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# Inertial reflected Krasnosel'skiĭ-Mann iteration with applications to monotone inclusion problems

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**ABSTRACT**
This paper studies an inertial reflected method for the approximation of fixed points (assuming existence) of nonexpansive mappings in Hilbert spaces. The proposed method is a combination of Krasnosel'skiĭ-Mann iteration, inertial extrapolation step and reflected step. The aim is to improve and complement various versions of inertial Krasnosel'skiĭ-Mann iterations and the recently proposed reflected Krasnosel'skiĭ-Mann iteration in the literature for approximating fixed points of nonexpansive mappings in Hilbert spaces. Some applications to problems of finding zeros of the sum of monotone operators are given. Finally, numerical simulations including image restoration problems are performed using standard test examples to show the superiority of our proposed method.

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## 1. Introduction and preliminaries

In this paper, let us take $H$ to be a real Hilbert space with the inner product $\langle .,. \rangle$ and the induced norm $\|\cdot\|$. A mapping $T : H \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in H.$$

Approximation of fixed of nonexpansive mappings has been a topic of interest for researchers in continuous optimization and fixed point theory. Some of the many important results on fixed point problems for nonexpansive mappings can be found in the literature (e.g. [1–5] and references therein). One of the most important iteration methods to approximate a fixed point of a nonexpansive mapping $T$ is the Krasnosel'skiĭ-Mann iteration [6–8], which is of the form: $x_1 \in H$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \quad \forall\, n = 1, 2, \ldots \tag{1}$$

where $\{\alpha_n\} \subset (0, 1)$. Several conditions have been imposed on $\{\alpha_n\}$ in order to obtain weak convergence results to a fixed point of nonexpansive mapping $T$ for

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the sequence $\{x_n\}$ generated by Krasnosel'skiĭ-Mann iteration (1). For some of these results, please see, for example, [8–19] and the references contained therein.

Recently, the study of Krasnosel'skiĭ-Mann iteration (1) with inertial extrapolation step for approximation of fixed of nonexpansive mapping $T$ has been of interest to many researchers. This is because it has been shown in applications to optimization problems, image restoration, signal processing, to mention but a few, that the presence of inertial extrapolation step improves the efficiency and increases the convergence speed of Krasnosel'skiĭ-Mann iteration (1) in terms of number of required iterations and the CPU time. Some of these related results are found, for example, in [20–32].

In the recent paper of Dong [33], Krasnosel'skiĭ-Mann iteration (1) with inertial extrapolation step was proposed and studied for finding a fixed point of a nonexpansive mapping $T$: $x_0, x_1 \in H$,

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}) \\ x_{n+1} = y_n + \alpha_n(Ty_n - y_n), \end{cases} \tag{2}$$

where $\delta_n \in [0,1)$, $\alpha_n \in (0,1)$ and the step $y_n = x_n + \delta_n(x_n - x_{n-1})$ is the inertial extrapolation step. Weak convergence results are obtain in [33] using (2) under some appropriate conditions on $\{\alpha_n\}$ and $\{\delta_n\}$ (see [33, Theorem 3.1] for more details). The results of Dong [33] improved on the earlier results of inertial Krasnosel'skiĭ-Mann iteration obtained in Bot et al. [34] and Maingé [30] by providing new conditions on $\{\delta_n\}$ to obtain weak convergence and depend only on the iteration coefficients.

Dong et al. [35] studied the following general inertial Krasnosel'skiĭ-Mann iteration for nonexpansive mapping $T$ in Hilbert spaces:

$$\begin{cases} z_n = x_n + \delta_n(x_n - x_{n-1}) \\ y_n = x_n + \beta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Tz_n, \end{cases} \tag{3}$$

where $\{\beta_n\}$ and $\{\delta_n\}$ are chosen in $[0,1)$ satisfying other conditions and $\{\alpha_n\} \subset (0,1)$. Using (3), Dong et al. [35] obtained weak convergence results for fixed point problem involving nonexpansive mapping $T$ and gave some numerical applications. It can be seen that general inertial Krasnosel'skiĭ-Mann iteration (3) extends inertial Krasnosel'skiĭ-Mann iteration (2) (observe that (3) reduces to (2) when $\beta_n = \delta_n$).

In [36], Dong et al. proposed the following inertial Krasnosel'skiĭ-Mann iteration for approximating fixed point of a nonexpansive mapping $T$: $x_0, y_0, z_0 \in H, \lambda_0 > 0$,

$$\begin{cases} x_{n+1} = (1-\lambda_n)y_n + \lambda_n Tz_n \\ z_{n+1} = x_{n+1} + \delta(x_{n+1} - x_n) \\ y_{n+1} = x_n + \dfrac{\theta - \lambda_n\delta}{1-\lambda_n}(x_n - x_{n-1}), \end{cases} \tag{4}$$

where $\delta \in (1, \infty)$ and $\theta \in [0, \frac{\delta(\delta-1)}{3\delta^2-1})$. They obtained weak convergence results for approximating fixed point of nonexpansive mapping $T$ in Hilbert space. It can be seen that inertial Krasnosel'skiĭ-Mann iteration (4) is a special case of general inertial Krasnosel'skiĭ-Mann iteration (3).

Furthermore, it is observed that the choice $\delta_n = 1$ is not considered in general inertial Krasnosel'skiĭ-Mann iteration (3) and also we have from (4) that the choice $\delta = 1$ is not considered, since $\delta \in (1, \infty)$. It has recently been shown by Iyiola et al. [37] that the choice $\delta_n = 1$ in (2) improves the numerical computations in terms of number of iterations and CPU times for inertial Krasnosel'skiĭ-Mann iteration (2). Therefore, this natural question arises.

*Question:* Can we obtain weak convergence results for general inertial Krasnosel'skiĭ-Mann iteration (3) with $\delta_n = 1$ and inertial Krasnosel'skiĭ-Mann iteration (4) with $\delta = 1$?

Our contribution in this paper is to answer the above question in the affirmative. Thus, we obtain weak convergence results for general inertial Krasnosel'skiĭ-Mann iteration (3) with $\delta_n = 1$ and inertia Krasnosel'skiĭ-Mann iteration (4) with $\delta = 1$. In summary,

- We give weak convergence analysis of general inertial Krasnosel'skiĭ-Mann iteration (3) with $\delta_n = 1$ and inertial Krasnosel'skiĭ-Mann iteration (4) when $\delta = 1$, otherwise referred to as inertial reflected Krasnosel'skiĭ-Mann iteration. Our results improves on the results of Dong et al. [35–37];
- We apply our inertial reflected Krasnosel'skiĭ-Mann iteration to monotone inclusions;
- Numerical implementations are given to justify the efficiency of our proposed method over some inertial Krasnosel'skiĭ-Mann iterations in [33,34] and image restoration.

Let us organize our paper as follows: We first introduce our proposed method from the point of view of dynamical system in Section 2. Then our main results are given in Section 3, where we give the weak convergence analysis of our proposed method. Applications of our results to finding the sum of three operators in which two are maximal monotone and the third is cocoercive operator are given in Section 4. We give numerical implementations and comparisons of our method with other related methods in the literature in Section 5. We conclude the paper in Section 6 by giving some final remarks and our future projects.

We give below some results which would be needed in the convergence analysis of our proposed method.

**Lemma 1.1:** *The following identities hold in H:*

(i) $\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2,\ \forall\, u, v \in H;$
(ii) $2\langle u - v, u - w\rangle = \|u - v\|^2 + \|u - w\|^2 - \|v - w\|^2,\ \forall\, u, v, w \in H;$

(iii) $\|\kappa u+(1-\kappa)v\|^2=\kappa\|u\|^2+(1-\kappa)\|v\|^2-\kappa(1-\kappa)\|u-v\|^2, \quad \forall u,v\in H, \forall\kappa\in\mathbb{R}$.

**Lemma 1.2 (Alvarez and Attouch [20]):** *Let $\{\eta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ be sequences in $[0,+\infty)$ such that*

$$\eta_{n+1}\le\eta_n+\lambda_n(\eta_n-\eta_{n-1})+\gamma_n, \quad \forall\, n\ge 1, \sum_{n=1}^{+\infty}\gamma_n<+\infty,$$

*and there exists a real number $\lambda$, where $0\le\lambda_n\le\lambda<1$ for all $n\in\mathbb{N}$. Then we have the following:*

(i) $\sum_{n=1}^{+\infty}[\eta_n-\eta_{n-1}]_+<+\infty$, *where* $[t]_+:=\max\{0,t\}$;
(ii) *there exists* $\eta^*\in[0,+\infty)$ *such that* $\lim_{n\to\infty}\eta_n=\eta^*$.

