &= \prod_{i=1}^{s} \frac{\sin(n_i w q(x_i))}{\sqrt{2w}\sin(wq(x_i))} \quad ; \quad n_i = 1, 2, \ldots \end{aligned} \tag{6.24}$$

Up to this point, we have completely ignored the term in the Hamiltonian (equation (6.6)) that deals with the second matrix. It is not difficulty to correct for this however. $\mathrm{Tr}(B\frac{\partial}{\partial B})$ is of the form of a number operator – it will simply count the number of B impurities that are present, namely s. Adding this to the energy eigenvalues gives

$$\epsilon_{\{n_i\}} = w\sum_{i=1}^{s} n_i \tag{6.25}$$

Rodrigues [8] noticed that the eigenfunctions are of the form of Chebyshev polynomials of the second kind. These are defined by

$$U_0(x) = 1 \; ; \quad U_1(x) = 2 \; ; \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \tag{6.26}$$

They can equivalently be described in terms of trigonometric ratios as

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{6.27}$$

We therefore write (6.24) as

$$\begin{aligned} \prod_{i=1}^{s} \frac{\sin(n_i w q(x_i))}{\sqrt{2w}\sin(wq(x_i))} &= \frac{1}{\sqrt{2w}} \prod_{i=1}^{s} U_{n_i-1}(\cos(wq_i)) \\ &= \frac{1}{\sqrt{2w}} \prod_{i=1}^{s} U_{n_i-1}\left(-\sqrt{\frac{w}{2}}x_i\right) \\ &\equiv \prod_{i=1}^{s} u_{n_i-1}(x_i) \end{aligned} \tag{6.28}$$

where the u have been defined in terms of the U to simplify notation.

Going back to (6.23), and using the density description of ψ (equation 6.15) the functionals

are of the form

$$\begin{aligned}\langle\psi_s|\{x_i\}\rangle &= \int\dots\int dx_1\dots dx_s\left(\prod_{i=1}^{s}u_{n_i-1}(x_i)\right)Tr\left(B\delta(x_1-M)\dots B\delta(x_s-M)\right)\\ &= \mathrm{Tr}(Bu_{n_1-1}(M)Bu_{n_2-1}(M)\dots Bu_{n_s-1}(M))\end{aligned}\tag{6.29}$$

These functions form a complete orthonormal set with respect to the measure (see [8] for more details)

$$\int[d\psi]\exp\left[-\int\frac{dx_1}{\pi\phi_0}\dots\int\frac{dx_s}{\pi\phi_0}|\psi_s(\{x_i\})|^2\right]\tag{6.30}$$

To simplify the notation, we take $(n_i-1)\to j_i$, where $j_i=0,1,2,\dots$, leaving states of the form

$$\mathrm{Tr}(Bu_{j_1}Bu_{j_2}\dots Bu_{j_s})\tag{6.31}$$

which have energy equal to $w(\sum_i j_i+s)$.

We have thus found both the states associated with the collective field theory of two matrices, and their associated energies. These states provide a useful way of dealing with two matrix models. The dependence on the first matrix M is encoded in the polynomials u_{n_i-1}, and the second matrix appears as creation operators sandwiched in between the polynomials. We can now go on to introduce g_{YM} interactions, which is done in the following chapter. The states that have been derived in this chapter will be the basic building blocks for this.

Note that these states are similar to the spin chain states discussed in section 3.2, where trace has been used to place the first impurity at the beginning of the chain. The B impurities play the role of the ϕ impurities in section 3.1. The parameters j_1 and j_2 are analogous to the number of Zs that lie in between impurities. We can think of the spin chain "length" as being $J+s$, which is also the free energy.

Chapter 7

g_{YM} interaction

We now introduce corrections induced by the g_{YM}^2 SYM interaction term to the results of chapter 6. We will only look at corrections that are small compared to the overall energy, so we are still looking at states that are in some sense close to 1/2 BPS states.

The interaction includes all six SYM scalars and is of the form

$$-g_{YM}^2 \sum_{i<j} \mathrm{Tr}\left(\left[\phi_i, \phi_j\right]^2\right) \quad , \quad i,j = 1,\ldots,6 \tag{7.1}$$

It is at this point where the identification of variables becomes important. In the section below, we will consider the angular momentum eigenstate representation from section 6.1. We give the physical justification, and reviewing some results obtained in [8]. Later, in section 8.4, we will consider directly modelling ϕ_1 and ϕ_2 (as described in section 6.2) to obtain an energy spectrum that is valid to all orders.

7.1 Angular momentum eigenstate model

The six SYM scalars are complexified in groups of two:

$$Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \quad Y = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)\,, \quad X = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$$

Equation (7.1) has two types of terms, commutators between adjacent fields that are complexified together, such as ϕ_1 and ϕ_2, and commutators between fields from different complexi-

fications, such as ϕ_1 and ϕ_3. The former all have simple expressions, for example

$$\mathrm{Tr}\left(\big[\phi_1,\phi_2\big]^2\right) = \mathrm{Tr}\left(\left[\frac{1}{\sqrt{2}}(Z+\bar{Z}),\frac{1}{\sqrt{2}i}(Z-\bar{Z})\right]^2\right) = \mathrm{Tr}\left(\big[Z,\bar{Z}\big]^2\right) \tag{7.2}$$

Commutators between terms from different complexifications. We will work out all the terms involving both Z and Y here. The terms involving Z and X, and Y and X, are similar. The commutator between the first and third scalars is

$$\mathrm{Tr}\left(\big[\phi_1,\phi_3\big]^2\right) = \frac{1}{4}\mathrm{Tr}\left\{\big(\big[Z,Y\big]+\big[Z,\bar{Y}\big]+\big[\bar{Z},Y\big]+\big[\bar{Z},\bar{Y}\big]\big)^2\right\} \tag{7.3}$$

There will be four of these terms, which we will denote by the numbers of the scalar fields that are commuted. For example, the above term is $(1,3)$. When going from the trace between scalars $(1,3)$ to $(1,4)$, $(2,3)$ or $(2,4)$, the following rules can be observed:

$$\begin{aligned}
\text{Going from}\quad (1,3)\to(1,4): &\quad \times(-1)\ , \qquad \bar{Y}\to-\bar{Y}\\
(1,3)\to(2,3): &\quad \times(-1)\ , \qquad \bar{Z}\to-\bar{Z}\\
(1,3)\to(2,4): &\quad \bar{Y}\to-\bar{Y}\ , \quad \bar{Z}\to-\bar{Z}
\end{aligned}$$

Any term which does not involve all four of the complex fields, Z, $\bar{Z}$, Y and $\bar{Y}$ will be positive in two of the commutators, and negative in two, for example any $ZYZY$ term will be positive in $(1,3)$ and $(2,4)$ and negative in $(1,4)$ and $(2,3)$. As a result, terms of this nature will cancel, and we only need to consider the terms with all four fields, which are positive for all the commutators.

Working from equation (7.3), the terms we need are

$$\begin{aligned}
&4.\frac{1}{4}\mathrm{Tr}\Big\{\big[Z,Y\big]\big[\bar{Z},\bar{Y}\big]+\big[Z,\bar{Y}\big]\big[\bar{Z},Y\big]+\big[\bar{Z},Y\big]\big[Z,\bar{Y}\big]+\big[\bar{Z},\bar{Y}\big]\big[Z,Y\big]\Big\}\\
&=2\mathrm{Tr}\Big\{2\big[Z,Y\big]\big[\bar{Z},\bar{Y}\big]-\big(\big[Z,Y\big]\big[\bar{Z},\bar{Y}\big]+\big[Z,Y\big]\big[\bar{Z},\bar{Y}\big]\big)\Big\}
\end{aligned}$$

Using the identity

$$\mathrm{Tr}\left(\big[a,c\big]\big[b,d\big]\right) = \mathrm{Tr}(\big[a,b\big]\big[c,d\big]) + \mathrm{Tr}(\big[a,d\big]\big[b,c\big]) \tag{7.4}$$

this becomes

$$2\mathrm{Tr}\Big\{2\big[Z,Y\big]\big[\bar{Z},\bar{Y}\big]-\big[Z,\bar{Z}\big]\big[Y,\bar{Y}\big]\Big\} \tag{7.5}$$

Adding together the terms of the form of equation (7.5) and those of the form of equation (7.2) gives the result

$$\begin{aligned} H_{\rm int} = 4g_{YM}^2\Big(\quad & \tfrac{1}{4}\big[Z,\bar{Z}\big]\big[Z,\bar{Z}\big] + \tfrac{1}{4}\big[Y,\bar{Y}\big]\big[Y,\bar{Y}\big] + \tfrac{1}{4}\big[X,\bar{X}\big]\big[X,\bar{X}\big] \\ & + \tfrac{1}{2}\big[Z,\bar{Z}\big]\big[Y,\bar{Y}\big] + \tfrac{1}{2}\big[Z,\bar{Z}\big]\big[X,\bar{X}\big] + \tfrac{1}{2}\big[Y,\bar{Y}\big]\big[X,\bar{X}\big] \\ & - \big[\bar{Z},\bar{X}\big]\big[Z,X\big] - \big[\bar{Y},\bar{X}\big]\big[Y,X\big] - \big[\bar{Z},\bar{Y}\big]\big[Z,Y\big] \ \Big) \end{aligned}$$

This clearly shows a split into D terms, involving commutators between a field and its complex conjugate, and F terms, involving different fields.

