$U \subset \mathbb{R}^m$ is open and $f \in C^r(U, \mathbb{R}^n)$. The $r$-jet of $f$ at $x \in U$ has a canonical representation, namely the Taylor Polynomial of $f$ of order $r$ at $x$. This polynomial map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is uniquely determined by the list of derivatives up to order $r$ of $f$ at $x$. This list belongs to the vector space

$$P^r(m,n) = \mathbb{R}^n \times \prod_{k=1}^{r} L^k_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n)$$

where $L^k_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n)$ denotes the vector space of symmetric $k$-linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Conversely, any element of $P^r(m,n)$ comes from a unique jet in $J^r_x(m,n)$. In this way we have the identifications

$$J^r_x(m,n) = P^r(m,n)$$

and

$$J^r(m,n) = \mathbb{R}^m \times P^r(m,n).$$

In particular, $J^r(m,n)$ is a finite dimensional vector space (for $r$ finite). If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets then $J^r(U,V)$ is an open set of $J^r(m,n)$.

If $M, N$ are manifolds of dimension $m, n$ with $\partial M = \partial N = \emptyset$ and charts $(\varphi, U)$, $(\psi, V)$, the map

$$\theta : J^r(U,V) \to J^r(\varphi U, \psi V)$$
$$\theta : j^r_x f \mapsto j^r_y(\psi f \varphi^{-1})$$

where $y = f(x)$, is a bijection. Thus $\theta$ sends each jet to the jet of its local representation. Since $J^r(\varphi U, \psi V)$ is an open set in the vector space $J^r(m,n)$ (which is isomorphic to a Euclidean space), we can view $(\theta, J^r(U,V))$ as a chart on $J^r(M,N)$. The topology on $J^r(M,N)$ is determined by these charts. In this way $J^r(M,N)$ is a $C^0$ manifold[3].

For each $C^r$ map $f : M \to N$ define a map

$$j^r f : M \to J^r(M,N)$$
$$x \mapsto j^r f(x).$$

This *r-prolongation* of $f$ is continuous and is clearly injective.

**Theorem 2.10.** The image of

$$j^r : C^r(M,N) \to C(M, J^r(M,N))$$

is closed in the weak topology.

*Proof.* We prove that the image is closed under uniform convergence on compact sets. It suffices to consider convex compact subsets of co-ordinate charts. Ultimately, we must prove that if $U \subset \mathbb{R}^m$ is open and $\{f_n\}$ is a sequence such that for each $k = 0, \ldots, r$ the sequence $\{D^k f_n(x)\}$ converges uniformly on $U$ to a continuous map $g_k : U \to L^k(\mathbb{R}^m, \mathbb{R}^n)$, then $g_k = D^k g_0$ (the $k$-th derivative of $g_0$). This is proved by induction on $k$. The inductive step is the same as the case for $k = 1$.

[3] If $M, N$ are $C^{r+s}$ manifolds, $J^r(M,N)$ is a $C^s$ manifold.

If $Df_n$ converges uniformly to $g_1$ and $f_n$ converges uniformly to $g_0$, we have that for $x, x+y \in U$

$$\begin{aligned} g_0(x+y) &= \lim_{n\to\infty} f_n(x+y) \\ &= \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} \int_0^1 Df_n(x+ty)y\,dt \\ &= g_0(x) + \int_0^1 g_1(x+ty)y\,dt \end{aligned}$$

by uniform convergence. It follows that $g_1 = Dg_0$ □

The topology on $C^r(M, N)$ induced by $j^r$ from the weak (resp. strong) topology on $C(M, J^r(M, N))$ is called the *weak* (resp. *strong* ) topology on $C^r(M, N)$ and is denoted $C^r_w(M, N)$ (resp. $C^r_s(M, N)$).

From theorems 2.7, 2.8 and 2.10 we obtain

**Theorem 2.11.** The strong and weak topologies have the following properties :

1. $C^r_w(M, N)$ has a complete metric.
2. Every weakly closed subspace of $C^r_s(M, N)$ is a Baire space (in the strong topology).

In the general case where $M, N$ are allowed to have boundaries the definition of the $r$-jet remains unchanged. We do however have to examine the topology on $J^r(M, N)$.

Let $U, V$ be open sets in the halfspaces $H^m \subset \mathbb{R}^m$, $H^n \subset \mathbb{R}^n$. For each $(x, y) \in U \times V$ there are canonical identifications (for $r < \infty$)

$$J^r_{x,y}(U, V) = \prod_{k=1}^{r} L^k_{sym}(\mathbb{R}^m, \mathbb{R}^n) = J^r_{0,0}(m, n)$$

consequently, $J^r(U, V) = U \times V \times J^r_{0,0}(m, n)$.

If either $\partial U$ or $\partial V$ is empty this is an open subset of a halfspace. If $\partial U \neq \emptyset$ and $\partial V \neq \emptyset$ it is not. It is however homeomorphic to an open subset of a halfspace; this follows from the same property for $U \times V$ which in turn follows from the homeomorphism

$$[0, \infty) \times [0, \infty) \cong \mathbb{R} \times [0, \infty).$$

thus again $J^r(M, N)$ has a $C^0$ manifold structure and the preceding development goes through[4].

## 2.3. Transversality Theory.

We turn our attention now to issues of stability. Particularly useful from the viewpoint of applications is the existence of properties which do not change at all, at least locally, when the original system is perturbed slightly. If a mathematical model describes a physical system, its parameters are usually only known with limited precision. If a particular property is observed for an open set of dynamical systems and hence an open set of parameters, then it is likely to reflect the true properties of the system.

[4] If $M, N$ are $C^{r+r}$ manifolds, both with boundary, $J^r(M, N)$ has no natural $C^r$ structure. In the case in which we shall work, $\partial N = \emptyset$ and so there will be a natural manifolds structure on the jet space.

Structural stability is a good example of such a situation. Any topological property for a structurally stable diffeomorphism is preserved under small perturbations. In the nonstructurally stable case however, the most one can hope for is to say something about "most" systems, that is to describe a "typical" system. For a family of systems depending smoothly on a finite number of parameters, there are two competing notions of "typical" :

1. Using the natural Lebesgue measure on the space of parameters. A property is typical in a certain domain $D$ of parameters, if it holds for all parameter values in $D$ except a set of measure zero.
2. If we consider open sets as large, a property is called typical or *generic* if it holds for a set of parameters which is a countable intersection of open dense subsets of $D$. Such a set is called *residual in* $D$.

