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2022
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The Complex-Valued Class of Korteweg-de Vries Equations

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UNIVERSITY OF THE
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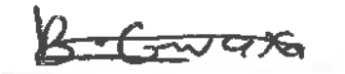
JOHANNESBURG

2021

DECLARATION

"I declare that the contents of this dissertation are original, except where due references have been made. It is submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution."



Signed at Johannesburg on the 1st day of November 2021.

Publications

Details of contributions to publications that form part of the research presented in this dissertation are:

B. Gwaxa and S. Jamal, Symmetries and conservation laws of a class of complex-valued Korteweg-de Vries Equations, submitted, 2021.

Abstract

In this dissertation, we apply Lie theory to determine the one-parameter point transformations which leave complex-valued Korteweg-de Vries equations invariant. The conserved vectors of the systems are constructed. We provide travelling wave reductions that lead to third-order ordinary differential equations. These equations are highly nonlinear to solve directly, and therefore we first establish their first integrals. The latter is of second-order and facilitates the solution of the complex-valued system.

ACKNOWLEDGEMENTS

I have run out of words to express my gratitude to my supervisor Professor Sameerah Jamal for her continuous support. For the last two years she has shared and demonstrated her vast knowledge that assisted me with my mathematics research for my honours and masters. Without her help, I would not have achieved this great accomplishment.

I would also like to thank my mother Nomapa Valencia Gwaxa and my aunt Nomzamo Ngxale for assisting me through my academic journey.

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Chapter 1

Introduction

The Korteweg-de Vries (KdV) equation has been obtained as a fundamental model for wave spread in a number of different physical conditions. Research on pole dynamics of KdV equations and work related to it, started with remarks of Kruskal in the 1970s [1], and subsequently, extensive work was then carried out at a later stage [2, 3, 4, 5, 6]. We will concentrate on the research of generalised KdV equations, all under a complex-valued sense. Such equations, in real-valued terms, have been actively studied in terms of its solitons and solutions, but to a lesser extent as complex-valued.

Recently, some studies have paid attention to construct the complex solutions by different methods [7, 8]. It is predicted that the complex KdV equation, shares the same properties of the real-valued KdV class of equations [9], i.e. the integrable properties, the infinitely many symmetries and infinitely many conservation laws [10]. More properties include the inverse scattering transformation [11], Hirota bilinear method [12] and Darboux transformation [13]. In addition, the resulting

coupled system, that has been decomposed into real and imaginary parts, is a special case of the general coupled KdV systems, connected to the two layer fluid dynamical systems and two-component Bose-Eubstein condensates [14]. Such equations are important in dynamics and physics.

Sophus Lie, during the nineteenth century, established an important result that became a foundation for the study of the theory of groups of transformations that leaves differential equations invariant. Lie sought to develop a theory of integrating ordinary differential equations (ODEs) that is equivalent to the Abelian theory of computing algebraic equations. Lie's group analysis classifies ODEs in terms of the symmetry group admitted by them. The concept of invariance of a differential equation under a set of infinitesimal transformations, enables the construction of solutions of the given differential equation. Recent uses of the Lie symmetry method can be found in the literature [15, 16, 17].

Our interest is to find out about the symmetry, conservation law and solution properties of the complex-valued KdV class of equations. We aim for significant and new information regarding these equations.

1.1 Outline of Chapters

Chapter 2 provides the notation and theory that we utilize to determine Lie point symmetries and conservation laws.

In Chapter 3, we investigate the symmetries of KdV equations, where the equation is assumed to be complex-valued. We construct solutions of these equations and find conservation laws associated with multipliers of the system of equations. We

also identify the graphical behaviour of the solutions.

Chapter 4 considers the symmetries of modified KdV equations, where again the equation is assumed to be complex-valued. The conservation laws are constructed via multipliers and solutions are obtained by reduction and first integrals of these equations. Some plots of the solutions are provided.

Chapter 2

Notation and Theory

2.1 Lie Point Symmetries of Differential Equations

The procedure for determining point symmetries for an arbitrary system of equations is as follows [18].

Definition 2.1.1 *Consider q unknown functions u^α which depend on p independent variables x^i, i.e. $u = (u^1, \ldots, u^q)$, $x = (x^1, \ldots, x^p)$, with indices $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$.*

Let

$$G_\alpha \left(x, u^{(k)} \right) = 0, \tag{1}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the kth derivative of u with respect to x.

Definition 2.1.2 *A one-parameter Lie group of transformations (ϵ as the group parameter) that is invariant under (1) is given by*

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \tag{2}$$

Invariance of (1) under the transformation (2) implies that any solution $u = \Theta(x)$ of (1) maps into another solution $v = \Psi(x; \epsilon)$ of (1). Expanding (2) around the identity $\epsilon = 0$, generates the following infinitesimal transformations:

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon \xi^i(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u}^\alpha &= u^\alpha + \epsilon \eta^\alpha(x, u) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{4}$$

Definition 2.1.3 *The infinitesimal criterion for invariance is given by*

$$X \left[G_\alpha \left(x, u^{(k)} \right) \right] = 0, \quad \textit{when} \quad G_\alpha \left(x, u^{(k)} \right) = 0, \tag{5}$$

where X is extended to all derivatives appearing in the equation through an appropriate prolongation.

Definition 2.1.4 *The operator in Eq. (4) can be used to define the Lagrange system*

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whose solution provides the zero-order invariants

$$W^{[0]}(x^i, u^\alpha). \tag{6}$$

These invariants can be used in order to reduce the order of the PDE.

2.2 Conservation Laws

Definition 2.2.1 *The total differentiation operator D_i with respect to x^i is given by*

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n. \tag{7}$$

Definition 2.2.2 *The Euler operator or variational derivative is given by*

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}. \tag{8}$$

Definition 2.2.3 *A current $T = (T^1, \dots, T^n)$ is conserved if it satisfies*

$$D_i T^i = 0, \tag{9}$$

along the solutions of (1). Eq. (9) is called a local conservation law.

Definition 2.2.4 *The multipliers Λ^α of (1) satisfies the relation [19]*

$$D_i T^i = \Lambda^\alpha G^\alpha, \tag{10}$$

for all functions u^α and the overdetermined equations for Λ^α are

$$\frac{\delta}{\delta u^\alpha} \Big[\Lambda^\alpha G^\alpha \Big] = 0. \tag{11}$$

Thus, a knowledge of each multiplier leads to a conserved vector determined by, as an example, a Homotopy operator.

