Symmetries and conservation laws of some classes of partial differential equations

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01 December 2022

Submitted in fulfillment of a master's degree in mathematics at the University of the Witwatersrand

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

Declaration

I, the undersigned, hereby declare that the work contained in this dissertation is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Tebogo Doctor Malatsi, 01 December 2022

Abstract

Using underlying invariance/symmetry properties and related/associated conservation laws, we investigate some 'high' order nonlinear equations. The multiplier method is mainly used to construct conserved vectors for these equations. This method results in conservation laws that are useful in a wide range of ways. When the partial differential equations are reduced to nonlinear ordinary differential equation (NLODE), exact solutions for the ODEs are constructed and graphical representations of the resulting solutions are provided. In some cases, the solutions obtained are the Jacobi elliptic cosine function and the solitary wave solutions. Firstly, we study the third-order 'equal width equation' followed by a new fourth-order nonlinear partial differential equation (NLPDE), which was recently established in the literature.

We employ Lie symmetry methods and various other techniques to construct closed-form solutions of the NLPDE, which we then visualize in 3D and 2D plots using Mathematica. Secondly, we establish conservation laws using the multiplier method.

Finally, we study the Korteweg-de Vries (KdV) equation having three dispersion sources. The exact solutions and conservation laws for this equation are discovered. The multipliers technique is used to derive conservation laws for this equation. The KdV equation is further reduced into a first-order NLODE using the travelling wave reduction method with conservation laws. The Jacobi ellliptic cosine function and solitary wave solutions are established.

Acknowledgements

I want to acknowledge AIMS (SA) and its funders for supporting this work, as well as my supervisor, Prof Abdul Kara for his unwavering support throughout my studies. In addition, I would like to thank my family and friends for their encouragement throughout my master's program.

Contents

1. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used to explain complex physical phenomena in a range of fields, including mathematical physics, engineering sciences, and other domains, therefore finding their exact solutions is a crucial component of research. This is a tough task as there are no systematic specialized tools or procedures for finding exact solutions to NLPDEs. Researchers and scientists have devised a number of approaches for obtaining exact solutions to NLPDEs over the years. For example, the (G'/G)-expansion method (Wang et al., 2008), Lie symmetry method (Olver, 1986; Ibragimov, 1995, 1999; Stephani, 1989; Bluman and Kumei, 2013), Bäcklund transformation (Gu, 1990), the simplest equation method (Kudryashov, 2005), the Hirota's method (Hirota, 1971), the Jacobi elliptic function method (Liu et al., 2001), Kudryashov's method (Kudryashov, 2012), the inverse scattering method (Ablowitz et al., 1991), and many other methods. In this dissertation, we will concentrate on the Kudryashov approach. The Kudryashov method enables us to efficiently solve high order nonlinear evolution equations. This method is applicable to both non-integrable and integrable equations.

A well-known approach for determining exact solutions to differential equations (DEs) is the symmetry method. Sophus Lie (Lie, 1895), a Norwegian mathematician, pioneered the idea of continuous groups, now referred to as Lie groups, when investigating the classification of the solutions of DEs. If a system of partial differential equations (PDEs) is invariant under a Lie group of point transformations, special solutions known as invariant solutions can be obtained (Bluman and Kumei, 2013). These solutions are invariant under a subgroup of the full group admitted by the system, and they can be used to reduce the underlying system to system of ODEs that, when solved, produce exact solutions.

Conservation laws play an important role in the study of DEs. They are employed in the development of numerical methods and also for analyzing existence, uniqueness, and stability of solutions. Emmy Noether (Noether, 1918) established the relationship between symmetry and conservation laws. She proved that if a system of DEs admits a variational principle (VP), then any local transformation group that leaves the action integral for its Lagrangian density invariant, yields a local conservation law (Bluman et al., 2010). As Noether's approach requires the existence of a VP, researchers worked on to develop new approaches for finding conservation laws for PDEs that do not have a VP. For example, the multiplier method (Olver, 1986; Steudel, 1962) and the direct method which were introduced by Laplace (Anco and Bluman, 2002) do not require a Lagrangian of DE. Also, two other methods for deriving conserved vectors without the usage of a Lagrangian were recently presented in (Ibragimov, 2007; Kara and Mahomed, 2006).

Conservation laws can also be used to reduce the order of PDEs so as to construct their solutions. To obtain group-invariant solutions in explicit form, reduction of order may be performed via first integrals, which lead to quadrature. A double-reduction approach for non-variational PDEs has been given in (Sjöberg, 2007, 2009). Many other applications of double reduction for PDEs have been investigated in the literature (Sjöberg, 2007, 2009; Khalique et al., 2008; Han et al., 2013; Naz et al., 2013; Buhe et al., 2015; Gandarias and Bruzón, 2017). The starting step for these methods, like many others, is a PDE with a known conservation law and symmetry group. Recently, a symmetry multi-reduction approach with conservation laws was proposed in (Anco and Gandarias, 2020). This method generalizes the well-known double reduction methods in the literature. By using multipliers, the requirement of symmetry-invariance for a conservation law was improved, enabling the direct derivation of conservation laws that are invariant under a particular symmetry without the need to first obtain conservation laws and test their invariance. The dissertation outline is as follows:

In Chapter 2, we provide an overview of the core principles of Lie group analysis of DEs and conservation laws. We provide a description of the Kudryashov's method for constructing exact solutions of NLPDEs.

In Chapter 3, we study the equal width (EW) equation. Lie symmetries of the equal width equation are presented. Symmetry reductions will be carried out using these symmetries. Moreover, the multiplier method will be used to obtain conservation laws for the EW equation. Finally, we test for symmetry invariance for each conservation law, and for conservation laws that are translation invariant, we perform travelling-wave reduction and obtain functionally independent first integrals, which are combined to give a first-order NLODE, which we then solve to obtain travelling wave solutions.

In Chapter 4, we study a new nonlinear fourth-order PDE by using its Lie symmetries to reduce the PDE into an ODE. Closed-form solutions are obtained by the integration of the ODE and by employing Kudryashov's method. Conservation laws of this PDE are derived via the multiplier method. We examine each conservation law for symmetry-invariance, and first integrals are obtained for the translation-invariant conservation laws. The obtained first integrals are combined to give a second-order NLODE, after which it is solved to produce a solitary wave solution.

In Chapter 5, we investigate the Korteweg-de Vries (KdV) equation with three dispersion sources. The travelling wave hypothesis is used to transform the KdV equation to a third-order NLODE. Kudryashov's method will be employed to solve this NLODE. Next, we establish conservation laws by use of the multiplier method. Finally, the PDE is reduced to a first-order NLODE using the travelling wave reduction method with conservation laws. This NLODE will be integrated, and exact solutions in the form of the Jacobi elliptic cosine function and the solitary wave will be provided.

2. Preliminaries

The preliminaries and results that are required for our study of NLPDEs in this dissertation are recalled in this chapter. The fundamental concepts of Lie group analysis of DEs and conservation laws are briefly discussed. We also give a short summary of the approaches that we shall utilize to obtain exact solutions. See (Olver, 1986; Ibragimov, 1995, 1999; Stephani, 1989; Bluman and Kumei, 2013; Ibragimov et al., 1998; Wang et al., 2008; Kudryashov, 2012; Ibragimov, 2009; Steudel, 1962) for further details on the concepts covered in this chapter.

2.1 Continuous one-parameter group of transformations.

Let $w = (w^1, \dots, w^n)$ be n independent variables with coordinates w^i, and let $v = (v^1, \dots, v^m)$ be m dependent variables with coordinates v^α. Consider a change of the variables w and v involving a real parameter ε:

$$T_\varepsilon: \ \bar{w}^i = g^i(w, v, \varepsilon), \ \bar{v}^\alpha = \Phi^\alpha(w, v, \varepsilon), \ i = 1, \dots, n, \ \alpha = 1, \dots, m, \tag{2.1.1}$$

where ε continuously ranges in values from a neighbourhood $\mathcal{D}' \in \mathcal{D} \subset \mathbb{R}$ of $\varepsilon = 0$, and g^i and Φ^α are differentiable functions.

2.1.1 Definition. A set G of transformations (2.1.1) is called a continuous one-parameter Lie group of transformations in w and v if

(i) (Closure) For $T_\varepsilon, T_\delta \in G$, where $\varepsilon, \delta \in \mathcal{D}' \subset \mathcal{D}$ then $T_\delta T_\varepsilon = T_\sigma \in G, \ \sigma = \Phi(\varepsilon, \delta) \in \mathcal{D}$.

(ii) (Identity) $T_0 \in G$ if and only if $\varepsilon = 0$ such that $T_0 T_\varepsilon = T_\varepsilon T_0 = T_\varepsilon$.

(iii) (Inverse) For $T_\varepsilon \in G, \ \varepsilon \in \mathcal{D}' \subset \mathcal{D}, \ T_\varepsilon^{-1} = T_{\varepsilon^{-1}} \in G, \ \varepsilon^{-1} \in \mathcal{D}$ such that $T_\varepsilon T_{\varepsilon^{-1}} = T_{\varepsilon^{-1}} T_\varepsilon = T_0$.

It is worth noting that the associativity property is derived from (i).

2.2 Prolongations and group generators.

The derivates of v with respect to w are defined as

$$v_i^\alpha = D_i(v^\alpha), \ v_{ij}^\alpha = D_j D_i(v_i), \dots, \tag{2.2.1}$$

where

$$D_i = \frac{\partial}{\partial w^i} + v_i^\alpha \frac{\partial}{\partial v^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha} + \cdots, \ i = 1, \dots, n \tag{2.2.2}$$

is the total differentiation operator. The set of all first derivatives v_i^α is denoted by $v_{(1)}$, i.e.,

$$v_{(1)} = \{v_i^\alpha\}, \ \alpha = 1, \dots, m, \ i = 1, \dots, n. \tag{2.2.3}$$

Similarly,

$$v_{(2)} = \{v_{ij}^\alpha\}, \ \alpha = 1, \dots, m, \ i, j = 1, \dots, n, \tag{2.2.4}$$

$v_{(3)} = \{v_{ijk}^\alpha\}$ and so on. Since $v_{ij}^\alpha = v_{ji}^\alpha$, $v_{(2)}$ contains only v_{ij}^α for $i \leq j$. In the same manner $v_{(3)}$ has only terms $i \leq j \leq k$.

2.2.1 Remark. In Lie symmetry analysis, all variables $w, v, v_{(1)}, \dots$ are considered functionally independent variables related exclusively by differential relations (2.2.1), and hence v_s^α are referred to as differential variables.

2.2.2 Prolonged groups.

Consider a one-parameter group of transformations given by (2.1.1) such that

$$g^i(w, v, 0) = w^i,\ \Phi^\alpha(w, v, 0) = v^\alpha. \tag{2.2.5}$$

After expanding the functions g^i and Φ^α around the parameter $\varepsilon = 0$, we obtain

$$\bar{w}^i = w^i + \varepsilon \left(\frac{\partial g^i(w, v, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} \right) + O(\varepsilon^2),\ \bar{v}^\alpha = v^\alpha + \varepsilon \left(\frac{\partial \Phi^\alpha(w, v, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} \right) + O(\varepsilon^2). \tag{2.2.6}$$

Set

$$\xi^i(w, v) = \frac{\partial g^i(w, v, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0},\ \eta^\alpha(w, v) = \frac{\partial \Phi^\alpha(w, v, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0}. \tag{2.2.7}$$

The transformations

$$\bar{w}^i \approx w^i + \varepsilon \xi^i(w, v),\ \bar{v}^\alpha \approx v^\alpha + \varepsilon \eta^\alpha(w, v) \tag{2.2.8}$$

are called the infinitesimal transformations of the Lie group of transformations (2.1.1), the components of $\xi^i(w, v)$ and $\eta^\alpha(w, v)$ are called the infinitesimals of (2.1.1). The tangent vector field (2.2.7) can also be written as a differential operator

$$\mathcal{Z} = \xi^i(w, v) \frac{\partial}{\partial w^i} + \eta^\alpha(w, v) \frac{\partial}{\partial v^\alpha}. \tag{2.2.9}$$

2.2.3 Remark. The operator (2.2.9) is called the infinitesimal generator of a one-parameter group.

2.2.4 Remark. According to the Lie's theory, the construction of a one-parameter group G is equivalent to the determination of the corresponding infinitesimal transformations.

2.2.5 Remark. One can write the infinitesimal transformation (2.2.8) as

$$\bar{w}^i \approx (1 + \varepsilon \mathcal{Z}) w^i,\ \bar{v}^\alpha \approx (1 + \varepsilon \mathcal{Z}) v^\alpha, \tag{2.2.10}$$

where $\mathcal{Z}$ is given by (2.2.9).

