Algebraic Structures in the Counting and Construction of Primary Operators in Free Conformal Field Theory

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Declaration

I, the undersigned, declare that the work contained in this thesis is my own original work. It is being submitted for the Doctor of Philosophy in the University of Witwatersrand, Johannesburg. It has not previously in its entirety or part been submitted for any degree or examination from other University.

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Abstract

The AdS/CFT correspondence relates conformal field theories in d dimensions to theories of quantum gravity, on negatively curved spacetimes in $d+1$ dimensions. The correspondence holds even for free CFTs which are dual to higher spin theories. Motivated by this duality, we consider a systematic study of primary operators in free CFTs.

We devise an algorithm to derive a general counting formula for primary operators constructed from n copies of a scalar field in a 4 dimensional free conformal field theory (CFT4). This algorithm is extended to derive a counting formula for fermionic fields (spinors), $O(N)$ vector models and matrix models. Using a duality between primary operators and multi-variable polynomials, the problem of constructing primary operators is translated into solving for multi-variable polynomials that obey a number of algebraic and differential constraints. We identify a sector of holomorphic primary operators which obey extremality conditions. The operators correspond to polynomial functions on permutation orbifolds. These extremal counting of primary operators leads to palindromic Hilbert series, which indicates they are isomorphic to the ring of functions defined on specific Calabi-Yau orbifolds. The class of primary operators counted and constructed here generalize previous studies of primary operators.

The data determining a CFT is the spectrum of primary operators and the OPE coefficients. In this thesis we have determined the complete spectrum of primary operators in free CFT in 4 dimensions. This data may play a role in attempts to give a derivation of a holographic dual to CFT4. Another possible application of our results concern recent studies of the epsilon expansion, which relates explicit data of the combinatorics of primary fields and OPE coefficients to anomalous dimensions of an interacting fixed point.

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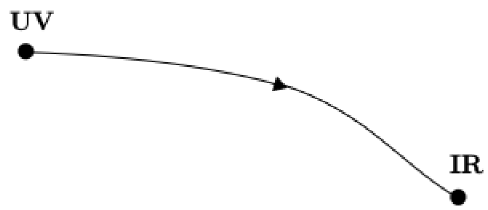
Chapter 1

Introduction

The duality between quantum gravity on asymptotically Anti-de Sitter (AdS) spacetime theory and Conformal Field Theory (CFT), which is known as the AdS/CFT correspondence, is one of the major breakthroughs to arise from string theory in recent years. The correspondence relates a strongly coupled quantum field theory to the classical dynamics of gravity living in one higher dimension. This kind of duality is sometimes referred to as the holographic duality or the gauge/gravity duality. To be more precise the correspondence is between a strongly coupled CFT_d in a large N limit and a semi-classical theory of gravity living in the AdS_{d+1} bulk spacetime. Every field in the bulk (gravity side) can be mapped into a primary field living on the boundary (CFT side). The correspondence is significant from both a conceptual and practical point of view. Not only does it give valuable physical insight into both sides of the correspondence, but it also provides new ways of performing calculations where conventional methods are intractable. The original correspondence is due to Maldacena[1]. It states that the 10-dimensional Type IIB superstring theory on the product space $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super Yang Mills (SYM) theory with gauge group $SU(N)$, living on the flat 4-dimensional boundary of AdS_5. This equivalence means that there is one-to-one correspondence between all aspects of the theories including the global symmetries, observables and correlation functions. Hence the theories are considered to be dual descriptions of each other. No form of the correspondence has been proven in a rigorous manner, leading it to be also known as the AdS/CFT conjecture.

This project was motivated by the idea of holography and the work carried out in [2] where primary operators were constructed from a product of two scalar fields operators. These primary operators constitute a tower of conserved higher spin currents. The descendents are derived by acting with spacetime derivatives on the primary operator, and the primary operator together with its descendents form an irreducible representation of the conformal group. The paper [2] then proceeds to calculate amplitudes (correlators) using Feynman diagrams for these conserved higher spin currents and manages to match them to the higher spin correlators of the dual gravity.

Apart from this motivation CFTs are interesting in their own rite. Conformal Field Theories (CFT) are a class of Quantum Field Theories (QFT) that enjoy a conformal symmetry. A conformal symmetry is a coordinate transformation which preserves the angles between any two vectors. Most QFT's that are scale invariant are also conformally invariant. Actually scale invariance often implies conformal invariance. This has been argued in $d = 2$[3] and almost argued in $d = 4$ dimensions [4][5] but a small possible loophole remains. In statistical systems/critical phenomena, scale invariance is realised when a system is at its critical point [6]. At the critical point the correlation length of the system becomes infinite and the system becomes scale invariant. Since there is no scale left to measure distances, the physics of the system looks the same at any length scale. In Quantum Field Theory an analogue of a critical point is a fixed point. QFT is basically the study of Renormalization Group (RG) flow, which is the flow of the theory from the high energy (UV) to the low energy (IR). The theory flows in a space of couplings [7]. During the flow the theory flows from a UV fixed point to an IR fixed point [8][9].



There are three possible phases in the IR region; (a) a phase with a mass gap, (b) a phase with massless particles, and (c) a Scale Invariant (SR) phase with a continous spectrum. Recently new phases, besides the three stated above, have been discovered [10, 11, 12].

Fixed points are points on a space of couplings where the beta functions vanish ($\beta(g) = 0$). When the beta function vanishes, the coupling strength doesn't evolve with the energy scale. Therefore a zero of the beta function means that the theory has evolved to scale invariance. Fixed points of the RG provide examples of conformal field theories. The fixed points can be in the UV or IR and we have no guarantee that a theory flowing from the UV or IR along an RG trajectory will end up in a fixed point. However this turns out to be the case in many physical systems.

At the fixed point the theory is conformal invariant. The possible macroscopic behaviour of the system at large scales is defined by its fixed points. At this point we can use techniques of conformal field theories to understand the general macroscopic features of theory that do not depend on the knowledge of microstates. Since CFT theories are mathematicaly controlable, this is helpful in the study of strongly coupled systems where perturbative techniques are of no use. Consider a simple field theory example which is a massless ϕ^4 scalar field theory with a Lagrangian

$$\mathcal{L} = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{4!}\phi^4 \right), \tag{1.1}$$

This theory is an IR free theory or trivial theory whose coupling become zero in the IR limit. When the theory flows to the UV region the coupling flow diverges, i.e. hitting the Landau pole. In the IR limit $g \to 0$, the theory is a free massless field. This point is called a Gaussian fixed point. The corresponding beta function at this point is $\beta(g = 0) = 0$. In d-dimensions the beta function is

$$\mu \frac{\partial g}{\partial \mu} = -(4 - d)g + 3\frac{g^2}{16\pi^2}. \tag{1.2}$$

Analyzing the beta function for different values of d. When $d > 4$ this function is positive and from dimensional analysis the coupling constant g is irrelevant, meaning the coupling will flow smoothly to zero at large distances, and the Lagrangian will flow to the free field fixed point. When $d = 4$ the same analysis takes place although from dimensional analysis the coupling constant g is marginal. However when $d < 4$, the first term in (1.2) on the RHS increases at large distance whilst the nonlinear second term in (1.2) decreases. There is a value of coupling $g = g^*$, where the increase and decrease effects of these terms come into balance giving a zero of the beta function

$$g^* = \frac{16\pi^2}{3}(4-d), \tag{1.3}$$

This is a nontrivial fixed point of the renormalization group flow in scalar field theory for $d < 4$. If we consider values of d close to 4, where $d = 4 - \epsilon$ and $\epsilon \to 0$, this fixed point occurs in a region where the coupling constant is small and we can use Feynman diagrams to study its properties. This point is called the Wilson Fischer point [7]. At this point the theory is scale invariant, which technically implies the theory is conformal invariant.

The conformal group is generated by the Poincare generators plus scaling and inversions generators. The conformal group generators are

$$P_\mu \qquad M_{\mu\nu} \qquad K_\mu \qquad D, \tag{1.4}$$

where P_μ is a translation generator, $M_{\mu\nu}$ is a Lorentz transformation generator, K_μ is a special conformal transformation (SCT) generator, and D is the dilatation (scaling) generator. For a theory with scalar operators ϕ having conformal scaling dimension Δ, these generators constrain the two and three point function up to the constant factor,

$$\langle\phi(x)\phi(y)\rangle = \frac{c}{|x-y|^{2\Delta}} \qquad \langle\phi(x)\phi(y)\phi(z)\rangle = \frac{\lambda_{\phi\phi\phi}}{|x-y|^{\Delta}|x-z|^{\Delta}|y-z|^{\Delta}}, \tag{1.5}$$

where c is constant and $\lambda_{\phi\phi\phi}$ is a structure constant. But conformal invarinace is not enough to constrain four point and higher point corelation functions. Therefore we cannot obtain all of the correlators of the theory just from imposing conformal invariance. Recently there has been an interest in the use of associativity of the Operator Product Expansion (OPE) and Bootstrap to study higher point correlation functions and also the spectrum of CFT's in strongly coupled regimes [13][14][15][16]. While Bootstrap is one of the techniques for understanding strongly coupled regimes, another popular technique for understanding this, is the AdS/CFT correspondence [1].

The objective of the project is to construct all possible local primary operators using a product of n copies of elementary fields, both in the free scalar field case and in the free fermion CFT. From here we use these primary operators to compute their corresponding correlators. We begin by focussing on a free field CFT in 4 dimensions (CFT4). Here the elementary field will transform in a representation of the conformal group $SO(4,2)$. Using the results obtained in [17] which give character formulae for the $SO(4,2)$ representation for the scalar field and the spinor field as

$$\chi_V(s,x,y) = s(1-s^2)\sum_{p,q}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \tag{1.6}$$

$$\chi_v(s,x,y) = s^{\frac{3}{2}}\sum_{q=0}^{\infty}\chi_{\frac{q+1}{2}}(x)\chi_{\frac{q}{2}}(y) \tag{1.7}$$

where $\chi_a(x)$ is the character for the left hand spin, and $\chi_a(y)$ is the character for the right hand spin and, s^b records the eigenvalue of the dilatation operator. We use the given characters to derive the characters for the symmetric product of n copies of scalar fields. Since the fermions (spinors) are Grassmen variables, we derive the characters for an antisymmetric product of n copies of the fermionic field. We then devise a general counting formula for the primary operators constructed from n copies of scalar or fermionic fields. Then using the duality between primary fields and multi-variable polynomials, we map the problem of constructing primary fields into a many-body quantum mechanics problem, where each primary corresponds to a multi-variable polynomial subjected to algebraic and differential constraints. One of the constraints, which comes from free field equation of motions, is a second order differential constraint which requires the polynomial function to be harmonic. Adopting an isomorphism between $\mathbb{R}^4$ and $\mathbb{C}^2 \times \mathbb{C}^2$ we are able to satisfy the harmonicity condition. We achieve this by working with holomorphic variables $(z, w, \bar{w}, \bar{z})$. For the scalar fields, selecting an extremal sector built out of (z, w) holomorphic variables results in a ring structure for the primary fields. This choice of isomorphism reduces the second-order differential condition to a first-order differential condition, which is a holomorphic condition.

The ring structure shows a palindromic Hilbert series property. The palindromicity implies that the primary fields correspond to functions on Calabi-Yau orbifolds which are

$$(\mathbb{C}^n/\mathbb{C} \times \mathbb{C}^n/\mathbb{C})/S_n = (\mathbb{C}^2)^n/(\mathbb{C}^2 \times S_n) \tag{1.8}$$

where n is the number of the elementary fields ϕ. Generalising to the $O(N)$ vector model gauge invariant primary fields correspond to functions on a Calabi-Yau orbifold with the geometry

$$(\mathbb{C}^2)^{2n}/(\mathbb{C}^2 \times S_n[S_2]) = (\mathbb{C}^{2n}/\mathbb{C} \times \mathbb{C}^{2n}/\mathbb{C})/S_n[S_2], \tag{1.9}$$

where $S_n[S_2]$ is a wreath product of S_n with S_2. We also consider a matrix model with fields ϕ_i^j transforming in the adjoint of the group $U(N)$. We find that the holomorphic primaries correspond to polynomial functions on the Calabi-Yau orbifolds

$$(\mathbb{C}^n/\mathbb{C} \times \mathbb{C}^n/\mathbb{C} \times S_n)/S_n = ((\mathbb{C}^2) \times S_n)/(\mathbb{C}^2 \times S_n). \tag{1.10}$$

The geometric structure found in these polynomial functions is novel and raises new questions about the geometry of the primary operators.

A general CFT is characterized by the CFT spectrum of primary operators$\{\Delta, \mathcal{R}\}$, where Δ is the scaling dimension of the local operator and $\mathcal{R}$ is the $SO(D)$ irreducible representation of the primary operator, and the OPE coefficients. We have managed to charaterize the free CFT4 spectrum of the primary operators in terms of $\{\Delta, j_L, j_R\}$ where j_L and j_R are respectively the left hand and right hand irreducible spin representation of the primary operators. Having the complete CFT data will help in the derivation of any postulated holographic dual to the CFT4. Another possible application of our results follows from [18] which relates explicit information of the combinatorics of primary fields and OPE coefficients of free CFT4 to observables in the epsilon expansion.

The thesis is organised as follows. In chapter 2 we give a brief introduction to CFT. We will introduce CFT symmetries and explain the consequences of these symmetries for correlation functions. We will discuss a special type of operator called a primary operator. In chapter 3 we will introduce and explain the AdS/CFT duality. We will show how the degrees of freedom are matched from the AdS and CFT side. Then we will describe the AdS/CFT dictionary and how it works. In chapter 4 we will give a summary of higher spin theory and how the AdS/CFT duality applies to higher spin theory. In chapter 5 we specifically talk about the paper published in [19], which presents novel results. We discuss the results obtained from constructing primary operators using n-copies of the free scalar field, and we extend this construction to $O(N)$ vector models and matrix models. In the last chapter 6, we extend the scalar field analysis of chapter 5 to fermions (spinors). We construct the primary operators using n-copies of left hand spinors. We observe that the extremal primary operators exhibit the same Calabi-Yau geometric structure as in the free scalar CFT. These geometric structures present in both the free scalar and fermion CFT were not observed before.

Chapter 2

CFT Background

In this chapter we will give a brief introduction to the basics of conformal field theory. The basic CFT ideas introduced here will provide the background for ideas covered in the following chapters. We will give a simple definition of what a conformal symmetry transformation is and discuss the consequences of these symmetries on the observables of the field theory. In particularly we will explain how they constrain correlation functions. We will discuss the conformal Killing vectors in any number of dimesnsions. After that we will introduce primary operators and discuss their properties. Then we consider the radial quantization of the CFT where we develop the idea of state-operator correspondence. Finally, we discuss the conditions on states/operators to ensure that the theory is unitary.

2.1 Basics of Conformal Field theory

A conformal transformation is a transformation that leaves the metric $g_{\mu\nu}$ invariant up to a scaling factor $\Omega(x)$ that depends on the spacetime coordinates,

$$\begin{aligned} \eta'_{\mu\nu}(x') =& \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial'^{\nu}} \eta_{\alpha\beta}(x) \\ =& \Omega(x) \eta_{\mu\nu}(x). \end{aligned} \tag{2.1}$$

From now on we specialize to flat spacetime. Considering an infinitesimal conformal transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$, $\Omega(x) = 1 + w(x)$, the above equation implies the conformal Killing vector equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}(\partial^\alpha \epsilon_\alpha). \tag{2.2}$$

Acting on (2.2) with the operator ∂_ρ and interchanging indices we obtain the following 3 equations

$$\begin{aligned}
\partial_\rho\partial_\mu \epsilon_\nu + \partial_\rho\partial_\nu \epsilon_\mu &= \frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial^\alpha \epsilon_\alpha) \\
\partial_\mu\partial_\nu \epsilon_\rho + \partial_\mu\partial_\rho \epsilon_\nu &= \frac{2}{d}\eta_{\rho\nu}\partial_\mu(\partial^\alpha \epsilon_\alpha) \\
\partial_\nu\partial_\mu \epsilon_\rho + \partial_\nu\partial_\rho \epsilon_\mu &= \frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial^\alpha \epsilon_\alpha).
\end{aligned} \tag{2.3}$$

Adding the first equation to the second equation and subtracting the third equation we obtain

$$\partial_\rho\partial_\nu \epsilon_\mu = \frac{1}{d}(\eta_{\mu\rho}\partial_\nu - \eta_{\nu\rho}\partial_\mu + \eta_{\mu\nu}\partial_\rho)\partial^\alpha \epsilon_\alpha. \tag{2.4}$$

Contracting the indices above with the metric $\eta^{\rho\nu}$ we get

$$\partial_\beta\partial^\beta \epsilon_\mu = \frac{2-d}{d}\partial_\mu(\partial^\alpha \epsilon_\alpha). \tag{2.5}$$

Acting on the equation above with the derivative ∂_ν we get

$$\partial_\nu\partial^\beta\partial_\beta \epsilon_\mu = \frac{2-d}{d}\partial_\nu\partial_\mu(\partial^\alpha \epsilon_\alpha). \tag{2.6}$$

Acting on the Killing vector equation (2.2) with the operator $\partial_\beta\partial^\beta$ we obtain

$$\partial_\mu\partial_\beta\partial^\beta \epsilon_\nu + \partial_\nu\partial_\beta\partial^\beta \epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\beta\partial^\beta(\partial^\alpha \epsilon_\alpha). \tag{2.7}$$

Symmetrizing equation (2.6) and comparing with the equation above we obtain

$$(2-d)\partial_\mu\partial_\nu f(x) = \eta_{\mu\nu}\partial_\beta\partial^\beta f(x), \tag{2.8}$$

where we have $f(x) = \partial^\alpha \epsilon_\alpha(x)$. Contracting with $\eta^{\mu\nu}$ gives

$$(d-1)\partial_\beta\partial^\beta f(x) = 0. \tag{2.9}$$

Now consider

$$\partial_\mu(\partial_\lambda\epsilon_\nu + \partial_\nu\epsilon_\lambda) = \frac{2}{d}\eta_{\nu\lambda}\partial_\mu(\partial\cdot\epsilon), \tag{2.10}$$

$$\partial_\lambda(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) = \frac{2}{d}\eta_{\mu\nu}\partial_\lambda(\partial\cdot\epsilon), \tag{2.11}$$

$$\partial_\nu(\partial_\mu\epsilon_\lambda + \partial_\lambda\epsilon_\mu) = \frac{2}{d}\eta_{\mu\lambda}\partial_\nu(\partial\cdot\epsilon). \tag{2.12}$$

Performing the sum, (2.10) + (2.12) − (2.11) implies

$$2\partial_\mu\partial_\nu\epsilon_\lambda = \frac{2}{d}(\eta_{\nu\lambda}\partial_\mu + \eta_{\mu\lambda}\partial_\nu - \eta_{\mu\nu}\partial_\lambda)\partial\cdot\epsilon. \tag{2.13}$$

Acting with ∂^λ on both sides yields

$$\begin{aligned} 2\partial_\mu\partial_\nu(\partial\cdot\epsilon) &= \frac{2}{d}(\partial_\nu\partial_\mu + \partial_\mu\partial_\nu - \underbrace{\eta_{\mu\nu}\partial\cdot\partial)}_{=0}\partial\cdot\epsilon \\ \Rightarrow \partial_\mu\partial_\nu(\partial\cdot\epsilon) &= 0. \end{aligned} \tag{2.14}$$

Therefore acting with ∂_ρ on (2.13) shows

$$\partial_\rho\partial_\mu\partial_\nu\epsilon_\lambda = 0. \tag{2.15}$$

When $d > 1$, the above equation implies that the conformal Killing vectors ϵ_μ must take the general form

$$\epsilon_\mu = c_\mu + a_{\mu\nu}x^\nu + b_{\mu\nu\rho}x^\nu x^\rho. \tag{2.16}$$

Equation (2.16) shows that the function $f(x) = \partial^\alpha\epsilon_\alpha(x)$ is linear in x. We make an ansatz that

$$\partial^{\alpha}\epsilon_{\alpha}(x) = -(2b \cdot x - \lambda)d, \tag{2.17}$$

which will be helpful shortly. Making the trick

$$\begin{aligned}
\partial_\rho(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu) =& \partial_\mu(\partial_\rho\epsilon_\nu + \partial_\nu\epsilon_\rho) - \partial_\nu(\partial_\rho\epsilon_\mu + \partial_\mu\epsilon_\rho) \\
=& \partial_\mu\left(\frac{2}{d}g_{\rho\nu}\partial\cdot\epsilon\right) - \partial_\nu\left(\frac{2}{d}g_{\rho\mu}\partial\cdot\epsilon\right) \\
=& \frac{2}{d}\Big(g_{\rho\nu}\partial_\mu(-(2b\cdot x - \lambda)d) - g_{\rho\mu}(\partial_\nu(-2b\cdot x - \lambda)d)\Big) \\
=& -4g_{\rho\nu}b_\mu + 4g_{\rho\mu}b_\nu.
\end{aligned} \tag{2.18}$$

Integrating the equation above, we obtain

$$\begin{aligned}
\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu =& 4(g_{\rho\mu}b_\nu x^\rho - g_{\rho\nu}b_\mu x^\rho) + 2w_{\mu\nu} \\
=& 4(b_\nu x_\mu - b_\mu x_\nu) + 2w_{\mu\nu},
\end{aligned} \tag{2.19}$$

which implies that

$$\begin{aligned}
\partial_\mu\epsilon_\nu =& w_{\mu\nu} + 2(b_\nu x_\mu - b_\mu x_\nu) - (2b\cdot x - \lambda)g_{\mu\nu} \\
\Rightarrow \epsilon_\nu =& a_\nu + w_{\mu\nu}x^\mu + b_\nu x^2 - 2b\cdot x x_\nu + \lambda x_\nu.
\end{aligned} \tag{2.20}$$

In d dimensions the conformal transformation has $\frac{(d+1)(d+2)}{2}$ transformation parameters, given by

$$\begin{aligned}
x'^\mu =& x^\mu + a^\mu, \text{ Translations has } (d+1) \text{ parameters} \\
x'^\mu =& x^\mu + \lambda x^\mu, \text{ Dilatations has 1 parameter} \\
x'^\mu =& x^\mu + \omega^\mu_\nu x^\nu, \text{Lorentz has } \frac{d(d-1)}{2} \text{ parameters} \\
x'^\mu =& x^\mu + 2(b_\alpha x^\alpha)x^\mu - x^2 b^\mu \text{ Special Conformal Transformation (SCT) has d parameters.}
\end{aligned} \tag{2.21}$$

There is one transformation missing which is not connected to the identity of the conformal group. It is a discrete transformation. This is the Inversion transformation and is denoted as follows

$$I : x'^{\mu} = \frac{x_\mu}{x^2}. \tag{2.22}$$

The special conformal transformation can also be obtained by performing a sequence of inversion, translation and inversion transformation. This is easily demostrated as follows

$$\begin{aligned} x'^{\mu} =&(Ie^{ib\cdot P}I)x_\mu \\ =&I(\frac{x_\mu}{x^2}+b_\mu) \\ =&\frac{\frac{x_\mu}{x^2}+b_\mu}{(\frac{x_\mu}{x^2}+b_\mu)^2} \\ =&\frac{x^\mu+b^\mu x^2}{1+2b\cdot x+b^2x^2}. \end{aligned} \tag{2.23}$$

Consider the conformal group element $g = e^{(ia^\mu P_\mu+\frac{i}{2}w^{\mu\nu}M_{\mu\nu}+i\lambda D+ib^\mu K_\mu)}$, obtained by taking the exponential of the conformal generators, $P_\mu, M_{\mu\nu}, D$ and K_μ. The conformal group element g, when it acts infinitesimally on the space-time vector x^α it transforms the vector in the following way

$$\begin{aligned} gx^\alpha =&e^{(ia^\mu P_\mu+\frac{i}{2}w^{\mu\nu}M_{\mu\nu}+i\lambda D+ib^\mu K_\mu)}x^\alpha \\ =&x^\alpha+\delta x^\alpha \\ =&x^\alpha+(ia^\mu P_\mu+\frac{i}{2}w^{\mu\nu}M_{\mu\nu}+i\lambda D+ib^\mu K_\mu)x^\alpha \\ =&x^\alpha+a^\alpha+w^{\alpha\beta}x_\beta+\lambda x^\alpha+x^2b^\alpha-2b\cdot xx^\alpha. \end{aligned} \tag{2.24}$$

We can get the explicit expression of generators from the equation above by comparing the terms in the last line with 3$^{\text{rd}}$ line terms. The comparison is from the equation

$$a^\alpha+w^{\alpha\beta}x_\beta+\lambda x^\alpha+x^2b^\alpha-2b\cdot xx^\alpha = (ia^\mu P_\mu+\frac{i}{2}w^{\mu\nu}M_{\mu\nu}+i\lambda D+ib^\mu K_\mu)x^\alpha. \tag{2.25}$$

Comparing the terms we find,

$$\begin{aligned} a^{\alpha} =& i a^{\mu} P_{\mu} x^{\alpha} \\ =& i a^{\mu} \delta^{\alpha}_{\mu} \\ =& i\Big(- i a^{\mu} \frac{\partial}{\partial x^{\mu}} \Big) x^{\alpha}, \end{aligned} \tag{2.26}$$

which implies that $P_{\mu} = -i\frac{\partial}{\partial x^{\mu}}$. Comparing the terms that involve Lorentz transformation we see that

$$\begin{aligned} w^{\alpha\beta} x_{\beta} =& \frac{i}{2} w^{\mu\nu} M_{\mu\nu} x^{\alpha} \\ =& \frac{i}{2} w^{\mu\nu} \left(\delta^{\alpha}_{\mu} x_{\nu} - \delta^{\alpha}_{\nu} x_{\mu} \right) \\ =& \frac{i}{2} w^{\mu\nu} \Big(x_{\nu} \frac{\partial}{\partial x^{\mu}} - x_{\mu} \frac{\partial}{\partial x^{\nu}} \Big) x^{\alpha} \\ \Rightarrow M_{\mu\nu} =& i\Big(x_{\mu} \frac{\partial}{\partial x^{\nu}} - x_{\nu} \frac{\partial}{\partial x^{\mu}} \Big). \end{aligned} \tag{2.27}$$

For the Dilatation operator we have,

$$\begin{aligned} \lambda x^{\alpha} =& i\lambda D x^{\alpha} \\ =& i\lambda x^{\mu} \delta^{\alpha}_{\mu} \\ =& i\lambda \Big(x^{\mu} \frac{\partial}{\partial x^{\mu}} \Big) x^{\alpha} \\ =& i\lambda (x \cdot \partial) x^{\alpha} \\ \Rightarrow D =& - i x \cdot \partial. \end{aligned} \tag{2.28}$$

For the Special Conformal operator K_{μ} we have

$$x^2 b^{\alpha} - b \cdot x x^{\alpha} = i b^{\mu} K_{\mu} x^{\alpha}. \tag{2.29}$$

The special conformal operator is

$$K_{\mu} = x^{\mu} + b^{\mu} x^2 - 2 x^{\mu} b \cdot x. \tag{2.30}$$

In $d = 2$, the Killing vector equation (2.2) becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = g_{\mu\nu}(\partial_\alpha \epsilon^\alpha). \tag{2.31}$$

We take the coordinates as (x_1, x_2). When $\mu = 1$ and $\nu = 1$, the metric $g_{\mu\nu} = 1$ and the Killing vector becomes

$$\begin{aligned} \partial_1 \epsilon_2 + \partial_2 \epsilon_1 &= 0 \\ \Rightarrow \frac{\partial \epsilon_2}{\partial x^1} &= -\frac{\partial \epsilon_1}{\partial x^2}. \end{aligned} \tag{2.32}$$

This equation is one of the Cauchy-Riemann equations. Similarly when $\mu = \nu$ we obtain

$$\frac{\partial \epsilon_1}{\partial x^1} = \frac{\partial \epsilon_2}{\partial x^2}. \tag{2.33}$$

A function that satisfies the Cauchy-Riemann equations is an analytic function. Going to complex co-ordinates $(z, \bar{z})$, the conformal transformations in two dimension are represented by analytic reparametrizations of the form

$$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z}). \tag{2.34}$$

Analytic functions satisfy the Cauchy-Riemann equations, which can compactly be written as

$$\frac{\partial}{\partial \bar{z}} f(z) = 0. \tag{2.35}$$

To obtain a basis for the conformal transformations, we consider co-ordinate transformations of the form

$$\begin{aligned} z \to z' &= z - \epsilon_n z^{n+1} \qquad \text{and} \qquad & \bar{z} \to \bar{z}' &= \bar{z} - \epsilon_n \bar{z}^{n+1} \\ &= z + \delta z & &= \bar{z} + \delta \bar{z} \end{aligned} \tag{2.36}$$

The generators of the transformation are specified by

$$l_n z = \delta z \qquad \text{and} \qquad \bar{l}_n \bar{z} = \delta \bar{z} \tag{2.37}$$

so that we identify the generators as $l_n = -z^{n+1}\partial_z$ and $\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$. These generators satisfy the well known Virasoro algebra

$$[l_m, l_n] = (m-n)l_{m+n}. \tag{2.38}$$

2.2 Conformal Algebra

The generators of the conformal group $SO(d,2)$ satisfy certain commutation relations which give the conformal algebra of the group. The conformal group algebra is (for $d > 2$)

$$[M_{\mu\nu}, P_\rho] = i(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu) \tag{2.39}$$

$$[M_{\mu\nu}, K_\rho] = i(\delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu) \tag{2.40}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\rho} - \delta_{\mu\sigma}M_{\rho\nu}) \tag{2.41}$$

$$[D, P_\mu] = iP_\mu \tag{2.42}$$

$$[D, K_\mu] = -iK_\mu \tag{2.43}$$

$$[K_\mu, P_\nu] = -2i(\delta_{\mu\nu}D - M_{\mu\nu}). \tag{2.44}$$

The first three commutation relations show that the operators $M_{\mu\nu}$ generate rotations in $SO(d)$ and the operators K_μ and P_μ transforms as vectors. The fourth and fifth commutation relations shows that the operator P_μ can be viewed as the raising operator and the operator K_μ can be viewed as lowering operator. For $d = 2$ the generators close the Virasoro algebra as commented in the last section.

2.2.1 Primary Operators

The representations of the conformal algebra we are interested in, are those in terms of local fields. Each irreducible representation of the conformal algebra has a unique primary field. The conformal algebra of the generators on the field $\mathcal{O}_\Delta(x)$ with dimension Δ is

$$[P_\mu, \mathcal{O}_\Delta(x)] =- i\partial_\mu \mathcal{O}_\Delta(x) \tag{2.45}$$
$$[D, \mathcal{O}_\Delta(x)] =- i(\Delta + x^\mu\partial_\mu)\mathcal{O}_\Delta(x) \tag{2.46}$$
$$[M_{\mu\nu}, \mathcal{O}_\Delta(x)] =i(x_\mu\partial_\nu - x_\nu\partial_\mu + M^R_{\mu\nu})\mathcal{O}_\Delta(x) \tag{2.47}$$
$$[K_\mu, \mathcal{O}_\Delta(x)] =i(2x_\mu\Delta + 2x^\alpha M^R_{\mu\alpha} + 2x_\mu x\cdot\partial - x^2\partial_\mu)\mathcal{O}_\Delta(x). \tag{2.48}$$

The action of the special conformal generator lowers the dimension of the operator, since

$$\begin{aligned} DK_\mu\mathcal{O}_\Delta(x) =&[D, K_\mu]\mathcal{O}_\Delta(x) + iK_\mu D\mathcal{O}_\Delta(x) \\ =&- iK_\mu\mathcal{O}_\Delta(x) + \Delta K_\mu\mathcal{O}_\Delta(x) \\ =&i(\Delta - 1)K_\mu\mathcal{O}_\Delta(x). \end{aligned} \tag{2.49}$$

If we keep on acting with the operator K_μ on the operator $\mathcal{O}_\Delta(x)$, the dimension of the operator will keep on decreasing. Since dimensions are bounded from below in physically sensible theories, the process of acting with K_μ's should terminate at some point. Operators that are annihilated by the action of the conformal generator K_μ are called primary operators. They are defined by the property

$$[K_\mu, \mathcal{O}_\Delta(0)] = 0. \tag{2.50}$$

This is the standard defining property of a primary operator. We create the descendents of the primary operator $\mathcal{O}_\Delta$ by acting with the momentum generator on the operator $\mathcal{O}_\Delta(x)$, which increases the dimension of the operator since

$$\begin{aligned} DP_\mu\mathcal{O}_\Delta(x) =&[D, P_\mu]\mathcal{O}_\Delta(x) + P_\mu\mathcal{O}_\Delta(x) \\ =&iP_\mu\mathcal{O}_\Delta(x) + i\Delta P_\mu\mathcal{O}_\Delta(x) \\ =&iP_\mu(\Delta + 1)\mathcal{O}_\Delta(x). \end{aligned} \tag{2.51}$$

We can fill out the multiplet of the representation of the primary operator $\mathcal{O}_\Delta(x)$ with its descendents. Schematically

$$\mathcal{O}_\Delta(x) \overbrace{\longrightarrow}^{P_\mu} \mathcal{O}_{\Delta+1}(x) \overbrace{\longrightarrow}^{P_\mu} \mathcal{O}_{\Delta+2}(x) \longrightarrow \cdots \overbrace{\longrightarrow}^{P_\mu} \mathcal{O}_{\Delta+n}(x). \tag{2.52}$$

This representation is analogous to an irreducible representation of $SU(2)$, where we raise the weight of the state by acting with the angular momentum raising operator. In our case the raising operator is represented by the momentum operator P_μ and, the lowering operator is represented by the special conformal operator K_μ.

It is straightforward to argue that invariance under the action of P_μ, $M_{\mu\nu}$, D constrain the two point correlation function of primary operators $\mathcal{O}_\Delta(x)$ to be

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle\rangle = \frac{c}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{2.53}$$

We have not accounted for the constraint imposed by the special conformal transformation (SCT) operator K_μ. We can apply the SCT to constrain c. The action of the special conformal operator on spacetime co-ordinate x is

$$x^\mu \to \tilde{x}^\mu = \frac{x^\mu}{x \cdot x}. \tag{2.54}$$

A useful identity is

$$\frac{1}{|x_1 - x_2|} = \frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1^2 - \tilde{x}_2^2|}. \tag{2.55}$$

The primary operator transforms as follows under a conformal transformation

$$\tilde{O}_\Delta(\tilde{x}) = \frac{1}{(\tilde{x})^\Delta}\mathcal{O}_\Delta(x). \tag{2.56}$$

Therefore the action of the SCT generator on the two point function is

$$\begin{aligned}\langle \tilde{O}_{\Delta_1}(\tilde{x}_1)\tilde{O}_{\Delta_2}(\tilde{x}_2)\rangle =& \frac{1}{(\tilde{x}_1^2)^{\Delta_1}}\frac{1}{(\tilde{x}_2^2)^{\Delta_2}}\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle \\ \Rightarrow \frac{1}{|\tilde{x}_1-\tilde{x}_2|^{\Delta_1+\Delta_2}} =& \frac{1}{(\tilde{x}_1^2)^{\Delta_1}}\frac{1}{(\tilde{x}_2^2)^{\Delta_2}}\frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}}.\end{aligned} \tag{2.57}$$

The last line implies that

$$\begin{aligned}\frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}} =& \frac{(\tilde{x}_1^2)^{\Delta_1}(\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1-\tilde{x}_2|^{\Delta_1+\Delta_2}} \\ \Rightarrow \Delta_1 =& \Delta_2.\end{aligned} \tag{2.58}$$

Since $\Delta_1 = \Delta_2$ this shows that $c = \delta_{\Delta_1,\Delta_2}$, and consequently the two point function becomes

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}. \tag{2.59}$$

Following a similar procedure for the three point function, conformal invariance constrains the three point function to be

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{\lambda_{\mathcal{OOO}}}{|x_1-x_2|^{\Delta_1+\Delta_3-\Delta_2}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{2.60}$$

A conformally symmetric 4 point function is constructed as follows

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle = F(u,v)\prod_{i<j}^{4}|x_{ij}|^{2\gamma_{ij}} \tag{2.61}$$

where $\gamma_{ij} = \gamma_{ji}$, $\sum_{i\neq j}\gamma_{ij} = -2\Delta_i$ and $F(u,v)$ is an arbitrary function of variables u and v,

$$u = \frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|} \qquad v = \frac{|x_{12}||x_{34}|}{|x_{23}||x_{14}|}, \tag{2.62}$$

where $x_{ij} = x_i - x_j$. These variables, called conformal cross ratios, are invariant under conformal transformation. We can decompose the 4-point correlation function by using conformal blocks,

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle = \sum_{\bar{\mathcal{O}}}(\lambda_{\mathcal{O}\bar{\mathcal{O}}\bar{\mathcal{O}}})^2 C_{\mathcal{O}}(x_{14}\partial_4)C_{\mathcal{O}}(x_{23},\partial_2)\langle \bar{\mathcal{O}}_{\Delta_{\mathcal{O}}}(x_2)\bar{\mathcal{O}}_{\Delta_{\mathcal{O}}}(x_4)\rangle \tag{2.63}$$

$$= \sum_{\bar{\mathcal{O}}}(\lambda_{\bar{\mathcal{O}}\bar{\mathcal{O}}\bar{\mathcal{O}}})^2 \frac{G_{\mathcal{O}}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}}x_{34}^{2\Delta_{\mathcal{O}}}}$$

where $G_{\mathcal{O}}(u,v)$ is a conformal block $\lambda_{\mathcal{O}\bar{\mathcal{O}}\bar{\mathcal{O}}}$ is an operator product coefficient and

$$C_{\mathcal{O}}(x,\partial_y) = \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}}(1+\sigma x_\mu\partial_\mu + \alpha x^\mu x^\nu \partial_\mu\partial_\nu + \beta x^2\partial^2 + \cdots) \tag{2.64}$$

where σ, α and β are fixed by conformal invariance. The conformal block $G_{\mathcal{O}}(u,v)$ collects the contribution from the primary $\mathcal{O}$ and all of its descendents, to the four point function.