**Lemma 1.3 (Opial [18]):** *Let $C$ be a subset of $H$, with $C\neq\emptyset$ and let $\{y_n\}$ be a sequence in $H$ that satisfies the following conditions:*

(i) $\lim_{n\to\infty}\|y_n-y\|$ *exists for any* $y\in C$;
(ii) *every sequential weak limit point of* $\{y_n\}$ *belongs to* $C$.
*Then* $\{y_n\}$ *converges weakly to a point in* $C$.

**Definition 1.4:** A mapping $T:H\to H$ is said to be demiclosed if for any sequence $\{y_n\}\subset H$ converges weakly to $y\in H$ and $\{Ty_n\}$ converges strongly to $z\in H$, then $T(y)=z$.

**Lemma 1.5 (Browder [38]):** *Let $C\subset H$ be nonempty, closed and convex and $T: C\to C$ be a nonexpansive mapping. Then $I-T$ is demiclosed at* 0.

## 2. Perspectives from dynamical systems

Let us first introduce our method in this paper from the point of view of dynamical system.

Let $T$ be a nonexpansive mapping and let us consider the following inexplicit first-order dynamical system:

$$\dot{x}(t)+\alpha(t)x(t)=\alpha(t)T(\dot{x}(t)+x(t)), \tag{5}$$

where $\alpha:[0,\infty)\to[0,\infty)$ is a Lebesgue measurable function. Now, set $\dot{x}(t)\approx\frac{x_{n+1}-x_n}{h}$ on the left-hand side and $\dot{x}(t)\approx\frac{x_n-x_{n-1}}{h}$ on the right-hand side of (5), we have

$$\frac{x_{n+1}-x_n}{h}+\alpha_n x_n=\alpha_n T\left(x_n+\frac{x_n-x_{n-1}}{h}\right). \tag{6}$$

Let us take $h = 1$ in (6), then we obtain

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(2x_n - x_{n-1}). \quad (7)$$

The Equation (7) is the reflected Krasnosel'skiĭ-Mann iteration studied in [37].

Now, let us consider the following inexplicit second-order dynamical system

$$\ddot{x}(t) + (\gamma(t) + \alpha(t)(1-\gamma(t)))\dot{x}(t) + \alpha(t)x(t) = \alpha(t)T(\dot{x}(t) + x(t)), \quad (8)$$

where $\alpha, \gamma : [0,\infty) \to [0,\infty)$ are Lebesgue measurable functions. Observe that (8) is the same as

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) = \alpha(t)T(\dot{x}(t) + x(t)) - \alpha(t)(1-\gamma(t))\dot{x}(t) - \alpha(t)x(t). \quad (9)$$

Discretizing, we get from (9) that

$$\begin{aligned}&\frac{x_{n+1} - 2x_n + x_{n-1}}{h_n^2} + \frac{\gamma_n(x_n - x_{n-1})}{h_n}\\ &\quad = \alpha_n T\left(x_n + \frac{x_n - x_{n-1}}{h}\right) - \frac{\alpha_n(1-\gamma_n)(x_n - x_{n-1})}{h_n} - \alpha_n x_n. \quad (10)\end{aligned}$$

If we take $h_n = 1$ in (10), we obtain from (10) that

$$\begin{aligned}x_{n+1} - 2x_n + x_{n-1} + \gamma_n(x_n - x_{n-1}) &= \alpha_n T(2x_n - x_{n-1})\\ &\quad - \alpha_n(1-\gamma_n)(x_n - x_{n-1}) - \alpha_n x_n. \quad (11)\end{aligned}$$

Thus,

$$\begin{aligned}x_{n+1} - x_n - (1-\gamma_n)(x_n - x_{n-1}) &= \alpha_n T(2x_n - x_{n-1})\\ &\quad - \alpha_n(1-\gamma_n)(x_n - x_{n-1}) - \alpha_n x_n. \quad (12)\end{aligned}$$

If we take $\theta_n = 1 - \gamma_n$. Then (12) becomes

$$\begin{aligned}x_{n+1} &= x_n + \theta_n(x_n - x_{n-1}) + \alpha_n T(2x_n - x_{n-1})\\ &\quad - \alpha_n(x_n + \theta_n(x_n - x_{n-1}))\\ &= (1-\alpha_n)(x_n + \theta_n(x_n - x_{n-1})) + \alpha_n T(2x_n - x_{n-1}). \quad (13)\end{aligned}$$

This is an inertial reflected Krasnosel'skiĭ-Mann iteration we intend to study in the next section of this paper.

## 3. Main results

In this section, we introduce our proposed inertial reflected Krasnosel'skiĭ-Mann iteration and obtain weak convergence analysis of our method for finding fixed point of a nonexpansive mapping $T$ under some appropriate conditions on the iterative parameters. For the rest of this paper, $F(T) := \{x \in H : Tx = x\}$, the set of fixed points of $T$. We give our main convergence result below.

**Theorem 3.1:** *Assume that $T : H \to H$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence in $H$ which is generated by: $x_0, x_1 \in H$,*

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T z_n, \end{cases} \tag{14}$$

*where $\{\alpha_n\}$ and $\theta$ satisfy the following conditions:*

(i) $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} \leq \frac{1}{1+\epsilon}$, $\epsilon \in (2, \infty)$;
(ii) $0 \leq \theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$.

*Then $\{x_n\}$ converges weakly to a fixed point of $T$.*

***Proof:*** Pick $z \in F(T)$, then

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1-\alpha_n)(y_n - z) + \alpha_n(Tz_n - z)\|^2 \\ &= (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|Tz_n - z\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - Tz_n\|^2 \\ &\leq (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z_n - z\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - Tz_n\|^2. \end{aligned} \tag{15}$$

From $x_{n+1} = (1-\alpha_n)y_n + \alpha_n T z_n$, implies $x_{n+1} - y_n = \alpha_n(Tz_n - y_n)$. So,

$$\|y_n - Tz_n\| = \frac{1}{\alpha_n}\|x_{n+1} - y_n\|. \tag{16}$$

Thus,

$$\|x_{n+1} - z\|^2 \leq (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z_n - z\|^2 - \frac{(1-\alpha_n)}{\alpha_n}\|y_n - x_{n+1}\|^2. \tag{17}$$

By Lemma 1.1(iii), we get

$$\begin{aligned} \|z_n - z\|^2 &= \|2x_n - x_{n-1} - z\|^2 \\ &= \|2(x_n - z) - (x_{n-1} - z)\|^2 \\ &= 2\|x_n - z\|^2 - \|x_{n-1} - z\|^2 + 2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{18}$$

Similarly, using Lemma 1.1(iii), one has

$$\begin{aligned} \|y_n - z\|^2 &= \|(1+\theta)(x_n - z) - \theta(x_{n-1} - z)\|^2 \\ &= (1+\theta)\|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 + \theta(1+\theta)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{19}$$

Also,

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &= \|x_{n+1} - (x_n + \theta(x_n - x_{n-1}))\|^2 \\
&= \|(x_{n+1} - x_n) - \theta(x_n - x_{n-1})\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \theta^2\|x_n - x_{n-1}\|^2 - 2\theta\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\
&\geq \|x_{n+1} - x_n\|^2 + \theta^2\|x_n - x_{n-1}\|^2 \\
&\quad - 2\theta\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\
&\geq (1-\theta)\|x_{n+1} - x_n\|^2 + (\theta^2 - \theta)\|x_n - x_{n-1}\|^2. \qquad (20)
\end{aligned}$$