2.2.6 Theorem. *Given infinitesimal transformations (2.1.1), or generator $\mathcal{Z}$, then the corresponding one-parameter group G is defined by integrating the following system of ordinary differential equations called Lie equations:(Ibragimov, 2009)*

$$\begin{aligned} \frac{d\bar{w}^i}{d\varepsilon} &= \xi^i(\bar{w}, \bar{v}),\ \bar{w}^i|_{\varepsilon=0} = w^i, \\ \frac{d\bar{v}^\alpha}{d\varepsilon} &= \eta^\alpha(\bar{w}, \bar{v}),\ \bar{v}^\alpha|_{\varepsilon=0} = v^\alpha. \end{aligned} \tag{2.2.11}$$

Now we look at how the derivatives are transformed. The D_i transforms as

$$D_i = D_i(g^j) \bar{D}_j, \tag{2.2.12}$$

where $\bar{D}_j$ are the total derivatives in transformed variables $\bar{w}^i$. So

$$\bar{v}_i^\alpha = \bar{D}_j(v^\alpha),\ \bar{v}_{ij}^\alpha = \bar{D}_j(\bar{v}_i^\alpha) = \bar{D}_i(\bar{v}_j^\alpha), \ldots. \tag{2.2.13}$$

Now, applying (2.1.1) together with (2.2.5) and (2.2.12), we have

$$\begin{aligned} D_i(\Phi^\alpha) &= D_i(g^j)\bar{D}_j(\bar{v}^\alpha) \\ &= D_i(g^j)\bar{v}_j^\alpha. \end{aligned} \tag{2.2.14}$$

Thus

$$\left(\frac{\partial g^j}{\partial w^i} + v_i^\beta \frac{\partial g^j}{\partial v^\beta}\right)\bar{v}_j^\alpha = \frac{\partial \phi^\alpha}{\partial w^i} + v_i^\beta \frac{\partial \phi^\alpha}{\partial v^\beta}. \tag{2.2.15}$$

The quantities $\bar{v}_j^\alpha$ can be represented as functions of $w, v, v_{(1)}$, for small ε, i.e., (2.2.15) is locally invertible:

$$\bar{v}_i^\alpha = \psi_i^\alpha(w, v, v_{(1)}, \varepsilon),\ \psi^\alpha|_{\varepsilon=0} = v_i^\alpha. \tag{2.2.16}$$

The transformations in $w, v, v_{(1)}$ space given by (2.1.1) together with (2.2.5) and (2.2.12) form a one-parameter group called the first prolongation of the group G and is denoted by $G^{[1]}$. We let

$$\bar{v}_i^\alpha \approx v_i^\alpha + \varepsilon\zeta_i^\alpha \tag{2.2.17}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (2.2.8) and (2.2.17). Higher-order prolongations of G, namely, $G^{[2]}, G^{[3]}$ can be obtained by derivatives of (2.2.14).

2.2.7 Prolonged generators.

Using (2.2.14) together with (2.2.8) and (2.2.17) we obtain

$$\begin{aligned} D_i(g^j)(\bar{v}_j^\alpha) &= D_i(\Phi^\alpha) \\ D_i(w^j + \varepsilon\xi^j)(v_j^\alpha + \varepsilon\zeta_j^\alpha) &= D_i(v^\alpha + \varepsilon\eta^\alpha) \\ v_i^\alpha + \varepsilon\zeta_i^\alpha + \varepsilon v_j^\alpha D_i\xi^j &= v_i^\alpha + \varepsilon D_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - v_j^\alpha D_i(\xi^j),\ (\text{ sum on } j). \end{aligned} \tag{2.2.18}$$

This is referred to as the first prolongation formula. Similarly, the second prolongation can be obtained, namely (Ibragimov, 1999)

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - v_{ik}^\alpha D_j(\xi^k),\ (\text{ sum on } j). \tag{2.2.19}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^\alpha = D_{i_k}(\zeta_{i_1,i_2,\ldots,i_{k-1}}^\alpha) - v_{i_1,i_2,\ldots,i_{k-1}j}^\alpha D_{i_p}(\xi^j),\ (\text{ sum on } j). \tag{2.2.20}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \ldots, G^{[k]}$. The corresponding prolonged generators are

$$\begin{aligned} \mathcal{Z}^{[1]} &= \mathcal{Z} + \zeta_i^\alpha \frac{\partial}{\partial v_i^\alpha}\ (\text{ sum on } i, \alpha) \\ &\vdots \\ \mathcal{Z}^{[k]} &= \mathcal{Z}^{[k-1]} + \zeta_{i_1,\ldots,i_k}^\alpha \frac{\partial}{\partial v_{i_1,\ldots,i_k}^\alpha},\ k \geq 1, \end{aligned}$$

where $\mathcal{Z}$ is given by (2.2.9).

2.3 Groups admitted by a PDE.

Let us consider a system of kth-order PDEs given by

$$\mathcal{J}_\alpha(w, v, v_{(1)}, \ldots, v_{(k)}) = 0, \;\; \alpha = 1, \ldots, m. \tag{2.3.1}$$

2.3.1 Definition. The vector field

$$\mathcal{Z} = \xi^i(w, v)\frac{\partial}{\partial w^i} + \eta^\alpha(w, v)\frac{\partial}{\partial v^\alpha} \tag{2.3.2}$$

is a point symmetry of the kth-order PDEs, if

$$\mathcal{Z}^{[k]}(\mathcal{J}_\alpha) = 0, \tag{2.3.3}$$

whenever $\mathcal{J}_\alpha = 0$. This can also be written as

$$\mathcal{Z}^{[k]}\mathcal{J}_\alpha|_{\mathcal{J}_\alpha=0} = 0, \tag{2.3.4}$$

where the symbol $|_{\mathcal{J}_\alpha=0}$ means evaluated on $\mathcal{J}_\alpha = 0$.

2.3.2 Definition. Equation (2.3.3) is called the determining equation of (2.3.1) because it determines all the infinitesimal symmetries of (2.3.1).

2.3.3 Definition. A one-parameter group G of transformations (2.1.1) is called a symmetry group of equations (2.3.1) if (2.3.1) has the same form in the new variables $\bar{w}$ and $\bar{v}$, i.e.,

$$\mathcal{J}_\alpha(\bar{w}, \bar{v}, \bar{v}_{(1)}, \ldots, \bar{v}_{(k)}) = 0, \tag{2.3.5}$$

where $\mathcal{J}_\alpha$ is the same as in (2.3.1).

2.4 Invariant functions.

2.4.1 Definition. A function $\mathcal{G}(w, v)$ is an invariant of the group G of transformations (2.1.1) if

$$\mathcal{G}(\bar{w}^i, \bar{v}^\alpha) \equiv \mathcal{G}(g^i(w, v, \varepsilon), \Phi^\alpha(w, v, \varepsilon)) = \mathcal{G}(w, v) \tag{2.4.1}$$

identically in the variables w, v and the group parameter ε.

2.4.2 Theorem (Infinitesimal criterion of invariance). *A function $\mathcal{G}(w, v)$ is an invariant of the group G if and only if it solves the following linear partial differential equation: (Ibragimov, 1999)*

$$\mathcal{Z}\mathcal{G} \equiv \xi^i(w, v)\frac{\partial \mathcal{G}}{\partial w^i} + \eta^\alpha(w, v)\frac{\partial \mathcal{G}}{\partial v^\alpha} = 0. \tag{2.4.2}$$

Invariant solutions.

From the above theorem it follows that every one-parameter group of point transformations (2.1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$\mathcal{I}_1(w, v) = c_1, \ldots, \mathcal{I}_{n-1}(w, v) = c_{n-1}, \tag{2.4.3}$$

which are obtained by solving the linear PDE given by (2.4.2), or its characteristic system

$$\frac{dw^1}{\xi^1(w, v)} = \cdots = \frac{dw^n}{\xi^n(w, v)} = \frac{dv^1}{\eta^1(w, v)} = \cdots = \frac{dv^m}{\eta^m(w, v)}. \tag{2.4.4}$$

Then one of the invariants is designated as a function of the others.

2.5 Kudryashov method.

We present the algorithm for computing exact solutions of NLPDEs via Kudryashov's method. This method was presented in (Kudryashov, 2012). Below we describe the method.

Consider the PDE in polynomial form

$$N_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0. \tag{2.5.1}$$

Step 1. The reduction of a NLPDE to a NLODE is performed by taking the travelling wave solutions into account by assuming

$$u(t, x) = \mathcal{F}(\xi), \ \ \xi = kx + \omega t,$$

where k and ω are parameters. As a result, the following NLODE with ω and k is obtained:

$$N_2(\mathcal{F}, \omega\mathcal{F}', \omega^2\mathcal{F}'', k^2\mathcal{F}'', \dots) = 0. \tag{2.5.2}$$

Step 2. Suppose exact solution of equation (2.5.2) is of the form

$$\mathcal{F}(\xi) = \sum_{m=0}^{M} a_m P^m(\xi), \tag{2.5.3}$$

where a_m $(m = 0, 1, 2, \dots, M)$ are constants to be determined, with $a_M \neq 0$, and $P(\xi)$ is the solution of the first-order ODE

$$P'(\xi) = P^2(\xi) - P(\xi), \tag{2.5.4}$$

which is given by

$$P(\xi) = \frac{1}{1 + \exp(\xi)}. \tag{2.5.5}$$

Step 3. We utilize equation (2.5.4) to generate an equation involving powers of $P(\xi)$ by substituting the value of $\mathcal{F}(\xi)$ into equation (2.5.2).

Step 4. Equating the coefficients of different powers of $P(\xi)$ to zero, we obtain the system of algebraic equations

$$\mathcal{P}_m(a_M, a_{M-1}, \dots, a_0, k, \omega, \dots) = 0, \ \ m = 0, 1, 2, \cdots, M. \tag{2.5.6}$$

Step 5. The solution of system of algebraic equations gives the values of coefficients $a_0, a_1, \dots, a_{M-1}, a_M$ and relations for parameters of equation (2.5.2). As a result, exact solutions of equation (2.5.2) in the form (2.5.3) are obtained.

2.6 Conservation laws.

Let $w = (w^1, \dots, w^n)$ be n independent variables with coordinates w^i and let $v = (v^1, \dots, v^m)$ be m dependent variables with coordinates v^α. Consider a kth order system of PDEs

$$\mathcal{E}^\alpha(w, v, v_{(1)}, v_{(2)}, \dots, v_{(k)}) = 0, \ \ \alpha = 1, \dots, m. \tag{2.6.1}$$

2.6.1 Definition. A conservation law of (2.6.1) is a tuple $T = (T^1, \dots, T^n), T^j = T^j(w, v, v_{(1)}, \dots, v_{(k-1)}) \in \mathcal{A}$ ($\mathcal{A}$ is the universal space of differential functions), $j = 1, \dots, n$, such that

$$D_i(T^i) = 0 \tag{2.6.2}$$

is satisfied for all solutions of (2.6.1).

2.6.2 Remark. When definition (2.6.1) is satisfied, (2.6.2) is called a conservation law for (2.6.1).

2.6.3 Fundamental operators.
The operators introduced below are defined in the space $\mathcal{A}$, the universal space of differential functions.

2.6.4 Definition. The Euler-Lagrange operator is defined by

$$\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial v^\alpha_{i_1\dots i_s}}, \ \alpha = 1,\dots,m. \tag{2.6.3}$$

2.6.5 Definition. The Lie-Bäcklund operator is given by

$$\mathcal{Z} = \xi^i\frac{\partial}{\partial w^i} + \eta^\alpha\frac{\partial}{\partial v^\alpha} + \sum_{s\geq 1}\zeta^\alpha_{i_1\dots i_s}\frac{\partial}{\partial v^\alpha_{i_1\dots i_s}} \tag{2.6.4}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are determined by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j v^\alpha_{ij}, \\ \zeta^\alpha_{i_1\dots i_s} &= D_{i_1}\cdots D_{i_s}(W^\alpha) + \xi^j v^\alpha_{ji_1\cdots i_s}, \ s > 1, \end{aligned} \tag{2.6.5}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j v^\alpha_j. \tag{2.6.6}$$

2.6.6 Definition. The Noether operator associated with the Lie-Bäcklund operator $\mathcal{Z}$ given by (2.6.4) is defined by

$$N^i = \xi^i + W^\alpha\frac{\delta}{\delta v^\alpha_i} + \sum_{s\geq 1} D_{i_1}\dots D_{i_s}(W^\alpha)\frac{\delta}{\delta v^\alpha_{ii_1\dots i_s}}, \ i = 1,\dots,n. \tag{2.6.7}$$

Noether's approach:

2.6.7 Definition. A Lie-Bäcklund operator $\mathcal{Z}$ of the form (2.6.4) is called a Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$ if there exists a vector $\mathcal{B} = (\mathcal{B}^1,\dots,\mathcal{B}^n)$, $\mathcal{B}^i \in \mathcal{A}$, such that

$$\mathcal{Z}(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i\left(\mathcal{B}^i\right). \tag{2.6.8}$$

2.6.8 Theorem (Noether (1918))**.** *For any Noether generator $\mathcal{Z}$ associated with a given Lagrangian $\mathcal{L} \in \mathcal{A}$, there corresponds a vector $T = (T^1,\dots,T^n)$, $T^i \in \mathcal{A}$, defined by*

$$T^i = N^i(\mathcal{L}) - \mathcal{B}^i, \quad i = 1,2,\dots,n, \tag{2.6.9}$$

which is a conserved vector of the Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta v^\alpha} = 0, \quad \alpha = 1,\dots,m, \tag{2.6.10}$$

where $\delta/\delta v^\alpha$ is the Euler-Lagrange operator given by (2.6.3), i.e., $D_i(T^i) = 0$ on the solutions of (2.6.1).

Multiplier's approach:

The concept of multipliers as well as their application in deriving conservation laws for differential equations was presented in (Olver, 1986; Steudel, 1962). Below we give an outline of the method.

The multipliers $\mathcal{Q}^\alpha$ of (2.6.1) has the property

$$D_i T^i = \mathcal{Q}^\alpha \mathcal{E}^\alpha. \tag{2.6.11}$$

Equation (2.6.11) holds for all functions v^α. The determining equations of $\mathcal{Q}^\alpha$ are obtained by taking the variational derivative of (2.6.11) (Olver, 1986):

$$\frac{\delta}{\delta v^\alpha}(\mathcal{Q}^\alpha \mathcal{E}^\alpha) = 0. \tag{2.6.12}$$

The conserved vectors can be obtained systematically using (2.6.11) once the multipliers have been computed from (2.6.12). A homotopy operator can also be used to compute the conserved quantities. See (**?**) for more details.

