

Group invariant solutions for the unsteady magnetohydrodynamic flow of a fourth grade fluid in a porous medium

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Research submitted to the Faculty of Science, University of the
Witwatersrand, Johannesburg for the Degree of Master of
Science by Dissertation.



May 29, 2014

Abstract

The effects of non-Newtonian fluids are investigated by means of two appropriate models studying a third and fourth grade fluid respectively. The geometry of both these models is described by the unsteady unidirectional flow of an incompressible fluid over an infinite flat rigid plate within a porous medium. The fluid is electrically conducting in the presence of a uniform applied magnetic field that occurs in the normal direction to the flow.

The classical Lie symmetry approach is undertaken in order to construct group invariant solutions to the governing higher-order non-linear partial differential equations. A three-dimensional Lie algebra is acquired for both fluid flow problems.

In each case, the invariant solution corresponding to the non-travelling wave type is considered to be the most significant solution for the fluid flow model under investigation since it directly incorporates the magnetic field term. A numerical solution to the governing partial differential equation is produced and a comparison is made with the results obtained from the analytical approach.

Finally, a graphical analysis is carried out with the purpose of observing the effects of the emerging physical parameters. In particular, a study is carried out to examine the influences of the magnetic field parameter and the non-Newtonian fluid parameters.

Declaration

I declare that this is my own, unaided work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature. It is being submitted as fulfilment of the Degree of Master of Science by Dissertation to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

Abdul Hamid Carrim

May 29, 2014

Dedication

To my family.

Acknowledgements

First and foremost, I would like to thank the Almighty for all the blessings he has bestowed upon me.

A sincere thank you to my supervisor, Professor F.M. Mahomed for your determination and unrelenting patience. I really appreciate the time and guidance you extended to me during the course of this dissertation.

This dissertation would not have been possible without the invaluable help and insights of Mr. Taha Aziz. Your unyielding encouragement, support and time throughout the course of this work is deeply appreciated.

Most importantly, I would like to thank my parents, siblings, family and friends for their tremendous and continued support, patience and understanding.

Lastly, the University of the Witwatersrand and the National Research Foundation are thanked for their financial support.

Contents

1	Introductory remarks	1
1.1	Introduction	1
1.2	Objectives	3
1.3	Outline	4
2	Theoretical considerations	5
2.1	Introduction	5
2.2	Algebraic approaches to differential equations	5
2.2.1	Lie group theory	6
2.2.2	Lie symmetry reduction methods and group invariant solutions	8
3	The unsteady MHD flow of a third grade fluid in a porous medium	10
3.1	Introduction	10
3.2	Mathematical formulation of the model	11
3.3	Lie point symmetry analysis	19
3.4	Classification of group invariant solutions	27
3.5	Group invariant and numerical solutions to the problem	31
3.5.1	Invariant solution via subgroup generated by X_3	31
3.5.2	Steady state solution	34
3.5.3	Travelling wave solution	37
3.5.4	Numerical solution of the governing model	40
3.6	Graphical analysis	41
3.7	Conclusion	48

4	The unsteady MHD flow of a fourth grade fluid in a porous medium	50
4.1	Introduction	50
4.2	Mathematical formulation of the model	51
4.3	Lie point symmetry analysis	60
4.4	Classification of group invariant solutions	65
4.5	Group invariant and numerical solutions to the problem	69
4.5.1	Invariant solution via subgroup generated by X_3	69
4.5.2	Steady state solution	74
4.5.3	Travelling wave solution	77
4.5.4	Numerical solution of the governing model	78
4.6	Graphical analysis	79
4.7	Conclusion	86
5	Concluding remarks	88
5.1	Conclusions	88
5.2	Further research possibilities	89
	Bibliography	90

List of Figures

3.2.1 Graphical representation of the third grade non-Newtonian fluid flow model.	12
3.6.1 Velocity field of the analytical solution (3.5.15) for varying values of time t , where $\phi_* = 1$, $M_* = 0.5$, $\beta_* = 1$, $\beta = 1.5$ are fixed.	42
3.6.2 Velocity field of the analytical solution (3.5.15) for varying values of the magnetic field parameter M_* , where $\phi_* = 1$, $t = \pi/2$, $\beta_* = 1.5$, $\beta = 2.5$ are fixed.	43
3.6.3 Velocity field of the analytical solution (3.5.15) for varying values of the third grade fluid parameter β , where $\phi_* = 1$, $M_* = 0.5$, $t = \pi/2$, $\beta_* = 1.5$ are fixed.	43
3.6.4 Velocity field of the analytical solution (3.5.15) for varying values of the third grade fluid parameter β_* , where $\phi_* = 1$, $M_* = 0.5$, $t = \pi/2$, $\beta = 1.5$ are fixed.	44
3.6.5 Numerical solution of the reduced ordinary differential equation (3.5.21) subject to the boundary conditions (3.5.22) and (3.5.23) for varying values of the magnetic field parameter M_* , where $\mu_* = 0.5$, $\beta = 1.5$, $\beta_* = 1$, $\phi_* = 0.5$ are fixed.	45
3.6.6 Velocity field of the analytical solution (3.5.56) for varying values of time t , where $\beta_* = 1.5$, $c = \beta = 1$ are fixed.	45
3.6.7 Velocity field of the analytical solution (3.5.56) for varying values of the wave speed c , where $\beta_* = 1.5$, $t = \pi/2$, $\beta = 1$ are fixed.	46

3.6.8 Numerical solution of the partial differential equation (3.2.39) subject to conditions (3.5.58)-(3.5.60) for varying values of time t , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\phi_* = 0.8$, $M_* = 0.5$ are fixed.	46
3.6.9 Numerical solution of the partial differential equation (3.2.39) subject to conditions (3.5.58)-(3.5.60) for varying values of the magnetic field parameter M_* , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\phi_* = 0.8$, $t = 0.5$ are fixed.	47
4.2.1 Graphical representation of the fourth grade non-Newtonian fluid flow model.	51
4.6.1 Velocity field of the analytical solution (4.5.24) for varying values of time t , where $\beta_* = 1$, $\beta = 1.5$, $\gamma_* = 0.5$, $\Gamma_* = 1$, $M_* = 0.5$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.	81
4.6.2 Velocity field of the analytical solution (4.5.24) for varying values of the magnetic field parameter M_* , where $t = \pi/2$, $\beta_* = 1$, $\beta = 1.5$, $\gamma_* = 0.5$, $\Gamma_* = 1$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.	81
4.6.3 Velocity field of the analytical solution (4.5.24) for varying values of the fourth grade fluid parameter β_* , where $t = \pi/2$, $\beta = 1.5$, $\gamma_* = 0.5$, $\Gamma_* = 1$, $M_* = 0.5$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.	82
4.6.4 Velocity field of the analytical solution (4.5.24) for varying values of the fourth grade fluid parameter Γ_* , where $t = \pi/2$, $\beta_* = 1$, $\beta = 1.5$, $\gamma_* = 0.5$, $M_* = 0.5$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.	82
4.6.5 Numerical solution of the reduced ordinary differential equation (4.5.30) subject to the boundary conditions (4.5.31) and (4.5.32) for varying values of the magnetic field parameter M_* , where $\mu_* = 1.5$, $\beta_* = 0.5$, $\kappa_1 = 0.9$, $\kappa_3 = 2.5$ are fixed.	83
4.6.6 Numerical solution of the reduced ordinary differential equation (4.5.50) subject to the boundary conditions (4.5.51)-(4.5.53) for varying values of the magnetic field parameter M_* , where $c = 0.5$, $\mu_* = 1$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 0.8$, $\gamma_* = 1$, $\Gamma_* = 0.1$, $\kappa_1 = 0.2$, $\kappa_2 = 2$, $\kappa_3 = 1$, $\kappa_4 = 0.8$, $\kappa_5 = 0.5$ are fixed.	84

-
- 4.6.7 Numerical solution of the partial differential equation (4.2.43)
 subject to conditions (4.5.54)-(4.5.58) for varying values of time
 t , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\gamma_* = 0.8$, $\Gamma_* = 0.5$,
 $\kappa_1 = 1.2$, $\kappa_2 = 0.2$, $\kappa_3 = 1.9$, $\kappa_4 = 1$, $\kappa_5 = 2.2$, $M_* = 0.5$ are fixed. 85
- 4.6.8 Numerical solution of the partial differential equation (4.2.43)
 subject to conditions (4.5.54)-(4.5.58) for varying values of time
 M_* , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\gamma_* = 0.8$,
 $\Gamma_* = 0.5$, $\kappa_1 = 1.2$, $\kappa_2 = 0.2$, $\kappa_3 = 1.9$, $\kappa_4 = 1$, $\kappa_5 = 2.2$, $t = 0.5$
 are fixed. 85

List of Tables

3.4.1 Commutator table of the three-dimensional Lie algebra (3.3.61)	29
3.4.2 Adjoint table for the symmetry generators (3.3.61)	30
4.4.1 Commutator table of the three-dimensional Lie algebra (4.3.38)	66
4.4.2 Adjoint table for the symmetry generators (4.3.38)	67

Chapter 1

Introductory remarks

1.1 Introduction

A fluid can be described as either a Newtonian or a non-Newtonian fluid based on their respective properties. Non-Newtonian fluids exhibit flow behaviour different from Newtonian fluids and have a vast number of material properties. The main distinction however is in the way they describe viscosity. For a Newtonian fluid, the viscosity remains constant since there exists a linear relationship between the shear stress and the shear rate. The shear rate is the rate at which deformation occurs in a material. The most common examples of Newtonian fluids are water and air. On the contrary, a non-Newtonian fluid demonstrates a variable viscosity since there is a non-linear relationship between the shear stress and shear rate. Non-Newtonian fluids are a common occurrence in the world around us. Substances such as mud, paint and blood and foods such as mayonnaise and custard are all classified as non-Newtonian fluids.

The study of non-Newtonian fluid flow problems has piqued recently due to its wide range of engineering and industrial applications. With them arise some uncharacteristic features of non-Newtonian fluids such as shear thickening and shear thinning effects. Shear thickening is the process whereby the viscosity of the fluid increases with an increase in the shear rate. An example of such a fluid is cornstarch. Conversely, the case when the viscosity decreases as the shear rate increases is known as shear thinning. An example of a fluid that exhibits this behaviour is lava.

Currently, there is no distinct mathematical model that completely describes all the properties of non-Newtonian fluids owing to the complexity of the physical nature of non-Newtonian fluids. However, one of the most widely recognized models is the class of Rivlin-Ericksen fluids. It is one of the foremost models that describes the flow properties of a non-Newtonian fluid [1]. The simplest such subclass of the Rivlin-Ericksen fluids is known as a second grade fluid. Much work has been done in the literature with regards to fluid flow problems of the second grade since the governing equations tend to be linear in nature and thus there is reasonably less difficulty in obtaining an analytical solution [2, 3]. The defining property of a second grade fluid is that it is capable of determining normal stress differences. However, it is vastly lacking since it fails to describe shear thinning and thickening effects which are inherent in many fluids. It is for this reason that a further subclass of the Rivlin-Ericksen fluids, a third grade fluid, is considered. Similar to a second grade fluid, a third grade fluid also attempts to describe differences in normal stresses. The stark contrast however is that a third grade fluid succeeds in encapsulating shear thinning and thickening effects and work has been done in the literature in this regard [4, 5, 6]. Notwithstanding, a third grade fluid still has inadequacies since it does not capture all properties of a non-Newtonian fluid. The most generalized subclass of the Rivlin-Ericksen fluids is the fourth grade non-Newtonian fluid. Its focal feature is that it simultaneously describes most properties of a non-Newtonian fluid. It has been found that in general, the governing equations associated with fourth grade fluid flow problems are complicated higher-order non-linear equations. It is precisely for this reason that a minimal amount of work has been carried out with regards to this particular flow problem and even less so with regards to exact solutions of such models [7, 8].

Other physical occurrences can also be of importance in the study of non-Newtonian fluid flow. Magnetohydrodynamic (MHD) flow describes the occurrence of a magnetic field acting on the flow of a non-Newtonian fluid. The study of the behaviour of non-Newtonian fluids in the occurrence of a magnetic field has received a fair amount of interest recently due to its many applications. Some of these include blood flow, plasma flow, flow of mercury amalgams and the flow of liquid metals and alloys [9].

Furthermore, the flow of a non-Newtonian fluid through a porous medium

is central to applications in nature as well as in science and engineering. Some useful applications include oil recovery and the movement of water, pollutants and nutrients into plants [10].

The major obstacle faced with non-Newtonian fluid flow problems is that the governing equations tend to be higher-order non-linear in nature and as such, obtaining and analysing an analytical solution proves to be quite difficult. Exact solutions of problems described by non-Newtonian fluid flow are highly significant since they enable us to verify any numerical solutions we might obtain. An efficient tool used to acquire exact solutions of non-linear problems described by differential equations is outlined by the Lie symmetry approach. It was introduced in the 1870's by the Norwegian mathematician Marius Sophus Lie and is extensively used today in a vast array of fields [11, 12]. We can use this approach to solve such problems since it has been shown to be a useful tool in the solution of non-linear problems that have been devised in terms of differential equations.

Based on the aforementioned facts, we are encouraged to model and solve non-Newtonian fluid flow problems. We consider two problems dealing with a third and fourth grade non-Newtonian fluid. The focus of this dissertation is on the unsteady flow of a non-Newtonian fluid over an infinite flat rigid plate in a porous medium while in the presence of a constant applied magnetic field. We then employ the Lie group theory to find exact solutions to the governing model. We then carry out a graphical analysis in order to infer the behaviour and effects of various pertinent parameters on the fluid flow.

1.2 Objectives

For this dissertation, the key objectives to be undertaken for both a third and fourth grade fluid begin with modelling the fluid flow problem under consideration. We then obtain the Lie point symmetries of the governing equation by implementing the classical Lie symmetry approach. Thereafter, we will classify the family of invariant solutions by means of finding the optimal system for the obtained Lie algebra. This will be followed by the construction of the corresponding group invariant solutions. We will resort to numerical methods in cases where we fail to obtain a closed-form solution. The governing model

will also be solved numerically and a comparison of the two approaches will be made. We will end off by using graphical methods in order to analyse the influences of the significant physical parameters and the effects they might have on the velocity profiles of the flow of fluid.

1.3 Outline

We will consider the non-Newtonian fluid flow problem where the geometry of the model is described by unsteady flow over an infinite flat rigid plate along a porous medium in the presence of a magnetic field. Chapter 2 contains basic equations and techniques that will be referred to throughout this dissertation. In Chapter 3, we provide a review of the modelling and solutions for a third grade fluid as proposed by Aziz et al. [6] as well as the calculation of the optimal system which was not considered by the authors. Next, we will extend this work for a fourth grade fluid under the selfsame geometry in Chapter 4. To the best of our knowledge, the study of this particular fluid model has not been considered before. We then present some concluding remarks in Chapter 5.

Chapter 2

Theoretical considerations

2.1 Introduction

In this chapter, we focus on solution methods of partial differential equations. In particular, we provide a description of the classical Lie group approach. Further, we provide methods of reduction for partial differential equations.

The classical Lie symmetry approach is a theory that is known for being a systematic search method for exact solutions of differential equations. It is commonly employed today in the fields of mathematics, physics and mechanics. The main objective of the Lie symmetry approach to differential equations as set out by Sophus Lie is to obtain one or several parameters local continuous transformations that leave the equations invariant. These are then exploited to obtain the so-called group invariant solutions.

2.2 Algebraic approaches to differential equations

In this section, we present a technique for solving partial differential equations described by the classical Lie group approach and set out a method by which analytical solutions to partial differential equations may be obtained.

2.2.1 Lie group theory

We begin by laying out the classical Lie symmetry approach for one dependent variable and n independent variables. We wish to study the invariance properties of a k th-order partial differential equation with one dependent variable $u = u(x)$ and n independent variables $x = (x_1, \dots, x_n)$ given by

$$F(x, u_{(1)}, \dots, u_{(k)}) = 0, \quad (2.2.1)$$

where

$$u_{(j)} = \left\{ \frac{\partial^j u}{\partial x_{i_1} \dots \partial x_{i_j}} \right\}, \quad (2.2.2)$$

is the set of all j th-order partial derivatives of u such that $j = 0, \dots, k$ and $i_l = 1, \dots, n$ for $l = 1, \dots, j$. In order to obtain the Lie point symmetries of the partial differential equation (2.2.1), we search for the group transformations that will leave equation (2.2.1) invariant.

Definition 2.2.1 (Lie group of transformations). Consider the Lie group of transformations defined by

$$\begin{aligned} \bar{x}^i &= f^i(x, u, \epsilon), \\ \bar{u} &= g(x, u, \epsilon), \end{aligned} \quad (2.2.3)$$

where ϵ is the group parameter and $i = 1, \dots, n$. These transformations form a one-parameter group since the closure, identity, inverse and associative properties hold.

These transformations can be reworked in terms of their Taylor series expansions since ϵ is a small parameter.

Definition 2.2.2 (Infinitesimal transformations). The infinitesimal transformations of the Lie group of transformations (2.2.3) is

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \\ \bar{u} &= u + \epsilon \eta(x, u) + O(\epsilon^2), \end{aligned} \quad (2.2.4)$$

where the components ξ^i and η are known as the infinitesimals of (2.2.3).

The one-parameter group of transformations (2.2.3) can be recovered from the infinitesimal transformations (2.2.4) by solving the Lie equations

$$\begin{aligned}\frac{d\bar{x}^i}{d\epsilon} &= \xi^i(\bar{x}, \bar{u}), \\ \frac{d\bar{u}}{d\epsilon} &= \eta(\bar{x}, \bar{u}),\end{aligned}\tag{2.2.5}$$

subject to the initial conditions

$$\begin{aligned}\bar{x}^i|_{\epsilon=0} &= x^i, \\ \bar{u}|_{\epsilon=0} &= u.\end{aligned}\tag{2.2.6}$$

As a consequence, the one-parameter Lie group of transformations can also be represented by its infinitesimal generator.

Definition 2.2.3 (Infinitesimal generator). The infinitesimal generator of the Lie group of transformations (2.2.3) is the operator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u},\tag{2.2.7}$$

where ξ^i and η are the coefficient functions of the operator X .

Since the partial differential equation (2.2.1) is k th-order, the infinitesimal transformations in equation (2.2.4) must be extended to the same order. This results in the formal definition of the extended infinitesimal generator.

Definition 2.2.4 (Extended infinitesimal generator). The k th extension of the infinitesimal transformations (2.2.4) corresponding to the k th extended infinitesimal generator is defined by

$$X^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta^i \frac{\partial}{\partial u_i} + \dots + \zeta^{i_1, \dots, i_k} \frac{\partial}{\partial u_{i_1, \dots, i_k}}, \quad k \geq 1,\tag{2.2.8}$$

where the extended infinitesimals ζ satisfy

$$\begin{aligned}\zeta^i &= D_i \eta - u_j D_i (\xi^j), \\ \zeta^{i_1, \dots, i_k} &= D_{i_k} (\zeta^{i_1, \dots, i_{k-1}}) - u_{i_1, \dots, i_{k-1}, j} D_{i_k} (\xi^j), \quad k \geq 2,\end{aligned}\tag{2.2.9}$$

and the total derivative operator D_i is

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_i} + \dots \quad i = 1, 2, \dots, n. \quad (2.2.10)$$

Now, the operator (2.2.7) is a Lie symmetry generator of the k th-order partial differential equation (2.2.1) if the invariance criterion holds.

Definition 2.2.5 (Invariance criterion). The invariance criterion states that a Lie point symmetry is admitted by the k th-order partial differential equation (2.2.1) if and only if

$$X^{[k]}(F)|_{F=0} = 0. \quad (2.2.11)$$

Implementation of the invariance criterion results in a polynomial comprising of derivatives of the dependent variable u . Since the coefficients of X in equation (2.2.11) do not consist of derivatives, we can separate with respect to the derivatives of u and its powers. This results in an overdetermined system of linear homogeneous partial differential equations known as the determining equations. These can then be solved for the coefficient functions ξ^i and η which will result in the acquisition of the Lie point symmetries of the partial differential equation (2.2.1).

2.2.2 Lie symmetry reduction methods and group invariant solutions

Lie symmetry reduction methods are of critical importance since they provide a robust way of solving higher-order non-linear partial differential equations. The foremost purpose of the obtained Lie point symmetries is to reduce the number of independent variables of the partial differential equation under study. This is done in anticipation that the partial differential equation may be reduced to an analytically solvable ordinary differential equation.

A noteworthy method that can be executed to achieve these objectives concerns seeking out a group invariant solution. A group invariant solution relating to a subgroup of the Lie symmetry group is an exact solution that remains unchanged under all possible transformations of said subgroup.

Definition 2.2.6 (Group invariant solution). $u = u(x)$ is a group invariant solution of the k th-order partial differential equation (2.2.1) with its corre-

sponding Lie symmetry generator (2.2.7) if and only if

$$X(u - u(x))|_{u-u(x)=0} = 0. \quad (2.2.12)$$

Otherwise stated,

$$\xi^i(x, u) \frac{\partial u}{\partial x^i} = \eta(x, u), \quad (2.2.13)$$

which is known as the invariant surface condition. This is then solved in order to achieve the group invariant solution and we do so by solving the corresponding system of characteristic curves given by the Lagrangian system

$$\frac{dx^1}{\xi^1(x, u)} = \dots = \frac{dx^n}{\xi^n(x, u)} = \frac{du}{\eta(x, u)}. \quad (2.2.14)$$

This means that we can use Lie theory to determine the infinitesimal transformations that leave the partial differential equation (2.2.1) invariant and its corresponding generator can then be used to construct the group invariant solutions by means of their characteristic equations.

Chapter 3

The unsteady MHD flow of a third grade fluid in a porous medium

3.1 Introduction

The purpose of this chapter is to present an in depth review of the work authored by Aziz et al. [6] wherein their objective was to find analytical solutions to mathematical models involving non-Newtonian fluids of the third grade. More specifically, they carried out an investigation with regards to the influence of an unsteady magnetic field on the flow of an incompressible third grade non-Newtonian fluid bound by an infinite flat rigid plate. We also present the calculation of the optimal system in order to classify the group invariant solutions. This approach has not been carried out by Aziz et al. for the specific fluid flow model under review.

The outline of this chapter is as follows: We begin in Section 3.2 by modelling the fluid flow problem under consideration through the equations of mass and momentum. Section 3.3 involves obtaining the Lie point symmetries of the governing equation by implementing the classical Lie point symmetry approach as set out in the preceding chapter. Thereafter, we will classify the group invariant solutions in Section 3.4 by means of finding the optimal system for the obtained Lie algebra. This optimal system will then be implemented in Section 3.5 to construct the corresponding group invariant solutions. In cases

where obtaining an exact solution proves impossible, a numerical solution will be found. A numerical solution to the governing equation will also be sought. A comparison will be made of the results acquired through the analytical and numerical approaches for the underlying model. In Section 3.6, a graphical investigation will be carried out on the obtained analytical and numerical solutions to inspect the influences of the relevant physical parameters on the velocity of the fluid flow coupled with a comprehensive analysis. Finally, we draw our conclusions in Section 3.7.

3.2 Mathematical formulation of the model

The primary aim of this section is to find a relevant mathematical model for the physical problem under consideration by making use of appropriate equations that describe the motion of a fluid. The flow models considered in this dissertation are contingent upon the laws of conservation of mass and momentum. Firstly, the continuity equation is a partial differential equation that describes conservation of mass and it is defined by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (3.2.1)$$

where $\partial/\partial t$ is partial differentiation with respect to time t , ρ the fluid density, $\mathbf{V}$ the velocity vector and ∇ known as nabla or del is the gradient vector operator defined as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}, \quad (3.2.2)$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors. Additionally, the dot $\cdot$ denotes the scalar product. Secondly, the momentum equation is a partial differential equation derived by Cauchy. It describes conservation of momentum and is given by

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{F}, \quad (3.2.3)$$

where d/dt is the material time derivative, $\mathbf{T}$ the Cauchy stress tensor and $\mathbf{F}$ the body force.