Chapter 3

The Complex-Valued KdV Equations

First, we consider the complex-valued KdV-Burgers' equation

$$\phi_t = 2\phi\phi_x + \phi_{xxx} + \alpha\phi_{xx}, \tag{12}$$

where $\phi = \phi(x,t)$ and α is a constant. For $\alpha = 0$, we get the complex-valued KdV equation.

The complex equation can be decomposed into real and imaginary parts if we suppose that

$$\phi(x,t) = u(x,t) + iv(x,t),$$

is a complex-valued dependent variable. Then Eq. (12) is equivalent to the coupled system

$$\begin{aligned} -u_t + 2uu_x - 2vv_x + u_{xxx} - \alpha u_{xx} = 0, \\ -v_t + 2uv_x + 2vu_x + v_{xxx} - \alpha v_{xx} = 0. \end{aligned} \tag{13}$$

Suppose that the Lie point symmetry is of the general form

$$X = \xi^1(x,t,u,v)\partial_x + \xi^2(x,t,u,v)\partial_t + \eta^1(x,t,u,v)\partial_u + \eta^2(x,t,u,v)\partial_v,$$

then solving (5) gives the system

$$\begin{gathered}
\eta^1_v = 0,\\
\xi^1_v = 0,\\
\xi^2_v = 0,\\
\eta^2_{vv} = 0,\\
\eta^2_u = 0,\\
\xi^1_u = 0,\\
\xi^2_u = 0,\\
\eta^1_{uu} = 0,\\
\xi^2_x = 0,\\
-2\alpha\eta^1_{xu} + 3\eta^1_{xxu} + \alpha\xi^1_{xx} - \xi^1_{xxx} + \xi^1_t - 2u\xi^1_x + 2u\eta^1_u + 2\eta^1 = 0,\\
-2\alpha\eta^2_{xv} + 3\eta^2_{xxv} + \alpha\xi^1_{xx} - \xi^1_{xxx} + \xi^1_t + 2u\eta^2_v - 2u\xi^1_x + 2\eta^1 = 0,\\
-\alpha\eta^2_{xx} + \eta^2_{xxx} - \eta^2_t + 2u\eta^2_x + 2v\eta^1_x = 0,\\
-\alpha\eta^1_{xx} + \eta^1_{xxx} - \eta^1_t + 2u\eta^1_x - 2v\eta^2_x = 0,\\
3\eta^1_{xu} - 3\xi^1_{xx} - \alpha\eta^1_u + 2\alpha\xi^1_x = 0,\\
3\eta^2_{xv} - 3\xi^1_{xx} - \alpha\eta^2_v + 2\alpha\xi^1_x = 0,\\
-v\xi^1_x + v\eta^2_v + \eta^2 = 0,\\
v\eta^1_u - v\xi^1_x + \eta^2 = 0,\\
\eta^1_u - 3\xi^1_x = 0,\\
\eta^2_v - 3\xi^1_x = 0,\\
\xi^2_t - \eta^1_u = 0,\\
\xi^2_t - \eta^2_v = 0.
\end{gathered}$$

The above system was solved using software such as Mathematica. Next we conduct

a detailed symmetry analysis of this system of equations.

3.1 Case A: $\alpha \neq 0$

If we solve the above determining system we obtain the symmetry coefficients

$$\eta^2 = 0,\ \eta^1 = \frac{-k_3}{2},\ \xi^1 = k_2 + k_3 t,\ \xi^2 = k_1. \tag{14}$$

Hence, using the constants k_i the individual symmetries of (13) are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial_x, \\ X_3 &= t\partial_x - \frac{1}{2}\partial_u, \end{aligned}$$

with Lie bracket relations $[A, B] = AB - BA$ in Table 3.1.

Table 3.1: Lie brackets of (13) with $\alpha \neq 0$

$[,]$	X_1	X_2	X_3
X_1	0	0	$-X_2$
X_2	0	0	0
X_3	X_2	0	0

3.1.1 Conserved Vectors

Suppose that the variable $\bar{\phi}$ denotes the conjugate of ϕ. Each conservation law is presented in the variables u, v and the original variable ϕ.

As defined in the previous chapter, for us to calculate the conserved vector we use the multiplier approach. Therefore, if (T^t, T^x) is a conserved vector of a conservation law, then

$$D_t T^t + D_x T^x = 0,$$

along the solutions of the differential equation (the system (13) in this case). Moreover, we find that there exists nontrivial differential functions $\Lambda_1(x,t,u,v)$ and $\Lambda_2(x,t,u,v)$, called multipliers, such that

$$\begin{aligned} &\Lambda_1(\text{LHS of the first equation of (13)}) \\ +&\Lambda_2(\text{LHS of the second equation of (13)}) \\ &= D_t T^t + D_x T^x, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \frac{\delta}{\delta u}(\text{LHS of (15)}) = 0, \\ \frac{\delta}{\delta v}(\text{LHS of (15)}) = 0, \end{aligned} \tag{16}$$

where $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ are the standard Euler operators which annihilate divergence expressions.

In each case, T^t is the conserved density and T^x is the conserved flux. The same approach is used in subsequent cases of this dissertation. This case admits two conservation laws, where the multipliers $\Lambda_1(x,t,u,v) = 1$ and $\Lambda_2(x,t,u,v) = 0$, provide the conservation law

$$T_1^t = u = \frac{1}{2}(\phi + \bar{\phi}),$$

$$T_1^x = \alpha u_x - u^2 + v^2 - u_{xx} = \frac{\alpha}{2}(\phi_x + \bar{\phi_x}) - \frac{1}{2}(\phi_{xx} + \bar{\phi_{xx}}) - \phi^2,$$

and the multipliers $\Lambda_1(x,t,u,v) = 0$ and $\Lambda_2(x,t,u,v) = 1$, provide the second conservation law

$$T_2^t = v = \sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi + \bar{\phi})^2)^2},$$

$$T_2^x = \alpha v_x - 2uv - v_{xx} = \alpha\sqrt{\phi_x\bar{\phi}_x - \frac{1}{2}(\phi + \bar{\phi}_x)} - \sqrt{\phi_{xx}\bar{\phi}_{xx} - \frac{1}{2}(\phi_{xx} + \bar{\phi}_{xx})} - \sqrt{(\bar{\phi}^2 - \phi\bar{\phi})^2}.$$

3.1.2 Symmetry Reductions and Solutions

In this subsection, we use the method of Lie invariants to firstly calculate invariant functions to reduce our system of equations. We find that the reduced equations are a system of third-order ordinary differential equations which cannot be solved further using symmetries or directly via software such as Maple or Mathematica. Instead, we find first integrals of the reduced system in the hopes of solving via these expressions.