2.7 Symmetry-invariance of a conservation law.

Let

$$\mathcal{G}(t, x, u, u_t, u_x, \ldots) = 0 \tag{2.7.1}$$

be a PDE admitting the point symmetry

$$\mathcal{Z} = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{2.7.2}$$

Symmetry-invariance of a conservation law is equivalent to the multiplier condition (Anco and Kara, 2018; Anco and Gandarias, 2020)

$$\tilde{\mathcal{Q}}|_\varepsilon = 0. \tag{2.7.3}$$

The multiplier $\mathcal{Q} \to \tilde{\mathcal{Q}}$ is defined by

$$\tilde{\mathcal{Q}} = \text{pr}\mathcal{Z}(\mathcal{Q}) + (R_{\mathcal{Z}} + D_x\xi + D_t\tau)\mathcal{Q} \tag{2.7.4}$$

where $R_{\mathcal{Z}}$ is defined by $\text{pr}\mathcal{Z}(\mathcal{G}) = R_{\mathcal{Z}}\mathcal{G}$.

2.8 Travelling wave reduction with conservation laws.

In this section, we outline the steps for determining first integrals using the travelling wave reduction approach with conservation laws. The presentation is based on the method presented in (Anco and Gandarias, 2020).

Consider a PDE given by (2.7.1) and the travelling wave symmetry

$$\mathcal{Z} = \frac{\partial}{\partial t} + \nu\frac{\partial}{\partial x} \tag{2.8.1}$$

whose invariants are $\xi = x - \nu t$ and $u = \mathcal{F}$, and whose invariant solution is given by $u = \mathcal{F}(\xi)$.

Step 1. Reduce (2.7.1) into the travelling wave ODE

$$\bar{\mathcal{G}}\left(\xi, \mathcal{F}, \frac{d\mathcal{F}}{d\xi}, \ldots\right) = 0. \tag{2.8.2}$$

Step 2. Solve the multiplier determining equation for $\mathcal{Q}$:

$$\text{pr}\mathcal{Z}(\mathcal{Q}) = \mathcal{Q}_t + \nu\mathcal{Q}_x = 0,\ \frac{\delta}{\delta u}\left[\mathcal{Q}\mathcal{G}\right] = 0 \tag{2.8.3}$$

where $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator.

Step 3. Derive the corresponding translation-invariant conservation laws for each multiplier $\mathcal{Q}$.

Step 4. Determine the first integrals in terms of the conserved currents (T^t, T^x) arising from the multiplier $\mathcal{Q}$ using the formula:

$$\Psi\left(\xi, \mathcal{F}, \frac{d\mathcal{F}}{d\xi}, \dots\right) = \bar{T}^t + \int \partial_\rho \bar{T}^x d\xi \tag{2.8.4}$$

which involves only the current

$$\left(\bar{T}^t, \bar{T}^x\right) = \left(T^t - \frac{1}{\nu}T^x, \frac{1}{\nu}T^x\right)\Big|_{u(t,x)=\mathcal{F}(\xi), t=\frac{1}{\nu}(\rho-\xi), x=\rho}$$

Alternatively, one can use the formula

$$\Psi\left(\xi, \mathcal{F}, \frac{d\mathcal{F}}{d\xi}\right) = \int \bar{\mathcal{Q}}\bar{\mathcal{G}} d\xi \tag{2.8.5}$$

directly in terms of the reduced multiplier

$$\bar{\mathcal{Q}} = |J|\mathcal{Q}\Big|_{u(t,x)=\mathcal{F}(\xi), t=\frac{1}{\nu}(\rho-\xi), x=\rho} \tag{2.8.6}$$

where $|J| = \left|\frac{\partial(\xi,\rho)}{\partial(t,x)}\right| = \frac{1}{\nu}$ is a Jacobian factor arising from the point transformation $(t, x, u) \to (\xi, \rho, \mathcal{F})$.

2.9 Concluding remarks.

In this chapter, we reviewed some fundamental concepts of Lie symmetry techniques for DEs and conservation law. The multiplier approach for obtaining conservation laws was discussed briefly . Furthermore, Kudryashov method for finding exact solutions to nonlinear partial differential equations was presented. Finally, we presented the travelling wave reduction method with conservation laws and stated the symmetry-invariance condition of a conservation law.

3. Conservation laws and symmetry multi-reduction method for the equal-width equation

In this chapter, we investigate a third-order equation that defines 'equal-width'. Firstly, we compute its Lie point symmetries and use these symmetries to reduce the order of the PDE. Secondly, the multiplier approach is used to derive conservation laws for the underlying equation. The equation will then be transformed into a first-order ODE using first integrals which will subsequently be solved to obtain exact solutions.

The EW equation is given by

$$u_t + 2\alpha u u_x - \beta u_{txx} = 0, \quad \alpha, \beta \neq 0, \tag{3.0.1}$$

where α is the nonlinearity parameter and β is the dispersion parameter. This equation was initially proposed by (Morrison et al., 1984) and used to explain nonlinear dispersive waves such as those generated in a shallow water channel. Several approaches have been used to find solutions to this equation. For example, the authors of (Gardner and Gardner, 1992) utilized the least-squares method to find numerical solutions and in (Gardner et al., 1997), Petrov-Galerkin method utilizing quadratic B-spline spatial finite elements was used to derive solutions for this equation.

3.1 Lie point symmetries of the equal-width equation.

In this section, we determine the Lie point symmetries for the EW equation (3.0.1). The vector field

$$\mathcal{Z} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.1.1}$$

is a Lie point symmetry of (3.0.1) if and only if

$$\mathcal{Z}^{[3]}\left(u_t + 2\alpha u u_x - \beta u_{txx}\right)\Big|_{(3.0.1)} = 0, \tag{3.1.2}$$

where $\mathcal{Z}^{[3]}$ is the third prolongation of the vector field $\mathcal{Z}$. From equation (3.1.2), we obtain

$$\zeta_t + 2\alpha u \zeta_x + 2\alpha u_x \eta - \beta \zeta_{txx}\Big|_{(3.0.1)} = 0.$$

Substituting the values of ζ_t, ζ_x, and ζ_{txx} in the above equation and splitting on different derivatives of u yields the following system of linear PDEs:

$$\begin{aligned}
&\tau_x = 0,\ \tau_u = 0,\ \xi_u = 0, \xi_t = 0,\ \eta_{tu} = 0, \eta_{uu} = 0,\ \xi_{xx} - 2\eta_{xu} = 0,\ 2\xi_x - \beta\eta_{xxu} = 0,\\
&\alpha\eta + \alpha u \xi_x + \alpha u \tau_t = 0,\ \eta_t + 2\alpha u \eta_x - \beta\eta_{txx} = 0.
\end{aligned}$$

Solving the above system for τ, ξ and η gives

$$\tau = C_1 t + C_3,\ \ \xi = C_2,\ \ \eta = -C_1 u.$$

As a result, the EW equation (3.0.1) has three Lie point symmetries

$$\mathcal{Z}_1 = \frac{\partial}{\partial t},\ \ \mathcal{Z}_2 = \frac{\partial}{\partial x},\ \ \mathcal{Z}_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

3.1.1 Remark. The symmetry $\mathcal{Z}_1$ is the translation in t whereas $\mathcal{Z}_2$ is translation in x, while $\mathcal{Z}_3$ is the scaling symmetry.

(i). We now consider the operator $\mathcal{Z}_3$, whose invariants are $J_1 = x$ and $J_2 = ut$. Its group-invariant solution is $u(t,x) = \frac{H(x)}{t}$, where $H(x)$ is an arbitrary function of x. Substituting the value of u into (3.0.1) reduces equation (3.0.1) into a second-order nonlinear ordinary differential equation (NLODE)

$$\beta H'' + 2\alpha H H' - H = 0. \tag{3.1.3}$$

Equation (3.1.3) admits the vector field $X = \frac{\partial}{\partial x}$. The 0th and 1st invariants of X are given by $H = y$ and $\omega = H'$. The second order invariant is given by

$$\frac{d\omega}{dy} = \frac{d\omega/dx}{dy/dx} = \frac{H''}{H'}.$$

Thus,

$$\frac{d\omega}{dy} = \frac{\frac{1}{\beta}(H - 2\alpha H H')}{H} = \frac{\frac{1}{\beta}(y - 2\alpha\omega y)}{\omega}. \tag{3.1.4}$$

Integrating equation (3.1.4) gives

$$-\frac{\ln(1-2\alpha\omega)}{4\alpha^2} - \frac{\omega}{2\alpha} = \frac{1}{2\beta}y^2 + K,$$

where K is an arbitrary constant of integration. Substituting back the values of ω and y, we have

$$-\frac{\ln(1-2\alpha H')}{4\alpha^2} - \frac{H'}{2\alpha} = \frac{1}{2\beta}H^2 + K.$$

Hence we have reduced equation (3.1.3) into a first-order ODE.

(ii). Considering the travelling wave symmetry, $\mathcal{Z} = \mathcal{Z}_1 + \nu\mathcal{Z}_2$, the EW equation reduces to a third-order nonlinear ODE

$$\mathcal{G} \equiv \beta\nu\mathcal{F}'''(\xi) + 2\alpha\mathcal{F}(\xi)\mathcal{F}'(\xi) - \nu\mathcal{F}'(\xi) = 0 \tag{3.1.5}$$

for travelling wave solutions $u = \mathcal{F}(\xi),\ \xi = x - \nu t$.

3.2 Conservation laws of the equal-width equation.

Here we use the multiplier approach to construct conservation laws for the EW equation (3.0.1).

3.2.1 Conservation laws of (3.0.1) using the multiplier approach.

A function $\mathcal{Q} = \mathcal{Q}(t,x,u,\ldots,u_{tx})$ is a second-order multiplier of equation (3.0.1) if and only if the Euler operator acting on the multiplication $\mathcal{Q}(t,x,u,\ldots,u_{tx})\{u_t + 2\alpha u u_x - \beta u_{txx}\}$ yields zero for any $u = u(t,x)$, i.e.,

$$\frac{\delta}{\delta u}\left[\mathcal{Q}(t,x,u,\ldots,u_{tx})\left\{u_t + 2\alpha u u_x - \beta u_{txx}\right\}\right] = 0, \tag{3.2.1}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_t D_x^2\frac{\partial}{\partial u_{txx}}\cdots$$

is the Euler-Lagrange operator. Following the procedure described in section (2.6), we obtain the following multipliers

$$\mathcal{Q}_1 = 1,\ \ \mathcal{Q}_2 = u,\ \ \mathcal{Q}_3 = -\alpha u^2 + \beta u_{tx}.$$

We now find the conserved vectors corresponding to these three multipliers. Recall that

$$\mathcal{Q}\{u_t + 2\alpha u u_x - \beta u_{txx}\} = D_t T^t + D_x T^x, \tag{3.2.2}$$

where $T^t = T^t(t, x, u, u_x)$ is the density and $T^x = T^x(t, x, u, u_x, u_{tx})$ is the flux.

Case 1. For the first multiplier $\mathcal{Q}_1 = 1$, equation (3.2.2) becomes

$$u_t + 2\alpha u u_x - \beta u_{txx} = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx} + T^x_{u_{tx}} u_{txx}.$$

Splitting the above equation by third derivatives of u, we have

$$u_{txx} : T^x_{u_{tx}} = -\beta, \tag{3.2.3}$$
$$\text{Rest} : u_t + 2\alpha u u_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}. \tag{3.2.4}$$

Integration of (3.2.3) with respect to u_{tx} yields

$$T^x = -\beta u_{tx} + A(t, x, u, u_x),$$

where $A(t, x, u, u_x)$ is an arbitrary function. Replacing the value of T^x into (3.2.4) gives

$$u_t + 2\alpha u u_x = T^t_t + T^t_u u_x t + T^t_{u_x} u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx}.$$

Separating the above equation on second derivatives of u yields

$$u_{tx} : T^t_{u_x} = 0, \tag{3.2.5}$$
$$u_{xx} : A_{u_x} = 0, \tag{3.2.6}$$
$$\text{Rest} : u_t + 2\alpha u u_x = T^t_t + T^t_u u_t + A_x + A_u u_x. \tag{3.2.7}$$

From (3.2.6), we get

$$A = A(t, x, u),$$

where $A(t, x, u)$ is an arbitrary function. Integrating equation (3.2.5) with respect to u_x, we obtain

$$T^t = B(t, x, u),$$

where $B(t, x, u)$ is an arbitrary function. Substituting the value of T^t into (3.2.7) results in

$$u_t + 2\alpha u u_x = B_t + B_u u_t + A_x + A_u u_x. \tag{3.2.8}$$

Splitting (3.2.8) on first derivatives of u yields

$$u_t : B_u = 1, \tag{3.2.9}$$
$$u_x : A_u = 2\alpha u, \tag{3.2.10}$$
$$\text{Rest} : A_x + B_t = 0. \tag{3.2.11}$$

Integrating (3.2.9) with respect to u, we obtain

$$B = u + C(t, x),$$

where $C(t, x)$ is an arbitrary function. Integration of (3.2.10) with respect to u yields

$$A = \alpha u^2 + D(t, x), \tag{3.2.12}$$

where $D(t,x)$ is an arbitrary function of t and x. Inserting the values of A and B into (3.2.11), we get $D_x + C_t = 0$. We take C and D to be zero since they contribute to a trivial component of conservation law. Thus, the conservation law with components T^t and T^x corresponding to the multiplier $\mathcal{Q}_1 = 1$ is

$$\left(T^t, T^x\right) = \left(u,\ \alpha u^2 - \beta u_{tx}\right).$$

It should be noted that when the multiplier is equal to 1, it means that the given equation is already in conservation form.