For this formulation, the Cartesian coordinate system xyz is used. Consider the unidirectional time-dependent flow of a third grade non-Newtonian

fluid that is bounded by a flat rigid plate. This plate is infinite in the xz -plane and coincides with the x -axis at $y = 0$. The fluid is incompressible and electrically conducting in the presence of a magnetic field and fills the porous half-space $y > 0$. The geometry of the model is depicted in Figure 3.2.1 below:

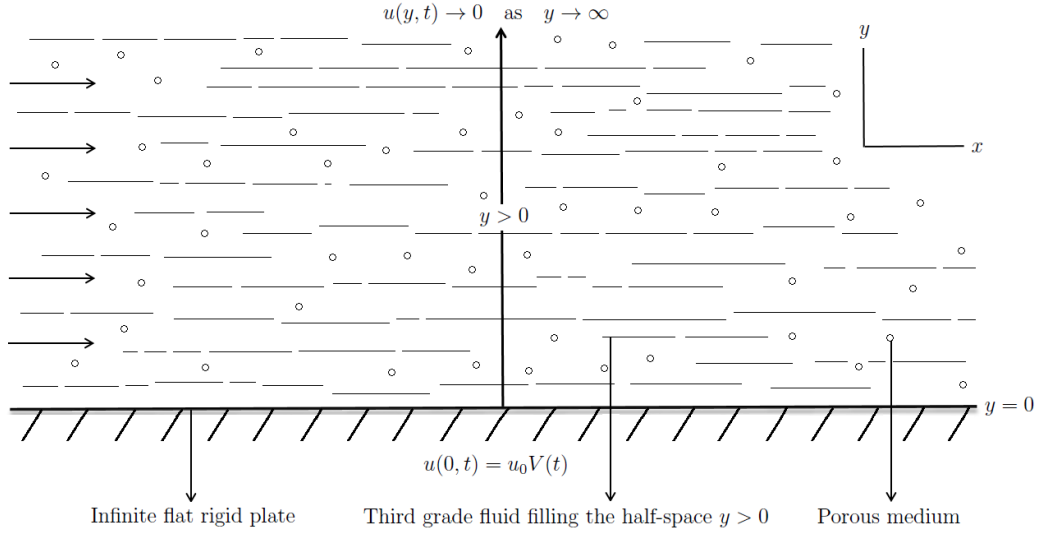


Figure 3.2.1: Graphical representation of the third grade non-Newtonian fluid flow model.

As such, the fluid flow problem is governed by the continuity and momentum equations respectively, defined as

$$\nabla \cdot \mathbf{V} = 0, \quad (3.2.4)$$

where $\mathbf{V} = (u, v, w)$ is the velocity vector and

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} + (\mathbf{J} \times \mathbf{B}) + \mathbf{R}, \quad (3.2.5)$$

where d/dt is the material time derivative defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla), \quad (3.2.6)$$

and the cross $\times$ denotes the vector product. Furthermore, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{J}$ the current density, $\mathbf{B}$ the magnetic induction and $\mathbf{R} = (R_x, R_y, R_z)$ Darcy's resistance due to the porous medium.

All the appropriate physical quantities except the pressure p are dependent

on y only since the plate is infinite in the xz -plane. As such, the velocity field $\mathbf{V}$ for the fluid flow problem under investigation should be of the form

$$\mathbf{V} = [u(y, t), 0, 0], \quad (3.2.7)$$

where $u(y, t)$ represents the velocity field in the x -direction. This velocity field identically satisfies the continuity equation (3.2.4). Otherwise stated, the incompressibility condition should be satisfied for incompressible fluid flow. Thus, the disturbance in the fluid u is a function of y and t only. In so doing, the fluid exhibits a unidirectional and a one-dimensional flow.

The flow of an electrically conducting fluid is described in terms of a magnetohydrodynamic flow and its presence is represented in the momentum equation (3.2.5) via the $(\mathbf{J} \times \mathbf{B})$ term. The current density $\mathbf{J}$ is defined by

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \quad (3.2.8)$$

where σ is the electrical conductivity, $\mathbf{E}$ the imposed electric field and the magnetic induction is defined such that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ where $\mathbf{B}_0$ is the applied magnetic field and $\mathbf{b}$ the induced magnetic field. For the fluid flow problem under examination, the fluid is electrically conducting due to the application of a uniform magnetic field in the transverse direction of the flow. Thus, the effects of the applied magnetic field $\mathbf{B}_0$ is applicable and considered to be the external body force and the effects of the imposed electric field $\mathbf{E}$ and the induced magnetic field $\mathbf{b}$ are negligible for small magnetic Reynolds number flow. In this instance, the current density reduces to $\mathbf{J} = \sigma(\mathbf{V} \times \mathbf{B}_0)$ and as such, the magnetic force becomes

$$\mathbf{J} \times \mathbf{B} = \sigma(\mathbf{V} \times \mathbf{B}_0) \times \mathbf{B}_0 = -\sigma B_0^2 \mathbf{V}. \quad (3.2.9)$$

Based on equation (3.2.7), the x -component of the applied magnetic field for this flow model becomes

$$(\mathbf{J} \times \mathbf{B})_x = -\sigma B_0^2 u. \quad (3.2.10)$$

The Cauchy stress tensor for a third grade fluid [4] is of the form

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \quad (3.2.11)$$

where p is fluid pressure, $\mathbf{I}$ the identity tensor, μ the dynamic fluid viscosity, α_i ($i = 1, 2$) and β_i ($i = 1, 2, 3$) the material constants, tr the trace which is the sum of the elements on the diagonal of a square matrix and $\mathbf{A}_i$ ($i = 1, 2$) the Rivlin-Ericksen tensors defined by the following equations:

$$\mathbf{A}_1 = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T, \quad (3.2.12)$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_{n-1}, \quad n > 1, \quad (3.2.13)$$

where T denotes the transpose of the resultant matrix. The Cauchy stress tensor (3.2.11) can be modified for an incompressible third grade fluid since it is thermodynamically compatible [13, 14, 15] if

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \quad (3.2.14)$$

and thus its constitutive relation has the form

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \quad (3.2.15)$$

The component form of the Cauchy stress tensor is a matrix given by

$$\mathbf{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}, \quad (3.2.16)$$

where each τ_{ij} represents a fluid flow in the i -direction and a disturbance of the fluid flow in the j -direction such that $\tau_{ij} = \tau_{ji}$. Subsequently, the component form of the momentum equation (3.2.5) is obtained by applying the velocity

field (3.2.7). This results in the set of governing equations given by

$$\rho \frac{du}{dt} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \sigma B_0^2 u + R_x, \quad (3.2.17)$$

$$0 = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + R_y, \quad (3.2.18)$$

$$0 = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + R_z, \quad (3.2.19)$$

where R_x , R_y and R_z represent the components of Darcy's resistance in the x , y and z directions respectively. Using equations (3.2.12) and (3.2.13), the following expressions are obtained for the Rivlin-Ericksen tensors $\mathbf{A}_i$ ($i = 1, 2$):

$$\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.2.20)$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & \frac{\partial^2 u}{\partial y \partial t} & 0 \\ \frac{\partial^2 u}{\partial y \partial t} & 2 \left(\frac{\partial u}{\partial y} \right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.2.21)$$

Each component of the Cauchy stress tensor τ_{ij} illustrated in equation (3.2.16) is then computed using equation (3.2.15) along with the Rivlin-Ericksen tensors $\mathbf{A}_i$ ($i = 1, 2$) in equations (3.2.20) and (3.2.21) above.

$$\begin{aligned} \tau_{xx} &= -p + \alpha_2 \left(\frac{\partial u}{\partial y} \right)^2, \\ \tau_{xy} &= \tau_{yx} = \mu \frac{\partial u}{\partial y} + \alpha_1 \left(\frac{\partial^2 u}{\partial y \partial t} \right) + 2\beta_3 \left(\frac{\partial u}{\partial y} \right)^3, \\ \tau_{xz} &= \tau_{zx} = 0, \\ \tau_{yy} &= -p + (2\alpha_1 + \alpha_2) \left(\frac{\partial u}{\partial y} \right)^2, \\ \tau_{yz} &= \tau_{zy} = 0, \\ \tau_{zz} &= -p. \end{aligned} \quad (3.2.22)$$

The only component of the Cauchy stress tensor with a physical significance

to the fluid flow problem under consideration is that of τ_{xy} since the fluid flow is in the x -direction and the disturbance of the fluid flow is in the y -direction. The relevant tensors are then imposed on the x -component of the momentum equation (3.2.17) which results in

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \left(\frac{\partial^3 u}{\partial y^2 \partial t} \right) + 6\beta_3 \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} - \sigma B_0^2 u + R_x. \quad (3.2.23)$$

The porous medium is represented in the momentum equation via the $\mathbf{R}$ term. The effects of the pores on the velocity field is incorporated by implementing Darcy's resistance in its generalized form. According to Darcy's law, we have

$$\mathbf{R} = \nabla p = -\frac{\phi}{\kappa}(\text{apparent viscosity})\mathbf{V}, \quad (3.2.24)$$

where ϕ is the porosity of the porous medium, κ the permeability of the porous medium and the apparent viscosity is defined as

$$\text{apparent viscosity} = \frac{\text{shear stress}}{\text{shear rate}}. \quad (3.2.25)$$

This law is consistent with the thermodynamics of the flow model and it specifies the relationship between the pressure drop and the velocity. It is a measure of the resistance of the flow within the bulk of the porous medium. It is evident that the apparent viscosity differs in relation to the non-Newtonian fluid flow model under consideration. The apparent viscosity of the unsteady unidirectional flow of a third grade fluid over a flat rigid plate is given by

$$\begin{aligned} \text{apparent viscosity} &= \frac{\tau_{xy}}{\frac{\partial u}{\partial y}}, \\ &= \mu + \alpha_1 \left(\frac{\partial}{\partial t} \right) + 2\beta_3 \left(\frac{\partial u}{\partial y} \right)^2. \end{aligned} \quad (3.2.26)$$

Substituting the apparent viscosity above into equation (3.2.24) yields the x -component of Darcy's resistance R_x :

$$\begin{aligned} R_x &= -\frac{\phi}{\kappa}(\text{apparent viscosity})u, \\ &= -\frac{\phi}{\kappa} \left[\mu u + \alpha_1 \left(\frac{\partial u}{\partial t} \right) + 2\beta_3 u \left(\frac{\partial u}{\partial y} \right)^2 \right]. \end{aligned} \quad (3.2.27)$$

It is clear that both the y and z -components $R_y = R_z = 0$ since $v = w = 0$.

In the absence of a modified pressure gradient, the following governing partial differential equation in u is obtained by implementing equation (3.2.27) into equation (3.2.23) to obtain

$$\begin{aligned} \rho \frac{\partial u}{\partial t} = & \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \left(\frac{\partial^3 u}{\partial y^2 \partial t} \right) + 6\beta_3 \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \\ & - \frac{\phi}{\kappa} \left[\mu u + \alpha_1 \left(\frac{\partial u}{\partial t} \right) + 2\beta_3 u \left(\frac{\partial u}{\partial y} \right)^2 \right] - \sigma B_0^2 u. \end{aligned} \quad (3.2.28)$$

In order to solve the above fifth-order non-linear partial differential equation, the relevant boundary and initial conditions need to be imposed. These are given by:

Boundary conditions:

$$u(0, t) = u_0 V(t), \quad t > 0, \quad (3.2.29)$$

$$u(\infty, t) = 0, \quad t > 0, \quad (3.2.30)$$

Initial condition:

$$u(y, 0) = g(y), \quad y > 0, \quad (3.2.31)$$

where u_0 is the reference velocity and $V(t)$ and $g(y)$ are functions that are yet to be determined. The first boundary condition (3.2.29) represents the no-slip condition while the second boundary condition (3.2.30) points to the fact that the mainstream velocity is zero. The initial condition (3.2.31) shows that at initiation, the fluid is flowing with some non-uniform velocity $g(y)$.

The dimensionless form of the governing model is obtained by defining the following transformations:

$$\bar{u} = \frac{u}{u_0}, \quad \bar{y} = \frac{u_0 y}{\nu}, \quad \bar{t} = \frac{u_0^2 t}{\nu}, \quad (3.2.32)$$

where ν is the kinematic viscosity defined by μ/ρ . Substituting these trans-

formations into equation (4.2.28) yields

$$\begin{aligned} \frac{\partial \bar{u}}{\partial \bar{t}} &= \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{\alpha}_1 \frac{\partial^3 \bar{u}}{\partial \bar{y}^2 \partial \bar{t}} + 3\bar{\beta}_3 \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \left[\bar{u} + \bar{\alpha}_1 \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{\beta}_3 \bar{u} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right] \bar{\phi} \\ &\quad - \bar{M}^2 \bar{u}, \end{aligned} \quad (3.2.33)$$

with

$$\begin{aligned} \bar{\alpha}_1 &= \frac{\alpha_1 u_0^2}{\rho \nu^2}, & \bar{\beta}_3 &= \frac{2\beta_3 u_0^4}{\rho \nu^3}, \\ \bar{\phi} &= \frac{\phi \nu^2}{\kappa u_0^2}, & \bar{M}^2 &= \frac{\sigma B_0^2 \nu}{\rho u_0^2}. \end{aligned} \quad (3.2.34)$$

Furthermore, the dimensionless boundary and initial conditions are given by

Boundary conditions:

$$\bar{u}(0, \bar{t}) = V(\bar{t}), \quad \bar{t} > 0, \quad (3.2.35)$$

$$\bar{u}(\infty, \bar{t}) = 0, \quad \bar{t} > 0, \quad (3.2.36)$$

Initial condition:

$$\bar{u}(\bar{y}, 0) = G(\bar{y}), \quad \bar{y} > 0, \quad (3.2.37)$$

where

$$G(\bar{y}) = \frac{g(\bar{y})}{u_0}. \quad (3.2.38)$$

It is worthwhile to note that the functions $V(\bar{t})$ and $G(\bar{y})$ are arbitrary. For simplicity of notation, the overhead bars from all the dimensionless quantities will be neglected in the subsequent analysis.

To end, we rewrite equation (3.2.33) in its compact form to obtain the final form of the governing equation in u which is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} - \beta_* u \left(\frac{\partial u}{\partial y} \right)^2 \\ &\quad - (\phi_* + M_*^2) u, \end{aligned} \quad (3.2.39)$$

subject to the boundary and initial conditions

$$u(0, t) = V(t), \quad t > 0, \quad (3.2.40)$$

$$u(\infty, t) = 0, \quad t > 0, \quad (3.2.41)$$

$$u(y, 0) = G(y), \quad y > 0, \quad (3.2.42)$$

where

$$\begin{aligned} \mu_* &= \frac{1}{1 + \alpha_1 \phi}, & \alpha_* &= \frac{\alpha_1}{1 + \alpha_1 \phi}, \\ \beta &= \frac{3\beta_3}{1 + \alpha_1 \phi}, & \beta_* &= \frac{\beta_3 \phi}{1 + \alpha_1 \phi}, \\ \phi_* &= \frac{\phi}{1 + \alpha_1 \phi}, & M_*^2 &= \frac{M^2}{1 + \alpha_1 \phi}. \end{aligned} \quad (3.2.43)$$

The parameters above manage to encompass the physics of the problem. In particular, μ_* is the viscosity parameter, α_* the second grade parameter, β and β_* the third grade parameters, ϕ_* the porosity parameter and M_*^2 the magnetic field parameter.

3.3 Lie point symmetry analysis

This section deals with the implementation of the classical Lie theory to the governing partial differential equation (3.2.39) as set out in Section 2.2. The main objective of the classical Lie symmetry approach is to obtain a transformation which leaves the governing equation invariant. These transformations or symmetries can then be utilised to obtain their corresponding group invariant solutions. It is found that two cases arise and as a result, the model equation admits two sets of Lie algebras.

We begin by pursuing transformations of the dependent variable u and the independent variables t and y that have the form

$$\begin{aligned} \bar{t} &= \bar{t}(t, y, u, \epsilon), \\ \bar{y} &= \bar{y}(t, y, u, \epsilon), \\ \bar{u} &= \bar{u}(t, y, u, \epsilon). \end{aligned} \quad (3.3.1)$$

This set of transformations characterize a one-parameter group with group parameter ϵ . What is more is that they are chosen such that the governing equation (3.2.39) remains invariant. The infinitesimal transformations can then be achieved by considering the Taylor series expansions of the Lie group of transformations (3.3.1) and these can be written as

$$\begin{aligned}\bar{t} &\simeq t + \epsilon \xi^1(t, y, u), \\ \bar{y} &\simeq y + \epsilon \xi^2(t, y, u), \\ \bar{u} &\simeq u + \epsilon \eta(t, y, u),\end{aligned}\tag{3.3.2}$$

with the operator

$$X = \xi^1(t, y, u) \frac{\partial}{\partial t} + \xi^2(t, y, u) \frac{\partial}{\partial y} + \eta(t, y, u) \frac{\partial}{\partial u}.\tag{3.3.3}$$

This operator is a Lie point symmetry generator of the governing equation (3.2.39) if and only if the invariance criterion holds:

$$X^{[3]}(\text{equation (3.2.39)}) \Big|_{\text{equation (3.2.39)}=0} = 0.\tag{3.3.4}$$

We act on equation (3.2.39) with the third prolongation $X^{[3]}$ of the operator X since the equation under examination is of the third order and this is given by

$$X^{[3]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} + \zeta^t \frac{\partial}{\partial u_t} + \zeta^y \frac{\partial}{\partial u_y} + \zeta^{yy} \frac{\partial}{\partial u_{yy}} + \zeta^{tyy} \frac{\partial}{\partial u_{tyy}},\tag{3.3.5}$$

where the coefficient functions of the extended infinitesimals ζ are explicitly given by

$$\zeta^t = D_t \eta - u_t D_t (\xi^1) - u_y D_t (\xi^2),$$

$$\zeta^y = D_y \eta - u_t D_y (\xi^1) - u_y D_y (\xi^2),$$

$$\zeta^{yy} = D_y \zeta^y - u_{ty} D_y (\xi^1) - u_{yy} D_y (\xi^2),$$

$$\zeta^{t yy} = D_t \zeta^{yy} - u_{t yy} D_t (\xi^1) - u_{y yy} D_t (\xi^2). \quad (3.3.6)$$

The total derivative operators are defined as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{ty} \frac{\partial}{\partial u_y} + \dots, \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \dots \end{aligned} \quad (3.3.7)$$

The employment of the third prolongation $X^{[3]}$ onto equation (3.2.39) yields the determining equation

$$\begin{aligned} & \left(\zeta^t - \mu_* \zeta^{yy} - \alpha_* \zeta^{t yy} - \beta (u_y)^2 \zeta^{yy} - 2\beta u_y u_{yy} \zeta^y + \beta_* (u_y)^2 \eta \right. \\ & \left. + 2\beta_* u u_y \zeta^y + (\phi_* + M_*^2) \eta \right) \Big|_{\text{equation (3.2.39)}=0} = 0. \end{aligned} \quad (3.3.8)$$

The application of the operators in equation (3.3.7) onto the extended infinitesimals ζ defined in equation (3.3.6) leads to

$$\zeta^t = \eta_t + (\eta_u - \xi_t^1) u_t - \xi_t^2 u_y - \xi_u^1 (u_t)^2 - \xi_u^2 u_t u_y, \quad (3.3.9)$$

$$\zeta^y = \eta_y - \xi_y^1 u_t + (\eta_u - \xi_y^2) u_y - \xi_u^2 (u_y)^2 - \xi_u^1 u_t u_y, \quad (3.3.10)$$

$$\begin{aligned} \zeta^{yy} &= \eta_{yy} - \xi_{yy}^1 u_t + (2\eta_{yu} - \xi_{yy}^2) u_y + (\eta_{uu} - \xi_{yu}^2) (u_y)^2 - 2\xi_{yu}^1 u_t u_y \\ &\quad - \xi_{uu}^2 (u_y)^3 - \xi_{uu}^1 u_t (u_y)^2 + (\eta_u - 2\xi_y^2) u_{yy} - 2\xi_y^1 u_{ty} - \xi_u^1 u_t u_{yy} \\ &\quad - 3\xi_u^2 u_y u_{yy} - 2\xi_u^1 u_y u_{ty}, \end{aligned} \quad (3.3.11)$$

$$\begin{aligned} \zeta^{t yy} &= \eta_{t yy} + (\eta_{y y u} - \xi_{t yy}^1) u_t + (2\eta_{t y u} - \xi_{t yy}^2) u_y - \xi_{y y u}^1 (u_t)^2 \\ &\quad + (\eta_{t u u} - \xi_{t y u}^2) (u_y)^2 + (2\eta_{y u u} - 2\xi_{t y u}^1 - \xi_{y y u}^2) u_t u_y - \xi_{t u u}^2 (u_y)^3 \\ &\quad + (\eta_{u u u} - \xi_{t u u}^1 - \xi_{y u u}^2) u_t (u_y)^2 - \xi_{u u u}^2 u_t (u_y)^3 - 2\xi_{y u u}^1 (u_t)^2 u_y \\ &\quad - \xi_{u u}^1 (u_t)^2 (u_y)^2 - \xi_{y y}^1 u_{tt} + (2\eta_{y u} + (\eta_{t u} - 2\xi_{t y}^2) u_{yy} - 2\xi_{t y}^1 - \xi_{y y}^2) u_{ty} \\ &\quad - 2\xi_u^1 (u_{ty})^2 + (\eta_{u u} - 2\xi_{t y u}^1 - 2\xi_{y u}^2) u_t u_{yy} - 4\xi_{y u}^1 u_t u_{ty} - 3\xi_{u u}^2 u_t u_y u_{yy} \\ &\quad - \xi_{u u}^1 (u_t)^2 u_{yy} - 4\xi_{u u}^1 u_t u_y u_{ty} - 2\xi_{y u}^1 u_y u_{tt} - 3\xi_{t u}^2 u_y u_{yy} \\ &\quad + (2\eta_{u u} - 2\xi_{t u}^1 - 2\xi_{y u}^2) u_y u_{ty} - \xi_{u u}^1 (u_y)^2 u_{tt} - 3\xi_{u u}^2 (u_y)^2 u_{ty} \\ &\quad - \xi_u^1 u_{tt} u_{yy} - 3\xi_u^2 u_{yy} u_{ty} - 2\xi_y^1 u_{t ty} + (\eta_u - \xi_t^1 - 2\xi_y^2) u_{t yy} - \xi_t^2 u_{y yy} \\ &\quad - 2\xi_u^1 u_t u_{t yy} - \xi_u^2 u_t u_{y yy} - 2\xi_u^1 u_y u_{t ty} - 3\xi_u^2 u_y u_{t yy}. \end{aligned} \quad (3.3.12)$$