2.3 Radial Quantization and State Operator Correspondence

The Hilbert space in QFT can be constructed by foliating spacetime in d dimensions with $d-1$ dimensional spacelike surfaces. Each surface has its own Hilbert space and these surfaces are all equivalent since they are related by a unitary transformation. States are created on these surfaces by inserting local operators. Usually the foliation is choosen to respect the symmetry of the theory. In a system with Poincare symmetry, the spacetime is foliated with surfaces of equal time and the states $|\psi_{in}\rangle$ are defined in the past of the surface and $|\psi_{out}\rangle$ are states defined in the future of the surface. An overlap of these states living on the same surface is

$$\langle \psi_{in}|\psi_{out}\rangle. \tag{2.65}$$

The unitarity evolution operator U is used to write the overlap as

$$\langle \psi_{in}|U|\psi_{out}\rangle = \langle \psi_{in}|e^{iH\Delta t}|\psi_{out}\rangle, \tag{2.66}$$

where $H = P^0$ is the Hamiltonian and Δt is the time. In a Poincare invariant theory, states can be characterized by their energy and momenta,

$$P^\mu|p\rangle = p^\mu|p\rangle. \tag{2.67}$$

In CFT we apply a foliation process called radial quantization. In d-dimensional CFT we foliate spacetime with S^{d-1} surfaces. This is related to a more conventional quantization by a conformal transformation from $\mathbb{R}^d$ to $\mathbb{R} \times S^{d-1}$. In radial quantization, the dilatation operator D is used to move between the S^{d-1} surfaces and the evolution operator is

$$U = e^{iD\tau} \tag{2.68}$$

with $\tau = \log(r)$ where τ is the time. States living on these spheres can be characterised in terms of their scaling dimensions

$$D|\Delta\rangle = i\Delta|\Delta\rangle, \tag{2.69}$$

and the $SO(d)$ spin l

$$M_{\mu\nu}|\Delta, l\rangle = M^R_{\mu\nu}|\Delta, l\rangle. \tag{2.70}$$

In the context of radial quantization we have a correspondence named the state operator correspondence, which asserts that for each state there is local operator corresponding to the state and the converse is true. Assume in radial quantization there are no operator insertions. This correspond to the vacuum state $|0\rangle$, invariant under all global conformal transformation. The dilatation operator gives

$$D|0\rangle = 0. \tag{2.71}$$

Inserting the spinless local operator $\mathcal{O}_\Delta(x=0)$ at the origin creates the state

$$\mathcal{O}_\Delta(0)|0\rangle = |\Delta\rangle \tag{2.72}$$

since

$$\begin{aligned} D\mathcal{O}_\Delta(0)|0\rangle =& \Delta\mathcal{O}_\Delta(0)|0\rangle \\ \Rightarrow \mathcal{O}_\Delta(0)|0\rangle =& |\Delta\rangle. \end{aligned} \tag{2.73}$$

Inserting the operator $\mathcal{O}_\Delta(x)$ at the position x, we obtain the states

$$\begin{aligned} \mathcal{O}_\Delta(x)|0\rangle =& e^{iPx}\mathcal{O}_\Delta(0)e^{-iPx} \\ =& e^{iPx}\mathcal{O}_\Delta(0)|0\rangle \\ =& e^{iPx}|\Delta\rangle \\ =& \sum_n \frac{(iPx)^n}{n!}|\Delta\rangle, \end{aligned} \tag{2.74}$$

which is a superposition of states with different eigenvalues. Therefore local operator $\mathcal{O}_\Delta(x)$ is not an eigenstate of operator D.

2.3.1 Unitarity

Unitarity requires that all states in the Hilbert space have positive norm, which leads to a bound on the scaling dimensions of the operators of the theory. This means that the scaling dimension of the operators must be above a certain value. Consider the kets $|\{s\}\rangle$, where $\{s\} = \{s_1, s_2 \cdots, s_n\}$ are $SO(d)$ weights and $n = \frac{d}{2}$. The kets $|\{s\}\rangle$ represent the irreducible representation that contains the lowest weight state. We will impose unitarity on the states $|\{s\}\rangle = |l, \triangle\rangle$, where l is the spin representation and $\triangle$ is the dimension of the primary state. The action of the dilatation operator D on these states is

$$\begin{aligned} iD|\{s\}\rangle &= iD|l, \triangle\rangle \\ &= \triangle|l, \triangle\rangle. \end{aligned} \tag{2.75}$$

The state $P_\mu|l, \triangle\rangle$ has dimension $\triangle + 1$. Since $P_\mu^\dagger = K_\mu$, the requirement of unitarity is equivalent to

$$\begin{aligned}\langle\{s\}|K_\mu P_\nu|\{s\}\rangle =&\langle\triangle, l|K_\mu P_\nu|l, \triangle\rangle \geq 0 \\ =&\langle\triangle, l|i(\delta_{\mu\nu}D - M_{\mu\nu})|l, \triangle\rangle \geq 0 \\ =&\langle\triangle, l|(\triangle\delta_{\mu\nu} - iM_{\mu\nu})|l, \triangle\rangle \geq 0.\end{aligned} \tag{2.76}$$

For a positive norm state,

$$\triangle\delta_{\mu\nu} \geq \langle\triangle, l|iM_{\mu\nu}|l, \triangle\rangle. \tag{2.77}$$

The task that is left is to determine when the condition above is satisfied. To compute $\langle\triangle, l|iM_{\mu\nu}|l, \triangle\rangle$, we will use methods usually used to treat the spin-orbit interaction. Towards this end, we write $iM_{\mu\nu}$ as follows,

$$\begin{aligned}iM_{\mu\nu} =&\frac{1}{2}i(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})M_{\alpha\beta} \\ =&(V \cdot M)_{\mu\nu},\end{aligned} \tag{2.78}$$

where $(V_{\alpha\beta})_{\mu\nu} = -i(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})$ are the $SO(d)$ generators in the vector representation. $(V \cdot M)_{\mu\nu}$ is a tensor product of two representation spaces, the vector space and the spin representation space. We manipulate the operator $(V \cdot M)_{\mu\nu}$ as follows

$$V \cdot M = \frac{1}{2}\Big((V + M) \cdot (V + M) - V \cdot V - M \cdot M\Big). \tag{2.79}$$

This is an analogue of a spin-orbit interaction in quantum mechanics

$$L \cdot S = \frac{1}{2}\Big((L + S)^2 - L^2 - S^2\Big). \tag{2.80}$$

The operator L in quantum mechanics is an analogue of $V_{\alpha\beta}$ and the operator S in quantum mechanics is an analogue of $M_{\alpha\beta}$. We know that the operators S^2 and L^2 are Casimirs and their eigenvalues are

$$\frac{s(s+1)}{2} \qquad \text{and} \qquad \frac{l(l+1)}{2} \tag{2.81}$$

respectively. We also know that the operator $(L+S)^2$ has Casimirs in the tensor product $l \otimes s$ and the eigenvalues are

$$\frac{j(j+1)}{2}, \tag{2.82}$$

where $j = |l-s|, \cdots, l+s$. The same treatment applies for the operator

$$V \cdot M = \frac{1}{2}\Big((V+M)\cdot(V+M) - V\cdot V - M\cdot M\Big), \tag{2.83}$$

if we move to a Clebsch Gordon coupled basis in the tensor product space. On this basis, $(V+M)^2$, V^2, M^2 are good quantum numbers. If M transforms in the representation R, then V^2 and M^2 have Casimirs $c_2(V)$ and $c_2(R=l)$ and the operator $(V+M)^2$ has Casimir in the tensor product $V \times R$. Denote by R' ($R' \in V \otimes R$) the representation with smallest quadratic Casimir. We choose the Casimir $c_2(R')$ so that the we obtain the strictest bound. The equation (2.77) becomes

$$\triangle \geqslant \frac{1}{2}\Big(c_2(R) + c_2(V) - c_2(R')\Big) \tag{2.84}$$

From [20], in an arbitrary dimension d, special representations obey

$$\triangle \geqslant 0 \qquad \text{scalar}$$

$$\triangle \geqslant \frac{1}{2}(d-1) \qquad \text{spinor}$$

$$\triangle \geqslant (d-1) \qquad \text{vector.}$$

The results found above are sensible since, the identity is a scalar representation with $\triangle = 0$. For the spinors, the bound is saturated by the free Dirac field, since

$$[\gamma_\mu P^\mu, \psi] = 0$$

gives the smallest Dirac representation and the scaling dimension bound $\triangle \geqslant \frac{1}{2}(d-1)$ has the canonical dimension of the Dirac field. For the vector field, the vector operators that saturate the bound above satisfy

$$[P^\mu, \psi_\mu] = 0.$$

An example of operators which satisfy this condition are conserved currents. The vector field A_μ from Maxwell theory is not constrained to have positive norm, because it is not gauge invariant. Its canonical dimension violates the bound on the scaling dimension of vector fields.

Equation (2.84) can be phrased in terms of spin l and scaling dimension Δ as

$$\begin{aligned}\triangle &\geqslant \frac{1}{2}\Big(c_2(R)+c_2(V)-c_2(R')\Big) \\ &\geqslant \frac{1}{2}\Big((d-1)+l(l+d-2)-(l-1)(l-1+d-2)\Big) \\ &\geqslant d-2+l.\end{aligned} \tag{2.85}$$

So far the bounds we have found are from the norm of states which are descendents of the primary. We can find new bounds which are from second descendents by computing

$$\langle\triangle, l|K_\mu K_\nu P_\alpha P_\sigma|\triangle, l\rangle,$$

and demanding that we get positve norm states as follows,

$$\begin{aligned}0 &\leq \langle\triangle, l|K_\mu K_\nu P_\alpha P_\sigma|\triangle, l\rangle \\ &= \langle\triangle, l|K_\mu([K_\nu, P_\alpha]P_\sigma + P_\alpha[K_\nu, P_\sigma])|l, \triangle\rangle \\ &= \langle\triangle, l|2iK_\mu(\delta_{\nu\alpha}D - M_{\nu\alpha})P_\sigma + 2iK_\mu P_\alpha(\delta_{\nu\sigma}D - M_{\nu\sigma})|l, \triangle\rangle.\end{aligned} \tag{2.86}$$

We will compute results for $l = 0$. For $l = 0$, $M_{\mu\nu}|l, \triangle\rangle = 0$. Therefore

$$\begin{aligned}0 &\leq 2(\triangle+1)\delta_{\nu\alpha}\langle\triangle|K_\mu P_\sigma|\triangle\rangle - 2i\langle\triangle|K_\mu M_{\nu\alpha}P_\sigma|\triangle\rangle + 2\triangle\delta_{\nu\sigma}\langle\triangle|K_\mu P_\alpha|\triangle\rangle \qquad (2.87)\\ &= 2(\triangle+1)\delta_{\nu\alpha}\langle\triangle|2i(\delta_{\mu\sigma}D - M_{\mu\sigma})|\triangle\rangle + 2\triangle\delta_{\nu\sigma}\langle\triangle|2i(\delta_{\mu\alpha}D - M_{\mu\alpha})|\triangle\rangle \\ &\quad - 2i\langle\triangle|K_\mu[M_{\nu\alpha}, P_\sigma]|\triangle\rangle \\ &= 4\triangle(\triangle+1)\delta_{\nu\alpha}\delta_{\mu\sigma} + 4\triangle^2\delta_{\nu\sigma}\delta_{\mu\alpha} - 2i\langle\triangle|K_\mu(-i(\delta_{\sigma\alpha}P_\nu - \delta_{\sigma\nu}P_\alpha))|\triangle\rangle \\ &= 4\triangle(\triangle+1)\delta_{\nu\alpha}\delta_{\mu\sigma} + 4\triangle^2\delta_{\nu\sigma}\delta_{\mu\alpha} + 4\triangle\delta_{\sigma\alpha}\delta_{\mu\nu} - 4\triangle\delta_{\sigma\nu}\delta_{\mu\alpha},\end{aligned}$$

taking the trace by setting $\mu = \nu$ and $\alpha = \sigma$, we obtain

$$\begin{aligned}0 &\leq 4\triangle(\triangle+1)d + 4\triangle^2 d - 4\triangle d^2 + 4\triangle d \\ &\leq 2\triangle + 2 - d \\ \Rightarrow \triangle &\geq \frac{d-2}{2}.\end{aligned} \tag{2.88}$$

This is a sensible bound for a spinless operator or state. It correspond to the dimension of the free scalar.

We have seen that unitarity bounds of the scaling dimensions Δ in terms of the spin s of the primary operator. Primary operators with scaling dimension below the unitarity bound will have negative norm states, which violates unitarity.

In radial quantization, overlaps and states can be interpreted in a quantum mechanical sense, where states evolve using a unitary operator U. Further, overlaps of states map to correlation functions of operators. This implies useful parallel between CFT and Quantum mechanics. This completes our introduction to CFT. In the next chapter we develop some of the ideas behind the AdS/CFT correspondence.

Chapter 3

Basic Introduction to AdS/CFT

In this chapter we will unpack and discuss the idea of AdS/CFT duality and the AdS/CFT dictionary. We begin by discussing the geometry of AdS spacetime. Since the AdS gravity and the CFT theories live in different dimensions, it is not obvious that the two describe the same degrees of freedom. We show how the number of degrees of freedom are matched by relating the entropies between the two theories. In order to illustrate the AdS/CFT dictionary we discuss the well studied example of the AdS/CFT correspondence between the $\mathcal{N} = 4$ super Yang Mills theory and the type IIB string theory on $AdS_5 \times S^5$. We will discuss the matching of parameters between these two theories

3.0.1 AdS/CFT

The AdS/CFT correspondence states that a Quantum field theory which is a Conformal Field Theory living in d spacetime dimensions can be described by a quantum gravity theory on $d+1$ dimensional AdS spacetime. This CFT lives on the boundary of the bulk gravity theory. The duality between these theories is a weak/strong relation. When the CFT is strongly coupled, the gravity side is weakly coupled and the converse is also true. This implies that the correspondence relates strongly coupled QFT to a classical description of gravity on the AdS_{d+1} spacetime. The duality can also be used to handle a generic strongly coupled conformal field theory living on the boundary of a classical AdS_{d+1} bulk. The most well studied example of this correspondence is between $\mathcal{N} = 4$ SYM with gauge group $SU(N)$, in the large N and type IIB superstring theory on $AdS_5 \times S^5$.

3.1 Anti-de Sitter space

Finding the dual geometry associated with the given QFT is not trivial in general. But if the theory is at a fixed point, the β- function vanishes and the theory enjoys conformal invariance, which means at the fixed point the QFT becomes a CFT and thanks to the extra symmetry we can now easily find the metric for the theory. For a QFT in d-dimensional spacetime, the most general Poincare invariance metric is

$$ds^2 = \Omega^2(z)(-dt^2 + d\vec{x}^2 + dz^2) \tag{3.1}$$

where $(\vec{x} = x_1, x_2, \cdots, x_{d-1})$, z is the coordinate of the holographic-dimension and $\Omega(z)$ is determined by enforcing conformal invariance. When the theory is conformally invariant, under the transformation $(t, \vec{x}) \to \lambda(t, \vec{x})$ and $z \to \lambda z$, $\Omega(z)$ transform as

$$\Omega(z) \to \lambda^{-1}\Omega(z), \tag{3.2}$$

which fixes

$$\Omega(z) = \frac{L}{z}, \tag{3.3}$$

where L is a constant. Thus, the metric becomes

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2). \tag{3.4}$$

This is the line element of AdS in (d+1)-spacetime dimensions, which is denoted as AdS_{d+1}, The constant L is the Anti-de Sitter radius. In (3.4) the conformal boundary of the AdS space is at $z = 0$. The metric is singular at this point, which means we will have to introduce a regularization scheme in order to physically define the observable quantities in the boundary of the AdS. The AdS metric is a solution to the equation of motion of a gravity theory with action

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\Big(-2\Lambda + R + aR^2 + bR^3 + \cdots\Big), \tag{3.5}$$

where G_N is the Newton constant, a, b are constants, and $g = \det\{g_{\mu\nu}\}$. R is the Ricci scalar ($R = g^{\mu\nu}R_{\mu\nu}$) and Λ is the cosmological constant. When $a = b = \cdots = 0$, the action in (3.5) becomes the Einstein-Hilbert action of general relativity with a cosmological constant. In this scenario the equations of motion are just the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu} \tag{3.6}$$

where $R_{\mu\nu}$ is the Rici tensor. AdS is a special case of the known maximally symmetric spacetimes for which

$$R_{\alpha\beta\gamma\rho} = -\frac{1}{L^2}(g_{\alpha\gamma}g_{\beta\rho} - g_{\alpha\rho}g_{\beta\gamma}). \tag{3.7}$$

The reason for why the underlying spacetime of the dual bulk gravity theory is Anti de-Sitter is hidden in the symmetry structure of the spacetime. To see the isometries it is useful to construct the AdS$_{d+1}$ spacetime by embedding it in $\mathbb{R}^{d,2}$ with the metric

$$ds^2 = -dy_0^2 + \sum_{i=1}^{d} dy_i^2 - dy_{d+1}^2, \tag{3.8}$$

where $g_{\alpha\beta} = \mathrm{diag}[-,+,\cdots,+,-]$ and the AdS_{d+1} spacetime is described by the surface

$$-y_0^2 + \sum_{i=1}^{d} dy_i - y_{d+1}^2 = -L^2. \tag{3.9}$$

Using this metric different forms of the AdS metric can be obtained by using particular transformations. For example, we can obtain a global AdS by making the following transformations

$$y_0 = L(1+r^2)^{1/2}\cos(\tau), \qquad y_{d+1} = L(1+r^2)^{1/2}\sin(\tau), \qquad y_i = Lr\Omega_{d-1}^2 \tag{3.10}$$

where $r \in (0, \infty)$ and $\tau \in [0, 2\pi]$.

3.1.1 AdS and CFT Degrees of Freedom

The gauge/gravity duality raises many conceptual questions, including the matching of the number of degrees of freedom on both sides of the correspondence. The number of degrees of freedom in a system is measured by its entropy. On the QFT side in d-spacetime dimensions, the entropy is an extensive quantity, proportional to the volume of the system. If R_{d-1} is a (d-1)-dimensional spatial region of the QFT at a fixed time, the entropy is

$$S_{QFT} \propto Vol(R_{d-1}). \tag{3.11}$$

On the gravity side, the theory lives in a (d+1)-dimensional spacetime. It sounds a bit absurb that a theory in (d+1)-dimension contains the same entropy as its dual with a lower dimension. The entropies are the same because the entropy of the gravity theory is subextensive. On the gravity side, the entropy in a volume is bounded by the entropy of a black hole that fits inside that volume and, the entropy is proportional to the area of the surface of the blackhole horizon. According to the Bekenstein-Hawking formula

$$S = \frac{A}{4G_N}, \tag{3.12}$$

where A is the area of the event horizon and G_N is the Newton constant. We now want to explain how to match the entropy in (3.11) and (3.12). Let R_d be a spatial region in the (d+1) dimensional space time where the gravity theory lives. Then R_d is bounded by a (d-1)-dimensional manifold R_{d-1} ($R_{d-1} = \partial R_d$). From (3.12) the gravity entropy is proportional to

$$S(R_d) \propto Area(\partial R_d) \propto Vol(R_{d-1}) \tag{3.13}$$

which roughly agrees with the entropy result of QFT in (3.11).

More concretely, on the QFT side, we can regulate the theory by putting in both a UV and an Infrared (IR) regulator. We place the theory in a spatial box, which serves as an IR regulator. We then descretize the system by putting it on a lattice with spacing ϵ, which serves as a UV regulator. In d-spacetime dimensions the system has $(\frac{R}{\epsilon})^{d-1}$ cells. If we identify c_{QFT} to be the number of degrees of freedom per lattice site, where c_{QFT} is the central charge, then the total number of degrees of freedom is

$$S_{QFT} = \left(\frac{R}{\epsilon}\right)^{d-1} c_{QFT}. \tag{3.14}$$

The number of degrees of freedom on the gravity side, which is AdS_{d+1} is given by the Bekenstein-Hawking formula,

$$S_{AdS} = \frac{A_\partial}{4G_N}, \tag{3.15}$$

where A_∂ is the area of the region of AdS_{d+1} at the boundary when $z \to 0$. A_∂ is found by integrating the volume element corresponding to the AdS metric, $ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$ sliced at $z = \epsilon \to 0$,

$$A_\partial = \int_{\mathbb{R}^{d-1}, z=\epsilon} d^{d-1}x\sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \int_{\mathbb{R}^{d-1}, z=\epsilon} d^{d-1}x. \tag{3.16}$$

The integral on the RHS is the volume element of $\mathbb{R}$, and it is infinite. We regulate the infinity by placing the system in a box of size R, the same way we did on the QFT side, so that

$$\int d^{d-1}x = R^{d-1}. \tag{3.17}$$

Therefore

$$A_\partial = \left(\frac{L}{\epsilon}\right)^{d-1} R^{d-1}. \tag{3.18}$$

If we introduce the Planck length l_P and the Planck mass M_P for a gravity theory in $d+1$ spacetime dimesnions, and identify

$$G_N = (l_P)^{d-1} = \frac{1}{(M_P)^{d-1}}, \tag{3.19}$$

then the number of degrees of freedom in an AdS_{d+1} space is

$$S_{AdS} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_P}\right). \tag{3.20}$$

Comparing the entropy of the QFT (3.20) and AdS (3.14) side we conclude that

$$c_{QFT} = \frac{1}{4}\left(\frac{L}{l_P}\right)^{d-1}, \tag{3.21}$$

which shows that S_{AdS} and S_{QFT} scale in the same way with the IR and UV cutoffs R and ϵ.

This is the matching condition between gravity and QFT. The action of the gravity in the AdS_{d+1} space of radius L, has a factor of $\frac{L^{d-1}}{G_N} = (\frac{L}{l_P})^{d-1}$ multiplying it. We know that a theory is semi-classical when the coefficient multiplying its action is large, so that the theory is dominated by a saddle point. This means that the classical gravity theory is reliable when

$$\rightarrow \left(\frac{L}{l_P}\right)^{d-1} \gg 1. \tag{3.22}$$

This happens when the AdS radius is large compared to the Planck length l_P and, since the scalar curvature goes like $1/L^2$, the curvature is small compared to the Planck length. This means QFT has a gravity dual when the central charge c_{QFT} is large, so that the number of degrees of freedom per unit volume is huge.

3.2 Large N Limit

The dual stringy description of a gauge theory can be perfectly illustrated by looking at a $U(N)$ Yang-Mills theory with Lagrangian

$$\mathcal{L} = -\frac{1}{g_{YM}^2} F_{\mu\nu a} F^{a\mu\nu}, \tag{3.23}$$

where $F^a_{\mu\nu}$ is the non-Abelian gauge field strength written as follows

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^a_{bc}[A^b_\mu, A^c_\nu] \tag{3.24}$$

$$A^a_\mu \to A'^a_\mu = U^\dagger A^a_\mu U - iU^\dagger \partial_\mu U, \tag{3.25}$$

where f^a_{bc} is a structure constant of the $SU(N)$ group and A^a_μ, $a = 1, \cdots, N^2$ is the gauge field. It can also be written as an $N \times N$ matrix $[A_\mu]_{\alpha\beta}$. Introducing the 't Hooft coupling $\lambda = g_{YM}^2 N$, the Lagrangian is written as

$$\mathcal{L} = -\frac{N}{\lambda} Tr[F_{\mu\nu} F^{\mu\nu}]. \tag{3.26}$$

Perfoming a 't Hooft expansion [21], λ is kept fixed and the expansion of the amplitudes is a double expansion in powers of N^{-2} and λ. Using the double line notation (Feynman rules), each Feynman diagram triangulates a surface and different powers of N corresponds to different topologies of the surface.

Computing Feynman diagrams for a matrix model we realise that every index loop contributes with a power of N to the diagram amplitude. In general the amplitude is given by

$$A_{amplitude} \sim \left(\frac{\lambda}{N}\right)^E \left(\frac{N}{V}\right)^V N^F, \tag{3.27}$$

where E is the number of propagators (edges), F is the number of loops (faces) and, V is the number of vertices. The sum of terms

$$F - E + V = \chi, \tag{3.28}$$

is called the Euler characteristic, and it depends only on the topology of the surface associated to the Feynman diagram. For diagrams which triangulate a surface with h handles

$$\chi = 2 - 2h. \tag{3.29}$$

The perturbative expansion (genus expansion) of a generic amplitude takes the form

$$\begin{aligned} F_N(\lambda) =& \sum_{h,f} C_{h,f} N^{2-2h} \lambda^{2h-2+f} \\ =& \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda), \end{aligned} \tag{3.30}$$

where $f_h(\lambda) = \sum \lambda^{2h-2+f} C_{h,f}$ and $f_h(\lambda)$ is the sum of all the diagrams with h handles. In summary, correlation functions of gauge invariant operators can be arranged as an expansion in $\frac{1}{N}$.

3.3 Dictionary of AdS/CFT

The most well studied version of the correspondence is between $\mathcal{N} = 4$ SYM and IIB string theory on $AdS_5 \times S^5$. The action for the CFT is

$$\begin{aligned} \mathcal{L} =& tr\Big(- \frac{1}{2g_{YM}^2} F^{\mu\nu} F_{\mu\nu} + \frac{i\theta}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i\bar{\psi}^i_\alpha (\sigma^\mu D_\mu)_{\alpha\beta} \psi^i_\beta - \sum_{M=1}^{6} (D_\mu \phi^M)(D^\mu \phi^M) \\ & + g_{YM} C^{ij}_M \psi_i [\phi^M, \psi_j] + \frac{1}{2} g^2_{YM} \sum_{M,N} [\phi^M, \phi^N]^2 \Big) \end{aligned} \tag{3.31}$$

where $\bar{\psi}^i_\alpha$ and ψ^i_β are Weyl fermions C^{ij}_M are the structure constants of the R-symmetry group $SU(4)$, ϕ^M are the scalar fields and $F_{\mu\nu}$ is the non-Abelian field strength. We can often use the supergravity approximation to string theory. The action for supergravity is

$$\begin{aligned} S_{IIB} =& \frac{1}{16\pi G} \int d^{10}x \sqrt{g} \Big(R - \frac{1}{2(Im\tau)^2} \frac{\partial_\mu \tau}{\partial^\mu \tau} - \frac{1}{4} |F_1|^2 - |G_3|^2 - \frac{1}{2} |F_5|^2 \Big) \\ & - \frac{1}{32\pi i G} \int d^{10}x A_4 \wedge \bar{G}_3 \wedge G_3 + fermions. \end{aligned} \tag{3.32}$$

G_3 is a 3 form field strength, F_1 is a 1 form field strength, F_5 is a self dual 5 form field strength, $\bar{G}_3$ is a 3 form field strength which is dual to G_3, R is the Ricci scalar and τ is a complex scalar field. The dictionary of the correspondence is partly determined by matching

1. Parametrs
2. Spectrum
3. Correlators.

Each of these three aspect will be discussed below.

3.3.1 Parameters

The CFT living on the boundary has two parameters, λ and N. λ determines the strength of the interactions. This parameter translate as follows[1, 22, 23]

$$\lambda = g_{YM}^2 N = \left(\frac{L}{l_s}\right)^4, \tag{3.33}$$

where l_s is the string length and L is the radius of curvature of the AdS spacetime. The above equation can be rewritten as[1, 22, 23]

$$\frac{l_s^2}{L^2} = \frac{1}{\sqrt{\lambda}}. \tag{3.34}$$

When

$$\lambda \gg 1 \Rightarrow L \gg l_s, \tag{3.35}$$

and string corrections can be neglected since curvatures are small. The classical supergravity action can be used. The 10-dimensional Newton constant will now be related to string parameters as follows

$$16\pi G = (2\pi)^7 g_s^2 l_s^2, \tag{3.36}$$

with $g_{YM}^2 = 4\pi g_s$, where g_s is the string coupling strength. Using this the equation above becomes[1, 22, 23]

$$G = l_p^8 = \frac{\pi^4}{2} g_{YM}^4 l_s^8. \tag{3.37}$$

Finally the relation between the Planck length l_p and the radius of the AdS space is given by[1, 22, 23]

$$\left(\frac{l_p}{L}\right)^8 = \frac{\pi^4}{2N^2}. \tag{3.38}$$

Spectrum

In gauge theory the observables are gauge invariant operators which can be both single trace and multi traces. Consider local single trace operators of the form

$$tr[\phi^M(x)\phi^N(x)], \quad tr[\phi^{M_1}(x)\cdots\phi^{M_n}(x)]. \tag{3.39}$$

These single trace local operators correspond to the fields on the gravity side[2, 24]. Any field in 10 dimensional supergravity on the $AdS_5 \times S^5$ background can be decomposed into an infinite set of fields on AdS$_5$. This technique is referred to as Kaluza-Klein decomposition. Compactification on S^5 creates a discrete set of modes in the spectrum with only the zeroth mode surviving in the low energy effective action. Here is an example of the correspondence between operators in the CFT and fields in string theory [2, 24]

$$\begin{aligned} &T_{\mu\nu}(x) \text{ (Stress energy tensor)} \leftrightarrow g_{\mu\nu}(x) \\ &J^{ij}_\mu(x)\text{(Conserved current)} \leftrightarrow A^{ij}_\mu(x) \text{ (Gauge field in AdS)}. \end{aligned} \tag{3.40}$$

3.3.2 Correlators

The boundary of AdS_{d+1} spacetime is a conformally flat d-dimensional spacetime on which the CFT is formulated. String fields in the bulk spacetime are fixed to value J at the boundary of AdS_{d+1}. The boundary values J behave as sources for the CFT operators. This is possible because for every string observable at the boundary of AdS_{d+1} there is a corresponding observable in the CFT. More concretely the CFT Lagrangian with the source term $J(x)$ is

$$\mathcal{L} \to \mathcal{L} + J(x)\mathcal{O}(x) = \mathcal{L} + \mathcal{L}_J, \tag{3.41}$$

where $\mathcal{O}(x)$ is the CFT operator and the corresponding generating function is

$$Z_{CFT}[J] = \langle exp[\int L_J]\rangle_{CFT}. \tag{3.42}$$

The connected correlators are obtained from (3.42), by taking derivatives of the logarithmic of Z_{CFT} as follows

$$\langle \prod_i \mathcal{O}(x_i)\rangle = \prod_i \frac{\delta}{\delta J(x_i)} \log Z_{QFT}[J]\bigg|_{J=0}. \tag{3.43}$$

On the gravity side we have a bulk field $\phi(z, x)$ fluctuating in AdS with $\phi_0(x)$ being the boundary value of $\phi(z, x)$,

$$\begin{aligned} \phi_0(x) =& \phi(z \to 0, x) \\ =& \phi\big|_{\partial ADS}(x). \end{aligned} \tag{3.44}$$

As mentioned earlier, the ϕ_0 field is related to a source $J(x)$ for the dual operator $\mathcal{O}$ in the CFT. The AdS/CFT correspondence implies that the generating function is given by [25][26],

$$Z_{CFT}[\phi_0] = \langle exp[\int \phi_0 \mathcal{O}]\rangle_{CFT} = Z_{gravity}[\phi \to \phi_0] \tag{3.45}$$

The most important take away from this chapter is the AdS/CFT dictionary. A correspondence was formulated using the parameters between the two theories. With this correspondence we can now see clearly the interplay between weak and strong coupling. In the next chapter we will consider higher spin theory and its duality to the free $O(N)$ vector model. This is the example of AdS/CFT of most relevance for this thesis.

Chapter 4

Lightning Review of Higher Spin Theories

In this chapter we explain the relevance of studying higher spin theories for understanding AdS/CFT. We will review relevant work that has contributed in the field of higher spin theories. We will then discuss some interesting results that have been achieved in the free field CFT. Through the discussion we will see what kind of higher spin operators have been constructed thus far. This chapter connects the objective of this project with higher spin theories and more generally with AdS/CFT. It clarifies the purpose of this project.

4.0.1 Higher Spin/CFT duality

Since the AdS/CFT duality was proposed by Juan Maldacena, there has been no formal mathematical proof for the duality. A significant task is left to prove and understand the duality at a mathematicaly rigorous level. Studying Higher Spin (HS) gravitational theories might lead us to a better understanding of how the AdS/CFT duality works. The higher spin theories are favourable in helping understand the duality because they have the right structure to be dual to a free vector model CFT at the boundary of AdS.

Higher spin theories were conjectured to be dual to a vector $O(N)$ model by Klebanov and Polyakov [27]. This conjecture was followed by a $\mathcal{N} = 1$ supersymmetric generalization by E. Sezgin and P. Sundell [28] and [29]. This was then followed by conjectures relevant to Chern-Simons gauge theories coupled to vector models [30, 31], and the 3d bosonization duality [32, 33] relating scalar and fermionic theories coupled to Chern-Simons. Before AdS/CFT was conjectured, a Russian Physicist, Vasiliev, constructed a fully non-linear theory of interacting higher spins in AdS [34] and exact non-linear equations of motion for the theory. In higher spin theory, the interactions are in the form of higher derivatives and the spectrum contains gauge fields starting with spin $s = 2$ followed by an infinite tower of HS fields. Since HS theory always contain gravitons $s = 2$, this means that they are a theory of quantum gravity. The higher derivative (interactions) are in inverse powers of a cosmological constant. Because of their infinite dimensional HS symmetry, the theory can be identified as a UV complete theory of quantum gravity.

4.0.2 Free Field CFT with Higher Spin Currents

[35] studied a simple free field CFT with the action

$$S = \frac{1}{2}\int d^dx(\partial_\mu\phi)^2. \tag{4.1}$$

From ordinary standard QFT we know that the theory has a symmetric traceless conserved stress-energy tensor

$$T_{\mu\nu} = 4(d-1)\partial_\mu\phi\partial_\nu\phi - ((d-2)\partial_\mu\partial_\nu + g_{\mu\nu}\partial^2)\phi^2. \tag{4.2}$$

This stress energy tensor has spin $s = 2$ and, dimension $\triangle = 2(\frac{d}{2} - 1) + s = d$. The above CFT has a much larger symmetry than just conformal invariance which is realised through the construction of the conserved HS currents that are bilinear in scalar fields and have spin s [35],

$$J_s = (z\cdot(\partial_1+\partial_2))^s C_s^{d/2-3/2}\left(\frac{z\cdot(\partial_1-\partial_2)}{z\cdot(\partial_1+\partial_2)}\right)\phi(x_1)\phi(x_2)\Big|_{x_x,x_2\to x}. \tag{4.3}$$

$C_s^{d/2-3/2}(x)$ is a Gegenbauer polynomial and z is a polarization vector. This tower of higher spin currents is conserved $\partial^\mu J_{\mu\mu_2\cdots\mu_s} = 0$, symmetric and traceless which corresponds to an irreducible representation of $SO(d)$ of spin

s. In the $SO(d)$ representation the spin operators satisfy the unitary bound

$$\triangle \leq d - 2 + s, \tag{4.4}$$

which means that for the currents above the unitary boundary is saturated. For $s = 2$ spin currents, we can verify that $J_{\mu\nu}$ is equal to the stress energy tensor (4.2). The conformal algebra for a $s = 2$ theory is the normal conformal algebra with the conformal generators of the group being

$$P_\mu, \qquad M_{\mu\nu}, \qquad K_\mu, \qquad D, \tag{4.5}$$

where P_μ is the translation generator, $M_{\mu\nu}$ Lorentz generator, K_μ special conformal generator and D is the dilatation generator. These generators can be recovered from a CFT argument. For a CFT with a conformal Killing vector ζ^μ, satisfying the conformal Killing vector equation $\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = \frac{2}{d} g_{\mu\nu}(\partial_\rho \zeta^\rho)$, we can construct a conserved current $J^\zeta_\mu = T_{\mu\nu}\zeta^\nu$ from $T_{\mu\nu}$ (stress-energy tensor). Using standard QFT techniques we can obtain the conserved charges from the conserved currents. The conserved charges are the generators of the symmetries of the theory. The Killing vectors are in one-to-one correspondence with the generators.

When we go to higher spin theory, since the theory (CFT) has conserved higher spins currents, the theory has infinte dimensional extension of the conformal algebra, which is called the HS algebra. One needs an infinte tower of charges to close the algebra.