Case 2. For the multiplier $\mathcal{Q}_2 = u$, equation (3.2.2) takes the form

$$u\left\{u_t + 2\alpha u u_x - \beta u_{txx}\right\} = D_t T^t + D_x T^x,$$

where $T^t = T^t(t,x,u,u_x)$ is the density and $T^x = T^x(t,x,u,u_x,u_{tx})$ is the flux. Applying the total derivatives on the above equation yields

$$u u_t + 2\alpha u^2 u_x - \beta u u_{txx} = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx} + T^x_{u_{tx}} u_{txx}. \tag{3.2.13}$$

Now applying the same procedure as in case 1, we find that the conserved vector (T^t, T^x) corresponding to multiplier $\mathcal{Q}_2 = u$ is given by

$$\left(T^t, T^x\right) = \left(\frac{1}{2}\beta u_x^2 + \frac{1}{2}u^2,\ \frac{2}{3}\alpha u^3 - \beta u u_{tx}\right).$$

Case 3. The second-order multiplier,

$$\mathcal{Q}_3 = -\alpha u^2 + \beta u_{tx}$$

gives rise to the conserved vector

$$\begin{aligned} T^t &= 4\alpha u^3 - \beta^2 u u_{xxxt} + \beta^2 u_x u_{xxt} + 2\beta^2 u_{xx} u_{xt} - 3\beta u u_{xt} - 3\beta u_x u_t, \\ T^x &= 6\alpha^2 u^4 - 12\alpha\beta u^2 u_{xt} + \beta^2 u u_{xxtt} - 2\beta^2 u_x u_{xtt} + \beta^2 u_t u_{xxt} + 4\beta^2 u_{xt}^2 + 3\beta u u_{tt} - 3\beta u_t^2. \end{aligned}$$

3.3 Travelling wave solutions.

In this section, we employ the travelling wave reduction method described in section (2.8) to obtain first integrals which reduces the ODE (3.1.5) to first-order. The reduced travelling wave ODE will be solved to find travelling wave solutions.

We begin by examining symmetry-invariance for each of the conservation laws derived in the preceding section. The condition for an operator $\mathcal{Z}$ to be associated with a conservation law is stated below.

An operator $\mathcal{Z}_i$ is associated with $\mathcal{Q}_i$ if (Anco and Kara, 2018; Anco and Gandarias, 2020)

$$\tilde{\mathcal{Q}}_i = 0. \tag{3.3.1}$$

The multiplier $\tilde{\mathcal{Q}}$ is given by (2.7.4).

As an example, we check for symmetry-invariance of the conservation law given by the multiplier $\mathcal{Q}_3 = -\alpha u^2 + \beta u_{tx}$, under $\mathcal{Z}_3$. For $\mathcal{Z}_3$, we have

$$\text{pr}\mathcal{Z}_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial x} - 2u_t\frac{\partial}{\partial u_t} - u_x\frac{\partial}{\partial u_x} - 2u_{txx}\frac{\partial}{\partial u_{txx}},$$

hence pr$\mathcal{Z}_3(\mathcal{Q}_3) = -2\mathcal{Q}_3$ and pr$\mathcal{Z}_3(\mathcal{G}) = -2\mathcal{G}$. Thus,

$$\begin{aligned}\tilde{\mathcal{Q}} &= \text{pr}\mathcal{Z}_3(\mathcal{Q}_3) + (R_{\mathcal{Z}_3} + D_x\xi + D_t\tau)\mathcal{Q}_3 \\ &= -3\mathcal{Q}_3 \\ &\neq 0.\end{aligned}$$

Hence, $\mathcal{Q}_2$ is not invariant under $\mathcal{Z}_3$, i.e, $\mathcal{Z}_3$ is not associated with the conservation law arising from $\mathcal{Q}_3$. Following the same procedure, we obtain the results given in table (3.1).

Table 3.1: symmetry-invariance of conservation laws.

$\tilde{\mathcal{Q}}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$
$\mathcal{Z}$	0	0	0
$\mathcal{Z}_1$	0	0	0
$\mathcal{Z}_2$	0	0	0
$\mathcal{Z}_3$	$-\mathcal{Q}_1$	$-2\mathcal{Q}_2$	$-3\mathcal{Q}_3$

The following observations are made from table (3.1):

- $\mathcal{Q}_i(i = 1, 2, 3)$ are strictly invariant under $\mathcal{Z}$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$.
- $\mathcal{Q}_i(i = 1, 2, 3)$ are not invariant under $\mathcal{Z}_3$ but rather the action of $\mathcal{Z}_3$ on each of the $\mathcal{Q}_i's$ leads to another multiplier, i.e., conformal invariance exists.

By utilizing the definition of symmetry-invariance of a conservation law, we see that $\mathcal{Z}$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are associated with conservation laws arising from multipliers $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$. Thus, we can use the travelling wave reduction method with conservation laws to perform reduction since $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are strictly invariant under the travelling wave symmetry, $\mathcal{Z}$. Using conservation laws corresponding to $\mathcal{Q}_1$,$\mathcal{Q}_2$ and $\mathcal{Q}_3$, we obtain the first integrals for the travelling wave ODE (3.1.5) by making use of formula (2.8.4):

Case 1. $\mathcal{Q}_1 : (T_1^t, T_1^x)$.

Because $\bar{T}_1^x$ does not explicitly include t and x, the first integral is given by $\bar{T}_1^t$. Hence,

$$\begin{aligned}\Psi_1 &= \left(T_1^t - \frac{1}{\nu}T_1^x\right)\Big|_{u(t,x)=\mathcal{F}(\xi),t=\frac{1}{\nu}(\rho-\xi),x=\rho} \\ &= \mathcal{F} - \frac{1}{\nu}\left(\beta\nu\mathcal{F}'' + \alpha\mathcal{F}^2\right) \\ &= -\beta\mathcal{F}'' - \frac{\alpha}{\nu}\mathcal{F}^2 + \mathcal{F} = C_1\end{aligned}$$

Case 2. $\mathcal{Q}_2 : (T_2^t, T_2^x)$.

Similar to Case 1, Ψ_2 is determined by $\bar{T}_2^t$,

$$\begin{aligned}\Psi_2 &= \left(T_2^t - \frac{1}{\nu}T_2^x\right)\Big|_{u(t,x)=\mathcal{F}(\xi),t=\frac{1}{\nu}(\rho-\xi),x=\rho} \\ &= \frac{1}{2}\beta\mathcal{F}'^2 + \frac{1}{2}\mathcal{F}^2 - \frac{1}{\nu}\left(\beta\nu\mathcal{F}\mathcal{F}'' + \frac{2}{3}\alpha\mathcal{F}^3\right) \\ &= -\beta\mathcal{F}\mathcal{F}'' + \frac{1}{2}\beta\mathcal{F}'^2 - \frac{2\alpha}{3\nu}\mathcal{F}^3 + \frac{1}{2}\mathcal{F}^2 = C_2.\end{aligned}$$

Case 3. $\mathcal{Q}_3 : (T_3^t, T_3^x)$.

Since $\bar{T}_3^x$ does not contain t and x explicitly, the first integral is given by $\bar{T}_3^t$. Thus,

$$\begin{aligned}\Psi_3 &= \left(T_3^t - \frac{1}{\nu}T_3^x\right)\Big|_{u(t,x)=\mathcal{F}(\xi),t=\frac{1}{\nu}(\rho-\xi),x=\rho} \\ &= \beta^2\nu\mathcal{F}''''\mathcal{F} - 2\beta^2\nu\mathcal{F}''^2 + 3\beta\nu\mathcal{F}\mathcal{F}'' + 3\beta\nu\mathcal{F}'^2 - \beta^2\nu\mathcal{F}'''\mathcal{F}' + 4\alpha\mathcal{F}^3 \\ &- \frac{1}{\nu}\{6\alpha^2\mathcal{F}^4 - 3\beta\nu^2\mathcal{F}'^2 + 3\beta\nu^2\mathcal{F}\mathcal{F}'' + \beta^2\nu^2\mathcal{F}\mathcal{F}'''' + 4\beta^2\nu^2\mathcal{F}''^2 \\ &- \beta^2\nu^2\mathcal{F}'''\mathcal{F}' + 12\alpha\beta\nu\mathcal{F}^2\mathcal{F}''\} \\ &= -12\alpha\beta\mathcal{F}^2\mathcal{F}'' - 6\beta^2\nu\mathcal{F}''^2 + 6\beta\nu\mathcal{F}'^2 - \frac{6\alpha^2}{\nu}\mathcal{F}^4 + 4\alpha\mathcal{F}^3 = C_3.\end{aligned}$$

Combining Ψ_1 and Ψ_2 result in the following reduced first-order NLODE

$$\mathcal{F}'^2 = \frac{1}{\beta}\mathcal{F}^2 - \frac{2\alpha}{3\beta\nu}\mathcal{F}^3 - \frac{2C_1}{\beta}\mathcal{F} + \frac{2C_2}{\beta} = f. \tag{3.3.2}$$

Hence, we have obtained a triple reduction for the EW equation under the travelling wave symmetry.

(i). Considering $C_1 = C_2 = 0$ in (3.3.2), we obtain the ODE

$$\mathcal{F}'^2 = \frac{1}{\beta}\mathcal{F}^2 - \frac{2\alpha}{3\beta\nu}\mathcal{F}^3. \tag{3.3.3}$$

Integrating the above ODE and choosing the integration constant to zero gives the solution

$$\mathcal{F}(\xi) = \frac{3\nu}{2\alpha}\operatorname{sech}^2\left(\frac{\xi}{2\sqrt{\beta}}\right). \tag{3.3.4}$$

Reverting back to original variables, we get a special kind of a solitary wave solution known as bright solution

$$u(t,x) = \frac{3\nu}{2\alpha}\operatorname{sech}^2\left[\frac{1}{2\sqrt{\beta}}(x-\nu t)\right]. \tag{3.3.5}$$

Figure (3.1) graphically displays solution (3.3.5) for parametric values $\alpha = 1.5$, $\beta = 0.20$ and $\nu = 1$.

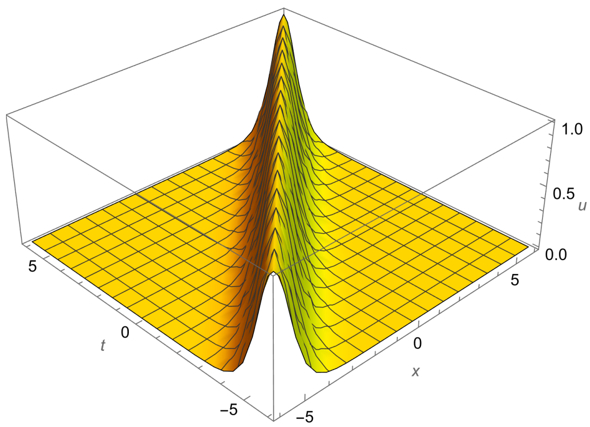

Figure 3.1: Wave profile of solution (3.3.5).

(ii). Solution of (3.3.2) with $C_1 \neq 0$ and $C_2 \neq 0$.

To solve equation (3.3.2), we assume that σ_1, σ_2 and σ_3 ($\sigma_1 > \sigma_2 > \sigma_3$) are roots of the cubic polynomial f, and that f can be factorized into

$$\frac{2\alpha}{3\beta\nu}(\mathcal{F}-\sigma_1)(\mathcal{F}-\sigma_2)(\mathcal{F}-\sigma_3) \tag{3.3.6}$$

Thus, equation (3.3.2) can be written as

$$\mathcal{F}'^2 = \frac{2\alpha}{3\beta\nu}(\mathcal{F}-\sigma_1)(\mathcal{F}-\sigma_2)(\mathcal{F}-\sigma_3), \tag{3.3.7}$$

consequently the solution is given in terms of the Jacobi elliptic function as (Billingham and King, 2000; Kudryashov, 2004)

$$\mathcal{F}(\xi) = \sigma_2 + (\sigma_1 - \sigma_2)\,\mathrm{cn}^2\left(\sqrt{\frac{\alpha}{6\beta\nu}(\sigma_1-\sigma_3)}\xi, P^2\right), \tag{3.3.8}$$

where $\xi = x - \nu t$, $P^2 = \frac{\sigma_1-\sigma_2}{\sigma_1-\sigma_3}$ and cn is the elliptic cosine function. Therefore, the travelling wave solution for the EW equation (3.0.1) is

$$u(t,x) = \sigma_2 + (\sigma_1 - \sigma_2)\,\mathrm{cn}^2\left(\sqrt{\frac{\alpha}{6\beta\nu}(\sigma_1-\sigma_3)}\,(x-\nu t), P^2\right). \tag{3.3.9}$$

Figure (3.2) graphically displays solution (3.3.9) for parametric values $\sigma_1 = 1$, $\sigma_2 = 0$, $\sigma_3 = 1.5$, $\alpha = -1.5$, $\beta = 0.5$ and $\nu = 1$.

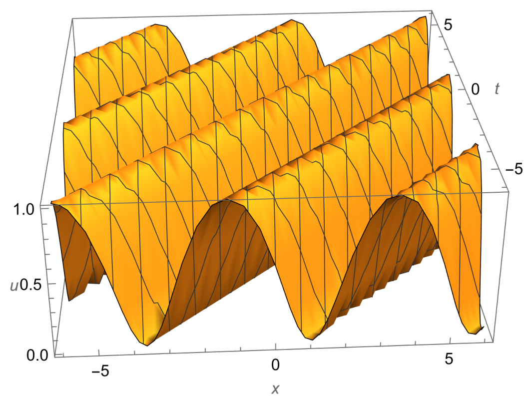

Figure 3.2: Wave profile of the elliptic solution (3.3.9).

3.4 Concluding remarks.

In this chapter, we investigated a third-order equal-width equation. We began by computing its Lie point symmetries, which we then used to perform symmetry reduction. Furthermore, the multiplier method was employed to derive conservation laws for (3.0.1). Three multipliers were obtained, resulting in three local conservation laws. Finally, the travelling wave reduction method with conservation laws was used to reduce the equation into a first-order NLODE, which was solved to construct travelling wave solutions. The type of solutions secured are the solitary wave and the Jacobi elliptic cosine function solutions.

4. Exact solutions and conservation laws of a new nonlinear fourth-order integrable equation

In this Chapter, we investigate a new nonlinear fourth-order integrable PDE. We first determine its Lie point symmetries, and then use them to compute the one-parameter group of transformations and thereafter perform symmetry reductions. This equation will be reduced to a NLODE and solved by direct integration and Kudryashov's method. The solutions will be graphically illustrated. We conclude the chapter by utilizing the multiplier approach to derive conservation laws for this PDE.