We then substitute the expansions for the coefficient functions obtained in

equations (3.3.9)-(3.3.12) into the determining equation (3.3.8). The unknown functions ξ^1 , ξ^2 and η are independent of the derivatives of u as well as powers and products of their derivatives. Hence, we can separate the resulting determining equation with respect to the powers and products of the derivatives of u . Applying this process and making further simplifications leads to an overdetermined system of linear homogeneous partial differential equations given by

$$u_{tty} : \xi_y^1 = 0, \quad (3.3.13)$$

$$u_{tt}u_{yy} : \xi_u^1 = 0, \quad (3.3.14)$$

$$u_{yyy} : \xi_t^2 = 0, \quad (3.3.15)$$

$$u_{tyy} : \xi_y^2 = 0, \quad (3.3.16)$$

$$u_{yy}u_{yyy} : \xi_u^2 = 0, \quad (3.3.17)$$

$$u_yu_{yy} : \eta_y = 0, \quad (3.3.18)$$

$$u_yu_{ty} : \eta_{uu} = 0, \quad (3.3.19)$$

$$u_{yy} : \mu_*\xi_t^1 + \alpha_*\eta_{tu} = 0, \quad (3.3.20)$$

$$1 : (\phi_* + M_*^2) \eta + (\phi_* + M_*^2) u\xi_t^1 + \eta_t - (\phi_* + M_*^2) u\eta_u = 0, \quad (3.3.21)$$

$$(u_y)^2 : \eta + u\xi_t^1 + u\eta_u = 0, \quad (3.3.22)$$

$$(u_y)^2 u_{yy} : \xi_t^1 + 2\eta_u = 0, \quad (3.3.23)$$

with the remainder of the equations being satisfied. From equations (3.3.13) and (3.3.14),

$$\xi^1 = a(t), \quad (3.3.24)$$

where a is some arbitrary function of t . From equations (3.3.15), (3.3.16) and (3.3.17),

$$\xi^2 = c_1, \quad (3.3.25)$$

where c_1 is an arbitrary constant. From equations (3.3.18) and (3.3.19),

$$\eta = c(t)u + d(t), \quad (3.3.26)$$

where c and d are arbitrary functions of t . By using equations (3.3.24) and

(3.3.26), we can rewrite the determining equations (3.3.20)-(3.3.23) as

$$\mu_* a'(t) + \alpha_* c'(t) = 0, \quad (3.3.27)$$

$$(\phi_* + M_*^2) d(t) + (\phi_* + M_*^2) a'(t)u + c'(t)u + d'(t) = 0, \quad (3.3.28)$$

$$2c(t)u + d(t) + a'(t)u = 0, \quad (3.3.29)$$

$$a'(t) + 2c(t) = 0. \quad (3.3.30)$$

Here, ' represents differentiation with respect to t . Since the functions c and d are dependent on t only, we can split equation (3.3.28) with respect to powers of u to get

$$u^1 : (\phi_* + M_*^2) a'(t) + c'(t) = 0, \quad (3.3.31)$$

$$u^0 : (\phi_* + M_*^2) d(t) + d'(t) = 0. \quad (3.3.32)$$

From equation (3.3.31),

$$a'(t) = -\frac{c'(t)}{(\phi_* + M_*^2)}. \quad (3.3.33)$$

Using equation (3.3.33) in equation (3.3.27) gives that

$$-\frac{\mu_* c'(t)}{(\phi_* + M_*^2)} + \alpha_* c'(t) = 0. \quad (3.3.34)$$

We can rearrange equation (3.3.34) so that it is of the form

$$c'(t) \left[(\phi_* + M_*^2) - \frac{\mu_*}{\alpha_*} \right] = 0. \quad (3.3.35)$$

For equation (3.3.35) to hold, $c'(t) = 0$ or $(\phi_* + M_*^2) - \mu_*/\alpha_* = 0$ and thus two cases arise for the solution to the above system of determining equations:

Case 1: $(\phi_* + M_*^2) \neq \frac{\mu_*}{\alpha_*}, \quad c'(t) = 0$

If $c'(t) = 0$, then equation (3.3.27) reduces to

$$a'(t) = 0, \quad (3.3.36)$$

and thus by integrating both sides with respect to t , we have that

$$a(t) = c_2, \quad (3.3.37)$$

where c_2 is a constant. Based on equation (3.3.36), the infinitesimal ξ^1 in equation (3.3.24) becomes

$$\xi^1 = c_2, \quad (3.3.38)$$

and we can simplify equation (3.3.29) to be

$$2c(t)u + d(t) = 0. \quad (3.3.39)$$

Again, we can split with respect to powers of u to obtain

$$u^1 : c(t) = 0, \quad (3.3.40)$$

$$u^0 : d(t) = 0, \quad (3.3.41)$$

and hence, based on the above, the infinitesimals for the partial differential equation (3.2.39) are

$$\begin{aligned} \xi^1 &= c_2, \\ \xi^2 &= c_1, \\ \eta &= 0. \end{aligned} \quad (3.3.42)$$

The preceding coefficients of the infinitesimals ξ^1 , ξ^2 and η found by solving the determining equations are substituted into the infinitesimal generator defined in equation (3.3.3) to obtain

$$X = c_2 \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial y}. \quad (3.3.43)$$

By setting each constant $c_i = 1$ with $c_j = 0$ ($i, j = 1, 2$) in the infinitesimal generator in equation (3.3.43), a two-dimensional Lie algebra is obtained, generated by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \end{aligned} \quad (3.3.44)$$

where X_1 is the time-translational symmetry generator and X_2 the space-translational symmetry generator in y .

Case 2: $(\phi_* + M_*^2) = \frac{\mu_*}{\alpha_*}$, $c'(t) \neq 0$

Now, we can split equation (3.3.29) with respect to powers of u to get

$$u^1 : 2c(t) + a'(t) = 0, \quad (3.3.45)$$

$$u^0 : d(t) = 0, \quad (3.3.46)$$

and from equation (3.3.46), the infinitesimal η in equation (3.3.26) reduces to

$$\eta = c(t)u. \quad (3.3.47)$$

By equation (3.3.45),

$$a'(t) = -2c(t). \quad (3.3.48)$$

Implementing this in equation (3.3.27) yields that

$$-2\mu_*c(t) + \alpha_*c'(t) = 0, \quad (3.3.49)$$

and rearranging gives the variable separable ordinary differential equation

$$\frac{dc(t)}{c(t)} = 2 \left(\frac{\mu_*}{\alpha_*} \right) t. \quad (3.3.50)$$

Integrating equation (3.3.50) with respect to t leads to

$$c(t) = c_2 e^{2\left(\frac{\mu_*}{\alpha_*}\right)t}, \quad (3.3.51)$$

where c_2 is a constant of integration. We now implement the imposed condition on the second case, viz.

$$(\phi_* + M_*^2) = \frac{\mu_*}{\alpha_*}, \quad (3.3.52)$$

and so equation (3.3.51) becomes

$$c(t) = c_2 e^{2(\phi_* + M_*^2)t}. \quad (3.3.53)$$

Hence, the infinitesimal η becomes

$$\eta = c_2 e^{2(\phi_* + M_*^2)t} u. \quad (3.3.54)$$

Furthermore, equation (3.3.48) becomes

$$a'(t) = -2c_2 e^{2\left(\frac{\mu_*}{\alpha_*}\right)t}, \quad (3.3.55)$$

which leads to a variable separable ordinary differential equation in $a(t)$ given by

$$da(t) = -2c_2 e^{2\left(\frac{\mu_*}{\alpha_*}\right)t} dt, \quad (3.3.56)$$

which can be integrated with respect to t to obtain

$$a(t) = \frac{-c_2 e^{2\left(\frac{\mu_*}{\alpha_*}\right)t}}{\left(\frac{\mu_*}{\alpha_*}\right)} + c_3, \quad (3.3.57)$$

where c_3 is a constant of integration. Again the condition (3.3.52) leads equation (3.3.57) to become

$$a(t) = -\left(\frac{c_2}{\phi_* + M_*^2}\right) e^{2(\phi_* + M_*^2)t} + c_3, \quad (3.3.58)$$

By equations (3.3.24), (3.3.25) and (3.3.54), the infinitesimals for the partial differential equation (3.2.39) are

$$\begin{aligned} \xi^1 &= -\left(\frac{c_2}{\phi_* + M_*^2}\right) e^{2(\phi_* + M_*^2)t} + c_3, \\ \xi^2 &= c_1, \\ \eta &= c_2 e^{2(\phi_* + M_*^2)t} u. \end{aligned} \quad (3.3.59)$$

The coefficients of the infinitesimals ξ^1 , ξ^2 and η above are substituted into the Lie point symmetry generator defined in equation (3.3.3) to obtain

$$X = \left[-\left(\frac{c_2}{\phi_* + M_*^2}\right) e^{2(\phi_* + M_*^2)t} + c_3 \right] \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial y} + c_2 e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u}. \quad (3.3.60)$$

By setting each constant $c_i = 1$ with $c_j = 0$ ($i, j = 1, 2, 3$) in the infinitesimal generator in equation (3.3.60), we obtain a three-dimensional Lie algebra which

is generated by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= -\left(\frac{1}{\phi_* + M_*^2}\right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u}, \end{aligned} \quad (3.3.61)$$

where X_1 is the time-translational symmetry generator, X_2 the space-translational symmetry generator in y and the generator X_3 has path curves equivalent to a combination of translations in t and scaling in u .

3.4 Classification of group invariant solutions

In this section, we focus on the application of a method known as the optimal system. This technique helps us in classifying group invariant solutions of differential equations and it was first considered by Ovsiannikov [16]. We will follow the approach as set out by Olver in [12].

In theory, a family of group invariant solutions that correspond to a subgroup of the symmetries admitted by a differential equation can be constructed [17]. However, the thought of listing all the group invariant solutions seems impractical since the possible number of those subgroups is infinite in most cases. In view of that, we seek to obtain optimal systems of subalgebras of the principle Lie algebra which provide a useful and methodical way of classifying these group invariant solutions.

We begin by computing the commutators of the three-dimensional Lie algebra spanned by the base vectors X_1 , X_2 and X_3 specified in equation (3.3.61). These are obtained by applying the commutation relation

$$[X_i, X_j] = X_i X_j - X_j X_i, \quad i, j = 1, \dots, n, \quad (3.4.1)$$

where

$$[X_i, X_j] = -[X_j, X_i], \quad (3.4.2)$$

and

$$[X_i, X_i] = 0. \quad (3.4.3)$$

This results in

$$\begin{aligned}
[X_1, X_2] &= X_1X_2 - X_2X_1 \\
&= \frac{\partial}{\partial t} \left(\frac{\partial}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial}{\partial t} \right) \\
&= 0,
\end{aligned} \tag{3.4.4}$$

and

$$\begin{aligned}
[X_2, X_3] &= X_2X_3 - X_3X_2 \\
&= \frac{\partial}{\partial y} \left\{ - \left(\frac{1}{\phi_* + M_*^2} \right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u} \right\} \\
&\quad - \left\{ - \left(\frac{1}{\phi_* + M_*^2} \right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u} \right\} \left(\frac{\partial}{\partial y} \right) \\
&= 0,
\end{aligned} \tag{3.4.5}$$

and

$$\begin{aligned}
[X_1, X_3] &= X_1X_3 - X_3X_1 \\
&= \frac{\partial}{\partial t} \left\{ - \left(\frac{1}{\phi_* + M_*^2} \right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u} \right\} \\
&\quad - \left\{ - \left(\frac{1}{\phi_* + M_*^2} \right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u} \right\} \left(\frac{\partial}{\partial t} \right) \\
&= 2(\phi_* + M_*^2) \left\{ - \left(\frac{1}{\phi_* + M_*^2} \right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u} \right\} \\
&= 2(\phi_* + M_*^2) X_3.
\end{aligned} \tag{3.4.6}$$

For convenience, we will display the commutators of the Lie algebra in the form of a commutator table where the (i, j) th entry is $[X_i, X_j]$. This is shown in Table 3.4.1 below:

Next, we calculate the adjoint representation using the commutator table shown in Table 3.4.1 as well as the Lie series [12] given by

$$\begin{aligned}
\text{Ad}(\exp(\epsilon X_i))X_j &= X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_i, [X_i, [X_i, X_j]]] + \dots, \quad i, j = 1, \dots, n,
\end{aligned} \tag{3.4.7}$$

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$2(\phi_* + M_*^2)X_3$
X_2	0	0	0
X_3	$-2(\phi_* + M_*^2)X_3$	0	0

Table 3.4.1: Commutator table of the three-dimensional Lie algebra (3.3.61)

where

$$\text{Ad}(\exp(\epsilon X_i))X_i = X_i. \quad (3.4.8)$$

As examples,

$$\begin{aligned}
\text{Ad}(\exp(\epsilon X_1))X_3 &= X_3 - \epsilon[X_1, X_3] + \frac{1}{2!}\epsilon^2[X_1, [X_1, X_3]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_1, [X_1, [X_1, X_3]]] + \dots \\
&= X_3 - 2\epsilon(\phi_* + M_*^2)X_3 + \frac{1}{2!}(2\epsilon(\phi_* + M_*^2))^2X_3 \\
&\quad - \frac{1}{3!}(2\epsilon(\phi_* + M_*^2))^2X_3 + \dots \\
&= \exp(-2\epsilon(\phi_* + M_*^2))X_3,
\end{aligned} \quad (3.4.9)$$

and

$$\begin{aligned}
\text{Ad}(\exp(\epsilon X_3))X_1 &= X_1 - \epsilon[X_3, X_1] + \frac{1}{2!}\epsilon^2[X_3, [X_3, X_1]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_3, [X_3, [X_3, X_1]]] + \dots \\
&= X_1 + 2\epsilon(\phi_* + M_*^2)X_3.
\end{aligned} \quad (3.4.10)$$

The remainder of the adjoint representations are found in a similar fashion. We construct an adjoint table for the symmetry generators (3.3.61) where the (i, j) th entry indicates $\text{Ad}(\exp(\epsilon X_i))X_j$. As a result, we have the adjoint representation given by Table 3.4.2 below:

This is followed by considering the non-zero vector

$$X = a_1X_1 + a_2X_2 + a_3X_3, \quad a_i \in \mathbb{R}. \quad (3.4.11)$$

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$e^{-2\epsilon(\phi_* + M_*^2)} X_3$
X_2	X_1	X_2	X_3
X_3	$X_1 + 2\epsilon(\phi_* + M_*^2) X_3$	X_2	X_3

Table 3.4.2: Adjoint table for the symmetry generators (3.3.61)

We are tasked with simplifying the coefficients a_i as much as we can through a number of well-judged applications of adjoint maps to X . With reference to Table 3.4.2, we can act on X by $\text{Ad}(\exp(\epsilon X_3))$ which returns

$$X = a_1 X_1 + a_2 X_2 + (2\epsilon(\phi_* + M_*^2) a_1 + a_3) X_3. \quad (3.4.12)$$

To start with, we consider the scenario when $a_1 \neq 0$. We can choose $a_1 = 1$ for the purposes of scaling and so X is now

$$X = X_1 + a_2 X_2 + (2\epsilon(\phi_* + M_*^2) + a_3) X_3. \quad (3.4.13)$$

Further, we can make the coefficient a_3 disappear for the distinct value of $\epsilon = -a_3/2(\phi_* + M_*^2)$. Accordingly,

$$X = X_1 + a_2 X_2. \quad (3.4.14)$$

On the other hand, choosing $a_1 = 0$ in the vector (3.4.11) gives that

$$X = a_2 X_2 + a_3 X_3. \quad (3.4.15)$$

We now consider all possible situations for the coefficient a_2 . We can first set $a_2 \neq 0$ and again, by means of scaling $a_2 = 1$, the vector X directly above becomes

$$X = X_2 + a_3 X_3. \quad (3.4.16)$$

Using Table 3.4.2, we can now act on this vector X by $\text{Ad}(\exp(\epsilon X_1))$ to get that

$$X = X_2 + a_3 e^{-2\epsilon(\phi_* + M_*^2)} X_3. \quad (3.4.17)$$

Specifying that $a_3 \neq 0$ gives rise to two cases: $a_3 > 0$ or $a_3 < 0$ and as such,

$$X = X_2 \pm X_3. \quad (3.4.18)$$

However, if we had chosen $a_3 = 0$, then the vector X in equation (3.4.15) would be

$$X = X_2. \quad (3.4.19)$$

The last choice is $a_2 = 0$ and this results in

$$X = X_3. \quad (3.4.20)$$

All things considered, an optimal system of one-dimensional subalgebras of the Lie algebra (3.3.61) is given by those generated by

$$\begin{aligned} &X_2, \\ &X_3, \\ &X_2 \pm X_3, \\ &X_1 + aX_2. \end{aligned} \quad (3.4.21)$$

3.5 Group invariant and numerical solutions to the problem

This section makes use of the optimal system (3.4.21) obtained in the preceding section to obtain varying types of group invariant solutions for the model equation (3.2.39). Some of the closed-form solutions are contained hereunder. Also, a numerical solution to the governing model is set up along with appropriate boundary and initial conditions.

3.5.1 Invariant solution via subgroup generated by X_3

We begin by finding group invariant solutions corresponding to the operator X_3 which is given by

$$X_3 = - \left(\frac{1}{\phi_* + M_*^2} \right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + e^{2(\phi_* + M_*^2)t} u \frac{\partial}{\partial u}. \quad (3.5.1)$$

The Lagrangian system provides the corresponding differential equations of characteristic curves which are given by

$$\frac{dt}{-\left(\frac{1}{\phi_* + M_*^2}\right)} = \frac{dy}{0} = \frac{du}{u}. \quad (3.5.2)$$

We see that y is invariant and so we can write that

$$y = k_1, \quad (3.5.3)$$

where k_1 is constant.

Note: The term $dy/0$ in equation (3.5.2) does not represent division by 0. Instead, it is written purely to note that the variable y is invariant for the characteristic system under consideration.

Now, the first and last terms of the characteristic system in equation (3.5.2) can be written as a variable separable ordinary differential equation given by

$$\frac{du}{u} = -(\phi_* + M_*^2) dt. \quad (3.5.4)$$

Integrating both sides gives

$$\begin{aligned} \ln u &= -(\phi_* + M_*^2) t + w(y), \\ u(y, t) &= w(y) \exp[-(\phi_* + M_*^2) t], \end{aligned} \quad (3.5.5)$$

which is the form of the invariant solution where $w(y)$ is an as yet to be determined function of y .

We now substitute equation (3.5.5) into the governing third-order non-linear partial differential equation (3.2.33) to obtain the ordinary differential equation for $w(y)$:

$$\begin{aligned} & -(\phi_* + M_*^2) \exp[-(\phi_* + M_*^2) t] w = \mu_* \exp[-(\phi_* + M_*^2) t] \frac{d^2 w}{dy^2} \\ & -\alpha_* (\phi_* + M_*^2) \exp[-(\phi_* + M_*^2) t] \frac{d^2 w}{dy^2} + \beta \exp[-3(\phi_* + M_*^2) t] \left(\frac{dw}{dy}\right)^2 \frac{d^2 w}{dy^2} \\ & -\beta_* \exp[-3(\phi_* + M_*^2) t] \left(\frac{dw}{dy}\right)^2 w - (\phi_* + M_*^2) \exp[-(\phi_* + M_*^2) t] w. \end{aligned} \quad (3.5.6)$$

Again, we can implement the condition in equation (3.3.52) to obtain

$$\begin{aligned} & \mu_* \exp [-(\phi_* + M_*^2) t] \frac{d^2 w}{dy^2} - \alpha_* \left(\frac{\mu_*}{\alpha_*} \right) \exp [-(\phi_* + M_*^2) t] \frac{d^2 w}{dy^2} \\ & + \beta \exp [-3(\phi_* + M_*^2) t] \left(\frac{dw}{dy} \right)^2 \frac{d^2 w}{dy^2} - \beta_* \exp [-3(\phi_* + M_*^2) t] \left(\frac{dw}{dy} \right)^2 w = 0, \end{aligned} \quad (3.5.7)$$

and further simplification results in

$$\exp [-3(\phi_* + M_*^2) t] \left(\frac{dw}{dy} \right)^2 \left[\beta \frac{d^2 w}{dy^2} - \beta_* w \right] = 0. \quad (3.5.8)$$

We note that the above equation holds true only if

$$\begin{aligned} \beta \frac{d^2 w}{dy^2} - \beta_* w &= 0, \\ \frac{d^2 w}{dy^2} - \frac{\beta_*}{\beta} w &= 0, \quad \beta \neq 0. \end{aligned} \quad (3.5.9)$$

It is imperative to note that β_* is a positive real number and it is not possible for it to be less than zero since we would obtain complex solutions which are not physically meaningful. Furthermore, we can use equation (3.5.5) to rewrite the boundary conditions in equations (3.2.40) and (3.2.41) for the second-order ordinary differential equation (3.5.9) as

$$w(0) = 1, \quad (3.5.10)$$

$$w(y) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \quad (3.5.11)$$

where

$$V(t) = \exp [-(\phi_* + M_*^2) t]. \quad (3.5.12)$$

The above boundary value problem is then solved in the instance when β_*/β is positive. Hence, the solution for $w(y)$ is written in the form

$$w(y) = a \exp \left(\sqrt{\frac{\beta_*}{\beta}} y \right) + b \exp \left(-\sqrt{\frac{\beta_*}{\beta}} y \right). \quad (3.5.13)$$

By applying the first boundary condition (3.5.10), we have that $a + b = 1$. The second boundary condition (3.5.11) requires $a = 0$ for a bounded solution and

so it follows that $b = 1$. Thus, the solution for $w(y)$ is

$$w(y) = \exp \left(-\sqrt{\frac{\beta_*}{\beta}} y \right), \quad (3.5.14)$$

and we obtain a solution for $u(y, t)$ of the form

$$\begin{aligned} u(y, t) &= \exp \left(-\sqrt{\frac{\beta_*}{\beta}} y \right) \exp [- (\phi_* + M_*^2) t], \\ u(y, t) &= \exp \left[- \left\{ (\phi_* + M_*^2) t + \sqrt{\frac{\beta_*}{\beta}} y \right\} \right]. \end{aligned} \quad (3.5.15)$$

Furthermore, we note that the solution of $u(y, t)$ in equation (3.5.15) above satisfies the boundary condition (3.2.40) and initial condition (3.2.42) with

$$\begin{aligned} V(t) &= \exp [- (\phi_* + M_*^2) t], \\ G(y) &= \left(-\sqrt{\frac{\beta_*}{\beta}} y \right), \end{aligned} \quad (3.5.16)$$

where the functions $V(t)$ and $G(y)$ depend on physical parameters of the fluid flow.

The closed-form solution in equation (3.5.15) is plotted in Figures 3.6.1-3.6.4 for varying values of the emerging parameters relevant to the fluid flow problem.

3.5.2 Steady state solution

Here, we consider the invariant solutions under the time-translational symmetry generator

$$X_1 = \frac{\partial}{\partial t}, \quad (3.5.17)$$

which has a characteristic system given by

$$\frac{dt}{1} = \frac{dy}{0} = \frac{du}{0}. \quad (3.5.18)$$

We see that both y and u are invariant and as a result

$$y = k_1 \quad \text{and} \quad u = k_2, \quad (3.5.19)$$

where k_1 and k_2 are constants. Writing $k_2 = P(k_1)$, we find that the invariant solution admitted by the generator X_1 is the steady-state solution

$$u(y, t) = P(y). \quad (3.5.20)$$

Implementing the invariant solution in equation (3.5.20) into the governing equation (3.2.39) results in the following second-order non-linear ordinary differential equation in $P(y)$:

$$\mu_* \frac{d^2 P}{dy^2} + \beta \left(\frac{dP}{dy} \right)^2 \frac{d^2 P}{dy^2} - \beta_* P \left(\frac{dP}{dy} \right)^2 - (\phi_* + M_*^2) P = 0. \quad (3.5.21)$$

The resulting invariant boundary conditions are obtained by applying the invariant solution in equation (3.5.20) onto equations (3.2.40) and (3.2.41) to get

$$P(0) = v_0, \quad (3.5.22)$$

$$P(y) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \quad (3.5.23)$$

where y can take on a sufficiently large value as possible and $V = v_0$ is a constant.