4.1 O(N) Vector Model

We will now consider a free $O(N)$ vector model with N massless fields. The model has a global $O(N)$ symmetry where ϕ^i transforms in the fundamental representation. The $O(N)$ vector model action is given as

$$\mathcal{S} = \frac{1}{2} \int d^d x \partial_\mu \phi^i \partial^\mu \phi^i \qquad\qquad i = 1, \cdots, N, \tag{4.6}$$

the equation of motion is pretty much the same as in the free scalar case,

$$\partial_\mu \partial^\mu \phi^i = 0. \tag{4.7}$$

The model has the same conserved higher spin (HS) currents, but now with additional $O(N)$ indices

$$\begin{aligned} J^{ij}(x,\epsilon) =& \sum_{k=0}^{s} c_{sk}((z\cdot\partial)^k \phi^i (z\cdot\partial)^{s-k}\phi^j \\ =& \phi^i \sum_{k=0}^{s} \frac{(-1)^k (z\cdot \overleftarrow{\partial})^k (z\cdot \overrightarrow{\partial})^{s-k}}{k!(k+\frac{d-4}{2})!(s-k+\frac{d-4}{2})!(s-k)!}\phi^j . \end{aligned} \tag{4.8}$$

These operators can be be decomposed into irreducible representations of $O(N)$

$$J_s^{ij} = J_s + J_s^{(ij)} + J_s^{[ij]} \tag{4.9}$$

where J_s are $O(N)$ singlets, $J_s^{(ij)}$ are symmetric traceless and $J_s^{[ij]}$ are anti-symmetric representation operators. Consider truncating to the $O(N)$ singlet sector which corresponds to taking the single and multi-trace operators. In the AdS/CFT correspondence, these single trace operators $\phi^i \partial_s \phi^i$ correspond to the single-particle states in AdS and the multi-trace operators $(\phi^{i_1}\partial_{s_1}\phi^{i_1})(\phi^{i_2}\partial_{s_1}\phi^{i_2})\cdots(\phi^{i_n}\partial_{s_n}\phi^{i_n})$ correspond to the multi-particle states in AdS. Single trace operators are bilinears in the fields ϕ^i. The full list of these single trace operators J_s includes $s = 0, 2, 4, 6, \cdots$. The operator $J_0 = \phi^i\phi^i$ and its dimension Δ and spin s are $(\Delta, s) = (d-2, 0)$. The singlet operators J_s have dimension Δ and spin s, being $(\Delta, s) = (d-2+s, s)$. The CFT single trace spectrum (Δ, s) should match the single particle spectrum of the bulk dual, and also the multi-trace operators spectrum $((\Delta, s))$ should match the multi-particle spectrum in the bulk dual (AdS). The conserved currents in the CFT correspond to the massless gauge fields in the AdS. A familiar example is the spin 1 conserved current in CFT corresponding to a gauge field in AdS, and the spin 2 stress energy tensor in CFT correspond to the graviton in AdS,

$$\partial \cdot J_s = 0 \qquad \Leftrightarrow \qquad \text{Massless HS gauge fields} \tag{4.10}$$

When interactions are switched on, the current operators are not conserved anymore and they now correspond to a massive gauge field ψ_s in AdS. Consider the scalar current operator $J_0 = \phi^i \phi^i$. Even though it appears in the free CFT, it is dual to the bulk scalar field ψ with mass m^2,

$$J_0 \quad \Leftrightarrow \quad \text{Scalar field } \psi \text{ with } m^2 = \Delta(\Delta - d)/l_{AdS}^2, \tag{4.11}$$

where Δ is the scaling dimesion of the operator J_0, so that $m^2 = -2(d-2)/l_{AdS}^2$ in the free CFT where $\Delta = d-2$. The global HS spin symmetry on the CFT side correspond to the HS gauge symmetry on the AdS side, generated by $s-1$ gauge parameters. This will be discussed in more detail in the next section, when we talk about the Fronsdal equations in AdS. The spectrum in (4.10) and (4.11) correspond to the minimal bosonic higher spin theory in AdS_{d+1} [36]. After matching the CFT operators with the bulk gauge fields, the next thing to do is to compute the correlators on both the CFT and AdS side. The CFT correlators are computed using J_s singlet current operators. As usual we sum the Wick contractions between the fields in these J_s currents. The results from the three point function computed in [24, 2, 37] for a normalized current $J_s \sim \frac{1}{\sqrt{N}} \phi^i \partial_s \phi^i$ give

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle \sim \frac{1}{\sqrt{N}}. \tag{4.12}$$

These correlators can be represented by a triangular diagram as shown below

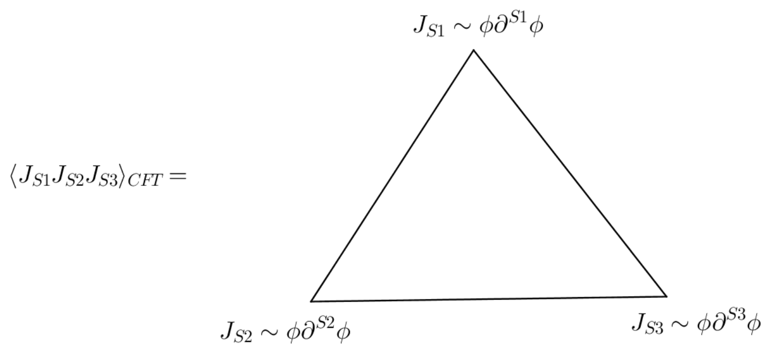
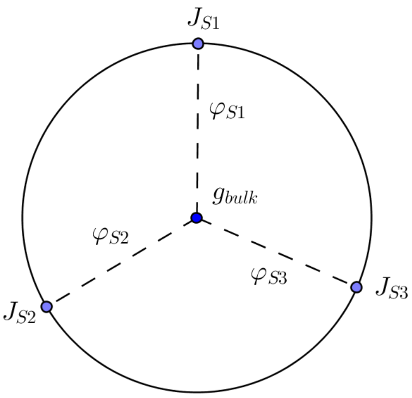


Figure 4.1: 3 point function of higher spin operators in CFT

In the diagram above each line represents a scalar propagator and the vertices contain appropriate derivatives. These derivatives of the AdS Vasiliev theory are linked to the interactions between the bulk dual fields. The correlator in (4.12) should be matched to the dual Witten diagrams [24, 2, 37] which are

$\langle J_{S1} J_{S2} J_{S3} \rangle_{CFT} =$



$+ \cdots$

Figure 4.2: Holographic 3 point function

Here the factor $\frac{1}{\sqrt{N}}$ sets the coupling of the higher spin gravity

$$g_{bulk} \sim \frac{1}{\sqrt{N}}. \tag{4.13}$$

Now the action in the bulk is written as follows

$$\frac{1}{G_N} \int d^{d+1} \mathcal{L}_{bulk} = \frac{1}{g_{bulk}^2} \int d^{d+1} x \mathcal{L}_{bulk} \tag{4.14}$$

with the Newton's constant $G_N \sim N^{-1}$, showing that the $\frac{1}{N}$ expansion on the CFT side is mapped to the pertubative expansion on the AdS gravity side, as powers of G_N. This non-trivial computation [24, 2, 37] shows that the free $O(N)$ vector model (singlet sector) is dual to the HS gravity in AdS, sometimes referred to as higher spin/vector model duality. One can generalize the $O(N)$ vector model to the $U(N)$ vector model by working with complex scalar fields and develop the same ideas and computations [35].

4.2 Fronsdal Higher Spins Equations in AdS

We now focus on the AdS side. We will discuss what has been achieved for the massless higher spin fields. These are the fields related to CFT conserved

HS spin primary operators. We will first discuss the Frondal equations in flat spacetime. We start by discussing the dynamics (equations of motion) of the familiar spin $s = 1$ and $s = 2$ cases of HS fields, before considering the dynamics of general HS fields. Consider the gauge field A_μ that has spin $s = 1$ with gauge symmetry

$$\delta A_\mu = \partial_\mu \epsilon \tag{4.15}$$

where ϵ is the gauge parameter. The Lagrangian for this field A_μ is the known Maxwell Lagrangian with the action

$$\mathcal{S} = -\frac{1}{4}\int d^d x F_{\mu\nu}F^{\mu\nu} \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.16}$$

and the Maxwell equation of motion follows,

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{or} \quad \partial_\alpha \partial^\alpha A_\mu - \partial_\mu \partial^\nu A_\nu = 0. \tag{4.17}$$

For $s = 2$ we have the Einstein equation of motion with zero cosmological constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{4.18}$$

Expanding around the flat metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the linearized equations for $h_{\mu\nu}$ is

$$\partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} + \partial_\mu \partial_\nu h^\rho_\rho = 0. \tag{4.19}$$

This is invariant under the gauge transformation

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{4.20}$$

We now want to generalize this to the massless HS fields $\psi_{\mu_1\mu_2\cdots\mu_s}$. The HS field $\psi_{\mu_1\mu_2\cdots\mu_s}$ is required to be totally symmetric and double traceless

$$\psi^{\mu\ \nu}_{\ \mu\ \nu\mu_5\cdots\mu_s} = 0. \tag{4.21}$$

The gauge transformation of the HS field is

$$\delta\psi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)}, \tag{4.22}$$

where the closed brackets on the indices symmetrize the enclosed indices. For the HS field in (4.21) to be gauge invariant, the spin $s-1$ gauge parameter $\epsilon_{\mu_1\mu_2\cdots\mu_{s-1}}$ needs to obey the traceless constraint

$$\epsilon^{\mu}{}_{\mu\mu_3\cdots\mu_{s-1}} = 0. \tag{4.23}$$

Then, the general equation of motion for a HS gauge fields is

$$\mathcal{F}_{\mu_1\cdots\mu_s} = \partial_\alpha\partial^\alpha\psi_{\mu_1\cdots\mu_s} - s\partial_{(\mu_1}\partial^\mu\psi_{\mu_2\cdots\mu_s)\mu} + \frac{s(s-1)}{2}\partial_{(\mu_1}\partial_{\mu_2}\psi_{\mu_3\cdots\mu_s)\mu}{}^{\mu} = 0s. \tag{4.24}$$

These equations generalize the previous $s=1$ and $s=2$ cases. The Lagrangian for these HS gauge fields is

$$S = \int d^dx(\psi^{\mu_1\cdots\mu_s}\mathcal{F}_{\mu_1\cdots\mu_s} - \frac{1}{4}s(s-1)\psi_{\mu}{}^{\mu\mu_3\cdots\mu_s}\mathcal{F}^{\nu}{}_{\nu\mu_3\cdots\mu_s}). \tag{4.25}$$

To remove unphysical degrees of freedom from the HS gauge fields, the gauge is fixed by choosing $\psi_{\mu_1\cdots\mu_s}$ to be transverse and traceless. The equation of motion in (4.24) will reduce to the Fierz-Pauli equations

$$\begin{aligned} \partial^\alpha\partial_\alpha\psi_{\mu_1\cdots\mu_s} &= 0 \\ \partial^\mu\psi_{\mu\mu_2\cdots\mu_s} &= 0 \\ \psi^{\mu}{}_{\mu\mu_3\cdots\mu_s} &= 0. \end{aligned} \tag{4.26}$$

To completely fix the gauge, there are also gauge parameters $\epsilon_{\mu_1\cdots\mu_{s-1}}$ that need to be gauge fixed. They satisfy similar Fierz-Pauli equations

$$
\begin{aligned}
\partial_\alpha \partial^\alpha \epsilon_{\mu_1 \cdots \mu_{s-1}} &= 0 \\
\partial^\mu \epsilon_{\mu \mu_2 \cdots \mu_{s-1}} &= 0 \\
\epsilon^\mu{}_{\mu \mu_3 \cdots \mu_{s-1}} &= 0.
\end{aligned}
\tag{4.27}
$$

These are constraints on the Fronsdal field in flat spacetime. Next we will look at the Fronsdal equation in curved space.

4.2.1 HS gauge fields in AdS

To generalize the Fronsdal equation from flat to a curved space it is not straight forward. Simply replacing spacetime derivatives with covariant derivatives only works for minimally coupled (to gravity) HS fields. For a general curved background this procedure doesn't work since the covariant derivatives do not commute and the Fronsdal equations are not gauge invariant. A maximally symmetric background [38] with the Riemann tensor

$$
R_{\mu\nu\rho\sigma} = -\frac{1}{l_{AdS}^2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}),
\tag{4.28}
$$

solves this problem. The gauge invariant Fronsdal equation of motion in AdS is

$$
\begin{aligned}
&\nabla^2 \psi_{\mu_1 \cdots \mu_s} - s \nabla_{(\mu_1} \nabla^\mu \psi_{\mu_2 \cdots \mu_s)\mu} + \frac{s(s-1)}{2} \nabla_{(\mu_1} \nabla_{\mu_2} \psi_{\mu_3 \cdots \mu_s)\mu}{}^\mu \\
&- \frac{1}{l_{AdS}^2} \left(((s-2)(s+d-3) - s) \psi_{\mu_1 \cdots \mu_s} + \frac{s(s-1)}{4} g_{(\mu_1 \mu_2} \psi_{\mu_3 \cdots \mu_s)\mu} \right) = 0,
\end{aligned}
\tag{4.29}
$$

where the second term in the second line is the compensating term needed to make the equation of motion gauge invariant. The equation above is invariant under the gauge transformation

$$
\delta \psi_{\mu_1 \cdots \mu_s} = \nabla_{(\mu_1} \epsilon_{\mu_2 \cdots \mu_s)}.
\tag{4.30}
$$

Applying the gauge fixing by requiring the field $\psi_{\mu_1 \cdots \mu_s}$ to be transverse and traceless, the equation of motion then reduces to

$$\begin{aligned}
\left(\nabla^2 - \frac{(s-2)(s+d-3)-s}{l_{AdS}^2}\right)\psi_{\mu_1\cdots\mu_s} &= 0 \\
\nabla^\mu \psi_{\mu\mu_2\cdots\mu_s} &= 0 \\
\psi^\mu_{\mu\mu_3\cdots\mu_s} &= 0,
\end{aligned} \tag{4.31}$$

which are the Fierz-Pauli equations on a curved space. To completely fix the gauge, the gauge parameter $\epsilon_{\mu_1\cdots\mu_{s-1}}$ should satisfy the equations

$$\begin{aligned}
\left(\nabla^2 - \frac{(s-1)(s+d-3)}{l_{AdS}^2}\right)\epsilon_{\mu_1\cdots\mu_{s-1}} &= 0 \\
\nabla^\mu \epsilon_{\mu\mu_2\cdots\mu_{s-1}} &= 0 \\
\epsilon^\mu{}_{\mu\mu_3\cdots\mu_{s-1}} &= 0.
\end{aligned} \tag{4.32}$$

Fronsdal equations are linear and are based on a metric-like formulation. Next we will discuss HS gauge fields in the frame-like formulation. This is relevant for introducing interactions, as shown by Vasilliev.

4.2.2 HS Gauge Fields in Frame-Like Formulation

Vasiliev's non-linear HS theory is based on the frame-like formulation and generalizes the vielbein approach to gravity using the differential forms language. In gravity we can introduce respectively the vielbein and spin connection

$$e^a_\mu \qquad w^{ab}_\mu. \tag{4.33}$$

The vielbein is related to the metric by

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu. \tag{4.34}$$

Here the spin connection w^{ab}_μ correspond to a gauge field with the local Lorentz rotations acting as gauge symmetries. The vielbein e^a_μ and spin connection w^{ab}_μ can be combined as components of the one-forms,

$$e^a = e^a_\mu dx^\mu \qquad w^{ab} = w^{ab}_\mu dx^\mu. \tag{4.35}$$

These forms obey the Cartan structure equations

$$\begin{aligned} de^a + w^a_b \wedge e^b &= T^a \\ dw^{ab} + w^a_c \wedge w^{cd} &= R^{ab} \qquad\qquad R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \tag{4.36}$$

where T^a is the torsion two-form and R^{ab} is the Riemann tensor two-form. When $T^a = 0$ we have

$$R_{\mu\nu\rho\sigma} = R^{ab}_{\mu\nu} e_{\rho a} e_{\sigma b}. \tag{4.37}$$

Since w^{ab} is a gauge field of local Lorentz transformations and ϵ^a transforms as a vector under Lorentz transformations, the Lie algebra is of the form

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{bc} M_{ad} - \eta_{bd} M_{ca} - \eta_{ac} M_{bd} + \eta_{ad} M_{cb}) \\ [M_{ab}, P_c] &= i(\eta_{bc} P_a - \eta_{ac} P_b) \\ [P_a, P_b] &= \frac{i}{l^2_{AdS}} M_{ab}, \end{aligned} \tag{4.38}$$

where P_a corresponds to the generators of local translations. This is the AdS algebra $SO(d,2)$. We can build a one form W as follows

$$W = -i(e^a P_a + \frac{1}{2} w^{ab} M_{ab}), \tag{4.39}$$

which is intepreted as the gauge field of the Lie algebra in (4.38). The curvature of W is

$$\begin{aligned} dW + W \wedge W =& -i(de^a + w^a_b \wedge e^b) P_a + \frac{1}{2}(dw^{ab} + w^a_c \wedge w^{cb} + \frac{1}{l^2_{AdS}} e^a \wedge e^b) M_{ab}) \\ =& -i(T^a P_a + \frac{1}{2}(R^{ab} + \frac{1}{l^2_{AdS}} e^a \wedge e^b) M_{ab}). \end{aligned} \tag{4.40}$$

For a flat connection on W the above equation becomes

$$dW + W \wedge W = 0. \tag{4.41}$$

The solution to this equation is

$$T^a = 0 \quad \text{and} \quad R^{ab} = -\frac{1}{l_{AdS}^2} e^a \wedge e^b \Rightarrow R_{\mu\nu\rho\sigma} = -\frac{1}{l_{AdS}^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{4.42}$$

This shows that the flat connection is related to a maximally symmetric background. The gauge transformation for W is

$$\delta W = d\epsilon + [W, \epsilon] \tag{4.43}$$
$$\epsilon = -i(\epsilon^a P_a + \frac{1}{2}\epsilon^{ab} M_{ab}),$$

where ϵ^a and ϵ^{ab} are the gauge parameters for local translations and local Lorentz transformations respectively. The generators P_a, M_{ab} are combined into $\frac{d(d+1)}{2}$ generators T_{AB}, where $A, B = 0, \cdots, d$ and $W = -iw^{AB}T_{AB}$ [39]. Generalizing to higher spins, the vielbein is given by the one-form

$$e^{a_1, \cdots, a_{s-1}} = e_\mu^{a_1 \cdots a_{s-1}} dx^\mu, \tag{4.44}$$

where $e^{a_1 \cdots a_{s-1}}$ is totally symmetric and traceless in the indices $a_1 \cdots a_{s-1}$,

$$\eta_{ab} e^{aba_3 \cdots a_{s-1}} = 0. \tag{4.45}$$

We will use the Young tableau notation $[n_1, n_2, n_3, \cdots, n_k]$ where n_i is the number of boxes in the i^{th} row. The tensor $e^{a_1 \cdots a_{s-1}}$ is in a reducible $SO(d)$ representation and it decomposes as follows

$$[1, 0, 0, \cdots] \otimes [s-1, 0, \cdots] = [s, 0, \cdots] + [s-2, 0, \cdots] + [s-1, 1, 0, \cdots] \tag{4.46}$$

where the first two representation on the RHS represent a symmetric and double traceless HS field, which is the Fronsdal field ($\psi_{\mu_1 \cdots \mu_s}$). The third representation on the RHS is a hook Young diagram representation which corresponds to gauge redundancies, so the corresponding gauge field has to be of the form $w_\mu^{a_1 \cdots a_{s-1}, b}$. This is the general spin connection in HS field theory. Unlike the $s = 2$ case, solving the analog of (4.41) in HS theory does not give a unique spin connection $w_\mu^{a_1\cdots a_{s-1},b}$. Rather a tower of HS spin connections are required to fix this gauge redundancy. Therefore HS fields in the framelike description have the following vielbein and spin connections [39],

$$\begin{aligned} & e_\mu^{a_1\cdots a_s} \\ & w_\mu^{a_1\cdots a_s,b_1\cdots b_t} \qquad t=1,2,\cdots,s-1. \end{aligned} \tag{4.47}$$

The HS analog of (4.41), which will allow the HS spin connection to be solved in terms of an HS vielbein is given by the equations

$$\begin{aligned} R^{a_1\cdots a_{s-1}} =& 0 \\ R^{a_1\cdots a_{s-1},b} =& 0 \\ & \vdots \\ R^{a_1,\cdots a_{s-1},b_1\cdots b_{s-1}} =& e_{(0)_c}\wedge e_{(0)_d} C^{a_1\cdots a_{s-1}c,b_1\cdots b_{s-1}d} \end{aligned} \tag{4.48}$$

where $R^{a_1\cdots a_{s-1},b_1\cdots b_{s-1}}$ are curvature two-forms that generalize (4.36) and $C^{a_1\cdots a_{s-1}c,b_1\cdots b_{s-1}d}$ is the HS generalized Weyl tensor. It is built out of s derivatives of the Fronsdal field, and $e^a_{(0)}$ is the tree level vielbein field from the fluctuation $e^a_\mu = e^a_{(0)\mu} + \hat{e}^a_\mu$. Just as in the $s=2$ case, $e^{a_1\cdots a_{s-1}}$ and $w^{a_1\cdots a_{s-1},b_1\cdots b_t}$ can be combined into a gauge field $w^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}$ where $A,B=0,\cdots,d$. This gauge field is now in the representation $[s-1,s-1,0,\cdots]$ of the AdS algebra. Each gauge field can be associated to a generator $T_{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}$ in the same representation. These HS fields can be combined into one field

$$W = -i\sum_s w^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} T_{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}. \tag{4.49}$$

The generators $T_{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}$ are the same generators from the CFT HS algebra constructed in terms of Killing vectors. These generators on the AdS side form the gauge algebra. To linearize (4.49), expand around the AdS background to linear order in the fluctuations $w=w_0+w_1$, to obtain

$$W = -iw_0^{AB}T_{AB} - i\sum_s w_1^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} T_{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}. \tag{4.50}$$

The linearized gauge transformation becomes

$$\delta w_1 = d\epsilon + [w_0, \epsilon] \tag{4.51}$$

where $w_0 = w_0^{AB} T_{AB}$ and $\epsilon = \sum_s \epsilon^{A_1 \cdots A_{s-1}, B_1 \cdots B_{s-1}} T_{A_1 \cdots A_{s-1}, B_1 \cdots B_{s-1}}$. The linearized curvature in (4.48) becomes

$$dW + W \wedge W = dw_1 + [w_0, w_1]. \tag{4.52}$$

The Vasiliev HS gauge theory is non-linear. Its non-linear gauge symmetry is given by

$$\delta W = d\epsilon + [W, \epsilon]. \tag{4.53}$$

This is the extent to which we will discuss the HS gauge theory. If we are ever to reproduce the full non-linear structure of this theory from CFT, it is clear that we need a good understanding of the free CFT. This is a major motivation for this PhD.

We have seen that the free scalar field CFT has a tower of higher spins primary operators packaged in a Gegenbauer polynomial. These higher spin primary operators are constructed using bilinears (2 copies of) of the scalar fields. This does not exhaust the spectrum of primaries of the CFT. Indeed, as we will see in the next chapter, even the spectrum of a single free scalar field theory is much richer than this.

Chapter 5

Primary Fields in Free Scalar Conformal Field Theory in 4-dimensions

This long chapter will contain the work of the papers that were published in [19][40]. This work contains novel results for the free conformal field theory of a scalar field in 4 dimensions (CFT4). Using representation theory a general generating function for the number of primary operators constructed from n copies of the free scalar field is derived. The generating function yields the correct counting for the primary fields. The counting is then specialised to counting primaries which obey extremality conditions defined in terms of the dimensions and left or right spins (i.e. in terms of relations between the charges under the Cartan subgroup of $SO(4,2)$). The construction of primary fields for scalar field theory is mapped to a problem of determining multi-variable polynomials subject to a system of algebraic and differential constraints. For the extremal primaries, we give a construction in terms of holomorphic polynomial functions on permutation orbifolds, which are shown to be Calabi-Yau spaces.

5.1 Introduction

In [41] we showed that free scalar four dimensional conformal field theory can be formulated as an infinite dimensional associative algebra, admitting a decomposition into linear representations of SO(4,2), and equipped with a bilinear product satisfying a non-degeneracy condition. This algebraic structure gives a formulation of the CFT4 as a two dimensional topological field theory (TFT2) with SO(4,2) invariance, where crossing symmetry is expressed as associativity of the algebra. TFT2 structure had previously been identified as a unifying structure in the study of combinatorics and correlators in BPS sectors of $N = 4$ SYM, quiver gauge theories, matrix models, tensor models, and in Feynman graph combinatorics [42, 43, 44, 45]. The theme of TFT2 as a powerful unifying structure for QFT combinatorics was also developed in [41] in the context of counting primary fields. In this chapter we return to a systematic study of primaries in free field theories in four dimensions. We consider scalar, vector and matrix models. Another motivation for the detailed construction of primary fields in four dimensional scalar QFT is that free field calculations have been found to be useful in calculating the anomalous dimensions of operators at the Wilson-Fischer fixed point in the epsilon expansion [18, 46, 47, 48, 49].

We start by developing some explicit formulae for the counting of primary fields, using characters of representations of $so(4, 2)$. This makes extensive use of previous literature on the subject, notably [17]. This is followed by considering the problem of constructing the primary fields. A useful remark is that the algebraic problem of finding composite fields of the form

$$(\partial \cdots \partial\phi)(\partial \cdots \partial\phi) \cdots (\partial \cdots \partial\phi) \tag{5.1}$$

where there are n ϕ fields involved, can be conveniently rephrased in terms of a question about multi-variable polynomial functions of $4n$ variables : $\Psi(x_\mu)$ where μ runs over the space-time coordinates and I runs from 1 to n. This relies on a function space realisation of the conformal algebra. We explain how this function space realisation arises naturally in radial quantization. The question of constructing primaries, when phrased in terms of the functions $\Psi(x_\mu)$ can viewed as a many-body quantum mechanics problem, where F is a many-body wavefunction of n particles moving on $\mathbb{R}^4$. These many-body wavefunctions have to obey three simple conditions :

• They have to obey Laplace's equation in each of the variables x^I_μ for $I = 1 \cdots n$.

• They have to be invariant under the simultaneous translation $x^I_\mu \to x^I_\mu + a_\mu$, for $\mu = 1 \ldots 4$.

• They have to be invariant under permutations $x^I_\mu \to x^{\sigma(I)}_\mu$ for any permutation $\sigma \in S_n$.

An infinite class of solutions of the Laplacian condition are obtained by choosing a complex structure to identify $\mathbb{R}^4 = \mathbb{C}^2$ so that $x_\mu \to (z, w, \bar{z}, \bar{w})$ and considering holomorphic functions of z, w. These primaries correspond to holomorphic polynomial functions on

$$(\mathbb{C}^2)^n/(\mathbb{C}^2 \times S_n) \tag{5.2}$$

which can also be written as

$$(\mathbb{C}^n/\mathbb{C} \times \mathbb{C}^n/\mathbb{C})/S_n. \tag{5.3}$$

The modding out by $\mathbb{C}^2$ is the condition of invariance under the shift while the S_n invariance comes from the permutation symmetry. A special class of these primary fields correspond to functions of z only i.e functions on

$$(\mathbb{C}^n)/(\mathbb{C} \times S_n). \tag{5.4}$$

These primaries were constructed in [50] using an oscillator realization of the conformal algebra, which is close to the differential realization used here. An extensive study of the representation of $so(4,2)$ on function spaces with emphasis on relations to quarternions, is developed in [51].

The association of primaries to functions on the orbifold has several interesting consequences. Since the holomorphic polynomial functions form a ring, and a class of primaries are in $1-1$ correspondence with these functions, we are finding a ring structure on this subspace of primary operators. This ring structure is different from the algebra structure related to the operator product expansion. The interplay between this product and the OPE would be an interesting subject for future study. The Hilbert series of the polynomial ring (5.2) has a very interesting *palindromy property* which we prove. The proof relies on an interesting algebraic structure based on symmetric groups in the problem. For fixed number of primaries n, this is

$$\bigoplus_{k,l=0}^{\infty} \mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \otimes \mathbb{C}(S_l) \tag{5.5}$$

where $\mathbb{C}(S_n)$ is the group algebra of the symmetric group S_n. As recently discussed in the context of Hilbert series for moduli spaces of supersymmetric vacua of gauge theories [52, 53], the palindromy property of Hilbert series is indicative that the ring being enumerated is Calabi-Yau. The precise mathematical statement is due to Stanley [54]. We show that the orbifold (5.2) indeed admits a unique non-singular nowhere-vanishing top-dimensional holomorphic form, which is inherited from the covering space.

Our work involves an interesting interplay between representations of $so(4,2)$ and representations of symmetric groups. Let V_+ be the lowest weight representation corresponding to local operators built from derivatives acting on the field ϕ. The construction of primaries built from derivatives acting on n copies of ϕ, amounts to finding explicit formulae for the lowest weight states of irreducible representations in the symmetrized tensor product $Sym^n(V_+)$. If we consider the primaries which arise at dimension $n+k$, of the class associated to the geometry (5.4) this can be mapped to a problem about multiplicities of $S_n \times S_k$ irreps in $V_H^{\times k}$ where V_H is the $n-1$ dimensional representation of S_n. A formula for these multiplicities, derived in [50], is found to be useful in the study of the geometry of (5.2). The connection between representation theory of symmetric groups and that of non-compact groups has also been discussed in [55] in the context of higher spin theories.

We extend this approch to primary fields to the case of vector fields in four dimensions. The underlying orbifold geometry for holomorphic primaries in this case is

$$(\mathbb{C}^{2n}/\mathbb{C} \times \mathbb{C}^{2n}/\mathbb{C})/S_n[S_2]. \tag{5.6}$$

The group $S_n[S_2]$ is the group of S_n which is generated by the n pairwise permutations $(1,2),(3,4),\cdots,(2n-1,2n)$ along with the $n!$ permutations of these pairs. It is called the wreath product of S_n with S_2. We establish the palindromy property of the Hilbert series in this case. For the case of primary fields in the free theory of matrices in four dimensions, we again find the underlying orbifold geometry

$$((\mathbb{C}^2)^n \times S_n)/(\mathbb{C}^2 \times S_n) = (\mathbb{C}^n/\mathbb{C} \times \mathbb{C}^n/\mathbb{C} \times S_n)/S_n \tag{5.7}$$

with a palindromic Hilbert series.

The chapter is organised as follows. In section 2 we describe a realisation of the conformal algebra $so(4,2)$ in terms of differential operators acting on polynomial functions of spacetime coordinates x_μ in $\mathbb{R}^4$. This is related, by a duality which we explain, to the standard realization of the conformal algebra in terms of derivatives acting on a scalar field. In section 3 we obtain a number of useful general formulae for the counting of primary fields. The first step is to start from the character of the irrep V_+ of $so(4,2)$ which contains all the local operators consisting of derivatives acting on a single scalar field. This is a function of variables s, x, y which keep track of dimension, left spin and right spin i.e eigenvalues of D (the scaling operator) and J_L, J_R (the Cartan generators for the two $SU(2)$'s in $SO(4) = SU(2) \times SU(2)$). We then derive a generating function for the Cauchy identity. We describe a specialisation of these formulae relevant to what we call extremal primaries. These include the leading twist primaries studied in the context of deep inelastic scattering in QCD. Taylor expansion of the generating function leads to explicit results for $n = 3, 4$ which take the form of rational functions of s, x, y. In section 4 we describe the construction of the primary fields using the polynomials representations. A new counting formula for the extremal primaries is obtained by exploiting the permutation group algebras $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \otimes \mathbb{C}(S_l)$ in the problem of building primaries from n fields ϕ and corresponding to polynomials of degree k in one holomorphic variable and degree l in the other. This is shown to be consistent with the derivation based on the Taylor expansion method of the previous section. These primary fields form a ring and the counting is recognised as a Hilbert series, which encodes aspects of the generators and relations of the ring. This is a ring of functions of an orbifold which we identify. The counting formula based on $S_n \times S_k \times S_l$ symmetry for the extremal sector is shown to have a palindromy property indicative of a Calabi-Yau nature of the orbifold. As further support for the Calabi-Yau nature of the orbifold, we construct the explicit top-dimensional holomorphic form. In section 5 we extend the results on counting and construction of primaries, and the underlying Calabi-Yau orbifold geometries, to the case of a four dimensional vector model. In section 6 we develop the story for the case of free four dimensional 1-matrix theory.

5.2 Representation of $so(4,2)$ on multi-variable polynomials

The generators of $SO(4,2)$ form the algebra

$$\begin{aligned}
[K_\mu, P_\nu] &= 2M_{\mu\nu} - 2D\delta_{\mu\nu} \\
[D, P_\mu] &= P_\mu \\
[D, K_\mu] &= -K_\mu \\
[M_{\mu\nu}, K_\alpha] &= \delta_{\nu\alpha} K_\mu - \delta_{\mu\alpha} K_\nu \\
[M_{\mu\nu}, K_\alpha] &= \delta_{\nu\alpha} P_\mu - \delta_{\mu\alpha} P_\nu
\end{aligned} \tag{5.8}$$

The representations of this algebra play a central role when the constraints that conformal invariance places on the dynamics of a CFT are developed. To develop the representation theory, one uses the fact that there is a unique primary operator $\mathcal{O}$ for each irrep, formed by taking products of the fundamental fields of the theory and derivatives of these fields, with each other. The primary operator is distinguished because its dimension can not be lowered. Consequently, primaries are annihilated by the generator of special conformal transformations

$$[K_\mu, \mathcal{O}] = 0. \tag{5.9}$$

The complete irrep is then formed by acting on the primary $\mathcal{O}$ with traceless symmetric polynomials in the momenta P_μ. The spectrum of dimensions of the primaries and their OPE coefficients provide a list of data that completely determines the correlation functions of local operators. Clearly then, it is interesting to determine the spectrum of primary operators of a conformal field theory. Our goal is to determine this list for the free bosonic field ϕ in four dimensions. The states corresponding to ϕ and its derivatives in the operator-state correspondence consists of a lowest weight state $|v_+\rangle$

$$
\begin{aligned}
D|v_+\rangle &= |v_+\rangle \\
K_\alpha|v_+\rangle &= 0
\end{aligned} \tag{5.10}
$$

This state obeys

$$
K_\alpha(P_\mu P_\mu)|0\rangle) = 0 \tag{5.11}
$$

which means that $P_\mu P_\mu|0\rangle$ can be set to zero to give an irreducible representation. The states in this representation are of the form

$$
(S^{(l)})^{\nu_1,\mu_2,\cdots,\nu_l}_{\mu_1,\mu_2,\cdots,\mu_l} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l}|v_+\rangle \tag{5.12}
$$

where $(S^{(l)})^{\nu_1,\mu_2,\cdots,\nu_l}_{\mu_1,\mu_2,\cdots,\mu_l}$ is symmetric and traceless in both upper and lower indices.

Solving for primaries $\mathcal{O}$ is a representation theory problem of finding the decomposition of the symmetrized tensor product $\mathrm{Sym}^n(V_+)$ into irreducible representations. A particular convenient realization of V_+ is in terms of harmonic polynomials. Indeed polynomials of the form

$$
(S^{(l)})^{\nu_1,\mu_2,\cdots,\nu_l}_{\mu_1,\mu_2,\cdots,\mu_l} x_{\mu_1} \cdots x_{\mu_l} \tag{5.13}
$$

are annihilated by the Laplacian

$$
\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha} \tag{5.14}
$$

and hence are harmonic. The algebra $so(4,2)$ is realised on these polynomials as [41]

$$
\begin{aligned}
K_\mu &= \frac{\partial}{\partial x_\mu} \\
P_\mu &= (x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu) \\
D &= (x\cdot\partial + 1) \\
M_{\mu\nu} &= x_\mu\partial_\nu - x_\nu\partial_\mu
\end{aligned} \tag{5.15}
$$

Note that in radial quantization $(P_\mu)^\dagger = -K_\mu$ and $(K_\mu)^\dagger = -P_\mu$. Thinking of x_μ as the co-ordinate of a particle, this is a single particle representation. The tensor product $V_+^{\otimes n}$ can be realized on a many-particle space of functions $\Psi(x_\mu^I)$, where $1 \leq I \leq n$ labels the particle number. The generators of $so(4,2)$ now include

$$
\begin{aligned}
K_\mu &= \sum_{I=1}^{n} \frac{\partial}{\partial x_\mu^I} \\
P_\mu &= \sum_{I=1}^{n} \left(x^{I\rho} x_\rho^I \frac{\partial}{\partial x_\mu} - 2 x_\mu^I x_\rho^I \frac{\partial}{\partial x_\rho^I} - 2 x_\mu^I \right)
\end{aligned}
\tag{5.16}
$$

along with the many-particle versions of $D, M_{\mu\nu}$ of (5.15). In this polynomial representation, the state of the scalar field $\lim_{|x|\to 0} \phi(x)|0\rangle$ corresponds to the harmonic function (5.1).

This polynomial representation is naturally understood in the context of radial quantization. Towards this end, consider the mode expansion of the field

$$
\phi(x_\mu) = \sum_{l=0}^{\infty} \sum_{m \in V_l} a_{l;m}^\dagger Y_{l,m}(x) + \sum_{l=0}^{\infty} \sum_{m \in V_l} a_{l;m} |x|^{-2} Y_{l,m}(x') \tag{5.17}
$$

The sum over m is over the states of the symmetric traceless tensor irrep V_l of $SO(4)$. Acting on the vacuum, which is annihilated by the $a_{l;m}$'s, we have the usual operator-state correspondence. For example, we find

$$
\begin{aligned}
&\lim_{x\to 0} \phi(x)|0\rangle = a_{0;0}^\dagger |0\rangle \equiv |\phi\rangle \\
&\lim_{x\to 0} \partial_\mu \phi(x)|0\rangle = a_{1;\mu}^\dagger |0\rangle \equiv |\partial_\mu \phi\rangle \\
&\lim_{x\to 0} \partial_\mu \partial_\mu \phi(x)|0\rangle = 0
\end{aligned}
\tag{5.18}
$$

The last equation above is expected because the free scalar field is a representation with null states. It expresses the free equation of motion. The scalar field and all its derivatives as $x \to 0$ lead to states in an irreducible lowest weight representation V of $SO(4,2)$, consisting of a lowest weight state of dimension $\Delta = 1$ along with states with higher dimension.

Let us rewrite the positive part of the radial mode expansion

$$\phi^+(x)|0\rangle = \sum_{l=0}^{\infty} a^\dagger_{l;\mu_1,\cdots,\mu_l}(S^l)^{\mu_1,\cdots,\mu_l}_{\nu_1,\cdots,\nu_l} x^{\nu_1}\cdots x^{\nu_l}|0\rangle \tag{5.19}$$

where $S^{(l)}$ is a projector, projecting to symmetric traceless tensors. We take $a^\dagger_{l;\mu_1,\cdots\mu_l}$ to be symmetric and traceless in the μ indices. $S^{(l)}$ is symmetric and traceless in the μ as well as the ν indices. The operator state map identifies

$$\lim_{x\to 0}\partial_{\mu_1}\cdots\partial_{\mu_l}\phi(x)|0\rangle = (S^{(l)})^{\nu_1,\cdots,\nu_l}_{\mu_1,\cdots,\mu_l} a^\dagger_{l;\nu_1,\cdots,\nu_l}|0\rangle \tag{5.20}$$

Note that we have a duality

$$\begin{aligned}\left(((S^{(l)})^{\nu_1,\cdots,\nu_l}_{\mu_1,\cdots,\mu_l} a^\dagger_{l;\nu_1,\cdots\nu_l}|0\rangle)^\dagger, \phi(x)|0\rangle\right) =& \langle 0|a_{l;\nu_1,\cdots,\nu_l}(S^{(l)})^{\nu_1,\cdots,\nu_l}_{\mu_1,\cdots,\mu_l}\phi(x)|0\rangle \\ =& (S^{(l)})^{\alpha_1,\cdots,\alpha_l}_{\mu_1,\cdots,\mu_l} x_{\alpha_1}\cdots x_{\alpha_l}\end{aligned} \tag{5.21}$$

where we have used the projector property of $S^{(l)}$. Unpacking this a little, if we apply ∂_μ to the local operator, go to zero to get the corresponding state and then do the duality, we will get new polynomial as the outcome

$$\lim_{x\to 0}\partial_\mu\partial_{\mu_1}\cdots\partial_{\mu_l}\phi(x)|0\rangle = (S^{(l)})^{\nu,\nu_1,\cdots,\nu_l}_{\mu,\mu_1,\cdots,\mu_l} a^\dagger_{l+1;\nu,\nu_1\cdots\nu_l}|0\rangle \tag{5.22}$$

If we take the overlap of this with $\phi(x)|0\rangle$ then we get

$$(S^{(l)})^{\nu,\nu_1,\cdots,\nu_l}_{\mu,\mu_1,\cdots,\mu_l} x_\nu x_{\nu_1}\cdots \nu_{\nu_l} \tag{5.23}$$

This polynomial of degree one higher is related to the previous polynomial by applying $P_\mu = (x^2\partial_\mu - 2x_\mu(x\cdot\partial + 1))$. We have the following identifications between operators and states, and then states and polynomials

$$\begin{aligned}&\mathcal{O} \to |\mathcal{O}\rangle \to P_{\mathcal{O}}(x) \\ &\partial_\mu\mathcal{O} \to |\partial_\mu\mathcal{O}\rangle \to P_\mu P_{\mathcal{O}}(x)\end{aligned} \tag{5.24}$$

This provides a concrete correspondence between applying ∂_μ to local operators made from a scalar, and applying P_μ as the dual differential operator on dual polynomials.