We now introduce creation and annihilation operators for each of the complex fields (those for Z will be the same as those introduced earlier).

$$\begin{aligned} & Z \to \frac{1}{\sqrt{2w}}(A + B^\dagger)\ , \quad Y \to \frac{1}{\sqrt{2w}}(C + D^\dagger)\ , \quad X \to \frac{1}{\sqrt{2w}}(E + F^\dagger) \\ \Big(\ & \bar{Z} \to \frac{1}{\sqrt{2w}}(A^\dagger + B)\ , \quad \bar{Y} \to \frac{1}{\sqrt{2w}}(C^\dagger + D)\ , \quad \bar{X} \to \frac{1}{\sqrt{2w}}(E^\dagger + F) \ \Big) \end{aligned}$$

We are not interested in all the possibilities arising from this interaction though, but rather only the states which are close to BPS states. To stay within the subspace of loops that are near to chiral primary operators, we thus use the technique followed by [32, 31], and project

$$\begin{aligned} & A + B^\dagger \to A\ , \quad A^\dagger + B \to A^\dagger \\ & C + D^\dagger \to C\ , \quad C^\dagger + D \to C^\dagger \\ & E + F^\dagger \to E\ , \quad E^\dagger + F \to E^\dagger \end{aligned}$$

All the D terms are now trivial (to see this explicitly, go to the coherent state basis, e.g. $C \to \partial/\partial C^\dagger$), which leaves just the F terms.

$$H_{\rm int} = -\frac{g_{YM}^2}{w^2}\Big(\big[E^\dagger, A^\dagger\big]\big[E,A\big] + \big[C^\dagger, E^\dagger\big]\big[C,E\big] + \big[C^\dagger, A^\dagger\big]\big[C,A\big]\Big)$$

We will retain only one of these. The interaction Hamiltonian of interest is thus

$$H_{\rm int} = -\frac{g_{YM}^2}{w^2}{\rm Tr}\big(\big[A^\dagger, C^\dagger\big]\big[A,C\big]\big) \tag{7.6}$$

Again, we define M as the Hermitian matrix associated with the $A, A^\dagger$ operators, as in (6.4).

In terms M and its canonical conjugate P_M (using identity (7.4 once more),

$$\begin{aligned} H_{\text{int}} &= -\frac{g_{YM}^2 N}{2w}\text{Tr}\left(\left[M,C\right]\left[M,C^\dagger\right]\right) + i\frac{g_{YM}^2}{2w^2}\text{Tr}\left(\left[C,C^\dagger\right]\left[M,P_M\right]\right) \\ &\quad - \frac{g_{YM}^2}{2Nw^3}\text{Tr}\left(\left[P_M,C\right]\left[P_M,C^\dagger\right]\right) \\ &\equiv H_{\text{int}(1)} + H_{\text{int}(2)} + H_{\text{int}(3)} \end{aligned}$$

We can act this on the many-impurity collective field theory states found in section 6.4. An in depth treatment of this is beyond the scope of this dissertation ([8] contains some details) – we will concern ourselves more with the ϕ_1, ϕ_2 model, which is tackled in chapter 8. In short, $H_{\text{int}(2)}$ and $H_{\text{int}(3)}$ are of sub-leading order, so $H_{\text{int}(1)}$ determines the spectrum. To first order, when there are two impurities, the resulting states are linear combinations of the states from (6.31),

$$O_J^m = \frac{1}{\sqrt{J+1}}\sum_{j=0}^{J} e^{\frac{2\pi i m}{J+1}j}\text{Tr}(Cu_j(M)Cu_{J-j}(M)), \qquad m = 0,\ldots,J \tag{7.7}$$

This are of a similar structure to the BMN loops defined in (3.6). Indeed, the states from (6.31) are only useful in the BMN limit. Their energy is proportional to $\sum_i j_i + s = J + s$, so if we calculate $\Delta - J$, this will give the number of impurities, s. The BMN limit keeps $\Delta - J$ finite, which corresponds to only considering states with a finite number of impurities.

It is worth noting the form of the operator $H_{\text{int}(1)}$ – this same operator (up to a constant factor) will determine the spectrum in chapter 8.

Chapter 8

Two Higgs scalar model

We now develop further the identification of variables introduced in section 6.2. The two matrices that we will consider are two of the SYM scalars. This identification was first mentioned in [8], for two impurities, which gave energy results to first order by considering a subset of the Hamiltonian that contributes to first order, namely the part that preserves J. We will develop this further by introducing an ansatz which gives an analytic solution to the difference equation that results from the Hamiltonian. This captures the effect of all the terms in the difference equation, not only the ones that conserve J. We then give in detail the Bogoliubov transformation mentioned in [8]. For two impurities, the energies derived from the ansatz are then shown (using a result proved in appendix C) to completely determine the spectrum resulting from the g_{YM} interaction term to all orders, not just to first order. These results will be presented in [33].

8.1 g_{YM} interaction

We take the Hamiltonian from (6.2), and add the g_{YM} interaction term from (7.1).

$$H = \frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{1}{2}\mathrm{Tr}(P_2^2) + \frac{1}{2}w^2\mathrm{Tr}(\phi_1^2) + \frac{1}{2}w^2\mathrm{Tr}(\phi_2^2) - g_{YM}^2\mathrm{Tr}([\phi_1,\phi_2][\phi_1,\phi_2]) \tag{8.1}$$

We will treat ϕ_1 exactly, and ϕ_2 as an impurity introduced into the background created by ϕ_1. To this end, creation and annihilation operators are defined for ϕ_2:

$$\phi_2 = \frac{1}{\sqrt{2w}}(C + C^\dagger), \qquad P_2 = -i\sqrt{\frac{w}{2}}(C - C^\dagger)$$

and the Hamiltonian becomes

$$\begin{aligned} H &= \underbrace{\frac{1}{2}\mathrm{Tr}(P_1^2) + \frac{1}{2}w^2\mathrm{Tr}(\phi_1^2)}_{H_{\phi_1}} + w\mathrm{Tr}(C^\dagger C) \\ &\quad - \frac{g_{YM}^2}{2w}\mathrm{Tr}\big([\phi_1, C]^2 + 2[\phi_1, C^\dagger][\phi_1, C] + [\phi_1, C^\dagger]^2\big) \end{aligned} \tag{8.2}$$

Even though we will only consider bosonic degrees of freedom, we will assume that the theory is part of a supersymmetric theory, so that we do not need to be concerned with normal ordering terms.

The first two terms describe the background created by ϕ_1. This is not too interesting – we would rather see the fluctuations on top of this background that arise from the introduction of the g_{YM} interaction term. The Hamiltonian we will use is therefore not the full Hamiltonian, but rather

$$H_C = H - H_{\phi_1} = w\mathrm{Tr}(C^\dagger C) - \frac{g_{YM}^2}{2w}\mathrm{Tr}\big([\phi_1, C]^2 + 2[\phi_1, C^\dagger][\phi_1, C] + [\phi_1, C^\dagger]^2\big) \tag{8.3}$$

It is worth taking a moment to consider the physical implications of the choice of variables. In the collective field theory approach, the first matrix is treated in the large N limit. A finite number of impurities of the second matrix are then introduced. In the angular momentum eigenstate model, this corresponds to states with large J near the ϕ_1-ϕ_2 plane, i.e. states that fall within the BMN limit. In the ϕ_1, ϕ_2 model, the picture is different. The states under consideration are now states where ϕ_1 is large and ϕ_2 is small. By considering $H_C = H - H_{\phi_1}$, we are doing the analogue of considering $\Delta - J$ in the angular momentum eigenstate model.

We define $H_{(1)}$ as the term in (8.3) containing ϕ_1, C and $C^\dagger$, and in the spirit of (4.12), rescale $\phi_1 \to \sqrt{N}\phi_1$:

$$H_{(1)} = -\frac{g_{YM}^2 N}{w}\mathrm{Tr}\big([\phi_1, C^\dagger][\phi_1, C]\big) \tag{8.4}$$

Note that this term has the same form (up to a factor of 2) as $H_{\mathrm{int}(1)}$ from section 7.1. It turns out (after performing a Bogoliubov transformation, which will be done in section 8.4) that this term is sufficient to determine the spectrum for two impurities. The term involving only creation operators, $[\phi_1, C]^2$, and the term involving only annihilation operators, $[\phi_1, C^\dagger]^2$, will determine the factors in the Bogoliubov transformation.

8.2 Difference equation generated by $H_{(1)}$

We will derive here the spectrum of $H_{(1)}$. We move to the coherent state basis, letting $(C^\dagger)_{ij} \to C_{ij}$ and $(C)_{ij} \to \left(\frac{\partial}{\partial C}\right)_{ji}$. The two-impurity result is derived first and this is later generalised to more impurities.

For two impurities, the Hamiltonian will act on states of the form derived in the section 6.4.

$$\langle j_1, j_2|\psi\rangle = \mathrm{Tr}(Cu_{j_1}(M)Cu_{j_2}(M))$$

For notational convenience, we will denote these states by simply $|j_1, j_2\rangle$.

Consider the first term, $H_{(1)}$, in (7.7). After multiplying out the commutators,the cyclic freedom of the trace means the $\partial/\partial C$ terms can be moved to the end, yielding

$$H_{(1)}|j_1, j_2\rangle = -\frac{g_{YM}^2 N}{w}\mathrm{Tr}\left[\left(2MCM - M^2C - CM^2\right)\frac{\partial}{\partial C}\right]\mathrm{Tr}(Cu_{j_1}(M)Cu_{j_2}(M))$$

The action of a derivative with respect to C on one of the states is

$$\frac{\partial}{\partial C_{ij}}\mathrm{Tr}(Cu_{j_1}Cu_{j_2}) = (u_{j_1}Cu_{j_2})_{ji} + (u_{j_2}Cu_{j_1})_{ji} \tag{8.5}$$

yielding

$$H_{(1)}|j_1, j_2\rangle = -\frac{g_{YM}^2 N}{w}\mathrm{Tr}\Big[4CMu_{j_1}CMu_{j_2} - 2Cu_{j_1}CM^2u_{j_2} - 2CM^2u_{j_1}Cu_{j_2}\Big] \tag{8.6}$$

For any number of impurities, the notation becomes cumbersome. We thus label states by how many Ms follow each C:

$$\mathrm{Tr}(CM^a u_{j_1} CM^b u_{j_2} \dots CM^z u_{j_s}) \to (a, b, \dots, z)$$

Equation (8.6) generalises to

$$\begin{aligned} H_{(1)}(0,\dots,0) \;=\; & -2\frac{g_{YM}^2 N}{w}\mathrm{Tr}\Big\{\ (1,1,0,\dots,0) + (0,1,1,0,\dots,0) + \dots \text{(cyclic permutations)} \\ & - (2,0,\dots,0) - (0,2,0,\dots,0) - \dots \text{(cyclic permutations)}\ \Big\} \end{aligned} \tag{8.7}$$