We use the ansatz method to solve the above boundary value problem for $P(y)$ by assuming a solution of the form

$$P(y) = Ae^{By}, \quad (3.5.24)$$

where $A = B \neq 0$ are constants that are yet to be determined. The substitution of this ansatz into equation (3.5.21) gives

$$\mu_* AB^2 e^{By} + \beta A^3 B^4 e^{3By} - \beta_* A^3 B^2 e^{3By} - (\phi_* + M_*^2) Ae^{By} = 0. \quad (3.5.25)$$

We now can separate in terms of powers of e^{By} and e^{3By} to obtain

$$e^{3By} : \quad \beta A^3 B^4 - \beta_* A^3 B^2 = 0, \quad (3.5.26)$$

$$e^{By} : \quad \mu_* A B^2 - (\phi_* + M_*^2) A = 0. \quad (3.5.27)$$

From equation (3.5.26), we have that

$$A^3 B^2 (\beta B^2 - \beta_*) = 0, \quad (3.5.28)$$

and since $A^3 \neq 0$ and $B^2 \neq 0$, it follows that

$$B = \pm \sqrt{\frac{\beta_*}{\beta}}. \quad (3.5.29)$$

Since we require the solution of this boundary value problem to be zero at infinity as illustrated by the second boundary condition (3.5.23), we stipulate that

$$B = -\sqrt{\frac{\beta_*}{\beta}}. \quad (3.5.30)$$

Substituting this chosen function for B into equation (3.5.27) gives that

$$A \left(\mu_* \left(\frac{\beta_*}{\beta} \right) - (\phi_* + M_*^2) \right) = 0, \quad (3.5.31)$$

and since $A \neq 0$, it follows that

$$\mu_* \left(\frac{\beta_*}{\beta} \right) - (\phi_* + M_*^2) = 0. \quad (3.5.32)$$

Thus, the closed-form solution for $P(y)$ has the form

$$P(y) = A \exp \left(-\sqrt{\frac{\beta_*}{\beta}} y \right). \quad (3.5.33)$$

The application of the first boundary condition (3.5.22) yields that $A = v_0$

and it follows that the steady-state solution is given by

$$u = v_0 \exp \left(-\sqrt{\frac{\beta_*}{\beta}} y \right), \quad (3.5.34)$$

provided that equation (3.5.32) holds.

The reduced ordinary differential equation in equation (3.5.21) can also be solved numerically, subject to the boundary conditions (3.5.22) and (3.5.23). The solution is obtained by using the *Mathematica* solver **NDSolve** and the result is plotted in Figure 3.6.5.

3.5.3 Travelling wave solution

We can acquire a travelling wave solution for the governing model equation (3.2.39) since it admits both a time and space-translational symmetry generator. For this reason, we can search for an invariant solution under the operator

$$\begin{aligned} X_c &= X_1 + cX_2, \quad c > 0, \\ X_c &= \frac{\partial}{\partial t} + c \frac{\partial}{\partial y}. \end{aligned} \quad (3.5.35)$$

This denotes a wave-front type travelling wave solution and since the wave is of the forward type, it is extending away from the plate with a constant wave speed c . The characteristic system of the operator X_c in equation (3.5.35) is given by

$$\frac{dt}{1} = \frac{dy}{c} = \frac{du}{0}. \quad (3.5.36)$$

It is evident that u is invariant and so

$$u = k_1, \quad (3.5.37)$$

where k_1 is a constant. Solving the first two terms of the characteristic system in equation (3.5.36) gives a variable separable ordinary differential equation given by

$$cdt = dy, \quad (3.5.38)$$

and by integrating, we obtain the second invariant given by

$$y - ct = k_2, \quad (3.5.39)$$

where k_2 is a constant of integration. We can write one invariant in terms of the other as $k_1 = F(k_2)$, and so we have that the invariant solution admitted by the operator X_c is

$$u(y, t) = F(x_1), \quad \text{where } x_1 = y - ct. \quad (3.5.40)$$

Acting the above invariant solution onto the governing equation (3.2.39) gives a third-order non-linear ordinary differential equation for $F(x_1)$ given by

$$\begin{aligned} -c \frac{dF}{dx_1} = & \mu_* \frac{d^2 F}{dx_1^2} - \alpha_* c \frac{d^3 F}{dx_1^3} + \beta \left(\frac{dF}{dx_1} \right)^2 \frac{d^2 F}{dx_1^2} - \beta_* F \left(\frac{dF}{dx_1} \right)^2 \\ & - (\phi_* + M_*^2) F. \end{aligned} \quad (3.5.41)$$

We apply the ansatz method to the ordinary differential equation in $F(x_1)$ by assuming a solution of the form

$$F(x_1) = Ae^{Bx_1}, \quad (3.5.42)$$

where $A = B \neq 0$ are constants that will be determined in due course. The substitution of this ansatz into equation (3.5.41) gives

$$\begin{aligned} -cABe^{Bx_1} = & \mu_* AB^2 e^{Bx_1} - \alpha_* cAB^3 e^{Bx_1} + \beta A^3 B^4 e^{3Bx_1} - \beta_* A^3 B^2 e^{3Bx_1} \\ & - (\phi_* + M_*^2) Ae^{Bx_1}. \end{aligned} \quad (3.5.43)$$

We now can separate in terms of powers of e^{By} and e^{3By} to obtain

$$e^{3Bx_1} : \quad \beta A^3 B^4 - \beta_* A^3 B^2 = 0, \quad (3.5.44)$$

$$e^{Bx_1} : \quad cAB + \mu_* AB^2 - \alpha_* cAB^3 - (\phi_* + M_*^2) A = 0. \quad (3.5.45)$$

From equation (3.5.44), we have that

$$A^3 B^2 (\beta B^2 - \beta_*) = 0, \quad (3.5.46)$$

and since $A^3 \neq 0$ and $B^2 \neq 0$, it follows that

$$B = \pm \sqrt{\frac{\beta_*}{\beta}}. \quad (3.5.47)$$

Since we require the solution to be zero at infinity by the boundary condition (3.2.41), we stipulate that

$$B = -\sqrt{\frac{\beta_*}{\beta}}. \quad (3.5.48)$$

From equation (3.5.45),

$$A (cB + \mu_* B^2 - \alpha_* c B^3 - (\phi_* + M_*^2)) = 0, \quad (3.5.49)$$

and since $A \neq 0$, it follows that

$$cB + \mu_* B^2 - \alpha_* c B^3 - (\phi_* + M_*^2) = 0. \quad (3.5.50)$$

Substituting the chosen function for B from equation (3.5.48) into the above gives

$$\begin{aligned} -c\sqrt{\frac{\beta_*}{\beta}} + \mu_* \frac{\beta_*}{\beta} + \alpha_* c \frac{\beta_*}{\beta} \sqrt{\frac{\beta_*}{\beta}} - (\phi_* + M_*^2) &= 0, \\ \sqrt{\frac{\beta_*}{\beta}} \left[\alpha_* c \frac{\beta_*}{\beta} - c \right] + \left[\mu_* \frac{\beta_*}{\beta} - (\phi_* + M_*^2) \right] &= 0. \end{aligned} \quad (3.5.51)$$

Thus, the solution for $F(x_1)$ has the form

$$F(x_1) = A \exp \left(-\sqrt{\frac{\beta_*}{\beta}} x_1 \right), \quad (3.5.52)$$

provided that the condition (3.5.51) holds. Hence, the invariant solution $u(y, t)$ becomes

$$u(y, t) = A \exp \left(-\frac{\sqrt{\beta_*}(y - ct)}{\sqrt{\beta}} \right). \quad (3.5.53)$$

Furthermore, applying the above invariant solution for $u(y, t)$ on the boundary

condition (3.2.40) yields that

$$A \exp \left(c \frac{\sqrt{\beta_*}}{\sqrt{\beta}} t \right) = V(t), \quad (3.5.54)$$

and it follows that $A = 1$ when

$$V(t) = \exp \left(c \frac{\sqrt{\beta_*}}{\sqrt{\beta}} t \right). \quad (3.5.55)$$

Finally, the closed-form solution $u(y, t)$ satisfying the condition (3.5.51) is given by

$$u(y, t) = \exp \left(- \frac{\sqrt{\beta_*}(y - ct)}{\sqrt{\beta}} \right), \quad (3.5.56)$$

provided that the condition (3.5.51) holds. This solution also satisfies the initial condition (3.2.42) with

$$G(y) = \exp \left(- \frac{\sqrt{\beta_*}}{\sqrt{\beta}} y \right). \quad (3.5.57)$$

The closed-form solution in equation (3.5.56) is plotted in Figures 3.6.6 and 3.6.7 for time t and the wave speed parameter c respectively, while stipulating that the condition (3.5.51) is satisfied.

3.5.4 Numerical solution of the governing model

Here, we specify the numerical solution of the governing partial differential equation (3.2.39) subject to the suitably chosen initial and boundary conditions:

$$u(0, t) = V(t) = e^{-t}, \quad t > 0, \quad (3.5.58)$$

$$u(\infty, t) = 0, \quad t > 0, \quad (3.5.59)$$

$$u(y, 0) = G(y) = e^{-y}, \quad y > 0. \quad (3.5.60)$$

This is implemented in *Mathematica* using the built-in solver **NDSolve** and the resultant solution is shown graphically in Figure 3.6.8.

3.6 Graphical analysis

In this section, we illustrate the velocity profiles that have been obtained in the preceding sections and these are plotted against y in Figures 3.6.1-3.6.8. We study the behaviour of a variety of significant parameters on the form of the velocity field. We do this in the hope that we may glean a better understanding of their physical significance on the fluid flow model under investigation.

The group invariant solution (3.5.15) is plotted in Figures 3.6.1-3.6.4 in order to observe the effects of the magnetic field on the fluid flow. The effects of time t is shown in Figure 3.6.1. It depicts velocity profiles of various values of t while the values of all other parameters remain fixed. It is evident that the velocity decreases for increasing values of t . Next, we wish to examine the effects of the magnetic field. As such, the velocity profiles of varying values of the magnetic field parameter M_* are plotted in Figure 3.6.2. It shows that in the presence of a greater applied magnetic field, the velocity decreases. Lastly, we describe the influence of the third grade parameters β and β_* in the presence of an applied magnetic field. These are shown in Figures 3.6.3 and 3.6.4 respectively. From Figure 3.6.3, we see that the velocity profile increases as the third grade parameter β increases in magnitude. In view of that, it is clear that β signifies the property of shear thinning. More simply, the velocity increases with continued thinning of the fluid and this observation also correlates with the physical system. On the contrary, Figure 3.6.4 makes it clear that the velocity profile diminishes with an escalation of the third grade parameter β_* . Consequently, we can deduce that β_* captures the effects of the shear thickening behaviour of the fluid and again we can see that this concurs with the physics of the problem. These figures also bring to light the fact that β and β_* have opposing influences on the velocity field in spite of the occurrence of the magnetic field.

The numerical solutions of the reduced second-order ordinary differential equation (3.5.21) are plotted in Figure 3.6.5 for changing values of the magnetic field parameter M_* subject to the boundary conditions (3.5.22) and (3.5.23). It is abundantly clear that the effects of the behaviour of M_* on the velocity profiles coincide with those previously noted from the velocity profiles of the closed-form solutions.

The travelling wave solutions (3.5.56) are depicted for various values of time

t and the wave speed c in Figures 3.6.6 and 3.6.7 respectively. It is worthwhile to note that in no way do these solutions determine the effects of the magnetic field on the flow of the fluid. We inspect that both time t and the wave speed c have the same effect on the closed-form solution (3.5.56) in that they both bring about an increase in the structure of the velocity profile.

Finally, the numerical solution of the governing partial differential equation (3.2.39) is plotted in Figures 3.6.8 and 3.6.9 for small changes in time t and the magnetic field parameter M_* respectively, subject to an apt choice of initial and boundary conditions (3.5.58)-(3.5.60). It is noted that the velocity profile diminishes with an increase in time in Figure 3.6.8 and from Figure 3.6.9, we see that the velocity field decreases as a greater magnetic field is applied. Again, these are the selfsame observations that was rendered before when investigating the analytical results of the group invariant solution (3.5.15).

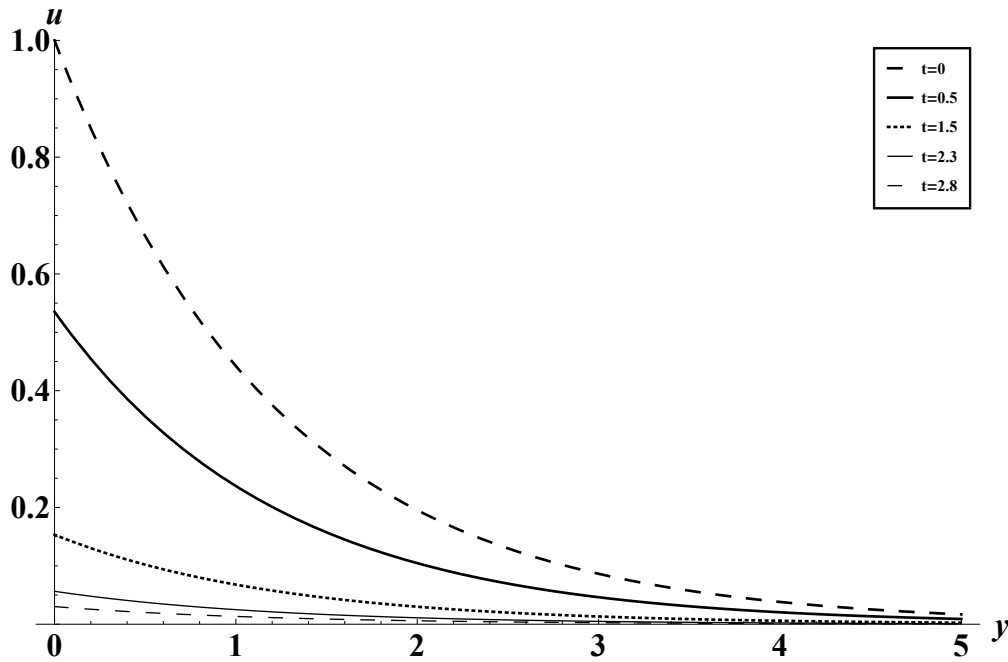


Figure 3.6.1: Velocity field of the analytical solution (3.5.15) for varying values of time t , where $\phi_* = 1$, $M_* = 0.5$, $\beta_* = 1$, $\beta = 1.5$ are fixed.

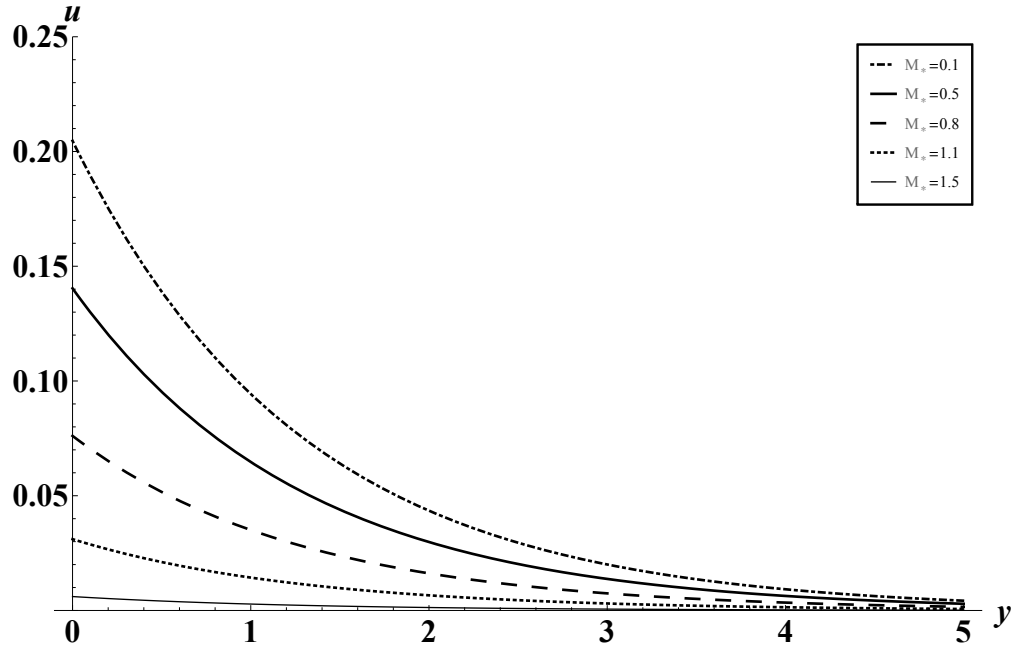


Figure 3.6.2: Velocity field of the analytical solution (3.5.15) for varying values of the magnetic field parameter M_* , where $\phi_* = 1$, $t = \pi/2$, $\beta_* = 1.5$, $\beta = 2.5$ are fixed.

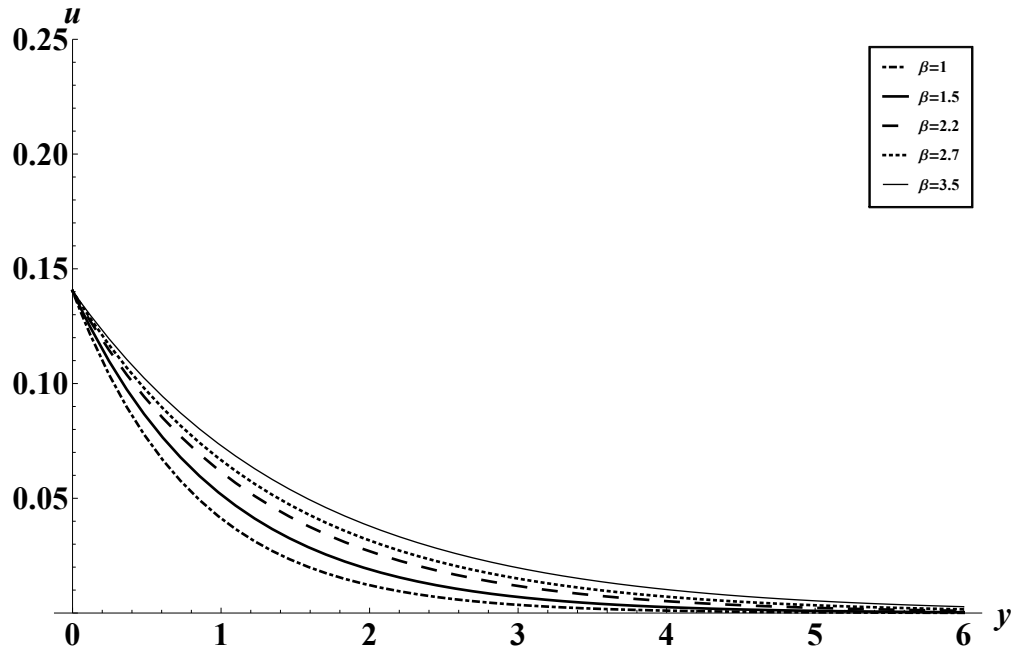


Figure 3.6.3: Velocity field of the analytical solution (3.5.15) for varying values of the third grade fluid parameter β , where $\phi_* = 1$, $M_* = 0.5$, $t = \pi/2$, $\beta_* = 1.5$ are fixed.

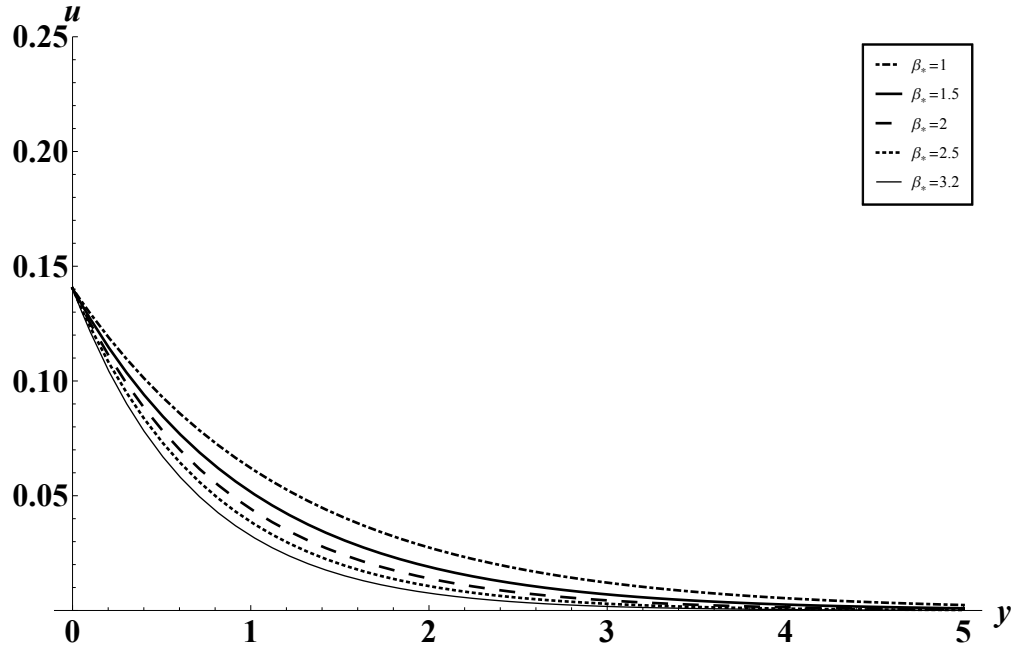


Figure 3.6.4: Velocity field of the analytical solution (3.5.15) for varying values of the third grade fluid parameter β_* , where $\phi_* = 1$, $M_* = 0.5$, $t = \pi/2$, $\beta = 1.5$ are fixed.

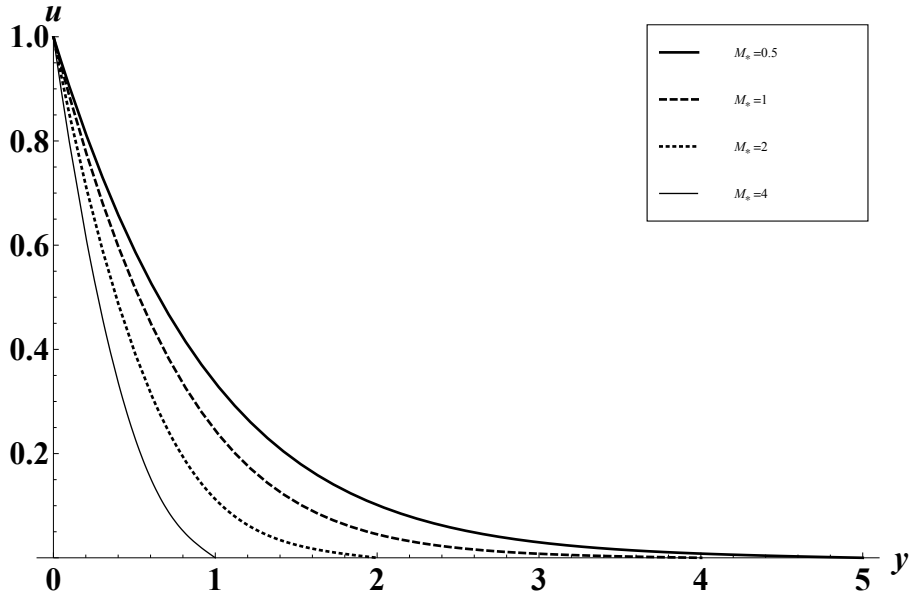


Figure 3.6.5: Numerical solution of the reduced ordinary differential equation (3.5.21) subject to the boundary conditions (3.5.22) and (3.5.23) for varying values of the magnetic field parameter M_* , where $\mu_* = 0.5$, $\beta = 1.5$, $\beta_* = 1$, $\phi_* = 0.5$ are fixed.