Suppose we consider the linear combination of symmetries $X_1 + aX_2$, a constant, the characteristic equations are

$$\frac{dt}{1} = \frac{dx}{a} = \frac{du}{0} = \frac{dv}{0}.$$

Solving these equations gives the invariant functions

$$\begin{aligned} \zeta &= -at + x, \\ v(x,t) &= F_1(\zeta), \\ u(x,t) &= F_2(\zeta). \end{aligned}$$

Hence the reduced system of equations corresponding to (13) are

$$\begin{aligned} -2F_1(\zeta)F_1'(\zeta) + (a + 2F_2(\zeta))F_2'(\zeta) - \alpha F_2''(\zeta) + F_2'''(\zeta) = 0, \\ (a + 2F_2(\zeta))F_1'(\zeta) + 2F_1(\zeta)F_2'(\zeta) - \alpha F_1''(\zeta) + F_1'''(\zeta) = 0. \end{aligned} \tag{17}$$

The above system is challenging to solve, so we find that the first integrals of (17)

are

$$I_1 = -\alpha F_1'(\zeta) + aF_1(\zeta) + 2F_1(\zeta)F_2(\zeta) + F_1''(\zeta), \tag{18}$$

$$I_2 = -\alpha F_2'(\zeta) + aF_2(\zeta) - F_1(\zeta)^2 + F_2(\zeta)^2 + F_2''(\zeta). \tag{19}$$

Since I_1 and I_2 are of second-order, we attempt to use the first integrals to solve for the unknown functions.

We find the solutions $F_1(\zeta) = 0$ and $F_2(\zeta) = \gamma$, where $F_2'(\zeta) = b(\gamma)$,

$$I_2 = -\alpha b(\gamma) + a\gamma + \gamma^2 + b(\gamma)b'(\gamma). \tag{20}$$

so that Eq. (20) is an Abel's ordinary differential equation of the second kind.

3.2 Case B: $\alpha = 0$

Similarly for $\alpha = 0$, that is, the complex-valued KdV equation, we solve the determining equations to find that the Lie point symmetry coefficients are

$$\eta^1 = \frac{-k_1}{2} - 2k_4u, \xi^1 = k_3 + k_1t + k_4x, \xi^2 = k_2 + 3k_4t, \eta^2 = -2k_4v. \tag{21}$$

From this, the individual symmetries of (13) are

$$X_1 = t\partial_x - \frac{1}{2}\partial_u, \tag{22}$$

$$X_2 = \partial_t, \tag{23}$$

$$X_3 = \partial_x, \tag{24}$$

$$X_4 = x\partial_x + 3t\partial_t - 2u\partial_u - 2v\partial_v, \tag{25}$$

with Lie brackets presented in Table 3.2.

Table 3.2: Lie brackets of (13) with $\alpha = 0$

$[,]$	X_1	X_2	X_3	X_4
X_1	0	X_3	0	$2X_1$
X_2	$-X_3$	0	0	$-3X_2$
X_3	0	0	0	$-X_3$
X_4	-2_1	3_2	X_3	0

3.2.1 Conserved Vectors

The conservation laws are the following six relations together with the corresponding multipliers. The multipliers $\Lambda_1(x,t,u,v) = 1$ and $\Lambda_2(x,t,u,v) = 0$, provide the conservation law

$$T_1^t = u = \frac{1}{2}(\phi + \bar{\phi}),$$

$$T_1^x = -u^2 + v^2 - u_{xx} = \frac{1}{2}(\phi_{xx} + \bar{\phi_{xx}}) - \phi^2,$$

and the multipliers $\Lambda_1(x,t,u,v) = 0$ and $\Lambda_2(x,t,u,v) = 1$, provide the conservation law

$$T_2^t = v = \sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi + \bar{\phi})^2)^2},$$

$$T_2^x = -2uv - v_{xx} = -\frac{1}{2}(\phi + \bar{\phi_x}) - \sqrt{\phi_{xx}\bar{\phi_{xx}} - \frac{1}{2}(\phi_{xx} + \bar{\phi_{xx}})} - \sqrt{(\bar{\phi}^2 - \phi\bar{\phi})^2}.$$

The multipliers $\Lambda_1(x,t,u,v) = v$ and $\Lambda_2(x,t,u,v) = u$, provide the conservation law

$$T_3^t = uv = \frac{1}{2}\sqrt{(\bar{\phi} - \phi\bar{\phi})^2},$$

$$
\begin{aligned}
T_3^x &= -2u^2v + \frac{2}{3}v^3 - uv_{xx} + v_xu_x - vu_{xx} \\
&= -(\phi+\bar{\phi})\sqrt{\phi\bar{\phi} - \left(\frac{1}{2}(\phi+\bar{\phi})\right)^2} + \frac{2}{3}\left(\sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi+\bar{\phi}))^2}\right)^3 \\
&\quad -\frac{1}{2}(\phi+\bar{\phi})\sqrt{\phi_{xx}\bar{\phi_{xx}} - \left(\frac{1}{2}(\phi+\bar{\phi})\right)^2} + \frac{1}{2}\sqrt{(\bar{\phi}-\phi\bar{\phi})^2} \\
&\quad -\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})^2\sqrt{\phi\bar{\phi} - \left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}.
\end{aligned}
$$