These two notions are quite distinct (see for example [15] exercises 7.1.4 and 7.1.5).

The reason for the importance of the second notion is the Baire theorem : *In a complete metric space a countable intersection of open dense sets is dense.*

In §3 we shall study certain generic properties of maps in the $C^r$ topology, $r \geq 1$. The key notion in dealing with generic properties in this topology is transversality. Assume, from now on, that unless otherwise specified, all manifolds are $C^\infty$

**Definition 2.12.** A point $x \in M$ is called *critical* for a $C^1$ map $f : M \to N$ if the linear map $T_x f : M_x \to N_y$ is not surjective. We denote by $\Sigma_f$ the set of critical points of $f$. The set $N \backslash f(\Sigma_f)$ is then called the set of *regular values* of $f$.

Let $f : M \to N$ be a $C^1$ map and $A \subset N$ a submanifold. If $K \subset M$, we write $f \pitchfork_K A$ to mean *$f$ is transverse to $A$ along $K$*, that is, whenever $x \in K$ and $f(x) = y \in A$, the tangent space $N_y$ is spanned by $A_y$ and the image $T_x f(M_x)$. When $K = M$ we simply write $f \pitchfork A$. Define

$$\pitchfork_K^r (M, N; A) = \{f \in C^r(M, N) \mid f \pitchfork_K A\}$$

$$\pitchfork^r (M, N; A) = \pitchfork_M^r (M, N; A).$$

**Theorem 2.13** (Inverse Function Theorem). If $y \in \mathbb{R}^n$ is a regular value of a $C^r$ map $f : U \to \mathbb{R}^n$, $U$ open in $\mathbb{R}^k$, then either $f^{-1}(y)$ is empty or it is a submanifold $V$ of $U$ of dimension $k - n$.

As an application of this theorem, we have the following regular value theorem for the case of manifolds with boundary.

**Theorem 2.14.** Let $M$ be a $C^r$ manifold with boundary and $N$ a $C^r$ manifold (with or without boundary), $r \geq 1$. Let $f : M \to N$ be a $C^r$ map. If $y \in N \backslash \partial N$ is a regular value for both $f$ and $f|\partial M$, then $f^{-1}(y)$ is a neat $C^r$ submanifold of $M$.

*Proof.* The property of being a neat $C^r$ submanifold has *local character*, that is, it is true of $A \subset M$ if and only if it is true of $A_i \subset M_i$ where $\{A_i\}$ is an open cover of $A$ and each $M_i$ is an open subset of $M$ containing $A_i$. This property is also *invariant under $C^r$ diffeomorphisms*, that is, $A \subset M$ is a neat submanifold if and only if $g(A) \subset M'$ is a $C^r$ submanifold where $g : M \to M'$ is a $C^r$ diffeomorphism (or even a $C^r$ embedding).

Using these properties, we now reduce the theorem to the case where $M$ is an open set of the $m$-halfspace $H^m \subset \mathbb{R}^m$ and $N = H^n$. Since $y \in N \backslash \partial N$ is a regular value for both $f$ and $f|\partial M$, the result follows from the Inverse Function Theorem (theorem 2.13).

□

**Theorem 2.15.** Let $A \subset N$ be a $C^r$ submanifold. Suppose that either

a. $A$ is neat or
b. $A \subset N \backslash \partial N$ or
c. $A \subset \partial N$.

If $f : M \to N$ is a $C^r$ map with both $f$ and $f|\partial M$ transverse to $A$, then $f^{-1}(A)$ is a $C^r$ submanifold and $\partial f^{-1}(A) = f^{-1}(\partial A)$.

*Proof.* We shall prove the theorem assuming (a), again working locally (the other cases are similar). Therefore, replace the pair $(N, A)$ by $(U \times V, U \times 0)$ where $U \times V \subset H^p \times H^q$ is an open neighbourhood of $(0, 0)$. The maps $f : M \to U \times V$ and $f|\partial M : \partial M \to \partial U \times V$ are transverse to $U \times 0$ and $\partial U \times 0$ if and only if the composite maps

$$g : M \xrightarrow{f} U \times V \xrightarrow{\pi} V$$

$$g|\partial M : \partial M \xrightarrow{f|\partial M} \partial U \times V \xrightarrow{\pi} V$$

have 0 for a regular value. Since $f^{-1}(U \times 0) = g^{-1}(0)$ and $(f|\partial M)^{-1}(\partial U \times 0) = (g|\partial M)^{-1}(0)$, the result follows from theorem 2.14. Also note that codimension $(A)$ = codimension $(f^{-1}A)$.

□

2.3.1. *The Morse-Sard Theorem.* An $n$ cube $C \subset \mathbb{R}^m$ of edge $\lambda > 0$ is a product

$$C = I_1 \times \ldots \times I_m \subset \mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^m$$

of closed intervals of length $\lambda$. The *measure* or *m-measure* of $C$ is $\mu(C) = \mu_m(C) = \lambda^m$.

A subset $X \subset \mathbb{R}^m$ has measure zero if, for every $\epsilon > 0$, $X$ can be covered by a family of $m$-cubes, the sum of whose measures is less than $\epsilon$. A countable union of sets of measure zero has measure zero. Therefore $X$ has measure zero if every point of $X$ has a neighbourhood in $X$ of measure zero.

**Lemma 2.16.** Let $U \subset \mathbb{R}^m$ be an open and $f : U \to \mathbb{R}^m$ a $C^1$ map. If $X \subset U$ has measure zero, so has $f(X)$.

*Proof.* Every point of $X$ belongs to an open ball $B \subset U$ such that $\|Df(x)\|$ is uniformly bounded on $B$, say by $k$. Then

$$|f(x) - f(y)| \leq k|x - y|$$

for all $x, y \in B$. It follows that if $C \subset B$ is an $m$-cube of edge $\lambda$, then $f(C)$ is contained in an $m$-cube $C'$ of edge less than $\sqrt{m}k\lambda = L\lambda$. Therefore $\mu(C') < L^m\mu(C)$.