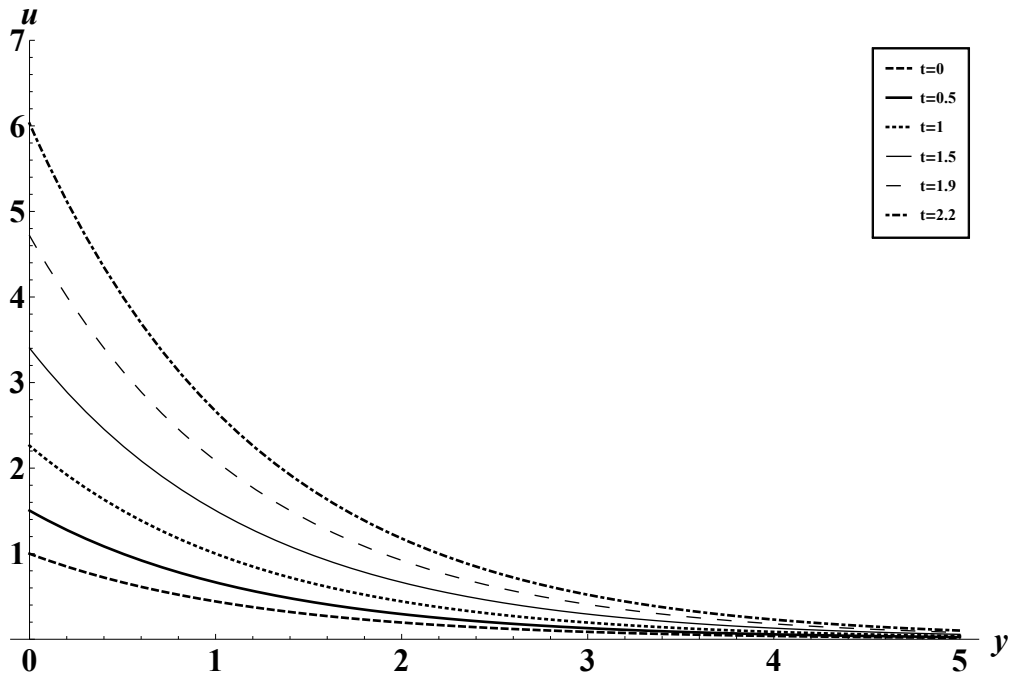


Figure 3.6.6: Velocity field of the analytical solution (3.5.56) for varying values of time t , where $\beta_* = 1.5$, $c = \beta = 1$ are fixed.

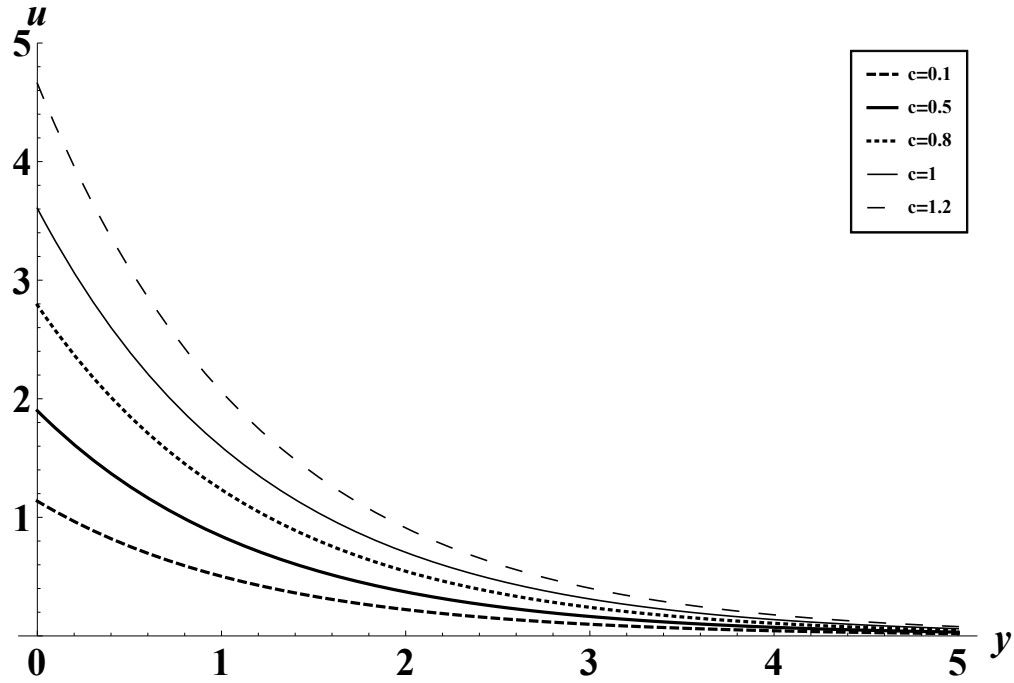


Figure 3.6.7: Velocity field of the analytical solution (3.5.56) for varying values of the wave speed c , where $\beta_* = 1.5$, $t = \pi/2$, $\beta = 1$ are fixed.

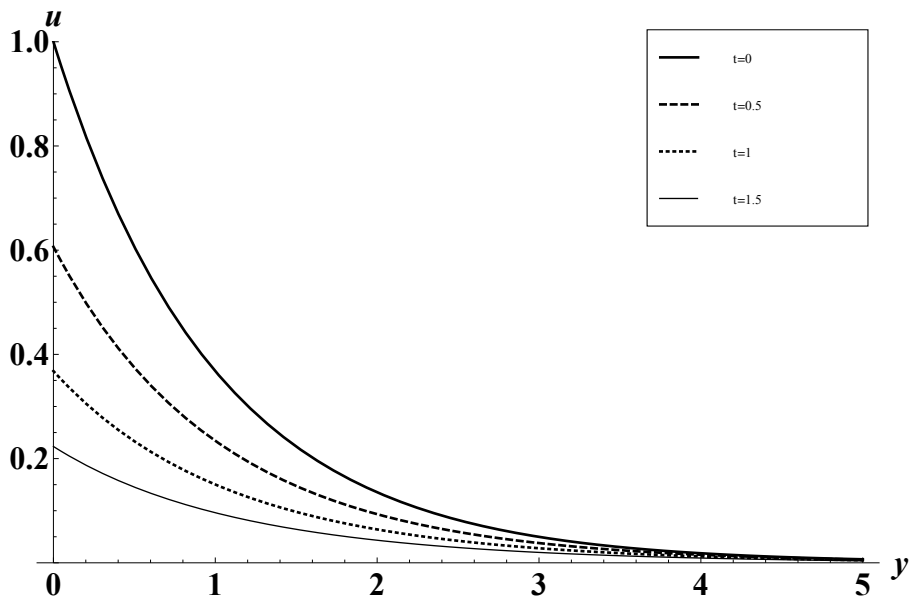


Figure 3.6.8: Numerical solution of the partial differential equation (3.2.39) subject to conditions (3.5.58)-(3.5.60) for varying values of time t , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\phi_* = 0.8$, $M_* = 0.5$ are fixed.

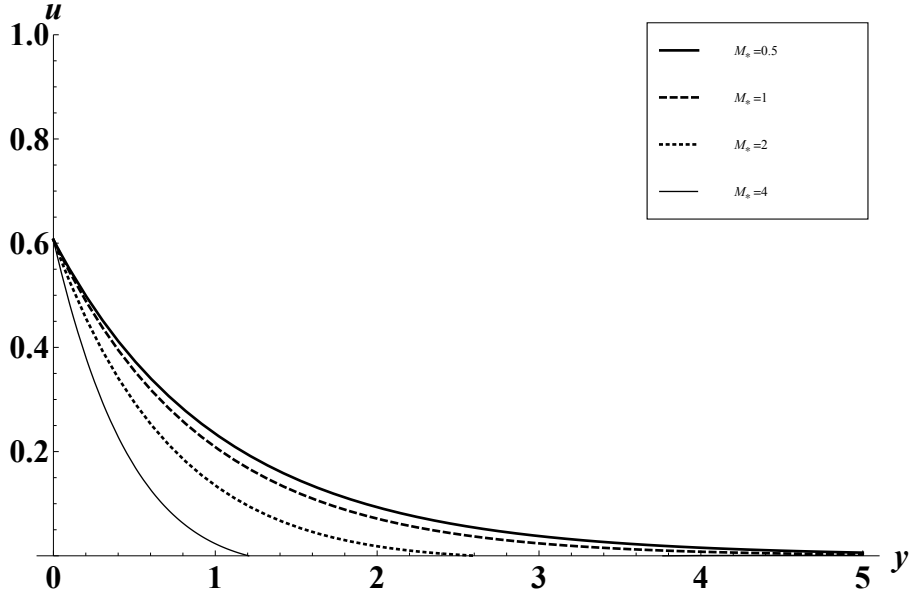


Figure 3.6.9: Numerical solution of the partial differential equation (3.2.39) subject to conditions (3.5.58)-(3.5.60) for varying values of the magnetic field parameter M_* , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\phi_* = 0.8$, $t = 0.5$ are fixed.

At this point, it is imperative to note the distinctness of the group invariant solution (3.5.15) since it single-handedly incorporates the magnetic field parameter M_* . In so doing, it is the only solution to directly capture the effects of the applied magnetic field on the physical problem. Conversely, the group invariant solutions (3.5.34) and (3.5.56) obtained from the steady state and travelling wave solutions respectively, do not directly involve the applied magnetic field parameter M_* . Both these solutions are only valid if their related imposing conditions (3.5.32) and (3.5.51) are also met. As a result, these solutions are inadequate since they only hold true for a certain value of the magnetic field term. In the main, we call attention to the fact that the most superior of these group invariant solutions is that of the non-travelling type via the subgroup generated by X_3 . It is significantly more meaningful since it achieves the feat of exhibiting the best characterization of the physics of the problem under examination.

3.7 Conclusion

In this chapter, we provided a detailed review of the paper composed by Aziz et al. [6] which focused on the modelling and solutions of a non-linear time-dependent fluid flow model.

The geometry of the model deals with the unsteady magnetohydrodynamic flow of a third grade non-Newtonian fluid over an infinite flat rigid plate within a porous medium. The fluid was such that it was incompressible and electrically conducting and filled up the region prescribed by $y > 0$. The relevant laws of conservation of mass and momentum were employed to gain the governing model which assumed the form of a third-order non-linear partial differential equation.

We then implemented a thorough Lie group analysis on the governing equation and we deduced a two-dimensional principal Lie algebra. Further, a three-dimensional Lie algebra was developed by invoking the restrictions on the physical parameters.

This was followed by the calculation of the optimal system of the admitted Lie point symmetry generators. This was done in order to investigate the relative significance of these symmetries in constructing their corresponding group invariant solutions. Such an approach was not previously studied by Aziz et al. [6].

The procedure of obtaining the group invariant solutions resulted in the original third-order partial differential equation to be reduced to various ordinary differential equations displaying differing levels of reduction. These were then solved by using various techniques in order to obtain their group invariant solutions. Three types of closed-form solutions were found with varying degrees of efficiency. In some cases, the reduced ordinary differential equations were also solved numerically. A numerical solution to the model equation was also presented with a specific choice of initial and boundary conditions. It is vital to note here that further significant solutions may be obtained for the velocity profile of the fluid flow according to the results of the optimal system. In particular, one may search for closed-form solutions under the linear combinations $X_2 \pm X_3$.

Finally, an investigation was carried out to inspect the influences of a number of wide-ranging physical parameters on the velocity of the fluid flow and

plots of numerous graphs aided in dissecting their behaviour. It was found that the non-travelling wave group invariant solution (3.5.15) generated by X_3 was the most meaningful solution since it best described the physical composition of the problem and unlike the other closed-form solutions, it succeeded in taking into account the effects of the applied magnetic field.

Chapter 4

The unsteady MHD flow of a fourth grade fluid in a porous medium

4.1 Introduction

In this chapter, we extend the work done in Chapter 3 for a fourth grade non-Newtonian fluid. Moreover, the geometry of the model remains unchanged. To the best of our knowledge, the study of this fluid flow model under this specific geometry has not been previously investigated.

The structure of this chapter is done in the same fashion as the preceding chapter. In Section 4.2, we model the relevant fluid flow problem with the aid of the equations of mass and momentum. We then find the symmetries of the model equation in Section 4.3 using the classical Lie symmetry approach. The optimal system for the obtained Lie algebra is then found in Section 4.4 in order to classify the group invariant solutions. In Section 4.5, we find the corresponding group invariant solutions by using the optimal system as well as setting up a numerical solution to the governing equation. We then make a note of the physical significance of the analytical and numerical solutions for varying values of meaningful parameters in Section 4.6 via graphs. To finish, we present some concluding remarks in Section 4.7.

4.2 Mathematical formulation of the model

This section involves finding a mathematical model to formulate the problem by implementing the fundamental fluid flow equations.

Once more, we consider a Cartesian coordinate system xyz . Consider the time-dependent magnetohydrodynamic flow of a non-Newtonian fluid of the fourth grade that is bounded by a flat rigid plate infinite in the xz -plane and this incompressible and electrically conducting fluid is contained within the porous half-space $y > 0$. This scenario is illustrated in Figure 4.2.1 below:

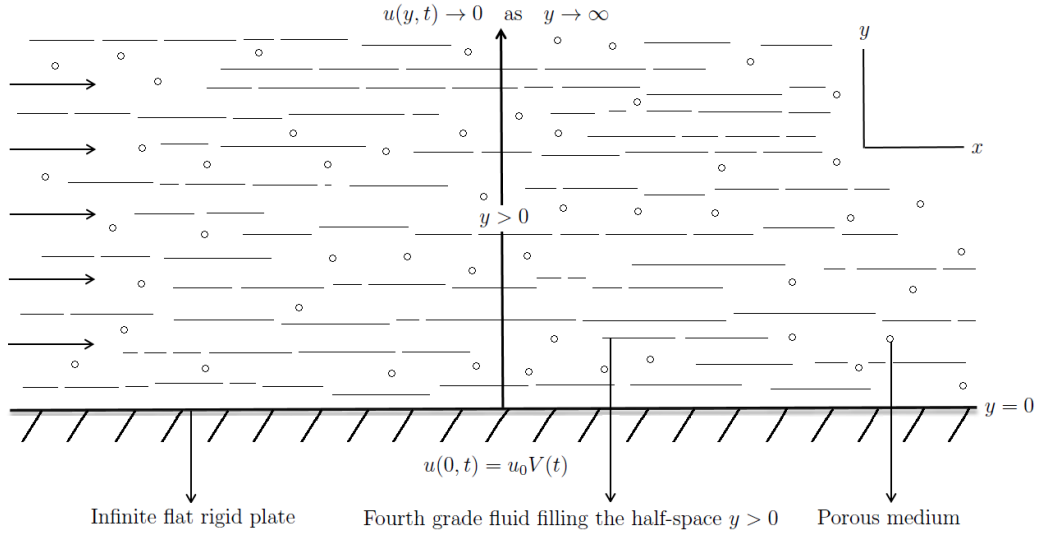


Figure 4.2.1: Graphical representation of the fourth grade non-Newtonian fluid flow model.

As before, the fluid flow problem is governed by the laws conservation of mass and momentum as defined in equations (3.2.4) and (3.2.5) respectively.

Due to the existence of an infinite plate in the xz -plane, all physical quantities except for the pressure p are y -dependent only. For unidirectional and a one-dimensional flow, the velocity field $\mathbf{V}$ for this particular fluid flow problem is of the form

$$\mathbf{V} = [u(y, t), 0, 0], \quad (4.2.1)$$

where $u(y, t)$ is the velocity field in the x -direction. This velocity field also identically satisfies the continuity equation (3.2.4).

The effects of the magnetohydrodynamic flow on the problem is defined as in equation (3.2.9). Using the velocity field as defined in equation (4.2.1), the

x -component of the applied magnetic field becomes

$$(\mathbf{J} \times \mathbf{B})_x = -\sigma B_0^2 u. \quad (4.2.2)$$

The Cauchy stress tensor for a fourth grade fluid [18] satisfies the constitutive equations:

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \sum_{i=1}^n \mathbf{S}_i \quad \text{with } n = 4, \\ &= -p\mathbf{I} + \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4, \end{aligned} \quad (4.2.3)$$

where p is the fluid pressure, $\mathbf{I}$ the identity tensor and $\mathbf{S}_i$ ($i = 1, 2, 3, 4$) the extra stress tensors defined by

$$\mathbf{S}_1 = \mu \mathbf{A}_1, \quad (4.2.4)$$

$$\mathbf{S}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \quad (4.2.5)$$

$$\mathbf{S}_3 = \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2) + \beta_3 (tr \mathbf{A}_1^2) \mathbf{A}_1, \quad (4.2.6)$$

$$\begin{aligned} \mathbf{S}_4 &= \gamma_1 \mathbf{A}_4 + \gamma_2 (\mathbf{A}_3 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_3) + \gamma_3 \mathbf{A}_2^2 + \gamma_4 (\mathbf{A}_2 \mathbf{A}_1^2 + \mathbf{A}_1^2 \mathbf{A}_2) \\ &\quad + \gamma_5 (tr \mathbf{A}_2) \mathbf{A}_2 + \gamma_6 (tr \mathbf{A}_2) \mathbf{A}_1^2 + [\gamma_7 (tr \mathbf{A}_3) + \gamma_8 (tr \mathbf{A}_2 \mathbf{A}_1)] \mathbf{A}_1, \end{aligned} \quad (4.2.7)$$

where μ is the dynamic fluid viscosity, α_i ($i = 1, 2$), β_i ($i = 1, 2, 3$) and γ_i ($i = 1, 2, \dots, 8$) are the material constants, tr the trace which is the sum of the elements on the diagonal of a square matrix and $\mathbf{A}_i$ ($i = 1, 2, 3, 4$) the Rivlin-Ericksen tensors as defined in equations (3.2.12) and (3.2.13).

The component form of the momentum equation (3.2.5) is found by using the velocity field (4.2.1) in conjunction with the component form of the Cauchy stress tensor as given in equation (3.2.16). This gives us the set of governing equations below:

$$\rho \frac{du}{dt} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \sigma B_0^2 u + R_x, \quad (4.2.8)$$

$$0 = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + R_y, \quad (4.2.9)$$

$$0 = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + R_z, \quad (4.2.10)$$

where R_x , R_y and R_z are the x , y and z components of Darcy's resistance.

Using equations (3.2.12) and (3.2.13), the following expressions are obtained for the Rivlin-Ericksen tensors $\mathbf{A}_i$ ($i = 1, 2, 3, 4$):

$$\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.11)$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & \frac{\partial^2 u}{\partial y \partial t} & 0 \\ \frac{\partial^2 u}{\partial y \partial t} & 2 \left(\frac{\partial u}{\partial y} \right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.12)$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & \frac{\partial^3 u}{\partial y \partial t^2} & 0 \\ \frac{\partial^3 u}{\partial y \partial t^2} & 4 \frac{\partial u}{\partial y} \left(\frac{\partial^2 u}{\partial y \partial t} \right) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.13)$$

$$\mathbf{A}_4 = \begin{pmatrix} 0 & \frac{\partial^4 u}{\partial y \partial t^3} & 0 \\ \frac{\partial^4 u}{\partial y \partial t^3} & 6 \frac{\partial u}{\partial y} \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.2.14)$$

The extra stress tensors $\mathbf{S}_i$ ($i = 1, 2, 3, 4$) are calculated using the definitions given by equations (4.2.4)-(4.2.7) and the Rivlin-Ericksen tensors found in equations (4.2.11)-(4.2.14) and are calculated to be

$$\mathbf{S}_1 = \begin{pmatrix} 0 & \mu \frac{\partial u}{\partial y} & 0 \\ \mu \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.15)$$

$$\mathbf{S}_2 = \begin{pmatrix} \alpha_2 \left(\frac{\partial u}{\partial y} \right)^2 & \alpha_1 \left(\frac{\partial^2 u}{\partial y \partial t} \right) & 0 \\ \alpha_1 \left(\frac{\partial^2 u}{\partial y \partial t} \right) & (2\alpha_1 + \alpha_2) \left(\frac{\partial u}{\partial y} \right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.16)$$

$$\mathbf{S}_3 = \begin{pmatrix} 2\beta_2 \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right) & \beta_1 \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^3 & 0 \\ \beta_1 \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^3 & 2(2\beta_1 + \beta_2) \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.17)$$

$$\mathbf{S}_4 = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2.18)$$

where

$$\begin{aligned} S_{11} &= 2\gamma_2 \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) + \gamma_3 \left(\frac{\partial^2 u}{\partial y \partial t} \right)^2 + 2\gamma_6 \left(\frac{\partial u}{\partial y} \right)^4, \\ S_{12} &= \gamma_1 \left(\frac{\partial^4 u}{\partial y \partial t^3} \right) + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \left(\frac{\partial u}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial y \partial t} \right), \\ S_{21} &= \gamma_1 \left(\frac{\partial^4 u}{\partial y \partial t^3} \right) + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \left(\frac{\partial u}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial y \partial t} \right), \\ S_{22} &= (6\gamma_1 + 2\gamma_2) \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) + \gamma_3 \left(\frac{\partial^2 u}{\partial y \partial t} \right)^2 \\ &\quad + (4\gamma_3 + 4\gamma_4 + 4\gamma_5 + 2\gamma_6) \left(\frac{\partial u}{\partial y} \right)^4. \end{aligned}$$

We then calculate the components of the Cauchy stress tensor τ_{ij} from equation (3.2.16) by using equation (4.2.3) along with the extra stress tensors $\mathbf{S}_i$ ($i = 1, 2, 3, 4$) in equations (4.2.15)-(4.2.18) above.

$$\begin{aligned} \tau_{xx} &= -p + \alpha_2 \left(\frac{\partial u}{\partial y} \right)^2 + 2\beta_2 \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right) + 2\gamma_2 \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) \\ &\quad + \gamma_3 \left(\frac{\partial^2 u}{\partial y \partial t} \right)^2 + 2\gamma_6 \left(\frac{\partial u}{\partial y} \right)^4, \end{aligned} \quad (4.2.19)$$

$$\begin{aligned}\tau_{xy} = \tau_{yx} = & \mu \frac{\partial u}{\partial y} + \alpha_1 \left(\frac{\partial^2 u}{\partial y \partial t} \right) + \beta_1 \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^3 \\ & + \gamma_1 \left(\frac{\partial^4 u}{\partial y \partial t^3} \right) + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \left(\frac{\partial u}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial y \partial t} \right),\end{aligned}\quad (4.2.20)$$

$$\tau_{xz} = \tau_{zx} = 0, \quad (4.2.21)$$

$$\begin{aligned}\tau_{yy} = & -p + (2\alpha_1 + \alpha_2) \left(\frac{\partial u}{\partial y} \right)^2 + 2(2\beta_1 + \beta_2) \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right) \\ & + (6\gamma_1 + 2\gamma_2) \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^3 u}{\partial y \partial t^2} \right) + \gamma_3 \left(\frac{\partial^2 u}{\partial y \partial t} \right)^2 \\ & + (4\gamma_3 + 4\gamma_4 + 4\gamma_5 + 2\gamma_6) \left(\frac{\partial u}{\partial y} \right)^4,\end{aligned}\quad (4.2.22)$$

$$\tau_{yz} = \tau_{zy} = 0, \quad (4.2.23)$$

$$\tau_{zz} = -p. \quad (4.2.24)$$

For the fluid flow problem under consideration, τ_{xy} is the only component of the Cauchy stress tensor with a physical significance since the fluid flow is in the x -direction and the disturbance of the fluid flow is in the y -direction. Thus, we substitute the appropriate tensors into the x -component of the momentum equation (4.2.8) which gives

$$\begin{aligned}\rho \frac{\partial u}{\partial t} = & \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \left(\frac{\partial^3 u}{\partial y^2 \partial t} \right) + \beta_1 \left(\frac{\partial^4 u}{\partial y^2 \partial t^2} \right) + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \left(\frac{\partial^5 u}{\partial y^2 \partial t^3} \right) \\ & + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial y \partial t} \right) \right] - \sigma B_0^2 u + R_x.\end{aligned}\quad (4.2.25)$$

We now consider the effects of the porous medium on the momentum equation (3.2.5). We have already established in Chapter 3 that the apparent viscosity is different for different non-Newtonian fluids. The apparent viscosity of the unsteady unidirectional flow of a fourth grade fluid over a flat rigid plate

is given by

$$\begin{aligned}
 \text{apparent viscosity} &= \frac{\tau_{xy}}{\frac{\partial u}{\partial y}}, \\
 &= \mu + \alpha_1 \left(\frac{\partial}{\partial t} \right) + \beta_1 \left(\frac{\partial}{\partial t} \right) + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 + \gamma_1 \left(\frac{\partial^3}{\partial t^3} \right) \\
 &\quad + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right). \quad (4.2.26)
 \end{aligned}$$

We then substitute this apparent viscosity into Darcy's law as specified in equation (3.2.24). This gives us that the x -component of Darcy's resistance for the fourth grade fluid model under consideration is

$$\begin{aligned}
 R_x &= -\frac{\phi}{\kappa} (\text{apparent viscosity})u, \\
 &= -\frac{\phi}{\kappa} \left[\mu u + \alpha_1 \left(\frac{\partial u}{\partial t} \right) + \beta_1 \left(\frac{\partial^2 u}{\partial t^2} \right) + 2(\beta_2 + \beta_3)u \left(\frac{\partial u}{\partial y} \right)^2 + \gamma_1 \left(\frac{\partial^3 u}{\partial t^3} \right) \right. \\
 &\quad \left. + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8)u \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right) \right]. \quad (4.2.27)
 \end{aligned}$$

We note here that the y and z -components of Darcy's resistance $R_y = R_z = 0$ since $v = w = 0$.