Primaries in the free theory are given by acting with traceless symmetric polynomials in momenta on the scalar field. Tracelessness is often implemented [56, 57] by using variables $z \cdot x^I = z^\mu x^I_\mu$ with z^μ a null vector, i.e. $z^\mu z_\mu = 0$. Thanks to the fact that z^μ is null, any polynomial in $z \cdot x^I$ automatically gives a traceless symmetric polynomials in x^I_μ after the z^μs are stripped away. In what follows we will solve the algebraic primary problem, to obtain a polynomial that corresponds to the primary. To obtain the primary operator written in terms of the original scalar field, we need to translate between the polynomials and operators. For the current polynomials, the translation between polynomials and operators is

$$(z \cdot \partial)^k \leftrightarrow (-1)^k 2^k k! (z \cdot x)^k \tag{5.25}$$

The construction is convenient because of its simplicity. However, it is not completely general, since there are primary operators that are not symmetric in their indices and hence can't be represented as a polynomial in $z \cdot x$. The general discussion makes use of projectors that project from symmetric tensors to traceless symmetric tensors. It is useful to consider a concrete example. The tensors of ranks 2 and 3 are given by

$$\begin{aligned}(S^{(2)})^{\alpha\beta}_{\mu\nu} &= \delta^\alpha_\mu \delta^\beta_\nu - \frac{1}{4}\delta_{\mu\nu}\delta^{\alpha\beta} \\ (S^{(3)})^{\alpha\beta\gamma}_{\mu\nu\rho} &= \delta^\alpha_\mu \delta^\beta_\nu \delta^\gamma_\rho - \frac{1}{6}(\delta_{\mu\nu}\delta^{\alpha\beta}\delta^\gamma_\rho + \delta_{\mu\rho}\delta^{\alpha\gamma}\delta^\beta_\nu + \delta^\alpha_\mu \delta^{\beta\gamma}\delta_{\nu\rho})\end{aligned} \tag{5.26}$$

These operators are projectors in the Brauer algebra of tensor operators that commute with $SO(4)$[58]

$$\begin{aligned}S^{(2)} &= 1 - \frac{C_{12}}{4} \\ S^{(3)} &= 1 - \frac{1}{6}(C_{12} + C_{13} + C_{23})\end{aligned} \tag{5.27}$$

The terms correcting the 1 above subtract off the trace of the tensors they act on. They satisfy

$$(S^{(n)})^2 P_n = S^{(n)} P_n \tag{5.28}$$

where P_n projects onto the totally symmetric polynomials of degree n

$$P_n = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \tag{5.29}$$

The multiplication (5.28) is in the Brauer algebra, where loops are assigned the value of 4. These elements of the Brauer algebra are completely determined by the projector property (5.28) and the property that they start with 1. In general

$$P_{\mu_1} \cdots P_{\mu_k} \cdot 1 = (-1)^k 2^k k! (S^{(k)})^{\nu_1 \dots \nu_k}_{\mu_1 \cdots \mu_k} x_{\nu_1} \cdots x_{\nu_k} \tag{5.30}$$

The above factor is easily obtained by deriving a recursion formula. Note that the term $x^2 \partial_\mu$ does not raise the rank of the tensor. The other two terms both raise the rank by one, which then leads to the recursion relation. In the many-particle realization such a traceless polynomial made of the I'th coordinates corresponds to derivatives acting on the I'th copy of ϕ in a sequence of n of these.

To construct primaries using n scalar fields we consider a multi-particle system with x^I_μ the coordinates of the n particles. Primaries at dimension $n+k$ are obtained by allowing k derivatives to act on the n fields. In the dual polynomial language, states at dimension $n+k$ in $V^{\otimes n}$ correspond to polynomials in x^I_μ of degree k. Primaries at dimension $n+k$ correspond to degree k polynomials $\Psi(x^I_\mu)$ that obey the conditions

$$\begin{aligned} K_\mu \Psi &= \sum_I \frac{\partial}{\partial x^I_\mu} \Psi = 0 \\ \mathcal{L}_I \Psi &= \sum_\mu \frac{\partial}{\partial x^I_\mu} \frac{\partial}{\partial x^I_\mu} \Psi = 0 \\ \Psi(x^I_\mu) &= \Psi(x^{\sigma(I)}_\mu) \end{aligned} \tag{5.31}$$

The first condition above is the familiar condition that the special conformal generator annihilates primary operators. The second condition implements the free scalar equation of motion which implies that the image of states like $P_\mu P_\mu$, with only μ summed, in the Fock space, is zero. This null state appears because the dimension of free scalar field saturates a unitarity bound. To see that the second constraint is indeed implementing the equation of motion, note that with the second of (5.15) we can calculate

$$P_\mu P_\mu = x^4 \partial_\mu \partial_\mu \tag{5.32}$$

Simplifying the product of differential operators, it is simple to verify that terms like x^2, $x^2 x \cdot \partial$ and $x^2 x_\mu x_\nu \partial_\mu \partial_\nu$ cancel out. The final condition in (5.31) above ensures that our polynomials are S_n invariant. By constructing S_n invariant polynomials, we are implementing the bosonic statistics of the scalar field.

In what follows we will focus on primaries (and hence polynomials) that transform in a definite representation of the $SO(4) = SU(2) \times SU(2)$ subgroup of $SO(4,2)$. To make the $SO(4)$ transformation properties of the polynomials more transparent, our construction makes use of the complex coordinates

$$\begin{aligned} z &= x_1 + i x_2 \qquad & w &= x_3 + i x_4 \\ \bar{z} &= x_1 - i x_2 \qquad & \bar{w} &= x_3 - i x_4 \end{aligned} \tag{5.33}$$

This amounts to choosing an isomorphism between $\mathbb{R}^4$ and $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. In our conventions, these coordinates have the following (j_L^3, j_R^3) charge assignments

$$\begin{aligned} z &\leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \qquad & \bar{z} &\leftrightarrow \left(-\frac{1}{2}, \frac{1}{2}\right) \\ w &\leftrightarrow \left(\frac{1}{2}, -\frac{1}{2}\right) \qquad & \bar{w} &\leftrightarrow \left(\frac{1}{2}, -\frac{1}{2}\right) \end{aligned} \tag{5.34}$$

We will construct a class of primaries corresponding to holomorphic polynomial functions on the orbifold

$$(\mathbb{C}^2)^n / (\mathbb{C}^2 \times S_n) \tag{5.35}$$

The division by $\mathbb{C}^2$ is a consequence of the first of (5.31). These will not form the complete set of primaries but a well-defined subspace of primaries, which we will call *extremal*. Before explaining this construction in more detail we show, in the next section, how characters of $so(4,2)$ representations can be used to get a complete counting of general primaries built from n fields. We will then specialize to the extremal primaries.

5.3 Counting with so(4,2) characters

In this section our goal is to enumerate the $SO(4,2)$ irreducible representations appearing among the composite fields made out of $n = 2, 3, \cdots$ fundamental fields. These multiplicities will, for example compute the spectrum of primary operators in the free CFT_4. This enumeration entails decomposing, into irreducible representations, the symmetrized tensor product $Sym^n(V_+)$, where $V_+ = D_{[1,0,0]}$ in the notion of [17]. The three integer labels in $D_{[\Delta,j_L,j_R]}$ are the dimension and two $SO(4)$ spins. After obtaining a general formula in terms of an infinite product, we specialize to primaries that obey extremality conditions, that relate their dimensions to their spin. For these primaries using results from [59], we find simple explicit formulas for the counting.

5.3.1 General Counting Formula

Consider a matrix M belonging to any matrix representation R of $SO(4,2)$. A key result for the analysis of this section is

$$\frac{1}{\det(1-tM)} = \sum_{n=0}^{\infty} t^n \chi_{Sym^n(R)}(M) \tag{5.36}$$

This is a special case of the Cauchy identity which states that

$$\prod_{i=1}^{N}\prod_{j=1}^{M} \frac{1}{(1-tx_iy_j)} = \sum_{n=0}^{\infty}\sum_{R\vdash n} t^n \chi_R(x)\chi_R(y) \tag{5.37}$$

where χ_R is a Schur polynomial in the N variables x_i and the M variables y_i, labelled by a Young diagram R with n boxes and height no larger than the minimum of M, N. When one of these variables is 1, then we sum over single-row Young diagrams. This formula (5.36) is easily proved by using the identity (this is just Wick's theorem)

$$
\begin{aligned}
(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n} =& \frac{1}{\pi^N}\int\prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_k\bar{z}^k}\frac{1}{n!}z_{i_1}\cdots z_{i_n}\bar{z}^{j_1}\cdots\bar{z}^{j_n} \\
=& \frac{1}{n!}\sum_{\sigma\in S_n}\delta^{j_1}_{i\sigma(1)}\delta^{j_2}_{i\sigma(2)}\cdots\delta^{j_n}_{i\sigma(n)}
\end{aligned}
\tag{5.38}
$$

to evaluate

$$
\frac{1}{\pi^N}\int\prod_{i=1}^{N} dz_k d\bar{z}_k e^{-\sum_{i,j} z_i(\delta^j_i - tM)\bar{z}^j} = \frac{1}{\det(1-tM)} \tag{5.39}
$$

Now, apply (5.36) to the case that

$$
M = s^D x^{J_{3,L}} y^{J_{3,R}} \tag{5.40}
$$

and specialize to the representation V_+ spanned by the free scalar and all the derivatives acting on it. Here we have chosen $D, J_{3,L}, J_{3,R}$ to span the Cartan subalgebra of $SO(4,2)$. It is straight forward to see that

$$
\frac{1}{\det(1-tM)} = \prod_{q=0}^{\infty}\prod_{a=-\frac{q}{2}}^{\frac{q}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}\frac{1}{1-ts^{q+1}x^a y^b} \tag{5.41}
$$

This generating function of the characters of the symmetrized tensor products of the free scalar representation will be denoted by $\mathbb{Z}(t,s,x.y)$. So we have

$$
\mathbb{Z}(t,s,x,y) = \prod_{q=0}^{\infty}\prod_{a=-\frac{q}{2}}^{\frac{q}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}\frac{1}{1-ts^{q+1}x^a y^b} = \sum_{n=0}^{\infty} t^n \chi_{Sym^n(V_+)}(s,x,y) \tag{5.42}
$$

where we have denoted $\chi_{Sym^n(V_+)}(M)$ by $\chi_{Sym^n(V_+)}(s,x,y)$. The characters for $Sym^n(V_+)$ follow by developing the infinite product above in a Taylor series in t. The decomposition of $Sym^n(V_+)$ into irreps is now achieved by writing $\chi_{Sym^n(V_+)}(s,x,y)$ as a sum of characters $\chi_{[\Delta,j_1,j_2]}(s,x,y)$ of M, in the irrep of dimension Δ and spins j_1, j_2

$$
\chi_{Sym^n(V_+)}(s,x,y) = \sum_{[\Delta,j_1,j_2]} N_{[\Delta,j_1,j_2]}\chi_{[\Delta,j_1,j_2]}(s,x,y) \tag{5.43}
$$

The coefficients $N_{[\Delta,j_1,j_2]}$ are non-negative integers, counting the number of times irrep $\mathcal{A}_{[\Delta,j_1,j_2]}$ (in the notation of [17]) appears in $Sym^n(V_+)$. If we restrict to the case that $n \geq 3$, the only characters $\chi_{[\Delta,j_1,j_2]}(s,x,y)$ which contribute are labeled by dimension Δ that do not saturate the unitarity bound and hence do not have any null states. In this case we have [17]

$$\chi_{[\Delta,j_1,j_2]}(s,x,y) = \frac{s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})} \tag{5.44}$$

It is useful to define

$$Z_n(s,x,y) \equiv \sum_{[\Delta,j_1,j_2]} N_{[\Delta,j_1,j_2]} s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y) \tag{5.45}$$

It follows that

$$Z_n(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\chi_{Sym^n(V)}(s,x,y) \tag{5.46}$$

The right hand side of this last equation is precisely a sum of (products of) $SU(2)$ characters, so we can treat this, following [60], using the orthogonality of $SU(2)$ characters. The result is mostly easily stated in terms of the generating function

$$G_n(s,x,y) = \sum_{d=0}^{\infty}\sum_{j_1,j_2} N_{[n+d,j_1,j_2]} s^{n+d} x^{j_1} y^{j_2} \tag{5.47}$$

which is given by

$$G_n(s,x,y) = \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)Z_n(s,x,y)\right]_{\geq} \tag{5.48}$$

where the subscript $\geq$ is a notation to indicate that the above function should first be expanded as a Laurent series in both x and y, and then negative powers of x and y should be discarded. The infinite product in the above formula makes it difficult to evaluate $G_n(s,x,y)$ in closed form. For that reason, in the next section, we focus on specific classes of primaries for which $G_n(s,x,y)$ can be evaluated.

To end this section let us explain how the above derivation is generalised when irreps that include null states appear in the tensor product $Sym^n(V_+)$. This is the case when $n=2$. Naively computing $G_2(s,x,y)$ using (5.48), we obtain the following terms

$$G_2(s,x,y) = s^2 + s^4 xy - s^5\sqrt{x}\sqrt{y} + s^6x^2y^2 - s^7x^{3/2}y^{3/2} + \cdots \tag{5.49}$$

The negative coefficients in the above expansion show this answer is manifestly wrong. The problem is that we have some null states that have not been removed correctly. There are two types of primaries that appear in the above sum. We have a primary with $\Delta = 2$ and $j_1 = j_2 = 0$ and primaries with $\Delta = 2+2j$ and $j_1 = i_2 = j$ for $j = 1,2,3\cdots$. The condition for a short multiplet[50] is that $\Delta = f(j_1) + f(j_2)$ with $f(j) = 0$ if $j = 0$ or $f(j) = j+1$ if $j > 0$. The primary with $\Delta = 2$ and $j_1 = j_2 = 0$ is not short and nothing needs to be subtracted. The primaries with $\Delta = 2+2j$ and $j_1 = j_2 = j$ are short irreps and hence have null states. These null states (and their descendants) must be removed. To understand how this is done, note that the primary with $\Delta = 2+2j$ and $j_1 = j_2 = j$ is a conserved higher spin current $J^{\mu_1\mu_2\cdots\mu_j}$ and the null state is nothing but the conservation law

$$\partial_\mu J^{\mu\mu_2\cdots\mu_j} = 0 \tag{5.50}$$

The null state thus has $\Delta = 3+2j$ and $j_1 = j - \frac{1}{2}$ and so the subraction of null states is achieved by removing the primary that does not need to be subtracted, dividing by $1 - s/\sqrt{xy}$ and then putting the original primary back in. In the end we have

$$\begin{aligned} G_2(s,x,y) =& \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)(Z_2(s,x,y)-s^2)\frac{1}{1-\frac{s}{\sqrt{xy}}}\right] + s^2 \\ =& \sum_{j=0}^{\infty} s^{2+2j}x^jy^j \end{aligned} \tag{5.51}$$

This is indeed the correct result.

5.3.2 Counting the Leading Twist Primaries

Consider the leading twist primaries, which have quantum numbers $[\Delta, j_1, j_2] = [n+q, \frac{q}{2}, \frac{q}{2}]$. Each such primary operator comes in a complete spin multiplet of $(q+1)^2$ operators. Choosing the operator with highest spin corresponds to studying polynomials constructed using only the single complex variable z, as we can see from (5.34). This corresponds to the fact that all primaries are constructed using a single component P_z of the momentum four vector operator. We will now count the leading twist primaries by counting this highest spin operator in each multiplet. Denote the corresponding generating function by $G_n^{\max}(s,x,y)$. To determine this generating function we will modify the above results in three ways:

1. We modify the formula (5.42) by replacing $\chi_{Sym^n(V)}(s,x,y)$ with a new function $\chi_n^{\max}(s,x,y)$ and we keep only the highest spin state in the product

$$\prod_{q=0}^{\infty} \frac{1}{1 - ts^{q+1}x^{\frac{q}{2}}y^{\frac{q}{2}}} = \sum_{n=0}^{\infty} t^n \chi_n^{\max}(s,x,y) \tag{5.52}$$

2. The leading twist primaries are all constructed using a single component of the momentum, that raises both the left and right spin maximally. Consequently in (5.48) we keep only the factor that corresponds to this component of the momentum, which amounts to replacing

$$\left(1 - s\sqrt{xy}\right)\left(1 - s\sqrt{\frac{x}{y}}\right)\left(1 - s\sqrt{\frac{y}{x}}\right)\left(1 - \frac{s}{\sqrt{xy}}\right) \to (1 - s\sqrt{xy}) \tag{5.53}$$

3. For each spin multiplet we keep only 1 state so there is no longer any need to replace the multiplet of spin states by a single state when we count. Thus in (5.48) we replace

$$\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) \to 1 \tag{5.54}$$

The final result is

$$G_n^{\max}(s,x,y) = \chi_n^{\max}(s,x,y)(1 - s\sqrt{xy}) \tag{5.55}$$

In this formula we don't need to track the dependence on x and y since for this class of primaries, once n and the dimension of the operator is specified, the spins are determined. For simplicity then, we will study

$$\sum_{n=0}^{\infty} t^n G_n^{\max}(s) = \sum_{n=0}^{\infty} t^n (1-s)\chi_n^{\max}(s) \tag{5.56}$$
$$= (1-s)\prod_{q=0}^{\infty} \frac{1}{1-ts^{q+1}}.$$

To extract $G_n^{\max}(s)$ we need to develop the infinite product above in a Taylor series in t. To do this we introduce the functions

$$F(t,s) = \prod_{q=0}^{\infty} \frac{1}{1-ts^{q+1}} \qquad \frac{\partial F}{\partial t} = f_1 F \qquad f_k = \frac{\partial^{k-1} f_1}{\partial t^{k-1}} \tag{5.57}$$

It is straightforward to find $F(0,s)=1$ and

$$f_k(t,s) = (k-1)!\sum_{a=0}^{\infty} \frac{s^{ka+k}}{(1-ts^{a+1})^k} \qquad f_k(0,s) = (k-1)!\frac{s^k}{1-s^k} \tag{5.58}$$

Using these quantities, we have

$$\frac{\partial^n F}{\partial t^n} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q} \frac{(n_1k_1+\cdots+n_qk_q)!}{n_1!\cdots n_q!(k_1!)^{n_1}\cdots(k_q!)^{n_q}} f_{k_1}^{n_1}\cdots f_{k_q}^{n_q}\delta_{n,n_1k_1+\cdots+n_qk_q}F \tag{5.59}$$

Inserting the formulas for the $f_k's$ we have

$$\left.\frac{\partial^n F}{\partial t^n}\right|_{t=0} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q} \frac{(n_1k_1+\cdots+n_qk_q)!}{n_1!\cdots n_q!k_1^{n_1}\cdots k_q^{n_q}}\left(\frac{s^{k_1}}{1-s^{k_1}}\right)^{n_1}\cdots\left(\frac{s^{k_q}}{1-s^{k_q}}\right)^{n_q}\delta_{n,n_1k_1+\cdots+n_qk_q} \tag{5.60}$$
$$= \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q} \frac{n!s^n}{n_1!\cdots n_q!k_1^{n_1}\cdots k_q^{n_q}}\left(\frac{s^{k_1}}{1-s^{k_1}}\right)^{n_1}\cdots\left(\frac{s^{k_q}}{1-s^{k_q}}\right)^{n_q}\delta_{n,n_1k_1+\cdots+n_qk_q}$$

Notice that this is a sum over conjugacy classes of S_n. The conjugacy class collects permutations with n_qk_q-cycles. This interpretation follows because the coefficient

$$\frac{n!}{n_1!\cdots n_q! k_1^{n_1}\cdots k_q^{n_q}} \tag{5.61}$$

is the order of the conjugacy class. There is a factor of $(1-s^k)^{-1}$ for each k-cycle in the permutation. Here are a few motivational examples

$$\begin{aligned}
\left.\frac{\partial F}{\partial t}\right|_{t=0} =& \frac{s}{1-s} \\
\left.\frac{\partial^2 F}{\partial t^2}\right|_{t=0} =& \frac{s^2}{(1-s)^2} + \frac{s^2}{1-s^2} = \frac{2s^2}{(1-s)(1-s^2)} \\
\left.\frac{\partial^3 F}{\partial t^3}\right|_{t=0} =& \frac{s^3}{(1-s)^3} + 3\frac{s^3}{(1-s)(1-s^2)} + \frac{2s^3}{1-s^3} = \frac{6s^3}{(1-s)(1-s^2)(1-s^3)}
\end{aligned} \tag{5.62}$$

It is easy to identify the above expressions: Recall the lowest weight discrete series irrep of $SL(2)$, denoted V_1, has character

$$\chi_1(s) = \mathrm{Tr}_{V_1}(s^{L_0}) = \frac{s}{1-s} \tag{5.63}$$

It then follows that ($P_{[n]}$ projects onto the symmetric irrep i.e. a single row of n boxes)

$$\begin{aligned}
\left.\frac{\partial F}{\partial t}\right|_{t=0} =& \frac{s}{1-s} \\
=& \chi_1(s)
\end{aligned} \tag{5.64}$$

$$\begin{aligned}
\frac{1}{2!}\left.\frac{\partial^2 F}{\partial t^2}\right|_{t=0} =& \frac{s^2}{2(1-s)^2} + \frac{s^2}{2(1-s^2)} = \mathrm{Tr}(P_{[2]}s^{L_0}) \\
=& \mathrm{Tr}_{Sym(V_1^{\otimes 2})}(s^{L_0})
\end{aligned} \tag{5.65}$$

$$\begin{aligned}
\frac{1}{3!}\left.\frac{\partial^3 F}{\partial t^3}\right|_{t=0} =& \frac{s^3}{3!(1-s)^3} + \frac{3s^2}{3!(1-s)(1-s^2)} + \frac{2s^3}{(1-s^3)} = \mathrm{Tr}(P_{[3]}s^{L_0}) \\
=& \mathrm{Tr}_{Sym(V_1^{\otimes 3})}(s^{L_0})
\end{aligned} \tag{5.66}$$

This interpretation follows for general n as proved in (5.60). Thus the general formula is

$$\frac{1}{n!}\frac{\partial^n F}{\partial t^n}\bigg|_{t=0} = \mathrm{Tr}(P_{[n]} s^{L_0}) = \frac{s^n}{(1-s)(1-s^2)(1-s^3)\cdots(1-s^n)} \tag{5.67}$$

where the last equality follows from (5.46) of [61], where these $SL(2)$ sector primaries were studied in the language of oscillators. Consequently we have

$$G_n^{\max}(s) = \frac{(1-s)}{n!}\frac{\partial^n F}{\partial t^n}\bigg|_{t=0} = \frac{s^n}{(1-s^2)(1-s^3)\cdots(1-s^n)} \tag{5.68}$$

Note the close connection between counting leading twist primaries and the multiplicities of $V^{SL(2)}_{\lambda=n+k} \otimes V^{S_n}_{[n]}$, which is given by the coefficient of q^k in

$$\prod_{i=2}^{n} \frac{1}{1-q^i} \tag{5.69}$$

The result (5.68) was also recently obtained in [62].
There are three other sectors of primaries that are closely related to this one: polynomials in $\bar{z}$ corresponds to primaries of the form $[n+q, -q, -q]$, polynomials in w to primaries of the form $[n+q, q, -q]$ and polynomials in $\bar{w}$ to primaries of the form $[n+q, -q, q]$.

5.3.3 Extremal Primaries

We now come to a more general class of primaries with charges

$$\Delta = n + q \quad ; \quad J_3^L = \frac{q}{2} \tag{5.70}$$

The charge J_3^R, which is part of $SU(2)_R$, is not constrained. These primary operators belong to complete multiplets of $SU(2)_R$. They correspond to polynomials constructed using the pair of complex variables z_I, w_I. This is clear from inspection of the charges in (5.34). Translating from the polynomial representation back to the usual scalar field representation, this corresponds to the fact that all primaries are constructed using only two components of the momentum four vector operator. The two components are complex linear combinations of the (hermitian) P_μ. Arguing as we did in the previous section, we introduce a generating function $G_n^{z,w}(s,x,y)$, which is now given by

$$G_n^{z,w}(s,x,y) = \left[\left(1-\frac{1}{y}\right)Z_n^{z,w}(s,x,y)\right]_{\geq} \tag{5.71}$$

where $Z_n(s,x,y)$ is obtained from

$$\prod_{q=0}^{\infty}\prod_{m=0}^{q}\frac{1}{(1-ts^{q+1}x^{\frac{q}{2}}y^{m-\frac{q}{2}})} = \sum_{n=0}^{\infty} t^n \chi_n(s,x,y) \tag{5.72}$$

$$Z_n^{z,w}(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{x/y})\chi_n(s,x,y) \tag{5.73}$$

The two brackets multiplying $Z_n(s,x,y)$ in (5.73) is a consequence of the fact that two components of the momentum four vector are used when constructing the primaries. From (5.72) it is clear that we are selecting the state from the $J_{3,L}$ multiplet (recorded using the variable x) with the highest spin. The product over m in (5.72) indicates that all the states in the $J_{3,R}$ multiplet are counted. The factor of $(1-\frac{1}{y})$ as well as the instruction (indicate with the subscript $\geq$ in (5.71)) to keep only positive powers of y ensures that we count each $SU(2)_R$ spin multiplet once. It is clear that the expansion of (5.71) has only positive powers of x. This is a consequence of the fact that we kept only one state from each $SU(2)_L$ multiplet.

It is again possible to derive closed expressions for the generating functions $Z_n^{z,w}(s,x,y)$ and $G_n^{z,w}(s,x,y)$. Introduce the functions

$$F_2(t,s,x,y) = \prod_{q=0}^{\infty}\prod_{m=0}^{q}\frac{1}{1-ts^{q+1}x^{\frac{q}{2}}y^{m-\frac{q}{2}}} = \sum_{n=0}^{\infty} t^n \chi_n(s,x,y) \tag{5.74}$$

$$\frac{\partial}{\partial t}F_2(t,s,x,y) = \sum_{q=0}^{\infty}\sum_{m=0}^{q}\frac{s^{q+1}x^{\frac{q}{2}}y^{m-\frac{q}{2}}}{1-ts^{q+1}x^{\frac{q}{2}}y^{m-\frac{q}{2}}}F_2(t,s,x,y) \tag{5.75}$$
$$\equiv f_1(t,s,x,y)F_2(t,s,x,y)$$

$$f_k(t,s,x,y) \equiv \frac{\partial^{k-1}f_1}{\partial t^{k-1}} = (k-1)!\sum_{q=0}^{\infty}\sum_{m=0}^{q}\frac{s^{kq+k}x^{\frac{qk}{2}}y^{km-\frac{kq}{2}}}{(1-ts^{q+1}x^{\frac{q}{2}}y^{m-\frac{q}{2}})^k} \tag{5.76}$$

It is simple to establish that $F_2(0,s,x,y)=1$ and

$$f_k(0,s,x,y) = s^k(k-1)!\frac{1}{1-s^kx^{\frac{k}{2}}y^{\frac{k}{2}}}\frac{1}{1-s^kx^{\frac{k}{2}}y^{-\frac{k}{2}}} \tag{5.77}$$

Exactly as above we have

$$\frac{\partial^n F}{\partial t^n} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{(n_1k_1+\cdots+n_qk_q)!}{n_1!\cdots n_q!(k_1!)^{n_1}\cdots(k_q!)^{n_q}}f_{k_1}^{n_1}\cdots f_{k_q}^{n_q}\delta_{n,n_1k_1+\cdots+n_qk_q} \tag{5.78}$$

Inserting the formulas for the f_k's and streamlining the notation by using $a=s\sqrt{xy}$ and $b=s\sqrt{\frac{x}{y}}$, we find

$$\begin{aligned}\frac{1}{n!}\frac{\partial^n F_2}{\partial t^n}\bigg|_{t=0} &= \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{s^n}{n_1!\cdots n_q!k_1^{n_1}\cdots k_q^{n_q}}\left(\frac{1}{(1-a^{k_1})(1-b^{k_1})}\right)^{n_1} \\ &\quad\cdots\left(\frac{1}{(1-a^{k_q})(1-b^{k_q})}\right)\delta_{n,n_1k_1+\cdots+n_qk_q} \\ &=\chi_n(s,x,y)\end{aligned} \tag{5.79}$$

The expression for $Z_n(s,x,y)$ now follows from (5.73).
It is not easy to proceed for general n, but it is straight forwards to obtain explicit formulas once a specific n is chosen. For example, the final result for

$n = 3$ fields is

$$Z_3^{z,w}(z,w) = \frac{s^3\left(s^6x^3 + s^4x^2 + x^2x + 1 + s^3x^{\frac{3}{2}}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)\right)}{(1 - s^2xy)(1 - s^3(xy)^{\frac{3}{2}})(1 - s^2\frac{x}{y})(1 - s^3(\frac{x}{y})^{\frac{3}{2}})} \tag{5.80}$$

To extract spin multiplets, we need to compute

$$G_3^{z,w}(z,w) = \left[Z_3(s,x,y)\left(1 - \frac{1}{y}\right)\right]_{\geq} = \frac{1}{2\pi i}\oint_C dz \frac{(1 - \frac{1}{z^2})Z_3(s,x,z^2)}{z - \sqrt{y}} \tag{5.81}$$

The contour C must have a radius larger than $\sqrt{y}$. We assume that s, x and y are all less than one so that the expansion of $Z_3^{z,w}(s,x,y)$ converges. Thus, we can take C to be the unit cicle. The integrand has poles at $z = \pm s\sqrt{x}$, $z = \sqrt{y}$, $z = \pm\frac{1}{s\sqrt{x}}$, $z = -\frac{s\sqrt{x}}{2}(1 \pm i\sqrt{3})$ and $z = -\frac{(1 \pm i\sqrt{3})}{2s\sqrt{x}}$. To compute the integral we need to pick up the residues from poles at $z = \pm s\sqrt{x}$, $z = \sqrt{y}$, and $z = -\frac{s\sqrt{x}}{2}(1 \pm i\sqrt{3})$. We obtain

$$G_3^{z,w}(z,w) = \frac{s^3(1 - s^{10}x^5y^3)}{(1 - s^4x^2)(1 - s^3\sqrt{x^3y^3})(1 - s^2xy)(1 - s^5x^{\frac{5}{2}}y^{\frac{5}{2}})} \tag{5.82}$$

It is easy to check, using mathematica, that this expression has the correct expansion. The check tests that the expansion, as a polynomial about $s = 0$, of the above generating function matches the counting following from the expansion of the function appearing in (5.48).

Consider next the final result for $n = 4$ fields, which is

$$\begin{aligned} Z_4^{z,w}(s,x,y) =& \frac{1}{4!}\frac{\partial^4 F_2}{\partial t^4}\Big|_{t=0} \qquad (5.83)\\ =& \frac{s^4 Q(s,x,y)}{(s^2x - y)^2(1 - s^2xy)^2(s^2x + y)(-s^3x^{\frac{3}{2}} + y^{\frac{3}{2}})(1 + s^2xy)(1 - s^3x^{\frac{3}{2}}y^{\frac{3}{2}})} \end{aligned}$$

$$\begin{aligned} Q(s,x,y) =& y^{\frac{7}{2}}(y + s^2xy + s^{10}x^5y + s^{12}x^6y + s^3x^{\frac{3}{2}}y^{\frac{1}{2}}(1 + y) \qquad (5.84)\\ &+ s^5x^{\frac{5}{2}}y^{\frac{1}{2}}(1 + y) + s^7x^{\frac{7}{2}}y^{\frac{1}{2}}(1 + y) + s^9x^{\frac{9}{2}}y^{\frac{1}{2}}(1 + y) + s^4x^2(1 + y)^2\\ &+ s^6x^3(1 + y)^2 + s^8x^4(1 + y)^2) \end{aligned}$$

To extract spin multiplets, we again need to compute

$$G_4^{z,w}(s,x,y) = \left[Z_4^{z,w}(s,x,y)\left(1-\frac{1}{y}\right)\right]_{\geq} = \frac{1}{2\pi i}\oint_C dz \frac{(1-\frac{1}{z^2})Z_4^{z,w}(s,x,z^2)}{z-\sqrt{y}} \tag{5.85}$$

The contour C must again have a radius larger than $\sqrt{y}$, so we again choose the unit circle $|z|=1$. The integrand has poles at $z=\pm s\sqrt{x}$, $z=\sqrt{y}$, $z=\pm\frac{1}{s\sqrt{x}}$, $z=\pm isx$, $z=-\frac{s\sqrt{x}}{2}i(1\pm i\sqrt{3})$, $z=\pm\frac{i}{sx}$ and $z=-\frac{(1\pm i\sqrt{3})}{2s\sqrt{x}}$. The integral above recieves contributions from poles at $z=\pm s\sqrt{x}$, $z=\sqrt{y}$, $z=\pm isx$, and $z=-\frac{s\sqrt{x}}{2}i(1\pm i\sqrt{3})$. We obtain

$$G_4^{z,w}(s,x,y) = \frac{s^4 R(s,x,y)}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)(1-s^4x^2)(1-s^6x^3)(1-s^8x^4)} \tag{5.86}$$

where

$$\begin{aligned} R(s,x,y) =& 1+s^5x^{\frac{5}{2}}(\sqrt{y}+s^3x^{\frac{3}{2}}y+s^5x^{\frac{5}{2}}y+y^3-s^6x^3y^{\frac{5}{2}}-s^8x^4y^{\frac{5}{2}}-s^{16}x^8y^{\frac{7}{2}} \\ &-s^{11}x^{\frac{11}{2}}y^2(1+y)+s^7x^{\frac{7}{2}}(1-y^2)+s^4x^2y^{\frac{3}{2}}(1-y^2)+s^2x\sqrt{y}(1+y^2) \\ &-s^9x^{\frac{9}{2}}y(1+y^2)-s^{10}x^5y^{\frac{3}{2}}(1+y-y^2)-s\sqrt{x}(1-y-y^2)) \end{aligned} \tag{5.87}$$

It is again easy to check, using mathematica, that this expression does indeed have the correct expansion.

There are other sectors of primaries that are slight variations of the extremal sector studied in this section. Polynomials in $z, \bar{w}_I$ correspond to primaries with $(\Delta = n+q, J_3^R = q)$. Polynomials in $\bar{z}_I, w_I$ correspond to primaries with $(\Delta = n+q, J_3^R = -q)$. Polynomials in $\bar{z}_I, \bar{w}_I$ correspond to primaries with $(\Delta = n+q, J_3^L = -q)$.

5.4 Construction with symmetric group

In this section we would like to provide construction formulae for the extremal primaries have counted in section 3. To accomplish this the polynomial representation of $SO(4,2)$ introduced in section 2 will play a central role. These polynomials are constructed using the coordinates x^I_μ, $I = 1, \cdots, n$ which admit a natural action of S_n. Constructing primaries then amounts to constructing polynomials that are consistent with (5.31). The first of (5.31) can be satisfied by constructing $n-1$ translationaly invariant "relative coordinates" out of the x^I_μ. This construction is not unique. Following [50], a particular convenient choice makes use of the variables

$$X^{(a)}_\mu = \frac{1}{\sqrt{a(a+1)}}(x^1_\mu + \cdots + x^a_\mu - a x^{a+1}_\mu) \tag{5.88}$$

These variables are in the $[n-1,1]$ irrep of S_n. To satisfy the second of (5.31) we need to build polynomials that are harmonic. In terms of complex coordinates the Laplacian is

$$\sum_\mu \frac{\partial}{\partial x^I_\mu}\frac{\partial}{\partial x^I_\mu} = \frac{\partial}{\partial z^I}\frac{\partial}{\partial \bar{z}^I} + \frac{\partial}{\partial w^I}\frac{\partial}{\partial \bar{w}^I} \tag{5.89}$$

It is clear that we can build harmonic polynomials by considering polynomials that are functions only of the z^I, which gives the leading twist primaries, or that are functions of the z^I and w^I, which gives the leading left twist primaries. Notice that the harmonic constraint is not a first order differential constraint. By replacing this with a holomorphic constraint, which are first order equations, the resulting problem entails finding families of polynomials that obey first order equations. This imploies that the problem will now have a natural ring structure, something which will be visible in our construction. The final constraint that needs to be obeyed is that the polynomials are S_n invariants. The counting formulas we derived in the previous section will give valuable insight into how to handle this final constraint.

5.4.1 Leading Twist Primaries

Specializing to $n=3$ and employing complex variables, we have

$$Z^{(1)} = \frac{z^1 - z^2}{\sqrt{2}} \qquad Z^{(2)} = \frac{z^1 + z^2 - 2z^3}{\sqrt{6}} \tag{5.90}$$

plus the obvious formulas for $\bar{Z}^{(a)}$, $\bar{W}^{(a)}$. The nice thing about these variables is that S_n acts on these variables with Young's orthogonal representation of $[n-1,1]$, i.e. for $n=3$ we have[63],

$$\Gamma_{\yng(2,1)}((12)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Gamma((23)) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \tag{5.91}$$

The remaining elements of the group can be generated using these two. When acting on a product of variables, say $Z^{(a_1)}Z^{(a_2)}\cdots Z^{(a_k)}$ we have

$$\Gamma_k(\sigma) = \Gamma_{\yng(2,1)}(\sigma) \times \cdots \times \Gamma_{\yng(2,1)}(\sigma) \tag{5.92}$$

Where we take a tensor product (the usual Kronecker product) of k copies of the matrices of the hook irrep. Any polynomial in the hook variables automatically obeys (5.31). Thus, all that is left is to project to S_n invariants in $V_H^{\otimes k}$. We can build these by acting with the projector from the tensor product of k copies of the hook onto the trivial irrep

$$P_{\yng(3)} = \frac{1}{3!} \sum_{\sigma \in S_3} \Gamma_k(\sigma) \tag{5.93}$$

Acting on $Z^{\otimes k}$ we obtain an expression of the form $\sum_i \hat{n}_i P_i(z)$ where $\hat{n}_i$ are unit vectors inside the carrier space of $\yng(2,1)^{\otimes k}$ and $P_i(z)$ are the polynomials that can be translated into primary operators.