The multipliers $\Lambda_1(x,t,u,v) = u$ and $\Lambda_2(x,t,u,v) = v$, provide the conservation law

$$
T_4^t = \frac{1}{2}(u^2-v^2) = \frac{1}{2}\left[\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2 - \sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi\bar{\phi}))^2}\right],
$$

$$
\begin{aligned}
T_4^x &= \frac{-2}{3}u^3 + 2v^2u - uu_{xx} + \frac{1}{2}u_x^2 + vv_{xx} - \frac{1}{2}v_x^2 \\
&= \frac{-2}{3}\left(\frac{1}{2}(\phi+\bar{\phi})\right)^3 + (\phi+\bar{\phi})(\phi\bar{\phi} - \left(\frac{1}{2}(\phi+\bar{\phi})\right)^2 - \frac{1}{2}(\phi+\bar{\phi})\left(\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})\right) \\
&\quad +\frac{1}{2}\left(\frac{1}{2}(\phi_x+\bar{\phi_x})\right)^2 + \left(\sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi+\bar{\phi}))^2}\right)\left(\sqrt{\phi_{xx}\bar{\phi_{xx}} - (\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}}))^2}\right) \\
&\quad -\frac{1}{2}\left(\sqrt{\phi_x+\bar{\phi_x} - \left(\frac{1}{2}(\phi_x+\bar{\phi_x})\right)^2}\right)^2,
\end{aligned}
$$

the multipliers $\Lambda_1(x,t,u,v) = tv$ and $\Lambda_2(x,t,u,v) = tu + \frac{1}{2}x$, provide the conservation law

$$
T_5^t = utv + \frac{1}{2}vx = \frac{t}{2}\sqrt{(\phi-\phi\bar{\phi})^2} + \frac{1}{2}\sqrt{\phi\bar{\phi} - \left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}x,
$$

$$
\begin{aligned}
T_5^x &= -2u^2tv + \frac{2}{3}v^3t - utv_{xx} + tu_xv_x - u_{xx}tv - uxt + \frac{1}{2}v_x - \frac{1}{2}xv_{xx} \\
&= -2t\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2} + \frac{2t}{3}\left(\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\right)^2 \\
&\quad -t\left(\frac{1}{2}(\phi+\bar{\phi})\right)\sqrt{\phi_{xx}+\bar{\phi_{xx}}-\left(\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})\right)} + t\frac{1}{2}\sqrt{(\bar{\phi_x}-\phi_x\bar{\phi_x})^2} \\
&\quad -t\left(\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})\right)\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2} - t\frac{1}{2}\sqrt{(\bar{\phi}-\phi\bar{\phi})^2} + \\
&\quad \frac{1}{2}\sqrt{\phi_x+\bar{\phi_x}-\left(\frac{1}{2}(\phi_x+\bar{\phi_x})\right)^2} - \frac{x}{2}\sqrt{\phi_{xx}+\bar{\phi_{xx}}-\left(\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})\right)^2},
\end{aligned}
$$

and finally, the multipliers $\Lambda_1(x,t,u,v) = 2tu + x$ and $\Lambda_2(x,t,u,v) = -2tv$, provide the conservation law

$$
T_6^t = t(u^2-v^2) + ux = t\left(\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2 - \left(\phi\bar{\phi} - \frac{1}{2}(\phi+\bar{\phi})\right)^2\right) + \frac{1}{2}(\phi+\bar{\phi})x,
$$

$$
\begin{aligned}
T_6^x &= \frac{-4}{3}u^3t + 4^2tu - xu^2 - 2utu_{xx} + ux^2t + xv^2 + 2vtv_{xx} - vtx^2 + u_x - xu_{xx} \\
&= \frac{-4t}{3}\left(\frac{1}{2}(\phi+\bar{\phi})\right)^3 + 4t\frac{1}{2}(\phi+\bar{\phi})\left(\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\right)^2 \\
&\quad -2t\left(\frac{1}{2}(\phi+\bar{\phi})\right)\left(\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})\right) + x^2t\left(\frac{1}{2}(\phi+\bar{\phi})\right) \\
&\quad +2t\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\sqrt{\phi_{xx}+\bar{\phi_{xx}}-\left(\frac{1}{2}(\phi_{xx}+\bar{\phi_{xx}})\right)^2} \\
&\quad -t\sqrt{\phi+\bar{\phi}-\frac{1}{2}(\phi+\bar{\phi})^2}\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2 + \frac{1}{2}(\phi_x+\bar{\phi_x}) - x\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx}) \\
&\quad -x\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2 + x\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}. \qquad (26)
\end{aligned}
$$

3.2.2 Symmetry Reductions and Solutions

As before, we consider the symmetry combination $X_2 + aX_3$, that has characteristic equations

$$\frac{dt}{1} = \frac{dx}{a} = \frac{du}{0} = \frac{dv}{0}.$$

Solving these equations gives the invariant functions

$$\begin{aligned} \zeta &= -at + x, \\ v(x,t) &= F_1(\zeta), \\ u(x,t) &= F_2(\zeta). \end{aligned}$$

Therefore we find the reduced system for (13) as

$$\begin{aligned} -2F_1(\zeta)F_1^{'}(\zeta) + (a + 2F_2(\zeta))F_2^{'}(\zeta) + F_2^{'''}(\zeta) = 0, \\ (a + 2F_2(\zeta))F_1^{'}(\zeta) + 2F_1(\zeta)F_2^{'}(\zeta) + F_1^{'''}(\zeta) = 0. \end{aligned} \tag{27}$$

The first integrals of (27) are second-order, and they are

$$\begin{aligned} I_1 &= -\frac{1}{2}F_1'(\zeta)^2 + \frac{1}{2}F_2'(\zeta)^2 + F_1(\zeta)F_1''(\zeta) - F_2(\zeta)F_2''(\zeta)\frac{a}{2}F_1(\zeta)^2 \\ &\quad +2F_2(\zeta)F_1(\zeta)^2 - \frac{2}{3}F_2(\zeta)^3 - \frac{a}{2}F_2(\zeta)^2, \\ I_2 &= -F_2'(\zeta)F_1'(\zeta) + F_1''(\zeta)F_2(\zeta) + F_1(\zeta)F_2''(\zeta) - \frac{2}{3}F_1(\zeta)^3 \\ &\quad +aF_1(\zeta)F_2(\zeta) + 2F_2(\zeta)^2F_1(\zeta), \\ I_3 &= cF_1(\zeta) + 2F_1(\zeta)F_2(\zeta) + F_1''(\zeta), \\ I_4 &= aF_2(\zeta) - F_1(\zeta)^2 + F_2(\zeta)^2 + F_2''(\zeta). \end{aligned}$$

A special solution exists here.