We get the following governing partial differential equation in u by substituting equation (4.2.27) into equation (4.2.25) in the absence of a modified pressure gradient:

$$\begin{aligned}
 \rho \frac{\partial u}{\partial t} &= \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \left(\frac{\partial^3 u}{\partial y^2 \partial t} \right) + \beta_1 \left(\frac{\partial^4 u}{\partial y^2 \partial t^2} \right) + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \\
 &\quad + \gamma_1 \left(\frac{\partial^5 u}{\partial y^2 \partial t^3} \right) + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \right. \\
 &\quad \left. \left(\frac{\partial^2 u}{\partial y \partial t} \right) \right] - \frac{\phi}{\kappa} \left[\mu u + \alpha_1 \left(\frac{\partial u}{\partial t} \right) + \beta_1 \left(\frac{\partial^2 u}{\partial t^2} \right) + 2(\beta_2 + \beta_3)u \left(\frac{\partial u}{\partial y} \right)^2 \right. \\
 &\quad \left. + \gamma_1 \left(\frac{\partial^3 u}{\partial t^3} \right) + 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8)u \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right) \right] - \sigma B_0^2 u. \quad (4.2.28)
 \end{aligned}$$

We impose the relevant boundary and initial conditions so that the aforementioned fifth-order non-linear partial differential equation can be solved and they are given as

Boundary conditions:

$$u(0, t) = u_0 V(t), \quad t > 0, \quad (4.2.29)$$

$$u(\infty, t) = 0, \quad t > 0, \quad (4.2.30)$$

Initial conditions:

$$u(y, 0) = f(y), \quad y > 0, \quad (4.2.31)$$

$$\frac{\partial u}{\partial t}(y, 0) = g(y), \quad y > 0, \quad (4.2.32)$$

$$\frac{\partial^2 u}{\partial t^2}(y, 0) = h(y), \quad y > 0, \quad (4.2.33)$$

where u_0 is the reference velocity and $V(t)$, $f(y)$, $g(y)$ and $h(y)$ are as yet to be determined functions. The first boundary condition (4.2.29) is the no-slip condition. The second boundary condition (4.2.30) specifies that the main-stream velocity is zero. The first initial condition (4.2.31) shows that initially, the fluid is flowing with some non-uniform velocity $f(y)$. The remaining two derivative initial conditions (4.2.32) and (4.2.33) are imposed such that the problem is well-posed.

We define the transformations below in order to find the dimensionless form of the governing model:

$$\bar{u} = \frac{u}{u_0}, \quad \bar{y} = \frac{u_0 y}{\nu}, \quad \bar{t} = \frac{u_0^2 t}{\nu}, \quad (4.2.34)$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The implementation of these transformations onto equation (4.2.28) returns

$$\begin{aligned} \frac{\partial \bar{u}}{\partial \bar{t}} = & \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{\alpha} \frac{\partial^3 \bar{u}}{\partial \bar{y}^2 \partial \bar{t}} + \bar{\beta}_1 \frac{\partial^4 \bar{u}}{\partial \bar{y}^2 \partial \bar{t}^2} + 3\bar{\beta} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{\gamma} \frac{\partial^5 \bar{u}}{\partial \bar{y}^2 \partial \bar{t}^3} + \bar{\Gamma} \frac{\partial}{\partial \bar{y}} \left[\left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right. \\ & \left. \left(\frac{\partial^2 \bar{u}}{\partial \bar{y} \partial \bar{t}} \right) \right] - \left[\bar{u} + \bar{\alpha} \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{\beta}_1 \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \bar{\beta} \bar{u} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \bar{\gamma} \frac{\partial^3 \bar{u}}{\partial \bar{t}^3} + \bar{\Gamma} \bar{u} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right) \left(\frac{\partial^2 \bar{u}}{\partial \bar{y} \partial \bar{t}} \right) \right] \bar{\phi} - \bar{M}^2 \bar{u}, \end{aligned} \quad (4.2.35)$$

with

$$\begin{aligned}
\bar{\alpha} &= \frac{\alpha_1 u_0^2}{\rho \nu^2}, & \bar{\beta}_1 &= \frac{\beta_1 u_0^4}{\rho \nu^3}, \\
\bar{\beta} &= 2(\beta_2 + \beta_3) \frac{u_0^4}{\rho \nu^3}, & \bar{\gamma} &= \frac{\gamma_1 u_0^6}{\rho \nu^4}, \\
\bar{\Gamma} &= 2(2\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 2\gamma_7 + \gamma_8) \frac{u_0^6}{\rho \nu^4}, \\
\bar{\phi} &= \frac{\phi \nu^2}{\kappa u_0^2}, & \bar{M}^2 &= \frac{\sigma B_0^2 \nu}{\rho u_0^2}.
\end{aligned} \tag{4.2.36}$$

We also use the transformations to obtain the dimensionless form of the boundary and initial conditions and they become

Boundary conditions:

$$\bar{u}(0, \bar{t}) = V(\bar{t}), \quad \bar{t} > 0, \tag{4.2.37}$$

$$\bar{u}(\infty, \bar{t}) = 0, \quad \bar{t} > 0, \tag{4.2.38}$$

Initial conditions:

$$\bar{u}(\bar{y}, 0) = F(\bar{y}), \quad \bar{y} > 0, \tag{4.2.39}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}}(\bar{y}, 0) = G(\bar{y}), \quad \bar{y} > 0, \tag{4.2.40}$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2}(\bar{y}, 0) = H(\bar{y}), \quad \bar{y} > 0, \tag{4.2.41}$$

where

$$F(\bar{y}) = \frac{f(\bar{y})}{u_0}, \quad G(\bar{y}) = \frac{\nu g(\bar{y})}{u_0^3}, \quad H(\bar{y}) = \frac{\nu^2 h(\bar{y})}{u_0^5}. \tag{4.2.42}$$

We take notice of the fact that the functions $V(\bar{t})$, $F(\bar{y})$, $G(\bar{y})$ and $H(\bar{y})$ are arbitrary. For simplicity of notation, the overhead bars from all the dimensionless quantities will be neglected in the subsequent analysis.

To end with, we rewrite equation (4.2.35) in its compact form to obtain

the final form of the governing equation in u which is given by

$$\begin{aligned} \frac{\partial u}{\partial t} = & \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_* \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \\ & + \gamma_* \frac{\partial^5 u}{\partial y^2 \partial t^3} + \Gamma_* \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial y \partial t} \right) \right] - \kappa_1 u - \kappa_2 \frac{\partial^2 u}{\partial t^2} \\ & - \kappa_3 u \left(\frac{\partial u}{\partial y} \right)^2 - \kappa_4 \frac{\partial^3 u}{\partial t^3} - \kappa_5 u \left(\frac{\partial u}{\partial y} \right) \frac{\partial^2 u}{\partial y \partial t} - M_*^2 u, \end{aligned} \quad (4.2.43)$$

subject to the boundary and initial conditions

$$u(0, t) = V(t), \quad t > 0, \quad (4.2.44)$$

$$u(\infty, t) = 0, \quad t > 0, \quad (4.2.45)$$

$$u(y, 0) = F(y), \quad y > 0, \quad (4.2.46)$$

$$\frac{\partial u}{\partial t}(y, 0) = G(y), \quad y > 0, \quad (4.2.47)$$

$$\frac{\partial^2 u}{\partial t^2}(y, 0) = H(y), \quad y > 0, \quad (4.2.48)$$

where

$$\frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \left(\frac{\partial^2 u}{\partial y \partial t} \right) \right] = 2 \left(\frac{\partial u}{\partial y} \right) \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 u}{\partial y \partial t} + \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}, \quad (4.2.49)$$

and

$$\begin{aligned} \mu_* &= \frac{1}{1 + \alpha\phi}, & \alpha_* &= \frac{\alpha}{1 + \alpha\phi}, & \beta &= \frac{\beta_1}{1 + \alpha\phi}, \\ \beta_* &= \frac{3\beta}{1 + \alpha\phi}, & \gamma_* &= \frac{\gamma}{1 + \alpha\phi}, & \Gamma_* &= \frac{\Gamma}{1 + \alpha\phi}, \\ \kappa_1 &= \frac{\phi}{1 + \alpha\phi}, & \kappa_2 &= \frac{\beta_1\phi}{1 + \alpha\phi}, & \kappa_3 &= \frac{\beta\phi}{1 + \alpha\phi}, \\ \kappa_4 &= \frac{\gamma\phi}{1 + \alpha\phi}, & \kappa_5 &= \frac{\Gamma\phi}{1 + \alpha\phi}, & M_*^2 &= \frac{M^2}{1 + \alpha\phi}, \end{aligned} \quad (4.2.50)$$

Each of the defined parameters incorporate a physical significance. More specifically, μ_* is the viscosity parameter, α_* the second grade parameter, β and β_* the third grade parameters, γ_* and Γ_* the fourth grade parameters, κ_1 - κ_5 the porosity parameters and M_*^2 the magnetic field parameter.

4.3 Lie point symmetry analysis

In this section, we apply the Lie theory as specified in Section 2.2 to the governing partial differential equation (4.2.43) to obtain its corresponding Lie point symmetries.

Again, we start by looking for transformations of the dependent variable u and the independent variables t and y of the form

$$\begin{aligned}\bar{t} &= \bar{t}(t, y, u, \epsilon), \\ \bar{y} &= \bar{y}(t, y, u, \epsilon), \\ \bar{u} &= \bar{u}(t, y, u, \epsilon),\end{aligned}\tag{4.3.1}$$

which represent a one-parameter group with group parameter ϵ . These are chosen in such a way that the governing equation (4.2.43) is invariant. Their corresponding infinitesimal transformations are found via a Taylor series expansion and are found to be

$$\begin{aligned}\bar{t} &\simeq t + \epsilon \xi^1(t, y, u), \\ \bar{y} &\simeq y + \epsilon \xi^2(t, y, u), \\ \bar{u} &\simeq u + \epsilon \eta(t, y, u).\end{aligned}\tag{4.3.2}$$

The related operator is given by

$$X = \xi^1(t, y, u) \frac{\partial}{\partial t} + \xi^2(t, y, u) \frac{\partial}{\partial y} + \eta(t, y, u) \frac{\partial}{\partial u}.\tag{4.3.3}$$

The following invariance criterion must hold for the above operator to be considered as a Lie point symmetry generator of the model equation (4.2.43):

$$X^{[5]}(\text{equation (4.2.43)}) \Big|_{\text{equation (4.2.43)}=0} = 0,\tag{4.3.4}$$

where $X^{[5]}$ is the fifth prolongation of the operator X defined as

$$\begin{aligned}X^{[5]} &= \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} + \zeta^t \frac{\partial}{\partial u_t} + \zeta^y \frac{\partial}{\partial u_y} + \zeta^{tt} \frac{\partial}{\partial u_{tt}} + \zeta^{yy} \frac{\partial}{\partial u_{yy}} \\ &\quad + \zeta^{ty} \frac{\partial}{\partial u_{ty}} + \zeta^{ttt} \frac{\partial}{\partial u_{ttt}} + \zeta^{tyy} \frac{\partial}{\partial u_{tyy}} + \zeta^{ttyy} \frac{\partial}{\partial u_{ttyy}} + \zeta^{tttyy} \frac{\partial}{\partial u_{tttyy}}.\end{aligned}\tag{4.3.5}$$

The coefficient functions of the extended infinitesimals ζ are given by equation (3.3.6) as well as

$$\begin{aligned}
\zeta^{tt} &= D_t \zeta^t - u_{tt} D_t (\xi^1) - u_{ty} D_y (\xi^2), \\
\zeta^{ty} &= D_t \zeta^y - u_{ty} D_t (\xi^1) - u_{yy} D_t (\xi^2), \\
\zeta^{ttt} &= D_t \zeta^{tt} - u_{ttt} D_t (\xi^1) - u_{tty} D_t (\xi^2), \\
\zeta^{ttyy} &= D_t \zeta^{tyy} - u_{ttyy} D_t (\xi^1) - u_{tyyy} D_t (\xi^2), \\
\zeta^{tttyy} &= D_t \zeta^{ttyy} - u_{tttyy} D_t (\xi^1) - u_{ttyyy} D_t (\xi^2).
\end{aligned} \tag{4.3.6}$$

The total derivative operators are the same as those formulated in equation (3.3.7). Acting the fifth prolongation $X^{[5]}$ onto equation (4.2.43) results in the determining equation

$$\begin{aligned}
&\left(\zeta^t - \mu_* \zeta^{yy} - \alpha_* \zeta^{tyy} - \beta_* (u_y)^2 \zeta^{yy} - 2\beta_* u_y u_{yy} \zeta^y - \gamma_* \zeta^{tttyy} \right. \\
&\quad - 2\Gamma_* u_y u_{yy} \zeta^{ty} - 2\Gamma_* u_y u_{ty} \zeta^{yy} - 2\Gamma_* u_{yy} u_{ty} \zeta^y - \Gamma_* (u_y)^2 \zeta^{tyy} \\
&\quad - 2\Gamma_* u_y u_{t yy} \zeta^y + \kappa_1 \eta + \kappa_2 \zeta^{tt} + 2\kappa_3 u u_y \zeta^y + \kappa_3 (u_y)^2 \eta + \kappa_4 \zeta^{ttt} \\
&\quad \left. + \kappa_5 u u_y \zeta^{ty} + \kappa_5 u u_{ty} \zeta^y + \kappa_5 u_y u_{ty} \eta + M_*^2 \eta \right) \Big|_{\text{equation (4.2.43)}=0} = 0.
\end{aligned} \tag{4.3.7}$$

We then substitute the expansions for the coefficient functions defined in equations (3.3.6) and (4.3.6) into the determining equation above. Since the unknown functions ξ^1 , ξ^2 and η are independent of the derivatives of u as well as powers and products of their derivatives, we can separate the resulting determining equation with respect to the powers and products of the derivatives of u . In so doing, we find an overdetermined system of linear homogeneous

partial differential equations given by

$$\xi_u^1 = 0, \quad (4.3.8)$$

$$\xi_y^1 = 0, \quad (4.3.9)$$

$$\xi_u^2 = 0, \quad (4.3.10)$$

$$\xi_y^2 = 0, \quad (4.3.11)$$

$$\xi_t^2 = 0, \quad (4.3.12)$$

$$\eta_{uu} = 0, \quad (4.3.13)$$

$$\eta_y = 0, \quad (4.3.14)$$

$$\xi_t^1 + \eta_u = 0, \quad (4.3.15)$$

$$3\beta_*\xi_t^1 + 2\beta_*\eta_u + 3\Gamma_*\eta_{tu} = 0, \quad (4.3.16)$$

$$\eta + 2u\xi_t^1 + u\eta_u = 0, \quad (4.3.17)$$

$$\beta\xi_t^1 - 3\gamma_*\xi_{tt}^1 + 3\gamma_*\eta_{tu} = 0, \quad (4.3.18)$$

$$\kappa_2\xi_t^1 - 3\kappa_4\xi_{tt}^1 + 3\kappa_4\eta_{tu} = 0, \quad (4.3.19)$$

$$\kappa_3\eta + 3\kappa_3u\xi_t^1 + \kappa_3u\eta_u + \kappa_5u\eta_{tu} = 0, \quad (4.3.20)$$

$$2\alpha_*\xi_t^1 - \beta\xi_{tt}^1 - \gamma_*\xi_{ttt}^1 + 2\beta\eta_{tu} + 3\gamma_*\eta_{ttu} = 0, \quad (4.3.21)$$

$$2\xi_t^1 - \kappa_2\xi_{tt}^1 - \kappa_4\xi_{ttt}^1 + 2\kappa_2\eta_{tu} + 3\kappa_4\eta_{ttu} = 0, \quad (4.3.22)$$

$$3\mu_*\xi_t^1 + \alpha_*\eta_{tu} + \beta\eta_{ttu} + \gamma_*\eta_{tttu} = 0, \quad (4.3.23)$$

$$(\kappa_1 + M_*^2)(\eta + 3u\xi_t^1 - u\eta_u) + \eta_t + \kappa_2\eta_{tt} + \kappa_4\eta_{ttt} = 0. \quad (4.3.24)$$

From equations (4.3.8) and (4.3.9),

$$\xi^1 = a(t), \quad (4.3.25)$$

where a is some arbitrary function of t . From equations (4.3.10), (4.3.11) and (4.3.12),

$$\xi^2 = c_1, \quad (4.3.26)$$

where c_1 is an arbitrary constant. From equations (4.3.13) and (4.3.14),

$$\eta = c(t)u + d(t), \quad (4.3.27)$$

where c and d are arbitrary functions of t . Using the infinitesimals in equations

(4.3.25) and (4.3.27), we can deduce from (4.3.15) that

$$c(t) = -a'(t). \quad (4.3.28)$$

Hence the infinitesimal in equation (4.3.27) becomes,

$$\eta = -a'(t)u + d(t). \quad (4.3.29)$$

Substituting the infinitesimals given in equations (4.3.25) and (4.3.29) into equation (4.3.16) yields

$$a''(t) - \left(\frac{\beta_*}{3\Gamma_*} \right) a'(t) = 0. \quad (4.3.30)$$

This can be rewritten in compact form as

$$\frac{d}{dt} \left[a'(t) - \left(\frac{\beta_*}{3\Gamma_*} \right) a(t) \right] = 0. \quad (4.3.31)$$

We can integrate both sides to get

$$a'(t) - \left(\frac{\beta_*}{3\Gamma_*} \right) a(t) = c_2, \quad (4.3.32)$$

where c_2 is an arbitrary constant of integration. This is now a linear ordinary differential equation in t with integrating factor given by $\exp\left(-\frac{\beta_*}{3\Gamma_*}t\right)$. Solving the ordinary differential equation gives that

$$a(t) = -\left(\frac{3\Gamma_*}{\beta_*} \right) c_2 + c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t \right), \quad (4.3.33)$$

where c_3 is an arbitrary constant. The infinitesimal (4.3.25) now becomes

$$\xi^1 = -\left(\frac{3\Gamma_*}{\beta_*} \right) c_2 + c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t \right) \quad (4.3.34)$$

and the infinitesimal (4.3.29) now becomes

$$\eta = -\left(\frac{\beta_*}{3\Gamma_*} \right) c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t \right) u + d(t). \quad (4.3.35)$$

It only remains to solve for $d(t)$ and by substituting the infinitesimals (4.3.34) and (4.3.35) into equation (4.3.17), it is clear that $d(t) = 0$. Thus the infinitesimals for the partial differential equation (4.2.43) are

$$\begin{aligned}\xi^1 &= -\left(\frac{3\Gamma_*}{\beta_*}\right) c_2 + c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t\right), \\ \xi^2 &= c_1, \\ \eta &= -\left(\frac{\beta_*}{3\Gamma_*}\right) c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u.\end{aligned}\tag{4.3.36}$$

The coefficients of the infinitesimals ξ^1 , ξ^2 and η in equation (4.3.36) found by solving the determining equations are substituted into the Lie point symmetry generator defined in equation (4.3.3) to obtain

$$X = \left[-\left(\frac{3\Gamma_*}{\beta_*}\right) c_2 + c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t\right)\right] \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial y} - \left(\frac{\beta_*}{3\Gamma_*}\right) c_3 \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u}.\tag{4.3.37}$$

By setting each constant $c_i = 1$ with $c_j = 0$ ($i, j = 1, 2, 3$) in the infinitesimal generator in equation (4.3.37), we obtain a three-dimensional Lie algebra which is generated by the operators

$$\begin{aligned}X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right) \frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right) \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u},\end{aligned}\tag{4.3.38}$$

where X_1 is the time-translational symmetry generator, X_2 the space-translational symmetry generator in y and the generator X_3 has path curves equivalent to a combination of translations in t and scaling in u . The remaining seven determining equations (4.3.18)-(4.3.24) give rise to a set of conditions that need to be upheld. Consequently, the Lie point symmetries in equation

(4.3.38) hold provided that

$$6\gamma_* \left(\frac{\beta_*}{3\Gamma_*} \right) - \beta = 0, \quad (4.3.39)$$

$$6\kappa_4 \left(\frac{\beta_*}{3\Gamma_*} \right) - \kappa_2 = 0, \quad (4.3.40)$$

$$\kappa_5 \left(\frac{\beta_*}{3\Gamma_*} \right) - \kappa_3 = 0, \quad (4.3.41)$$

$$4\gamma_* \left(\frac{\beta_*}{3\Gamma_*} \right)^2 + 3\beta \left(\frac{\beta_*}{3\Gamma_*} \right) - 2\alpha_* = 0, \quad (4.3.42)$$

$$4\kappa_4 \left(\frac{\beta_*}{3\Gamma_*} \right)^2 + 3\kappa_2 \left(\frac{\beta_*}{3\Gamma_*} \right) - 2 = 0, \quad (4.3.43)$$

$$\gamma_* \left(\frac{\beta_*}{3\Gamma_*} \right)^3 + \beta \left(\frac{\beta_*}{3\Gamma_*} \right)^2 + \alpha_* \left(\frac{\beta_*}{3\Gamma_*} \right) - 3\mu_* = 0, \quad (4.3.44)$$

$$\kappa_4 \left(\frac{\beta_*}{3\Gamma_*} \right)^3 + \kappa_2 \left(\frac{\beta_*}{3\Gamma_*} \right)^2 + \left(\frac{\beta_*}{3\Gamma_*} \right) - 3(\kappa_1 + M_*^2) = 0. \quad (4.3.45)$$

4.4 Classification of group invariant solutions

In this section, we endeavour to acquire the optimal system in order to classify the group invariant solutions of the partial differential equation (4.2.43).