Write $X = \bigcup_{j=1}^{\infty} X_j$ where each $X_j$ is a compact subset of a ball $B$ as above. For each $\epsilon > 0$, $X_j \subset \bigcup_k C_k$ where each $C_k$ is an $m$-cube and $\sum \mu(C_k) < \epsilon$. It follows that $f(X_j) \subset \bigcup_k C'_k$ where the sum of the measures of the $m$-cubes is less than $L^m\epsilon$. Hence, each $f(X_j)$ has measure zero and so $X$ has measure zero. □

Now let $M$ be a ($C^\infty$) $m$-dimensional manifold. A subset $X \subset M$ is said to have measure zero if, for each chart $(\varphi, U)$, the set $\varphi(U \cap X) \subset \mathbb{R}^m$ has measure zero. By lemma 2.16, this will be true provided there is some atlas of charts with this property.

Since a cube does not have measure zero, a subset of measure zero in $\mathbb{R}^m$ has an empty interior. It follows that a closed measure zero subset of $\mathbb{R}^m$ or a manifold $M$ is nowhere dense. More generally, suppose $X \subset M$ is measure zero and is $\sigma$-compact, that is, $X$ is a countable collection of compact sets. Each of these subsets is nowhere dense and so $X$ is nowhere dense by the Baire category theorem. The complement of $X$ is residual and therefore dense.

We state now the theorem of Morse and Sard. Since the proof involves methods that are not used at all in this survey and indeed throughout differential topology, it is not presented here.

**Theorem 2.17** (Morse and Sard). Let $M, N$ be manifolds of dimension $m, n$ and $f : M \to N$, a $C^r$ map. If $r > \max\{0, m-n\}$, then $f(\Sigma_f)$ has zero measure in $N$. The set of regular values is residual and therefore dense.

Immediate consequences of Sard's theorem show that there is no retraction $D^n \to S^{n-1}$ and the same argument implies that if $M$ is any compact smooth manifold, there is no retraction $M \to \partial M$. Given the correct boundary conditions for an excess demand function, this result can be used to prove that there are equilibrium points for an economy. See Smale [31].

The main result of this section is :

**Theorem 2.18** (Transversality Theorem). Let $M, N$ be manifolds and $A \subset N$ a submanifold. Let $1 \leq r \leq \infty$ then

1. $\pitchfork^r (M, N; A)$ is residual and therefore dense in $C^r_s(M, N)$;
2. If, in addition, $A$ is closed in $N$ and $L \subset M$ is closed then $\pitchfork^r_L (M, N; A)$ is dense and open in $C^r_s(M, N)$.

We shall prove this theorem locally and then globalize using a standard globalization technique. Since this technique will be used in both theorems 2.18 and 2.23 it will be presented abstractly below.

Let $M$ and $N$ be $C^r$ manifolds $0 \leq r \leq \infty$. By a $C^r$ *mapping class* on $(M, N)$ we mean a function $\mathcal{X}$ of the following type. The domain of $\mathcal{X}$ is the set of triples $(L, U, V)$ where $U \subset M, V \subset N$ are open, $L \subset M$ is closed and $L \subset U$. To each triple, $\mathcal{X}$ assigns a set of maps $\mathcal{X}_L(U, V) \subset C^r(U, V)$. In addition, $\mathcal{X}$ satisfies the following localisation axiom :

**Axiom.** Given triples $(L, U, V)$, a map $f \in C^r(U, V)$ is in $\mathcal{X}_L(U, V)$ provided there are triples $(L_i, U_i, V_i)$ and maps $f_i \in \mathcal{X}_{L_i}(U_i, V_i)$ such that $L \subset \cup L_i$ and $f = f_i$ in a neighbourhood of $L_i$ for all $i$.

The first example we shall use of a $C^r$ mapping class is $\mathcal{X}_L(U,V) = ⋔^r_L(U,V;V\cap A)$.

A mapping class $\mathcal{X}$ is called *rich* if there are open covers $\mathcal{U},\mathcal{V}$ of $M,N$ such that whenever open sets $U\subset M, V\subset N$ are subsets of elements of $\mathcal{U},\mathcal{V}$ and $L\subset U$ is compact, then $\mathcal{X}_L(U,V)$ is dense and open in $C^r_w(U,V)$.

**Theorem 2.19** (Globalisation Theorem). Let $\mathcal{X}$ be a rich $C^r$ mapping functor on $(M,N)$, $0\le r\le\infty$. For every closed set $L\subset M$, $\mathcal{X}_L(M,N)$ is dense and open in $C^r_s(M,N)$.

*Proof.* Fix $L$ and $f\in C^r(M,N)$. In what follows, $i$ runs over a countable index set $\Lambda$. Let $\Phi=\{\varphi_i,U_i\}$ be a locally finite atlas on $M$, $K_i\subset U_i$ compact sets such that $L=\bigcup K_i$ and $\Psi=\{\psi_i,V_i\}$ a family of charts on $N$ such that $f(K_i)\subset V_i$. Since $\mathcal{X}$ is rich, we can choose $U_i,V_i$ such that $\mathcal{X}_{K_i}(E,V_i)$ is dense and open in $C^r_w(E,V_i)$ for every open set $E\subset U_i$ which contains $K_i$.

Define $\mathcal{M}\subset C^r(M,N)$ to be the set of all $g\in C^r(M,N)$ such that $g\mid U_i\in\mathcal{X}_{K_i}(U_i,V_i)$, for all $i\in\Lambda$ The localisation axiom implies $\mathcal{M}\subset\mathcal{X}_L(M,N)$.

Suppose $f\in\mathcal{X}_L(M,N)$ (and hence in $\mathcal{M}$ by localisation). The assumption that each $\mathcal{X}_{K_i}(U_i,V_i)$ is weakly open implies $\mathcal{M}$ is strongly open (because $\Phi$ is locally finite).

Now we drop the assumption that $f\in\mathcal{X}_L(M,N)$ and proceed to the density part of the theorem.

For each $i$, let $\epsilon_i>0$ be given. Then there is a strong basic neighbourhood

$$\mathcal{N}=\mathcal{N}^r(f,\Phi,\Psi,K,\epsilon)$$

where $K=\{K_i\}_{i\in\Lambda}$ and $\epsilon=\{\epsilon_i\}_{i\in\Lambda}$.