Putting (18)–(20) into (17), we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq\ & (1-\alpha_n)(1+\theta)\|x_n - z\|^2 - \theta(1-\alpha_n)\|x_{n-1} - z\|^2 \\
& + (1-\alpha_n)\theta(1+\theta)\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - z\|^2 \\
& - \alpha_n\|x_{n-1} - z\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
& - \frac{(1-\alpha_n)}{\alpha_n}(1-\theta)\|x_{n+1} - x_n\|^2 \\
& - \frac{(1-\alpha_n)}{\alpha_n}(\theta^2 - \theta)\|x_n - x_{n-1}\|^2 \\
=\ & ((1-\alpha_n)(1+\theta) + 2\alpha_n)\,\|x_n - z\|^2 \\
& - (\theta(1-\alpha_n) + \alpha_n)\,\|x_{n-1} - z\|^2 \\
& + \left[(1-\alpha_n)\theta(1+\theta) + 2\alpha_n - \frac{(1-\alpha_n)}{\alpha_n}(\theta^2 - \theta)\right]\|x_n - x_{n-1}\|^2 \\
& - \frac{(1-\alpha_n)}{\alpha_n}(1-\theta)\|x_{n+1} - x_n\|^2 \\
=\ & (1 + \alpha_n + \theta(1-\alpha_n))\|x_n - z\|^2 \\
& - (\alpha_n + \theta(1-\alpha_n))\|x_{n-1} - z\|^2 \\
& - \rho_n\|x_{n+1} - x_n\|^2 + \sigma_n\|x_n - x_{n-1}\|^2, \qquad (21)
\end{aligned}$$

where

$$\rho_n := \frac{(1-\alpha_n)}{\alpha_n}(1-\theta),$$

and

$$\sigma_n := (1-\alpha_n)\theta(1+\theta) + 2\alpha_n - \frac{(1-\alpha_n)}{\alpha_n}(\theta^2 - \theta).$$

Define

$$\Gamma_n = \|x_n - z\|^2 - (\alpha_n + \theta(1-\alpha_n))\|x_{n-1} - z\|^2 + \sigma_n\|x_n - x_{n-1}\|^2.$$

Then,

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n &= \|x_{n+1} - z\|^2 - (\alpha_{n+1} + \theta(1 - \alpha_{n+1}))\|x_n - z\|^2 \\
&\quad + \sigma_{n+1}\|x_{n+1} - x_n\|^2 - \|x_n - z\|^2 \\
&\quad + (\alpha_n + \theta(1 - \alpha_n))\|x_{n-1} - z\|^2 \\
&\quad - \sigma_n\|x_n - x_{n-1}\|^2 \\
&\leq (1 + \alpha_n + \theta(1 - \alpha_n))\|x_n - z\|^2 - (\alpha_n + \theta(1 - \alpha_n))\|x_{n-1} - z\|^2 \\
&\quad - \rho_n\|x_{n+1} - x_n\|^2 + \sigma_n\|x_n - x_{n-1}\|^2 \\
&\quad - (\alpha_{n+1} + \theta(1 - \alpha_{n+1}))\|x_n - z\|^2 \\
&\quad + \sigma_{n+1}\|x_{n+1} - x_n\|^2 - \|x_n - z\|^2 \\
&\quad + (\alpha_n + \theta(1 - \alpha_n))\|x_{n-1} - z\|^2 \\
&\quad - \sigma_n\|x_n - x_{n-1}\|^2 \\
&\leq -\rho_n\|x_{n+1} - x_n\|^2 + \sigma_{n+1}\|x_{n+1} - x_n\|^2 \\
&= -(\rho_n - \sigma_{n+1})\|x_{n+1} - x_n\|^2. \qquad (22)
\end{aligned}$$

Now,

$$\begin{aligned}
\rho_n - \sigma_{n+1} &= \frac{(1 - \alpha_n)}{\alpha_n}(1 - \theta) - (1 - \alpha_{n+1})\theta(1 + \theta) \\
&\quad - 2\alpha_{n+1} + \frac{(1 - \alpha_{n+1})}{\alpha_{n+1}}(\theta^2 - \theta) \\
&\geq \epsilon(1 - \theta) + \epsilon(\theta^2 - \theta) - (1 - \alpha_{n+1})\theta(1 + \theta) - 2\alpha_{n+1} \\
&> \epsilon(1 - \theta) + \epsilon(\theta^2 - \theta) - 2(1 - \alpha_{n+1}) - 2\alpha_{n+1} \\
&= \epsilon(1 - \theta) + \epsilon(\theta^2 - \theta) - 2 \\
&= \epsilon\theta^2 - 2\epsilon\theta + \epsilon - 2. \qquad (23)
\end{aligned}$$

Since $\theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$, we have that $\epsilon\theta^2 - 2\epsilon\theta + \epsilon - 2 > 0$. Combining (22) and (23), we get

$$\Gamma_{n+1} - \Gamma_n \leq -\delta\|x_{n+1} - x_n\|^2, \qquad (24)$$

where $\delta := \epsilon\theta^2 - 2\epsilon\theta + \epsilon - 2$. Hence, $\Gamma_{n+1} \leq \Gamma_n$. So, $\{\Gamma_n\}$ is non-increasing. Consequently,

$$\begin{aligned}
\Gamma_n &= \|x_n - z\|^2 - (\alpha_n + \theta(1 - \alpha_n))\|x_{n-1} - z\|^2 + \sigma_n\|x_n - x_{n-1}\|^2 \\
&\geq \|x_n - z\|^2 - (\alpha_n + \theta(1 - \alpha_n))\|x_{n-1} - z\|^2.
\end{aligned}$$

So,

$$\begin{aligned}
\|x_n - z\|^2 &\le (\alpha_n + \theta(1-\alpha_n))\|x_{n-1} - z\|^2 + \Gamma_n \\
&\le \left(\frac{1}{1+\epsilon} + \theta(1-\alpha)\right)\|x_{n-1} - z\|^2 + \Gamma_n \\
&= \mu\|x_{n-1} - z\|^2 + \Gamma_n \\
&\le \mu\|x_{n-1} - z\|^2 + \Gamma_1 \\
&\vdots \\
&\le \mu^n\|x_0 - z\|^2 + (1 + \mu + \cdots + \mu^{n-1})\Gamma_1 \\
&\le \mu^n\|x_0 - z\|^2 + \frac{\Gamma_1}{1-\mu},
\end{aligned} \tag{25}$$

where $\mu := \frac{1}{1+\epsilon} + \theta(1-\alpha) \in (0,1)$ since $\theta < \frac{\epsilon}{(1+\epsilon)(1-\alpha)}$. blueFurthermore, observe that $\theta < \frac{\epsilon}{(1+\epsilon)(1-\alpha)}$ because $\theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon} < \frac{\epsilon}{(1+\epsilon)(1-\alpha)}$ for $\alpha \in (0,1)$ and $\epsilon \in (2,\infty)$. Therefore, $\{\|x_n - z\|\}$ is bounded and so is $\{x_n\}$. Furthermore,

$$-\mu\|x_{n-1} - z\|^2 \le \|x_n - z\|^2 - \mu\|x_{n-1} - z\|^2 \qquad \le \Gamma_n \le \Gamma_1.$$

Now,

$$\begin{aligned}
\Gamma_{n+1} &= \|x_{n+1} - z\|^2 - (\alpha_{n+1} + \theta(1-\alpha_{n+1}))\|x_n - z\|^2 \\
&\quad + \sigma_{n+1}\|x_{n+1} - x_n\|^2 \\
&\ge -(\alpha_{n+1} + \theta(1-\alpha_{n+1}))\|x_n - z\|^2.
\end{aligned} \tag{26}$$

Using (25) and (26) give

$$\begin{aligned}
-\Gamma_{n+1} &\le (\alpha_{n+1} + \theta(1-\alpha_{n+1}))\|x_n - z\|^2 \\
&\le \mu\|x_n - z\|^2 \\
&\le \mu^{n+1}\|x_0 - z\|^2 + \frac{\mu\Gamma_1}{1-\mu}.
\end{aligned} \tag{27}$$

By (24), one gets

$$\delta\|x_{n+1} - x_n\|^2 \le \Gamma_n - \Gamma_{n+1}$$

and thus (using (27)),

$$\begin{aligned}
\delta\sum_{k=1}^{n}\|x_{k+1} - x_k\|^2 &\le \Gamma_1 - \Gamma_{n+1} \\
&\le \Gamma_1 + \mu^{n+1}\|x_0 - z\|^2 + \frac{\mu\Gamma_1}{1-\mu} \\
&= \mu^{n+1}\|x_0 - z\|^2 + \frac{\Gamma_1}{1-\mu}.
\end{aligned} \tag{28}$$