To begin, we compute the commutators of the three-dimensional Lie algebra in equation (4.3.38). This is done by using the commutation relations set out in equations (3.4.1)-(3.4.3) and this gives the Lie brackets as produced in equations (4.4.1)-(4.4.3) below:

$$\begin{aligned} [X_1, X_2] &= X_1 X_2 - X_2 X_1 \\ &= \frac{\partial}{\partial t} \left(\frac{\partial}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial}{\partial t} \right) \\ &= 0, \end{aligned} \quad (4.4.1)$$

and

$$\begin{aligned}
[X_2, X_3] &= X_2X_3 - X_3X_2 \\
&= \frac{\partial}{\partial y} \left\{ -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right) \frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right) \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u} \right\} \\
&\quad - \left\{ -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right) \frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right) \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u} \right\} \left(\frac{\partial}{\partial y}\right) \\
&= 0,
\end{aligned} \tag{4.4.2}$$

and

$$\begin{aligned}
[X_1, X_3] &= X_1X_3 - X_3X_1 \\
&= \frac{\partial}{\partial t} \left\{ -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right) \frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right) \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u} \right\} \\
&\quad - \left\{ -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right) \frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right) \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u} \right\} \left(\frac{\partial}{\partial t}\right) \\
&= \frac{\beta_*}{3\Gamma_*} \left\{ -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right) \frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right) \exp\left(\frac{\beta_*}{3\Gamma_*}t\right) u \frac{\partial}{\partial u} \right\} \\
&= \frac{\beta_*}{3\Gamma_*} X_3.
\end{aligned} \tag{4.4.3}$$

The commutator table is shown in Table 4.4.1 below:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{\beta_*}{3\Gamma_*} X_3$
X_2	0	0	0
X_3	$-\frac{\beta_*}{3\Gamma_*} X_3$	0	0

Table 4.4.1: Commutator table of the three-dimensional Lie algebra (4.3.38)

We then need to work out the adjoint representations. This is done by making use of the Lie series stipulated in equations (3.4.7) and (3.4.8) as well

as the commutator table in Table 4.4.1 above. For instance,

$$\begin{aligned}
\text{Ad}(\exp(\epsilon X_1))X_3 &= X_3 - \epsilon[X_1, X_3] + \frac{1}{2!}\epsilon^2[X_1, [X_1, X_3]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_1, [X_1, [X_1, X_3]]] + \dots \\
&= X_3 - \epsilon\frac{\beta_*}{3\Gamma_*}X_3 + \frac{1}{2!}\left(\epsilon\frac{\beta_*}{3\Gamma_*}\right)^2 X_3 \\
&\quad - \frac{1}{3!}\left(\epsilon\frac{\beta_*}{3\Gamma_*}\right)^2 X_3 + \dots \\
&= \exp\left(-\epsilon\frac{\beta_*}{3\Gamma_*}\right)X_3,
\end{aligned} \tag{4.4.4}$$

and

$$\begin{aligned}
\text{Ad}(\exp(\epsilon X_3))X_1 &= X_1 - \epsilon[X_3, X_1] + \frac{1}{2!}\epsilon^2[X_3, [X_3, X_1]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_3, [X_3, [X_3, X_1]]] + \dots \\
&= X_1 + \epsilon\frac{\beta_*}{3\Gamma_*}X_3.
\end{aligned} \tag{4.4.5}$$

In a similar manner, the remainder of the adjoint representations are found and they are indicated in Table 4.4.2 below:

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$\exp\left(-\epsilon\frac{\beta_*}{3\Gamma_*}\right)X_3$
X_2	X_1	X_2	X_3
X_3	$X_1 + \epsilon\frac{\beta_*}{3\Gamma_*}X_3$	X_2	X_3

Table 4.4.2: Adjoint table for the symmetry generators (4.3.38)

For a non-zero vector

$$X = a_1X_1 + a_2X_2 + a_3X_3, \quad a_i \in \mathbb{R}, \tag{4.4.6}$$

the coefficients a_i need to be simplified as much as possible by applying adjoint

maps to X . First off, we act on X by $\text{Ad}(\exp(\epsilon X_3))$ by implementing Table 4.4.2 and we get

$$X = a_1 X_1 + a_2 X_2 + \left(\epsilon \frac{\beta_*}{3\Gamma_*} a_1 + a_3 \right) X_3, \quad (4.4.7)$$

Now suppose that $a_1 \neq 0$. By scaling X , we can assume that $a_1 = 1$ and so

$$X = X_1 + a_2 X_2 + \left(\epsilon \frac{\beta_*}{3\Gamma_*} + a_3 \right) X_3. \quad (4.4.8)$$

We can then get the coefficient a_3 to vanish by making $\epsilon = -3a_3\Gamma_*/\beta_*$ and as such

$$X = X_1 + a_2 X_2. \quad (4.4.9)$$

However, if $a_1 = 0$, then the vector (4.4.6) becomes

$$X = a_2 X_2 + a_3 X_3. \quad (4.4.10)$$

If we now consider $a_2 \neq 0$, then by scaling $a_2 = 1$, the vector X above becomes

$$X = X_2 + a_3 X_3. \quad (4.4.11)$$

By applying $\text{Ad}(\exp(\epsilon X_1))$ onto the vector X in equation (4.4.11), we have that

$$X = X_2 + a_3 \exp\left(-\epsilon \frac{\beta_*}{3\Gamma_*}\right) X_3. \quad (4.4.12)$$

If $a_3 \neq 0$, it follows that either $a_3 > 0$ or $a_3 < 0$. As a consequence,

$$X = X_2 \pm X_3. \quad (4.4.13)$$

On the other hand, if $a_3 = 0$, the vector X in equation (4.4.11) becomes

$$X = X_2. \quad (4.4.14)$$

Finally, if $a_2 = 0$, then

$$X = X_3. \quad (4.4.15)$$

To summarize, we have obtained the optimal system of one-dimensional sub-

algebras of the Lie algebra (4.3.38) spanned by

$$\begin{aligned} X_2, \\ X_3, \\ X_2 \pm X_3, \\ X_1 + aX_2. \end{aligned} \tag{4.4.16}$$

4.5 Group invariant and numerical solutions to the problem

In this section, we strive to obtain group invariant solutions to the governing model (4.2.43) by using the techniques as set out in Section 2.2 as well as the findings of the calculation of the optimal system (4.4.16).

4.5.1 Invariant solution via subgroup generated by X_3

We consider the invariant solutions under the operator

$$X_3 = -\exp\left(\frac{\beta_*}{3\Gamma_*}t\right)\frac{\partial}{\partial t} + \left(\frac{\beta_*}{3\Gamma_*}\right)\exp\left(\frac{\beta_*}{3\Gamma_*}t\right)u\frac{\partial}{\partial u}. \tag{4.5.1}$$

The Lagrangian system provides the corresponding differential equations of characteristic curves which are given by

$$\frac{dt}{-\exp\left(\frac{\beta_*}{3\Gamma_*}t\right)} = \frac{dy}{0} = \frac{du}{\left(\frac{\beta_*}{3\Gamma_*}\right)\exp\left(\frac{\beta_*}{3\Gamma_*}t\right)u}. \tag{4.5.2}$$

It is clear that y is invariant and as such

$$y = k_1, \tag{4.5.3}$$

where k_1 is constant. The first and last terms of the characteristic system in equation (4.5.2) can be written as a variable separable ordinary differential equation given by

$$\frac{du}{u} = -\left(\frac{\beta_*}{3\Gamma_*}\right)dt. \tag{4.5.4}$$

Integrating both sides of equation (4.5.4) gives

$$\begin{aligned}\ln u &= -\left(\frac{\beta_*}{3\Gamma_*}\right)t + W(y), \\ u(y, t) &= W(y) \exp\left(-\frac{\beta_*}{3\Gamma_*}t\right),\end{aligned}\quad (4.5.5)$$

which is the form of the invariant solution where $W(y)$ is an as yet to be determined function.

We now substitute equation (4.5.5) into the governing fifth-order non-linear partial differential equation (4.2.43) and simplify using the condition stipulated by equation (4.3.41) to obtain the ordinary differential equation for $W(y)$:

$$\frac{d^2W}{dy^2} - \left[\frac{\kappa_1 + M_*^2 - \left(\frac{\beta_*}{3\Gamma_*}\right) + \kappa_2 \left(\frac{\beta_*}{3\Gamma_*}\right)^2 - \kappa_4 \left(\frac{\beta_*}{3\Gamma_*}\right)^3}{\mu_* - \alpha_* \left(\frac{\beta_*}{3\Gamma_*}\right) + \beta \left(\frac{\beta_*}{3\Gamma_*}\right)^2 - \gamma_* \left(\frac{\beta_*}{3\Gamma_*}\right)^3} \right] W = 0. \quad (4.5.6)$$

The form of this ordinary differential equation can be further reduced by implementing the conditions on the Lie point symmetries given by equations (4.3.39)-(4.3.45). From equation (4.3.45),

$$-\kappa_4 \left(\frac{\beta_*}{3\Gamma_*}\right)^3 - \left(\frac{\beta_*}{3\Gamma_*}\right) = \kappa_2 \left(\frac{\beta_*}{3\Gamma_*}\right)^2 - 3(\kappa_1 + M_*^2). \quad (4.5.7)$$

Substituting equation (4.5.7) in equation (4.5.6) gives

$$\frac{d^2W}{dy^2} - \left[\frac{-2 \left(\kappa_1 + M_*^2 - \kappa_2 \left(\frac{\beta_*}{3\Gamma_*}\right)^2 \right)}{\mu_* - \alpha_* \left(\frac{\beta_*}{3\Gamma_*}\right) + \beta \left(\frac{\beta_*}{3\Gamma_*}\right)^2 - \gamma_* \left(\frac{\beta_*}{3\Gamma_*}\right)^3} \right] W = 0. \quad (4.5.8)$$

From equation (4.3.41),

$$\frac{\beta_*}{3\Gamma_*} = \frac{\kappa_3}{\kappa_5}. \quad (4.5.9)$$

Substituting equation (4.5.9) in equation (4.5.8) gives

$$\frac{d^2W}{dy^2} - \left[\frac{-2 \left(\kappa_1 + M_*^2 - \kappa_2 \left(\frac{\kappa_3}{\kappa_5} \right)^2 \right)}{\mu_* - \alpha_* \left(\frac{\beta_*}{3\Gamma_*} \right) + \beta \left(\frac{\beta_*}{3\Gamma_*} \right)^2 - \gamma_* \left(\frac{\beta_*}{3\Gamma_*} \right)^3} \right] W = 0. \quad (4.5.10)$$

From equation (4.3.44),

$$-\gamma_* \left(\frac{\beta_*}{3\Gamma_*} \right)^3 - \alpha_* \left(\frac{\beta_*}{3\Gamma_*} \right) = \beta \left(\frac{\beta_*}{3\Gamma_*} \right)^2 - 3\mu_*. \quad (4.5.11)$$

Substituting equation (4.5.11) in equation (4.5.10) gives

$$\frac{d^2W}{dy^2} - \left[\frac{-2 \left(\kappa_1 + M_*^2 - \kappa_2 \left(\frac{\kappa_3}{\kappa_5} \right)^2 \right)}{-2 \left(\mu_* - \beta \left(\frac{\beta_*}{3\Gamma_*} \right)^2 \right)} \right] W = 0. \quad (4.5.12)$$

Finally, from equation (4.3.39),

$$\frac{\beta_*}{3\Gamma_*} = \frac{\beta}{6\gamma_*}. \quad (4.5.13)$$

Substituting equation (4.5.13) in equation (4.5.12) gives the final reduced second-order ordinary differential equation for $W(y)$ given by

$$\frac{d^2W}{dy^2} - \left[\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)} \right] W = 0. \quad (4.5.14)$$

Furthermore, the boundary conditions in equations (4.2.44) and (4.2.45) can be rewritten for the reduced ordinary differential equation (4.5.14) using equation (4.5.5) as

$$W(0) = 1, \quad (4.5.15)$$

$$W(y) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \quad (4.5.16)$$

where

$$V(t) = \exp\left(-\frac{\beta_*}{3\Gamma_*}t\right). \quad (4.5.17)$$

The above ordinary differential equation (4.5.14) is solved for the three following cases:

$$\textbf{Case 1:} \left[\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)} \right] = 0.$$

It follows that

$$\frac{d^2W}{dy^2} = 0. \quad (4.5.18)$$

The solution of equation (4.5.18) subject to the boundary conditions in equations (4.5.15) and (4.5.16) is

$$W(y) = C, \quad (4.5.19)$$

where C is a constant. According to equation (4.5.5), the solution for $u(y, t)$ in this case can be written as

$$u(y, t) = C \exp\left(-\frac{\beta_*}{3\Gamma_*}t\right). \quad (4.5.20)$$

Thus, it is evident from equation (4.5.20) that a time-dependent solution has been obtained.

$$\textbf{Case 2:} \left[\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)} \right] < 0.$$

In this case, the solution for $W(y)$ can be written as

$$W(y) = a \cos \left[\sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}} y \right] + b \sin \left[\sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}} y \right]. \quad (4.5.21)$$

We get from the first boundary condition (4.5.15) that $b = 0$ but since the second boundary condition (4.5.16) states that $W(\infty) = 0$, we obtain an unbounded solution for $W(y)$. Hence, no solution exists for $u(y, t)$ in this case.

$$\text{Case 3: } \left[\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)} \right] > 0.$$

By its characteristic equation, the solution for $W(y)$ is written in the form

$$W(y) = a \exp \left[\sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)}} y \right] + b \exp \left[- \sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)}} y \right]. \quad (4.5.22)$$

By applying the first boundary condition (4.5.15), we have that $a + b = 1$. The second boundary condition (4.5.16) requires $a = 0$ for a bounded solution and so it follows that $b = 1$. Thus, the solution for $W(y)$ is

$$W(y) = \exp \left[- \sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)}} y \right], \quad (4.5.23)$$

and we obtain a solution for $u(y, t)$ of the form

$$u(y, t) = \exp \left[- \sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)}} y \right] \exp \left(- \frac{\beta_*}{3\Gamma_*} t \right),$$

$$u(y, t) = \exp \left[- \left\{ \left(\frac{\beta_*}{3\Gamma_*} t \right) + \sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2 \kappa_3^2}{\kappa_5^2} \right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2} \right)}} y \right\} \right]. \quad (4.5.24)$$

Furthermore, the solution of $u(y, t)$ in equation (4.5.24) above satisfies the

boundary condition (4.2.44) and initial conditions (4.2.46)-(4.2.48) with

$$\begin{aligned}
 V(t) &= \exp\left(-\frac{\beta_*}{3\Gamma_*}t\right), \\
 F(y) &= \exp\left(-\sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2\kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}y\right), \\
 G(y) &= -\left(\frac{\beta_*}{3\Gamma_*}\right)\exp\left(-\sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2\kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}y\right), \\
 H(y) &= \left(\frac{\beta_*}{3\Gamma_*}\right)^2\exp\left(-\sqrt{\frac{\left(\kappa_1 + M_*^2 - \frac{\kappa_2\kappa_3^2}{\kappa_5^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}y\right), \quad (4.5.25)
 \end{aligned}$$

where $V(t)$, $F(y)$, $G(y)$ and $H(y)$ are all dependent on physical parameters of the fluid flow.

The group invariant solution (4.5.24) is plotted for various parameters that are pertinent to the fourth grade fluid model and these are successively illustrated in Figures 4.6.1-4.6.4.

4.5.2 Steady state solution

Here, we consider the invariant solutions under the time-translational symmetry generator

$$X_1 = \frac{\partial}{\partial t}, \quad (4.5.26)$$

which has a characteristic system given by

$$\frac{dt}{1} = \frac{dy}{0} = \frac{du}{0}. \quad (4.5.27)$$

We see that both y and u are invariant and as a result

$$y = k_1 \quad \text{and} \quad u = k_2, \quad (4.5.28)$$

where k_1 and k_2 are constants. Writing $k_2 = J(k_1)$, we find that the invariant solution admitted by the generator X_1 is the steady-state solution

$$u(y, t) = J(y). \quad (4.5.29)$$

Implementing the invariant solution in equation (4.5.29) into the governing equation (4.2.43) results in the following second-order non-linear ordinary differential equation in $J(y)$:

$$\mu_* \frac{d^2 J}{dy^2} + \beta_* \left(\frac{dJ}{dy} \right)^2 \frac{d^2 J}{dy^2} - \kappa_3 J \left(\frac{dJ}{dy} \right)^2 - (\kappa_1 + M_*^2) J = 0. \quad (4.5.30)$$

The resulting invariant boundary conditions are obtained by applying the invariant solution in equation (4.5.29) onto equations (4.2.44) and (4.2.45) to get

$$J(0) = 1, \quad (4.5.31)$$

$$J(y) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \quad (4.5.32)$$

where $V = 1$.

We use the ansatz method to solve the above boundary value problem for $J(y)$ by assuming a solution of the form

$$J(y) = Ae^{By}, \quad (4.5.33)$$

where $A = B \neq 0$ are yet to be determined constants. The substitution of this ansatz (4.5.33) into equation (4.5.30) gives

$$\mu_* AB^2 e^{By} + \beta_* A^3 B^4 e^{3By} - \kappa_3 A^3 B^2 e^{3By} - (\kappa_1 + M_*^2) A e^{By} = 0. \quad (4.5.34)$$

We can separate equation (4.5.34) in terms of powers of e^{By} and e^{3By} to obtain

$$e^{3By} : \quad \beta_* A^3 B^4 - \kappa_3 A^3 B^2 = 0, \quad (4.5.35)$$

$$e^{By} : \quad \mu_* AB^2 - (\kappa_1 + M_*^2) A = 0. \quad (4.5.36)$$

From equation (4.5.35), we have that

$$A^3 B^2 (\beta_* B^2 - \kappa_3) = 0, \quad (4.5.37)$$

and since $A^3 \neq 0$ and $B^2 \neq 0$, it follows from equation (4.5.37) that

$$B = \pm \sqrt{\frac{\kappa_3}{\beta_*}}. \quad (4.5.38)$$

However, we choose

$$B = -\sqrt{\frac{\kappa_3}{\beta_*}}, \quad (4.5.39)$$

since we require the solution to be zero at infinity due to the second boundary condition (4.5.32). Substituting the obtained function for B from equation (4.5.39) into equation (4.5.36) gives that

$$A \left(\mu_* \left(\frac{\kappa_3}{\beta_*} \right) - (\kappa_1 + M_*^2) \right) = 0, \quad (4.5.40)$$

and since $A \neq 0$, it follows from equation (4.5.40) that

$$\mu_* \left(\frac{\kappa_3}{\beta_*} \right) - (\kappa_1 + M_*^2) = 0. \quad (4.5.41)$$

Thus, the exact solution for $J(y)$ is of the form

$$J(y) = A \exp \left(-\sqrt{\frac{\kappa_3}{\beta_*}} y \right). \quad (4.5.42)$$

Applying the first boundary condition (4.5.31) gives that $A = 1$ and so the steady-state solution is given by

$$u = \exp \left(-\sqrt{\frac{\kappa_3}{\beta_*}} y \right), \quad (4.5.43)$$

provided that equation (4.5.41) holds.

We can also use the *Mathematica* solver **NDSolve** to obtain the numerical solution of the ordinary differential equation (4.5.30) subject to the boundary conditions (4.5.31) and (4.5.32). This is implemented in order to observe the influence of the applied magnetic field parameter M_* and the resultant plot is

included in Figure 4.6.5.

4.5.3 Travelling wave solution

We can acquire a travelling wave solution for the governing model equation (4.2.43) since it admits both a time and space-translational symmetry generator. For this reason, we can search for an invariant solution under the operator

$$\begin{aligned} X_c &= X_1 + cX_2, \quad c > 0, \\ X_c &= \frac{\partial}{\partial t} + c \frac{\partial}{\partial y}. \end{aligned} \quad (4.5.44)$$

This denotes a wave-front type travelling wave solution and since the wave is of the forward type, it is extending away from the plate with a constant wave speed c . The characteristic system of the operator X_c in equation (4.5.44) is given by

$$\frac{dt}{1} = \frac{dy}{c} = \frac{du}{0}. \quad (4.5.45)$$

It is evident that u is invariant and so

$$u = k_1, \quad (4.5.46)$$

where k_1 is a constant. Solving the first two terms of the characteristic system in equation (4.5.45) gives a variable separable ordinary differential equation given by

$$cdt = dy, \quad (4.5.47)$$

and by integrating, we obtain the second invariant given by

$$y - ct = k_2, \quad (4.5.48)$$

where k_2 is a constant of integration. We can write one invariant in terms of the other as $k_1 = K(k_2)$, and so we have that the invariant solution admitted by the operator X_c is

$$u(y, t) = K(\psi), \quad \text{where } \psi = y - ct. \quad (4.5.49)$$

Acting the invariant solution in equation (4.5.49) above onto the govern-

ing equation (4.2.43) results in a fifth-order non-linear ordinary differential equation for $K(\psi)$ given by

$$\begin{aligned}
0 = & c \frac{dK}{d\psi} + \mu_* \frac{d^2 K}{d\psi^2} - \alpha_* c \frac{d^3 K}{d\psi^3} + \beta c^2 \frac{d^4 K}{d\psi^4} + \beta_* \left(\frac{dK}{d\psi} \right)^2 \frac{d^2 K}{d\psi^2} - \gamma_* c^3 \frac{d^5 K}{d\psi^5} \\
& - \Gamma_* c \left(\frac{dK}{d\psi} \right)^2 \frac{d^3 K}{d\psi^3} - 2\Gamma_* c \frac{dK}{d\psi} \left(\frac{d^2 K}{d\psi^2} \right)^2 - \kappa_1 K - \kappa_2 c^2 \frac{d^2 K}{d\psi^2} - \kappa_3 K \left(\frac{dK}{d\psi} \right)^2 \\
& + \kappa_4 c^3 \frac{d^3 K}{d\psi^3} + \kappa_5 c K \left(\frac{dK}{d\psi} \right) \frac{d^2 K}{d\psi^2} - M_*^2 K.
\end{aligned} \tag{4.5.50}$$

Further, applying equation (4.5.49) onto equations (4.2.44)-(4.2.48) results in the transformed boundary conditions

$$K(0) = v_0, \tag{4.5.51}$$

$$K(\psi) \rightarrow 0 \quad \text{as} \quad \psi \rightarrow \infty, \tag{4.5.52}$$

$$\frac{d^n K}{d\psi^n} \rightarrow 0 \quad \text{as} \quad \psi \rightarrow \infty, \quad \text{for} \quad n = 1, 2, 3, \tag{4.5.53}$$

where $V = v_0$ is a constant and ψ can take on as sufficiently large a value as possible.

The ansatz method was applied to the above boundary value problem in $K(\psi)$ but we failed to obtain a closed-form solution since the equation in (4.5.50) is non-linear and complicated in nature.

However, we can endeavour to solve the ordinary differential equation (4.5.50) numerically, subject to the boundary conditions (4.5.51)-(4.5.53). We include the plot of the obtained numerical solution in Figure 4.6.6.

4.5.4 Numerical solution of the governing model

Here, the numerical solution of the governing partial differential equation (4.2.43) is presented subject to the appropriately chosen initial and bound-

ary conditions stipulated by

$$u(0, t) = V(t) = e^{-t}, \quad t > 0, \quad (4.5.54)$$

$$u(\infty, t) = 0, \quad t > 0, \quad (4.5.55)$$

$$u(y, 0) = F(y) = e^{-y^2}, \quad y > 0, \quad (4.5.56)$$

$$\frac{\partial u}{\partial t}(y, 0) = G(y) = e^{-y}, \quad y > 0, \quad (4.5.57)$$

$$\frac{\partial^2 u}{\partial t^2}(y, 0) = H(y) = e^{-2y}, \quad y > 0. \quad (4.5.58)$$

This is solved using the *Mathematica* solver **NDSolve** with the result plotted in Figure 4.6.7.

4.6 Graphical analysis

The acquired velocity profiles from the applicable sections are contained here-with by means of graphical plots versus y and these are demonstrated in Figures 4.6.1-4.6.6. The objective of such an enterprise is to study the behaviour of a number of meaningful parameters relative to fourth grade fluid flow on the structure of the velocity field. In so doing, we would like to make some inferences and observations with regards to their physical significance for the fourth grade fluid flow model.