Fix $j\in\Lambda$. Let $E=U_j\cap f^{-1}(V_j)$; then $K_j\subset E$. Since $\mathcal{X}$ is rich, $\mathcal{X}_{K_j}(E,V_j)$ is dense. Let $\lambda:E\to[0,1]$ be a $C^r$ map with compact support, such that $\lambda=1$ near $K_j$.

To simplify notation, identify $V_j$ with an open subset of a halfspace via $\psi_j$ so that vector operations make sense on elements of $V_j$.

If $g\in C^r_w(E,V_j)$ is sufficiently close to $f|E$, the following map $\Gamma(g)=h\in C^r(M,N)$ is well defined

$$h(x)=\begin{cases} f(x)+\lambda(x)[g(x)-f(x)] & \text{if } x\in E\\ f(x) & \text{if } x\in M\backslash E.\end{cases}$$

Moreover, as $g$ tends to $f|E$ in the weak topology $h\to f$ in the *strong* topology. Since $\mathcal{X}_{K_j}(E,V_j)$ is dense, we can choose a $g\in\mathcal{X}_{K_j}(E,V_j)$ so close to $f|E$ that $h\in\mathcal{N}$. Since $h=g$ near $K_j$ it follows that $h\in\mathcal{X}_{K_j}(M,N)$.

This shows that for all $i\in\Lambda$ the set $\mathcal{X}_{K_j}(M,N)$ is dense in $C^r_s(M,N)$. We already know that it is open and therefore from Baire we know that $\bigcap_i\mathcal{X}_{K_j}(M,N)$ is strongly dense. Since this set contains $\mathcal{X}(M,N)$, the latter is strongly dense. $\square$

The proof of theorem 2.18 is based on the following result :

**Lemma 2.20.** Let $K$ be a compact set in a manifold $U$, $\mathbb{R}^a\subset\mathbb{R}^n$ a linear subspace, $V\subset\mathbb{R}^n$ an open set. Then $⋔^r_K(U,V;\mathbb{R}^a\cap V)$ is dense and open in $C^r_w(U,V)$, $1\le r\le\infty$.

*Proof.* Since $C^r(U,V)$ is open in $C^r_w(U,\mathbb{R}^n)$, it suffices to take $V=\mathbb{R}^n$. Let $\pi:\mathbb{R}^n \to \mathbb{R}^n \setminus \mathbb{R}^a$ be the projection. If $f \in C^r(U,\mathbb{R}^n)$ and $x \in U$, then $f \pitchfork_x \mathbb{R}^a$ if and only if either

i. $f(x) \notin \mathbb{R}^a$; or
ii. $f(x) \in \mathbb{R}^a$ and $x$ is a regular point for $\pi f : U \to \mathbb{R}^n \setminus \mathbb{R}^a$.

Suppose $f \pitchfork_K \mathbb{R}^a$. Then, each $y \in K$ has a compact neighbourhood $K_y \subset K$ such that either (i) holds for all $x \in K_y$ or (ii) holds for all $x \in K_y$. Fix such a $K_y$. In either of the two cases, the subset of $C^r_w(U,\mathbb{R}^n)$ for which $f \pitchfork_{K_y} \mathbb{R}^a$ is open. Since $K$ is covered by a finite set $K_1,\dots,K_s$ of such neighbourhoods, it follows that $\pitchfork^r_K (U,\mathbb{R}^n;\mathbb{R}^a)$ is open.

To prove the denseness, we use the result (not proved here, see for example [12]) that $C^\infty$ maps are dense in $C^r_w(U,\mathbb{R}^n)$. It therefore suffices to show that an arbitrary $C^\infty$ map $g:U\to\mathbb{R}^n$ is in the closure of $\pitchfork^r_K (U,\mathbb{R}^n;\mathbb{R}^a)$.

Let $\{y_k\}$ be a sequence in $\mathbb{R}^n$ tending to 0 such that each $\pi(y_k)$ is a regular value of $\pi g : U \to \mathbb{R}^n \setminus \mathbb{R}^a$. Define

$$g_k : U \to \mathbb{R}^n$$
$$g_k(x) = g(x) - y_k.$$

Then $g_k \to g$ in $C^r_w(U,\mathbb{R}^n)$. Since $g_k \pitchfork \mathbb{R}^a$, $\pitchfork^r_K (U,\mathbb{R}^n;\mathbb{R}^a)$ is dense in $C^r_w(U,\mathbb{R}^n)$ □

*Proof of transversality theorem.* Assume initially that $A$ is closed and $\partial N = \emptyset$ and we prove the second part first.

From lemma 2.20, it follows that, for $1 \le r \le \infty$, the function

$$\mathcal{X} : (L,U,V) \to \pitchfork^r_L (U,V;A\cap V)$$

is a rich[5] $C^r$ mapping class on $(M,N)$. From the Globalisation Theorem 2.19, it follows that $\pitchfork^r_L (M,N;A)$ is strongly dense and open.

Now suppose $A$ is still closed but $\partial N \neq 0$. We may assume $N \subset \mathbb{R}^q$ is a closed submanifold. Then the weak and strong topologies on $C^r(M,N)$ are those it inherits from the weak and strong topologies on $C^r(M,\mathbb{R}^q)$. We have already shown that $\pitchfork^r (M,\mathbb{R}^q;A)$ is strongly open since $A \subset \mathbb{R}^q$ is closed and $\partial\mathbb{R}^q = \emptyset$. Therefore, the equation

$$\pitchfork^r (M,N;A) = C^r(M,N) \cap \pitchfork^r (M,\mathbb{R}^q;A)$$

shows that $\pitchfork^r (M,N;A)$ is strongly open in $C^r(M,N)$. A similar proof shows that $\pitchfork^r_L (M,N;A)$ is strongly open.

To prove density, let $N_0 = N \setminus \partial N$, $A_0 = A \setminus \partial N$ so that $\partial N_0 = \emptyset$ and $A_0 \subset N_0$ is a closed submanifold.. Then $\pitchfork^r_L (M,N_0;A_0)$ is strongly dense in $C^r(M,N_0)$. Also, $C^r(M,N_0) \subset C^r(M,N)$ is strongly dense. Therefore, $\pitchfork^r_L (M,N_0;A_0) \subset \pitchfork^r_L (M,N;A)$ is strongly dense. This completes the proof of the second part of the theorem.