Thus,

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 \leq \frac{\Gamma_1}{\delta(1-\mu)} < \infty. \tag{29}$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From (21) we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 + \alpha_n + \theta(1-\alpha_n))\|x_n - z\|^2 \\ &\quad - (\alpha_n + \theta(1-\alpha_n))\|x_{n-1} - z\|^2 \\ &\quad - \rho_n\|x_{n+1} - x_n\|^2 + \sigma_n\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - z\|^2 + (\alpha_n + \theta(1-\alpha_n))(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ &\quad + \sigma_n\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - z\|^2 + (\alpha_n + \theta(1-\alpha_n))(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ &\quad + \Bigg[(1-\alpha)\theta(1+\theta) + \frac{2}{1+\epsilon} \\ &\quad + \frac{(1-\alpha)}{\alpha}(\theta - \theta^2)\Bigg]\|x_n - x_{n-1}\|^2, \end{aligned} \tag{30}$$

where

$$\sigma_n \leq (1-\alpha)\theta(1+\theta) + \frac{2}{1+\epsilon} + \frac{(1-\alpha)}{\alpha}(\theta - \theta^2), \quad \forall\, n \geq 1.$$

Observe that

$$\alpha_n + \theta(1-\alpha_n) \leq \frac{1}{1+\epsilon} + \theta(1-\alpha) < 1.$$

Using Lemma 1.2 and (29) in (30), we get

$$\lim_{n\to\infty} \|x_n - z\| \text{ exists.}$$

From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we obtain

$$\begin{aligned} \|x_{n+1} - z_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &= \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| \to 0, \; n \to \infty. \end{aligned}$$

Also,

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - y_n\| \\ &= \|x_{n+1} - x_n\| + \theta\|x_n - x_{n-1}\| \to 0, n \to \infty. \end{aligned}$$

Now,

$$\|x_{n+1} - y_n\| = \alpha_n \|y_n - Tz_n\| \geq \alpha \|y_n - Tz_n\|,$$

which means that

$$\lim_{n\to\infty} \|y_n - Tz_n\| = 0. \tag{31}$$

Furthermore,

$$\begin{aligned}\|y_n - z_n\| &= \|x_n + \theta(x_n - x_{n-1}) - 2x_n + x_{n-1}\| \\ &= \|x_{n-1} - x_n + \theta(x_n - x_{n-1})\| \\ &\leq \|x_{n-1} - x_n\| + \theta\|x_n - x_{n-1}\| \to 0, n \to \infty,\end{aligned} \tag{32}$$

and by (31) and (32),

$$\|z_n - Tz_n\| \leq \|y_n - Tz_n\| + \|y_n - z_n\| \to 0, \ n \to \infty. \tag{33}$$

From $z_n = 2x_n - x_{n-1}$, we have that $x_n - z_n \to 0, n \to \infty$. As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some point $p \in H$. Since $x_n - z_n \to 0, n \to \infty$, we have that $\{z_{n_j}\}$ converges weakly to $p \in H$. By Lemma 1.5, we have $p \in F(T)$. Since the two assumptions of Lemma 1.3 are verified, it follows that $\{x_n\}$ converges weakly to a point in $F(T)$. ■

**Remark 3.2:** (a) Observe that if $\theta = 0$ in condition (ii) of Theorem 3.1, then our proposed method (14) reduces to [37, (11)].

(b) In [39], a reflected inertial Krasnoselskii-type algorithm of the form

$$x_{n+1} = \lambda x_{n-1} + (1 - 2\lambda)x_n + \lambda T(2x_n - x_{n-1}) \tag{34}$$

with $\lambda \in (0, \frac{\sqrt{2}-1}{2})$ was studied. Our proposed inertial reflected Krasnosel'skiĭ-Mann iteration in (14) can be re-written as

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Tz_n \\ &= (1 - \alpha_n)(x_n + \theta(x_n - x_{n-1})) + \alpha_n T(2x_n - x_{n-1}) \\ &= -\theta(1 - \alpha_n)x_{n-1} + (1 - \alpha_n)(1 + \theta)x_n + \alpha_n T(2x_n - x_{n-1}).\end{aligned} \tag{35}$$

Even though both (34) and (35) are reflected inertial Krasnoselskii-type methods, the choices of iterative parameters (control sequences) in (34) and (35) are different. For example, one observes that the coefficient of $x_{n-1}$ in (34) is $\lambda \in (0, \frac{\sqrt{2}-1}{2})$ while the coefficient of $x_{n-1}$ in (35) is $-\theta(1 - \alpha_n) \in (-1, 0]$.

**Remark 3.3:** Observe in our proposed inertial reflected Krasnosel'skiĭ-Mann iteration (14) that as $\epsilon$ increases, so is $\theta$ approaching 1 since $\lim_{\epsilon\to\infty} \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon} = 1$.

Furthermore, as $\epsilon$ increases, $\alpha_n$ reduces in our inertial reflected Krasnosel'skiĭ-Mann iteration (14). Conversely, as $\epsilon$ approaches 2, $\theta$ approaches zero and $\alpha_n$ approaches $\frac{1}{3}$.

**Remark 3.4:** As noted by one of the referees, the proposed inertial reflected Krasnosel'skiĭ-Mann iteration (14) can also be expressed in terms of $x_n$ and $z_n$ alone. This is done by adding and subtracting terms to obtain an error that is bounded by the common bound $(1-\theta)\|x_n - x_{n-1}\|$. Consequently, the dissymmetry between $y_n$ and $z_n$ in (14) disappears, and this suggests, on the one hand, that the roles of $y_n$ and $z_n$ can be interchanged, and that the result can be obtained by analyzing a perturbed algorithm in only one scale.

**Remark 3.5:** In Theorem 3.1, the relaxation parameter $\alpha_n$ must be below $\frac{1}{3}$, which is very restrictive. In earlier papers by Shehu [40], and Iyiola and Shehu [41], whose analysis is based on Boț et al. [34], the relationship between the extrapolation and relaxation parameters is more complex but also more flexible. However, the methods in [40,41] do not have reflected step and the nonexpansive operator is not evaluated at the reflected step as we constructed in our (14). Furthermore, the method in [42] by Maulen et al. only involves inertial Krasnosel'skiĭ-Mann iteration (without reflected step) for a family of quasi-nonexpansive mappings.

Suppose $S : H \to H$ is $\beta$-averaged mapping (A mapping $S : H \to H$ is said to be a $\beta$-averaged mapping if and only if $S = (1-\beta)I + \beta T$; where $\beta \in (0, 1)$ and $T : H \to H$ is a nonexpansive mapping), we have the following corollary from Theorem 3.1.

**Corollary 3.6:** *Suppose $S : H \to H$ is a $\beta$-averaged mapping and $F(S) \neq \emptyset$. Let $\{x_n\}$ be generated by: $x_0, x_1 \in H$,*

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n\beta)y_n - \alpha_n(1-\beta)z_n + \alpha_n S z_n, \end{cases} \tag{36}$$

*where $\{\alpha_n\}$ and $\theta$ satisfy the following conditions:*

(i) $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} < \frac{1}{\beta(1+\epsilon)}, \epsilon \in (2, \infty)$;
(ii) $0 \leq \theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$.

*Then $\{x_n\}$ converges weakly to a fixed point of S.*

***Proof:*** Since $S$ is $\beta$-averaged, then $S = (1-\beta)I + \beta T$, where $T$ is nonexpansive. Hence, (36) is equivalent to

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n\beta)y_n + \alpha_n\beta T z_n, \end{cases}$$

with $F(S) = F(T)$. Consequently, the proof follows from Theorem 3.1. ■

## 4. Applications

In this section, we apply the results in Section 3 to solve monotone inclusion problems. We first recall some definitions below.