It is useful to consider a few examples. Acting with the projector (5.93) on the tensor product of k copies of the hook, we find

$$P_{a_1 a_2 \cdots a_k} = \sum_{\sigma \in S_3} \Gamma_k(\sigma)_{a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_k} Z^{(b_1)} Z^{(b_2)} \cdots Z^{(b_k)} \tag{5.94}$$

It is simple to implement this projector in mathematica. For $k = 1$ we find $P_{a_1} = 0$. For $k = 2$ the projector is

$$P_{a_1 a_2} = ((z_1 - z_2))^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2) \begin{bmatrix} \frac{1}{6} \\ 0 \\ 0 \\ \frac{1}{6} \end{bmatrix} \tag{5.95}$$

so the invariant polynomial is

$$P(z) = ((z_1 - z_2))^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2) \tag{5.96}$$

By inspection, this obviously obeys (5.31). For $k = 3$ the projector is

$$P_{a_1 a_2 a_3} = (z_1 + z_2 - 2z_3)(z_1 + z_3 - 2z_2)(z_2 + z_3 - 2z_1) \begin{bmatrix} 0 \\ -\frac{1}{6\sqrt{6}} \\ -\frac{1}{6\sqrt{6}} \\ 0 \\ -\frac{1}{6\sqrt{6}} \\ 0 \\ 0 \\ \frac{1}{6\sqrt{6}} \end{bmatrix} \tag{5.97}$$

so the invariant polynomial is

$$P_{a_1 a_2 a_3} = (z_1 + z_2 - 2z_3)(z_1 + z_3 - 2z_2)(z_2 + z_3 - 2z_1) \tag{5.98}$$

This polynomial again obeys (5.31). Finally, for $k = 4$ the projector is

$$P_{a_1 a_2 a_3 a_4} = ((z_1 - z_2)^4 + (z_1 - z_3)^4 + (z_2 - z_3)^4) \begin{bmatrix} \frac{1}{12} \\ 0 \\ 0 \\ \frac{1}{36} \\ 0 \\ \frac{1}{36} \\ \frac{1}{36} \\ 0 \\ 0 \\ \frac{1}{36} \\ \frac{1}{36} \\ 0 \\ 0 \\ \frac{1}{12} \end{bmatrix} \tag{5.99}$$

so the invariant polynomial is

$$P(z) = ((z_1 - z_2)^4 + (z_1 - z_3)^4 + (z_2 - z_3)^4) \tag{5.100}$$

This clearly obeys (5.31), so this is again the correct answer.

The polynomials we constructed in this way will obey the conditions spelled out in (5.31). In fact, they obey an even stronger linear condition

$$\partial_{\bar{z}^I} P(z) = 0 = \partial_{\bar{w}^I}(z) \tag{5.101}$$

which imply the Laplacian constraint. As a result, taking all possible values of k we find that the polynomials constructed exhibit a highly non-trivial structure enjoyed by the leading twist primaries: the polynomials $P_i(z)$ are a finitely generated polynomial ring. The counting formula (5.68) gives the Hilbert series for holomorphic functions on $(\mathbb{C}^n/\mathbb{C})/S_n$. The quotient by $\mathbb{C}$ sets the center of mass momentum of the many body wave function to zero as dictated by the first of (5.31). The orbifold by S_n implements the last of (5.31). The counting formula (5.68) implies that the ring has $n-1$ generators. These generators are given by constructing the $n-1$ possible independent S_n invariants out of the hook variables introduced in (5.88). For example, for $n = 2$ fields the polynomials are generated by $(z_1 - z_2)^2$. The polynomials corresponding to primaries are

$$(z_1 - z_2)^{2k} \tag{5.102}$$

Using (5.25) it is easy to see that (these vanish if s is odd)

$$\begin{aligned} O_s =& (z_1 - z_2)^s \\ \leftrightarrow& \frac{s!}{2^s} \sum_{k=0}^{s} \frac{(-1)^k}{(k!(s-k)!)^2} \partial_z^{s-k}\phi \partial_z^k \phi \end{aligned} \tag{5.103}$$

reproducing the higher spin currents, given for example in[35]. For $n = 3$ fields the ring of polynomials that correspond to primary operators is generated by

$$(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2 \tag{5.104}$$

and

$$(z_1 + z_2 - 2z_3)(z_3 + z_2 - 2z_1)(z_1 + z_3 - 2z_2) \tag{5.105}$$

In general, the generators of the ring are a product of the variables $Z^{(a)}$ introduced above, such that the product is S_n invariant. For $n = 4$ the ring is generated by $(z_1 - z_2)^2 + \cdots, (z_1 + z_2 + 2z_3)(z_3 + z_2 - 2z_1)(z_1 + z_3 - 2z_2) + \cdots$ and $(z_1 + z_2 + z_3 - 3z_4)(z_3 + z_2 + z_4 - 3z_1)(z_1 + z_3 + z_4 - 3z_2)(z_1 + z_2 + z_4 - 3z_3)$, where $\cdots$ stand for terms that must be summed to obtain an S_4 invariant. The ring structure that has appeared is rather interesting. The product on the ring is simply multiplication of polynomials. This is a natural product in the polynomial language, but is highly non-trivial in the original CFT description. A natural guess would be that this is somehow connected to the OPE of primaries, which is the natural product on the primaries of the CFT. However, this cannot be correct because the polynomial ring exists for a fixed number n. Thus, in terms of the CFT language, the ring multiplication is a product between two primaries, each of which has n fields, and the result is again a primary with n fields. The operator product of two local operators, each containing n fields, is a sum of operators containing $2n - 2k$ fields with $k = 0, 1, \cdots, n$. For odd n the product of elements of the ring gives an operator with an even number of fields. This product can therefore not even be a subalgebra of the CFT operator product algebra. This product and the associated ring structure of primary fields in free CFT4 appears to be a genuinely new structure, not previously noticed. A natural question to ask is whether or not these primary operators are orthogonal. We can translate any polynomial into an operator and then compute the two point function of the operator. The computation can also be carried out by a judicious choice of an inner product for the polynomial. For example, consider the correlator

$$\langle \partial_z^k \phi(x) \partial_{z'}^l \phi(x') \rangle = (-1)^k (k+l)! \frac{(\bar{z} - \bar{z}')^{k+l}}{(|z - z'|^2 + |w - w'|^2)^{k+l+1}} \tag{5.106}$$

Everything in the above result is determined by conformal invariance, except the overall number $= (-1)^k (k+l)!$. Recalling that z^n translates into $\frac{1}{n!}\partial_z^n$, this number can be computed if we use the following inner product for the polynomials

$$\langle z^k z'^l \rangle = (-1)^k \frac{(k+l)!}{k!l!} \tag{5.107}$$

Notice that the norm following from this inner product is not positive definite. For n fields we have polynomials in z_k for the primary at x and in z'_k for the primary at x' , with $k = 1, \cdots, n$. In this more general setting, the inner product is

$$\langle \prod_{k=1}^{n} z_k^{p_k} \prod_{l=1}^{n} z_l'^{q_l} \rangle_p = \prod_{k=1}^{n} (-1)^{p_k} \frac{(p_k + q_k)!}{p_k! q_k!} \tag{5.108}$$

In addition, due to Wick's theorem, there are a total of $n!$ Wick contractions contributing, which introduces a factor of $n!$. In the end, if polynomials P_i of degree k_i in n variables translate into primaries $\mathcal{O}_i$ constructed from n fields with dimension $n + k_i$, then we have

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(x') \rangle = \frac{c_{ij} (\bar{z} - \bar{z}')^{k_i + k_j}}{(|z - z'|^2 + |w - w'|^2)^{k_i + k_j + n}} \tag{5.109}$$

with

$$c_{ij} = n! \langle P_i(z_k) P_j(z'_k) \rangle_p \tag{5.110}$$

Using the above formulas, it is easy to check that primary operators with different dimensions are orthogonal, as they must be. Further, we also see that although our ring of primaries is a basis, the operators in the basis are not orthogonal.

5.4.2 Extremal Primaries

The above construction is easily extended to the other classes of extremal primaries we have counted. The leading left or right twist class is provided by polynomials in two holomorphic coordinates, z and w. Consider polynomials of degree k in Z and of degree l in W , with Z, W the hook variables transforming in the hook representation V_H of S_n , described by a Young diagram with row lengths $[n-1,1]$. These polynomials belong to a subspace of $V_H^{\otimes k} \otimes V_H^{\otimes l}$ of S_n . To characterize this subspace using representation theory, start with the decompositions in terms of $S_n \times S_k$ irreps

$$\begin{aligned} V_H^{\otimes k} &= \bigoplus_{\Lambda_1 \vdash n, \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)} \\ V_H^{\otimes l} &= \bigoplus_{\Lambda_3 \vdash n, \Lambda_4 \vdash l} V_{\Lambda_3}^{(S_n)} \otimes V_{\Lambda_4}^{(S_k)} \otimes V_{\Lambda_3,\Lambda_4}^{Com(S_n \times S_l)} \end{aligned} \tag{5.111}$$

$Com(S_n \times S_k)$ is the algebra of linear operators on $V_H^{\otimes k}$ which commute with $S_n \times S_k$. The tensor product $V_H^{\otimes k} \otimes V_H^{\otimes l}$ is a representation of

$$\mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_l) \tag{5.112}$$

These decompositions (5.111) have been studied in detail in [50] where they were used to construct BPS states of $\mathcal{N} = 4$ SYM. In the application we consider here, the Z and W variables are commuting which implies that they are in the trivial rep $\Lambda_2 \otimes \Lambda_4 = [k] \otimes [l]$ of $S_k \times S_l$. The multiplicity with which a given $S_n \times S_k$ irrep (Λ_1, Λ_2) appears is given by the diemension of the irrep of the commutants $Com(S_n \times S_k)$ in $V_H^{\otimes k}$. We want to project to states in $V_H^{\otimes k} \otimes V_H^{\otimes l}$ which are invariant under the diagonal $\mathbb{C}(S_n)$ in the algebra (5.112). This constrains $\Lambda_3 = \Lambda_1$. Thus we find that the number of $S_k \times S_l \times S_n$ invariants is

$$\sum_{\Lambda_1 \vdash n} \text{Mult}(\Lambda_1, [k]; S_n \times S_k)\text{Mult}(\Lambda_1, [l]; S_n \times S_l) \tag{5.113}$$

The generating functions for these multiplicities have been derived in [61]. $\text{Mult}(\Lambda_1, [k]; S_n \times S_k)$ is the coefficient of q^k in

$$\begin{aligned} Z_{SH}(q; \Lambda_1) =& (1-q) q^{\frac{\sum_i c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})} \\ =& \sum_k q^k Z_{SH}^k(\lambda_1) \end{aligned} \tag{5.114}$$

Here c_i is the length of the i'th column in Λ_1, b runs over boxes in the Young diagram Λ_1 and h_b is the hook length of the box b. Thus, for the number of primaries constructed from z_i, w_i we get

$$\sum_{\Lambda_1 \vdash n} Z^k_{SH}(\Lambda_1) Z^l_{SH}(\Lambda_1) \tag{5.115}$$

The above integer gives the number of primaries in the free scalar theory, of weight $n+k+l$, with spin $(J^L_3, J^R_3) = (\frac{k+l}{2}, \frac{k-l}{2})$. For the generating function $Z^{z,w}_n(s,x,y)$ which encodes all k, l, we have

$$Z^{z,w}_n(s,x,y) = s^n \sum_{\Lambda_1 \vdash n} Z_{SH}(s\sqrt{xy}, \Lambda_1) Z_{SH}(s\sqrt{\frac{x}{y}}, \Lambda_1) \tag{5.116}$$

where Λ_1 is a partition of n and we can use the formula (5.114).

We can in fact see that the above discussion is consistent with the Taylor expansion formula (5.79). We can recognise this formula as $\mathrm{Tr}(P_{[n]} a^{L_0} b^{L_0})$ where the trace is being taken in

$$\bigoplus_{k,l=0}^{\infty} \mathrm{Sym}^{k+l}(V_H) \tag{5.117}$$

which can be identified with a tensor product of discrete irreps of $SL(2)$, which we may denote as $V^{\otimes n}_{SL(2)} \otimes V^{\otimes n}_{SL(2)}$; one factor corresponds to the z variables and another to the w variables. $P_{[n]}$ is the projector for the symmetric irrep of S_n. Factor out the trace into the separate $SL(2)$ factors to get (see (5.73))

$$\frac{1}{n!} \frac{\partial^n F_2}{\partial t^n}\bigg|_{t=0} = \mathrm{Tr}(P_{[n]} a^{L_0} b^{L_0}) \tag{5.118}$$

$$= \sum_{\Lambda_1 \vdash n} \mathrm{Tr}(P_{\Lambda_1} a^{L_0}) \mathrm{Tr}(P_{\Lambda_1} b^{L_0}) \tag{5.119}$$

Note also that

$$\frac{1}{1-a} Z_{SH}(a, \Lambda) = \mathrm{Tr}(P_\Lambda a^{L_0}) \tag{5.120}$$

which follows by recognising that the raising operators of the $SL(2)$ representation on $z_1 \cdots z_n$ can be separated into a weight one centre of mass coordinate and the differences which span the hook representation of S_n . This demonstrates the equivalence between the Taylor expansion formula (5.79) and the $S_n \times S_k \times S_l$ formula (5.114). It is important to note that this is a non-trivial equivalence: both formulae are self-contained ways of calculating the multiplicities.

We have thus re-expressed our earlier Taylor expansion in a way that makes the representation theory content of the counting manifest. This structure in the counting problem can be used to provide an explicit construction formula. First we need to decompose the Z and W polynomials into definite S_n irreps. The projector onto irrep r from the tensor product of k copies of the hook is

$$P^{r,k} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_r(\sigma) \Gamma_k(\sigma) \tag{5.121}$$

We also need the projection onto the symmetric irrep

$$P^{k+l} = \frac{1}{n!} \sum_{\sigma \in S_n} \Gamma_{k+l}(\sigma) \tag{5.122}$$

Using these two projectors, the polynomials corresponding to primaries constructed using two holomorphic variables are now given by

$$\sum_A P_A(z,w) \vec{n}^A = P^{k+l} \sum_{r \vdash n} (P^{r,l} \times P^{r,k}) Z^{\otimes k} W^{\otimes l} \tag{5.123}$$

where $\vec{n}^A$ are unit vectors inside the carrier space of $\yng(2,1)^{\otimes k+l}$ and $P_A(z,w)$ are the polynomials we want. In fact, the construction formula given in (5.123) constructs a larger class of polynomials than those counted in (5.71). This is because the polynomials counted in (5.71) are extremal and hence they are annihilated by J_+^R . We will return to this point in the discussion below. The construction formula that has been sketched above can easily be implemented numerically. To implement (5.123), we need the projector onto irrep r in the space obtained by taking the tensor product of k copies of the hook

$$P^r_{a_1 \cdots a_k, b_1 \cdots b_k} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_r(\sigma) \Gamma_k(\sigma)_{a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_k} \tag{5.124}$$

and we need the projection onto the symmetric irrep

$$P_{a_1\cdots a_n,b_1\cdots b_n}=\frac{1}{n!}\sum_{\sigma\in S_n}\Gamma_k(\sigma)_{a_1\cdots a_n,b_1\cdots b_n} \tag{5.125}$$

We find that (5.123) is now given by

$$\sum_A P_A(z,w)\vec{n}^A_{e_1\cdots e_{k+l}}=P_{e_1\cdots e_{k+l},a_1\cdots a_k,c_1\cdots c_l}P^r_{a_1\cdots a_k,b_1\cdots b_k}P^r_{c_1\cdots c_l,d_1\cdots d_l}Z^{(b_1)}\cdots Z^{(b_k)}W^{(d_1)}\cdots W^{(d_l)} \tag{5.126}$$

where $\vec{n}^A$ are unit vectors and $P_A(z,w)$ are the polynomials we want. To start, consider $k=l=1$. We find

$$\tilde{P}_{e_1e_2}=(-w_3(z_1+z_2-2z_3)+w_1(2z_1-z_2-z_3)-w_2(z_1-2z_2+z_3))\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \tag{5.127}$$

so that the invariant polynomial is

$$P(z,w)=-w_3(z_1+z_2-2z_3)+w_1(2z_1-z_2-z_3)-w_2(z_1-2z_2+z_3) \tag{5.128}$$

This polynomial is not extremal. This is easily verified by computing

$$-J^R_+P(z,w)=z_i\frac{\partial}{\partial w_i}P(z,w)=(z_1-z_2)^2+(z_1-z_3)^2+(z_2-z_3)^2 \tag{5.129}$$

so that this is another state in the multiplet of the $k=2$ primary we built in the last section.

To focus on the extremal polynomials counted in (5.71) we must implement the constraint that these polynomials are annihilated by J^R_+ . Towards this end, note that the polynomials in Z, W carry a representation of $SU(2)_R$, so that we can further decompose the polynomials according to their $SU(2)_R$

quantum numbers. Z, W form an $SU(2)$ doublet with Z the $+\frac{1}{2}$ state and W the $-\frac{1}{2}$ state. There is an action of S_{k+l} on these polynomials that commutes with $SU(2)_R$. This S_{k+l} action acts to permute the $W^{(a)}$ and $Z^{(a)}$ factors. Denote the matrix representing $\sigma \in S_{k+l}$ by $\Gamma(\sigma)$. This rep is generated by the adjacent permutations which are easy to build. Towards this end, note that swapping two factors in the tensor product is accomplished by the permutation P which obeys $Px \otimes y = y \otimes x$, i.e. we have

$$P \begin{bmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_2 y_1 \\ x_1 y_2 \\ x_2 y_2 \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.130}$$

Using the adjacent permutations we can construct any $\Gamma(\sigma)$ and then any projector

$$K^R = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \chi_R(\sigma) \Gamma(\sigma) \tag{5.131}$$

with $\chi_R(\sigma)$ a symmetric group character. The label R is a Young diagram with at most 2 rows. The spin of the $SU(2)$ irrep that K^R projects to is given by $(R_1 - R_2)/2$ where R_1 and R_2 are the lengths of the rows of R. As an example, consider $k = 2 = l$. The rep of S_4 we need is generated by ($\mathbf{1}$ is the 2×2 identity)

$$\Gamma((12)) = P \otimes \mathbf{1} \otimes \mathbf{1} \quad \Gamma((23)) = \mathbf{1} \otimes P \otimes \mathbf{1} \quad \Gamma((34)) = \mathbf{1} \otimes \mathbf{1} \otimes P \tag{5.132}$$

To construct the primary corresponding to $s^7 x^2$ we need to project on the $SU(2)_R$ irrep with spin zero. This is accomplished by using the projector

$$K^{\yng(2,2)}_{a_1 a_2 a_3 a_4, b_1 b_2 b_3 b_4} = \frac{1}{4!} \sum_{\sigma \in S_4} \chi_{\yng(2,2)}(\sigma) \Gamma_{a_1 a_2 a_3 a_4, b_1 b_2 b_3 b_4}(\sigma) \tag{5.133}$$

It is simple to compute

$$K^{\boxplus}_{a_1a_2a_3a_4,b_1b_2b_3b_4}\tilde{P}_{b_1b_2b_3b_4} = (w_1(z_2 - z_3) + w_2(z_3 - z_1) + w_3(z_1 - z_2))^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{4}{3} \\ 0 \\ -\frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ 0 \\ -\frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{4}{3} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.134}$$

Thus the invariant polynomial is

$$P(z, w) = (w_1(z_2 - z_3) + w_2(z_3 - z_1) + w_3(z_1 - z_2))^2 \tag{5.135}$$

By inspection it is obvious that this polynomial obeys the conditions (5.31) and further that it is a highest weight of $SU(2)_R$, i.e $J^R_+(z, w) = 0$. The above polynomial suggests a natural generalization: consider the family of polynomials indexed by the integer n

$$\Psi_n = \left(w^{(3)}(\bar{z}^{(2)} - \bar{z}^{(1)}) + w^{(2)}(\bar{z}^{(1)} - \bar{z}^{(3)}) + w^{(1)}(\bar{z}^{(3)} - \bar{z}^{(2)})\right)^{2n} \tag{5.136}$$

It is obvious that they obey (5.31) and hence that these polynomials do correspond to primary operators. It is also clear that they are extremal, i.e. $J_+\Psi_n = 0$. These primaries have spin $[2n, 0]$ and dimension $\Delta = 3 + 4n$. The translation into the free field language is

$$\mathcal{O}_{[2n,0]}^{\Delta=4n+3} = \sum_{r=0}^{2n} \sum_{s=0}^{2n-r} \sum_{t=0}^{r} \sum_{u=0}^{s} \sum_{v=0}^{2n-r-s} \frac{(2n)!(-1)^{t+u+v}}{(r-t)!t!(s-u)!u!(2n-r-s-v)!v!} \times \tag{5.137}$$

$$(P_w^{2n-r-s} P_{\bar{z}}^{t+s-u} \phi)(P_w^s P_{\bar{z}}^{r-t+v} \phi)(P_w^r P_{\bar{z}}^{2n_u-r-s-v} \phi)$$

The polynomials we have constructed in (5.123) obey all of the conditions spelled out in (5.31). In fact, they again obey an even stronger linear condition

$$\partial_{\bar{z}^I} P(\vec{z}, \vec{w}) = 0 = \partial_{w^I} P(\vec{z}, \vec{w}) \tag{5.138}$$

which imply the Laplacian constraint. As a result, taking all possible values of k, l we find that the polynomials $P_A(z, w)$ are again a finitely generated polynomial ring. This is a consequence of the Leibnitz rule for the derivatives of a product of functions. The ring of polynomials that correspond to extremal primaries is the polynomial ring of holomorphic functions for

$$(\mathbb{C}^2)^n / (\mathbb{C}^2 \times S_n) \tag{5.139}$$

In (5.82), we have computed the Hilbert series for the polynomials in two holomorphic variables, that correspond to extremal primary operators built using two scalar fields. Using generalities about Hilbert series for algebraic varieties (see [52, 53] for applications in the context of moduli spaces of SUSY gauge theories), we know that if the ring is generated by h homogeneous elements of positive degrees $d_1, \cdots d_h$, then the Hilbert series is a rational fraction

$$H_S(t) = \frac{Q(t)}{\prod_{i=1}^{h}(1 - t^{d_i})} \tag{5.140}$$

where Q is a polynomial with integer coefficients. Thus, we see from (5.82) that for $n = 3$ the polynomials $P_A(z, w)$ are a finitely generated polynomial ring with 4 generators and one relation and that this space of polynomials is a complete intersection and it is 3 dimensional. Using this Hilbert series and the explicit constructions described above, we can identify the generators ($z_{ij} = z_i - z_j$)

$$G_1 = (z_{12})^2 + (z_{13})^2 + (z_{23})^2 \leftrightarrow s^2xy \tag{5.141}$$

$$G_2 = (z_{13} + z_{23})(z_{31} + z_{21})(z_{12} + z_{32}) \leftrightarrow s^3\sqrt{x^3y^3} \tag{5.142}$$

$$G_3 = \begin{vmatrix} w_1 & w_2 & w_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix}^2 \leftrightarrow s^4x^2 \tag{5.143}$$

$$G_4 = \begin{vmatrix} z_1^2 & z_2^2 & z_3^2 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix} \times \begin{vmatrix} w_1 & w_2 & w_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix} \leftrightarrow s^5x^{\frac{5}{2}}y^{\frac{3}{2}} \tag{5.144}$$

of this ring. Consider the last generator above: either of the determinants being multiplied is antisymmetric under permuting 1, 2 or 1, 3 or 2, 3 so that the product is symmetric. The relation obeyed by these generators is easily identified

$$27(G_4)^2 + G_3\Big((G_2)^2 - \frac{1}{2}(G_1)^3\Big) = 0 \tag{5.145}$$

Once again the ring structure exhibited by the polynomials implies a genuinely new structure for the extremal primary operators that was not previously recognized. The Hilbert series in more complicated situations encodes detailed information about the generators of the ring, relations between these generators, relations between the relations and so on. An example of this structure is given in Appendix I.

The Hilbert series we have computed so far exhibit a palindromic property of the numerators. This can be verified for $Z_3^{z,w}(s,x,y)$ and $Z_4^{z,w}(s,x,y)$. A general property of the numerators

$$Q_n(s,x,y) = \sum_{k=0}^{D} a_k(x,y)s^k \tag{5.146}$$

is that $a_{D-k}(x,y) = a_k(x.y)$. A theorem due to Stanley[54] suggests that this palindromic property of the numerators implies the Calabi-Yau property of the underlying orbifolds. It is fascinating that non-trivial properties of the combinatorics of primary fields in four dimensional scalar field theory is related to the geometry of Calabi-Yau orbifolds (5.139). Motivated by this connection, we will prove this palindromic property of the numerators in the next section.

To obtain $G_n^{z,w}(s,x,y)$ from $Z_n(s,x,y)$, we have kept only the highest weight operator (under $SU(2)$) from a complete spin multiplet of primary operators. Geometrically, this can be viewed as modding out by the action of G_+, generated by the $SU(2)$ raising operator J^+ , i.e. G_+ is the unipotent group of upper triangular 2×2 matrices with 1 on the diagonal. Consequently, the Hilbert series $G_n(s,x,y)$ is the polynomial ring of functions for

$$\frac{(\mathbb{C}^2)^n}{(\mathbb{C}^2 \times G_+ \times S_n)} \tag{5.147}$$

5.4.3 Palindromy properties

The palindromic property of the Hilbert series can be stated as follows

$$Z_n^{z,w}(q_1^{-1}, q_2^{-1}) = (q_1 q_2)^{n-1} Z_n^{z,w}(q_1, q_2) \tag{5.148}$$

In this section we will prove that our Hilbert series $Z_n^{z,w}(q_1,q_2)$ do indeed enjoy this transformation property.

Our starting point is the formula

$$Z_n^{z,w}(q_1,q_2) = s^n \sum_{\Lambda \vdash n} Z_{SH}(q_1,\Lambda) Z_{SH}(q_2,\Lambda) \tag{5.149}$$

where $q_1 = s\sqrt{xy}$, $q_2 = s\sqrt{\frac{x}{y}}$. This has the property $Z_n^{z,w}(q_1,q_2) = Z_n^{z,w}(q_2,q_1)$.

The exchange of q_1, q_2 amounts to the inversions of y. Now, observe that

$$Z_{SH}(q^{-1}, \Lambda) = (-q)^{n-1} Z_{SH}(q, \Lambda^T) \tag{5.150}$$

This is easily demonstrated using the explicit formula (5.114) and the identity

$$\begin{aligned}\sum_b h_b =& \frac{1}{2}\Big(\sum_i c_i(c_i+1) + \sum_i r_i(r_i+1)\Big) - n \\ =& \frac{1}{2}\Big(\sum_i c_i^2 - \sum_i r_i^2\Big)\end{aligned} \tag{5.151}$$

Here c_i is the length of the i'th column and r_i is the length of i'th row. Also note that the row lengths of Λ^T are the column lengths of Λ and vice versa. The identity can be understood as follows. As we sum over hook lengths, for each column of length c_i we have a contribution to the sum of $1+2+\cdots+c_i$ as we start from the bottom and go up to the top. For each row, we can similarly sum $1+2+\cdots+r_i$, but this over counts 1 for each box. Hence the identity. Using this result

$$\begin{aligned}Z_n^{z,w}(q_1^{-1}, q_2^{-1}) =& s^n (q_1 q_2)^{n-1} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda^T) Z_{SH}(q_2, \Lambda^T) \\ =& s^n (q_1 q_2)^{n-1} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda) Z_{SH}(q_2, \Lambda) \\ =& (q_1 q_2)^{n-1} Z_n^{z,w}(q_1, q_2)\end{aligned} \tag{5.152}$$

In the last step, we used the fact that transposition is a symmetry of the set of Young diagrams. Summing over Λ^T is the same as summing over Λ.

The Hilbert series $G_n^{z,w}(s, x, y)$ also exhibit the palindromy property. We know

$$Z_n^{z,w}(s^{-1}, x^{-1}, y^{-1}) = s^{2n-2} x^{n-1} Z_n^{z,w}(s, x, y) \tag{5.153}$$

Also (CCW for counterclockwise and CW for clockwise)

$$G_n^{z,w}(s, x, y) = \frac{1}{2\pi i} \oint_{CCW} dz \Big(1 - \frac{1}{z^2}\Big) Z_n^{z,w}(s, x, z^2) \frac{1}{z - \sqrt{y}} \tag{5.154}$$

We will study $\sqrt{y}G_n^{z,w}(s,x,y)$ which can be written in two equivalent ways

$$\sqrt{y}G_n^{z,w}(s,x,y) = \frac{1}{2\pi i}\oint_{CCW} dz\Big(1-\frac{1}{z^2}\Big)Z_n^{z,w}(s,x,z^2)\frac{\sqrt{y}}{z-\sqrt{y}}, \tag{5.155}$$

since the integrand doesn't have a simple pole at $z = 0$, we perform the following manipulation,

$$\begin{aligned}\sqrt{y}G_n^{z,w}(s,x,y) =& \frac{1}{2\pi i}\oint_{CCW} dz\Big(1-\frac{1}{z^2}\Big)Z_n^{z,w}(s,x,z^2)\frac{\sqrt{y}}{z-\sqrt{y}}, \\ =& \frac{1}{2\pi i}\oint_{CCW} dz\Big(1-\frac{1}{z^2}\Big)Z_n^{z,w}(s,x,z^2)\frac{z}{z-\sqrt{y}}\end{aligned} \tag{5.156}$$

Both of the representations will be needed below. Now, study

$$\begin{aligned}\frac{1}{\sqrt{y}}G_n^{z,w}(s^{-1},x^{-1},y^{-1}) =& \frac{1}{2\pi i}\oint_{CCW} dz\Big(1-\frac{1}{z^2}\Big)Z_n^{z,w}(s^{-1},x^{-1},z^2)\frac{\frac{1}{\sqrt{y}}}{z-\frac{1}{\sqrt{y}}} \\ =& \frac{1}{2\pi i}\oint_{CCW} dz\Big(1-\frac{1}{z^2}\Big)Z_n^{z,w}(s^{-1},x^{-1},z^2)\frac{1}{z\sqrt{y}-1}\end{aligned} \tag{5.157}$$

Now change integration variables from z to $w = \frac{1}{z}$ to find

$$\begin{aligned}\frac{1}{\sqrt{y}}G_n^{z,w}(s^{-1},x^{-1},y^{-1}) =& \frac{1}{2\pi i}\oint_{CCW} \frac{dw}{w^2}(1-w^2)Z_n^{z,w}(s^{-1},x^{-1},w^{-2})\frac{w}{\sqrt{y}-w} \\ =& \frac{s^{2n-2}x^{n-1}}{2\pi i}\oint_{CCW} \frac{dw}{w^2}(1-w^2)Z_n^{z,w}(s,x,w)\frac{w}{w-\sqrt{y}} \\ =& s^{2n-2}x^{n-1}\sqrt{y}G_n^{z,w}(s,x,y)\end{aligned} \tag{5.158}$$

5.4.4 Gorenstein, Calabi-Yau and top-forms

In this section we would like to return to the issue of the Calabi-Yau property for the permutation orbifolds relevant for the combinatorics of the primaries. Stanley's theorem[54] tells us that a Cohen Macauly ring that is an integral domain and has a palindromic Hilbert series, is a Gorenstein ring. Further, since our rings are defined over an affine space the canonical bundle in this case is trivial, establishing the Calabi-Yau property. According to [64], the rings that we consider are Cohen Macaulay because they are the quotient of a Noetherian ring $(\mathbb{C}^2)^n/\mathbb{C}^2$ by a reductive group S_n. However, in general, the relevant rings are not an integral domain. It is therefore not clear that we can apply Stanley's theorem to conclude that our permutation orbifolds are Calabi-Yau.

An alternative approach to demonstrating the Calabi-Yau property, is to construct a nowhere vanishing top form. To motivate the general formula, it is useful to start with some simple cases. For $n = 2$ the top form

$$\Omega^{(n-1)}(dz) = dz_{12} = dz_1 - dz_2 \tag{5.159}$$

is clearly a translation invariant form on $\mathbb{C}^2$ so it is clearly a top form on the quotient $\mathbb{C}^2/\mathbb{C}$. It is odd under S_2 . For $n = 3$, a translation invariant, S_n-odd top form is given by

$$\Omega^{(n-1)}(dz) = dz_{12} \wedge d_{23} = dz_1 \wedge dz_2 - dz_1 \wedge dz_3 + dz_2 \wedge dz_3 \tag{5.160}$$

For general n, we have

$$\Omega^{(n-1)}(dz) = dz_{12} \wedge dz_{23} \wedge \cdots \wedge dz_{n-1,n} = \sum_{k=1}^{n} I_{\partial_k} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \tag{5.161}$$

The operator I_{∂_k} removes the dz_k in the n-form and leaves an $(n-1)$-form, with a sign $(-1)^{k-1}$. In terms of these, the top forms for the orbifolds relevant for the extremal primary primary are

$$\Omega^{(n-1)}(dz) \wedge \Omega^{(n-1)}(dw) \tag{5.162}$$

5.5 Vector Model Primaries: Symmetry breaking $S_{2n} \to S_n[S_2]$

Up to now we have considered a single real scalar field. However, the methods we have developed readily apply in more general settings. For applications to holography[27], it is natural to consider the free gauged $O(N)$ vector model, conjectured to be dual to higher spin gravity[65]. The scalar field is now an

$O(N)$ vector and primaries must be $O(N)$ gauge invariants. In this section we will explain how the techniques we have developed in this chapter apply to the counting and construction of primaries in the gauged $O(N)$ vector model. To obtain a gauge invariant, all vector indices must be contracted. Thus, to construct a primary, we now distribute the derivatives among

$$\phi_{I_1}\phi_{I_1}\phi_{I_2}\phi_{I_2}\cdots\phi_{I_n}\phi_{I_n} \tag{5.163}$$

where the vector indices I are summed from 1 to N . We no longer have an S_{2n} symmetry acting to swap the bosonic fields. The symmetry is broken to a smaller group which can swap the fields in a given contracted pair, or it can swap the pairs. This symmetry group is the wreath product $S_n[S_2]$. Thus, we don't want to project $V_+^{\times 2n}$ onto the trivial of S_{2n} (i.e. $\mathrm{Sym}(V_+^{\otimes 2n})$), we rather want to project onto the trivial of $S_n[S_2]$. We will restrict attention to the case where $2n < N$. This avoids subtleties due to finite N relations, associated with the stringy exclusion principle in the context of matrix invariants. These can be dealt with using a Young diagram basis, which is left for a future discussion.

We know the character for the fundamental representation V_+ of $SO(4,2)$. To repeat the analysis we carried out for the free scalar, we need the character for the tensor product of $2n$ fields, after projecting to the trivial of $S_n[S_2]$. This gives

$$\chi_{\mathcal{H}_n}(s,x,y) = \frac{1}{2^n n!}\sum_{\sigma\in S_n[S_2]} \mathrm{Tr}_{V^{\otimes 2n}}(\sigma M^{\otimes 2n}) \tag{5.164}$$

where M is again given by $s^{\Delta}x^{J_3^L}y^{J_3^R}$. This is equal to

$$\begin{aligned}\chi_{\mathcal{H}_n}(s,x,y) &= \sum_{p\vdash 2n} Z_p^{S_n[S_2]}\prod_{i=1}(\mathrm{Tr}M^i)^{p_i} \\ &= \sum_{p\vdash 2n} Z_p^{S_n[S_2]}\prod_{i=1}\Big(\sum_a m_a^i\Big)^{p_i}\end{aligned} \tag{5.165}$$

where m_a are the eigenvalues of M and $Z_p^{S_n[S_2]}$ is the cycle index, which gives the number of permutations in $S_n[S_2]$ with cycle structure specified by p_i. The generating function for these cycle indices is known (see e.g. [59]) and can be used to find the following generating function for the characters

$$\sum_{n=0}^{\infty} t^n \mathrm{Tr}_{\mathcal{H}_n}(M) = \prod_a \frac{1}{\sqrt{1-tm_a^2}} \prod_{a\neq b} \frac{1}{\sqrt{(1-tm_am_b)}} \tag{5.166}$$
$$= \prod_a \frac{1}{\sqrt{1-tm_a^2}} \prod_{a>b} \frac{1}{(1-tm_am_b)}$$

We can now argue as we did in section 3. Using the known eigenvalues of M the generalization of (5.42) is given by

$$\mathcal{Z}(s,x,y) = \sum_{n=0}^{\infty} t^n \chi_{\mathcal{H}_n}(s,x,y) \tag{5.167}$$
$$= \prod_{q=0}^{\infty} \prod_{a=-\frac{q}{2}}^{\frac{q}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{\sqrt{1-ts^{2q+2}x^{2a}y^{2b}}}$$
$$\times \prod_{q_2=0}^{\infty} \prod_{a_2=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{b_2=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{(q_1,a_1,b_1)<(q_2,a_2,b_2)} \frac{1}{(1-ts^{q_1+q_2+2}x^{a_1+a_2}y^{b_1+b_2})}$$

This can be simplified further. We can order the triples (q,a,b) as follows: The inequality $(q_1,a_1,b_1) < (q_2,a_2,b_2)$ means: $q_1 < q_2$ or $q_1 = q_2$, $a_1 < a_2$ or $q_1 = q_2$, $a_1 = a_2$, $b_1 < b_2$. Alternatively, we can write

$$\mathcal{Z}(s,x,y) = \prod_{q=0}^{\infty} \prod_{a=-\frac{q}{2}}^{\frac{q}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{\sqrt{1-ts^{2q+2}x^{2a}y^{2b}}} \tag{5.168}$$
$$\times \prod_{q_2=0}^{\infty} \prod_{a_2=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{b_2=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{q_1=0}^{\infty} \prod_{a_1=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{b_1=-\frac{q_2}{2}}^{\frac{q_2}{2}} \frac{1}{(1-ts^{q_1+q_2+2}x^{a_1+a_2}y^{b_1+b_2})}$$

We can now define the generating function (here we take $n > 1$ to avoid complications with null states)

$$G_{2n}^{O(N)}(s,x,y) = \sum_{d=0}^{\infty} \sum_{j_1,j_2} N_{[2n+d,j_1,j_2]}^{O(N)} s^{2n+d} x^{j_1} x^{j_1} \tag{5.169}$$

which is given by

$$G_{2n}^{O(N)}(s,x,y) = \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)Z_{2n}(s,x,y)\right]_{\geq} \tag{5.170}$$

where

$$\begin{aligned} Z_{2n}(s,x,y) =& \chi_{\mathcal{H}_n}(s,x,y)\left(1-s\sqrt{xy}\right)\left(1-s\sqrt{\frac{x}{y}}\right)\left(1-s\sqrt{\frac{y}{x}}\right)\left(1-\frac{s}{\sqrt{xy}}\right) \\ =& \sum_{d=0}^{\infty}\sum_{j_1,j_2} N^{O(N)}_{[2n+d,j_1,j_2]} s^{2n+d}\chi_{j_1}(x)\chi_{j_2}(y) \end{aligned} \tag{5.171}$$

For $n = 1$ we need to subtract out the null states that are present since the primaries being counted include conserved higher spin currents.