A new nonlinear integrable equation was developed in (Wazwaz, 2018), which reads

$$u_{tt}+u_{txxx}+\alpha(u_x u_t)_x+\beta u_{xx}=0, \tag{4.0.1}$$

where α and β are nonzero constants. Similar to the Boussinesq equation, equation (4.0.1) is used to model both right- and left-going waves. Painlevé analysis was used to examine the integrability of equation (4.0.1) and the simplified Hirota's method was employed to derive its multiple soliton solutions (Wazwaz, 2018). Furthermore, complex simplified Hirota's method was invoked and multiple complex soliton solutions were determined.

In our work we shall study a generalization of equation (4.0.1), namely

$$u_{tt}+u_{txxx}+\alpha u_t u_{xx}+\beta u_x u_{tx}+\gamma u_{xx}=0, \tag{4.0.2}$$

where α, β and γ are nonzero constants.

4.1 Solutions of a new nonlinear fourth-order integrable equation (4.0.2).

Equation (4.0.2) will admit the vector field

$$\mathcal{Y}=\tau(t,x,u)\frac{\partial}{\partial t}+\xi(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u} \tag{4.1.1}$$

on condition that

$$\mathcal{Y}^{[4]}\left(u_{tt}+u_{txxx}+\alpha u_t u_{xx}+\beta u_x u_{tx}+\gamma u_{xx}\right)\Big|_{(4.0.2)}=0, \tag{4.1.2}$$

where $\mathcal{Y}^{[4]}$ is the fourth prolongation of (4.1.1) given by

$$\mathcal{Y}^{[4]}=\mathcal{Y}+\zeta_t\frac{\partial}{\partial t}+\zeta_x\frac{\partial}{\partial x}+\zeta_{tx}\frac{\partial}{\partial u_{tx}}+\zeta_{tt}\frac{\partial}{\partial u_{tt}}+\zeta_{xx}\frac{\partial}{\partial u_{xx}}+\zeta_{txxx}\frac{\partial}{\partial u_{txxx}}.$$

After expanding (4.1.2), we obtain

$$\zeta_{tt}+\zeta_{txxx}+\alpha u_{xx}\zeta_t+\alpha u_t\zeta_{xx}+\beta u_{tx}\zeta_x+\beta u_x\zeta_{tx}+\gamma\zeta_{xx}\Big|_{(4.0.2)}=0. \tag{4.1.3}$$

Substituting the values of $\zeta_t,\zeta_x,\zeta_{tx},\zeta_{tt},\zeta_{xx}$, and ζ_{txxx} into equation (4.1.3), and splitting the resulting equation on appropriate derivatives of u, the following set of linear PDEs are obatined:

$$\xi_u=0,\xi_t=0,\tau_u=0,\tau_x=0,\eta_{tu}=0,\eta_{uu}=0,\alpha\eta_{xx}+\eta_{xxxu}-\tau_{tt}=0,\beta\eta_{tx}+2\eta_{xu}-\gamma\xi_{xx}=0,3\xi_x-\tau_t=0,$$

$\gamma\xi_x + \gamma\tau_t + \alpha\eta_t = 0, \xi_x + \eta_u = 0, 2\eta_{xxu} - \xi_{xxx} + \beta\eta_x = 0, 2\alpha\eta_{xu} + \beta\eta_{xu} - \alpha\xi_{xx} = 0, \eta_{xu} - \xi_{xx} = 0,$ $\eta_{tt} + \eta_{txxx} + \gamma\eta_{xx} = 0.$

After some long calculations, we obtain

$$\tau = 3C_1 t + C_3,\;\; \xi = C_1 x + C_2,\;\; \eta = -C_1 u - \frac{4\gamma C_1}{\alpha} t + C_4.$$

Thus, the corresponding Lie point symmetries of (4.0.2) are given by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t},\;\; \mathcal{Y}_2 = \frac{\partial}{\partial x},\;\; \mathcal{Y}_3 = \frac{\partial}{\partial u},\;\; \mathcal{Y}_4 = 3\alpha t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} - (4\gamma t + \alpha u)\frac{\partial}{\partial u}.$$

4.1.1 Group of transformations.

In this subsection we compute the one-parameter Lie group of transformations for equation (4.0.2). To get the group of transformations, we solve the Lie equations along with the initial conditions:

$$\frac{d\bar{t}}{da} = \tau(\bar{t},\bar{x},\bar{u}),\;\; \bar{t}|_{a=0} = t;\;\; \frac{d\bar{x}}{da} = \xi(\bar{t},\bar{x},\bar{u}),\;\; \bar{x}|_{a=0} = x;\;\; \frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{u}),\;\; \bar{u}|_{a=0} = u.$$

We compute the one-parameter group T_{a_i} generated by each infinitesimal generator $\mathcal{Y}_i,\; i = 1, 2, 3, 4.$ After solving the above system for each $\mathcal{Y}_i$, we obtain the one-parameter group of transformations as

$$\begin{aligned}
&T_{a_1} : (\bar{t},\bar{x},\bar{u}) \longrightarrow (t + a, x, u)\,, \quad T_{a_2} : (\bar{t},\bar{x},\bar{u}) \longrightarrow (t, x + a, u)\,,\\
&T_{a_3} : (\bar{t},\bar{x},\bar{u}) \longrightarrow (t, x, u + a)\,, \quad T_{a_4} : (\bar{t},\bar{x},\bar{u}) \longrightarrow \left(te^{3\alpha a}, xe^{\alpha a}, ue^{-\alpha a} + \frac{\gamma t}{\alpha}e^{-\alpha a} - \frac{\gamma t}{\alpha}e^{3\alpha a}\right).
\end{aligned}$$

Since each group T_{a_i} is a symmetry group, if $u = f(t, x)$ is a solution of equation (4.0.2), then so are the functions

$$\begin{aligned}
&u_1 = f(t - a, x), \quad u_2 = f(t, x - a), \quad u_3 = a + f(t, x),\\
&u_4 = e^{-\alpha a} f\left(te^{-3\alpha a}, xe^{-\alpha a}\right) + \frac{\gamma t}{\alpha}e^{-4\alpha a} - \frac{\gamma t}{\alpha}.
\end{aligned}$$

4.1.2 Symmetry reduction of (4.0.2).

In this subsection we perform symmetry reductions of (4.0.2) by utilizing its time and space translation symmetries. Consider the combination $\mathcal{Y} = \mathcal{Y}_1 + \nu\mathcal{Y}_2$ with ν a constant, which has two invariants

$$J_1 = x - \nu t,\;\; J_2 = u. \tag{4.1.4}$$

Writing the dependent invariant as a function of the independent invariant gives

$$u = \mathcal{F}(x - \nu t). \tag{4.1.5}$$

Substituting equation (4.1.5) into (4.0.2), we obtain the following fourth-order NLODE:

$$(\nu^2 + \gamma)\mathcal{F}''(\xi) - (\alpha + \beta)\nu\mathcal{F}'(\xi)\mathcal{F}''(\xi) - \nu\mathcal{F}''''(\xi) = 0, \tag{4.1.6}$$

where $\xi = x - \nu t$.

Solution of (4.1.6) by direct integration.

Integrating equation (4.1.6) with respect to ξ and then multiplying the resulting equation by its integrating factor $\mathcal{F}''(\xi)$, we obtain

$$a\mathcal{F}'\mathcal{F}'' - \frac{b}{2}\mathcal{F}'^2\mathcal{F}'' - \nu\mathcal{F}''\mathcal{F}''' = K_1\mathcal{F}'', \tag{4.1.7}$$

where $a = \nu^2 + \gamma$, $b = (\alpha + \beta)\nu$ and K_1 an arbitrary constant of integration. Integrating equation (4.1.7) with respect to ξ yields

$$\frac{a}{2}\mathcal{F}'^2 - \frac{b}{6}\mathcal{F}'^3 - \frac{\nu}{2}\mathcal{F}''^2 = K_1\mathcal{F}' + K_2, \tag{4.1.8}$$

where K_2 is an arbitrary constant of int. Setting $\mathcal{F}' = \Psi$, we obtain the following

$$\Psi'^2 = \frac{a}{\nu}\Psi^2 - \frac{b}{3\nu}\Psi^3 + \frac{2K_1}{\nu}\Psi + \frac{2K_2}{\nu} = f_1. \tag{4.1.9}$$

To solve equation (4.1.9), we assume that r_1, r_2 and r_3 ($r_1 > r_2 > r_3$) are roots of the polynomial f_1, and that f_1 can be factorized into the following:

$$\frac{b}{3\nu}(\Psi - r_1)(\Psi - r_2)(\Psi - r_3).$$

Thus, equation (4.1.9) now takes the form

$$\Psi'^2 = \frac{b}{3\nu}(\Psi - r_1)(\Psi - r_2)(\Psi - r_3),$$

whose solution is given by the Jacobi elliptic function (Billingham and King, 2000; Kudryashov, 2004)

$$\Psi(\xi) = r_2 + (r_1 - r_2)\,\mathrm{cn}^2\left(\sqrt{\frac{\alpha+\beta}{12}(r_1 - r_3)}\xi, R^2\right), \tag{4.1.10}$$

where $R^2 = \frac{r_1 - r_2}{r_1 - r_3}$, $\xi = x - \nu t$, with cn denoting the cosine elliptic function. Since $\mathcal{F}' = \Psi$, we integrate equation (4.1.10) and getting back to the variables t and x, we have the periodic solution

$$\begin{aligned} u(t,x) = {} & \sqrt{\frac{12(r_1 - r_2)^2}{(\alpha+\beta)(r_1 - r_3)R^8}}\left\{\mathrm{EllipticE}\left[\mathrm{sn}\left(\frac{(\alpha+\beta)(r_1 - r_3)}{12}\xi, R^2\right), R^2\right]\right\} \\ & + \left\{r_2 - (r_1 - r_2)\frac{1 - R^4}{R^4}\right\}\xi + C_3, \end{aligned} \tag{4.1.11}$$

where $\xi = x - \nu t$, C_3 an arbitrary constant of integration, and

$$\mathrm{EllipticE}[q, w] = \int_0^q \sqrt{\frac{1 - w^2 m^2}{1 - m^2}}dm \tag{4.1.12}$$

is the incomplete elliptic integral (Abramowitz and Stegun (1972)). We exhibit the elliptic solution (4.1.11) with the periodic soliton wave revealed in Figure 4.1 with unalike parametric values $\alpha = 4$, $\beta = 0.2$, $c = 1$, $r_1 = 100$, $r_2 = 50.05$, $r_3 = -60$ and $C_3 = 0$ in the range $-10 \le t, x \le 10$.

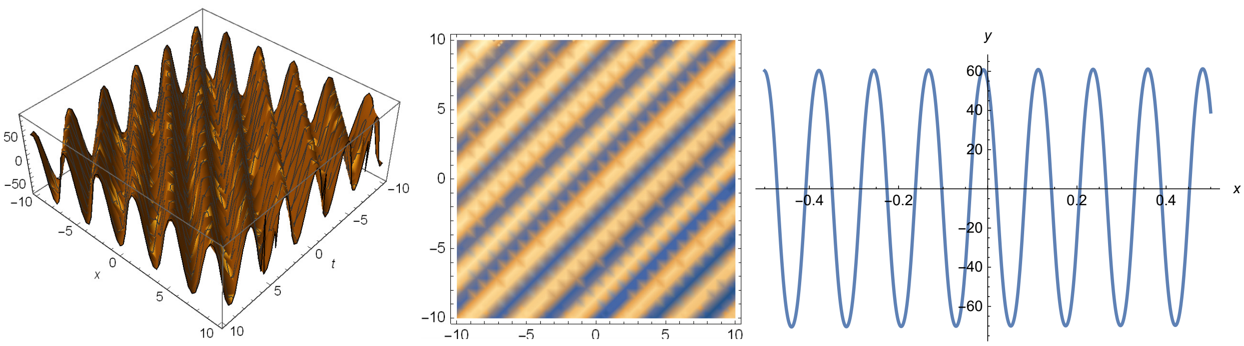

Figure 4.1: Wave profile of the elliptic solution (4.1.11).

Solution of (4.1.6) by Kudryashov's method.

We now solve equation (4.1.6) using Kudryashov's approach. We start by assuming that the solution of equation (4.1.6) is of the form

$$\mathcal{F}(\xi) = \sum_{i=0}^{M} A_i \Phi^i(\xi), \tag{4.1.13}$$

where $\Phi(\xi)$ satisfies the Riccati equation

$$\Phi'(\xi) = \Phi(\xi)^2 - \Phi(\xi), \tag{4.1.14}$$

which has the solution

$$\Phi(\xi) = \frac{1}{1+\exp(\xi)}. \tag{4.1.15}$$

From the balancing procedure (Kudryashov (2012)) we get $M = 1$ for (4.1.6). Thus, the solution can be written as

$$\mathcal{F}(\xi) = A_0 + A_1\Phi(\xi). \tag{4.1.16}$$

Substituting equation (4.1.16) into (4.1.6) and using (4.1.14), we obtain

$$\begin{aligned} &2(\nu^2+\gamma)A_1\Phi(\xi)^3 - 3(\nu^2+\gamma)A_1\Phi(\xi)^2 + (\nu^2+\gamma)A_1\Phi(\xi) - 2(\alpha+\beta)\nu A_1^2\Phi(\xi)^5 \\ &+ 5(\alpha+\beta)\nu A_1^2\Phi(\xi)^4 - 4(\alpha+\beta)\nu A_1^2\Phi(\xi)^3 + (\alpha+\beta)\nu A_1^2\Phi(\xi)^2 - 24\nu A_1\Phi(\xi)^5 \\ &+ 60\nu A_1\Phi(\xi)^4 - 50\nu A_1\Phi(\xi)^3 + 15\nu A_1\Phi(\xi)^2 - \nu A_1\Phi(\xi) = 0. \end{aligned} \tag{4.1.17}$$

Splitting equation (4.1.17) on powers of $\Phi(\xi)$, we obtain

$$\begin{aligned} \Phi(\xi)^5 &: (\alpha+\beta)\nu A_1^2 + 12\nu A_1 = 0, \\ \Phi(\xi)^4 &: (\alpha+\beta)\nu A_1^2 + 12\nu A_1 = 0, \\ \Phi(\xi)^3 &: (\nu^2+\gamma)A_1 - 2(\alpha+\beta)\nu A_1^2 - 25\nu A_1 = 0, \\ \Phi(\xi)^2 &: (\alpha+\beta)\nu A_1^2 + 15\nu A_1 - 3(\nu^2+\gamma)A_1 = 0, \\ \Phi(\xi) &: (\nu^2+\gamma)A_1 - \nu A_1 = 0. \end{aligned}$$

Solving the above algebraic equations we obtain

$$A_0 = A_0,\ A_1 = -\frac{12}{\alpha+\beta},\ \nu = \frac{1 \pm \sqrt{1-4\gamma}}{2}. \tag{4.1.18}$$

Therefore, the general solution of (4.0.2) corresponding to the constants given by (4.1.18) is given by

$$u(t,x) = A_0 - \frac{12}{(\alpha+\beta)(1+\exp(x-\nu t))}, \tag{4.1.19}$$

where A_0 is an arbitrary constant. The profile of solution (4.1.19) gives the anti-kink wave demonstrated in Figure 4.2 with dissimilar constant values $\alpha = 2,\ \beta = 1,\ \nu = 2,\ A_0 = 10$ in the range $-8 \le t, x \le 8$.