The recurrence relation that defines the Chebyshev polynomials (6.26) means that

$$\begin{aligned} MU_j &= \frac{1}{2}\left(U_{j+1}+U_{j-1}\right) \\ M^2U_j &= \frac{1}{4}\left(U_{j+2}+2U_j+U_{j-2}\right) \end{aligned}$$

From the definition of u in terms of U, contained in (6.28), it follows that

$$\begin{aligned} Mu_j &= -\frac{1}{\sqrt{2w}}\left(u_{j+1}+u_{j-1}\right) \\ M^2u_j &= \frac{1}{2w}\left(u_{j+2}+2u_j+u_{j-2}\right) \end{aligned} \tag{8.8}$$

which allows us to absorb the extra factors of M, and recover states of the original form. This leads to a difference equation. For two impurities,

$$\begin{aligned} H_{(1)}|j_1,j_2\rangle &= \frac{g_{YM}^2N}{w^2}\Big\{\ 4|j_1,j_2\rangle-2|j_1+1,j_2-1\rangle-2|j_1-1,j_2+1\rangle \\ &\quad -2|j_1+1,j_2+1\rangle+|j_1+2,j_2\rangle+|j_1,j_2+2\rangle \\ &\quad -2|j_1-1,j_2-1\rangle+|j_1-2,j_2\rangle+|j_1,j_2-2\rangle\ \Big\} \end{aligned} \tag{8.9}$$

This generalises for more impurities to

$$\begin{aligned} H_{(1)}|j_1,j_2,\ldots,j_s\rangle &= \frac{g_{YM}^2N}{w^2}\Big\{\ (2s)|j_1,j_2,\ldots,j_s\rangle-|j_1+1,j_2+1,\ldots\rangle-|j_1+1,j_2-1,\ldots\rangle \\ &\quad -\ |j_1-1,j_2+1,\ldots\rangle-\ |j_1-1,j_2-1,\ldots\rangle \\ &\quad +\ |j_1+2,j_2,\ldots\rangle+|j_1-2,j_2,\ldots\rangle \\ &\quad +\ \ldots\ \text{(all cyclic permutations, } j_1\to j_2,\ j_2\to j_3 \text{ etc.)}\Big\} \end{aligned} \tag{8.10}$$

It is interesting to see how this difference equation affects the "length" of the states. To show this explicitly for two impurities, we take $j_1\to j$ and $j_2\to J-j$, and move from j_1,j_2 space to J,j space by defining $|J,j\rangle\equiv\langle J,j|\psi\rangle=\mathrm{Tr}(CMu_jCMu_{J-j})$. In terms of these states,

$$\begin{aligned} H_{(1)}|J,j\rangle &= \frac{g_{YM}^2N}{w^2}\Big\{\ 4|J,j\rangle-2|J,j+1\rangle-2|J,j-1\rangle \\ &\quad -2|J+2,j+1\rangle+|J+2,j+2\rangle+|J+2,j\rangle \\ &\quad -2|J-2,j-1\rangle+|J-2,j-2\rangle+|J-2,j\rangle\ \Big\} \end{aligned} \tag{8.11}$$

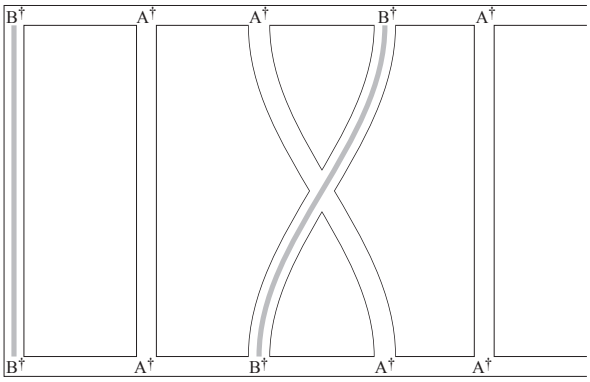


Figure 8.1: An operator that takes $|2, j\rangle \to |1, j+1\rangle$, preserving J.

This shows how the interaction Hamiltonian splits into three parts. The first line consists of terms that preserve J. The second and third lines consist of terms that move into a different J subspace, adding or subtracting two respectively. We will call the J-preserving terms the "diagonal" part, $H^D_{(1)}$, and the other ("non-diagonal") terms $H^+_{(1)}$ and $H^-_{(1)}$. Thus we can write (8.11) as $H_{(1)} = H^D_{(1)} + H^+_{(1)} + H^-_{(1)}$.

Operators that act on states $|j_1, j_2\rangle$ can be understood in a diagrammatic sense, imported from spin chains. The spin chain analogous to $u_j(A^\dagger)$ is a string of j $A^\dagger$s. $B^\dagger$ impurities are placed between these. For example, the state $|2, 1\rangle \equiv \mathrm{Tr}(B^\dagger u_2 B^\dagger u_1)$ would be represented as $B^\dagger A^\dagger A^\dagger B^\dagger A^\dagger$.

To see the effect of an operator on a state then, the initial state is placed above the final state, and lines are drawn in corresponding to the summation of indices. For example, an operator that takes $|2, j\rangle \to |1, j+1\rangle$ would be represented by the diagram in figure 8.1. The J preserving terms in the difference equation (8.11) will all look similar to this.

The terms that do not preserve J are not something that would typically be seen in a spin chain however. For J to change, it means that new $A^\dagger$s must be being created or annihilated. The term that takes $J \to J + 2$ would then have to look something like figure 8.2. There is nothing preventing these diagrams in the ϕ_1, ϕ_2 model. In the angular momentum model however, these are problematic, as the angular momentum charge must be preserved. We cannot therefore have an operator that creates new $A^\dagger$s, and diagrams like figure 8.2 are forbidden.

Equation (8.11) is the analogue of the difference equations treated in the literature, given by equations (3.7) and (3.8).

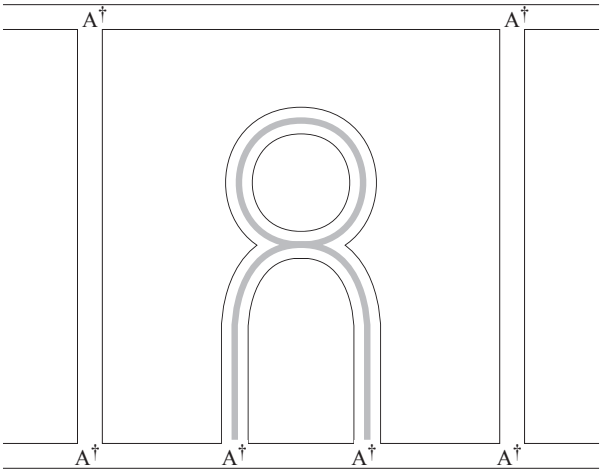


Figure 8.2: An operator that takes $J \to J+2$.

8.3 Eigenstate ansatz and energy spectrum

The two impurity case is treated first. In order to solve equation (8.9) we introduce energy eigenstates $f_{\lambda_1,\lambda_2}(j_1,j_2)$ (or just $f(j_1,j_2)$ for short) by making the ansatz

$$f_{\lambda_1,\lambda_2}(j_1,j_2) \propto \lambda_1^{j_1}\lambda_2^{j_2}$$

we will require these states to satisfy the difference equation in (8.9), so that

$$\begin{aligned} E_{\lambda_1,\lambda_2}(j_1,j_2)f(j_1,j_2) &= \frac{g_{YM}^2 N}{w^2}\Big\{\, 4f(j_1,j_2) - 2f(j_1+1,j_2-1) - 2f(j_1-1,j_2+1) \\ &\quad -2f(j_1+1,j_2+1) + f(j_1+2,j_2) + f(j_1,j_2+2) \\ &\quad -2f(j_1-1,j_2-1) + f(j_1-2,j_2) + f(j_1,j_2-2) \,\Big\} \end{aligned} \tag{8.12}$$

which leads to

$$E_{\lambda_1,\lambda_2}f(j_1,j_2) = \frac{g_{YM}^2 N}{w^2}\left\{4 - 2\frac{\lambda_1}{\lambda_2} - 2\frac{\lambda_2}{\lambda_1} - 2\lambda_2\lambda_1 + \lambda_1^2 + \lambda_2^2 - 2\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right\} f(j_1,j_2)$$

By demanding Hermiticity of the eigenvalues, we can see that two conditions which must be satisfied are

$$\lambda_1^2 + \frac{1}{\lambda_1^2} = \lambda_1^{*2} + \frac{1}{\lambda_1^{*2}} \qquad \text{and} \qquad \lambda_2^2 + \frac{1}{\lambda_2^2} = \lambda_2^{*2} + \frac{1}{\lambda_2^{*2}}$$

We thus ensure that $\lambda_1^* = \frac{1}{\lambda_1}$ and $\lambda_2^* = \frac{1}{\lambda_2}$ by setting

$$\lambda_1 = e^{i\alpha} \qquad \lambda_2 = e^{i\beta}$$

The energy eigenvalues are then given by

$$\begin{aligned} E_{\alpha,\beta} &= \frac{2g_{YM}^2 N}{w^2}\big(2 - 2\cos(\alpha-\beta) - 2\cos(\alpha+\beta) + \cos(2\alpha) + \cos(2\beta)\big) \\ &= \frac{4g_{YM}^2 N}{w^2}\big(\cos^2\alpha + \cos^2\beta - \cos^2\alpha\ \cos^2\beta - \cos^2\beta\ \cos^2\alpha\big) \end{aligned} \tag{8.13}$$

This is the form that generalises most easily to more impurities. For example, the 3-impurity ansatz is

$$f_{\lambda_1,\lambda_2,\lambda_3}(j_1,j_2,j_3) \to f_{\alpha,\beta,\gamma}(j_1,j_2,j_3) \propto e^{i\alpha j_1} e^{i\beta j_2} e^{i\gamma j_3}$$

and the solution to the difference equation in (8.10) is

$$E_{\alpha,\beta,\gamma} = \frac{4g_{YM}^2 N}{w^2}\big(\cos^2\alpha + \cos^2\beta + \cos^2\gamma - \cos^2\alpha\ \cos^2\beta - \cos^2\beta\ \cos^2\gamma - \cos^2\gamma\ \cos^2\alpha\big)$$