For $I_1 = I_4 = 0$, we have $F_1(\zeta) = 0$, and

$$F_2(\zeta) = -\frac{3\,a}{2}\left(\tan\left(\frac{C_1}{2}\sqrt{a} - \frac{\zeta}{2}\sqrt{a}\right)\right)^2 - \frac{3\,a}{2}. \tag{28}$$

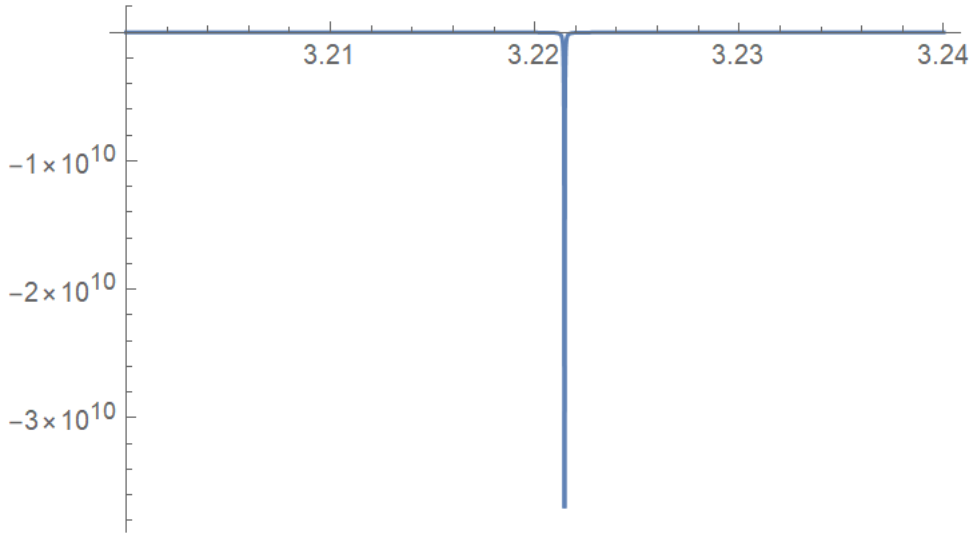

Figure 3.1: The evolution of (28) for $C_1 = 1,\ a = 2$

Thus, $v(x,t) = 0$ and

$$u(x,t) = -\frac{3\,a}{2}\left(\tan\left(\frac{C_1}{2}\sqrt{a} - \frac{-at+x}{2}\sqrt{a}\right)\right)^2 - \frac{3\,a}{2}. \tag{29}$$

Since $\phi = u + iv$, then

$$\phi(x,t) = -\frac{3\,a}{2}\left(\tan\left(\frac{C_1}{2}\sqrt{a} - \frac{-at+x}{2}\sqrt{a}\right)\right)^2 - \frac{3\,a}{2}. \tag{30}$$

The symmetry X_4 leads to the characteristic equations,

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{du}{-2u} = \frac{dv}{-2v},$$

which when solved gives the relations

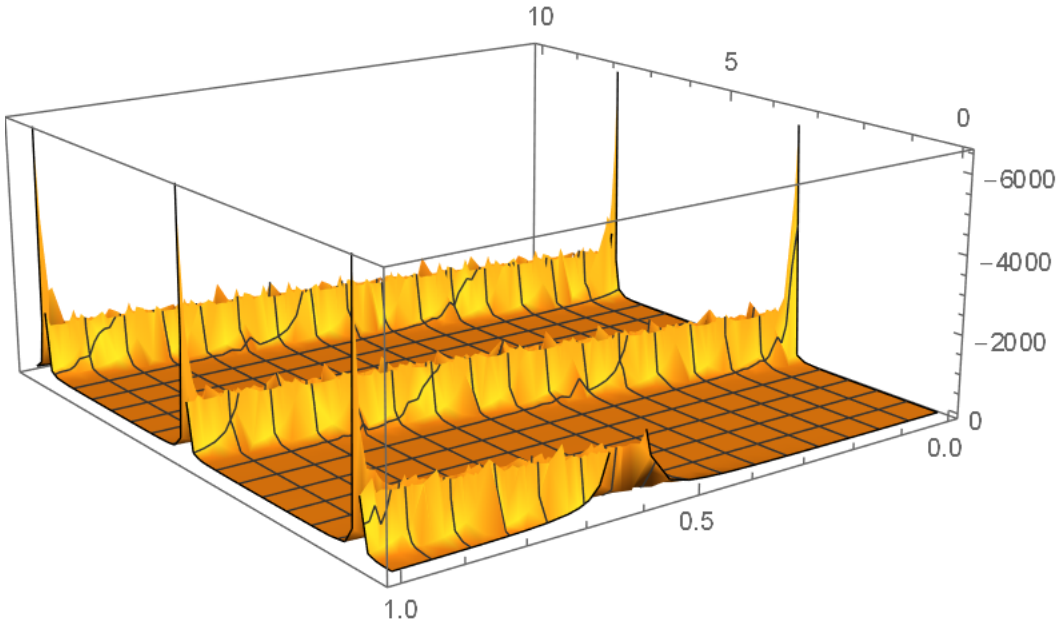

Figure 3.2: A 3D plot of (30) where the free parameters are chosen to be $C_2 = 1,\ a = 2$

$$\zeta = \frac{x}{\sqrt[3]{t}},$$
$$v(x,t) = \frac{1}{\sqrt[3]{t}} F_1(\zeta),$$
$$u(x,t) = \frac{1}{\sqrt[3]{t}} F_2(\zeta).$$

The above transformation leads to the reduced system of equations

$$\begin{aligned} &\frac{-6}{\zeta} F_1(\zeta) F_1'(\zeta) + F_2'(\zeta) + \frac{2}{\zeta} F_2(\zeta)(1 + 3F_2'(\zeta)) + \frac{3}{\zeta} F_2'''(\zeta) = 0, \\ &(1 + \frac{6}{\zeta} F_2(\zeta)) F_1'(\zeta) + \frac{2}{\zeta} F_1(\zeta)(1 + 3F_2'(\zeta)) + \frac{3}{\zeta} F_1'''(\zeta) = 0, \end{aligned} \tag{31}$$

which unfortunately could not be solved analytically.

3.3 Discussion and Conclusion

We decomposed the complex-valued equations into real and imaginary parts, and analyzed the equations in terms of Lie point symmetries and conservation laws. For $\alpha \neq 0$ we obtained three Lie points symmetries. We also used the multiplier approach to calculate conservation laws with $\alpha \neq 0$ having two conservation laws. For $\alpha = 0$, we have four Lie point symmetries and the conservation laws follows as six relations. We calculated invariant functions to reduce the systems of equations. The reduced systems of equations are third-order ordinary differential equations.