[5]Take $\mathcal{U}$ to be any open cover of $M$ and $\mathcal{V}$ be an atlas of co-ordinate domains on $N$ that come from submanifold charts for $(N,A)$ and apply lemma 2.20.

To complete the proof, assume $A$ is not closed. Let $A_k$ be a countable family of compact co-ordinate disks on $A$. Then

$$\pitchfork (M, N; A) = \bigcap_{k=1}^{\infty} \pitchfork^r (M, N; A_k).$$

Since $A_k$ is closed, $\pitchfork (M, N; A_k)$ is strongly dense and open which proves $\pitchfork^r (M, N; A)$ is residual.

$\square$

Oin our applications we shall not deal with all $C^r$ maps from $M$ to $N$ but only with a family of maps parametrised by another manifold $V$. In this case we have a map $F : V \to C^r(M, N)$ and a submanifold $A \subset N$ and we wish to determine the size of the set $\{v \in V | F(v) \pitchfork A\}$, where $F(v) = F_v : M \to N$.

**Remark 2.21.** The next two theorems are stated and proved in terms of manifolds without boundaries. The generalisation to the case with boundaries (where the boundary of the submanifold satisfies conditions (a) to (c) of theorem 2.15) is done in the same way as the previous proof.

**Theorem 2.22** (Parametric Transversality Theorem). Let $V, M, N$ be $C^r$ manifolds without boundary and $A \subset N$ a $C^r$ submanifold. Let $F : V \to C^r(M, N)$ satisfy :

1. the *evaluation map* $F^{ex} : V \times M \to N : (v, x) \mapsto F_v(x)$ is $C^r$ ; and
2. $F^{ex}$ is transverse to $A$; and
3. $r > \max\{0, \dim N + \dim A - \dim M\}$.

Then the set $\pitchfork (F; A) = \{v \in V : F_v \pitchfork A\}$ is residual and therefore dense. If $A$ is closed in $N$ and $F$ is continuous for the strong topology on $C^r(M, N)$ then $\pitchfork (F; A)$ is also open.

*Proof.* The last statement follows from the openness of $\pitchfork^r (M, N; A)$. To prove the rest of the theorem let $W = (F^{ev})^{-1}(A) \subset V \times M$. By (1), $W$ is a $C^r$ submanifold of $V \times M$. Let $\pi : V \times M \to V$ be the projection. It is readily verified that $F_v \pitchfork A$ if and only if $v \in V$ is a regular value for the $C^r$ map $\pi | W : W \to V$. The dimension of $W$ is $\dim V + \dim M - \dim N + \dim A$. The theorem follows from the Morse-Sard (theorem 2.17).

$\square$

The situation in §3 requires a natural generalisation of this result to examine the jet map

$$j^r : C^s(M, N) \to C^{s-r}(M, J^r(M, N))$$

$1 \leq r < s \leq \infty$. Given a submanifold $A \subset J^r(M, N)$ we try to approximate a $C^s$ map $g : M \to N$ by another $C^s$ map $h$ whose prolongation $j^r h : M \to J^r(M, N)$ is transverse to $A$. Denote the set of such maps $h$ by $\pitchfork^s (M, N; j^r, A)$. The following theorem is also called the *Jet Transversality Theorem*.

**Theorem 2.23** (Thom Transversality Theorem). Let $M$ and $N$ be $C^\infty$ manifolds without boundary (see remark 2.21) and $A \subset J^r(M, N)$ a $C^\infty$ submanifold. Let $1 \leq r < s \leq \infty$ then $\pitchfork^s (M, N; j^r, A)$ is residual and therefore dense in $C^s_S(M, N)$. It is open if $A$ is closed.

*Proof.* Suppose $A$ is closed. Openness follows from the openness of

$$C^{s-r}(M, J^r(M, N, ; A)).$$

To prove density let $U \subset M$, $V \subset N$ be open sets and let $L \subset U$ be closed in $M$. Define

$$\pitchfork_L (U, V) = \{f \in C^s(U, V) : j^r f \pitchfork_L A\}.$$

One easily sees that $\mathcal{X} : (L, U, V) \mapsto \pitchfork_L (U, V)$ is a $C^s$ mapping class on $(M, N)$. By the Globalisation Theorem (theorem 2.19), it suffices to prove $\mathcal{X}$ is rich. For the covering $\mathcal{U}$, $\mathcal{V}$ in the definition of rich choose any open coverings by co-ordinate domains. It now suffices to prove that if $U \subset \mathbb{R}^m$ is an open subset and $A \subset J^r(U, \mathbb{R}^n)$ is a closed submanifold, then $\pitchfork^s (U, \mathbb{R}^n; j^r, A)$ is open and dense in $C^s_w(U, \mathbb{R}^n)$. Moreover, it is enough to prove this for $s$ finite, $s > r$. Fix $f \in C^s(U, \mathbb{R}^n)$. Openness is obvious. To prove denseness we find a $C^\infty$ manifold $X$ and a map $\alpha : X \to C^s_w(U, \mathbb{R}^n)$ with $f \in \alpha(x)$ and then apply parametric transversality to the composition

$$F : X \xrightarrow{\alpha} C^s(U, \mathbb{R}^n) \xrightarrow{j^r} C^{s-r}(U, J^r(U, \mathbb{R}^n)).$$

This requires the evaluation map of $F$,

$$F^{ev} : X \times U \to J^r(U, \mathbb{R}^n)$$

to be transverse to $A$ and sufficiently differentiable ($F^{ev}$ will be a $C^\infty$ submersion).

Put $X = J^s_0(\mathbb{R}^m, \mathbb{R}^n)$. Every element of $X$ is the $s$-jet at 0 of a unique mapping $g : \mathbb{R}^m \to \mathbb{R}^n$ whose co-ordinate maps $g_1, \ldots, g_n$ are polynomials of degree $\leq s$ in the co-ordinates of $\mathbb{R}^m$. We identify the elements of $X$ with such maps $g$.