For n impurities, the energy is

$$E_{\{\alpha_i\}} = \frac{4g_{YM}^2 N}{w^2}\sum_{i=1}^{n}\big(\cos^2\alpha_i - \cos^2\alpha_i\ \cos^2\alpha_{i+1}\big); \qquad \alpha_{n+1} \equiv \alpha_i \tag{8.14}$$

The two-impurity case is of special interest, for reasons that will become apparent in section 8.4. For two impurities, equation (8.13) can be simplified further. We define $q = \alpha + \beta$, and $p = \alpha - \beta$. Then,

$$\begin{aligned} E_{p,q} &= \frac{2g_{YM}^2 N}{w^2}\big(2 - 2\cos q - 2\cos p + \cos(p+q) + \cos(p-q)\big) \\ &= \frac{2g_{YM}^2 N}{w^2}\Big(2 - 2\Big(1 - 2\sin^2\Big(\frac{q}{2}\Big)\Big) - 2\Big(1 - 2\sin^2\Big(\frac{p}{2}\Big)\Big) + \\ &\qquad + \ (\cos p\cos q - \sin p\sin q) + (\cos p\cos q + \sin p\sin q)\Big) \\ &= \frac{16g_{YM}^2 N}{w^2}\sin^2\Big(\frac{p}{2}\Big)\sin^2\Big(\frac{q}{2}\Big) \\ E_{p,\beta} &= \frac{16g_{YM}^2 N}{w^2}\sin^2\Big(\frac{p}{2}\Big)\sin^2\Big(\frac{p}{2}+\beta\Big) \end{aligned} \tag{8.15}$$

We can give a physical meaning to the variables p and β by returning to the eigenstates,

$$e^{i\alpha j_1}e^{i\beta j_2} = e^{ipj_1}e^{i\beta(j_1+j_2)} \equiv f_{\beta,p}(j_1, j_2) \tag{8.16}$$

In terms of J and j, rather than j_1 and j_2, this is $e^{ipj}e^{i\beta J}$.

For finite J, we can apply spin chain thinking. J is related to the length of the chain. We have used the cyclicity of the trace to lock the first impurity at the beginning of the chain. We can therefore think of the second impurity as free to go to any position within J, determined by the value of j. The states with $j = 0$ and $j = J$ are the same, so we can identify the two ends. The "chain" is then a circle, obeying periodic boundary conditions so that $j + J \leftrightarrow j$. This implies that $\exp[ipJ] = 1$, or $p = 2k\pi/J$, for integral k.

The eigenstate $f_{\beta,k}(j_1, j_2) \equiv e^{i2\pi kj_1}e^{i\beta(j_1+j_2)}$ thus has energy

$$E_{k,\beta} = \frac{16g_{YM}^2 N}{w^2}\sin^2\left(\frac{k\pi}{J}\right)\sin^2\left(\frac{k\pi}{J}+\beta\right)$$

In the large J limit, it is more sensible to think of p as a continuous variable with the energy given by (8.15).

8.4 Bogoliubov transformation

We now return to the Bogoliubov transformation mentioned in section 8.1, which was suggested in [8]. We can eliminate the term involving only creation operators, $[\phi_1, C]^2$, and the term involving only annihilation operators, $[\phi_1, C^\dagger]^2$, by taking

$$C_{ij} \to a_{(ij)}\tilde{C}_{ij} - b_{(ij)}\tilde{C}^\dagger_{ij}{}^1$$

Note that ϕ_1 is an Hermitian matrix, and can therefore be diagonlised by an unitary matrix U. If we denote the (real) eigenvalues of ϕ_1 by λ_i, then $[U^\dagger\phi_1 U, C]_{ij} = (\lambda_i - \lambda_j)C_{ij}$. We define

[1] There is no sum over the i and j indices here – $a_{(ij)}$ is understood as being the coefficient that multiplies the (i, j)th element of $\tilde{C}$

$\bar{C} \equiv U^\dagger \tilde{C} U$. Then,

$$\begin{aligned}
\mathrm{Tr}([\phi_1, \tilde{C}]^2) &= \mathrm{Tr}\big([U^\dagger \phi_1 U,\ U^\dagger \tilde{C} U]^2\big) \\
&= \sum_{i,j=1}^{N} -(\lambda_i - \lambda_j)^2 \bar{C}_{ij} \bar{C}_{ji} \\
\mathrm{Tr}([\phi_1, \tilde{C}^\dagger][\phi_1, \tilde{C}]) &= \sum_{i,j=1}^{N} -(\lambda_i - \lambda_j)^2 \bar{C}^\dagger_{ij} \bar{C}_{ji}
\end{aligned} \tag{8.17}$$

Putting these results into equation (8.3) (suppressing indices) gives

$$\begin{aligned}
& w\mathrm{Tr}(C^\dagger C) - \frac{g_{YM}^2 N}{2w} \mathrm{Tr}\big([\phi_1, C]^2 + 2[\phi_1, C^\dagger][\phi_1, C] + [\phi_1, C^\dagger]^2\big) \\
&= w\mathrm{Tr}((a\tilde{C} - b\tilde{C}^\dagger)(a\tilde{C}^\dagger - b\tilde{C})) \\
&\quad + \underbrace{\frac{g_{YM}^2 N}{2w}(\lambda_i - \lambda_j)^2}_{X}\ \mathrm{Tr}\big((a\tilde{C} - b\tilde{C}^\dagger)^2 + (a\tilde{C}^\dagger - b\tilde{C})(a\tilde{C} - b\tilde{C}^\dagger) + (a\tilde{C}^\dagger - b\tilde{C})^2\big)
\end{aligned} \tag{8.18}$$

We group the terms in terms of C and $C^\dagger$:

$$\begin{aligned}
& \big(-wab + Xaa - 2Xab + Xbb\big)\bar{C}\bar{C} \\
+\ & \big(-wab + Xaa - 2Xab + Xbb\big)\bar{C}^\dagger\bar{C}^\dagger \\
+\ & \big(waa + wbb + 2Xaa + 2Xbb - 4Xab\big)\bar{C}^\dagger\bar{C}
\end{aligned} \tag{8.19}$$

For the first two lines to disappear, we see that

$$X(a_{(ij)}a_{(ji)} + b_{(ij)}b_{(ji)}) = (w + 2X)a_{(ij)}b_{(ji)}$$

We take $a_{(ij)} \equiv \cosh(\theta_{ij})$ and $b_{(ij)} \equiv \sinh(\theta_{ij})$ to give

$$\tanh(2\theta_{ij}) = \frac{2X}{w + 2X} = \frac{\frac{g_{YM}^2 N}{w}(\lambda_i - \lambda_j)^2}{w + \frac{g_{YM}^2 N}{w}(\lambda_i - \lambda_j)^2} \tag{8.20}$$

The Hamiltonian is then [2]

$$H_C = \sum_{i,j=1}^{N} \Big[(w + 2X)\cosh(2\theta_{ij}) - 2X\sinh(2\theta_{ij})\Big]\bar{C}^\dagger_{ij}\bar{C}_{ji}$$

[2] Again neglecting normal ordering terms

From equation (8.20),

$$\cosh(2\theta_{ij}) = \frac{w+2X}{\sqrt{(w+2X)^2-(2X)^2}}$$
$$\sinh(2\theta_{ij}) = \frac{2X}{\sqrt{(w+2X)^2-(2X)^2}}$$

and the Hamiltonian becomes

$$\begin{aligned} H_C &= \sum_{i,j=1}^{N} \left[\frac{(w+2X)^2-(2X)^2}{\sqrt{(w+2X)^2-(2X)^2}} \right] \bar{C}^\dagger_{ij}\bar{C}_{ji} \\ &= \sum_{i,j=1}^{N} \sqrt{w^2+2g^2_{YM}N(\lambda_i-\lambda_j)^2}\bar{C}^\dagger_{ij}\bar{C}_{ji} \end{aligned} \tag{8.21}$$

The square root can be expanded: (in the following, A_n is the numerical constant preceding the nth term in the expansion i.e. $\sqrt{1+x} = 1 + A_1x + A_2x^2 + \ldots$)

$$\sum_{i,j=1}^{N} \sqrt{w^2+2g^2_{YM}N(\lambda_i-\lambda_j)^2}\bar{C}^\dagger_{ij}\bar{C}_{ji} = \sum_{n=0}^{\infty} A_n w \left(\frac{2g^2_{YM}N}{w^2}\right)^n \sum_{i,j=1}^{N} \bar{C}^\dagger_{ij}(\lambda_i-\lambda_j)^{2n}\bar{C}_{ji}$$

Repeated use of equation (8.17) makes this

$$\sum_{n=0}^{\infty} A_n w \left(\frac{2g^2_{YM}N}{w^2}\right)^n (-1)^n \quad \mathrm{Tr}\big([\phi_1,[\underbrace{\phi_1,\ldots[\phi_1,\tilde{C}^\dagger]\ldots}_{n \text{ times nested commutators}}]]\ [\phi_1,[\underbrace{\phi_1,\ldots[\phi_1,\tilde{C}]\ldots}_{n \text{ times nested commutators}}]]\big) \tag{8.22}$$

8.5 Two impurity spectrum

Up to this point, the derivation has been applicable to states with any number of impurities. The restriction is now made to states with two impurities. In this case, the action of this term on $|j_1, j_2\rangle \equiv \mathrm{Tr}(C^\dagger u_{j_1}(M) C^\dagger u_{j_2}(M))$ is shown in appendix C to be the same as the operator

$H_{(1)}$ (up to a constant) introduced in chapter 7, applied n times. Using equation (C.1),

$$\begin{aligned}
&w\left(\frac{2g_{YM}^2N}{w^2}\right)^n(-1)^n\,\mathrm{Tr}\big([\phi_1,[\phi_1,\dots[\phi_1,\tilde{C}^\dagger]\dots]][\phi_1,[\phi_1,\dots[\phi_1,\tilde{C}]\dots]]\big)|j_1,j_2\rangle\\
&=2w\left(\frac{2g_{YM}^2N}{w^2}\right)^n(-1)^n\left(\frac{1}{2}\mathrm{Tr}[M,\tilde{C}^\dagger][M,\tilde{C}]\right)^n|j_1,j_2\rangle\\
&=2w\left(\frac{1}{w}\right)^n\left(-\frac{g_{YM}^2N}{w}\mathrm{Tr}\big([\phi_1,\tilde{C}^\dagger][\phi_1,\tilde{C}]\big)\right)^n|j_1,j_2\rangle\\
&=2w\left(\frac{H_{(1)}}{w}\right)^n|j_1,j_2\rangle
\end{aligned}\tag{8.23}$$

for $n\geq q$.