We begin by plotting the analytical solution (4.5.24) in Figures 4.6.1-4.6.4 as we wish to study the influence of the magnetic field on the fluid flow. Figure 4.6.1 depicts the effects of time t on the velocity profiles and we examine various values of t while fixing the values of all other parameters. We note that the velocity decreases by increasing the value of t . Furthermore, the effects of the applied magnetic field are studied by plotting the velocity profiles of varying values of the magnetic field parameter M_* in Figure 4.6.2. Based on the outcome, we can reason that the velocity decreases when a greater magnetic field is applied. The final effects we wish to study are those of the fourth grade parameters β_* and Γ_* . These are carried out in the presence of an applied magnetic field and can be viewed in Figures 4.6.3 and 4.6.4 respectively. Figure 4.6.3 establishes that increasing the fourth grade parameter β_* causes the velocity profile to decrease. This exhibiting behaviour leads us to conclude that β_* is responsible for encapsulating the effects of shear thickening of the

fluid. The fourth grade parameter Γ_* has the opposite effect: the velocity profile increases with an increase of the parameter Γ_* . This behaviour can be inferred from Figure 4.6.4 and as such, we can say that Γ_* portrays the shear thinning property. At this point, it is clearly evident that the observations of both parameters agree with their physical significance to the fourth grade fluid flow model. Moreover, they have an opposite effect on the form of the velocity field despite the fact that a magnetic field is being applied.

The steady state solution results in a second-order ordinary differential equation (4.5.30). This was solved numerically subject to the boundary conditions (4.5.31) and (4.5.32) and the results are shown in Figure 4.6.5. The objective was to observe the effects of the applied magnetic field and hence the plots of velocity profiles for changing values of the magnetic field parameter M_* are shown. It can be seen that the velocity profiles decrease in the presence of an increasing magnetic field and this statement concurs with the aforementioned effects obtained from the analytical results.

Next, we attempt to obtain numerical solutions of the fifth-order non-linear ordinary differential equation (4.5.50) since an exact solution was not found. This is solved numerically subject to the boundary conditions (4.5.51)-(4.5.53) and the results are plotted in Figure 4.6.6. Again, we wish to directly observe the behaviour of the magnetic field parameter M_* and this is done by plotting the velocity profiles of various values. The output is unsurprising since it verifies the results obtained from the closed-form solution (4.5.24).

As a final point, we plot the numerical solution of the governing partial differential equation (4.2.43) in Figures 4.6.7 and 4.6.8 subject to the specific initial and boundary conditions (4.5.54)-(4.5.58) for varying values of time t and the magnetic field parameter M_* respectively. It is visible from Figure 4.6.7 that the velocity profile oscillates with an change in time. Figure 4.6.8 depicts that an increased applied magnetic field causes the velocity field to decrease which is consistent with the analytical results of the closed-form solution (4.5.24).

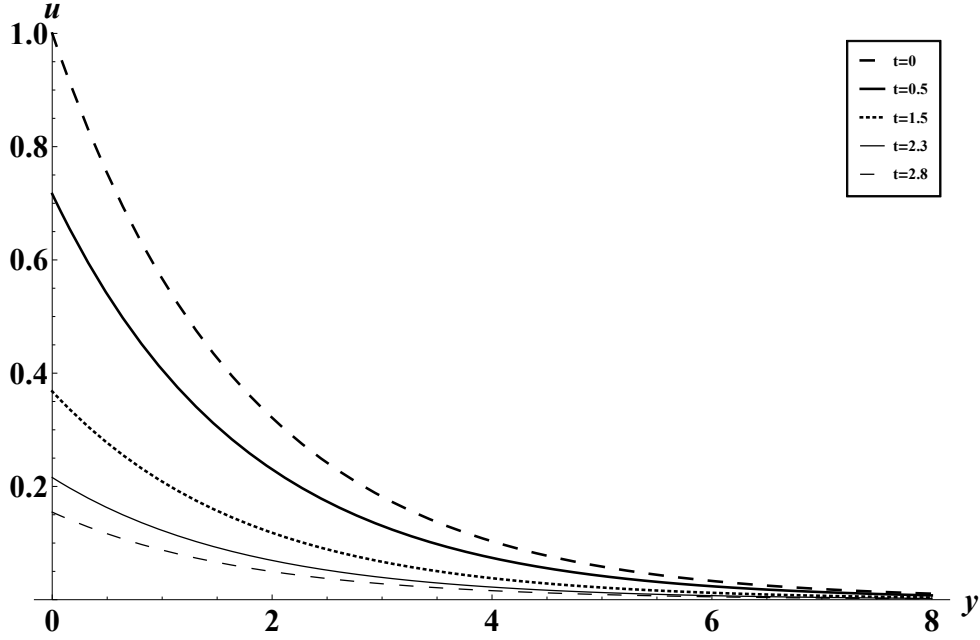


Figure 4.6.1: Velocity field of the analytical solution (4.5.24) for varying values of time t , where $\beta_* = 1$, $\beta = 1.5$, $\gamma_* = 0.5$, $\Gamma_* = 1$, $M_* = 0.5$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.

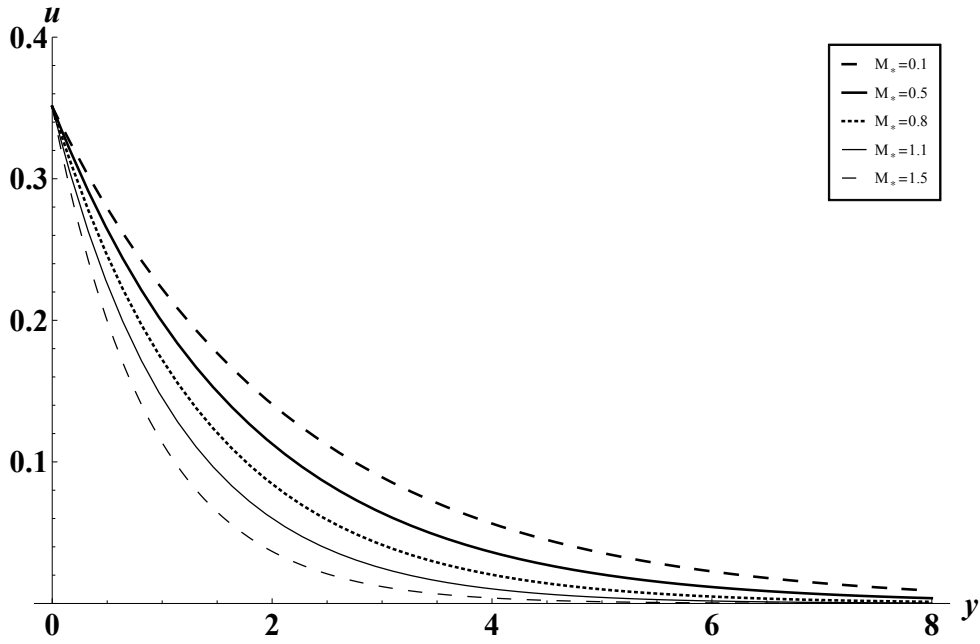


Figure 4.6.2: Velocity field of the analytical solution (4.5.24) for varying values of the magnetic field parameter M_* , where $t = \pi/2$, $\beta_* = 1$, $\beta = 1.5$, $\gamma_* = 0.5$, $\Gamma_* = 1$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.

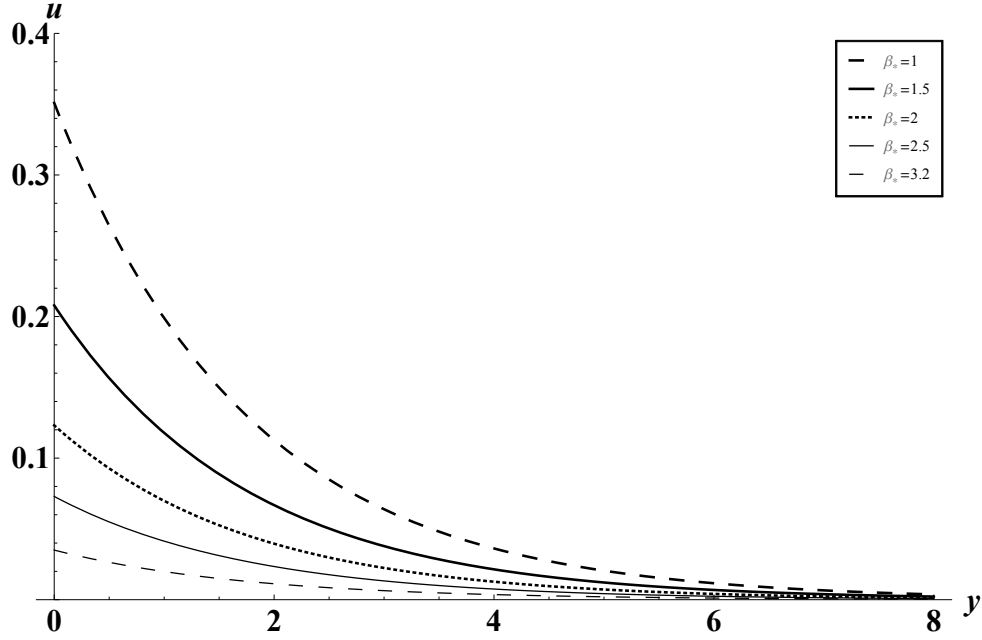


Figure 4.6.3: Velocity field of the analytical solution (4.5.24) for varying values of the fourth grade fluid parameter β_* , where $t = \pi/2$, $\beta = 1.5$, $\gamma_* = 0.5$, $\Gamma_* = 1$, $M_* = 0.5$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.

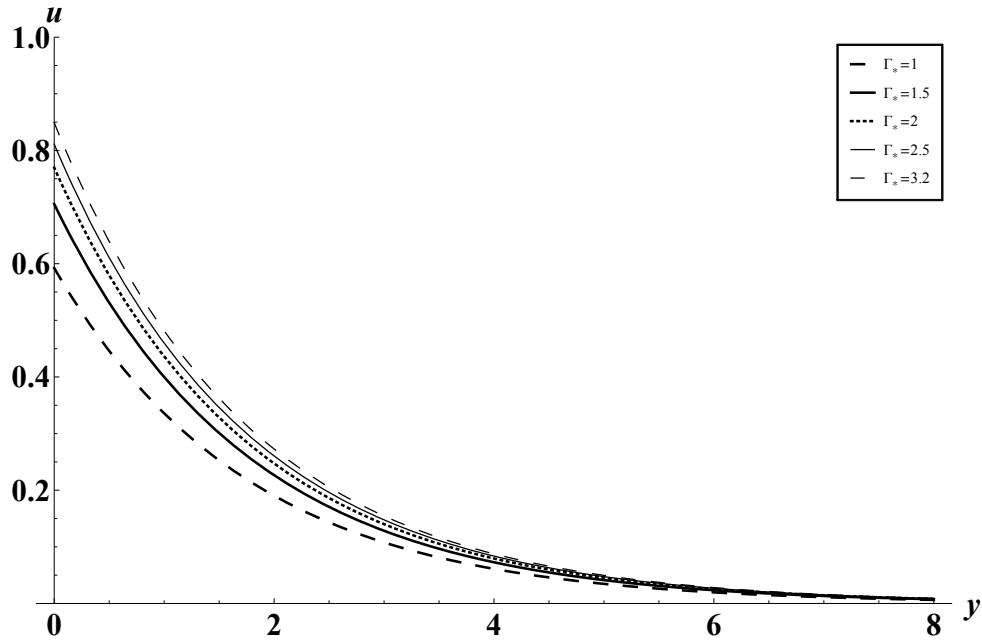


Figure 4.6.4: Velocity field of the analytical solution (4.5.24) for varying values of the fourth grade fluid parameter Γ_* , where $t = \pi/2$, $\beta_* = 1$, $\beta = 1.5$, $\gamma_* = 0.5$, $M_* = 0.5$, $\mu_* = 2.2$, $\kappa_1 = 1$, $\kappa_2 = 1.1$, $\kappa_3 = 1.8$, $\kappa_5 = 2.5$ are fixed.

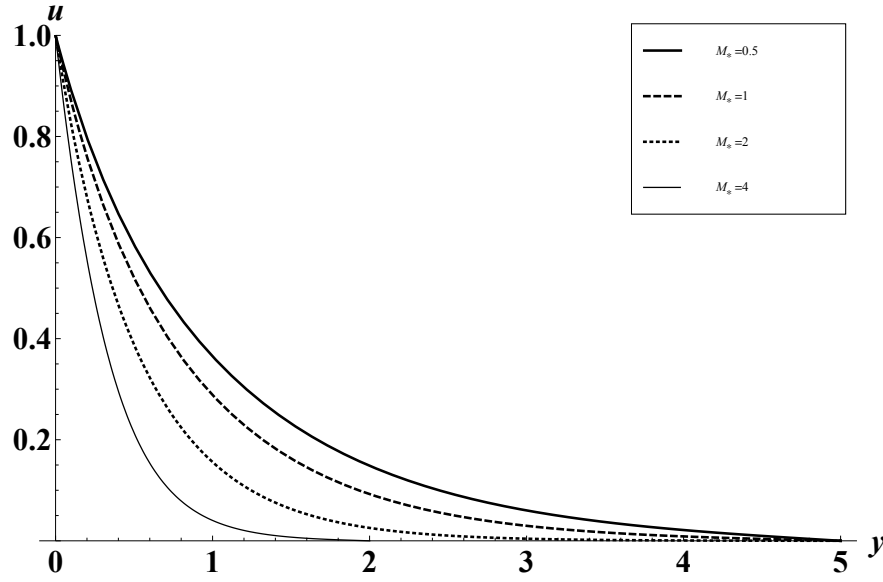


Figure 4.6.5: Numerical solution of the reduced ordinary differential equation (4.5.30) subject to the boundary conditions (4.5.31) and (4.5.32) for varying values of the magnetic field parameter M_* , where $\mu_* = 1.5$, $\beta_* = 0.5$, $\kappa_1 = 0.9$, $\kappa_3 = 2.5$ are fixed.

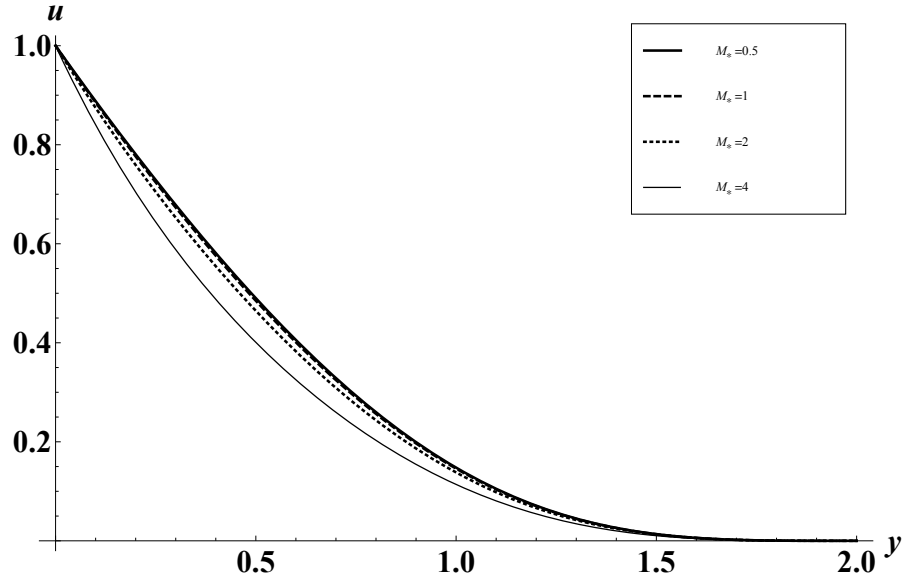


Figure 4.6.6: Numerical solution of the reduced ordinary differential equation (4.5.50) subject to the boundary conditions (4.5.51)-(4.5.53) for varying values of the magnetic field parameter M_* , where $c = 0.5$, $\mu_* = 1$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 0.8$, $\gamma_* = 1$, $\Gamma_* = 0.1$, $\kappa_1 = 0.2$, $\kappa_2 = 2$, $\kappa_3 = 1$, $\kappa_4 = 0.8$, $\kappa_5 = 0.5$ are fixed.

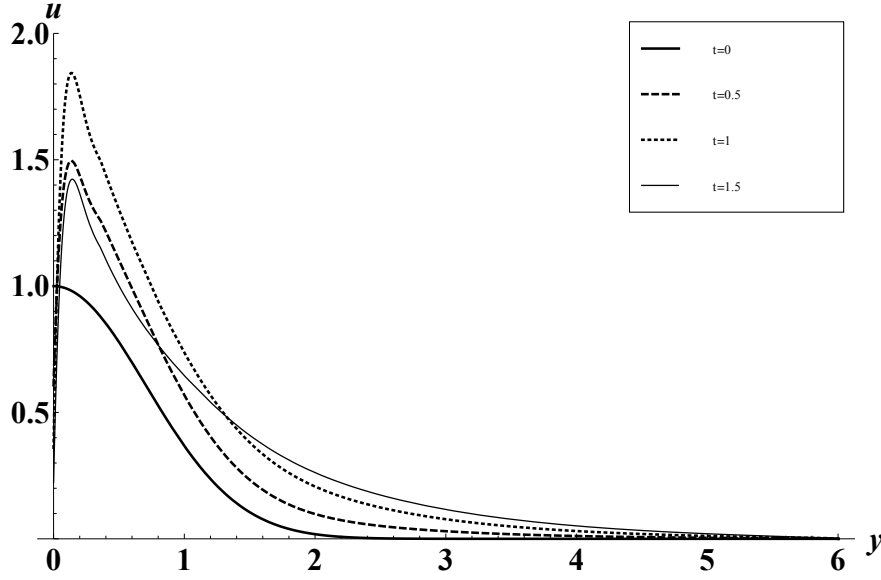


Figure 4.6.7: Numerical solution of the partial differential equation (4.2.43) subject to conditions (4.5.54)-(4.5.58) for varying values of time t , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\gamma_* = 0.8$, $\Gamma_* = 0.5$, $\kappa_1 = 1.2$, $\kappa_2 = 0.2$, $\kappa_3 = 1.9$, $\kappa_4 = 1$, $\kappa_5 = 2.2$, $M_* = 0.5$ are fixed.

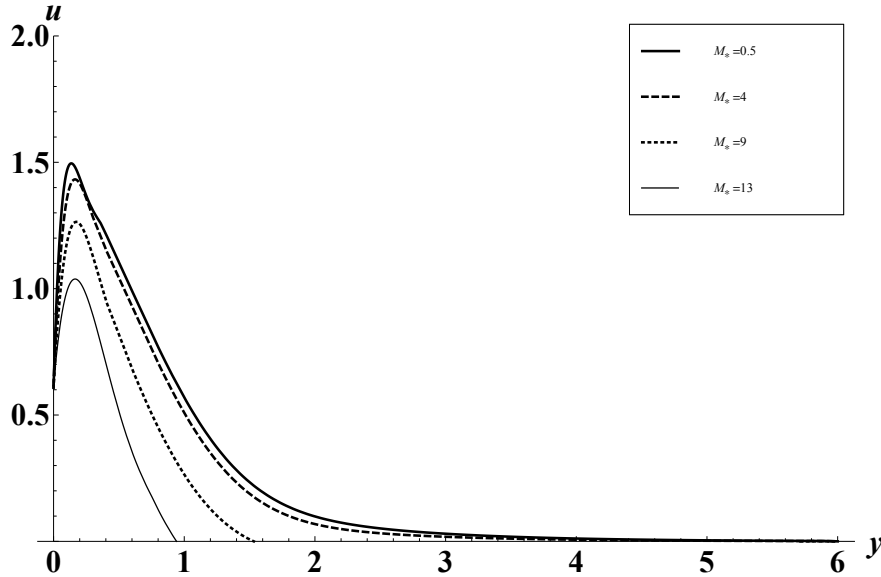


Figure 4.6.8: Numerical solution of the partial differential equation (4.2.43) subject to conditions (4.5.54)-(4.5.58) for varying values of time M_* , where $\mu_* = 2.5$, $\alpha_* = 2$, $\beta = 1.5$, $\beta_* = 2.6$, $\gamma_* = 0.8$, $\Gamma_* = 0.5$, $\kappa_1 = 1.2$, $\kappa_2 = 0.2$, $\kappa_3 = 1.9$, $\kappa_4 = 1$, $\kappa_5 = 2.2$, $t = 0.5$ are fixed.

It is abundantly clear through the analysis that the invariant solution generated by X_3 is the most convincing solution. This is due to the fact that it includes the magnetic field term M_* directly unlike the steady state solution which indirectly takes the magnetic field term into account through its imposing condition. Otherwise stated, we cannot precisely observe the effects of the applied magnetic field on the velocity profile of the flow of fluid since the solution only holds true for specific values of the magnetic field term.

4.7 Conclusion

This chapter dealt with developing a model for the unsteady non-linear fluid flow of a non-Newtonian fluid of the fourth grade as well as attempting to find closed-form solutions for the velocity of the fluid.

The geometry of the model that was studied was identical to that of Chapter 3 except that we were concerned with a fourth grade non-Newtonian fluid. A fifth-order non-linear partial differential equation was obtained as the governing equation when the relevant fluid flow equations were implemented.

A classical Lie point symmetry approach was carried out on this governing model which resulted in a three-dimensional Lie algebra along with a set of imposed conditions on the physical parameters of the problem.

We then classified the group invariant solutions of this Lie algebra by means of the so-called optimal system.

By using the Lie point symmetries to find group invariant solutions, the governing fifth-order partial differential equation was reduced to various ordinary differential equations. A number of techniques were used to solve these equations in the hope of finding an analytical solution and two such solutions were found. The travelling wave solution resulted in a fifth-order non-linear ordinary differential equation that we could not solve analytically and so we resorted to numerical methods in order to find its associated velocity field for the problem. We also specified a numerical solution to the model equation for a particular choice of boundary and initial conditions. Based on the findings of the optimal system, it is possible to find other important analytical solutions to the governing model. In particular, we could use the linear combinations of $X_2 \pm X_3$ to find other viable solutions.

We then used numerous plots to explain the behaviour and effects of a certain number of physical parameters on the velocity profile of the fluid flow. We discovered that the group invariant solution corresponding to the generator X_3 was the most practical solution since it directly took into consideration the effects of the applied magnetic field as well as other pertinent parameters.

Chapter 5

Concluding remarks

5.1 Conclusions

This dissertation has been concerned with gaining an understanding of the behaviour of non-Newtonian fluids. This has been achieved by developing mathematical models for two specific types of problems that occur in the field of non-Newtonian fluid flow, which at its core, focus on problems that exhibit an applied magnetic field. We reviewed the study of the unsteady flow of a third grade fluid in a porous medium in Chapter 3 as proposed by Aziz et al. In addition, we presented the calculation of the optimal system which was not considered by the authors. The unsteady flow of a fourth grade fluid in a porous medium was studied in Chapter 4. Here, we presented new analytical solutions for this particular fluid model.

In both cases we undertook the classical Lie symmetry approach. This helped in reducing the higher-order non-linear partial differential equations to possibly solvable ordinary differential equations. In some cases, we were able to find closed-form solutions for the velocity of the fluid using a variety of methods. We also found numerical solutions to the governing equations using the *Mathematica* solver **NDSolve**.

We undertook a graphical analysis of both the analytical and numerical approaches in order to observe the effects of, amongst others, the magnetic field parameter and the non-Newtonian fluid parameters. For both fluid models, we singled out the invariant solutions of the non-travelling wave type and considered them to be the most significant since they directly took the effects of the

applied magnetic field into account. We also observed that certain parameters exhibited the effects of shear thinning and thickening of non-Newtonian fluids. Furthermore, these parameters seem to have a marked effect on the solutions to the problem.

5.2 Further research possibilities

It was shown that the optimal system afforded us with other ways of obtaining closed-form solutions that may be meaningful to the specific problem. This can perhaps be looked at in further detail in the future. In deriving our models, we discarded the effects of the induced magnetic field. It might prove useful to capture these effects in the future. We also specifically focused on unidirectional fluid flows. Again, it might be worthwhile to extend this study for multidimensional flows. Lastly, we may be able to implement the conditional symmetry approach to find other significant solutions.

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