Define $\alpha : X \to C^s_w(U, \mathbb{R}^n)$, $g \mapsto f + g|U$ and

$$F = j^r \circ \alpha : X \to C^{s-r}(U, J^r(U, \mathbb{R}^n)).$$

Then $F(0) = f$. To compute $F^{ev}$ make the natural identification

$$J^r(U, \mathbb{R}^n) = J^r_0(U, \mathbb{R}^n) \times U.$$

Then the map

$$F^{ev} : J^s_0(\mathbb{R}^m, \mathbb{R}^n) \times U \to J^r_0(\mathbb{R}^m, \mathbb{R}^n) \times U$$

is given by

$$(j^s_0(g), x) \mapsto (j^r_0(g + f), x).$$

The map

$$\begin{aligned} \beta : J^s_0(\mathbb{R}^m, \mathbb{R}^n) &\to J^r_0(\mathbb{R}^m, \mathbb{R}^n) \\ j^s_0(g) &\mapsto j^r_0(g + f) \end{aligned}$$

is affine, hence $F$ is $C^\infty$. Moreover the derivative of $\beta$ at any point is the "forgetful" linear map

$$\begin{aligned} J^s_0(\mathbb{R}^m, \mathbb{R}^n) &\to J^r_0(\mathbb{R}^m, \mathbb{R}^n) \\ j^s_0(g) &\to j^r_0(g) \end{aligned}$$

which is surjective. Thus $F^{ev} \pitchfork A$; by theorem 2.22 it follows that

$$\{x \in X : j^r(\alpha(x)) \pitchfork A\}$$

is dense in $X$. Since

$$\alpha : X \to C^s_w(U, \mathbb{R}^n)$$

is continuous, it follows that $f$ is in the closure of

$$\{h \in C^s_w(U, \mathbb{R}^n) : j^r h \pitchfork A\}.$$

This proves that $\mathcal{X}$ is rich; hence for closed $A$ the theorem follows from the Globalisation Theorem.

If $A$ is not closed then write $A = \bigcup_{k=1}^{\infty} A_k$ where each $A_k$ is a compact co-ordinate disk in $A$. Then each $\pitchfork(M, N; j^r, A_k)$ is dense and open in $C^s_s(M, N)$. By the Baire intersection theorem, $\pitchfork(M, N; j^r, A)$ is dense.

□

2.4. **Lie Groups.** An important special class of differentiable manifolds are the Lie Groups. In sections to come, properties of certain well known Lie groups such as the general linear group, the special linear group and the orthogonal group will be used.

**Definition 2.24.** A *Lie group* $G$ is a differentiable manifold which is endowed with group structure such that the map $G \times G \to G$ defined by $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is $C^\infty$.

**Definition 2.25.** A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) such that for all $x, y, z \in \mathfrak{g}$,

1. $[x, y] = -[y, x]$ (anit-commutativity)
2. $\big[[x, y], z\big] + \big[[y, z], x\big] + \big[[z, x], y\big] = 0.$ (Jocobi identity)

Let $\sigma \in G$, a *left translation* by $\sigma$, $l_\sigma$, is a diffeomorphism of $G$ defined by $l_\sigma(\tau) = \sigma\tau$ for all $\tau \in G$. If $V$ is a subset of $G$ let $\sigma V$ denote the set $l_\sigma(V)$. A vector field $X$ (not necessarily assumed to be smooth) on $G$ is called *left invariant* if, for each $\sigma \in G$, $X$ is $l_\sigma$ related to itself, that is

$$dl_\sigma \circ X = X \circ l_\sigma. \tag{2.2}$$

**Proposition 2.26.** Let $G$ be a Lie group and let $\mathfrak{g}$ be its set of left invariant vector fields.

1. $\mathfrak{g}$ is a real vector space and the map $\alpha : \mathfrak{g} \to G_e (= T_e G)$ defined by $\alpha(X) = X(e)$ is an isomorphism of $\mathfrak{g}$ with the tangent space $G_e$ to $G$ at the identity. Consequently, $\dim \mathfrak{g} = \dim G_e = \dim G$.
2. Left invariant vector fields are smooth.
3. The Lie bracket of two left invariant vector fields is itself a left invariant vector field.
4. $\mathfrak{g}$ forms a Lie algebra under the Lie bracket operation on vector fields (see definition 2.4).

*Proof.* [39] page 85

□

We define the *Lie algebra of the Lie group* $G$ to be the Lie algebra $\mathfrak{g}$ of left invariant vector fields on $G$. Alternatively, we could take as the Lie Algebra of $G$ the tangent space $G_e$ at the identity with the Lie algebra structure induced by requiring the vector space isomorphism ((1) in the proposition 2.26) of $\mathfrak{g}$ with $G_e$ to be an isomorphism of Lie algebras.

The Lie algebras of the classical groups admit particularly elegant interpretations in terms of their tangent spaces at the identity. In particular, the $n^2$ dimensional vector space $gl(n, \mathbb{R})$ of all $n \times n$ real matrices forms a Lie algebra if we set $[A, B] = AB - BA$. The general linear group $GL(n, \mathbb{R})$ inherits its manifold structure as an open set of $gl(n, \mathbb{R})$ where the determinant function does not vanish and Lie group structure under matrix multiplication. Let $x_{ij}$ be the global co-ordinate function on $gl(n, \mathbb{R})$ which assigns to each matrix its $ij$-th entry. Then if $\sigma, \tau \in Gl(n, \mathbb{R})$, $x_{ij}(\sigma\tau^{-1})$ is a rational function of $\{x_{kl}(\sigma)\}$ and $\{x_{kl}(\tau)\}$ with nonzero denominator. This proves that the map $(\sigma, \tau) \to \sigma\tau^{-1}$ is $C^\infty$.

It can be shown that $gl(n, \mathbb{R})$ is the Lie algebra of $Gl(n, \mathbb{R})$ (see [39] page 86).

**Definition 2.27.** Let $G$ and $H$ be Lie groups. A map $\psi : H \to G$ is a *(Lie group) homomorphism* if $\psi$ is both $C^\infty$ and a group homeomorphism of the abstract group. If $G = Gl(n, \mathbb{R})$, then $\psi : H \to G$ is called a *representation* of the Lie group. $(H, \psi)$ is a *Lie subgroup* of the Lie group $G$ if

1. $H$ is a Lie group; and
2. $(H, \psi)$ is a submanifold of $G$; and
3. $\psi : H \to G$ is a group homomorphism.