A mapping $A : H \to H$ is said to be $\eta$-inverse strongly monotone($\eta$-cocoercive), if there exists a constant $\eta > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq \eta\|A(x) - A(y)\|^2, \quad \forall\, x, y \in H. \tag{37}$$

An operator $A : D(A) \subset H \to 2^H$ is said to be monotone, if

$$\langle u - v, x - y\rangle \geq 0 \quad \forall\, x, y \in D(A), \quad u \in Ax, v \in Ay,$$

where $D(A)$ means the domain of $A$. The operator $A$ is maximal monotone, if $A$ is monotone and its graph

$$G(A) := \{(x, y) : x \in D(A), y \in Ax\}$$

is not properly contained in the graph of any other monotone operator. As an example, the subdifferential of $g$, $\partial g$ $[\partial g(x) := \{s \mid g(y) \geq g(x) + \langle s, y - x\rangle\ \forall y\}]$ is a maximal monotone operator (see, e.g. [1]), where $g : H \to \mathbb{R}$ is a proper, convex and lower semicontinuous function.

Suppose $A$ and $B$ are maximal monotone operators on $H$ and $C$ is $\eta$-inverse strongly monotone (i.e. $\eta$-cocoercive ). Let us consider the following inclusion problem:

$$\text{Find } x \in H \text{ such that } 0 \in Ax + Bx + Cx. \tag{38}$$

Let denote by $\mathrm{zer}(A + B + C)$, the set of solutions to the inclusion problem (38). In the recent results given by Davis and Yin [43], the inclusion problem (38) is equivalently written as

$$\text{Find } x \in H \text{ such that } Sx = x \text{ with } S := J_{\gamma A} \circ (2J_{\gamma B} - I - \gamma C \circ J_{\gamma B}) + I - J_{\gamma B}, \tag{39}$$

where $\gamma > 0$ and $J_{\gamma A}(x) := (x + \gamma A(x))^{-1}, x \in H$. Furthermore, we have the following result regarding operator $S$ in (39).

**Proposition 4.1:** *[43, Proposition 2.1] Let $A$ and $B$ be maximal monotone operators and $C$ an $\eta$-cocoercive mapping. Let $\gamma \in (0, 2\eta)$. Then operator $S$ defined* *in* (39) *is a $\beta$-averaged mapping with* $\beta := \frac{2\eta}{4\eta-\gamma} < 1$ *and* $\operatorname{zer}(A+B+C) = J_{\gamma B}(F(S))$. *Furthermore, if* $\gamma \in (0, 2\eta\epsilon)$, *where* $\epsilon \in (0,1)$, *then for all* $w, z \in H$,

$$\|Sz - Sw\|^2 \leq \|z-w\|^2 - \frac{1-\beta}{\beta}\|(I-S)z - (I-S)w\|^2 - \gamma\left(2\eta - \frac{\gamma}{\epsilon}\right)\|C \circ J_{\gamma B}(z) - C \circ J_{\gamma B}(w)\|^2. \quad (40)$$

Using the definition of $S$ as defined as in (39) in (36) gives $x_0, x_1 \in H$,

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ u_n = J_{\gamma B}(z_n) \\ v_n = J_{\gamma A}(2u_n - z_n - \gamma C u_n) \\ x_{n+1} = y_n - \alpha_n \beta(y_n - z_n) + \alpha_n(v_n - u_n) \end{cases} \quad (41)$$

as a three-operator splitting method to solve the inclusion problem (38). Applying Corollary 3.6 with three-operator splitting method (41), we have the following weak convergence result for inclusion problem (38).

**Theorem 4.2:** *Assume zer*$(A+B+C) \neq \emptyset$. *Let* $\gamma \in (0, 2\eta\tau)$, *where* $\tau \in (0,1)$ *and*

(i) $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} < \frac{4\eta-\gamma}{2\eta(1+\epsilon)}, \epsilon \in (2, \infty)$;
(ii) $0 \leq \theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$.

*Choose* $x_0 = x_1 \in H$, *and let* $\{x_n\}$ *be the sequence generated by* (41). *Then* $\{x_n\}$ *weakly converges to a fixed point* $z^*$ *of S and* $J_{\gamma B}(z^*)$ *solves inclusion problem* (38).

***Proof:*** We obtain the desired conclusion by applying Corollary 3.6 and [43, Lemma 2.2]. ■

As a special case of inclusion problem (38), we give the following forward-backward splitting method with inertial reflection step, adapted from the three-operator splitting method (41).

**Proposition 4.3:** *Let* $A : H \to 2^H$ *be a maximal monotone operator and* $B : H \to H$ *be an* $\eta$*-inverse strongly monotone operator, for some* $\eta > 0$, *such that* $\operatorname{zer}(A+B) \neq \emptyset$. *Let* $\gamma \in (0, 2\eta]$. *Let the sequence* $\{x_n\}$ *in H be generated by:* $x_0, x_1 \in H$,

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n J_{\gamma A}(z_n - \gamma B(z_n)), \ \forall\, n \geq 1. \end{cases} \quad (42)$$

*where* $\{\alpha_n\}$ *and* $\theta$ *satisfy the following conditions:*

(i) $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} < \frac{1}{1+\epsilon}, \epsilon \in (2, \infty)$;
(ii) $0 \leq \theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$.

*Then* $\{x_n\}$ *converges weakly to a point in* $\operatorname{zer}(A+B)$.

***Proof:*** Define $T := J_{\gamma A}(I - \gamma B)$ (see, for example [44, Theorem 7]). It can easily be shown that $T$ is $\frac{2\eta}{4\eta-\gamma}$-averaged mapping (hence nonexpansive) and $\operatorname{zer}(A+B) = F(T)$. Then the conclusion follows from Theorem 3.1. ■

**Remark 4.4:** We can still obtain the conclusion of Proposition 4.3 if the iterative procedure (42) in Proposition 4.3 is replaced with the following method adapted from Corollary 3.6:

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ x_{n+1} = (1 - \alpha_n\beta)y_n - \alpha_n(1-\beta)z_n + \alpha_n J_{\gamma A}(z_n - \gamma B(z_n)), \end{cases} \tag{43}$$

where $\{\alpha_n\}$ and $\theta$ satisfy the following conditions:

(i) $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} < \frac{1}{\beta(1+\epsilon)}, \epsilon \in (2, \infty), \beta := \frac{2\eta}{4\eta-\gamma}$;
(ii) $0 \leq \theta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$.

Furthermore, we adapt the three-operator splitting method (41) to a Douglas-Rachford splitting method with inertial reflection step and obtain the following result.

**Proposition 4.5:** *Suppose* $A, B : H \to 2^H$ *are maximal monotone operators such that* $\operatorname{zer}(A+B) \neq \emptyset$. *Let* $\gamma > 0$. *Let the sequence* $\{x_n\}$ *in* $H$ *be generated by:* $x_0, x_1 \in H$,

$$\begin{cases} z_n = 2x_n - x_{n-1} \\ y_n = x_n + \theta(x_n - x_{n-1}) \\ u_n = J_{\gamma A}(z_n) \\ v_n = J_{\gamma B}(2u_n - z_n) \\ x_{n+1} = y_n - \dfrac{\alpha_n}{2}(y_n - z_n) + \alpha_n(v_n - u_n), \ \forall\, n \geq 1, \end{cases} \tag{44}$$

*where*

$$0 < \alpha \leq \alpha_n \leq \alpha_{n+1} < \frac{2}{1+\epsilon}, \quad \epsilon \in (2, \infty).$$

*Then* $\{x_n\}$ *converges weakly to some point* $y \in H$ *such that* $J_{\gamma B} y \in \operatorname{zer}(A+B)$.