Chapter 4

The Complex-Valued Modified KdV Equations

The complex-valued modified KdV equation (mKdV) equation is

$$\phi_t = \phi_{xxx} - 6|\phi|^2\phi_x - \alpha\phi_{xx}, \tag{35}$$

where $\phi = \phi(x,t)$ and α is a constant. Decomposing into real and imaginary parts as previously, we find the system of equations

$$\begin{aligned} -u_t + u_{xxx} - 6(u^2+v^2)u_x - \alpha u_{xx} = 0, \\ -v_t + v_{xxx} - 6(u^2+v^2)v_x - \alpha v_{xx} = 0. \end{aligned} \tag{36}$$

The Lie point symmetry coefficients are determined from the system

$$\eta^1_v = 0,$$
$$\xi^1_v = 0,$$
$$\xi^2_v = 0,$$
$$\eta^2_{vv} = 0,$$
$$\eta^2_u = 0,$$
$$\xi^1_u = 0,$$
$$\xi^2_u = 0,$$
$$\eta^1_{uu} = 0,$$
$$\xi^2_x = 0,$$
$$-2\alpha\eta^1_{xu} + 3\eta^1_{xxu} + \alpha\xi^1_{xx} - \xi^1_{xxx} + \xi^1_t + 6u^2\xi^1_x - 6u^2\eta^1_u$$
$$-6v^2\eta^1_u - 12u\eta^1 + 6v^2\xi^1_x - 12v\eta^2 = 0,$$
$$-2\alpha\eta^2_{xv} + 3\eta^2_{xxv} + \alpha\xi^1_{xx} - \xi^1_{xxx} + \xi^1_t - 6u^2\eta^2_v + 6u^2\xi^1_x$$
$$-12u\eta^1 + 6v^2\xi^1_x - 6v^2\eta^2_v - 12v\eta^2 = 0,$$
$$-\alpha\eta^2_{xx} + \eta^2_{xxx} - \eta^2_t - 6u^2\eta^2_x - 6v^2\eta^2_x = 0,$$
$$-\alpha\eta^1_{xx} + \eta^1_{xxx} - \eta^1_t - 6u^2\eta^1_x - 6v^2\eta^1_x = 0,$$
$$3\eta^1_{xu} - 3\xi^1_{xx} - \alpha\eta^1_u + 2\alpha\xi^1_x = 0,$$
$$3\eta^2_{xv} - 3\xi^1_{xx} - \alpha\eta^2_v + 2\alpha\xi^1_x = 0,$$
$$\eta^1_u - 3\xi^1_x = 0,$$
$$\eta^2_v - 3\xi^1_x = 0,$$
$$\xi^2_t - \eta^1_u = 0,$$
$$\xi^2_t - \eta^2_v = 0.$$

4.1 Case A: $\alpha \neq 0$

After solving the symmetry determining system, we find

$$\xi^1 = k_1,\ \xi^2 = k_2,\ \eta^1 = -k_3 v,\ \eta^2 = k_3 u. \tag{37}$$

Hence, the individual symmetries of (36) are

$$X_1 = \partial_x, \tag{38}$$

$$X_2 = \partial_t, \tag{39}$$

$$X_3 = -v\partial_u + u\partial_v, \tag{40}$$

and commutators are displayed in Table 4.1. and for the case $\alpha \neq 0$, we find no

Table 4.1: Lie brackets of (36) with $\alpha \neq 0$

$[,]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	0
X_3	0	0	0

multipliers and conserved vectors.

4.1.1 Symmetry Reductions and Solutions

As in the above section, we utilize the symmetries $X_2 + aX_1$, to find the transformation

$$\zeta = -at + x,$$
$$v(x,t) = F_1(\zeta),$$
$$u(x,t) = F_2(\zeta),$$

so that the system (36) reduces to

$$(a - 6F_1(\zeta)^2 - 6F_2(\zeta)^2)F_2^{'}(\zeta) - \alpha F_2^{''}(\zeta) + F_2^{'''}(\zeta) = 0,$$
$$(a - 6F_1(\zeta)^2 - 6F_2(\zeta)^2)F_1^{'}(\zeta) - \alpha F_1^{''}(\zeta) + F_1^{'''}(\zeta) = 0. \tag{41}$$

The first integral of the system (41) is only

$$I_1 = e^{-\alpha\zeta}(F_1'(\zeta)F_2''(\zeta) - F_1''(\zeta)F_2'(\zeta)), \tag{42}$$

which cannot be used to solve further. More first integrals are needed.

4.2 Case B: $\alpha = 0$

For $\alpha = 0$, in this case, we obtain the symmetry coefficients as

$$\xi^1 = k_2 + \frac{k_4 x}{3}, \xi^2 = k_3 + k_4 t, \eta^1 = \frac{-k_4 u}{3} + k_1 v, \eta^2 = -k_1 u - \frac{k_4 v}{3}. \tag{43}$$

The Lie point symmetries of (36) are

$$X_1 = v\partial_u - u\partial_t, \tag{44}$$

$$X_2 = \partial_x, \tag{45}$$

$$X_3 = \partial_t, \tag{46}$$

$$X_4 = \frac{x}{3}\partial_x + t\partial_t - \frac{u}{3}\partial_u - \frac{v}{3}\partial_t, \tag{47}$$

with Lie brackets in Table 4.2.

Table 4.2: Lie brackets of (36) with $\alpha = 0$

$[,]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	$-\frac{X_2}{3}$
X_3	0	0	0	$-X_3$
X_4	0	$\frac{X_2}{3}$	X_3	0

4.2.1 Conserved Vectors

The conservation laws are calculated to be only one. The multipliers $\Lambda_1(x,t,u,v) = u$ and $\Lambda_2(x,t,u,v) = v$, gives rise to the conservation law

$$T^t = \frac{1}{2}(u^2+v^2),$$
$$T^x = \frac{3}{2}u^4 + 3u^2v^2 + \frac{3}{2}v^4 - uu_{xx} + \frac{1}{2}u_x^2 - vv_{xx} + \frac{1}{2}v_x^2,$$

and in the variable $\phi(x,t)$, the conservation law is

$$T^t = \frac{1}{2}|\phi|^2,$$
$$T^x = \frac{3}{2}(|\phi|^2)^2 + \frac{1}{2}\phi_x\bar{\phi_x} + \frac{1}{2}(\phi\bar{\phi_{xx}} + \bar{\phi}\phi_{xx}).$$