Equation (8.21) can thus be expressed as

$$H_C=w\bar{C}_{ij}^\dagger\bar{C}_{ji}+2w\sum_{n=1}^{\infty}A_n\left(\frac{H_{(1)}}{w}\right)^n\tag{8.24}$$

The $\bar{C}_{ij}^\dagger\bar{C}_{ji}$ term is of the form of a number operator, and counts the impurities, which is 2 in this case. Applying the above equation to the energy eigenstates $f_{\beta,p}(j_1,j_2)$ (8.16) gives the spectrum

$$2w+2w\sum_{n=1}^{\infty}A_n\left(\frac{16g_{YM}^2N}{w^3}\sin^2\left(\frac{p}{2}\right)\sin^2\left(\beta+\frac{p}{2}\right)\right)^n\tag{8.25}$$

This can be re-expressed in terms of a square root:

$$E_{g_{YM}}=2w\sqrt{1+\frac{16g_{YM}^2N}{w^3}\sin^2\left(\frac{p}{2}\right)\sin^2\left(\beta+\frac{p}{2}\right)}\tag{8.26}$$

This approach does not work for more than two impurities, because in this case the operator equivalence proved in appendix C does not apply for all n.

8.6 Strong and weak coupling limits

In the strong coupling limit, where $g_{YM}^2N/w^2\gg1$, only the second term of (8.26) will contribute, and the two impurity energy is

$$E_{g_{YM}}\approx\left|\frac{4g_{YM}\sqrt{N}}{w}\sin\left(\frac{p}{2}\right)\sin\left(\beta+\frac{p}{2}\right)\right|$$

At weak coupling, where $g_{YM}^2N/w^2\ll1$, only the first term in the expansion contributes.

The two operators from appendix C are trivially the same when $n = 1$, for any number of impurities. We can therefore find the weal coupling spectrum for any number of impurities. The energies for respectively two impurities and many impurities are therefore

$$\begin{aligned} E_{g_{YM}\ (2\text{ imp.})} &\approx 2w + \frac{16 g_{YM}^2 N}{w^2} \sin^2\left(\frac{p}{2}\right) \sin^2\left(\beta + \frac{p}{2}\right) \\ E_{g_{YM}\ (n\text{ imp.})} &\approx 2w + \frac{4 g_{YM}^2 N}{w^2} \sum_{i=1}^{n} \left(\cos^2\alpha_i - \cos^2\alpha_i \ \cos^2\alpha_{i+1}\right); \quad \alpha_{n+1} \equiv \alpha_i \end{aligned} \tag{8.27}$$

8.7 Similarity to Giant Magnon results

In [34], Hofman and Maldacena found a bound state of two giant magnons at weak coupling that has energy (to first order in the coupling strength) given by

$$2 + \frac{\lambda}{2\pi^2} 2 \cos^2\left(\frac{p}{2}\right) \sin^2\left(\frac{p}{2}\right) \tag{8.28}$$

To first order in the 't Hooft coupling, the energy we have obtained (in (8.27)) is

$$2 + 16\lambda \sin^2\left(\frac{p}{2}\right) \sin^2\left(\beta + \frac{p}{2}\right)$$

where the 't Hooft coupling $\lambda = g_{YM}^2 N$ and $w \to 1$. For the particular value of $\beta = \pi/2$ (plus $n\pi$), this is

$$2 + 16\lambda \cos^2\left(\frac{p}{2}\right) \sin^2\left(\frac{p}{2}\right) \tag{8.29}$$

which is identical to (8.28) up to redefinition of the Yang-Mills coupling constant.

Chapter 9

Conclusions

This dissertation has had two major drives. The first has been to show the power of the collective field thechnique by reviewing the work that has been on solving matrices in the large N limit. This began with the relevant background within the context of the AdS/CFT correspondence. This correspondence, while not yet rigorously proven or completely understood, is one of the most interesting recent discoveries in high energy physics. It couples two very different theories, both of which have had much work on them. It therefore offers a unique window to study both string and gauge theories. A description of the theories on either side of this correspondence, and motivation for the correspondence were given. Some of the links between states on either side of the correspondence were mentioned. The work done by Berenstein, Maldacena and Nastase on the plane wave limit, and the analysis of spin chains, were of particular interest.

The collective field theory technique was developed in a systematic manner. The basic formalism was developed initially in general coordinates. This was applied to a single matrix with an harmonic potential, and the spectrum was found. The physical motivation for the study of matrix models in the context of the Higgs scalars in the Super-Yang-Mills theory was then explored. Two possible identifications of variables were explored. The first corresponds to angular momentum eigenstates. In this picture, 1/2 BPS states are described by a single matrix model. The results for 1/2 BPS states have been found to agree with those found by Lin, Lunin and Maldacena [5]. The second variable identification involves treating two Higgs directly, rather than complexifying and creating angular momentum eigenstates. While this approach has been mentioned in the literature, it has not been thoroughly developed.

In order to study states beyond 1/2 BPS states in either picture, it is necessary to be able

to solve a model with two matrices. The full two matrix solution is very difficult. It is here that the power of the collective field theory technique becomes apparent. The second matrix can be introduced in a creation/annihilation operator basis as impurities into the background created by the first matrix. Using this approach, the spectrum and the states that result from using this technique with two matrices and an harmonic potential were found.

The work that has been done on introducing a Super-Yang-Mills interaction term in the angular momentum eigenstate model was then reviewed. This used the states that result from the two matrix model, with two impurities. A Hamiltonian was found, and resulting first order states were given. These states resemble the structure of BMN loops.

The second goal of this dissertation was to cover new territory by exploring more deeply the model that results from considering two of the Higgs scalars, and introducing the Super-Yang-Mills interaction term. The advantage of using the two Higgs scalar model is that it provides a way to calculate the spectrum (once the background has been subtracted) of a set of states, when a g_{YM} interaction term has been included. The disadvantage is that the mechanics of the collective field theory technique mean that one matrix is treated in the large N limit, whereas only a finite number of impurities of the second matrix are included. This had a nice physical explanation in the angular momentum eigenstate representation, where the zero impurity states are 1/2 BPS states, and states with a finite number of impurities are therefore states that are close to BMN states, or close to the plane defined by the two Higgs scalars. In the two Higgs scalar model however, there is no prior reason to treat the two matrices asymmetrically. By considering only a finite number of one matrix therefore, we are thus restricting ourselves to states near the axis of that scalar.

Within the two Higgs model, the background was subtracted from the full Hamiltonian (analogously to considering $\Delta - J$). A Bogoliubov transformation was applied to the resulting Hamiltonian. The operator that determines the spectrum of this Hamiltonian is of the same form as the leading operator found in the angular momentum eigenstate model. This operator was shown to generate a difference equation, which can be solved for any number of impurities. In the case of two impurities, this determines the spectrum exactly to all orders. For more than two impurities, the operator does not determine the spectrum exactly beyond first order, hence the results will only be a good approximation in the weak coupling limit.

The spectrum is one of the most important properties of any theory, and there are no

indications that the spectrum that we have obtained is not positive definite. One may conjecture that the spectrum may be the same as that for angular momentum eigenstates in a non-supersymmetric setting. Some indication that this may be the case is provided by the fact that the spectrum that has been obtained can be made to agree with non-supersymmetric giant magnon results in the literature.

9.1 Directions for future research

There are several possibilities for further work in this area. The first is to use the results obtained for the two Higgs scalar model to derive the spectrum in the angular momentum eigenstate model. This model could potentially be more interesting than the two scalar model, as it ties in to the work has been done in the BMN limit. The similarity to giant magnon results suggest that the spectrum that has been found for the two scalar model could indeed be linked to the angular momentum eigenstate spectrum.

The second avenue for further research would be to understand the link to giant magnons more clearly. An in-depth comparison of the two models could lead to understanding the significance of the link. It could also possibly also explain why there is extra freedom in the spectrum that we have derived, before the arbitrary setting of a parameter to get agreement with magnon results.

Thirdly, one could consider higher order terms than merely quadratic fluctuations. It would be interesting to see how cubic or higher order terms impact the spectrum.

Lastly, one could look at a three matrix model, i.e. a model with two different types of impurities. This model would describe states in a three-dimensional space, rather than the two dimensional plane considered so far. This offers the possibility of a link to QCD which lives in 3+1 dimensions.