***Proof:*** The desired conclusion follows from Theorem 3.1, [1, Theorem 25.6] and [45, Corollary 5.2] with $T := R_{\gamma B} \circ R_{\gamma A}$ where $R_{\gamma A} := 2J_{\gamma A} - I$. ■

**Table 1.** Algorithm (42) with varying $\theta$, $\alpha_n$ and $\epsilon$ for Example 5.2.

| | | Cameraman | | MRI | |
|---|---|---|---|---|---|
| $\alpha_n$ | $\theta$ | CPU | SNR | CPU | SNR |
| $\frac{n-0.1}{15001n}$ | 0 | 1.4844 | 22.3929 | 0.2062 | 5.8597 |
| ($\epsilon = 15{,}000$) | 0.2 | 1.2690 | 23.2181 | 0.1605 | 7.3152 |
| | 0.5 | 1.2777 | 24.3278 | 0.1503 | 8.3337 |
| | 0.7 | 1.1329 | 24.6979 | 0.1466 | 8.7887 |
| | 0.9 | 1.1194 | 25.3058 | 0.1623 | 8.8549 |
| $\frac{n^2-0.5}{31n^2+1}$ | 0 | 1.2599 | 19.6138 | 0.2080 | 16.2555 |
| ($\epsilon = 30$) | 0.2 | 1.2216 | 21.1432 | 0.1606 | 17.0320 |
| | 0.5 | 1.2287 | 24.3613 | 0.1547 | 18.0313 |
| | 0.7 | 1.2590 | 26.2761 | 0.1495 | 17.1801 |
| $\frac{n-0.5}{11n+2}$ | 0 | 1.2923 | 18.3352 | 0.2027 | 7.9082 |
| ($\epsilon = 10$) | 0.2 | 1.2772 | 20.9131 | 0.2069 | 9.8334 |
| | 0.5 | 1.2594 | 25.4354 | 0.1425 | 15.3593 |

**Table 2.** Comparison of algorithms with $\epsilon = 15{,}000$ for Example 5.2.

| | Cameraman | | MRI | |
|---|---|---|---|---|
| Algorithms | CPU | SNR | CPU | SNR |
| Alg. (42) | 0.5249 | 24.9573 | 0.0765 | 8.7143 |
| Alg. (43) | 0.6536 | 23.8031 | 0.0910 | 8.6758 |
| Bot et al. Alg. | 1.6628 | 19.2891 | 0.1606 | 6.8789 |
| Dong et al. Alg. | 2.4122 | 10.6085 | 0.2043 | 6.1436 |
| Dong Alg. | 1.7536 | 19.2491 | 0.1693 | 6.6487 |
| Maingé Alg. | 1.4433 | 19.3465 | 0.1176 | 7.4563 |
| Vong & Liu Alg. | 1.2228 | 19.6949 | 0.1075 | 7.7724 |

## 5. Numerical experiments

In this section, our focus is on the numerical implementation of our proposed algorithm related to some of the applications we give in Section 4. This involves comparison with some other algorithms in the literature. For the numerical comparisons, we compare our methods with [30,33,34,46,47] using the examples below. Also, we consider different values of $\epsilon \in (2, \infty)$. The codes are written in MATLAB 2016 (b) running on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30 GHz and 8.00 Gb-RAM. In Tables 1–6, 'Iter', 'CPU' and 'Alg.' means the number of iterations, CPU time in seconds and Algorithm, respectively.

**Remark 5.1:** (a) For the numerical comparisons, we employ different choices of the parameters as chosen in [30,33,34,46,47]:

Bot et al. [34, Algorithm (5)]: $\theta_n = 0.11$ and $\alpha_n = 0.8$.

Dong et al. [46, Algorithm 3.1]: $\theta_n = 0.5$, $\mu = 0.4$, $\lambda = 0.5$, $\gamma_n = 0.9$ and $\beta_n = \frac{1}{(n+1)^2}$.

Dong [33, Algorithm 1 in Section 4]: $\theta_n = 0.21$ and $\alpha_n = 0.7$.

Maingé [30, Algorithm (1.2)]: $\theta_n = \min\{0.9, \frac{1}{n^{0.01}(n+1)||x_n - x_{n-1}||^2}\}$ and $\alpha_n = \frac{9n}{10n+1}$.

**Table 3.** Algorithm (42) with varying $\theta$, $\alpha_n$ and $\epsilon$ for Example 5.4.

| | | $N = 400$ | | $N = 2500$ | |
|---|---|---|---|---|---|
| $\alpha_n$ | $\theta$ | CPU | Iter | CPU | Iter |
| $\frac{n-0.1}{15001n}$ | 0 | 0.0392 | 318 | 0.1953 | 595 |
| ($\epsilon = 15{,}000$) | 0.2 | 0.0369 | 309 | 0.0971 | 582 |
| | 0.5 | 0.0336 | 298 | 0.0761 | 567 |
| | 0.7 | 0.0387 | 282 | 0.0605 | 537 |
| | 0.9 | 0.0344 | 272 | 0.0378 | 507 |
| $\frac{n^2-0.5}{31n^2+1}$ | 0 | 0.0541 | 186 | 0.0735 | 326 |
| ($\epsilon = 30$) | 0.2 | 0.0353 | 180 | 0.0524 | 319 |
| | 0.5 | 0.0278 | 171 | 0.0385 | 306 |
| | 0.7 | 0.0206 | 166 | 0.0246 | 294 |
| $\frac{n-0.5}{11n+2}$ | 0 | 0.0469 | 99 | 0.0409 | 134 |
| ($\epsilon = 10$) | 0.2 | 0.0163 | 95 | 0.0214 | 122 |
| | 0.5 | 0.0119 | 92 | 0.0274 | 112 |

**Table 4.** Comparison of algorithms with $\epsilon = 15{,}000$ for Example 5.4.

| | $N = 900$ | | $N = 1600$ | |
|---|---|---|---|---|
| Algorithms | CPU | Iter | CPU | Iter |
| Alg. (42) | 4.6415 | 119 | 14.8506 | 130 |
| Alg. (43) | 5.6431 | 143 | 17.5942 | 156 |
| Bot et al. Alg. | 7.8305 | 200 | 24.7048 | 218 |
| Dong et al. Alg. | 8.2625 | 210 | 25.7887 | 229 |
| Dong Alg. | 6.3265 | 162 | 19.0594 | 170 |
| Maingé Alg. | 7.5866 | 195 | 24.0534 | 214 |
| Vong & Liu Alg. | 7.0853 | 181 | 21.5196 | 192 |

**Table 5.** Algorithm (42) with varying $\theta$, $\alpha_n$ and $\epsilon$ for Example 5.5.

| | | Case 1 | | Case 2 | |
|---|---|---|---|---|---|
| $\alpha_n$ | $\theta$ | CPU | Iter | CPU | Iter |
| $\frac{n-0.1}{15001n}$ | 0 | 4.8898 | 138 | 108.2405 | 182 |
| ($\epsilon = 15{,}000$) | 0.2 | 3.7750 | 114 | 70.2063 | 150 |
| | 0.5 | 3.7554 | 113 | 65.6388 | 149 |
| | 0.7 | 3.6941 | 111 | 63.8221 | 148 |
| | 0.9 | 3.6632 | 110 | 63.5460 | 141 |
| $\frac{n^2-0.5}{31n^2+1}$ | 0 | 6.6878 | 149 | 161.3398 | 197 |
| ($\epsilon = 30$) | 0.2 | 5.2961 | 135 | 108.3675 | 181 |
| | 0.5 | 4.6268 | 132 | 94.6117 | 178 |
| | 0.7 | 5.0849 | 130 | 86.9076 | 177 |
| $\frac{n-0.5}{11n+2}$ | 0 | 14.1344 | 237 | 291.7402 | 323 |
| ($\epsilon = 10$) | 0.2 | 11.4538 | 224 | 265.0681 | 317 |
| | 0.5 | 11.5260 | 220 | 243.9152 | 312 |

Vong and Liu [47, (rmiMann)]: $\theta_n = 0.5$, $\mu = 0.4$, $\lambda = 0.5$, $\gamma_n = 0.9$, $\beta_n = \frac{1}{(n+1)^2}$ and $\xi_n = \frac{2}{(n+1)^{1+0.01}}$.

(b) We carry out several sensitivity analyzes on $\theta$, $\alpha_n$ and $\epsilon$ only for Algorithm (42) since we have similar results for Algorithms (43) and (44). The results of the sensitivity analyzes (see Tables 1,3,5) show that, as we make different choices of $\{\alpha_n\}$, our methods perform better as $\theta$ gets closer to 1.

**Table 6.** Comparison of algorithms with $\epsilon = 15{,}000$ for Example 5.5.

| | Case 1 | | Case 2 | |
|---|---|---|---|---|
| Algorithms | CPU | Iter | CPU | Iter |
| Alg. (42) | 4.2043 | 108 | 65.2372 | 146 |
| Alg. (43) | 5.1417 | 113 | 67.6933 | 149 |
| Bot et al. Alg. | 10.0795 | 168 | 156.4393 | 216 |
| Dong et al. Alg. | 7.9246 | 189 | 185.6493 | 275 |
| Dong Alg. | 7.2000 | 149 | 99.8749 | 153 |
| Maingé Alg. | 10.8218 | 155 | 116.1563 | 154 |
| Vong & Liu Alg. | 8.8705 | 152 | 110.1250 | 154 |

(c) The results of the comparisons (see Tables 2,4,6 and Figures 1–2) show that our methods perform better than other state-of-the-art methods for solving inclusion problems.