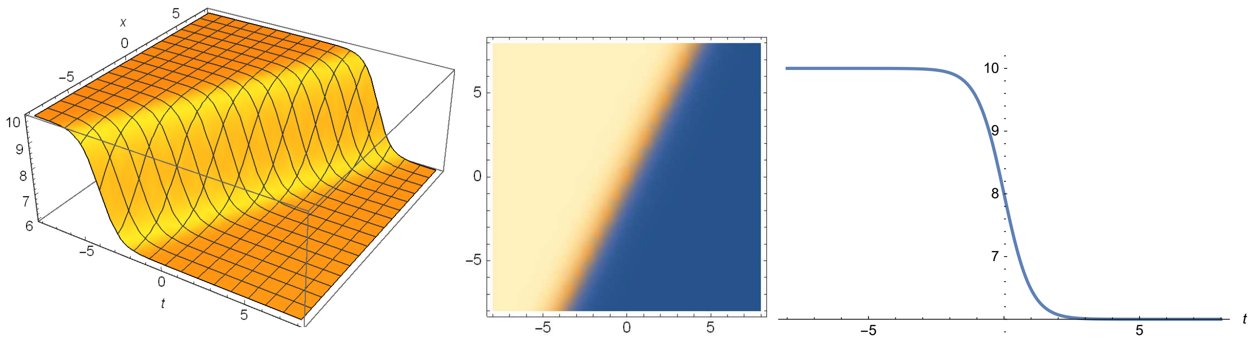

Figure 4.2: Wave profile of the exponential function solution (4.1.19).

4.2 Conservation laws of a new nonlinear fourth-order integrable equation (4.0.2).

In this section, we utilize the multipliers method to derive conservation laws for equation (4.0.2). We wish to derive multipliers of the form

$$\mathcal{Q} = \mathcal{Q}(t, x, u, u_t, u_x). \tag{4.2.1}$$

For multipliers of the form (4.2.1), the determining equation becomes

$$\frac{\delta}{\delta u}\left[\mathcal{Q}(t, x, u, u_t, u_x)\{u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\}\right] = 0, \tag{4.2.2}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_tD_x\frac{\partial}{\partial u_{tx}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_tD_x^3\frac{\partial}{\partial u_{txxx}} + \cdots$$

is the Euler operator. Expanding equation (4.2.2) yields

$$\begin{aligned}&\mathcal{Q}_u\left(u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\right) - D_t\left(\mathcal{Q}_{u_t}\left\{u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\right\}\right)\\ &- D_x\left(\mathcal{Q}_{u_x}\left\{u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\right\}\right) + D_t^2(\mathcal{Q}) + \beta D_tD_x(u_x\mathcal{Q}) + \gamma D_x^2(\mathcal{Q})\\ &+ \alpha D_x^2(u_t\mathcal{Q}) - \alpha D_t(u_{xx}\mathcal{Q}) - \beta D_x(u_{tx}\mathcal{Q}) + D_x^3D_t(\mathcal{Q}) = 0.\end{aligned} \tag{4.2.3}$$

Assuming $\beta \neq \alpha$, $\beta \neq 2\alpha$, and applying the total derivatives in equation (4.2.3), we obtain the determining equations

$$\mathcal{Q}_t = 0,\ \mathcal{Q}_x = 0,\ \mathcal{Q}_u = 0,\ \mathcal{Q}_{u_t} = 0,\ \mathcal{Q}_{u_xu_x} = 0. \tag{4.2.4}$$

Solving the above system of equations yields

$$\mathcal{Q} = C_1 + C_2u_x, \tag{4.2.5}$$

where C_1 and C_2 are constants. Thus, the two nontrivial conservation law multipliers are $\mathcal{Q}_1 = 1$ and $\mathcal{Q}_2 = u_x$. We now recall and apply the multiplier definition. A multiplier $\mathcal{Q}(t, x, u, u_t, u_x)$ has the property that

$$\mathcal{Q}(t, x, u, u_t, u_x)\left\{u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\right\} = D_tT^t + D_xT^x, \tag{4.2.6}$$

where $T^t = T^t(t,x,u,u_t,u_x)$ is the density and $T^x = T^x(t,x,u,u_t,u_x,u_{txx})$ is the flux.

Case 1. For the multiplier $\mathcal{Q}_1 = 1$, equation (4.2.6) becomes

$$\begin{aligned}&u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x} - T^x_x - u_x T^x_u \\ &- u_{tx} T^x_{u_t} - u_{xx} T^x_{u_x} - u_{txxx} T^x_{u_{txx}} = 0. \end{aligned} \tag{4.2.7}$$

Splitting equation (4.2.7) by fourth derivatives of u yields

$$u_{txxx} : T^x_{u_{txx}} = 1, \tag{4.2.8}$$

$$\begin{aligned}\text{Rest } : \; &u_{tt} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x} - T^x_x - u_x T^x_u \\ &- u_{tx} T^x_{u_t} - u_{xx} T^x_{u_x} = 0. \end{aligned} \tag{4.2.9}$$

Integrating equation (4.2.8) with respect to u_{txx} gives

$$T^x = u_{txx} + A(t,x,u,u_t,u_x), \tag{4.2.10}$$

where $A(t,x,u,u_t,u_x)$ is an arbitrary function. Substituting the value of T^x into equation (4.2.9), we obtain

$$\begin{aligned}&u_{tt} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x} - A_x - u_x A_u \\ &- u_{tx} A_{u_t} - u_{xx} A_{u_x} = 0. \end{aligned} \tag{4.2.11}$$

Splitting equation (4.2.11) on second derivatives of u gives

$$u_{tt} : T^t_{u_t} = 1, \tag{4.2.12}$$

$$u_{tx} : T^t_{u_x} + A_{u_t} = \beta u_x, \tag{4.2.13}$$

$$u_{xx} : A_{u_x} = \alpha u_t + \gamma, \tag{4.2.14}$$

$$\text{Rest: } \; T^t_t + u_t T^t_u + A_x + u_x A_u = 0. \tag{4.2.15}$$

Integrating equation (4.2.14) with respect to u_x gives that

$$A = \alpha u_t u_x + \gamma u_x + B(t,x,u,u_t), \tag{4.2.16}$$

where $B(t,x,u,u_t)$ is an arbitrary function. Substituting (4.2.16) into (4.2.13), and then integrate with respect to u_x, we obtain

$$T^t = \frac{1}{2}\beta u_x^2 - \frac{1}{2}\alpha u_x^2 - u_x B_{u_t} + C(t,x,u,u_t), \tag{4.2.17}$$

where $C(t,x,u,u_t)$ is an arbitrary function. After substituting (4.2.17) into (4.2.12), we obtain $-u_x B_{u_t u_t}(t,x,u,u_t) + C_{u_t}(t,x,u,u_t) = 1$, and splitting on u_x yields

$$u_x : B_{u_t u_t} = 0, \tag{4.2.18}$$

$$\text{Rest: } \; C_{u_t} = 1. \tag{4.2.19}$$

Integrating equation (4.2.18) with respect to u_t gives

$$B = D(t,x,u)u_t + E(t,x,u), \tag{4.2.20}$$

where $D(t,x,u)$ and $E(t,x,u)$ are arbitrary functions. Integrating (4.2.19) with respect to u_t yields

$$C = u_t + F(t,x,u), \tag{4.2.21}$$

where $F(t,x,u)$ is an arbitrary function. Thus, equations (4.2.17) and (4.2.16) becomes

$$T^t = \frac{1}{2}\beta u_x^2 - \frac{1}{2}\alpha u_x^2 - u_x D(t,x,u) + u_t + F(t,x,u) \tag{4.2.22}$$

and

$$A = \alpha u_t u_x + \gamma u_x + D(t,x,u)u_t + E(t,x,u), \tag{4.2.23}$$

respectively. Substituting (4.2.22) and (4.2.23) into (4.2.15) yields

$$-u_x D_t + F_t + u_t F_u + u_t D_x + E_x + u_x E_u = 0. \tag{4.2.24}$$

Splitting equation (4.2.24) on first derivatives of u gives

$$u_t: \ F_u + D_x = 0, \ u_x: \ E_u - D_t = 0, \quad \text{Rest:} \quad F_t + E_x = 0.$$

We consider $D, E,$ and F to be zero as they contribute to the trivial component of the conservation law. Hence, the conservation law corresponding to the multiplier $\mathcal{Q} = 1$ is given by

$$\left(T^t, T^x\right) = \left(u_t + \frac{1}{2}\beta u_x^2 - \frac{1}{2}\alpha u_x^2, \ u_{txx} + \alpha u_t u_x + \gamma u_x\right). \tag{4.2.25}$$

Case 2. For the multiplier $\mathcal{Q}_2 = u_x$, equation (4.2.6) takes the form

$$u_x\{u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\} = D_t T^t + D_x T^x, \tag{4.2.26}$$

where $T^t = T^t(t,x,u,u_t,u_x,u_{xx})$ is the density and $T^x = T^x(t,x,u,u_t,u_x,u_{txx})$ is the flux. After applying the total derivatives on equation (4.2.26), we have

$$\begin{aligned} &u_x\{u_{tt} + u_{txxx} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x} \\ &- u_{txx} T^t_{u_{xx}} - T^x_x - u_x T^x_u - u_{tx} T^x_{u_t} - u_{xx} T^x_{u_x} - u_{txxx} T^x_{u_{txx}} = 0. \end{aligned} \tag{4.2.27}$$

Splitting equation (4.2.27) on fourth derivatives of u yields

$$u_{txxx}: T^x_{u_{txx}} = u_x, \tag{4.2.28}$$

$$\begin{aligned} \text{Rest:} \quad &u_x\{u_{tt} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x} \\ &- u_{txx} T^t_{u_{xx}} - T^x_x - u_x T^x_u - u_{tx} T^x_{u_t} - u_{xx} T^x_{u_x} = 0. \end{aligned} \tag{4.2.29}$$

Integrating equation (4.2.28) with respect to u_{txx} gives

$$T^x = u_x u_{txx} + A(t,x,u,u_t,u_x), \tag{4.2.30}$$

where $A(t,x,u,u_t,u_x)$ is an arbitrary function. After substituting the value of T^x into (4.2.29), we obtain

$$\begin{aligned} &u_x\{u_{tt} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x} \\ &- u_{txx} T^t_{u_{xx}} - A_x - u_x A_u - u_{tx} A_{u_t} - u_{xx} u_{txx} - u_{xx} A_{u_x} = 0. \end{aligned} \tag{4.2.31}$$

Splitting equation (4.2.31) on third derivatives of u, we obtain the following

$$u_{txx}: T^t_{u_{xx}} = -u_{xx}, \tag{4.2.32}$$

$$\text{Rest:} \ u_x\{u_{tt} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\} - T^t_t - u_t T^t_u - u_{tt} T^t_{u_t} - u_{tx} T^t_{u_x}$$

$$- A_x - u_x A_u - u_{tx} A_{u_t} - u_{xx} A_{u_x} = 0. \tag{4.2.33}$$

Integrating equation (4.2.32) with respect to u_{xx} yields

$$T^t = -\frac{1}{2}u_{xx}^2 + B(t, x, u, u_t, u_x), \tag{4.2.34}$$

where $B(t, x, u, u_t, u_x)$ is an arbitrary function. Substituting the value of T^t into (4.2.33), we have

$$\begin{aligned} &u_x\{u_{tt} + \alpha u_t u_{xx} + \beta u_x u_{tx} + \gamma u_{xx}\} - B_t - u_t B_u - u_{tt} B_{u_t} - u_{tx} B_{u_x} \\ &- A_x - u_x A_u - u_{tx} A_{u_t} - u_{xx} A_{u_x} = 0. \end{aligned} \tag{4.2.35}$$

Splitting equation (4.2.35) on second derivatives of u gives

$$u_{tt} : B_{u_t} = u_x, \tag{4.2.36}$$
$$u_{tx} : B_{u_x} + A_{u_t} = \beta u_x^2, \tag{4.2.37}$$
$$u_{xx} : A_{u_x} = \alpha u_t u_x + \gamma u_x, \tag{4.2.38}$$
$$\text{Rest:} \quad B_t + u_t B_u + A_x + u_x A_u = 0. \tag{4.2.39}$$