4.2.2 Symmetry Reductions and Solutions

The symmetry $X_3 + aX_2$, leads to the reduced equations

$$(a - 6F_1(\zeta)^2 - 6F_2(\zeta)^2)F_2^{'}(\zeta) + F_2^{'''}(\zeta) = 0,$$
$$(a - 6F_1(\zeta)^2 - 6F_2(\zeta)^2)F_1^{'}(\zeta) + F_1^{'''}(\zeta) = 0, \tag{48}$$

where

$$\zeta = -at + x,$$
$$v(x,t) = F_1(\zeta),$$
$$u(x,t) = F_2(\zeta).$$

The system (48) possesses the first integrals

$$I_1 = -F_1'(\zeta)F_2''(\zeta) + F_1''(\zeta)F_2'(\zeta), \tag{49}$$
$$I_2 = F_1''(\zeta)F_1(\zeta) + F_2(\zeta)F_2''(\zeta) - \frac{1}{2}F_1'(\zeta)^2 - \frac{1}{2}F_2'(\zeta)^2 - \frac{3}{2}F_1(\zeta)^4 - 3F_2(\zeta)^2F_1(\zeta)^2 + \frac{1}{2}aF_1(\zeta)^2 - \frac{3}{2}F_2(\zeta)^4 + \frac{1}{2}aF_2(\zeta)^2. \tag{50}$$

For $I_1 = 0$, we have the solution $F_1(\zeta) = 0$ and the implicit solution

$$\pm \int^{F_2(\zeta)} \frac{1}{\sqrt{\gamma^4 - \gamma^2 a + C_1\,\gamma - 2\,I_2}} d\gamma - \zeta - C_2 = 0.$$

For $C_1 = I_2 = 0, a = 1$, the integral is evaluated to be

$$\pm \arctan\left(\frac{1}{\sqrt{F_2(\zeta)^2 - 1}}\right),$$

hence the explicit solution is

$$F_2(\zeta) = \pm \frac{1}{\tan(\zeta + C_2)} \sqrt{(\tan(\zeta + C_2))^2 + 1}. \tag{51}$$

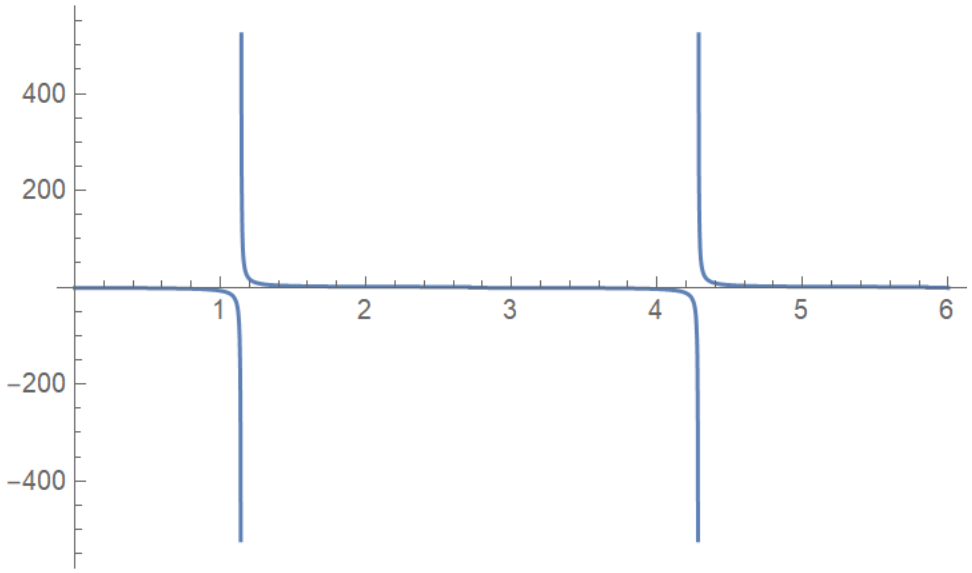

Figure 4.1: The progression of solution (51), taking the positive sign and $C_2 = 1$

Thus we have that $u(x,t)$ equals the solution (51) with $\zeta = -t + x$, $v(x,t)$ is zero and we may recover that

$$\phi(x,t) = \pm \frac{1}{\tan(-t + x + C_2)} \sqrt{(\tan(-t + x + C_2))^2 + 1}. \tag{52}$$

This solution is easily simplified to be written in terms of $\csc(-t+x+C_2)$, so that one can see the singularity for $-t+x+C_2=0$.

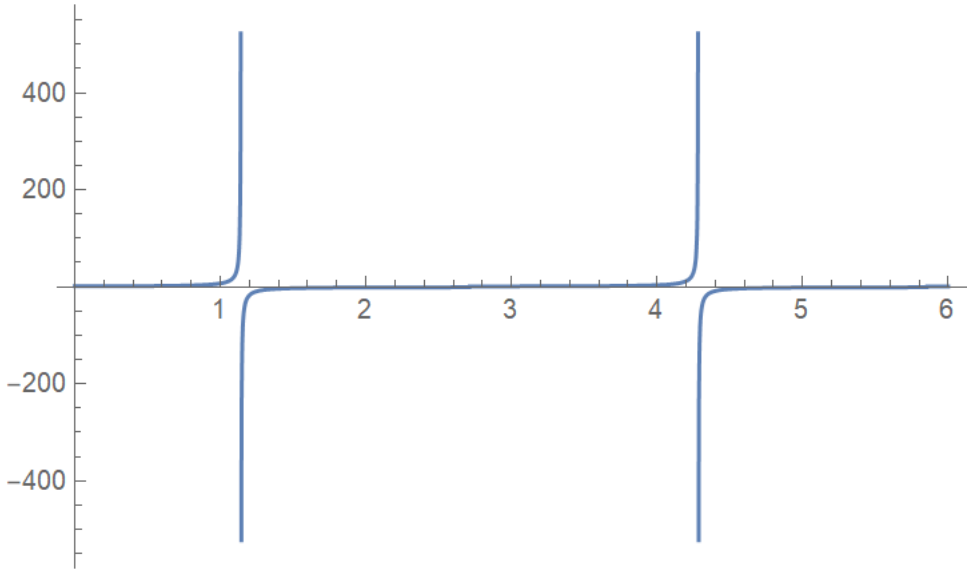

Figure 4.2: The progression of solution (51), taking the negative sign and $C_2 = 1$