**Example 5.2 (Image Restoration Problem):** Let us consider the following problem:

$$\min_{x \in \mathbb{R}^N} \{\|Dx - b\|_2^2 + \gamma \|x\|_1\}, \tag{45}$$

where $\gamma > 0$, $x \in \mathbb{R}^N$ is the original image to be recovered, $b \in \mathbb{R}^M$ is the observed image and $D : \mathbb{R}^N \to \mathbb{R}^M$ is the blurring operator. In order to solve the above problem using numerical computations, we choose the regularization parameter $\gamma = 5e^{-2}$. Furthermore, we consider the $256 \times 256$ Cameraman Image and the $128 \times 128$ Medical Resonance Imaging (MRI) from the MATLAB Image Processing Toolbox. Moreover, we use the Gaussian blur of size $9 \times 9$ and standard deviation $\sigma = 4$ to create the blurred and noisy image (observed image). Also, we measure the quality of the restored image using the signal-to-noise ratio defined by

$$\text{SNR} = 20 \times \log_{10}\left(\frac{\|x\|_2}{\|x - x^*\|_2}\right),$$

where $x$ is the original image and $x^*$ is the restored image. Note that the larger the SNR, the better the quality of the restored image. We also choose the initial values as $x_0 = \mathbf{0} \in \mathbb{R}^{N\times N}$, $x_1 = \mathbf{1} \in \mathbb{R}^{N\times M}$ with $M = N = 50$ for cameraman test and $M = N = 20$ for MRI test. The numerical results are reported in Figures 1, 3, Tables 1 and 2 below.

**Remark 5.3:** In Example 5.2, our model does not really deblur the images, but induces a sparse(r) representation in the canonical basis, in line with compressed sensing (it is a regularized LASSO). Deblurring is better achieved by using the total variation instead of the 1-norm.

**Example 5.4:** As shown in Proposition 4.5, the inertial reflected Krasnosel'skiĭ-Mann iteration (14) can be applied to the Douglas-Rachford splitting method in other to solve the monotone inclusion problem $0 \in A(x) + B(x)$ in Hilbert space

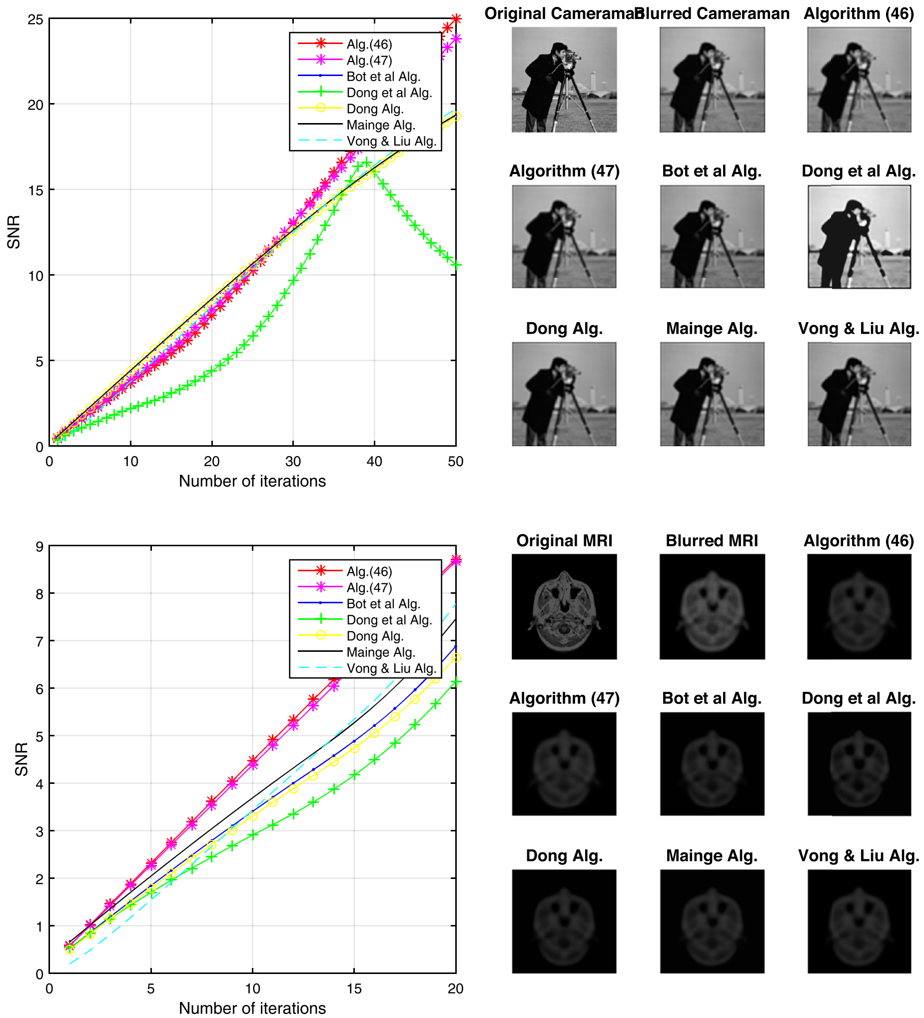


**Figure 1.** Comparison of algorithms with $\epsilon = 15{,}000$ for Example 5.2: Top: *Cameraman*; Bottom: *MRI*.

$H$, where $A$ and $B$ are maximal monotone operators on $H$ (see [1,15,48]). We implement the reflected Douglas-Rachford splitting method and replace $\gamma$ with $\gamma_n$. Now, concerning $\gamma_n$, we adopt the strategies in [33,49] to update $\gamma_n$. Also, we assume that $A$ is continuous, then at $n$th iterate, for a known $\gamma_n$, calculate first

$$\rho_n := \frac{\gamma_n \|A(x_n) - A(x_{n-1})\|}{\|x_n - x_{n-1}\|}.$$

Then we update $\gamma_n$ by

$$\gamma_{n+1} = \begin{cases} 1.5\gamma_n, \ \text{if } \rho_n \le 0.5, \\ \gamma_n, \ \text{if } 0.5 < \rho_n < 2, \\ 0.5\gamma_n, \ \text{if } \rho_n \ge 2. \end{cases} \tag{46}$$

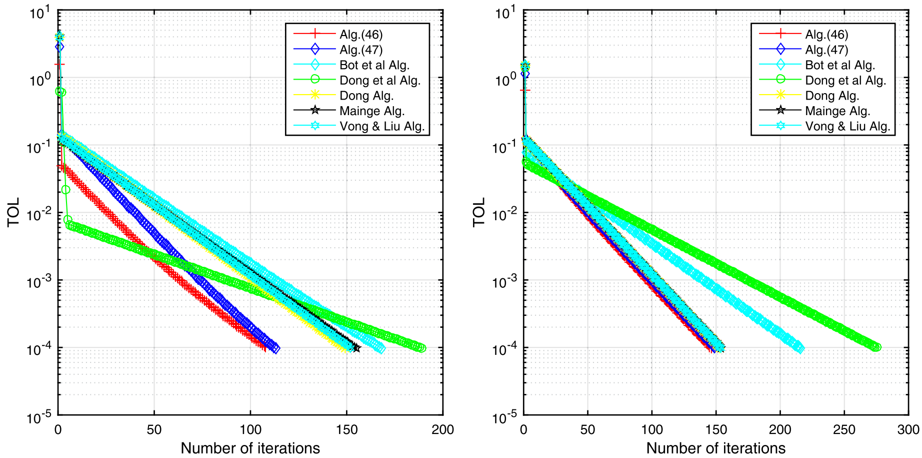


**Figure 2.** Comparison of algorithms with $\epsilon = 15{,}000$ for Example 5.5: Left: *Case 1*; Right: *Case 2*.

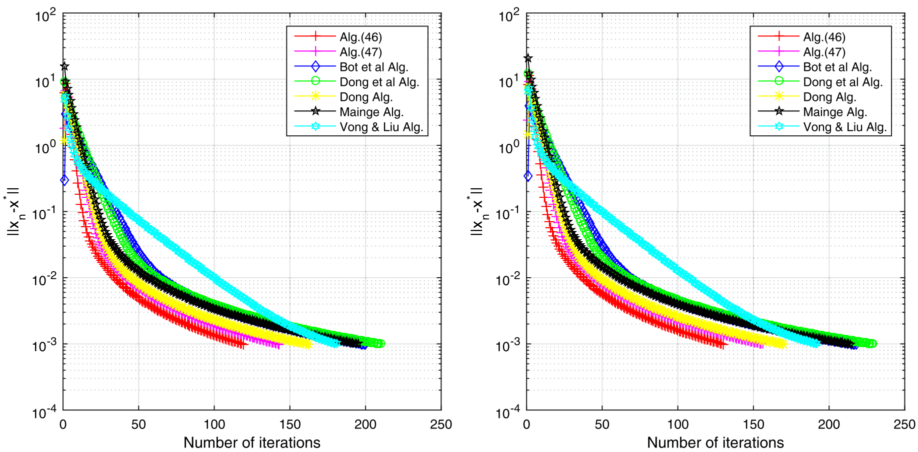


**Figure 3.** Comparison of algorithms with $\epsilon = 15{,}000$ for Example 5.4: Left: $\boldsymbol{N = 900}$; Right: $\boldsymbol{N = 1600}$.