Integrating equation (4.2.38) with respect to u_x gives that

$$A = \frac{1}{2}\alpha u_t u_x^2 + \frac{1}{2}\gamma u_x^2 + C(t, x, u, u_t), \tag{4.2.40}$$

where $C(t, x, u, u_t)$ is an arbitrary function. Substituting (4.2.40) into (4.2.37), and integrating with respect to u_x yields

$$B = \frac{1}{3}\beta u_x^3 - \frac{1}{6}\alpha u_x^3 - u_x C_{u_t} + D(t, x, u, u_t), \tag{4.2.41}$$

where $D(t, x, u, u_t)$ is an arbitrary function. Substituting equation (4.2.41) into equation (4.2.36), we have

$$- u_x C_{u_t u_t}(t, x, u, u_t) + D_{u_t}(t, x, u, u_t) = u_x, \tag{4.2.42}$$

and splitting on u_x gives

$$u_x : C_{u_t u_t} = -1, \tag{4.2.43}$$
$$\text{Rest} : D_{u_t} = 0. \tag{4.2.44}$$

Integrating equation (4.2.43) twice with respect to u_t yields

$$C = u_t E(t, x, u) + F(t, x, u) - \frac{1}{2}u_t^2. \tag{4.2.45}$$

Equation (4.2.44) implies that

$$D = G(t, x, u). \tag{4.2.46}$$

Thus, the values of A and B become

$$A = \frac{1}{2}\alpha u_t u_x^2 + \frac{1}{2}\gamma u_x^2 - \frac{1}{2}u_t^2 + u_t E(t, x, u) + F(t, x, u) \tag{4.2.47}$$

and

$$B = \frac{1}{3}\beta u_x^3 - \frac{1}{6}\alpha u_x^3 + u_t u_x - u_x E(t, x, u) + G(t, x, u), \tag{4.2.48}$$

respectively. Substituting the values of A and B into equation (4.2.39) yields

$$-u_xE_t + G_t + u_tG_u + u_tE_x + F_x + u_xF_u = 0 \tag{4.2.49}$$

and splitting on the first derivatives of u gives

$$u_t: \ G_u + E_x = 0, \ u_x: \ F_u - E_t = 0, \ \text{Rest}: \ G_t + F_x = 0.$$

We choose E, F and G equal to zero as they contribute to the trivial component of the conservation law. Hence, the conservation law corresponding to the multiplier $\mathcal{Q} = u_x$ is given by

$$\left(T^t, T^x\right) = \left(u_tu_x + \frac{1}{3}\beta u_x^3 - \frac{1}{6}\alpha u_x^3 - \frac{1}{2}u_{xx}^2, \ u_xu_{txx} + \frac{1}{2}\alpha u_tu_x^2 + \frac{1}{2}\gamma u_x^2 - \frac{1}{2}u_t^2\right). \tag{4.2.50}$$

4.3 Travelling wave reduction via conservation laws for a new nonlinear fourth-order integrable equation (4.0.2).

In this section, we apply the travelling wave reduction method with conservation laws to obtain two first integrals which reduces (4.0.2) to a second-order NLODE.

We begin by using multipliers to check each of the conservation laws discovered in the preceding section for symmetry invariance. The table in (4.1) is obtained by applying the definition of symmetry-invariance of a conservation law.

Table 4.1: symmetry-invariance of conservation laws.

$\tilde{\mathcal{Q}}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$
$\mathcal{Y}$	0	0
$\mathcal{Y}_1$	0	0
$\mathcal{Y}_2$	0	0
$\mathcal{Y}_3$	0	0
$\mathcal{Y}_4$	$-3\mathcal{Q}_1$	$-5\mathcal{Q}_2$

From table (4.1), we see that the $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are strictly invariant under $\mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{Y}_3$, but not invariant under $\mathcal{Y}_4$. In other words, $\mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{Y}_3$ are associated with the conservation laws arising from the multipliers $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Moreover, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are strictly invariant under the travelling wave symmetry, $\mathcal{Y}$, so we can apply the travelling wave reduction method with conservation laws to obtain first integrals for (4.1.6). The first integrals for (4.1.6) are given by (2.8.4) as follows:

Case 1. $\mathcal{Q}_1 : (T_1^t, T_1^x)$.

$$\begin{aligned}
\Psi_1 &= \left(T_1^t - \frac{1}{\nu}T_1^x\right)\Big|_{u(t,x)=\mathcal{F}(\xi), t=\frac{1}{\nu}(\rho-\xi), x=\rho} \\
&= -\frac{1}{2}\alpha\mathcal{F}'^2 + \frac{1}{2}\beta\mathcal{F}'^2 - \nu\mathcal{F}' - \frac{1}{\nu}\left(-\nu\mathcal{F}''' - \alpha\nu\mathcal{F}'^2 + \gamma\mathcal{F}'\right) \\
&= \mathcal{F}''' + \frac{1}{2}(\alpha+\beta)\mathcal{F}'^2 - \frac{\nu^2+\gamma}{\nu}\mathcal{F}' = C_1.
\end{aligned}$$

Case 2. $\mathcal{Q}_2 : (T_2^t, T_2^x)$.

$$
\begin{aligned}
\Psi_2 &= \left(T_2^t - \frac{1}{\nu}T_2^x\right)\Big|_{u(t,x)=\mathcal{F}(\xi),t=\frac{1}{\nu}(\rho-\xi),x=\rho} \\
&= -\frac{1}{2}\mathcal{F}''^2 - \frac{1}{6}\alpha\mathcal{F}'^3 + \frac{1}{3}\beta\mathcal{F}'^3 - \nu\mathcal{F}'^2 - \frac{1}{\nu}\left(-\frac{1}{2}\nu^2\mathcal{F}'^2 - \frac{1}{2}\alpha\nu\mathcal{F}'^3 + \frac{1}{2}\gamma\mathcal{F}'^2 - \nu\mathcal{F}'''\mathcal{F}'\right) \\
&= -\frac{1}{2}\mathcal{F}''^2 + \frac{1}{3}(\alpha+\beta)\mathcal{F}'^3 - \frac{\nu^2+\gamma}{2\nu}\mathcal{F}'^2 + \mathcal{F}'\mathcal{F}''' = C_2.
\end{aligned}
$$

Combining Ψ_1 and Ψ_2 yields a second-order NLODE

$$
\mathcal{F}''^2 + \frac{1}{3}(\alpha+\beta)\mathcal{F}'^3 - \frac{1}{\nu}(\nu^2+\gamma)\mathcal{F}'^2 - 2C_1\mathcal{F}' + 2C_2 = 0. \tag{4.3.1}
$$

Hence, we have obtained a triple reduction for (4.0.2) under the travelling wave symmetry. Setting $\mathcal{F}' = \mathcal{U}$ gives

$$
\mathcal{U}'^2 + \frac{1}{3}(\alpha+\beta)\mathcal{U}^3 - \frac{1}{\nu}(\nu^2+\gamma)\mathcal{U}^2 - 2C_1\mathcal{U} + 2C_2 = 0. \tag{4.3.2}
$$

Considering $C_1 = C_2 = 0$ yields the ODE

$$
\mathcal{U}'^2 + \frac{(\alpha+\beta)}{3}\mathcal{U}^3 - \frac{(\nu^2+\gamma)}{\nu}\mathcal{U}^2 = 0, \tag{4.3.3}
$$

whose solution can be expressed in terms of a trigonometric function as

$$
\mathcal{U} = \frac{3(\nu^2+\gamma)}{\nu(\alpha+\beta)}\operatorname{sech}^2\left(\frac{n}{2}\xi \pm \sqrt{\frac{3}{4}}c_1\right), \tag{4.3.4}
$$

where $n = \sqrt{\frac{\nu^2+\gamma}{\nu}}$ and c_1 is an arbitrary constant. Integrating (4.3.4), setting the integration constant to zero, and returning to the original variables yields the solution of (4.0.2) in the form

$$
u(t,x) = \frac{6n}{\alpha+\beta}\tanh\left[\frac{n}{2}(x-\nu t)\right]. \tag{4.3.5}
$$

4.3.1 Remark. The solution (4.3.5) is a special kind of solitary wave solution known as anti-kink solitary wave.

Figure (4.3) depicts the profile of solution (4.3.5) with distinct constant values $\alpha = 1.5,\ \beta = 0.50,\ \nu = 1$ and $\gamma = 0.3$.

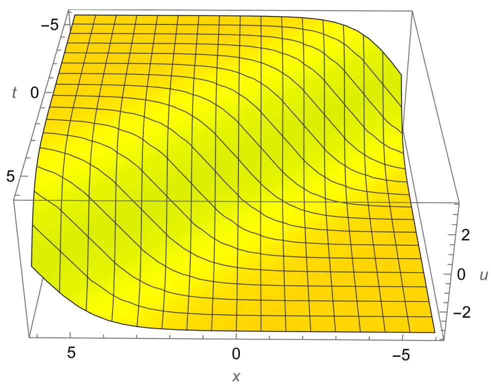

Figure 4.3: The 3D wave profile of the solitary wave solution (4.3.5).

4.4 Concluding remarks.

In this chapter, we investigated a new nonlinear fourth-order integrable PDE. Lie point symmetries of the equation were found, one-parameter group transformations were computed, symmetry reductions were performed and this equation was reduced to a fourth-order NLODE. By using direct integration and Kudryashov's method, we were able to find exact solutions to this NLODE. These solutions were graphically illustrated. Moreover, the multiplier method was invoked to derive conservation laws for (4.0.2). We found two multipliers, which yielded two local conservation laws for (4.0.2). Using the travelling wave reduction approach with conservation laws, two functionally independent first integrals which reduced (4.0.2) to a second-order NLODE were obtained. This NLODE was solved to give a solitary wave solution.

5. Exact solutions and conservation laws for the KdV equation having three dispersion sources

The Korteweg-de Vries (KdV) equation with three dispersion sources is studied in this chapter. The equation is as follows:

$$u_t + auu_x + b_1u_{xxx} + b_2u_{txx} + b_3u_{ttx} = 0, \tag{5.0.1}$$

where the coefficients $b_j(i = 1, 2, 3)$ are carried by the three dispersion terms. They are from triple spatial dispersion, Spatio-temporal dispersion, and the dual-temporal-spatial dispersion effects, respectively. This equation is an extension of the well-known KdV equation, which has been examined with third-order dispersion. This equation, along with the Gardener's equation, has been examined in (Biswas et al., 2022). The models' solitary wave solutions and conservation laws were recovered.

In this chapter, we use Kudryashov's method to generate exact solutions for equation (5.0.1). The travelling wave reduction method with conservation laws will be applied to reduce (5.0.1) to a first-order NLODE, which will then be solved to obtain exact solutions.

5.1 Solution via Kudryashov's method.

Performing the travelling wave transformation $u(t, x) = g(\xi)$, $\xi = x - \nu t$, where ν is the velocity of the wave, gives the following third-order NLODE:

$$-\nu g'(\xi) + ag(\xi)g'(\xi) + dg'''(\xi) = 0, \tag{5.1.1}$$

where $d = b_1 - \nu b_2 + \nu^2 b_3$. We assume that equation (5.1.1) has a solution of the form

$$g(\xi) = \sum_{j=0}^{M} B_j \Phi(\xi)^j, \tag{5.1.2}$$

where $\Phi(\xi)$ satisfies the Ricatti equation

$$\Phi'(\xi) = \Phi(\xi)^2 - \Phi(\xi), \tag{5.1.3}$$

which has the solution

$$\Phi(\xi) = \frac{1}{1 + \exp(\xi)}. \tag{5.1.4}$$

We find $M = 2$ after balancing the highest order derivative term $g'''(\xi)$ and the highest order nonlinear term $g(\xi)g'(\xi)$, therefore the solution of (5.1.1) may be expressed in the form

$$g(\xi) = B_0 + B_1\Phi(\xi) + B_2\Phi(\xi)^2. \tag{5.1.5}$$

Putting (5.1.5) into (5.1.1) and making use of (5.1.3) gives a polynomial in powers of $\Phi(\xi)$. The following set of algebraic equations is produced by grouping terms with the same power of $\Phi(\xi)$ and equating each coefficient to zero:

$$\begin{aligned}
&\Phi(\xi)^5 : aB_2^2 + 12B_2d = 0,\\
&\Phi(\xi)^4 : 3aB_1B_2 - 2aB_2^2 + 6B_1d - 54B_2d = 0,\\
&\Phi(\xi)^3 : aB_1^2 - 2B_2\nu + 2aB_0B_2 - 3aB_1B_2 - 12B_1d + 38B_2d = 0,
\end{aligned}$$

$$\Phi(\xi)^2 : aB_0A_1 - B_1\nu + 2B_2\nu - 2aB_0B_2 - aB_1^2 + 7B_1d - 8B_2d = 0,$$
$$\Phi(\xi) : B_1\nu - aB_0B_1 - B_1d = 0.$$

Solving the above algebraic equations for B_0, B_1 and B_2 yields the following constants

$$B_0 = \frac{\nu - d}{a}, \; B_1 = \frac{12d}{a}, \; B_2 = -\frac{12d}{a}, \tag{5.1.6}$$

which result in the travelling wave solution

$$u(t,x) = \frac{\nu - d}{a} + \frac{12d}{a(1 + \exp(x - \nu t))} - \frac{12d}{a(1 + \exp(x - \nu t))^2}. \tag{5.1.7}$$

Figure (5.1) depicts the profile of solution (5.1.7) with unalike constant values $a = 1,\; b_1 = 0.50,\; b_2 = -1,\; b_3 = 1$ and $\nu = 1$.

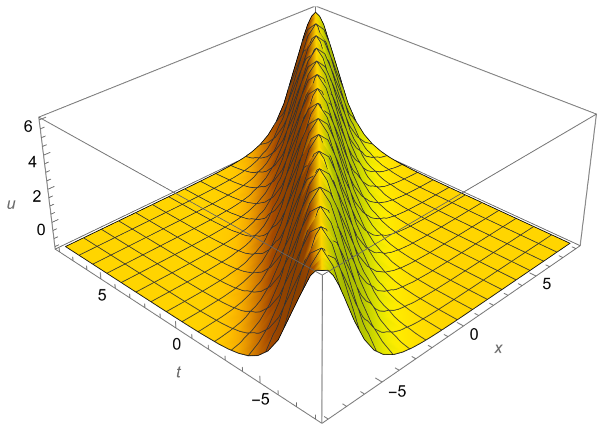

Figure 5.1: The 3D wave profile of the exponential solution (5.1.7).