We can again specialize to the counting of extremal primaries. For example, the leading twist primaries are counted by $G_{2n}^{O(N),\max}(s,x,y)$ where

$$\sum_{n=0}^{\infty} t^n G_{2n}^{O(N),\max}(s,x,y) = \sum_{n=0}^{\infty} t^n(1-s\sqrt{xy})\chi_{2n}^{O(N),\max}(s,x,y) \tag{5.172}$$

$$\sum_{n=0}^{\infty} \chi_{2n}^{O(N),\max}(s,x,y)t^n = \prod_{q=0}^{\infty}\frac{1}{\sqrt{1-ts^{2q+2}x^q y^q}}\prod_{q_1,q_2}^{\infty}\frac{1}{\sqrt{1-ts^{q_1+q_2+2}x^{\frac{q_1+q_2}{2}}y^{\frac{q_1+q_2}{2}}}} \tag{5.173}$$

It is now straightforward to obtain the Hilbert series for leading twist primaries built using 4 fields

$$G_4^{O(N),\max}(s,x,y) = \frac{s^4(1-s^6x^3y^3)}{(1-s^2xy)^2(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)} \tag{5.174}$$

This shows that there are 4 generators and a single relation, that this space of operators is a complete intersection and it is 3 dimensional. In a similar way we have

$$G_6^{O(N),\max}(s,x,y) = \frac{s^6(1 - s\sqrt{xy} + s^3x^{\frac{3}{2}} - s^7x^{\frac{7}{2}}y^{\frac{7}{2}} + s^9x^{\frac{9}{2}}y^{\frac{9}{2}} - s^{10}x^5y^5)}{(1 - s\sqrt{xy})(1 - s^2xy)^2(1 - s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1 - s^4x^2y^2)(1 - s^6x^3y^3)} \tag{5.175}$$

The Hilbert series for these primaries are again plindromic. For the case of one-complex variable that we are discussing, we have

$$G_{2n}^{O(N),\max}(q) = s^{2n} \sum_{\Lambda \vdash 2n, \Lambda even} Z_{SH}(q, \Lambda) \tag{5.176}$$

Using this formula and (5.150) we find

$$\begin{aligned} G_{2n}^{O(N),\max}(q^{-1}) =& s^{2n} \sum_{\Lambda \vdash 2n, \Lambda even} Z_{SH}(q^{-1}, \Lambda) \\ =& - q^{2n-1} s^{2n} \sum_{\Lambda \vdash 2n, \Lambda even} Z_{SH}(q, \Lambda^T) \\ =& - (q)^{2n-1} G_{2n}^{O(N),\max}(q) \end{aligned} \tag{5.177}$$

This demonstrates the palidromy property for the Hilbert series associated to the orbifold

$$(\mathbb{C})^{2n}/(\mathbb{C} \times S_n[S_2]) \tag{5.178}$$

Now consider the two complex variable case

$$\begin{aligned} \mathcal{Z}(s,x,y) =& \prod_{q=0}^{\infty} \frac{1}{\sqrt{1 - ts^{2q+2}x^qy^q}} \prod_{q_1,q_2}^{\infty} \frac{1}{\sqrt{1 - ts^{q_1+q_2+2}x^{\frac{q_1+q_2}{2}}y^{\frac{q_1+q_2}{2}}}} \\ =& \sum_{t=0}^{\infty} t^n \chi_{\mathcal{H}_n}^{z,w}(s,x,y) \end{aligned} \tag{5.179}$$

It is natural to consider the generating functions

$$Z_n^{O(N),zw}(s,x,y) = (1 - s\sqrt{xy})\left(1 - s\sqrt{\frac{x}{y}}\right)\chi_{\mathcal{H}_n}^{z,w}(s,x,y) \tag{5.180}$$

and

$$G_n^{O(N),zw} = \left[\left(1 - \frac{1}{y}\right) Z_n^{O(N),zw}(s,x,y)\right]_{\geq} \tag{5.181}$$

A straightforward computation gives

$$Z_4^{O(N),zw}(s,x,y) = \frac{g(s,x,y)}{(1-s\sqrt{\frac{x}{y}})^4(1-s\sqrt{xy})^4(1+s\sqrt{\frac{x}{y}})^2(1+s\sqrt{xy})^2(1+s^2\frac{x}{y})(1+s^2xy)} \tag{5.182}$$

where

$$\begin{aligned} g(s,x,y) =& s^4\bigg(1 - (s\sqrt{x} + s^3x^{\frac{3}{2}} + s^5x^{\frac{5}{2}} + s^7x^{\frac{7}{2}})(\sqrt{y} + \frac{1}{\sqrt{y}}) \\ &+ (s^8x^4 + s^4x^2 + 2s^2x + 2s^6x^3) + (s^4x^2 + s^2x + s^6x^3)(y + 1 + \frac{1}{y})\bigg) \end{aligned} \tag{5.183}$$

This result can be recovered by using the generating function

$$s^4 \sum_{\Lambda_1,\Lambda_2} (C(\ydiagram{4},\Lambda_1,\Lambda_2) + C(\ydiagram{2,2},\Lambda_1,\Lambda_2)) Z_{SH}(\Lambda_1, s\sqrt{xy}) Z_{SH}(\Lambda_2, s\sqrt{\frac{x}{y}}) \tag{5.184}$$

Recall that

$$Z_{SH}(\Lambda,q) = (1-q)q^{\sum_i \frac{c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})} \tag{5.185}$$

Formula (5.184) is a consequence of the fact that an irrep Λ of S_{2n} contains the trivial of $S_n[S_2]$ with multiplicity 1. For the example given above, using the fact that the non-zero terms are

$$
\begin{aligned}
&C(\yng(4),\Lambda_1,\Lambda_2) = \lambda_{\Lambda_1,\Lambda_2} \\
&C(\yng(4),\yng(2,2),\yng(2,2)) = C(\yng(2,2),\yng(4),\yng(4)) = 1 \\
&C(\yng(3,1),\yng(3,1),\yng(2,2)) = C(\yng(2,1,1),\yng(2,1,1),\yng(2,2)) = 1 \\
&C(\yng(2,1,1),\yng(3,1),\yng(2,2)) = C(\yng(3,1),\yng(2,1,1),\yng(2,2)) = 1 \\
&C(\yng(1,1,1,1),\yng(2,2),\yng(2,2)) = C(\yng(2,2),\yng(1,1,1,1),\yng(2,2)) = 1 \\
&C(\yng(2,2),\yng(2,2),\yng(2,2)) = 1
\end{aligned}
\tag{5.186}
$$

we obtain complete agreement between (5.182) and (5.180). The geometries associated to $Z_{2n}^{O(N),zw}(s,x,y)$ are

$$
\frac{(\mathbb{C}^2)^n}{(\mathbb{C}^2 \times S_n[S_2])} \tag{5.187}
$$

and, after we impose the G_+ condition, the geometries for $G_{2n}^{O(N),zw}(s,x,y)$ are

$$
\frac{(\mathbb{C}^2)^{2n}}{(G_+ \times S_n[S_2])} \tag{5.188}
$$

G_+ is the unipotent group of upper trianglar 2×2 matrices with 1 on the diagonal. For the 2-complex variables case, we have the Hilbert series

$$
Z_n^{O(N),zw}(q_1,q_2) = s^{2n} \sum_{\Lambda_1,\Lambda_2 \vdash 2n} \sum_{\Lambda \vdash 2n, \Lambda even} C(\Lambda_1,\Lambda_2,\Lambda) Z_{SH}(q_1,\Lambda_1) Z_{SH}(q_2,\Lambda_2) \tag{5.189}
$$

where $C(R,S,T)$ is the Kronecker coefficient giving the number of S_n invariants in the tensor product of three irreps R,S,T of S_n. Applying the inversion

$$
\begin{aligned}
Z_{2n}^{O(N),zw}(q_1^{-1},q_2^{-1}) =& s^{2n}\sum_{\Lambda_1,\Lambda_2\vdash 2n}\sum_{\Lambda\vdash 2n,\Lambda even} C(\Lambda_1,\Lambda_2,\Lambda)Z_{SH}(q_1^{-1},\Lambda_1)Z_{SH}(q_2^{-1},\Lambda_2) \\
=& s^{2n}(q_1q_2)^{2n-1}\sum_{\Lambda_1,\Lambda_2\vdash 2n}\sum_{\Lambda\vdash 2n,\Lambda even} C(\Lambda_1,\Lambda_2,\Lambda)Z_{SH}(q_1^{-1},\Lambda_1^T)Z_{SH}(q_2^{-1},\Lambda_2^T) \\
=& s^{2n}(q_1q_2)^{2n-1}\sum_{\Lambda_1,\Lambda_2\vdash 2n}\sum_{\Lambda\vdash 2n,\Lambda even} C(\Lambda_1^T,\Lambda_2^T,\Lambda)Z_{SH}(q_1^{-1},\Lambda_1)Z_{SH}(q_2^{-1},\Lambda_2) \\
=& s^{2n}(q_1q_2)^{2n-1}\sum_{\Lambda_1,\Lambda_2\vdash 2n}\sum_{\Lambda\vdash 2n,\Lambda even} C(\Lambda_1,\Lambda_2,\Lambda)Z_{SH}(q_1,\Lambda_1)Z_{SH}(q_2,\Lambda_2) \\
=& (q_1q_2)^{2n-1}Z_n^{O(N),zw}(q_1,q_2)
\end{aligned}
\tag{5.190}
$$

In going from the second to third line, we renamed $\Lambda_1 \to \Lambda_1^T$, $\Lambda_2 \to \Lambda_2^T$. In going from the third to fourth line, we used an invariance of the Kronecker multiplicity

$$
C(\Lambda_1,\Lambda_2,\Lambda) = C(\Lambda_1^T,\Lambda_2^T,\Lambda) \tag{5.191}
$$

which follows from

$$
C(\Lambda_1,\Lambda_2,\Lambda) = \frac{1}{(2n)!}\sum_{\sigma\in S_{2n}} \chi_{\Lambda_1}(\sigma)\chi_{\Lambda_2}(\sigma)\chi_{\Lambda}(\sigma) \tag{5.192}
$$

and

$$
\chi_{\Lambda^T}(\sigma) = (-1)^\sigma\chi_\Lambda(\sigma) \tag{5.193}
$$

where $(-1)^\sigma$ is the parity of σ. The formula (5.190) demonstrates that the palindromy property of the Hilbert series for the counting of vector model primaries.

5.6 Matrix Model Primaries

Another interesting generalization of the single real scalar field, is to a matrix scalar. We gauge the free theory. The net effect is that we look for primary operators with all indices contracted. There are many ways that the indices can be contracted, corresponding to the different possible multitrace structures that can be written down. Thus, generalizing to the matrix scalar introduces an interesting non-trivial structure to the problem.

The large N counting of gauge invariant functions of a single matrix, is achieved by integrating[66]

$$\mathcal{Z}(x) = \int dU e^{\sum_i \frac{x^i}{i}(trU^i)(trU^{\dagger i})} = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)} \tag{5.194}$$

For multi-matrices, the large N counting is [66]

$$\mathcal{Z}(x_i) = \int dU e^{\sum_i \frac{(\sum_a x_a^i)}{i}(trU^i)(trU^{\dagger i})} = \prod_{i=1}^{\infty} \frac{1}{(1-\sum_{a=1}^{M} x_a^i)} \tag{5.195}$$

where M is the number of matrices in the model. Specializing to the 2-matrix case, this is

$$\mathcal{Z}(x,y) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i-y^i)} \tag{5.196}$$

For the matrix scalar, we have matrix fields

$$\partial_{l,m}\phi^i_j \tag{5.197}$$

l denotes a symmetric traceless irrep of $SO(4)$ and m runs over the states in this irrep. There are known methods that can be used to write diagonal bases for the local operators of this theory[50, 67]. For the large N counting of gauge invariants built from derivatives of a single matrix, we have [68]

$$\mathcal{Z}(t,s,x,y) = \int dU e^{\sum_i^{\infty} \sum_{q=0}^{\infty} \sum_{a_q,b_q=-\frac{q}{2}}^{\frac{q}{2}} \frac{(ts^{(1+q)}x^{a_q}y^{b_q})^i}{i}(trU^i)(trU^{\dagger i})} \tag{5.198}$$

Note that this can also be written as

$$\mathcal{Z}(t,s,x,y) = \int dU e^{\sum_i \frac{t^i}{i}\chi_{V^+}(s^i,x^i,y^i)(trU^i)(trU^{\dagger i})} \tag{5.199}$$

By repeating steps similar to the ones we did for the integral encountered in case of multi-matrices, we get

$$\mathcal{Z}(t,s,x,y) = \prod_{i=1}^{\infty} \frac{1}{(1 - \sum_{q=0}^{\infty} \sum_{a_q,b_q=-\frac{q}{2}}^{\frac{q}{2}} s^{i+qi} x^{ia_q} y^{ib_q})} \tag{5.200}$$

To simplify this further, we will derive an identity quoted in [66]. The state space of a single scalar V_+ is obtained by acting on the ground state with products of the operators P_μ . This is a $4D$ irrep of $SO(4) = SU(2) \times SU(2)$ with spins $(1/2, 1/2)$. The equation of motion says that $P_\mu P_\mu$ acting on the ground state is zero. An immediate consequence is that the independent states in V_+ generated by q copies of P transform as the symmetric traceless irrep of $SO(4)$, corresponding to the Young diagram with a single row of length q. This irrep of $SO(4)$ is the $(q/2, q/2)$ irrep of $SU(2) \times SU(2)$. It immediately follows that

$$\begin{aligned}
\chi_{V_+}(s,x,y) =& \mathrm{tr}_{V_+}(s^D x^{J_L} y^{J_R}) \\
=& s \sum_{q=0}^{\infty} s^q \chi_{q/2}(x) \chi_{q/2}(y) \\
=& s \sum_{q=0}^{\infty} s^q \sum_{a_q=-\frac{q}{2}}^{\frac{q}{2}} x^{a_q} \sum_{b_q=-\frac{q}{2}}^{\frac{q}{2}} y^{b_q}
\end{aligned} \tag{5.201}$$

This character was used above in (5.198). The state space obtained by acting with all the P_μ's, without setting $P_\mu P_\mu = 0$ has character

$$\chi_{\tilde{V_+}}(s,x,y) = \mathrm{tr}_{V_+}(s^D x^{J_L} y^{J_R}) = s \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} s^{2p+q} \chi_{q/2}(x) \chi_{q/2}(y) \tag{5.202}$$

The p summation is over the number of powers of P^2. A basis in $\tilde{V}_+$ can be given by multiplying powers of P^2 with traceless products. Doing the sum over p, we find

$$\chi_{\tilde{V_+}}(s,x,y) = s(1 - s^2)^{-1} \chi_{V^+}(s,x,y) \tag{5.203}$$

so that

$$\chi_{V^+}(s,x,y) = s^{-1}(1-s^2)\chi_{\tilde{V}_+}(s,x,y) \tag{5.204}$$

Now by thinking about $\tilde{V}_+$ as isomorphic to the Fock space generated by four oscillators P_μ (which transform in the $(1/2,1/2)$ of $SU(2)\times SU(2)$) it is evident that

$$\chi_{\tilde{V}_+}(s,x,y) = \frac{s}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-\frac{s}{\sqrt{xy}})(1-s\sqrt{\frac{y}{x}})} \equiv sP(s,x,y) \tag{5.205}$$

and so we find

$$\chi_{V^+}(s,x,y) = (1-s^2)P(s,x,y) = s\sum_{q=0}^{\infty} s^q \sum_{a_q=-\frac{q}{2}}^{\frac{q}{2}} x^{a_q} \sum_{b_q=-\frac{q}{2}}^{\frac{q}{2}} y^{b_q} \tag{5.206}$$

Thus, we have the identity

$$\sum_{q=0}^{\infty} s^q \sum_{a_q=-\frac{q}{2}}^{\frac{q}{2}} x^{a_q} \sum_{b_q=-\frac{q}{2}}^{\frac{q}{2}} y^{b_q} = \frac{-(s-s^{-1})}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-\frac{s}{\sqrt{xy}})(1-s\sqrt{\frac{y}{x}})} \tag{5.207}$$

Using this identity, we can now rewrite (5.200) as

$$\mathcal{Z}(s,x,y) = \prod_{i=1}^{\infty}\left(1+\frac{(ts)^i(s^i-s^{-i})}{(1-s^i\sqrt{x^iy^i})(1-s^i\sqrt{\frac{x^i}{y^i}})(1-\frac{s^i}{\sqrt{x^iy^i}})(1-s^i\sqrt{\frac{y^i}{x^i}})}\right)^{-1} \tag{5.208}$$

$$= \sum_{n=0}^{\infty} t^n \chi_n(s,x,y)$$

As we did above, we can define two primary generating functions as follows

$$
\begin{aligned}
Z_n(s,x,y) =& \sum_{\Delta}\sum_{j_1,j_2} \mathcal{N}^{(n)}_{[\Delta,j_1,j_2]} s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y) \\
=& \chi_n(s,x,y)\left(1 - s\sqrt{xy}\right)\left(1 - s\frac{x}{y}\right)\left(1 - s\frac{y}{x}\right)\left(1 - \frac{s}{\sqrt{xy}}\right)
\end{aligned}
\tag{5.209}
$$

and

$$
\begin{aligned}
G_n(s,x,y) =& \sum_{\Delta}\sum_{j_1,j_2} \mathcal{N}^{(n)}_{[\Delta,j_1,j_2]} s^{\Delta}x^{j_1}y^{j_2} \\
=& \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)Z_n(s,x,y)\right]_{\geq}
\end{aligned}
\tag{5.210}
$$

Here $\mathcal{N}^{(n)}_{[\Delta,j_1,j_2]}$ counts the number of primaries of dimension Δ and spins (j_1,j_2) that can be constructed using n matrix fields. We can again specialize the counting to counting leading twist primaries, or to count extremal primaries. The relevant generating function for the counting of extremal primaries is given by

$$
Z^{zw}_n(s,x,y) = s^n \sum_{\Lambda_1,\Lambda_2 \vdash n}\sum_{R,\Lambda\vdash n} Z_{SH}(s\sqrt{xy},\Lambda_1)Z_{SH}(s\sqrt{\frac{x}{y}},\Lambda_2)C(\Lambda_1,\Lambda_2,\Lambda)C(R,R,\Lambda)
\tag{5.211}
$$

This follows from the general counting of matrix gauge invariants in the case where the matrices X_a transform under some global symmetry group G, given in [50]. The resulting Hilbert series, for $n=3$, is

$$
Z^{z,w}_3 = \frac{s^3 Y(s,x,y)}{(1-s\sqrt{\frac{x}{y}})^2(1+s\sqrt{\frac{x}{y}})(-1+s\sqrt{xy})^2(1+s\sqrt{xy})(s^2\frac{x}{y}+s\sqrt{\frac{x}{y}}+1)(1+s\sqrt{xy}+s^2xy)}
\tag{5.212}
$$

$$
\begin{aligned}
Y(s,x,y) =& 3 + 3s^6x^3 + (s\sqrt{x}+s^5x^{\frac{5}{2}})\left(\frac{1}{\sqrt{y}}+\sqrt{y}\right) + (s^2x+s^4x^2)\left(\frac{1}{y}+5+y\right) \\
&+ s^3x^{\frac{3}{2}}\left(\frac{1}{y^{\frac{3}{2}}}+\frac{5}{\sqrt{y}}+5\sqrt{y}+y^{\frac{3}{2}}\right)
\end{aligned}
\tag{5.213}
$$

This counts the total number of primaries we can build from 3 matrix fields. We can refine this counting by specifying the trace structure. Schematically, the primaries we study have the form

$$\mathcal{O} = \sum_{\vec{n},\vec{m}} c_{\vec{n},\vec{m}} \partial_{z_1}^{n_1} \partial_{w_1}^{m_1} \phi^{i_1}_{i_{\sigma(1)}} \partial_{z_2}^{n_2} \partial_{w_2}^{m_2} \phi^{i_2}_{i_{\sigma(2)}} \partial_{z_3}^{n_3} \partial_{w_3}^{m_3} \phi^{i_3}_{i_{\sigma(3)}} \Big|_{z_k=z, w_k=w} \tag{5.214}$$

i.e. they are specified by allowing derivatives to act on some gauge invariant operator specified by the permutation $\sigma \in S_n$. After we translate to the polynomial language, primaries are specified by polynomials in n variables z_i and w_i, as well as by the trace structure, i.e. they are functions on the space

$$\frac{(\mathbb{C}^2)^n}{\mathbb{C}^2} \times S_n \tag{5.215}$$

These functions have to be invariant under an action of $\gamma \in S_n$

$$\gamma : (w_I, z_J, \sigma) \to (w_{\gamma(I)}, Z_{\gamma(I)}, \gamma^{-1}\sigma\gamma) \qquad \gamma \in S_n \tag{5.216}$$

Modding out by this symmetry we find the primaries are functions on the space

$$\frac{((\mathbb{C}^2)^n \times S_n)}{(\mathbb{C}^2 \times S_n)} \tag{5.217}$$

We can also obtain a description by fixing a specific permutation, and then dividing by those permutations γ that fix σ. Lets work out this description for $n = 3$. For primaries obtained by acting with derivatives on $\mathrm{Tr}(\phi)^3$, $\sigma = (1)(2)(3)$ which is left invariant by $\gamma \in S_3$. Thus, we need to consider

$$\frac{(\mathbb{C}^2)^3}{(\mathbb{C}^2 \times S_3)} \tag{5.218}$$

We need to project to the trivial of S_3 and hence

$$Z^{zw}_{\mathrm{Tr}(\phi)^3} = s^3 \sum_{\Lambda \vdash 3} Z_{SH}(s\sqrt{xy}, \Lambda) Z_{SH}(s\sqrt{\frac{x}{y}}, \Lambda) \tag{5.219}$$

$$= \frac{s^3\left(1 + s^2 x + s^4 x^2 + s^6 x^3 + s^3 x^{\frac{3}{2}}\left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)\right)}{(1 - s\sqrt{xy})^2(1 + s\sqrt{xy})^2(-1 + s\sqrt{xy})^2(s^2\frac{x}{y} + s\sqrt{\frac{x}{y}} + 1)(1 + s\sqrt{xy} + s^2 xy)}$$

For primaries obtained by acting with derivatives on $\mathrm{Tr}(\phi^2)\mathrm{Tr}(\phi)$, we can choose $\sigma = (12)(3)$ which is left invariant by $S_2 \times S_1$. Thus we need to consider

$$\frac{(\mathbb{C}^2)^3}{(\mathbb{C}^2 \times S_2 \times S_1} \tag{5.220}$$

where S_2 contains permutations of (z_1, w_1) and (z_2, w_2). Thus, we need to project to the trivial $(\yng(2), \yng(1))$ of the $S_2 \times S_1$ subgroup. This representation is subduced once by $\yng(3)$ and once by $\yng(2,1)$. Thus

$$Z^{zw}_{\mathrm{Tr}(\phi^2)\mathrm{Tr}(\phi)} = s^3 Z_{SH}(s\sqrt{xy}, \yng(3) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(3)) + 2s^3 Z_{SH}(s\sqrt{xy}, \yng(2,1)) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(2,1)) \tag{5.221}$$

$$+ s^3 Z_{SH}(s\sqrt{xy}, \yng(1,1,1) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(1,1,1)) + s^3 Z_{SH}(s\sqrt{xy}, \yng(3) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(2,1))$$

$$+ s^3 Z_{SH}(s\sqrt{xy}, \yng(2,1) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(3)) + s^3 Z_{SH}(s\sqrt{xy}, \yng(2,1) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(1,1,1))$$

$$+ s^3 Z_{SH}(s\sqrt{xy}, \yng(1,1,1) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(2,1))$$

$$= \frac{s^3(1 + s^2 x)}{(1 - s\sqrt{xy})^2(1 + s\sqrt{xy})(1 + s\sqrt{xy})^2(1 + s\sqrt{xy})}$$

For primaries obtained by acting with derivatives on $\mathrm{Tr}(\phi^3)$, we can take $\sigma = (123)$ which is left invariant by Z_3. Thus, we need to consider

$$\frac{(\mathbb{C}^2)^3}{(\mathbb{C}^2 \times Z_3)} \tag{5.222}$$

where Z_3 is the group comprising $\{1, (123), (132)\}$. We need to project to the trivial of Z_3. The trivial of Z_3 is subduced once by $\yng(3)$ and once by, $\yng(1,1,1)$. Thus

$$
\begin{aligned}
Z^{zw}_{(\mathrm{Tr}\phi^3)} =& s^3 Z_{SH}(s\sqrt{xy}, \yng(3) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(3)) + 2s^3 Z_{SH}(s\sqrt{xy}, \yng(2,1)) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(2,1)) \qquad (5.223)\\
&+ s^3 Z_{SH}(s\sqrt{xy}, \yng(1,1,1) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(1,1,1)) + s^3 Z_{SH}(s\sqrt{xy}, \yng(3)) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(1,1,1))\\
&+ s^3 Z_{SH}(s\sqrt{xy}, \yng(1,1,1) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(3))\\
=& \frac{s^3\Big(1 + s^4x^2 - (s\sqrt{x} + s^3x^{\frac{3}{2}})(\frac{1}{\sqrt{y}} + \sqrt{y}) + s^2x(\frac{1}{y} + 3 + y)\Big)}{(1 - s\sqrt{\frac{x}{y}})^2(1 - s\sqrt{xy})^2(s^2\frac{x}{y} + s\sqrt{\frac{x}{y}} + 1)(1 + s\sqrt{xy} + s^2xy)}
\end{aligned}
$$

Note that

$$
Z^{zw}_3 = Z^{zw}_{\mathrm{Tr}(\phi)^3} + Z^{zw}_{\mathrm{Tr}(\phi^2)\mathrm{Tr}(\phi)} + Z^{zw}_{\mathrm{Tr}(\phi^3)} \tag{5.224}
$$

as it must be. The permutation quotient geometry which includes all trace structures is

$$
\frac{(\mathbb{C}^2)^n \times S_n}{(\mathbb{C}^2 \times S_n)} \tag{5.225}
$$

This has an $SU(2)$ action. We can again look at functions which are annihilated by J_+. Let G_+ be the subalgebra of $GL(2,\mathbb{C})$ generated by J_+. The Hilbert series in this case is G^{zw}_n. The algebra of functions annihilated by J_+ corresponds to functions on

$$
\frac{(\mathbb{C}^2)^n \times S_n}{(\mathbb{C}^2 \times S_n \times G_+)} \tag{5.226}
$$

It is again possible to establish the palidromic property for the Hilbert series relevant for the matrix case. In the matrix case, we have the counting function

$$
Z^{zw}_n(q_1, q_2) = s^n \sum_{\Lambda_1, \Lambda_2 \vdash n} \sum_{R \vdash n} C(\Lambda_1, \Lambda_2, \Lambda) C(R, R, \Lambda) Z_{SH}(q_1, \Lambda_1) Z_{SH}(q_2, \Lambda_2) \tag{5.227}
$$

The symmetry under $q_1 \leftrightarrow q_2$, equivalently $x \to x, y \to y^{-1}$ is clear. Now apply inversion

$$\begin{aligned}
Z_n^{zw}(q_1^{-1}, q_2^{-1}) =& s^n \sum_{\Lambda_1, \Lambda_1 \vdash n} \sum_{R \vdash n} C(\Lambda_1, \Lambda_2, \Lambda) C(R, R, \Lambda) Z_{SH}(q_1^{-1}, \Lambda_1) Z_{SH}(q_2^{-1}, \Lambda_2) \\
&\qquad (5.228) \\
=& s^n (q_1 q_2)^{n-1} \sum_{\Lambda_1, \Lambda_2 \vdash n} \sum_{R \vdash n} C(\Lambda_1, \Lambda_2, \Lambda) C(R, R, \Lambda) Z_{SH}(q_1, \Lambda_1^T) Z_{SH}(q_2, \Lambda_2^T) \\
=& s^n (q_1 q_2)^{n-1} \sum_{\Lambda_1, \Lambda_2 \vdash n} \sum_{R \vdash n} C(\Lambda_1^T, \Lambda_2^T, \Lambda) C(R, R, \Lambda) Z_{SH}(q_1, \Lambda_1) Z_{SH}(q_2, \Lambda_2) \\
=& s^n (q_1 q_2)^{n-1} \sum_{\Lambda_1, \Lambda_2 \vdash n} \sum_{R \vdash n} C(\Lambda_1, \Lambda_2, \Lambda) C(R, R, \Lambda) Z_{SH}(q_1, \Lambda_1) Z_{SH}(q_2, \Lambda_2) \\
=& (q_1 q_2)^{n-1} Z_n^{zw}(q_1, q_2).
\end{aligned}$$

5.7 Summary and Outlook

We mapped the algebraic problem of constructing primary fields in the quantum field theory of a free scalar field ϕ in four dimensions to one of finding polynomial functions on $(\mathbb{R}^4)^n$ subject to constraints involving Laplace's equation on each factor, a condition of invariance under translations by the diagonal $\mathbb{R}^4$ and an S_n symmetry related to the bosonic statistics of the elementary field (5.31). By considering holomophic solutions to the Laplacian conditions, we mapped the primary fields to functions on the complex orbifold

$$(\mathbb{C}^2)^n/(\mathbb{C}^2 \times S_n) \tag{5.229}$$

We showed that this space has a palindromic Hilbert series and is Calabi-Yau. We generalized the discussion to the quantum field theory of free vector fields $\phi_i^j(x)$ in the large N limit and found that the orbifold

$$(\mathbb{C}^2)^{2n}/(\mathbb{C}^2 \times S_n[S_2]) \tag{5.230}$$

plays an analogous role. We established the palindromy property. We then considered the free matrix scalar in four dimensions $\phi_i^j(x)$ again in the large N limit. The orbifold is now

$$((\mathbb{C}^2)^n \times S_n)/(\mathbb{C}^2 \times S_n) \tag{5.231}$$

We established the palindromy of the Hilbert series.

In this chapter we have focused on the explicit construction of extremal primary fields. However, the formulation of the problem of constructing general primary fields given in (5.31), as a system of equations for harmonic polynomal functions on $(\mathbb{R}^4)^n$, should be useful beyond the extremal sector. In this more general case, we have to include non-holomorphic solutions to the harmonic constraints-solving this simultaneously with the symmetry and translation constraints proves surprisingly tricky. In this case, we do not expect the ring structure of the extremal primaries to survive. Our preliminary investigations indicate that this most general problem has a graph-theoretic formulation, which will be interesting to exploit. At the level of counting these primaries, we still have the full expressions for the $so(4,2)$ characters of $\mathrm{Sym}^n(V_+)$ which, once expanded in terms of irreducible representations, will in principle yield the counting for the general case. However finding explicit expressions analogous to (5.82) or (5.86) looks challenging. It would very interesting to explore the possible application of the higher spin symmetries and twistor space variables of [69, 70] in shedding light on this problem. It is interesting to note that symmetric group representation theoretic questions close to (but not identical) to the ones we have used have played a role in the discussion of higher spin symmetries in [55]. Some recent mathematical results on these symmetric group multiplicities are in [71]. A number of immediate generalizations of the current work are: free fermions, gauge fields, the free limit of QCD and supersymmetric theories. Some of the early constructions of primary fields - in the $SL(2)$ sector which is a special case of the extremal operators we considered were done in the context of deep inelastic scattering in QCD (see for example the review [72]). It will be fascinating to explore QCD applications of the holomorphic primaries considered here. The explicit enumeration and construction of superconformal primary fields in $N = 4$ SYM will give a better understanding of the dual $AdS_5 \times S_5$ background. While the map between branes and geometries in the half-BPS sector of the bulk and the half-BPS states in $\mathcal{N} = 4$ SYM[73, 74, 75] is reasonably well understood, there are important open problems, most notably in the sector of sixteenth BPS states [76] but also in the quarter and eighth-BPS sectors (some progress on branes states in these sectors is in [77, 78, 61, 79, 80, 81, 82, 83]). A better understanding of operators with derivatives is a step in the direction of a more complete picture of the duality map in general. The construction of holomorphic primaries for the 1-matrix case should admit, without much diffculty, generalization to multi-matrix systems and more generally to quiver theories by combining the methods of the present chapter with those of [43, 50, 84, 85, 86, 87, 88]. Another natural direction is to consider correlators involving the extremal primary fields and the determination of anomalous dimensions for these fields at the Wilson-Fischer fixed point using the techniques of [18].

Chapter 6

Counting and Construction of Free Fermion Primary Operators in CFT4

This chapter is basically an extension of the previous chapter. We extend the analysis of the previous chapter to the fermion CFT. We follow the same approach, using representation theory to derive a general generating function for the number of primary operators constructed from using n-copies of the left hand or right hand spinors. We use this generating function to obtain the correct counting of primary operators. We then translate the problem of constructing primary operators from n-copies of the fundamental spinor, into a problem of determining a multi-variable polynomial that obeys a number of algebraic and differential constraints. Focussing on extremal primaries we find that these primary operators display the same Calabi-Yau geometries as in the free field scalar operator case. The work carried out in this chapter has been submitted for publication in [89].

6.1 Introduction

The remarkable success of the conformal bootstrap[90, 91, 92, 93] suggests that algebraic structures present in conformal field theory (CFT) can profitably be exploited to extract highly nontrivial information about the CFT. In the papers [41, 94] a systematic approach towards manifesting and exploiting some of these algebraic structures was outlined. The key result is that the algebraic structure of CFT defines a two dimensional topological field theory (TFT2) with $SO(4,2)$ invariance. Crossing symmetry is expressed as associativity of the algebra of local CFT operators. A basic observation which is at the heart of this result, is that the free four dimensional CFT of a scalar field can be formulated as an infinite dimensional associative algebra. This algebra admits a decomposition into linear representations of $SO(4,2)$, and is equipped with a non-degenerate bilinear product. A concrete application of these ideas has enabled a systematic study of primaries in bosonic free field theories in four dimensions, for scalar, vector and matrix models[40, 19]. For closely related ideas see [95].

We know from the AdS/CFT correspondence[1, 26, 25] that strongly coupled CFTs have a dual holographic gravitational description. The combinatorics of the matrix model Feynman diagrams plays an important role in holography. In this setting the TFT2 structure also appears as a powerful organizing structure, explicating algebraic structures that were not previously appreciated[42, 43, 44, 45]. Thus, it seems that the TFT2 idea is rich enough to incorporate the algebraic structure emerging both from the conformal symmetry, and from the color combinatorics.

In this chapter we extend the study of [40, 19] by carrying out a systematic study of primaries in free fermion field theories in four dimensions. In section 6.2 we obtain formulae for the counting of primary fields constructed from n copies of the fundamental fermion, using the characters of representations of $so(4,2)$. For a beautiful discussion of these characters, see [17]. By specializing the particular classes of primaries, we can make the counting formulae very explicit. These special classes of primaries obey extremality conditions stated using relations between the charges under the Cartan subgroup of SO(4, 2)). The construction of primary fields is then mapped to a problem of determining multi-variable polynomials subject to a system of algebraic and differential constraints. This relies on a function space realization of the conformal algebra, which is explained in section 6.3. We give concrete examples of polynomials obeying the constraints and the associated primary operators. Finally, in the last section we verify that the Hilbert series for the counting of extremal primaries are palindromic. The palindromy property of Hilbert series is indicative that the ring being enumerated is Calabi-Yau. It it interesting that palindromic Hilbert series also arise for moduli spaces of supersymmetric vacua of gauge theories, as found in [52, 53].

6.2 Counting Primaries

We enumerate the $SO(4,2)$ irreducible representations appearing among the composite fields made out of $n = 2, 3, \cdots$ copies of a free chiral fermion field. The fermions are Grassman fields, so there is a sign change when two fields are swapped. Consequently, we should be taking the antisymmetric product of the $SO(4,2)$ representations. Enumerating the primaries entails decomposing, the antisymmetrized tensor product $\mathrm{Asym}^n(W_+)$ into irreducible representations, where $W_+ = \mathcal{D}_{[\frac{1}{2},0]+}$ in the notation of [17]. After obtaining a general formula in terms of an infinite product, we specialize to primaries that obey extremality conditions, that relate their dimension to their spin. For these primaries using results from [50], we find simple explicit formulas for the counting.

6.2.1 Generalities

The basic formula we use in this section states

$$\det(1 + tM) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M) \tag{6.1}$$

where $\chi_{(1^n)}(M)$ is the trace over the antisymmetrized product of n copies of M. From formula (3.44) of [17] we know the character of a left handed Weyl fermion is

$$\begin{aligned}\chi_{W_+}(s,x,y) =& s^{\frac{3}{2}}(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y))P(s,x,y) \\ =& s^{\frac{3}{2}} \sum_{q=0}^{\infty} s^q \chi_{\frac{q+1}{2}}(x)\chi_{\frac{q}{2}}(y) \\ =& \mathrm{Tr}_{W_+}(M)\end{aligned} \tag{6.2}$$

with $M = s^D x^{J_{3,L}} y^{J_{3,R}}$. It is straightforward to verify that

$$\det(1 + tM) = \prod_{q=0}^{\infty} \prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} (1 + ts^{\frac{3}{2}+q} x^a y^b) \tag{6.3}$$

Applying (6.1) we find the generating function of the characters of the antisymmetrized tensor products of the free Weyl fermion representation

$$\mathcal{Z}(t,s,x,y) = \prod_{q=0}^{\infty} \prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} (1 + ts^{\frac{3}{2}+q} x^a y^b) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(s,x,y) \tag{6.4}$$

By expanding $\mathcal{Z}(t,s,x,y)$ as a series in t we can easily read off the character of the antisymmetrized tensor products of n copies of the free Weyl fermion representation $\chi_{(1^n)}(s,x,y)$, as the coefficient of t^n. To be completely clear, $\chi_{(1^n)}(s,x,y)$ is the character of M in the representation given by the antisymmetrized tensor product $\mathrm{Asym}^n(W_+)$. The next step is to decompose this into a sum of $SO(4,2)$ characters, for irreps of dimension Δ and spins j_L, j_R

$$\chi_{(1^n)}(s,x,y) = \sum_{[\Delta,j_L,j_R]} N_{[\Delta,j_L,j_R]}\chi_{[\Delta,j_L,j_R]}(s,x,y) \tag{6.5}$$

The coefficients $N_{[\Delta,j_1,j_2]}$ count how many times irrep $\mathcal{A}_{[\Delta,j_1,j_2]}$ (in the notation of [17]) appears in $\mathrm{Asym}^n(W_+)$. Hence, $N_{[\Delta,j_1,j_2]}$ are non-negative integers. The case that $n=2$ is subtle because some of irreps appearing in the above decomposition are short. We will consider $n=2$ separately in detail below. For $n \geq 3$ we have[17]

$$\chi_{[\Delta,j_1,j_2]}(s,x,y) = \frac{s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})} \tag{6.6}$$

It is useful to define

$$Z_n(s,x,y) \equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y) \tag{6.7}$$

so that

$$Z_n(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\ \chi_{(1^n)}(s,x,y) \tag{6.8}$$

The right hand side of (6.7) is a sum of (products of) $SU(2)$ characters. Following [60], it can be simplified by using the orthogonality of $SU(2)$ characters. The result is most easily stated in terms of the generating function

$$\begin{aligned} G_n(s,x,y) &= \left[(1-\frac{1}{x})(1-\frac{1}{y})Z_n(s,x,y)\right]_{\geq} \\ &= \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} x^{j_1} y^{j_2} \end{aligned} \tag{6.9}$$

The subscript $\geq$ is an instruction to keep only non negative powers of x and y.