Appendix A

Calculations of collective field theory parameters

A.1 Calculation of $\bar{\omega}$ in k space

Starting from the definition, (6.10), ω can be found by repeated application of (5.9)

$$
\begin{aligned}
&\omega(k_1,...,k_s) = \frac{\partial^2}{\partial M_{ij}\partial M_{ji}} Tr\left(Be^{ik_1M}...Be^{ik_sM}\right) \\
&= \int_0^1 d\alpha \frac{\partial}{\partial M_{ij}}\Big\{ \quad \left(e^{(1-\alpha)ik_1M}Be^{ik_2M} \ ... \ Be^{ik_sM}Be^{\alpha ik_1M}\right)_{ij} \\
&\qquad\qquad + \quad ... \\
&\qquad\qquad + \ \left(e^{(1-\alpha)ik_sM}Be^{ik_1M} \ ... \ Be^{ik_{s-1}M}Be^{\alpha ik_sM}\right)_{ij} \quad \Big\} \\
&= \int_0^1 d\beta \int_0^1 d\alpha \Big\{ \\
&\quad -(1-\alpha){k_1}^2 Tr\left(e^{\beta(1-\alpha)ik_1M}\right) Tr\left(e^{(1-\beta)(1-\alpha)ik_1M}Be^{ik_2M}...Be^{ik_sM}Be^{\alpha ik_1M}\right) \\
&\quad + \text{(terms that involve no impurity-free traces)} \\
&\quad -\alpha {k_1}^2 Tr\left(e^{(1-\alpha)ik_1M}Be^{ik_2M}...Be^{ik_sM}Be^{\beta\alpha ik_1M}\right) Tr\left(e^{(1-\beta)\alpha ik_1M}\right) \\
&\quad ... \\
&\quad -(1-\alpha){k_s}^2 Tr\left(e^{\beta(1-\alpha)ik_sM}\right) Tr\left(e^{(1-\beta)(1-\alpha)ik_sM}Be^{ik_1M}...Be^{ik_{s-1}M}Be^{\alpha ik_sM}\right) \\
&\quad + \text{(terms that involve no impurity-free traces)} \\
&\quad -\alpha {k_s}^2 Tr\left(e^{(1-\alpha)ik_sM}Be^{ik_1M}...Be^{ik_{s-1}M}Be^{\beta\alpha ik_sM}\right) Tr\left(e^{(1-\beta)\alpha ik_sM}\right) \quad \Big\}
\end{aligned}
$$

Only the terms with zero impurity loops are necessary, so all the terms that contain no impurity-free traces can be neglected. If one changes $\alpha \to (1-\alpha)$ in the first term for each k_i,

and $\beta \to (1-\beta)$ in the second term for each k_i, then

$$\begin{aligned}
&\omega(k_1,...,k_s) = \\
&= -2\int_0^1 d\alpha \int_0^1 d\beta \Big\{ \quad \alpha {k_1}^2 \psi_0(\beta\alpha k_1)\psi_s((1-\alpha\beta)k_1, k_2, ..., k_s) \\
&\qquad\qquad\qquad\qquad \cdots \\
&\qquad\qquad\qquad\qquad \alpha {k_s}^2 \psi_0(k_1)\psi_s((1-\alpha\beta)k_1, k_2, ..., (1-\alpha\beta)k_s) \Big\}
\end{aligned}$$

Working on the first term, and letting $k'' = \alpha k_1$ and $k' = \alpha\beta k_1$:

$$\begin{aligned}
&\int_0^1 d\alpha \int_0^1 d\beta \; \alpha {k_1}^2 \psi_0(\beta\alpha k_1)\psi_s((1-\alpha\beta)k_1, k_2, ..., k_s) \\
&= \int_0^{k_1} dk'' \int_0^{k''} dk' \; \psi_0(k')\psi_s(k_1 - k', k_2, ..., k_s)
\end{aligned}$$

To remove the k'' integral, we swop the order of integration and take $k' \to k_1 - k'$, obtaining

$$\begin{aligned}
&\int_0^{k_1} dk' \int_{k'}^{k_1} dk'' \; \psi_0(k')\psi_s(k_1 - k', k_2, ..., k_s) \\
&= \int_0^{k_1} dk' \; k'\psi_0(k_1 - k')\psi_s(k', k_2, ..., k_s)
\end{aligned}$$

The other terms work similarly to the k_1 term, so the final answer is

$$\bar{\omega}(k_1,...,k_s) = -2\sum_{i=1}^{s} \int_0^{k_i} dk' k' \psi_0(k_i - k')\psi_s(k_1, ..., k_{i-1}, k', k_{i+1}, ..., k_s) \tag{A.1}$$

A.2 Calculation of Ω in k space

$$\begin{aligned}
&\Omega_{0;s}(k_0; k_1, ..., k_s) = Tr\left(\frac{\partial\psi_0(k_0)}{\partial M_{ij}} \frac{\partial\psi_s(k_1, ..., k_s)}{\partial M_{ji}}\right) \\
&= \frac{\partial}{\partial M_{ij}} Tr\left(e^{ik_0 M}\right) \frac{\partial}{\partial M_{ji}} Tr\left(Be^{ik_1 M} ... Be^{ik_s M}\right) \\
&= ik_0 \left(e^{ik_0 M}\right)_{ji} \int_0^1 d\alpha \; \left(e^{(1-\alpha)ik_1 M} Be^{\alpha ik_2 M} ... Be^{ik_s M} Be^{\alpha ik_1 M}\right)_{ij} \\
&\quad + \cdots \\
&\quad + ik_0 \left(e^{ik_0 M}\right)_{ji} \int_0^1 d\alpha \; \left(e^{(1-\alpha)ik_s M} Be^{\alpha ik_1 M} ... Be^{ik_{s-1} M} Be^{\alpha ik_s M}\right)_{ij} \\
&= -k_0 \left(\sum_{i=1}^{s} k_i \psi_s(k_1, ..., k_i + k_0, ..., k_s)\right)
\end{aligned} \tag{A.2}$$

A.3 Calculation of ω in x space

To calculate the value of ω in position space, it is first useful to express ψ in a different form. Because M is Hermitian, it can be diagonalised, such that $V^\dagger MV = \Lambda$, where Λ is a diagonal matrix containing the eigenvalues of M:

$$V^\dagger MV = \Lambda = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ & & \ddots & \\ 0 & 0 & ... & \lambda_s \end{pmatrix} \tag{A.3}$$

It follows from (A.3) that

$$\left(V^\dagger e^{ikM} V\right)_{nm} = \delta_{nm} e^{ik\lambda_n} \tag{A.4}$$

This allows one to express ψ as follows:

$$\begin{aligned} \psi_s(k_1, ..., k_s) &= Tr\left(V^\dagger BV\ V^\dagger e^{ik_1 M} V\ ...\ V^\dagger BV\ V^\dagger e^{ik_s M} V\right) \\ &= \sum_{n_1=1}^{N} ... \sum_{n_s=1}^{N} (V^\dagger BV)_{n_1 n_2} e^{ik_1\lambda_{n_1}}\ ...\ (V^\dagger BV)_{n_s n_1} e^{ik_s\lambda_{n_s}} \\ \psi_s(x_1, ..., x_s) &= \sum_{n_1=1}^{N} .... \sum_{n_s=1}^{N} \delta(x_1 - \lambda_{n_1})(V^\dagger BV)_{n_1 n_2}\ \times ... \times \\ &\quad \times \delta(x_s - \lambda_{n_s})(V^\dagger BV)_{n_s n_1} \end{aligned} \tag{A.5}$$

Therefore, from (6.14),

$$
\begin{aligned}
&\bar{\omega}(\{k_i\}) = \\
&= -2\sum_{i=1}^{s}\int_0^{k_i} dk'k' \sum_m e^{i(k_i-k')\lambda_m} \times \\
&\quad \times \sum_{n_1,...,n_s} \left(V^\dagger BV\right)_{n_1n_2} e^{ik_1\lambda_{n_1}} \ ... \ \left(V^\dagger BV\right)_{n_in_{i+1}} e^{ik'\lambda_{n_i}} \ ... \ \left(V^\dagger BV\right)_{n_sn_1} e^{ik_s\lambda_{n_s}} \\
&= -2\sum_{i=1}^{s}\int_0^{k_i} dk'k' \int dz \int dy_1 ... \int dy_s \ e^{i(k_i-k')z} \sum_m \delta(z-\lambda_m) \times \\
&\quad \times \sum_{n_1=1}^{N} .... \sum_{n_s=1}^{N} \delta(y_1-\lambda_{n_1})(V^\dagger BV)_{n_1n_2} \ \ ... \ \delta(y_s-\lambda_{n_s})(V^\dagger BV)_{n_sn_1} \times \\
&\quad \times e^{ik_1y_1} \ ... \ e^{ik_{i-1}y_{i-1}} \ e^{ik'y_i} \ e^{ik_{i+1}y_{i+1}} \ ... \ e^{ik_sy_s} \\
&= -2\sum_{i=1}^{s}\int dz \int dy_1 ... \int dy_s \ \int_0^{k_i} dk'k' e^{-ik'(z-y_i)} e^{ik_iz} \times \\
&\quad \times \psi_0(z) \ \psi_s(y_1,...,y_s) \prod_{\substack{j=1 \\ (j\neq i)}}^{N} e^{k_jy_j}
\end{aligned}
\tag{A.6}
$$

The k' integration can now be done using integration by parts.

$$
\int_0^{k_i} dk'k' e^{-ik'(z-y_i)} e^{ik_iz} = \frac{ik_ie^{ik_iy_i}}{(z-y_i)} + \frac{e^{ik_iy_i}}{(z-y_i)^2} - \frac{e^{ik_iz}}{(z-y_i)^2} \tag{A.7}
$$

This finally leaves us in the position where we can apply a Fourier transformation, which converts the exponentials to delta functions. Integrals should obey the principal value prescription in what follows.

$$
\begin{aligned}
\bar{\omega}(\{x_i\}) &= \int \frac{dk_1}{2\pi} e^{-ik_1x_1} \ ... \ \int \frac{dk_s}{2\pi} e^{-ik_sx_s} \ \bar{\omega}(k_i) \\
&= -2\sum_{i=1}^{s}\int dk_1 ... \int dk_s \int dz \int dy_1 ... \int dy_s \ \psi_0(z) \ \psi_s(y_1,...,y_s) \times \\
&\quad \times \left[\frac{\frac{\partial}{\partial y_i}\delta(x_i-y_i)}{(z-y_i)} + \frac{\delta(x_i-y_i)}{(z-y_i)^2} - \frac{\delta(x_i-z)}{(z-y_i)^2}\right] \prod_{\substack{j=1 \\ (j\neq i)}}^{N} \delta(x_j-y_j) \\
&= -2\sum_{i=1}^{s}\int dz\psi_0(z)\frac{\partial}{\partial x_i}\left(\frac{\psi_s(\{x_i\})}{x_i-z}\right) \\
&\quad -2\sum_{i=1}^{s}\int dz\psi_0(z)\left[\frac{\psi_s(\{x_i\})}{(x_i-z)^2} - \delta(z-x_i)\int dy_i \frac{\psi(x_1,...,y_i,...,x_s;s)}{(y_i-z)^2}\right]
\end{aligned}
$$