Let

$$Q := \{x \;:\; x_i \geq 0,\ i = 1, \ldots, N\}, \quad \text{and}$$

$$A_1 = \begin{pmatrix} D & -I & & & \\ -I & D & -I & & \\ & \cdots & \cdots & & \\ & & \cdots & \cdots & \\ & & -I & D & -I \\ & & & -I & D \end{pmatrix}, \quad A_2 = \frac{hc}{2}\begin{pmatrix} O & I & & & \\ -I & O & I & & \\ & \cdots & \cdots & & \\ & & \cdots & \cdots & \\ & & -I & O & I \\ & & & -I & O \end{pmatrix},$$

be two $N \times N$ block tridiagonal matrices, where $N = m^2, h = \frac{1}{m+1}, c = 100$, $I$ is an $m \times m$ identity matrix and $D$ is an $m \times m$ matrix taking the form $4I + M + M^T$, where $M_{i,j} = 1$ if $j = i + 1,\ i = 1, \ldots, m-1$; otherwise $M_{i,j} = 0$.

Now, let $A = A_1 + A_2$. Then we consider the following complementarity problem which is also given in [33, Section 5]:

$$0 \in Ax - b + \partial\delta_Q(x),$$

where $\partial\delta_Q(x)$ is the sub-differential of the indicator function

$$\delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{if } x \notin Q. \end{cases}$$

Setting $b = Ae_1$, where $e_1$ is the first column of the identity matrix with order $N \times 1$ corresponding to the matrix $A$, as seen in [33, Section 5], $x^* = e_1$ is the unique solution of the complementarity problem. Thus, we set

$$b := Ae_1, \quad A(x) := Ax - b, \quad B(x) := \partial\delta_Q(x).$$

Using this test example, we implement our method (44) in $\mathbb{R}^N$ by choosing $x_1 = \text{ones}(N, 1)$ and $x_0 = \text{zeros}(N, 1)$ with $N \in \{400, 900, 1600, 2500\}$ and use $v_n = P_Q(2u_n - z_n) = \max\{0, 2u_n - z_n\}$ and $u_n = (I + \gamma_n A)\backslash(z_n + \gamma_n b)$. For the numerical implementation of this example, we use the stopping criterion $\|e_n\|_2 := \|x_n - x^*\| < 10^{-3}$. Our numerical results for this example are reported in Figure 3, Tables 3 and 4 below.

**Example 5.5:** Suppose $\Omega \subset \mathbb{R}^N, N \geq 1$ is a domain and $H := L^2(\Omega) := \{f : \Omega \to \mathbb{R} : \int_\Omega |f(x)|^2 \, \mathrm{d}x < +\infty\}$ is endowed with the scalar product $\langle f, g \rangle := \int_\Omega f(x)g(x) \, \mathrm{d}x, f, g \in L^2(\Omega)$ and induced norm $\|f\| := (\int_\Omega |f(x)|^2)^{\frac{1}{2}}$. Let us consider the following optimization problem (see [50]):

$$\min_{v \in H} \left\{ \frac{\lambda}{2} \int_\Omega ((Kv)(x) - b(x))^2 \, \mathrm{d}x + \frac{1}{2} \int_\Omega F(x, v(x)) \, \mathrm{d}x \right\}, \tag{47}$$

where $\lambda > 0, b \in L^2(\Omega)$,

$$K : H \to H, (Kv)(x) := \int_\Omega \kappa(x, y)v(y) \, \mathrm{d}y,$$

with $\kappa \in L^2(\Omega \times \Omega)$, and $F : \Omega \times H \to \mathbb{R}$ a measurable function. The parameter $\lambda$ controls the trade-off between a good fit of $v$ and a smoothness requirement due to the regularization term, while the integral operator $K$ models the deblurring of the original image $v$. The linear operator $K$ is bounded with $\|K\| \leq \|\kappa\|_{L^2(\Omega \times \Omega)}$ and its adjoint operator $K^* : H \to H$ is given by $(K^*v)(x) := \int_\Omega \kappa(y, x)v(y) \, \mathrm{d}y$ for any $x \in \Omega$.

In this numerical simulation, let us take $\Omega := (0,1), F(x, v(x)) := v^2, x \in \Omega$ and

$$\kappa(x,y) := \begin{cases} 1, & y \le x \\ 0, & y > x. \end{cases}$$

For this choice, $K$ is given by

$$(Kv)(x) = \int_0^x v(y)\, \mathrm{d}y.$$

It can be seen that (see [50])

$$\|K\| \le \|\kappa\|_{L^2(\Omega\times\Omega)} = \frac{1}{2}$$

and the adjoint of $K$ is given by

$$(K^*v)(x) := \int_x^1 v(y)\, \mathrm{d}y.$$

In this case, Problem (47) reduces to

$$\min_{v\in H} \left\{ \frac{\lambda}{2} \int_\Omega ((Kv)(x) - b(x))^2 \, \mathrm{d}x + \frac{1}{2} \int_{(0,1)} (v(x))^2 \, \mathrm{d}x \right\}. \tag{48}$$

Now, define $f(v) := \frac{1}{2} \int_{(0,1)} (v(x))^2 \, \mathrm{d}x, v \in H$ and $g(v) := \frac{\lambda}{2} \int_\Omega ((Kv)(x) - b(x))^2 \, \mathrm{d}x, v \in H$. Then

$$B(v) := \nabla g(v) = \lambda K^*(Kv - b), \quad v \in H,$$
$$J_{\gamma A}(v) := \mathrm{Prox}_{\gamma f}(v) = \frac{v}{1+\gamma}$$

and $\nabla g$ is $\frac{\lambda}{4}$-Lipschitz continuous.

For this example, we consider the following cases for the numerical implementation:

*Case* 1: $x_1 = 1 + t^2$ and $x_0 = 5 + t$.

*Case* 2: $x_1 = 4t + t^2$ and $x_0 = 2 + e^t$.

Furthermore, at the $n$th iterative step, we define the function $\mathrm{TOL}_n$ by $\mathrm{TOL}_n := ||x_n - J_{\gamma A}(x_n - \gamma B(x_n))||$, and use the stopping criterion $\mathrm{TOL}_n < \tau$ for the iterative processes, where $\tau$ is the predetermined error. Here we take $\tau = 10^{-4}$. We give numerical results of this example in Figure 2, Tables 5 and 6 below.

## 6. Final remarks

We obtain weak convergence analysis for finding a fixed point of a nonexpansive mapping using our proposed inertial reflected Krasnosel'skiǐ-Mann iteration. We gave some applications to monotone inclusion problems involving three-operator splitting method, forward-backward splitting method and Douglas-Rachford splitting method. Numerical implementations show numerical advantage gained by our proposed inertial reflected Krasnosel'skiǐ-Mann iteration over other inertial Krasnosel'skiǐ-Mann iterations proposed in the literature. Future projects include applications of our results to solving linearly constrained separable convex optimization problems and primal-dual algorithm versions of our method to solve structured convex optimization problems. We will also consider, in the future, the weak convergence analysis of the discrete version of our method proposed in Section 2.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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