It is easy to check that this agrees with standard character computations.

For example, the expansion

$$G_3(s,x,y) = s^{\frac{11}{2}}x\sqrt{y} + s^{\frac{13}{2}}x^{\frac{5}{2}} + s^{\frac{13}{2}}x^{\frac{3}{2}}y + s^{\frac{15}{2}}y^{\frac{3}{2}} + s^{\frac{15}{2}}x^3y^{\frac{3}{2}} + s^{\frac{15}{2}}x^2y^{\frac{3}{2}} + s^{\frac{17}{2}}x^{\frac{7}{2}}y \\ + s^{\frac{17}{2}}x^{\frac{3}{2}}y^2 + s^{\frac{17}{2}}x^{\frac{5}{2}}y^2 + s^{\frac{19}{2}}x^4y^{\frac{3}{2}} + s^{\frac{19}{2}}xy^{\frac{5}{2}} + 2s^{\frac{19}{2}}x^3y^{\frac{5}{2}} + s^{\frac{19}{2}}x^4y^{\frac{5}{2}} + ... \tag{6.10}$$

can be reproduced using characters. The relevant Schur polynomial for this case is calculated as follows

$$\chi_{(1^3)}(s,x,y) = \frac{1}{6}\left[(\chi_L(s,x,y))^3 - 3\chi_L(s^2,x^2,y^2)\chi_L(s,x,y) + 2\chi_L(s^3,x^3,y^3)\right] \tag{6.11}$$

Using Mathematica, we find the following terms

$$\begin{aligned}\chi_{(1^3)}(s,x,y) = {} & \mathcal{A}_{[\frac{11}{2},1,\frac{1}{2}]} + \mathcal{A}_{[\frac{13}{2},\frac{5}{2},0]} + \mathcal{A}_{[\frac{13}{2},\frac{3}{2},1]} \\ & + \mathcal{A}_{[\frac{15}{2},0,\frac{3}{2}]} + \mathcal{A}_{[\frac{15}{2},2,\frac{3}{2}]} + \mathcal{A}_{[\frac{15}{2},3,\frac{3}{2}]} \\ & + \mathcal{A}_{[\frac{17}{2},\frac{7}{2},1]} + \mathcal{A}_{[\frac{17}{2},\frac{3}{2},2]} + \mathcal{A}_{[\frac{17}{2},\frac{5}{2},2]} \\ & + \mathcal{A}_{[\frac{19}{2},4,\frac{3}{2}]} + +\mathcal{A}_{[\frac{19}{2},1,\frac{5}{2}]} + 2\mathcal{A}_{[\frac{19}{2},3,\frac{5}{2}]} + \mathcal{A}_{[\frac{19}{2},4,\frac{5}{2}]} \\ & + \mathcal{A}_{[\frac{21}{2},\frac{9}{2},0]} + \mathcal{A}_{[\frac{21}{2},\frac{9}{2},2]} + \mathcal{A}_{[\frac{21}{2},\frac{3}{2},3]} + \mathcal{A}_{[\frac{21}{2},\frac{5}{2},3]} + \mathcal{A}_{[\frac{21}{2},\frac{7}{2},3]} \\ & + \mathcal{A}_{[\frac{21}{2},\frac{9}{2},3]} + ...\end{aligned} \tag{6.12}$$

in complete agreement with (6.10).

The case that $n = 2$ is complicated by the fact that representations that include null states appear in the decomposition. The condition for a short multiplet[20] is $\Delta = f(j_1) + f(j_2)$ with $f(j) = 0$ if $j = 0$ or $f(j) = j + 1$ if $j > 0$. For $n = 2$ the decomposition includes a primary with $\Delta = 3$ and $j_1 = j_2 = 0$ which is not short, as well as primaries with $\Delta = 2j$ $j_1 = (2j-1)/2$ and $j_2 = (2j-3)/2$ which are short representations and hence have null states. These null states (and their descendants) must be removed. These short representations arise because their primary operators are conserved higher spin currents

$$\partial_\mu J^{\mu\mu_2\cdots\mu_j} = 0 \tag{6.13}$$

The subtraction of null states is achieved by removing the $\Delta = 3$ primary that does not need to be subtracted, dividing by $1 - s/\sqrt{xy}$ which removes the null descendents and then putting the original primary back in. In the end we have

$$G_2(s,x,y) = \left[(1-\frac{1}{x})(1-\frac{1}{y})\left(Z_2(s,x,y) - s^3\right)\frac{1}{1-\frac{s}{\sqrt{xy}}}\right]_{\geq} + s^3$$
$$= \sum_{j=0}^{\infty} s^{3+2j} x^{\frac{3}{2}+j} y^{\frac{1}{2}+j} \tag{6.14}$$

This is indeed the correct result[96].

6.2.2 Leading Twist Primaries

By restricting to well defined classes of primaries, we can significantly simplify the counting formulas of the previous section. The biggest simplification comes from focusing on the leading twist primaries, which have quantum numbers $[\Delta, j_1, j_2] = [\frac{n(n+2)}{2} + q, \frac{n(n+1)}{4} + \frac{q}{2}, \frac{n(n-1)}{4} + \frac{q}{2}]$. Each such primary operator comes in a complete spin multiplet of $(\frac{n(n+1)}{2} + q + 1)(\frac{n(n-1)}{2} + q + 1)$ operators. Choosing the operator with highest spin corresponds to studying primaries constructed using a single component P_z of the momentum four vector operator. To count the leading twist primaries we can count this highest spin operator in each multiplet. The corresponding generating function is $G_n^{\max}(s,x,y)$. This generating function is obtained after a simple modification of the results of the previous section. First, we replace $\chi_{Asym^n(V)}(s,x,y)$ with a new function $\chi_n^{\max}(s,x,y)$, by keeping only the highest spin state from each multiplet in the product

$$\prod_{q=0}^{\infty}(1 + t s^{\frac{3}{2}+q} x^{q+\frac{1}{2}} y^q) = \sum_{n=0}^{\infty} t^n \chi_n^{\max}(s,x,y) \tag{6.15}$$

The leading twist primaries are constructed using a single component of the momentum, that raises left and right spin maximally. Consequently in (6.8) we replace

$$(1 - s\sqrt{xy})(1 - s\sqrt{\frac{x}{y}})(1 - s\sqrt{\frac{y}{x}})(1 - \frac{s}{\sqrt{xy}}) \to (1 - s\sqrt{xy}) \tag{6.16}$$

Finally, for each spin multiplet we keep only 1 state so there is no longer any need to replace the multiplet of spin states by a single state when we count.

The final result is

$$\begin{aligned}G_n^{\max}(s,x,y) =& (1-s\sqrt{xy})\chi_n^{\max}(s,x,y) \\ =& \sum_{\Delta,j_1,j_2} N^{\max}_{[\Delta,j_1,j_2]} s^{\Delta} x^{j_1} y^{j_2}\end{aligned} \tag{6.17}$$

where $N^{\max}_{[\Delta,j_1,j_2]}$ is the number of leading twist primaries of dimension Δ and spin (j_1,j_2). For the leading twist primaries, once n and the dimension of the operator is specified, the spin of the primary is fixed. Consequently, we need not track the x and y dependence. This leads to the formula

$$\sum_{n=0}^{\infty} t^n G_n^{\max}(s) = (1-s)\prod_{q=0}^{\infty}(1+ts^{\frac{3}{2}+q}) \equiv (1-s)F(t,s) \tag{6.18}$$

We can obtain explicit expressions for $G_n^{\max}(s)$ by developing $F(t,s)$ in a Taylor series. Define

$$f_q(t,s) = \frac{\partial^q}{\partial t^q}\log F(t,s) \tag{6.19}$$

Straight forward computation gives

$$f_q(t,s) = \sum_{k=0}^{\infty} \frac{(-1)^{q+1}(q-1)! s^{\frac{3q}{2}+kq}}{(1+ts^{k+\frac{3}{2}})^q} \tag{6.20}$$

so that, after reinstating x and y, we have

$$f_k(0,s,x,y) = (k-1)!(-1)^{k-1}\frac{s^{\frac{3k}{2}}x^{\frac{k}{2}}}{1-s^k x^{\frac{k}{2}} y^{\frac{k}{2}}} \tag{6.21}$$

Explicit expressions for $G_n^{\max}$ are now easily obtained. For example

$$\begin{aligned}G_3^{\max}(s,x,y) =& \frac{1}{3!}(1-s\sqrt{xy})\frac{\partial^3 F}{\partial t^3}\Big|_{t=0} \\ =& \frac{1}{3!}(1-s\sqrt{xy})(f_3+3f_1f_2+f_1^3) \\ =& \frac{s^{\frac{15}{2}}x^3y^{\frac{3}{2}}}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})}\end{aligned} \tag{6.22}$$

Similarly

$$G_4^{\max}(s,x,y) = \frac{s^{12}x^5y^3}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)} \tag{6.23}$$

It is possible to obtain a general closed formula for $G_n^{\max}(s)$. To make the argument as transparent as possible, again set $x = 1 = y$. Evaluate the derivative

$$\frac{\partial^n F}{\partial t^n} = \sum_{n_1,\cdots,n_q} \sum_{k_1,\cdots,k_q} \frac{(n_1 k_1 + \cdots + n_q k_q)!}{n_1! \cdots n_q! (k_1!)^{n_1} \cdots (k_q!)^{n_q}} f_{k_1}^{n_1} \cdots f_{k_q}^{n_q} \delta_{n, n_1 k_1 + \cdots n_q k_q} F \tag{6.24}$$

and use the formulas for the f_k's to find

$$\frac{\partial^n F}{\partial t^n}\Big|_{t=0} = \sum_{n_1,\cdots,n_q} \sum_{k_1,\cdots,k_q} \frac{(-1)^{n-\sum_i n_i} n! s^{\frac{3n}{2}}}{n_1! \cdots n_q! \, k_1^{n_1} \cdots k_q^{n_q}} \left(\frac{s^{\frac{3k_1}{2}}}{1-s^{k_1}}\right)^{n_1} \cdots \left(\frac{s^{\frac{3k_q}{2}}}{1-s^{k_q}}\right)^{n_q} \delta_{n, n_1 k_1 + \cdots n_q k_q} \tag{6.25}$$

Notice that this is a sum over conjugacy classes of S_n. The conjugacy class collects permutations with n_q k_q-cycles. This interpretation follows because the coefficient

$$\frac{n!}{n_1! \cdots n_q! \, k_1^{n_1} \cdots k_q^{n_q}} \tag{6.26}$$

is the order of the conjugacy class. Each conjugacy class is weighted by the factor $(-1)^{n-\sum_i n_i}$ which is the signature of the permutation with n_q k_q-cycles. There is a factor of $\frac{s^{\frac{3k}{2}}}{1-s^k}$ for each k-cycle in the permutation. The lowest weight discrete series irrep of $SL(2)$, built on a ground state with dimension $\frac{3}{2}$ has character

$$\chi_1(s) = \mathrm{Tr}_{V_1}(s^{L_0}) = \frac{s^{\frac{3}{2}}}{1-s} \tag{6.27}$$

Denote this irrep by W_1. It then follows that ($P_{[1^n]}$ projects onto the antisymmetric irrep i.e. a single column of n boxes)

$$\frac{1}{n!} \frac{\partial^n F}{\partial t^n}\Big|_{t=0} = \mathrm{Tr}_{W_1}(P_{[1^n]} s^{L_0}) = \frac{s^{\frac{n}{2}(n+2)}}{(1-s)(1-s^2)(1-s^3)\cdots(1-s^n)} \tag{6.28}$$

where the last equality follows from eqn (49) of [50], where these $SL(2)$ sector primaries were studied in the language of oscillators. We now easily find

$$G_n^{\max}(s,x,y) = (s\sqrt{xy})^{\frac{n(n-1)}{2}}(s^{\frac{3}{2}}\sqrt{x})^n \prod_{k=2}^{n} \frac{1}{1-(s\sqrt{xy})^k} \tag{6.29}$$

6.2.3 Extremal Primaries

We now consider the class of primaries with charges

$$\Delta = \frac{3n}{2} + q \ \ ; \ \ J_3^L = \frac{n}{2} + \frac{q}{2} \tag{6.30}$$

The charge J_3^R, which is part of $SU(2)_R$, is not constrained. These primaries fill out complete multiplets of $SU(2)_R$. They are constructed using two components of the momentum four vector operator which are complex linear combinations of the (hermitian) P_μ. Introduce a generating function $G_n^{z,w}(s,x,y)$, given by

$$G_n^{z,w}(s,x,y) = \left[\left(1-\frac{1}{y}\right) Z_n^{z,w}(s,x,y)\right]_{\geq} \tag{6.31}$$

where $Z_n(s,x,y)$ is defined by

$$Z_n^{z,w}(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{x/y})\chi_n(s,x,y) \tag{6.32}$$

with

$$\sum_{n=0}^{\infty} t^n \chi_n(s,x,y) = \prod_{q=0}^{\infty} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} (1+ts^{\frac{3}{2}+q}x^{\frac{q+1}{2}}y^b) \equiv F_2(t,s,x,y) \tag{6.33}$$

It is again possible to derive closed expressions for the generating functions $Z_n^{z,w}(s,x,y)$ and $G_n^{z,w}(s,x,y)$. Introduce the functions

$$\begin{aligned} f_k(t,s,x,y) \equiv& \frac{\partial^{k-1}}{\partial t^{k-1}} \log F_2 \\ =&(-1)^{k-1}(k-1)! \sum_{q=0}^{\infty} \sum_{m=-\frac{q}{2}}^{\frac{q}{2}} \frac{s^{kq+\frac{3k}{2}} x^{\frac{(q+1)k}{2}} y^{km}}{(1+ts^{q+\frac{3}{2}}x^{\frac{q+1}{2}}y^m)^k} \end{aligned} \tag{6.34}$$

It is simple to establish that

$$f_k(0,s,x,y) = (-1)^{k-1}(k-1)!\frac{s^{\frac{3k}{2}}x^{\frac{k}{2}}}{(1-s^k x^{\frac{k}{2}}y^{\frac{k}{2}})(1-s^k x^{\frac{k}{2}}y^{-\frac{k}{2}})} \tag{6.35}$$

Exactly as above we have

$$\frac{\partial^n F_2}{\partial t^n}\Big|_{t=0} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{(n_1k_1+\cdots+n_qk_q)!}{n_1!\cdots n_q!(k_1!)^{n_1}\cdots(k_q!)^{n_q}} f_{k_1}^{n_1}\cdots f_{k_q}^{n_q}\delta_{n,n_1k_1+\cdots n_qk_q} \tag{6.36}$$

Inserting the formulas for the f_k's expressions for the $Z_n(s,x,y)$ now follows from (6.32). To extract spin multiplets, we need to compute

$$G_n^{z,w}(z,w) = \left[Z_n(s,x,y)\left(1-\frac{1}{y}\right)\right]_{\geq} = \frac{1}{2\pi i}\oint_C dz\frac{\left(1-\frac{1}{z^2}\right)Z_n(s,x,z^2)}{z-\sqrt{y}} \tag{6.37}$$

As an example, the generating functions counting the extremal primaries constructed from 3 fields are given by

$$Z_3^{z,w}(s,x,y) = s^{\frac{9}{2}}x^{\frac{3}{2}}\frac{s^3x^{\frac{3}{2}}y^{\frac{3}{2}}(1+s\sqrt{\frac{x}{y}}+s^2\frac{x}{y}) + s^3\frac{x^{\frac{3}{2}}}{y^{\frac{3}{2}}}(1+s\sqrt{xy}+s^2xy)}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-\frac{s^2x}{y})(1-\frac{s^3x^{\frac{3}{2}}}{y^{\frac{3}{2}}})} \tag{6.38}$$

$$\begin{aligned}G_3^{z,w}(s,x,y) =& \frac{s^{\frac{13}{2}}x^{\frac{5}{2}}(1+s\sqrt{x}y^{\frac{3}{2}})}{(1-s^4x^2)(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})}\\ =& s^{\frac{13}{2}}x^{\frac{5}{2}} + s^{\frac{15}{2}}x^3y^{\frac{3}{2}} + s^{\frac{17}{2}}x^{\frac{7}{2}}y + s^{\frac{19}{2}}x^4y^{\frac{3}{2}} + s^{\frac{19}{2}}x^4y^{\frac{5}{2}} + s^{\frac{21}{2}}x^{\frac{9}{2}}\\ &+ s^{\frac{21}{2}}x^{\frac{9}{2}}y^2 + s^{\frac{21}{2}}x^{\frac{9}{2}}y^3 + \cdots\end{aligned} \tag{6.39}$$

6.3 Construction

In this section we will explain how the counting of the previous section can be used to provide concrete formulas for the construction of the primary operators in the free fermion CFT. For the leading twist counting this is manifest.

For the counting of extremal primaries, we will argue that our formulas can naturally be phrased as counting the multiplicities of the symmetric groups representations. The quantities being counted are then easily constructed using projectors onto these representations. In this analysis, a polynomial representation of $SO(4,2)$ will play an important role. This representation is described in the next subsection, after which we describe the construction of leading twist primaries and then extremal primaries.

6.3.1 Polynomial rep

We use the following representation of $SO(4,2)$

$$K_\mu = \frac{\partial}{\partial x^\mu} \tag{6.40}$$

$$D = \left(x \cdot \frac{\partial}{\partial x} - \frac{3}{2}\right) \tag{6.41}$$

$$M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + \mathcal{M}_{\mu\nu} \tag{6.42}$$

$$P_\mu = (x^2 \frac{\partial}{\partial x^\mu} - 2x_\mu x \cdot \frac{\partial}{\partial x} + 3x_\mu + 2x^\nu \mathcal{M}_{\mu\nu}) \tag{6.43}$$

In the formula above we should replace $\mathcal{M}_{\mu\nu}$ by the relevant matrix representing the spin part of the conformal group. In Minkowski spacetime we have (the two possibilities correspond to taking either a left handed $(\frac{1}{2}, 0)$ or a right handed $(0, \frac{1}{2})$ spinor)

$$\mathcal{M}^{\mu\nu} = \sigma^{\mu\nu} \qquad \text{or} \qquad \bar{\sigma}^{\mu\nu} \tag{6.44}$$

where

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)_\alpha{}^\beta \tag{6.45}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{6.46}$$

and

$$\sigma^{\mu}{}_{\alpha\dot{\beta}} = (\mathbf{1}, \vec{\sigma}) \qquad\qquad \bar{\sigma}^{\mu\dot{\beta}\alpha} = (\mathbf{1}, -\vec{\sigma}) \tag{6.47}$$

In Euclidean space we have

$$\mathcal{M}^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{1}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}\right) \tag{6.48}$$

or

$$\mathcal{M}^{\mu\nu} = \bar{\sigma}^{\mu\nu} \equiv \frac{1}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right) \tag{6.49}$$

where now

$$\sigma^{\mu} = (-i\vec{\sigma}, \mathbf{1}) \qquad\qquad \bar{\sigma}^{\mu} = (i\vec{\sigma}, \mathbf{1}) \tag{6.50}$$

The generators in Minkowski space close the algebra

$$[M_{\rho\sigma}, M_{\phi\theta}] = \eta_{\theta\rho}M_{\phi\sigma} + \eta_{\phi\sigma}M_{\theta\rho} - \eta_{\theta\sigma}M_{\phi\rho} - \eta_{\phi\rho}M_{\theta\sigma}$$

$$[P_\mu, P_\nu] = 0 = [K_\mu, K_\nu] \qquad [P_\beta, K_\alpha] = 2\eta_{\alpha\beta}D - 2M_{\alpha\beta}$$

$$[M_{\beta\rho}, K_\alpha] = \eta_{\alpha\rho}K_\beta - \eta_{\alpha\beta}K_\rho \qquad [M_{\beta\rho}, P_\alpha] = \eta_{\alpha\rho}P_\beta - \eta_{\alpha\beta}P_\rho$$

$$[D, P_\mu] = P_\mu \qquad [D, K_\mu] = -K_\mu \qquad [D, M_{\mu\nu}] = 0 \tag{6.51}$$

The Euclidean generators obey the same algebra with $\eta_{\mu\nu}$ replaced with $\delta_{\mu\nu}$. States in this representation correspond to polynomials in the spacetime coordinates x_μ times a constant spinor ζ_α, which transforms in the $(\frac{1}{2}, 0)$ if we study the theory of a left handed fermion, or in the $(0, \frac{1}{2})$ if we study a right handed fermion. The 2×2 matrix $\mathcal{M}_{\mu\nu}$ acts on this constant spinor. Further, ζ_α is Grassman valued to account for the fact that the fermions are anticommuting fields. Concretely, each operator corresponds to a state (by the state operator correspondence) and each state corresponds to a polynomial times the spinor (thanks to the representation we have just described)

$$x_{\mu_1} \cdots x_{\mu_k} \zeta_\alpha \tag{6.52}$$

To deal with operators constructed from a product of n copies of the basic fermion field, we consider a "multiparticle system". When we move to the multiparticle system, we have polynomials on the n particle coordinates x^I_μ, times the n particle spinor, obtained by taking the tensor product of n copies of ζ_α

$$(\zeta \otimes \zeta \otimes \cdots \otimes \zeta)_{\alpha_1 \alpha_2 \cdots \alpha_n} \tag{6.53}$$

To write the generator of the conformal group, for the multiparticle system, we need the matrices

$$\mathcal{M}^{(I)}_{\mu\nu} = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathcal{M}_{\mu\nu} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \tag{6.54}$$

where the matrix $\mathcal{M}_{\mu\nu}$ on the right hand side is the 2×2 matrix we introduced above and it appears as the Ith factor on the right hand side. In total $\mathcal{M}^{(I)}_{\mu\nu}$ has n factors. The n-particle representation of $SO(4,2)$ includes

$$K_\mu = \sum_{I=1}^{n} \frac{\partial}{\partial x^I_\mu} \tag{6.55}$$

$$P_\mu = \sum_{I=1}^{n} ((x^{I\rho} x^I_\rho \frac{\partial}{\partial x^I_\mu} - 2x^I_\mu x^I \cdot \frac{\partial}{\partial x^I} + 3x^I_\mu + 2x^{I\,\nu} \mathcal{M}^{(I)}_{\mu\nu}) \tag{6.56}$$

The representations introduced above all have null states. This is to be expected, since the dimension of the free fermion field saturates the unitarity bound. For the $(\frac{1}{2}, 0)$ field in Minkowski space, for example, the null state is exhibited by verifying that

$$\bar{\sigma}^\mu P_\mu \zeta = 0 \tag{6.57}$$

for any choice of ζ. Let us now spell out the conditions that the polynomial $P_\mathcal{O}$ corresponding to an operator $\mathcal{O}$ must obey if the operator $\mathcal{O}$ is a primary operator. The general polynomial $P_\mathcal{O}$ will have spinor indices (it is constructed from a tensor product of copies of ζ) as well as four vector indices inherited from the spacetime coordinates. There are three conditions that must be imposed: Primaries are annihilated by the special conformal generator K_μ

$$[K_\mu, \mathcal{O}] = 0 \tag{6.58}$$

This implies that the corresponding polynomial is translation invariant

$$\sum_{I=1}^{n} \frac{\partial}{\partial x_{\mu}^{I}} P_{\mathcal{O}} = 0 \tag{6.59}$$

Secondly, the equation of motion must be obeyed by each fermionic field. Finally, we require that the polynomials are in the antisymmetric representation of S_n. Since the ζs are Grassman variables, we must impose this condition if we are to get a non-zero primary upon translating back to the language of the fermion field theory.

The above set of constraints on the polynomials corresponding to primaries is not yet very useful. To obtain a more manageable set of constraints, we will motivate replacing the constraint coming from the equation of motion with a constraint that simply requires that each polynomial is holomorphic. Our first observation is that the operator $\bar{\sigma}^{\mu} P_{\mu}$, known as the Cauchy-Fueter operator, has been used to define regular functions of a quarternionic variable. This theory of regular functions is well developed[97]. An important result, is Fueters Theorem[98], which gives a method for constructing Cauchy-Fueter regular functions in terms of holomorphic functions. In view of Fueter's theorem, we will replace the equation of motion constraint with the constraint that the polynomials are holomorphic. Thus, in the end we search for translation invariant, holomorphic polynomials that are in the antisymmetric representation of S_n. We will manage to test that the counting of these polynomials matches the counting of primaries in complete generality, and for a number of examples, we will construct the primary corresponding to a given polynomial and explicitly verify that it is annihilated by K_{μ}.

6.3.2 Leading Twist

The leading twist primaries are given by polynomials in a single complex variable z^{I}, $I = 1, 2, ..., n$. Any such polynomial is automatically holomorphic, so we need not worry about the equation of motion constraint. To solve the translation invariance condition, we work with the hook variables Z^{a}, $a = 1, 2, ..., n-1$ defined by

$$Z^{a} = \frac{1}{\sqrt{a(a+1)}} (z^{(1)} + z^{(2)} + \cdots + z^{(a)} - a z^{(a+1)}) \tag{6.60}$$

Our problem is now reduced to constructing antisymmetric polynomials from the hook variables. By construction, it is clear that the degree k polynomials belong to a subspace of $V_H^{\otimes k}$ of S_n. We can characterize the antisymmetric subspace, that we want to extract, using representation theory. Towards this end, consider the following decomposition in terms of $S_n \times S_k$ irreps

$$V_H^{\otimes k} = \bigoplus_{\Lambda_1 \vdash n,\ \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)} \tag{6.61}$$

In the above expression, $Com(S_n \times S_k)$ is the algebra of linear operators on $V_H^{\otimes k}$ that commute with $S_n \times S_k$. This decompositions has been studied in detail in [50]. The Z variables are commuting so that we need to consider the case that $\Lambda_2 = [k]$ the symmetric representation given by a Young diagram with a single row of k boxes. The resulting multiplicity is given by the coefficient of q^k in

$$\begin{aligned} Z_{SH}(q;\Lambda_1) &= (1-q)\, q^{\frac{\sum_i c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})} \\ &= \sum_k q^k Z_{SH}^k(\Lambda_1) \end{aligned} \tag{6.62}$$

Here c_i is the length of the i'th column in Λ_1, b runs over boxes in the Young diagram Λ_1 and h_b is the hook length of the box b. Evaluating this formula for the antisymmetric representations, for which Λ_1 is a single column, gives[50]

$$\frac{q^{\frac{n}{2}(n-1)}}{(1-q^2)\cdots(1-q^n)} \tag{6.63}$$

After accounting for the dimension of n elementary fermion fields, this is in complete agreement with (6.28).

Now that we have verified that the number of translation invariant, holomorphic polynomials in the antisymmetric representation of S_n agrees with the counting of leading twist primaries, we can move on to a construction formulas for these primaries. Indeed, the relevant polynomials are given by acting with a projector onto the antisymmetric representation, on the hook variables. This polynomial multiplies an anticommuting tensor product of Grassman valued constant spinors. The projector from the tensor product of

k copies of the hook onto the antisymmetric representation of S_n is

$$P_{(1^n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \Gamma_k(\sigma) \tag{6.64}$$

where $\mathrm{sgn}(\sigma)$ is the signature of permutation σ. When acting on a product of variables, say $Z^{(a_1)} Z^{(a_2)} \cdots Z^{(a_k)}$ we have

$$\Gamma_k(\sigma) = \Gamma_{(n-1,1)}(\sigma) \otimes \cdots \otimes \Gamma_{(n-1,1)}(\sigma) \tag{6.65}$$

where on the right hand side we take a tensor product (the usual Kronecker product) of k copies of the matrices of the hook representation of S_n, labeled by a Young diagram with $n-1$ boxes in the first row and 1 box in the second row. Our construction formula is

$$\frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \Gamma_k(\sigma)_{a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_k} Z^{b_1} Z^{b_2} \cdots Z^{b_k} (\zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_n)_{\alpha_1 \cdots \alpha_n} \tag{6.66}$$

The above formula produces an expression of the form $\sum_i \hat{n}_i P_i(Z)$ where $\hat{n}_i$ are unit vectors inside the carrier space of $V_H^{\otimes k}$ and $P_i(Z)$ are the polynomials that correspond to primary operators. To translate polynomials into momenta, the formula [40]

$$z^k \quad \leftrightarrow \quad \frac{(-1)^k P^k}{2^k k!} \tag{6.67}$$

is very useful. We will now gives some examples of polynomials obtained from formula (6.66). We will also translate these polynomials into primary operators.

If we consider $n = 2$ fields, there is a single hook variable given by $Z = z_1 - z_2$. To find a polynomial that is antisymmetric under swapping $1 \leftrightarrow 2$, we must raise Z to an odd power. Thus, we predict that primaries for the fermion fields correspond to the polynomials

$$(z_1 - z_2)^{2s+1} = \sum_{k=0}^{2s+1} \frac{(2s+1)!}{k!(2s-k+1)!} (-1)^k z_1^{2s-k+1} z_2^k \tag{6.68}$$

Translating the polynomial variables into momenta we find the following primary

$$|\psi\rangle = \sum_{k=0}^{2s+1} \frac{(-1)^k}{((2s-k+1)!k!)^2} P^k|\frac{3}{2},\frac{1}{2},0\rangle \otimes P^{2s-k+1}|\frac{3}{2},\frac{1}{2},0\rangle \tag{6.69}$$

where, because our fields are fermions, we have

$$|\frac{3}{2},\frac{1}{2},0\rangle_1 \otimes |\frac{3}{2},\frac{1}{2},0\rangle_2 = -|\frac{3}{2},\frac{1}{2},0\rangle_2 \otimes |\frac{3}{2},\frac{1}{2},0\rangle_1 \tag{6.70}$$

Thus, our expression for the fermionic primaries built from two fields are

$$\sum_{k=0}^{2s+1} \frac{(-1)^k}{((2s-k+1)!k!)^2} (\partial_1 + i\partial_2)^k \psi(x)(\partial_1 + i\partial_2)^{2s-k+1}\psi(x) \tag{6.71}$$

which exactly matches the form of the higher spin currents[99, 100].

For $n = 3$ fields it is easy to see that

$$(z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \tag{6.72}$$

is holomorphic, translation invariant and in the antisymmetric representation of S_3. The corresponding primary operator can be simplified to

$$\psi(x)(\partial_1 + i\partial_2)\psi(x)(\partial_1 + i\partial_2)^2\psi(x) \tag{6.73}$$

It is not difficult to see that this operator is indeed annihilated by K_μ.

6.3.3 Extremal Primaries

In this section we will consider the construction of extremal primaries, which correspond to polynomials in two holomorphic coordinates, z and w. We will characterize these polynomials by two degrees, one for Z and one for W. Polynomials of degree k in Z and of degree l in W belong to a subspace of $V_H^{\otimes k} \otimes V_H^{\otimes l}$ of S_n. The relevant decompositions in terms of $S_n \times S_k$ irreducible representations are

$$\begin{aligned} V_H^{\otimes k} &= \bigoplus_{\Lambda_1 \vdash n, \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)} \\ V_H^{\otimes l} &= \bigoplus_{\Lambda_3 \vdash n, \Lambda_4 \vdash l} V_{\Lambda_3}^{(S_n)} \otimes V_{\Lambda_4}^{(S_l)} \otimes V_{\Lambda_3,\Lambda_4}^{Com(S_n \times S_l)} \end{aligned} \tag{6.74}$$

The tensor product $V_H^{\otimes k} \otimes V_H^{\otimes l}$ is a representation of

$$\mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_l) \tag{6.75}$$

The Z and W variables are commuting so that $\Lambda_2 \otimes \Lambda_4 = [k] \otimes [l]$ is the trivial representation of $S_k \times S_l$. The multiplicity with which a given $S_n \times S_k$ irrep (Λ_1, Λ_2) appears is given by the dimension of the irreducible representation of the commutants $Com(S_n \times S_l)$ in $V_H^{\otimes k}$. Since our polynomials multiply a product of anticommuting Grassman spinors, we want to project to states in $V_H^{\otimes k} \otimes V_H^{\otimes l}$ which are in the totally antisymmetric irreducible representation of the diagonal $\mathbb{C}(S_n)$ in the algebra (6.75). This constrains $\Lambda_3 = \Lambda_1^T$. Thus we find that the number of $S_k \times S_l$ invariants and S_n antisymmetric representations is

$$\sum_{\Lambda_1 \vdash n} \text{Mult}(\Lambda_1^T, [k]; S_n \times S_k) \;\; \text{Mult}(\Lambda_1, [l]; S_n \times S_l) \tag{6.76}$$

Thus, for the number of primaries constructed from z_i, w_i we get

$$\sum_{\Lambda_1 \vdash n} Z_{SH}^k(\Lambda_1) Z_{SH}^l(\Lambda_1^T) \tag{6.77}$$

The above integer gives the number of primaries in the free fermion CFT, of weight $\frac{3n}{2} + k + l$, with spin $(J_3^L, J_3^R) = (\frac{k+l+n}{2}, \frac{k-l}{2})$. The generating function $Z_n^{z,w}(s, x, y)$ which encodes all k, l is given by

$$Z_n^{z,w}(s, x, y) = s^{\frac{3n}{2}} x^{\frac{n}{2}} \sum_{\Lambda \vdash n} Z_{SH}(s\sqrt{xy}, \Lambda) Z_{SH}(s\sqrt{\frac{x}{y}}, \Lambda^T) \tag{6.78}$$

where Λ is a partition of n and we can use the formula (6.62). It is straight forwards to check, for example, that

$$\begin{aligned} Z_n^{z,w}(s, x, y) =& s^{\frac{9}{2}} x^{\frac{3}{2}} \Big(Z_{SH}(s\sqrt{xy}, \yng(3)) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(1,1,1)) + Z_{SH}(s\sqrt{xy}, \yng(2,1)) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(2,1)) \\ & + Z_{SH}(s\sqrt{xy}, \yng(1,1,1)) Z_{SH}(s\sqrt{\frac{x}{y}}, \yng(3)) \Big) \end{aligned} \tag{6.79}$$

reproduces (6.38).

For $n = 3$ fields, it is easy to see that the polynomials

$$w_3(z_2 - z_1) + w_2(z_1 - z_3) + w_1(z_3 - z_2) \tag{6.80}$$

and

$$2w_1w_2z_1^2 - w_2^2z_1^2 - 2w_1w_3z_1^2 + w_3^2z_1^2 - 2w_1^2z_1z_2 + 2w_2^2z_1z_2 + 4w_1w_3z_1z_2$$

$$- 4w_2w_3z_1z_2 + w_1^2z_2^2 - 2w_1w_2z_2^2 + 2w_2w_3z_2^2 - w_3^2z_2^2 + 2w_1^2z_1z_3 - 4w_1w_2z_1z_3$$

$$+ 4w_2w_3z_1z_3 - 2w_3^2z_1z_3 + 4w_1w_2z_2z_3 - 2w_2^2z_2z_3 - 4w_1w_3z_2z_3 + 2w_3^2z_2z_3$$

$$- w_1^2z_3^2 + w_2^2z_3^2 + 2w_1w_3z_3^2 - 2w_2w_3z_3^2 \tag{6.81}$$

are holomorphic, translation invariant and in the antisymmetric representation of S_3. To translate these polynomials into primary operators, we use the dictionary

$$z^k \quad \leftrightarrow \quad \frac{(-1)^k P_z^k}{2^k k!} \qquad w^k \quad \leftrightarrow \quad \frac{(-1)^k P_w^k}{2^k k!} \tag{6.82}$$

where we have set $P_z = P_1 - iP_2$ and $P_w = P_3 - iP_4$. After a little work we finally obtain the following two primary operators

$$\psi(x)P_z\psi(x)P_w\psi(x) \tag{6.83}$$

and

$$2P_wP_z^2\psi(x)P_w\psi(x)\psi(x) + 2P_z\psi(x)P_w^2P_z\psi(x)\psi(x)$$

$$+P_w^2\psi(x)P_z^2\psi(x)\psi(x) + 4P_wP_z\psi(x)P_z\psi(x)P_w\psi(x) \tag{6.84}$$

6.4 Geometry

In this section we comment on the permutation orbifolds relevant for the combinatorics of the fermion primaries. The leading twist primaries are holomorphic polynomials in n complex variables. We mod out by translations and restrict to the antisymmetric representation of S_n, so that the leading twist primaries correspond to holomorphic polynomial functions on

$$(\mathbb{C})^n/(\mathbb{C} \times S_n) \tag{6.85}$$

A very similar argument shows that extremal primaries correspond to holomorphic polynomial functions on

$$(\mathbb{C})^{2n}/(\mathbb{C}^2 \times S_n) \tag{6.86}$$

We will now argue that the Hilbert series of the fermionic primaries are counted by palindromic Hilbert series, suggesting that they are Calabi-Yau. We leave a more detailed study of these issues for the future. A palindromic Hilbert series obeys

$$Z_n^{z,w}(q_1^{-1}, q_2^{-1}) = (q_1 q_2)^{n-1} Z_n^{z,w}(q_1, q_2) \tag{6.87}$$

Our Hilbert series $Z_n^{z,w}(q_1, q_2)$ enjoy this transformation property. To demonstrate this, our starting point is the formula

$$Z_n^{z,w}(q_1, q_2) = s^{\frac{3n}{2}} x^{\frac{n}{2}} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda) Z_{SH}(q_2, \Lambda^T) \tag{6.88}$$

where we have introduced the variables $q_1 = s\sqrt{xy}, q_2 = s\sqrt{x/y}$. This has the property $Z_n^{z,w}(q_1, q_2) = Z_n^{z,w}(q_2, q_1)$. This follows because exchange of q_1, q_2 amounts to the inversion of y, and by using the identity [40]

$$Z_{SH}(q^{-1}, \Lambda) = (-q)^{n-1} Z_{SH}(q, \Lambda^T) \tag{6.89}$$

Using this result

$$\begin{aligned} Z_n^{z,w}(q_1^{-1}, q_2^{-1}) =& s^n (q_1 q_2)^{n-1} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda^T) Z_{SH}(q_2, \Lambda) \\ =& s^n (q_1 q_2)^{n-1} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda) Z_{SH}(q_2, \Lambda^T) \\ =& (q_1 q_2)^{n-1} Z_n^{z,w}(q_1, q_2) \end{aligned} \tag{6.90}$$

The results of section (4.3) of [40] now imply that the Hilbert series $G_n^{z,w}(s, x, y)$ also exhibit the palindromy property.