A.4 Calculation of Ω in x space

Calculation of Ω from (6.14):

$$\begin{aligned}\Omega_{0;s}(z;x_1,...,x_s) &= \int \frac{dk_0}{2\pi} e^{-ik_0 z} \int \frac{dk_1}{2\pi} e^{-ik_1 x_1} ... \int \frac{dk_s}{2\pi} e^{-ik_s x_s} \Omega_{0;s}(k_0,k_1,...,k_s) \\ &= \int \frac{dk_0}{2\pi} e^{-ik_0 z} \int \frac{dk_1}{2\pi} e^{-ik_1 x_1} ... \int \frac{dk_s}{2\pi} e^{-ik_s x_s} \times \\ &\quad \times \; -k_0 \left(\sum_{i=1}^{s} k_i \psi_s(k_1,...,k_i+k_0,...,k_s) \right) \end{aligned} \tag{A.8}$$

Letting $k_i \to k_i + k_0$ yields

$$\begin{aligned}\Omega_{0;s}(z;x_1,...,x_s) &= \sum_{i=1}^{s} \frac{\partial}{\partial z} \frac{\partial}{\partial x_i} \int \frac{dk_0}{2\pi} e^{-ik_0(z-x_i)} \psi_s(x_1,...,x_s) \\ &= \sum_{i=1}^{s} \frac{\partial}{\partial z} \frac{\partial}{\partial x_i} \left(\delta(z-x_i) \psi_s(x_1,...,x_i,...,x_s) \right) \end{aligned} \tag{A.9}$$

Appendix B

Calculation of eigenvalues and eigenfunctions for many impurities

We derive here the eigenvalues and eigenfunctions for the multi-impurity kernel (equation 6.22) acting on the functionals given by equation (6.23). The derivation below follows the method used in [6]. When the kernel acts upon the functionals, the result is

$$\begin{aligned} & \sum_{i=1}^{s} \int dz \frac{\phi_0(z)}{(x_i - z)^2} (f(x_1, ..., x_s) - f(x_1, .., z, ..., x_s)) \qquad \text{(B.1)} \\ = & \sum_{i=1}^{s} \left[\left(-\frac{d}{dx_i} \int dz \frac{\phi_0(z)}{x_i - z} \right) f(x_1, .., x_s) + \frac{d}{dx_i} \int dz \frac{\phi_0(z) f(x_1, .., z, ..., x_s)}{x_i - z} \right] \end{aligned}$$

This is a sum of kernels of the form found by Marchesini and Onofri ([7]). All integrals in this section should be understood in the principal value prescription.

We will look at the action of the ith term in the sum, then multiply the eigenfunctions that result to obtain the complete eigenfunction. Looking at the first term, the equation we want to solve is

$$-\partial_{x_i} \int dz \frac{\phi_0(z)}{x_i - z} f = \epsilon_i f \qquad \text{(B.2)}$$

The quadratic potential that we are dealing with obeys the equation found by Brezin, Itzykson, Parisi and Zuber [25],

$$\int dz \frac{\phi_0(z)}{x_i - z} = wx \ . \qquad \text{(B.3)}$$

This can be seen by applying the identity in equation (5.17) to the background Hamiltonian

(5.24), so that the potential is

$$\frac{N^2}{2}\int dx\phi_0(x)\left[\left(\int dy\frac{\phi_0(y)}{x-y}\right)^2-\frac{w^2x^2}{2}\right]$$

and (B.3) results from minimisation with respect to ϕ_0. The solution of (B.2) is then trivial:

$$-w=\epsilon_i. \tag{B.4}$$

Turning to the second term in (B.1), the equation we want to solve is

$$\partial_{x_i}\int dz\frac{\phi_0(z)f(x_1,...,z,...,x_s)}{x_i-z}=\epsilon_i f(x_1,...,x_i) \tag{B.5}$$

In what follows, we will move to the "time of flight" coordinates introduced in section 5.4, so that

$$\frac{dx}{dq}=\pi\phi_0\ ,\quad x(q)=-\sqrt{\frac{2}{w}}\cos(wq)\quad\text{and}\quad\pi\phi_0=\sqrt{2w}\sin(wq)\ .$$

Consider now the following integral for $n>0$:

$$\int_{-\pi/w}^{\pi/w}\frac{dq}{\pi}\pi\phi_0(q)\frac{e^{in_iwq}}{x(q_0)-x(q)}=2\int_{-\pi}^{\pi}\frac{dq}{\pi}\sin(q)\frac{e^{in_iq}}{\cos(q)-\cos(q_0)}\ , \tag{B.6}$$

where n_i is a positive (non-zero) integer. This integral can be performed via complex analysis. We take the contour integral along the path shown in figure B.1. The function is periodic, with a period of 2π, so the vertical paths will cancel. Poles will occur at $q=\pm q_0+2k\pi$ where k is an integer. Only two of these poles will occur in any period of 2π. We choose the "time of flight" variables such that the poles occur at $q=\pm q_0$. These poles lie along the path of integration, so we will take the principal value prescription, picking up half the value of the residue of each pole.

The residues can easily be found by noting that (B.6) is equal to

$$2\int_{-\pi}^{\pi}\frac{dq}{\pi}\sin(q)\frac{e^{in_iq}}{-2\sin\left(\frac{q+q_0}{2}\right)\sin\left(\frac{q-q_0}{2}\right)}\ .$$

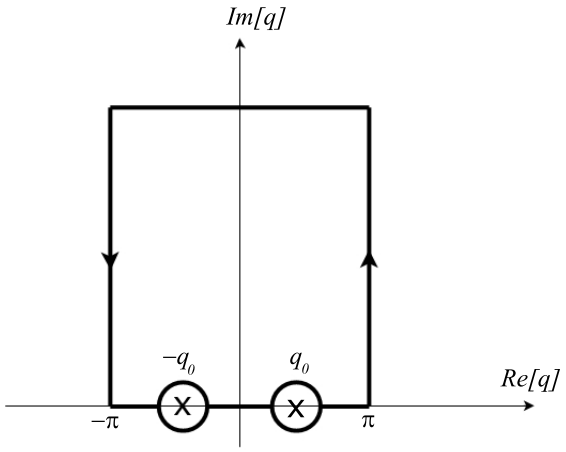


Figure B.1: The contour followed to solve equation B.6. The principal value prescription is taken at the poles, $-q_0$ and q_0.

The residues are thus

$$\begin{aligned} \text{res}(q_0) &= -\frac{1}{2}\sin(q_0)\frac{e^{in_i q_0}}{sin(q_0)} = -\frac{e^{in_i q_0}}{\pi} \ , \\ \text{res}(-q_0) &= -\frac{e^{-in_i q_0}}{\pi} \ . \end{aligned}$$

The solution to the integral is therefore

$$\pi i \left[\text{res}(q_0) + \text{res}(-q_0)\right] = -2i\cos(n_i q_0) \tag{B.7}$$

Returning to (B.6), note that if one expands $\exp[in_i q] = \cos(n_i q) + i\sin(n_i q)$, the cos term is antisymmetric and falls away, leaving

$$\int_0^{\pi/w} \frac{dq}{\pi}\pi\phi_0(q)\frac{\sin(n_i w q)}{x(q_0) - x(q)} = -\cos(n_i q_0)$$

We now move to x space, taking $x(q_0) \to z$ and $x(q) \to x_i$.

$$\int_{-\sqrt{2/w}}^{-\sqrt{2/w}} dz \frac{\sin(n_i q(z))}{x_i - z} = -\cos(n_i q(x_i)) \tag{B.8}$$

Taking a derivative with respect to x_i puts this into the same form as (B.5).

$$\begin{aligned}
\partial_{x_i} \int_{-\sqrt{2/w}}^{-\sqrt{2/w}} dz \frac{\sin(n_i q(z))}{x_i - z} &= n_i \sin(n_i q(x_i)) \partial_{x_i} q(x_i) \\
\partial_{x_i} \int_{-\sqrt{2/w}}^{-\sqrt{2/w}} dz \frac{\pi\phi_0 \frac{\sin(n_i q(z))}{\pi\phi_0}}{x_i - z} &= n_i \frac{\sin(n_i q(x_i))}{\pi\phi_0}
\end{aligned} \tag{B.9}$$

We thus identify a family of suitable eigenfunctions f, for each value of n_i

$$f_{n_i}(\{x_i\}) \equiv \frac{\sin(n_i q(x_i))}{\pi\phi_0} = \frac{\sin(n_i q(x_i))}{\sqrt{2}\sin(q(x_i))} \tag{B.10}$$

The eigenvalues are

$$\epsilon_i = n_i$$

Multiplying together all the eigenvalues from both the first and second term in equation (B.1) gives

$$\epsilon_{\{n_i\}} = w \left(\sum_{i=1}^{s} n_i - s \right) \tag{B.11}$$

The complete eigenfunctions are products of the eigenfunctions we have just found:

$$\prod_{i=1}^{s} \frac{\sin(n_i w q(x_i))}{\sqrt{2w}\sin(w q(x_i))} \tag{B.12}$$

These are the eigenvalues and eigenfunctions that appear in equation (6.24) in the main text.