5.2 Conservation laws

In this section, we derive conservation laws for (5.0.1) by means of the multiplier method. We seek for multipliers of the form $\mathcal{Q}(t, x, u, u_t, u_x, u_{xx}, u_{tx}, u_{xx})$. Following the procedure outlined in (2.6) yields three multipliers

$$\mathcal{Q}_1 = 1, \; \mathcal{Q}_2 = u, \; \mathcal{Q}_3 = \frac{au^2 + 2b_1u_{xx} + 2b_2u_{tx} + 2b_3u_{tx}}{2b_2}. \tag{5.2.1}$$

The corresponding conserved vectors are then given by

$$\begin{aligned} T_1^t &= \frac{2}{3}b_3u_{tx} + \frac{1}{3}b_2u_{xx} + u, \\ T_1^x &= \frac{1}{3}b_3u_{tt} + \frac{2}{3}b_2u_{tx} + b_1u_{xx} + \frac{1}{2}au^2; \end{aligned} \tag{5.2.2}$$

$$\begin{aligned} T_2^t &= \frac{2}{3}b_3uu_{tx} + \frac{1}{3}b_2uu_{xx} - \frac{1}{3}b_3u_tu_x - \frac{1}{6}b_2u_x^2 + \frac{1}{2}u^2, \\ T_2^t &= \frac{1}{3}au^3 + b_1uu_{xx} - \frac{1}{2}b_1u_x^2 + \frac{2}{3}b_2uu_{tx} - \frac{1}{3}b_2u_tu_x + \frac{1}{3}b_3uu_{tt} - \frac{1}{6}b_3u_t^2; \end{aligned} \tag{5.2.3}$$

$$
\begin{aligned}
T_3^t = \frac{1}{12b_2}[&2\,au^3 - b_1\,b_2\,uu_{xxxx} + b_1\,b_2\,u_xu_{xxx} + 2\,b_1\,b_2\,{u_{xx}}^2 - 2\,b_1\,b_3\,uu_{xxxt} \\
&+ 4\,b_1\,b_3\,u_tu_{xxx} - 2\,b_1\,b_3\,u_xu_{xxt} + 4\,b_1\,b_3\,u_{xt}u_{xx} - {b_2}^2uu_{xxxt} + {b_2}^2u_xu_{xxt} + 2\,{b_2}^2u_{xt}u_{xx} \\
&- 3\,b_2\,b_3\,uu_{xxtt} + 4\,b_2\,b_3\,u_tu_{xxt} + 2\,b_2\,b_3\,u_{tt}u_{xx} - b_2\,b_3\,u_xu_{xtt} + 4\,b_2\,b_3\,{u_{xt}}^2 \\
&- 2\,{b_3}^2uu_{xttt} + 4\,{b_3}^2u_tu_{xtt} + 4\,{b_3}^2u_{tt}u_{xt} - 2\,{b_3}^2u_{ttt}u_x + 6\,b_1\,uu_{xx} \\
&+ 3\,b_2\,uu_{xt} + 3\,b_2\,u_tu_x + 6\,b_3\,{u_t}^2], \\
T_3^x = \frac{1}{24b_2}[&3\,a^2u^4 + 12\,ab_1\,u^2u_{xx} + 12\,ab_2\,u^2u_{xt} + 12\,ab_3\,u^2u_{tt} \\
&+ 12\,{b_1}^2{u_{xx}}^2 + 2\,b_1\,b_2\,uu_{xxxt} + 2\,b_1\,b_2\,u_tu_{xxx} - 4\,b_1\,b_2\,u_xu_{xxt} + 20\,b_1\,b_2\,u_{xt}u_{xx} \\
&+ 4\,b_1\,b_3\,uu_{xxtt} - 4\,b_1\,b_3\,u_tu_{xxt} + 16\,b_1\,b_3\,u_{tt}u_{xx} + 2\,{b_2}^2uu_{xxtt} \\
&+ 2\,{b_2}^2u_tu_{xxt} - 4\,{b_2}^2u_xu_{xtt} + 8\,{b_2}^2{u_{xt}}^2 + 6\,b_2\,b_3\,uu_{xttt} - 2\,b_2\,b_3\,u_tu_{xtt} \\
&+ 12\,b_2\,b_3\,u_{tt}u_{xt} - 4\,b_2\,b_3\,u_{ttt}u_x + 4\,{b_3}^2uu_{tttt} - 4\,{b_3}^2u_tu_{ttt} + 4\,{b_3}^2{u_{tt}}^2 - 12\,b_1\,uu_{xt} \\
&+ 12\,b_1\,u_tu_x - 6\,b_2\,uu_{tt} + 6\,b_2\,{u_t}^2].
\end{aligned} \tag{5.2.4}
$$

5.3 Travelling wave reduction with conservation laws

In this section, we employ first integrals to reduce the order of equation (5.0.1). We acquire two functionally independent first integrals, which we combine to get a first-order NLODE.

Equation (5.0.1) admits the following Lie point symmetries:

$$\mathcal{X}_1 = \frac{\partial}{\partial t},\ \mathcal{X}_2 = \frac{\partial}{\partial x}.$$

The travelling wave symmetry, $\mathcal{X}$, is produced by a linear combination of these two symmetries and is given by $\mathcal{X} = \mathcal{X}_1 + \nu\mathcal{X}_2$. Using $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$, we test for symmetry invariance for each of the conservation laws obtained in the previous section. After a few straightforward computations, we arrive at the results shown in table (5.1).

Table 5.1: symmetry-invariance of conservation laws.

$\tilde{\mathcal{Q}}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$
$\mathcal{X}$	0	0	0
$\mathcal{X}_1$	0	0	0
$\mathcal{X}_2$	0	0	0

Table (5.1) shows that the travelling wave symmetry is associated with conservation laws arising from $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$, allowing us to apply the travelling wave reduction approach discussed in section (2.8). Following the approach given in (2.8), and using (5.2.2) and (5.2.3), we obtain the following two first integrals:

$$\Psi_1 = \mathcal{F} - \frac{1}{2\nu}a\mathcal{F}^2 + \left(b_2 - \frac{b_1}{\nu} - b_3\nu\right)\mathcal{F}'' = k_1. \tag{5.3.1}$$

$$\Psi_2 = \frac{1}{2}\mathcal{F}^2 - \frac{1}{3\nu}a\mathcal{F}^3 + \frac{1}{2\nu}\left(b_1 - b_2\nu + b_3\nu^2\right)\mathcal{F}'^2 + \left(b_2 - \frac{b_1}{\nu} - b_3\nu\right)\mathcal{F}\mathcal{F}'' = k_2. \tag{5.3.2}$$

Combining (5.3.1) and (5.3.2) gives the following first-order NLODE:

$$\mathcal{F}'^2 = \frac{\nu}{d}\mathcal{F}^2 - \frac{a}{3d}\mathcal{F}^3 - \frac{2k_1\nu}{d}\mathcal{F} + \frac{2k_2\nu}{d}, \tag{5.3.3}$$

where $d = b_1 - b_2\nu + b_3\nu^2$. As a result, we have a triple reduction for equation (5.0.1) using the travelling wave symmetry.

Case 1. Considering $k_1 = k_2 = 0$ leads to the NLODE

$$\mathcal{F}'^2 = \frac{\nu}{d}\mathcal{F}^2 - \frac{a}{3d}\mathcal{F}^3. \tag{5.3.4}$$

Separating variables, integrating, and then considering the integration constant to zero yields a special kind of solitary wave solution known as bright soliton (Biswas et al., 2022)

$$u(t,x) = \frac{3\nu}{a}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{\nu}{d}}(x-\nu t)\right]. \tag{5.3.5}$$

The 3D wave profile for the solitary wave solution (5.3.5) with parametric values $a = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, and $\nu = 1$ is shown in figure (5.2).

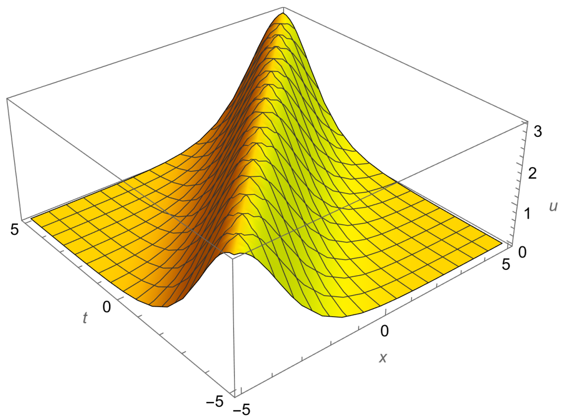

Figure 5.2: 3D wave profile of the solitoray wave solution (5.3.5).

Case 2. $k_1 \neq 0,\ k_2 \neq 0$

The right hand side of equation (5.3.3) is a cubic polynomial in $\mathcal{F}$, and we suppose that it can be factorized into the form

$$\frac{a}{3d}(\mathcal{F}-m_1)(\mathcal{F}-m_2)(\mathcal{F}-m_3),$$

where m_1, m_2 and m_3 $(m_1 > m_2 > m_3)$ are the roots of the polynomial. Thus, equation (5.3.3) can be written in the form

$$\mathcal{F}'^2 = \frac{a}{3d}(\mathcal{F}-m_1)(\mathcal{F}-m_2)(\mathcal{F}-m_3). \tag{5.3.6}$$

As a result, we get the solution of (5.0.1) in the form of the Jacobi elliptic cosine function (Billingham and King, 2000; Kudryashov, 2004)

$$u(t,x) = m_2 + (m_1 - m_2)\mathrm{cn}^2\left(\sqrt{\frac{a(m_1-m_3)}{12d}}(x-ct), N^2\right), \tag{5.3.7}$$

where $N^2 = \frac{m_1 - m_2}{m_1 - m_3}$ and cn is the elliptic cosine function. The 3D wave profile of solution (5.3.7) for distinct constant values $m_1 = 0.57$, $m_2 = 1.02$, $m_3 = 0$, $d = -0.144$, and $\nu = 0.70$ is shown in Figure (5.3).

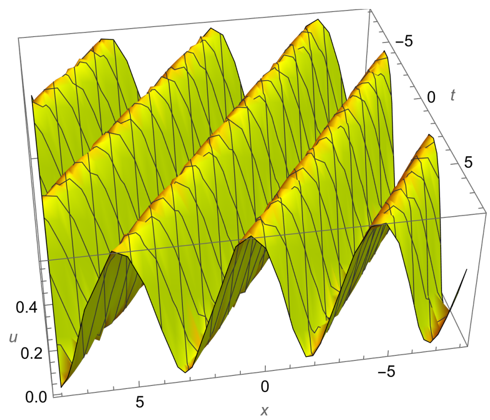

Figure 5.3: 3D wave profile of solution (5.3.7).

5.4 Concluding remarks

In this chapter, we studied the KdV equation with three dispersion sources. The travelling wave transformation was used to convert this equation to a third-order NLODE, which was then solved using Kudryashov's method. The multiplier method was used to derive the conserved vectors for (5.0.1). We discovered three multipliers, which resulted in three local conservation laws. Equation (5.0.1) was reduced to a first-order NLODE using the travelling wave reduction method with conservation laws. We solved this NLODE and obtained the Jacobi elliptic cosine and solitary wave solutions. These solutions' 3D wave profiles were plotted.

6. Conclusion

In Chapter 2, we discussed the core ideas of Lie symmetry analysis of differential equations and conservation laws . Kudryashov's approach for obtaining travelling wave solutions of NLPDEs was discussed briefly.

In Chapter 3, we examined the equal-width equation. Using the Lie group symmetry approach, we determined the Lie point symmetries admitted by this equation. Three Lie point symmetries were found, and symmetry reduction was performed. Furthermore, the multiplier approach was utilized to obtain conservation laws for the equal-width equation. This approach yielded three multipliers, resulting in three local conservation laws. We were able to get functionally independent first integrals, which were then combined to reduce equation (3.0.1) to a first-order NLODE. We then solved this NLODE, and obtained the Jacobi elliptic cosine and solitary wave solutions. For different parameters, graphical representations of these solutions were provided.

In Chapter 4, we investigated a new nonlinear fourth-order integrable equation from a group standpoint. Four Lie point symmetries, as well as their corresponding one-parameter group of transformations, were found. New transformed solutions were generated from known ones under the group of transformations. Using a linear combination of the time and space translation symmetries, equation (4.0.2) was reduced to a fourth-order NLODE. This NLODE was solved by direct integration and by using Kudryashov's method, and the corresponding wave profiles were plotted with the help of Mathematica. Moreover, conservation laws for (4.0.2) were derived using the multiplier approach. We found two multipliers, which resulted in two local conservation laws. Finally, we were able to reduce (4.0.2) to a second-order NLODE, which provided the solitary wave solution, using the travelling wave reduction method with conservation laws.

In Chapter 5, we examined the KdV equation with three dispersion sources. The travelling wave hypothesis was used to reduce this equation to a third-order NLODE. Kudryashov's method was used to solve this NLODE, yielding an exponential solution. Moreover, the multiplier approach was used to generate conservation laws for (5.0.1). Three multipliers were acquired, resulting in three local conservation laws. We obtained two functionally independent first integrals using the travelling wave reduction approach with conservation laws, which were then combined to produce a first-order NLODE. We were able to solve this NLODE and acquire the solitary wave and Jacobi elliptic cosine function solutions. Using Mathematica, we plotted the 3D wave profiles of these solutions.

In the future, rather than focusing on a single symmetry, we will take into account the reduction of a partial differential equation via conservation laws under an algebra of symmetries.

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