**Example 2.28.** Define $\psi : Gl(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ where $\mathrm{Sym}(n, \mathbb{R})$ is the vector space of all real symmetric $n \times n$ matrices by

$$\psi(A) = AA^T$$

where $A^T$ is the transpose of $A$.

Let $O = O(n) = \psi^{-1}(I)$, $I$ the identity matrix. $O(n)$ is a subgroup of $Gl(n, \mathbb{R})$, under matrix multiplication, called the *orthogonal group*. To apply the Implicit Function Theorem to conclude that $O(n)$ has a unique manifold structure such that $(O(n), i)$, $i$ the inclusion mapping, is a submanifold of $Gl(n, \mathbb{R})$ and that in this manifold structure $i$ is an imbedding and $O(n)$ has dimension $(n(n-1))/2$, one need only check that $d\psi_\sigma$ is surjective at each $\sigma \in O(n)$. For this it suffices to check that $d\psi_I$ is surjective since, whenever $\sigma \in O(n)$

$$\psi = \psi \circ r_\sigma$$

where $r_\sigma$ is the diffeomorphism of $Gl(n, \mathbb{R})$ defined by $r_\sigma(\tau) = \tau\sigma$ (right translation by $\sigma$). This is done via a directly calculation.

**Proposition 2.29** (and Definition). Let $H$ be a closed subgroup of a Lie group $G$ and let $G/H = \{\sigma H | \sigma \in G\}$. Let $\pi : G \to G/H$ denote the natural projection $\pi(\sigma) = \sigma H$. Then $G/H$ has a unique manifold structure such that

1. $\pi$ is $C^\infty$; and
2. there exist smooth sections of $G/H$ in $G$ (that is, if $\sigma H \in G/H$, there exists a neighbourhood $W$ of $\sigma H$ and a $C^\infty$ map $\tau : W \to G$ such that $\pi \circ \tau = 1_{id}$).

Manifolds of the form $G/H$ are called *homogeneous manifolds*.

**Remark 2.30.** $f$ is a $C^\infty$ function on $G/H$ if and only if $f \circ \pi$ is $C^\infty$ on $G$ for if $f$ is $C^\infty$ then certainly $f \circ \pi$ is $C^\infty$. Conversely if $f \circ \pi$ is $C^\infty$ then $f$, which can be locally represented as a composition of a smooth section of $G/H$ in $G$ with $f \circ \pi$, is also $C^\infty$.

**Example 2.31** (Homogeneous Manifolds). Let $e_i, i = 1, \ldots, n$ be the canonical basis of $\mathbb{R}^n$ where $e_i$ is the $n$-tuple consisting of all zeros except for a 1 in the $i$-th spot. Each matrix $\sigma \in Gl(n, \mathbb{R})$ uniquely determines a linear transformation on $\mathbb{R}^n$ which shall be denoted $\sigma$ by requiring that

$$\sigma(e_j) = \sum_i \sigma_{ij} e_j.$$

In other words, if we consider the $n$-tuples of $\mathbb{R}^n$ as $n \times 1$ matrices, then $\sigma$ acts on $\mathbb{R}^n$ in the natural way by matrix multiplication. The map, which we denote here by $\psi$, which sends $(\sigma, v)$ to $\sigma(v)$ gives an action of $Gl(n, \mathbb{R})$ on $\mathbb{R}^n$ on the left :

$$Gl(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n. \tag{2.3}$$

Let $\langle\, , \rangle$ denote the standard inner product on $\mathbb{R}^n$ with respect to which the basis $\{e_i\}$ is orthonormal. Then if $\sigma \in Gl(n, \mathbb{R})$,

$$\langle \sigma(v), w \rangle = \langle v, \sigma^T(w) \rangle. \tag{2.4}$$

If $\sigma \in O(n)$, the orthogonal group, then $\sigma^T\sigma = I$ and therefore it follows that

$$\langle \sigma(v), \sigma(v) \rangle = \langle v, \sigma^T\sigma(v) \rangle = \langle v, v \rangle;$$

thus $\sigma$ preserves lengths of vectors. Thus the action in (2.3) restricted to $O(n) \times S^{n-1}$ factors through the unit sphere $S^{n-1}$ (that is, in the following diagram $\psi(O(n) \times S^{n-1}) \subset i(S^{n-1})$, $i$ the identity map).

$$\begin{tikzcd}
O(n) \times S^{n-1} \arrow[r, "\psi"] \arrow[dr, "\eta"'] & \mathbb{R}^n \\
& S^{n-1} \arrow[u, "i"']
\end{tikzcd}$$

Obviously the map $\psi$ is an imbedding. It follows that $\eta$ is continuous.

**Proposition 2.32.** $\eta$ is $C^\infty$.

*Proof.* It suffices to show that $S^{n-1}$ can be covered by a co-ordinate system $(U, \tau)$ such that the map $\psi \circ \tau$ restricted to the open set $\eta^{-1}(U)$ is $C^\infty$. Let $p \in S^{n-1}$ and $(V, \gamma)$ a co-ordinate system on a neighbourhood of $i(p)$. As a consequence of the Inverse Function Theorem, there is a projection $\pi$ of $\mathbb{R}^n$ onto a suitable subspace (obtained by setting certain of the co-ordinate functions equal to 0) such that the map $\tau = \pi \circ \gamma \circ i$ yields a co-ordinate system on a neighbourhood $U$ of $p$. Then

$$\tau \circ \eta|_{\eta^{-1}(U)} = \pi \circ \gamma \circ i \circ \eta|_{\eta^{-1}(U)} = \pi \circ \gamma \circ \psi|_{\eta^{-1}(U)}$$

which is $C^\infty$. $\square$

So there is a natural $C^\infty$ action

(2.5) $$O(n) \times S^{n-1} \to S^{n-1}$$

of the orthogonal group $O(n)$ on the unit sphere $S^{n-1}$ on the left. The fact that this action is transitive (that is whenever $v, w \in S^{n-1}$, there is a $\sigma \in O(n)$ such that $\sigma(v) = w$) can be seen as follows :

If $v_1 \in S^{n-1}$, let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ containing $v_1$ as the first element. Let

$$v_i = \sum_j \sigma_{ji} e_j.$$

Then the matrix $\sigma$, whose entries are determined by this equation, is orthogonal and $\sigma(e_1) = v_1$. It follows that if $v$ and $w$ are any two points of $S^{n-1}$, there is an orthogonal matrix $\sigma$ such that $\sigma(v) = w$.