Appendix C

Equivalence of two operators for two impurities

This appendix contains the proof of the equivalence of two operators (when applied to the 2-impurity states that we are dealing with) that is used in section 8.4 (equation (8.23)). To simplify notation, we will denote the n times nested commutator of an operator by (n) after the commutator, for example

$$[M,\underbrace{[M,\ldots[M,C^\dagger]\ldots]}_{n \text{ times nested commutators}}]\;[M,\underbrace{[M,\ldots[M,C]\ldots]}_{n \text{ times nested commutators}}] \equiv [M,C^\dagger]_{(n)}\;[M,C]_{(n)}$$

The statement of the proof is then that

$$\mathrm{Tr}\big([M,C^\dagger]_{(n)}[M,C]_{(n)}\big)\mathrm{Tr}(C^\dagger M u_j C^\dagger M u_k) = 2\left(\frac{1}{2}\mathrm{Tr}[M,C^\dagger][M,C]\right)^n \mathrm{Tr}(C^\dagger M u_j C^\dagger M u_k) \tag{C.1}$$

for $n \geq 1$. In a coherent state basis, $C^\dagger \to C$, $C \to \partial/\partial C$,

$$\begin{aligned}\underbrace{\mathrm{Tr}\big([M,C]_{(n)}[M,\partial/\partial C]_{(n)}\big)}_{\hat{O}_n}\mathrm{Tr}(CMu_jCMu_k) &= 2\left(\underbrace{\frac{1}{2}\mathrm{Tr}[M,C][M,\partial/\partial C]}_{\hat{H}_1}\right)^n \mathrm{Tr}(CMu_jCMu_k)\\ \hat{O}_n\mathrm{Tr}(CMu_jCMu_k) &= 2\hat{H}_1{}^n\mathrm{Tr}(CMu_jCMu_k)\end{aligned} \tag{C.2}$$

C.1 Action of $\hat{O}_n$

First note that

$$\mathrm{Tr}[M,x][M,y] = -\mathrm{Tr}\big(\big[M,[M,x]\big]y\big)$$

This can be applied repeatedly to show that

$$\begin{aligned}\mathrm{Tr}[M,C]_{(n)}\left[M,\frac{\partial}{\partial C}\right]_{(n)} &= -\mathrm{Tr}[M,C]_{(n+1)}[M,\partial/\partial C]_{(n-1)} \\ &= (-1)^n\mathrm{Tr}\left([M,C]_{(2n)}\frac{\partial}{\partial C}\right) \end{aligned} \tag{C.3}$$

Secondly, it can be shown inductively that

$$[M,C]_{(k)} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}M^{k-i}CM^i \tag{C.4}$$

This clearly holds for $k=1$,

$$[M,C]_{(1)} = MC - CM$$

If one assumes that it holds for k, then

$$\begin{aligned}[M,C]_{(k+1)} &= \left[M,\sum_{i=0}^{k}(-1)^i\binom{k}{i}M^{k-i}CM^i(-1)^i\right] \\ &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\big(M^{k-i+1}CM^i - M^{k-i}CM^{i+1}\big) \\ &= \sum_{i=0}^{k+1}(-1)^iM^{k+1-i}CM^i\left[\binom{k}{i}+\binom{k}{i-1}\right] \\ &\quad \text{(where } \binom{a}{b} \text{ is understood as 0 for } b<0 \text{ or } b>a\text{)} \\ &= \sum_{i=0}^{k+1}(-1)^iM^{k+1-i}CM^i\binom{k+1}{i}\end{aligned}$$

Using (C.3) and (C.4), we see that

$$\mathrm{Tr}\big([M,C]_{(n)}[M,\partial/\partial C]_{(n)}\big) = (-1)^n\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\mathrm{Tr}(M^{2n-i}CM^i\partial/\partial C) \tag{C.5}$$

The above operator is then applied to a state $\mathrm{Tr}(Cu_jCu_k)$, using (8.5), to yield

$$\begin{aligned}
&\mathrm{Tr}\big([M,C]_{(n)}[M,\partial/\partial C]_{(n)}\big)\mathrm{Tr}(CMu_jCMu_k)\\
&=(-1)^n\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}(M^{2n-i}CM^i)_{ab}(u_jCu_k+u_kCu_j)_{ba}\\
&=2(-1)^n\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\mathrm{Tr}(CM^iu_jCM^{2n-i}u_k)
\end{aligned}\tag{C.6}$$

C.2 Action of $\hat{H}_1{}^n$

We prove inductively that

$$(\mathrm{Tr}([M,C][M,\partial/\partial C]))^k\,\mathrm{Tr}(CMu_jCMu_k)=(-1)^k2^k\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}\mathrm{Tr}(CM^iu_jCM^{2k-i}u_k)\tag{C.7}$$

For $k=1$, equation (8.6) shows that

$$\mathrm{Tr}([M,C][M,\partial/\partial C])\mathrm{Tr}(CMu_jCMu_k)=-2\mathrm{Tr}\big(Cu_jCM^2u_k-2CMu_jCMu_k+CM^2u_jCu_k\big)$$

To show that it holds for $k+1$:

$$
\begin{aligned}
&\mathrm{Tr}[M,C][M,\partial/\partial C](-1)^k 2^k \sum_{i=0}^{2k}(-1)^i \binom{2k}{i} \mathrm{Tr}(CM^i u_j CM^{2k-i}u_k) \\
&= (-1)^k 2^k \sum_{i=0}^{2k}(-1)^i \mathrm{Tr}\left((2MCM - MMC - CMM)\frac{\partial}{\partial C}\right) \mathrm{Tr}(CM^i u_j CM^{2k-i}u_k) \\
&= (-1)^k 2^k \sum_{i=0}^{2k}(-1)^i (2MCM - MMC - CMM)_{ab} \times \\
&\qquad \times \left((M^i u_j CM^{2k-i}u_k)_{ba} + (M^{2k-i}u_k)_{bq}(CM^i u_j)_{qa}\right) \\
&= 2(-1)^k 2^k \sum_{i=0}^{2k}(-1)^i \mathrm{Tr}(-CM^{i+2}u_j CM^{2k-i} + 2CM^{i+1}u_j CM^{2k-i+1} - CM^i u_j CM^{2k-i+2}) \\
&= (-1)^k 2^{k+1} \mathrm{Tr}\Bigg\{ \left[-\binom{2k}{0}\right] CM^0 u_j CM^{2k+2}u_k + (-1)\left[-2\binom{2k}{0} - \binom{2k}{1}\right] CM^1 u_j CM^{2k+1}u_k \\
&\qquad\qquad + (-1)^2 \left[-\binom{2k}{0} - 2\binom{2k}{1} - \binom{2k}{2}\right] CM^2 u_j CM^{2k}u_k + \ldots \Bigg\} \\
&= (-1)^k 2^{k+1} \sum_{i=0}^{2k+2}(-1)^i \left[-\binom{2k}{i-2} - 2\binom{2k}{i-1} - \binom{2k}{i}\right] \mathrm{Tr}(CM^i u_j CM^{2k+2-i}u_k) \\
&\qquad \text{(where } \binom{a}{b} \text{ is understood as 0 for } b<0 \text{ or } b>a) \\
&= (-1)^{k+1} 2^{k+1} \sum_{i=0}^{2k+2}(-1)^i \binom{2k+2}{i} \mathrm{Tr}(CM^i u_j CM^{2k+2-i}u_k) \qquad \text{(C.8)}
\end{aligned}
$$

which proves (C.7).

We then see that

$$
\left(\frac{1}{2}\mathrm{Tr}([M,C][M,\partial/\partial C])\right)^n \mathrm{Tr}(CMu_j CMu_k) = (-1)^n \sum_{i=0}^{2n}(-1)^i \binom{2n}{i} \mathrm{Tr}(CM^i u_j CM^{2n-i}u_k)
$$

which is half of (C.6). Thus result (C.2) is proved.

C.3 More than two impurities

The equality fails for more than two impurities. For three impurities, the results are different for $n \geq 3$. For four or more impurities, they are different for $n \geq 2$. This is shown in table C.3. The notation used is the same as that in section 8.2, where the numbers correspond to how many Ms appear after each C in the state $\mathrm{Tr}(CM^a u_{j_1} CM^b uj_2 \ldots)$. For $n = 1$, the two operators are trivially the same for any number of impurities.

(a) Three impurities, with $n = 3$

Term	$\mathbf{2\hat{H}_1{}^3}$	$\mathbf{\hat{O}_3}$
$(0,0,6)$	-2	-2
$(0,1,5)$	6	6
$(0,2,4)$	-12	-15
$(0,3,3)$	14	20
$(0,4,2)$	-12	-15
$(0,5,1)$	6	6
$(0,6,0)$	-2	-2
$(1,0,5)$	6	6
$(1,1,4)$	-6	0
$(1,2,3)$	6	0
$(1,3,2)$	6	0
$(1,4,1)$	-6	0
$(1,5,0)$	6	6
$(2,0,4)$	-12	-15
$(2,1,3)$	6	0
$(2,2,2)$	-18	0
$(2,3,1)$	6	0
$(2,4,0)$	-12	-15
$(3,0,3)$	14	20
$(3,1,2)$	6	0
$(3,2,1)$	6	0
$(3,3,0)$	14	20
$(4,0,2)$	-12	-15
$(4,1,1)$	-6	0
$(4,2,0)$	-12	-15
$(5,0,1)$	6	6
$(5,1,0)$	6	6
$(6,0,0)$	-2	-2

(b) Four impurities with $n = 2$

Term	$\mathbf{2\hat{H}_1{}^2}$	$\mathbf{\hat{O}_2}$
$(0,0,2,2)$	6	6
$(0,0,3,1)$	-4	-4
$(0,0,1,3)$	-4	-4
$(0,0,4,0)$	2	2
$(0,0,0,4)$	2	2
$(0,1,2,1)$	4	0
$(0,1,3,0)$	-4	-4
$(0,1,1,2)$	-4	0
$(0,2,1,1)$	-4	0
$(0,2,2,0)$	6	6
$(0,2,0,2)$	4	0
$(0,3,1,0)$	-4	-4
$(0,4,0,0)$	2	2
$(1,0,1,2)$	4	0
$(1,0,2,1)$	-4	0
$(1,0,0,3)$	-4	-4
$(1,1,1,1)$	8	0
$(1,1,2,0)$	-4	0
$(1,1,0,2)$	-4	0
$(1,2,1,0)$	4	0
$(1,2,0,1)$	-4	0
$(1,3,0,0)$	-4	-4
$(2,0,1,1)$	-4	0
$(2,0,2,0)$	4	0
$(2,0,0,2)$	6	6
$(2,1,1,0)$	-4	0
$(2,1,0,1)$	4	0
$(2,2,0,0)$	6	6
$(3,0,0,1)$	-4	-4
$(3,1,0,0)$	-4	-4
$(4,0,0,0)$	2	2

Table C.1: $2\hat{H_1}^n$ and $\hat{O}_n$ applied to states with three or four impurities, showing that the operators are not equal.

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