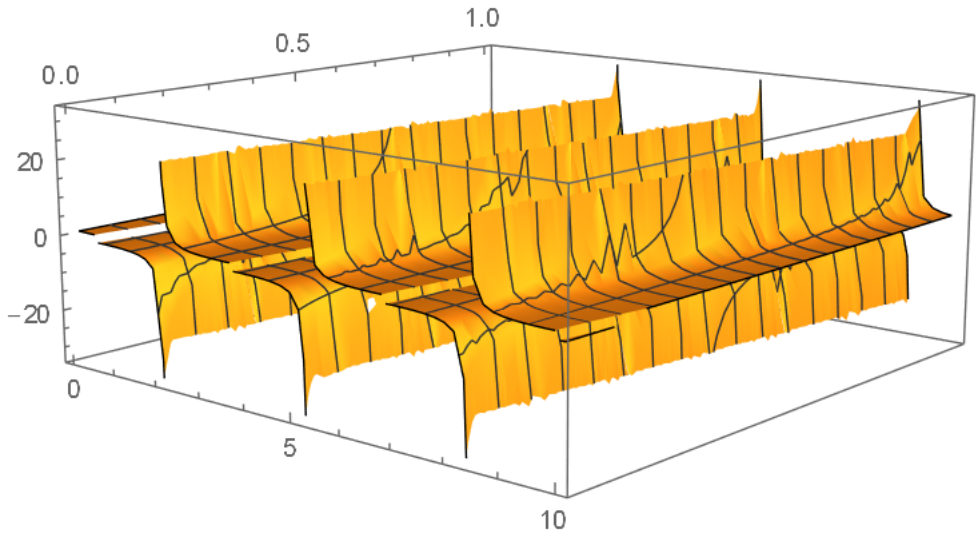

Figure 4.3: The function (52) with $C_2 = 1$, taking the positive sign

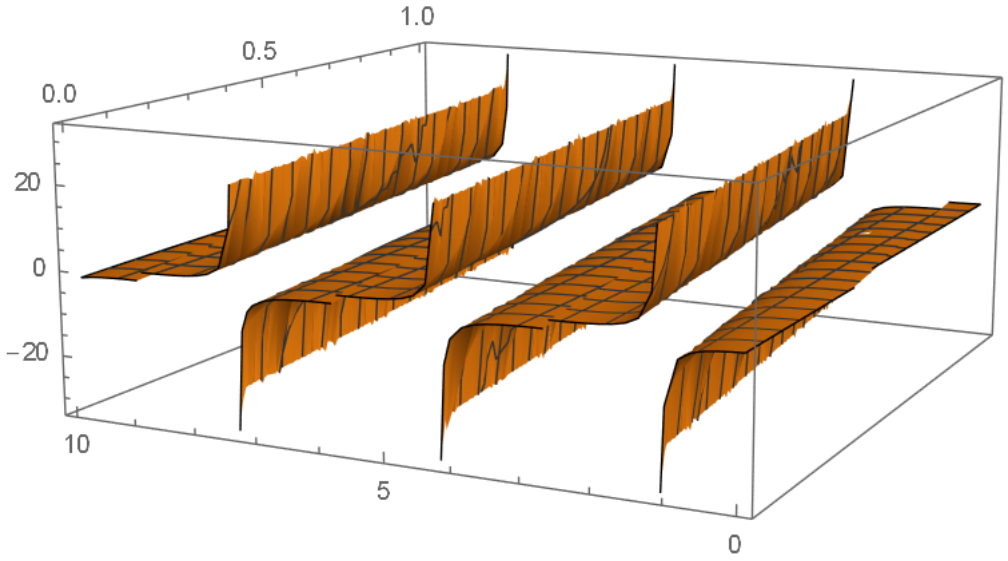

Figure 4.4: The function (52) with $C_2 = 1$, taking the negative sign

A second reduction can be performed with symmetry X_4, where we find for

$$\zeta = \frac{x}{\sqrt[3]{t}},$$
$$v(x,t) = \frac{1}{\sqrt[3]{t}} F_1(\zeta),$$
$$u(x,t) = \frac{1}{\sqrt[3]{t}} F_2(\zeta).$$

equation (36) reduces to the system

$$\begin{aligned} &\frac{-F_2(\zeta)}{\zeta} - F_2'(\zeta) - \frac{18}{\zeta} F_1(\zeta)^2 F_2'(\zeta) + \frac{18}{\zeta} F_2(\zeta)^2 F_2'(\zeta) - \frac{3}{\zeta} F_2'''(\zeta) = 0, \\ &-\frac{F_1(\zeta)}{\zeta} + \frac{18}{\zeta} F_1(\zeta)^2 F_1'(\zeta) - (1 - \frac{18}{\zeta} F_2(\zeta)^2) F_1'(\zeta) - \frac{3}{\zeta} F_1'''(\zeta) = 0, \end{aligned} \tag{53}$$

which unfortunately cannot be solved analytically.

4.3 Discussion and Conclusion

In this chapter, we also decomposed the equation into real and imaginary parts. For $\alpha \neq 0$ we found three Lie points symmetries with no conservation laws. For $\alpha = 0$ we obtained four Lie point symmetries and only one conservation law. Combinations of Lie point symmetries were used to reduce the KdV system of equations.

Chapter 5

Conclusion

KdV equations belong to fundamental models for wave spread in physical conditions. We concentrated on the KdV Burgers' and mKdV equations, that were considered in terms of a complex defined variable. These equations depended on a variable α that led to two cases for each equation. We decomposed each equation into real and imaginary parts to obtain a system of equations. This system was analysed in terms of its Lie point symmetries and conservation laws. The latter was calculated using the multiplier approach. In both systems, the case $\alpha \neq 0$ admits three Lie point symmetries and the case $\alpha = 0$ admits four Lie point symmetries. The KdV Burgers' system possesses conservation laws for all α, but the mKdV system only admits conservation laws for $\alpha = 0$. The Lie brackets corresponding to the Lie point symmetries of each case are presented in Tables.

We reduced each system to a system of third-order differential equations. These systems are highly nonlinear and are difficult to solve. Therefore, we resorted to finding first integrals of the reduced equations. Such first integrals are of second-

order and are easier to deal with than the third-order reduced system of equations. In one case, we showed that the solution is related to Abel's equation of the second kind. Two other solutions were found that involved trigonometric functions. Finally, we graphed each of the solutions.

The study of complex-valued KdV equations is far more involved than real-valued KdV models, but it is not surprising that complex-valued KdV equations share many properties with real-valued KdV equations i.e., integrable properties, symmetries and conservation laws. Our solutions involve trigonometric functions, but not the well-known solutions of the KdV equations containing hyperbolic trigonometric functions. The latter gives rise to solitonic solutions, where our solutions appear to be blow up solutions. In future work, we will extend this approach to other higher-order partial differential equations and systems.

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