Now the set of matrices in $O(n)$ of the form

$$\begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,(n-1)} & 0 \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,(n-1)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{(n-1),1} & \sigma_{(n-1),2} & \cdots & \sigma_{(n-1),(n-1)} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

forms a closed subgroup of $O(n)$. The matrices

$$\tilde{\sigma} = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,(n-1)} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{(n-1),1} & \sigma_{(n-1),2} & \cdots & \sigma_{(n-1),(n-1)} \end{bmatrix}$$

occurring in this subgroup are precisely the matrices in $O(n-1)$. So $O(n-1)$ sits in $O(n)$ as this natural closed subgroup. We claim that $O(n-1)$ is the isotopy group for the action (2.5) at $e_n \in S^{n-1}$ (If $v \in S^{n-1}$ and $H = \{\sigma \in O(n) | \sigma(v) = v\}$, then $H$ is a closed subgroup of $S^{n-1}$ called the *isotopy group* of $S^{n-1}$ at $v$).

*Proof.* Clearly the elements of $O(n)$ leave $e_n$ fixed. On the other hand, suppose that $\sigma \in O(n)$ and $\sigma(e_n) = e_n$. Since $\sigma(e_n) = \sum_i \sigma_{in} e_i$, it follows that $\sigma_{in} = 0$ for $i < n$ and $\sigma_{nn} = 1$. Since $\sigma$ is orthogonal, $\sigma\sigma^T = I$ and this implies that $\sum_i \sigma_{ni}^2 = 1$. Since $\sigma_{nn} = 1$ we must have that $\sigma_{ni} = 0$, $i < n$, thus $\sigma \in O(n-1)$. □

Finally, we claim that the map

$$\beta : O(n)/O(n-1) \to S^{n-1} \qquad \beta(\sigma O(n-1)) = \sigma(e_n)$$

is a diffeomorphism.

Before we can proceed with the proof of this statement, we introduce some machinery.

The Lie algebra associated with the Lie group $O(n)$ is the space $O(n)_e = \mathfrak{o}(n)$ of *skew symmetric matrices*, that is, $\mathfrak{o}(n) = \{A \in Gl(n) | A + A^T = 0\}$, define a mapping

$$\exp : \mathfrak{o}(n) \to O(n)$$

by

$$\exp tX = e^{tX} = \sum_{i=0}^{\infty} \frac{X^n}{n!}.$$

The next two result are quoted without proof. For a discussion of these results, see [39] pg. 104

**Theorem 2.33.** Let $i : O(n-1) \to O(n)$ be the inclusion homeomorphism. Then the following diagram is commutative.

$$\begin{tikzcd} O(n-1) \arrow[r, "i"] & O(n) \\ \mathfrak{o}(n-1) \arrow[u, "\exp"] \arrow[r, "di"] & \mathfrak{o}(n) \arrow[u, "\exp"'] \end{tikzcd}$$

**Proposition 2.34.** Let $X \in \mathfrak{o}(n)$. If $X \in di(\mathfrak{o}(n-1))$ then $\exp tX \in i(O(n-1))$ for all $t$. Conversely, if $\exp tX \in i(O(n-1))$ for $t$ in some open interval, then $X \in di(\mathfrak{o}(n))$.

*Proof that $\beta$ is a diffeomorphism.* Observe first that $\beta$ is well defined since

$$\beta(\sigma(\iota(e_n))) = \beta(\sigma(e_n))$$

for any $\iota \in O(n-1)$. $\beta$ is surjective since $O(n)$ acts transitively. It is injective since if

$$\beta(\sigma_1(O(n-1))) = \beta(\sigma_2(O(n-1)))$$

then $\sigma_2^{-1}\sigma_1(\epsilon_n) = \epsilon_n$ which implies that $\sigma_2^{-1}\sigma_1 \in O(n-1)$ or $\sigma_1 O(n-1) = \sigma_2 O(n-1)$.

To prove that $\beta$ is a diffeomorphism it suffices to prove that $\beta$ is $C^\infty$ and that $d\beta$ is non-singular at each point[6].

By remark 2.30, $\beta$ is $C^\infty$ if and only if $\beta \circ \pi$ is $C^\infty$ on $O(n)$ where $\pi$ is the natural projection of $O(n)$ onto $O(n)/O(n-1)$. Now $\beta \circ \pi = \eta \circ i_{\epsilon_n}$ where $i_{\epsilon_n}$ is the $C^\infty$ map of $O(n)$ into $O(n) \times S^{n-1}$ defined by $i_{\epsilon_n}(\sigma) = (\sigma, \epsilon_n)$ and $\eta$ is the transitive action of the Lie group $O(n)$ on $S^{n-1}$, that is, $\eta(\sigma, \epsilon_n) = \sigma(\epsilon_n)$. Thus $\beta$ is $C^\infty$.

To show that $d\beta$ is non-singular at each point. Let $\tilde{\beta} = \beta \circ \pi$. Since the kernel of $d\pi|O(n)_\sigma$ is

$$\left(\sigma O(n-1)\right)_\sigma \subset O(n)_\sigma,$$

in order to prove that

$$d\beta|(O(n)/O(n-1))_{\sigma O(n-1)}$$

is non-singular it is sufficient to prove that the kernel of $d\tilde{\beta}|O(n)_\sigma$ is also $(\sigma O(n-1))_\sigma$. Since for each $\sigma \in O(n)$

$$\tilde{\beta} = \eta_\sigma \circ \tilde{\beta} \circ l_{\sigma^{-1}}$$

($\eta_\sigma = \eta(\sigma, \cdot)$ the transitive action of the Lie group and $l_\sigma$ is left translation), it suffices to show that the kernel of $d\tilde{\beta}|O(n)_e$ is $O(n-1)_e$, $e$ the identity. Certainly $O(n-1)_e \subset \ker(d\tilde{\beta}|O(n)_e)$ so let $x \in \ker(d\tilde{\beta}|O(n)_e)$. To show $x \in O(n-1)_e$, let

[6] See appendix A.