6.5 Summary and Outlook

Previous studies [40] have explained how to map the algebraic problem of constructing primary fields in the quantum field theory of a free scalar field ϕ in four dimensions to one of finding polynomial functions on $(\mathbb{R}^4)^n$ that are harmonic, translation invariant and which are in the trivial representation of S_n. In this chapter, we have extended this construction to describe primary fields in the free quantum field theory of a single Weyl fermion. Concrete results achieved with this new point of view include a complete counting formula for the complete set primary fields, explicit counting formulas (Hilbert series) for counting special classes of primaries, as well as detailed construction formulas for these primary operators. We have also established the palindromy of the Hilbert series.

One weak point in our analysis, that warrants further study, is the treatment of the constraint coming from the equation of motion. Motivated by results for Cauchy-Fueter regular functions, we simply stated that we will consider holomorphic polynomials. This has been verified explicitly, by checking that this leads to the correct number of primaries and further that when the polynomials are translated back into the operator language, that we do indeed obtain operators annihilated by K_μ. It would however be nice to perform a detailed analysis of the equation of motion constraint, which has to be carried out before the complete class of primaries can be treated.

Immediate generalizations of the current work include studies of CFTs which include gauge fields. The free limit of QCD and supersymmetric theories would be good starting points. Indeed, early constructions of primary fields in the $SL(2)$ sector (leading twist primaries) were performed in the context of deep inelastic scattering in QCD (see for example the review [72]). Do the holomorphic primaries considered here have QCD applications? Explicit enumeration and construction of superconformal primary fields in $\mathcal{N} = 4$ SYM will give a better understanding of the dual $AdS_5 \times S^5$ background. Finally, another natural direction is to consider correlators involving the extremal primary fields and the determination of anomalous dimensions for these fields at the Wilson-Fischer fixed point using the techniques of [18, 46, 47, 48, 49].

Chapter 7

Conclusions

The thesis was motivated by the AdS/CFT correspondence. It focuses on the correspondence that relates a free conformal field theory in 4 dimensions to a quantum gravity (higher spin) theory in 5 dimensions, in a negatively curved spacetime. We preferred to study the CFT4 side of the correspondence since it is free and hence a solvable theory. Computations carried out in [2, 17] gave insight into how we should formulate the CFT4 problem. We begin with a counting of primary operators followed by the construction of primary operators. The construction translates a problem of constructing primary operators into a problem of constructing a mult-variable polynomial obeying algebraic and symmetry constraints.

For the counting, we start by writing the character of the free scalar field as a representation of $SO(4,2)$[17]. We start by taking the $SO(4,2)$ character of n of the fields and symmetrizing to get the $\text{Sym}^n(V_+)$ representation. This is done because the scalar fields obey bosonic statistics. Expanding this character as a sum of characters of irreducible representations, we obtain the counting for the primary operators. Analysing the spectrum of 2 copies of the scalar field, shows no degeneracies. However the spectrum of $n > 2$ copies of scalar fields contain primary operators which are degenerate. The degeneracy is between primary operators having the same scaling dimension, left and right hand spins. This indicates that the spectrum of $n > 2$ copies of the scalar field is much richer than that of $n = 2$ copies of the scalar field.

Focussing on the extremal or leading twist primaries, in (5.116) we obtained a symmetric group interpretation of the counting and demonstrated agreement with the group representation theory results. The generating function of the leading twist primaries forms a palindromic Hilbert series. The relevant primary operators with their corresponding harmonic and translation invariant functions, are functions on the Calabi-Yau orbifolds

$$(\mathbb{C}^n/\mathbb{C} \times \mathbb{C}^n/\mathbb{C})/S_n = (\mathbb{C}^2)^n/(\mathbb{C}^2 \times S_n). \tag{7.1}$$

It becomes difficult to obtain the whole counting of primary operators from (5.48) as n increases, especially when n is beyond $n = 5$. This could be attributed to the lack of effective mathematica code.

Employing the same counting and construction strategy developed for scalar fields to the vector models Φ^{I_i}, we find similar results. The generating function for the vector model primaries again form a palindromic Hilbert series and the primary operator polynomial function is a function on the Calabi-Yau orbifold

$$(\mathbb{C}^2)^{2n}/(\mathbb{C}^2 \times S_n[S_2]) = (\mathbb{C}^{2n}/\mathbb{C} \times \mathbb{C}^{2n}/\mathbb{C})/S_n[S_2]. \tag{7.2}$$

Employing the same counting and construction to matrix models ϕ_i^j we again find a palindromic Hilbert series. The primary operator polynomial functions are functions on a geometric Calabi-Yau orbifold of the form

$$(\mathbb{C}^n/\mathbb{C} \times \mathbb{C}^n/\mathbb{C} \times S_n)/S_n = ((\mathbb{C}^2) \times S_n)/(\mathbb{C}^2 \times S_n). \tag{7.3}$$

In the last chapter we extended these methods of counting and construction to the Weyl spinors. We formulate the counting problem by taking a tensor product of n copies of Weyl spinors. We then map these products of n copies into a totally antisymmetric subspace $\mathrm{Antisym}^n(V_+)$. This is done because the Weyl spinors obey fermi statistics. Decomposing in terms of irreducible representations produces the counting for the primary operators. Restricting ourselves to the extremal or leading twist primaries, in (6.78) we managed to again interpret the counting in terms of permutation algebras. Again we observe that the generating functions of the leading twist primaries form a palindromic Hilbert series. For the construction of the fermionic primaries, the algebraic constraints require the primary polynomial to be translation invariant and holomorphic. The holomorphic requirement is motivated by the Cauchy-Feuter Theorem[98]. The primary polynomial functions of the Weyl spinor CFT has a geometric Calabi-Yau structure given by

$$(\mathbb{C})^{2n}/(\mathbb{C}^2 \times S_n). \tag{7.4}$$

The weak part of this construction is that the general constraint implied by the equation of motion has not been understood. Further exploration of this point is needed. Translating the primary polynomial back to the operator language, the primary operator is indeed annihilated by the special conformal operator K_μ, which confirms the construction is correct.

The future direction of this work is to extend it to superconformal field theories ($\mathcal{N} = 4$ SYM) that is, to theories that contain gauge fields. By considering gauge theories such as $\mathcal{N} = 4$ SYM we will be moving closer to studying theories of nature and AdS/CFT. We can also extend this work to study gauge theories with interactions at the zero of the beta function where the theory will be conformal invariant. We could also extend this work to theories such as the UV fixed point of the Gross-Neveu model in $2 + \epsilon$ dimensions. Another good example of an interacting theory that we could consider is to consider correlators involving the extremal primary operators and determining the anomalous dimensions for these fields at the Wilson-Fischer fixed point using techniques of [18, 46, 47, 48, 49]. A more challenging future direction for this work is to consider these computation in general d dimension instead of 4 dimensions.

Appendix A

Introduction to Hilbert Series

Since Hilbert series may not be familiar to a physics audience, we will in this section introduce the basic ideas in a series of examples.

A.1 Hilbert Series

Define the polynomial ring $\mathbb{R}[X]$, in X over the field $\mathbb{R}$ as the set of the polynomials

$$P(X) = p_0 + p_1 X + p_2 X^2 + \cdots + p_q X^q + \cdots ; \tag{A.1}$$

where $p_a \in \mathbb{R}$, $a = 0, 1, \cdots, q \cdots$. If we take the variables X to be any m^{th} root of unity, we know that

$$X = e^{\frac{2\pi i}{m}} \qquad m \in \mathbb{Z}. \tag{A.2}$$

Thus $X^m = 1$. This relation severly limits the independent monomials we can form. Indeed, the complete set is given by

$$\{1, X, X^2, \cdots, X^{m-1}\}. \tag{A.3}$$

In this example we say the ring is generated by a single generator X subjected to a single relation $X^m = 1$. The Hilbert series of a graded ring is a function that counts the number of independent monomials that can be formed. For the free conformal field theory we have graded using dimension and the two

Lorentz spins. Here our grading is the degree of the monomial. The Hilbert series is given by a rational function whose numerator encodes constraints and whose denominator encodes the generators. For the case at hand

$$
\begin{aligned}
Hs(t) =& \frac{1-t^m}{1-t} \\
=& \sum_{n=0}^{\infty} c_n t^n,
\end{aligned}
\tag{A.4}
$$

where c_n counts the number of monomials of degree n. In our example

$$
Hs(t) = 1 + t + \cdots + t^{m-1} \tag{A.5}
$$

corresponding to the fact that there is a single monomial for each degree starting from 0 to $m-1$.

A.2 Hilbert series on S^1 and S^2

Functions defined on S^1 are functions of θ that are periodic

$$
h(\theta) = h(\theta + 2\pi), \tag{A.6}
$$

We can embed S^1 in $\mathbb{R}^2$ by using the co-ordinates

$$
x = \cos\theta \qquad y = \sin\theta. \tag{A.7}
$$

Equivalently that we can work on $\mathbb{R}^2$ as long as we impose the constraint $x^2 + y^2 = 1$. Noting that

$$
h_m = (x \pm iy)^m = e^{\pm im\theta}, \tag{A.8}
$$

we see that a complete set of monomials is given by

$$
1, (x \pm iy), (x \pm iy)^2, (x \pm iy)^3, \cdots, (x \pm iy)^m, \cdots. \tag{A.9}
$$

The generators of the ring are x and y, and they obey a single constraint (relation) which says

$$x^2 + y^2 = 1. \tag{A.10}$$

We will again grade the ring by the degree of the polynomial. Since we have two generators x and y, and one relation of degree 2, our Hilbert series in this case is

$$\begin{aligned} Hs(t) =& \frac{1-t^2}{(1-t)^2} \\ =& 1 + 2t + 2t^2 + 2t^3 + 2t^4 + \cdots \end{aligned} \tag{A.11}$$

The coefficients of 2's in the above expansion of the Hilbert series indicates that there are 2 independent monomials at each degree above zero. These monomials take the form

$$(x+iy)^m \qquad (x-iy)^m. \tag{A.12}$$

We will now determine the Hilbert series on a two sphere S^2. Functions defined on S^2 are functions of θ and ϕ that are periodic

$$f(\theta, \phi) = f(\theta + 2\pi, \phi). \tag{A.13}$$

We embed the sphere in $\mathbb{R}^3$ with co-ordinates

$$x = \sin\phi\cos\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \cos\phi. \tag{A.14}$$

that obey the constraint $x^2 + y^2 + z^2 = 1$. A basis for functions on the sphere is given by the spherical harmonics $Y_{ml}(\theta, \phi)$,

$$Y_{ml}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-1)!}{(l+1)!}}P_l^m(\cos\theta)e^{im\phi} \tag{A.15}$$

where for each value of l, $m = -l, l+1, \cdots, l-1, l$. The generators of the ring are x, y and z. The equation that constrains these generators is

$$x^2 + y^2 + z^2 = 1, \tag{A.16}$$

which is of degree 2. Since our ring is generated by 3 generators with a single degree 2 relation, therefore the graded ring Hilbert series is

$$\begin{aligned} Hs(t) =& \frac{1-t^2}{(1-t)^3} \\ =& 1 + 3t + 5t^2 + 7t^3 + 9t^4 + \cdots . \end{aligned} \tag{A.17}$$

We have anticipated this counting in our discussion of the spherical harmonics (Y_{ml}). We know that for each degree l there are $(2l+1)$ independent monomials we can construct. The table below shows, for finite l, the different kinds of monomials we can get.

Monomial degree	$l=0$	$l=1$	$l=2$	$l=3$
Number of monomials $(2l+1)$	1	3	5	7
Types of monomials	Y_{00}	Y_{11}, Y_{10}, Y_{1-1}	Y_{22} ,Y_{21} , Y_{20}, Y_{2-1}, Y_{2-2}	Y_{33}, Y_{32}, Y_{31}, Y_{30}, Y_{3-1}, Y_{3-2}, Y_{3-3}

A.3 Hilbert series on S^3

Considering the polynomial ring defined on S^3. There are $(\lambda+1)^2$ monomials for each degree λ. We can understand where the multiplicity $(\lambda+1)^2$ comes from, by counting the number of components of a symmetric traceless tensor $T^{\mu_1 \cdots \mu_\lambda}$ of degree λ in $d = 4$ dimensions. We use the Young diagrams to compute the number of components;

$$
\begin{aligned}
\text{Number of components} =& \frac{\begin{array}{|c|c|c|c|c|}\hline 4 & 4{+}1 & \cdots & \cdots & 4{+}\lambda{-}1 \\ \hline\end{array}}{\begin{array}{|c|c|c|c|c|}\hline \lambda & \lambda{-}1 & \cdots & \cdots & 1 \\ \hline\end{array}} - \frac{\begin{array}{|c|c|c|c|c|}\hline 4 & 4{+}1 & \cdots & \cdots & 4{+}\lambda{-}3 \\ \hline\end{array}}{\begin{array}{|c|c|c|c|c|}\hline \lambda{-}2 & \lambda{-}3 & \cdots & \cdots & 1 \\ \hline\end{array}} \qquad \text{(A.18)} \\
=& \frac{(4+\lambda-1)!}{(4-1)!\lambda!} - \frac{(4+\lambda-3)!}{(4-1)!(\lambda-2)!} \\
=& \frac{4(4+1)(4+2)\cdots(4+\lambda-3)}{3!(\lambda-2)!}\left(\frac{(2+\lambda)(3+\lambda)}{\lambda(\lambda-1)} - 1\right) \\
=& (\lambda+1)^2.
\end{aligned}
$$

We can embed S^3 in $\mathbb{R}^4$ as follows

$$
\begin{aligned}
x_1 =& \sin\theta_2 \sin\theta_1 \cos\phi \qquad \text{(A.19)} \\
x_2 =& \sin\theta_2 \sin\theta_1 \sin\phi \\
x_3 =& \sin\theta_2 \cos\phi \\
x_4 =& \cos\theta_2.
\end{aligned}
$$

These coordinates obey the constraint

$$
x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \qquad \text{(A.20)}
$$

The ring is generated by 4 generators x_1, x_2, x_3 and x_4, and these generators have a single degree 2 relation. Taking this into account, the Hilbert series is

$$
\begin{aligned}
Hs(t) =& \frac{1-t^2}{(1-t)^4} \qquad \text{(A.21)} \\
=& 1 + 4t + 9t^2 + 16t^3 + 25t^4 + 36t^5 + \cdots .
\end{aligned}
$$

The counting for the different kinds of monomials implied by the Hilbert series is the same counting that we obtained by counting the number of components in a symmetric traceless tensor.

We can now infer the Hilbert series in S^d. In S^d we have coordinates in

$\mathbb{R}^{d+1}$ that obey the constrain

$$x_1^2 + x_2^2 + \cdots + x_d^2 + x_{d+1}^2 = 1. \tag{A.22}$$

The ring is generated by $d+1$ generators $x_1, x_2, \cdots, x_{d+1}$. Since the ring is generated by $d+1$ generators with a single degree 2 relation, the Hilbert series

$$Hs(t) = \frac{1-t^2}{(1-t)^d}. \tag{A.23}$$

The number of monomials are determined from computing the number of components of a symmetric traceless tensor $T^{\mu_1\mu_2\cdots\mu_\lambda}$ in $d+1$ dimensions,

$$\begin{aligned} \text{Number of components} =& \frac{(d+\lambda)!}{d!\lambda!} - \frac{(d+\lambda-2)!}{d!(\lambda-2)!} \\ =& \frac{(d+\lambda-2)!}{(d-1)!\lambda!}\Big(d+2\lambda-1\Big) \end{aligned} \tag{A.24}$$

Hence

$$\begin{aligned} Hs(t) =& \frac{1-t^2}{(1-t)^d} \\ =& 1 + (d+1)t + \frac{d}{2}(d+3)t^2 + \cdots . \end{aligned} \tag{A.25}$$

Appendix B

Unpacking the Counting Formula $Z_n^{z,w}(s,x,y)$

In this appendix we are going to show how to compute (5.116) from chapter 5. Consider the equation

$$Z_n^{z,w}(s,x,y) = \sum_{\Lambda_1 \vdash n} Z_{SH}(s\sqrt{xy}, \Lambda_1) Z_{SH}(s\sqrt{\frac{x}{y}}, \Lambda_1), \tag{B.1}$$

where

$$Z_{SH}(q, \Lambda_1) = (1-q) q^{\sum_i \frac{c_i(c_i-1)}{2}} \prod_b \frac{1}{1-q^{h_b}}, \tag{B.2}$$

and c_i is the length of the Young diagram column, h_b is the hook length of the box number b in a Young diagram. For $n = 3$ (B.1) becomes

$$\begin{aligned} Z_3^{z,w}(s,x,y) =& Z_{SH}(\sqrt{xy}, \ydiagram{1,1,1}) Z_{SH}(\sqrt{\frac{x}{y}}, \ydiagram{1,1,1}) + Z_{SH}(\sqrt{xy}, \ydiagram{2,1}) Z_{SH}(\sqrt{\frac{x}{y}}, \ydiagram{2,1}) \\ &+ Z_{SH}(\sqrt{xy}, \ydiagram{3}) Z_{SH}(\sqrt{\frac{x}{y}}, \ydiagram{3}). \end{aligned} \tag{B.3}$$

Let $q_1 = s\sqrt{xy}$ and $q_2 = s\sqrt{\frac{x}{y}}$, then

$$\begin{aligned} Z_{SH}(q_1, \ydiagram{1,1,1}) &= (1-q_1) q_1^{\frac{3(3-1)}{2}} \prod_b \frac{1}{1-q_1^{h_b}} \\ &= (1-q_1) q_1^3 \frac{1}{1-q_1^3} \frac{1}{1-q_1^2} \frac{1}{1-q_1} \\ &= \frac{q_1^3}{1-q_1^3} \frac{1}{1-q_1^2}, \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} Z_{SH}(q_1, \ydiagram{2,1}) =& (1-q_1) q_1 \prod_b \frac{1}{1-q^{h_b}} \\ =& (1-q_1) q_1 \frac{1}{1-q_1^3} \left(\frac{1}{1-q_1} \right)^2 \\ =& \frac{q_1}{1-q_1^3} \frac{1}{1-q_1} \\ =& \frac{q_1(q_1+1)}{1-q_1^3} \frac{1}{1-q^2}. \end{aligned} \tag{B.5}$$

Then

$$\begin{aligned} Z_{SH}(q_1, \ydiagram{3}) =& (1-q_1) \prod_b \frac{1}{1-q_1^{h_b}} \\ =& (1-q_1) \frac{1}{1-q_1^3} \frac{1}{1-q_1^2} \frac{1}{1-q_1} \\ =& \frac{1}{1-q_1^3} \frac{1}{1-q_1^2}. \end{aligned} \tag{B.6}$$

Therefore

$$Z_3^{z,w}(s,x,y) = \left(\frac{q_1^3}{1-q_1^3} \frac{1}{1-q_1^2} \right) \left(\frac{q_2^3}{1-q_2^3} \frac{1}{1-q_2^2} \right) + \left(\frac{q_1(q_1+1)}{1-q_1^3} \frac{1}{1-q_1^2} \right) \left(\frac{q_2(q_2+1)}{1-q_2^3} \frac{1}{1-q_2^2} \right) \tag{B.7}$$

$$\begin{aligned} &+ \left(\frac{1}{1-q_1^3} \frac{1}{1-q_1^2} \right) \left(\frac{1}{1-q_2^3} \frac{1}{1-q_2^2} \right) \\ =& \frac{(s^6 x^3 + s^4 x^2 + s^2 x + 1 + s^3 x^{\frac{3}{2}} (\sqrt{y} + \frac{1}{\sqrt{y}}))}{(1-s^2 xy)(1-s^3 (xy)^{\frac{3}{2}})(1-s^2 \frac{x}{y})(1-s^3 (\frac{x}{y})^{3/2})}. \end{aligned}$$

Finally, we should multiply by s^3 to account for the fact that ϕ^3 has $\Delta = 3$.

Appendix C

Review of Characters Methods

In this Appendix we give a quick review of some background from the theory of characters. This will help to orient the reader for our counting methods in free CFT.

C.1 Example using $SU(2)$ characters

To illustrate the idea involved in computing the CFT characters we begin by deriving the usual rules for the addition of angular momentum in quantum mechanics. In non-relativistic quantum mechanics we know that the product of a wave function with spin j_1 with a wave function with spin j_2, gives a wave function with possible spins j in the range $|j_1 - j_2| \leq j \leq j_1 + j_2$,

$$j_1 \otimes j_2 = \oplus_{|j_1-j_2|}^{j_1+j_2} \; j$$

Rotations are generated by the three components of angular momentum. We call these the generators of angular momentum and we call the commutator algebra the Lie algebra of rotations

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k.$$

We obtain the elements of the group by exponentiating the group generators. Denote $g = e^{i\theta J_3}$ with $J_3 \in \mathrm{su}(2)$ and denote $x = e^{i\theta}$. From our knowledge of angular momentum in quantum mechanics, the character for the spin j irrep is

$$\chi_j(g) = Tr(e^{i\theta J}) \tag{C.1}$$

$$= \sum_{m=-j}^{j} \langle m|e^{i\theta J}|m\rangle \tag{C.2}$$

$$= \sum_{m=-j}^{j} \langle m|e^{i\theta m}|m\rangle \tag{C.3}$$

$$= x^j + x^{j-1} + \cdots + x^{-j+1} + x^{-j}$$

$$= \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \tag{C.4}$$

The product of two characters assuming $j_2 > j_1$, is given by

$$\begin{aligned}
\chi_{j_1}(x)\chi_{j_2}(x) =& (x^{j_1} + x^{j_1-1} + \cdots + x^{-j_1+1} + x^{-j_1})\frac{x^{j_2+\frac{1}{2}} - x^{-j_2-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
=& \sum_{k=-j_1}^{j_1} \frac{x^{j_2+k+\frac{1}{2}} - x^{-j_2+k-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
=& \sum_{k=-j_1}^{j_1} \frac{x^{j_2+k+\frac{1}{2}} - x^{-j_2-k-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
=& \sum_{k=j_2-j_1}^{j_2+j_1} \frac{x^{k+\frac{1}{2}} - x^{-k-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} = \sum_{k=j_2-j_1}^{j_2+j_1} \chi_k(x)
\end{aligned} \tag{C.5}$$

This illustrates the approach we will adopt to compute characters in a CFT. If we have the characters of $SO(4,2)$ we can easily compute the products of irreps in the CFT. In particular, denoting the representation that the free scalar field belongs to by V, we want to decompose the character for $\chi_{Sym(V^{\otimes n})}$ into a sum over characters of irreps. Here we have the symmetric product $Sym(V^{\otimes n})$ because we must respect the bosonic statistics of the field. We will project to the symmetric product by employing Young projectors. The projector to the symmetric product of n copies of V is

$$P = \frac{1}{n!}\sum_{\sigma\in S_n} \sigma$$

where σ acts on $V^{\otimes n}$ by permuting the factors in the tensor product and we have used the fact that the character for the symmetric representation is 1 for all group elements.

C.2 Characters of $SO(4,2)$

In the paper [17] Dolan has computed the characters we need. The basic result we will use is given by

$$\chi_V(s,x,y) = s(1-s^2)\sum_{p,q=0}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \tag{C.6}$$

The character is a trace of some group element. Lets start by spelling out what group element is being traced. Inside $SO(4,2)$ we have the maximal compact subgroup $SO(2)\times SO(4)$. The generator of $SO(2)$ is the dilatation operator D. We can decompose $SO(4)$ into $SU(2)_L \times SU(2)_R$. Extract $J_{3,L} \in SU(2)_L$ and $J_{3,R} \in SU(2)_R$. Using the generators $D, J_{3,L}, J_{3,R}$ define the group element

$$g = e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}, \tag{C.7}$$

every irrep of $SO(4,2)$ is built on a primary field and we label the irrep by the quantum numbers of the primary. The scalar field has $[D, J_{3,L}, J_{3,R}] = [1,0,0]$. The character quoted in (C.6) is for group element g (C.7). An extra point to be aware of is that since $SO(4,2)$ is not compact, it can have null states. The free scalar has a null state since $\partial^\mu\partial_\mu\phi = 0$. The factor $(1-s^2)$ subtracts this null state and its descendants out. Towards this end we show the maximal compact subgroup of $SO(4,2)$ is $SO(2)\times SO(4)$, and we can decompose $SO(4)$ into $SU(2)_L \times SU(2)_R$.

Consider the interval

$$ds^2 = dt_1^2 + dt_2^2 - (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \tag{C.8}$$

which is invariant under $SO(4,2)$. Since we boost to frames with $v < c$, any transformations (boosts) that mixes time components with spatial components will not form a compact subgroup of $SO(4,2)$. However, transformations that mix only time (or space) components will form compact subgroups. In particular, $SO(2)$ (which mixes the time components) and $SO(4)$ (which mixes the spatial components) will form compact subgroups. Therefore, the maximal compact subgroup of $SO(4,2)$ will be a direct product of these two subgroups.

We can decompose group elements in $SO(4)$ into group elements of $SU(2) \times SU(2)$. First, note that we can write any group element g in $SU(2)$ as

$$g = e^{-i\bar{\theta}\cdot\bar{J}}, \tag{C.9}$$

where $\bar{\theta}$ stands for three parameters and $J_1 = \frac{\sigma_1}{2}, J_2 = \frac{\sigma_2}{2}$, and $J_3 = \frac{\sigma_3}{2}$ (σ_i are the Pauli matrices). These generators close the Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{C.10}$$

Now, we can parametrize $SO(4)$ using the following six parameters and six generators:

$$r = e^{-i\bar{a}\cdot\bar{A} - i\bar{b}\cdot\bar{B}} \tag{C.11}$$

where

$$A_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, A_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{C.12}$$

$$A_3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, B_1 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \tag{C.13}$$

$$B_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, B_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \tag{C.14}$$

The benefit of using these generators is in their commutation relations:

$$[A_i, A_j] = i\epsilon_{ijk} A_k, [B_i, B_j] = i\epsilon_{ijk} B_k, [A_i, B_j] = 0. \tag{C.15}$$

These are the same commutation relations as $SU(2)$. Therefore, $\bar{A}$ and $\bar{B}$ generates subgroups of $SO(4)$ that are equivalent to $SU(2)$. $\bar{A}, \bar{B}$ are 4 dimensional representations (where as J_i is 2 dimensional). Through a change of basis using

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & -i & i & 0 \end{pmatrix}, S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & -1 & i \\ -1 & -i & 0 & 0 \end{pmatrix} \tag{C.16}$$

observe that

$$S^{-1}\bar{A}S = \bar{J} \otimes \mathbf{1}, \qquad S^{-1}\bar{B}S = \mathbf{1} \otimes \bar{J}, \tag{C.17}$$

where $\mathbf{1} \in SO(4)$. We can therefore rewrite r as

$$S^{-1}rS = e^{-i\bar{a}\cdot\bar{J}} \otimes e^{-i\bar{b}\cdot\bar{J}},$$

where $e^{-i\bar{a}\cdot\bar{J}}, e^{-i\bar{b}\cdot\bar{J}} \in SU(2)$.
The character is a trace of the group element g of the group. Thus it can be written as

$$\mathrm{Tr}(e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}) = \sum_i \langle i|e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}|i\rangle \tag{C.18}$$

$$= s(1-s^2) \sum_{p,q=0}^{\infty} s^{2p+q} \chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \tag{C.19}$$

One of the states appearing in the sum over i is the primary operator. The remaining states are from the descendents in the representation.

Expanding the character (C.18) by writing the sum out and expanding LHS and RHS up to order s^3, we can illustrate the equivalence of both equations. Begin by expansion on the RHS,

$$
\begin{aligned}
\chi_V(s,x,y) =& s(1-s^2)\sum_{p,q}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \\
=& s\sum_{q}^{\infty} s^{q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \\
=& s(1+s\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y)+s^2\chi_1(x)\chi_1(y)+s^3\chi_{\frac{3}{2}}(x)\chi_{\frac{3}{2}}(y)+\cdots) \\
=& s\Big(1+s(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+s^2(x^1+1+x^{-1})(y^1+1+y^{-1}) \\
& +s^3(x^{\frac{3}{2}}+x^{\frac{1}{2}}+x^{-\frac{1}{2}}+x^{-\frac{3}{2}})(y^{\frac{3}{2}}+y^{\frac{1}{2}}+y^{-\frac{1}{2}}+y^{-\frac{3}{2}})+\cdots\Big).
\end{aligned} \tag{C.20}
$$

Moving to the LHS equation we are now aware that the character is a trace over the group element $g = e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}$. It is written as follows

$$
\mathrm{Tr}(e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}) = \sum_i \langle i|e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}|i\rangle \tag{C.21}
$$

where i is a sum over states. Using the group element to act on the state of a primary field and its descendents, we can recover equation (C.20). The states of a primary field and its descendents are represented bythe quantum numbers $[D, J_{3,L}, J_{3,R}]$. The primary field has quantum numbers, $[D, J_{3,L}, J_{3,R}] = [1,0,0]$, which we represent as a state, $|1,0,0\rangle$. If we let $s = e^t$, $x = e^{i\theta_L}$ and $y = e^{i\theta_R}$, for the primary field state we have,

$$
\begin{aligned}
g|\phi\rangle =& e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}|1,0,0\rangle \\
=& e^t|\phi\rangle \\
=& s|\phi\rangle
\end{aligned} \tag{C.22}
$$

The state for the descendent $\partial_\mu\phi$ is a combination of the states,

$$
|\partial_\mu\phi\rangle \leftrightarrow \left\{\left|2,\frac{1}{2},\frac{1}{2}\right\rangle, \left|2,\frac{1}{2},-\frac{1}{2}\right\rangle, \left|2,-\frac{1}{2},\frac{1}{2}\right\rangle, \left|2,-\frac{1}{2},-\frac{1}{2}\right\rangle\right\}. \tag{C.23}
$$

Therefore

$$\{g|\partial_\mu\phi\rangle\} = \Big\{ e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, \tag{C.24}$$

$$e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle, e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}$$

$$= \Big\{ e^{2t+\frac{i\theta_L}{2}+\frac{i\theta_R}{2}}\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, e^{2t+\frac{i\theta_L}{2}-\frac{i\theta_R}{2}}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, e^{2t-\frac{i\theta_L}{2}+\frac{i\theta_R}{2}}\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle,$$

$$e^{2t-\frac{i\theta_L}{2}-\frac{i\theta_R}{2}}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}$$

$$= \Big\{ s^2x^{1/2}y^{1/2}\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, s^2x^{1/2}y^{-1/2}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, s^2x^{-1/2}y^{1/2}\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle,$$

$$s^2x^{-1/2}y^{-1/2}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}$$

Taking the trace we end up with

$$\langle\partial_\mu\phi|g|\partial_\mu\phi\rangle = s^2(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(y^{\frac{1}{2}}+y^{-\frac{1}{2}}) = s^2\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y). \tag{C.25}$$

For the descendent state $\partial_\mu\partial_\nu\phi$ we have,

$$\{g|\partial_\mu\partial_\nu\phi\rangle\} =\{g|3,1,1\rangle, g|3,1,0\rangle, g|3,1,-1\rangle, g|3,0,1\rangle, g|3,0,-1\rangle, g|3,0,0\rangle, \tag{C.26}$$

$$g|3,-1,1\rangle, g|3,-1,0\rangle, g|3,-1,-1\rangle\}$$

$$=\{s^3xy|3,1,1\rangle, s^3x|3,1,0\rangle, s^3xy^{-1}|3,1,-1\rangle, s^3y|3,0,1\rangle, s^3y^{-1}|3,0,-1\rangle,$$

$$s^3|3,0,0\rangle, s^3x^{-1}y|3,-1,1\rangle, s^3x^{-1}|3,-1,0\rangle, s^3x^{-1}y^{-1}|3,-1,-1\rangle\}$$

Taking trace we end up with

$$\langle\partial_\mu\partial_\nu\phi|g|\partial_\mu\partial_\nu\phi\rangle =s^3(xy+x+xy^{-1}+y+y^{-1}+1+x^{-1}y+x^{-1}+x^{-1}y^{-1}) \tag{C.27}$$

$$=s^3\chi_1(x)\chi_1(y).$$

Continuing in this way to higher descendents we learn that

$$\mathrm{Tr}(e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}) = \sum_i \langle i|e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}|i\rangle = \chi_V(s,x,y) \tag{C.28}$$

C.3 Product of Two copies of scalar Operators

Now we want to take a product of two copies of the representation that the free scalar field belongs to. By decomposing this into irreducible representations, we will learn what primary operators we can construct from a product of two scalar fields. The character we want to compute is

$$\chi_{Sym(V^{\otimes 2})} = \sum_{i,j} \langle i| \otimes \langle j| g \otimes g\, P|i\rangle \otimes |j\rangle \tag{C.29}$$

where $P = \frac{1}{2!}\sum_{\sigma \in S_n} \sigma$ projects us onto the symmetric subspace

$$P|i\rangle \otimes |j\rangle = \frac{1}{2}\left[|i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle\right] \tag{C.30}$$

Thus

$$\chi_{Sym(V^{\otimes 2})} = \frac{1}{2}\left(\mathrm{Tr}(g)^2 + \mathrm{Tr}(g^2)\right) \tag{C.31}$$

We can argue the equation above as follows: Denote the matrix representation M on the basis $|i\rangle$, for the group element g belonging to the group G as

$$\langle i|M|j\rangle = M_{ij}. \tag{C.32}$$

The trace of this matrix is

$$\sum_i \langle i|M|i\rangle = \sum_i M_{ii} \tag{C.33}$$

The tensor product has matrix elements

$$\langle i_1 i_2|M \otimes M|j_1 j_2\rangle = M_{i_1 j_1} M_{i_2 j_2}. \tag{C.34}$$

Thus, for example

$$\begin{aligned}
\mathrm{Tr}(M^{\otimes 2}) &= \sum_{i_1,i_2} \langle i_1, i_2 | MM | i_1, i_2 \rangle \\
&= \sum_{i_1,i_2} M_{i_1 i_1} M_{i_2 i_2} \\
&= \mathrm{Tr}(M)^2.
\end{aligned} \tag{C.35}$$

The symmetric character is defined as

$$\begin{aligned}
\chi_{Sym(V^{\otimes 2})} &= \mathrm{Tr}\Big(M^{\otimes 2} \frac{1}{2!} \sum_{\sigma \epsilon S_2} \sigma \Big) \\
&= \sum_{i_1,i_2} \Big\langle i_1, i_2 \Big| \Big(M \otimes M \frac{1}{2!} \sum_{\sigma \epsilon S_2} \sigma \Big) \Big| i_1, i_2 \Big\rangle,
\end{aligned} \tag{C.36}$$

where S_2 is the symmetric group of order 2. Since we are summing over the group elements we have

$$\chi_{Sym(V^{\otimes 2})} = \sum_{i_1,i_2} \Big\langle i_1, i_2 \Big| M \otimes M \frac{1}{2!} \Big(1 + (12) \Big) \Big| i_1, i_2 \Big\rangle. \tag{C.37}$$

The identity element leaves the state unchanged and the element (12) swaps the states i_1 and i_2. Therefore

$$\begin{aligned}
\chi_{Sym(V^{\otimes 2})} &= \frac{1}{2} \sum_{i_1,i_2} (M_{i_1 i_1} M_{i_2 i_2} + M_{i_1 i_2} M_{i_1 i_2}) \\
&= \frac{1}{2} (\mathrm{Tr}(M)^2 + \mathrm{Tr}(M^2)).
\end{aligned} \tag{C.38}$$

We can also compute

$$\begin{aligned}
\chi_{Sym(V^{\otimes 3})} &= \mathrm{Tr}\Big(\frac{1}{3!} \sum_{\sigma \epsilon S_3} \sigma M^{\otimes 3} \Big) \\
&= \frac{1}{3!} \sum_{i_1,i_2,i_3} \Big\langle i_1, i_2, i_3 \Big| M^{\otimes 3} \Big(1 + (12) + (23) + (13) + (123) + (132) \Big) \Big| i_1, i_2, i_3 \Big\rangle \\
&= \frac{1}{3!} \sum_{i_1,i_2,i_3} \Big(M_{i_1 i_1} M_{i_2 i_2} M_{i_3 i_3} + M_{i_1 i_2} M_{i_2 i_1} M_{i_3 i_3} + M_{i_1 i_1} M_{i_2 i_3} M_{i_3 i_2} \\
&\quad + M_{i_1 i_3} M_{i_2 i_2} M_{i_3 i_1} + M_{i_1 i_2} M_{i_2 i_3} M_{i_3 i_1} + M_{i_1 i_3} M_{i_2 i_1} M_{i_3 i_2} \Big) \\
&= \frac{1}{3!} \Big(\mathrm{Tr}(M)^3 + 3\mathrm{Tr}(M^2)\mathrm{Tr}(M) + 2\mathrm{Tr}(M^3) \Big).
\end{aligned} \tag{C.39}$$

We can generalise the computation above to

$$\chi_{Sym(V^{\otimes n})} = \mathrm{Tr}\bigg(\frac{1}{n!}\sum_{\sigma\epsilon S_n}\sigma M^{\otimes n}\bigg). \tag{C.40}$$

Now using the fact that $\mathrm{Tr}(g) = \chi_V(s,x,y)$ we can show that $\mathrm{Tr}(g^2) = \chi_V(s^2,x^2,y^2)$.

We know that

$$\mathrm{Tr}(g) = \mathrm{Tr}(e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}) = \sum_i \langle i|e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}|i\rangle. \tag{C.41}$$

Therefore

$$\mathrm{Tr}(g^2) = \mathrm{Tr}(e^{2tD+2i\theta_L J_{3,L}+2i\theta_R J_{3,R}}) = \sum_i \langle i|e^{2tD+2i\theta_L J_{3,L}+2i\theta_R J_{3,R}}|i\rangle. \tag{C.42}$$

We continue the same way as in the previous computation. We do an expansion on the states i containing a scalar primary and its descendents. We begin with the primary state

$$\begin{aligned} g^2|\phi\rangle &= e^{2tD+2i\theta_L J_{3,L}+2i\theta_R J_{3,R}}|1,0,0\rangle \\ &= s^2|1,0,0\rangle. \end{aligned} \tag{C.43}$$

Now consider the first descendent state $\partial_\mu\phi$,

$$\{g^2|\partial_\mu\phi\rangle\} = \Big\{e^{2tD+2i\theta_L J_{3,L}+2i\theta_R J_{3,R}}\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, \tag{C.44}$$
$$e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle, e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}$$
$$= \Big\{e^{4t+\frac{2i\theta_L}{2}+\frac{2i\theta_R}{2}}\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, e^{4t+\frac{2i\theta_L}{2}-\frac{2i\theta_R}{2}}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, e^{4t-\frac{2i\theta_L}{2}+\frac{2i\theta_R}{2}}\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle,$$
$$e^{4t-\frac{i\theta_L}{2}-\frac{i\theta_R}{2}}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}$$
$$= \Big\{s^4x^1y^1\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, s^4x^1y^{-1}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, s^4x^{-1}y^1\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle,$$
$$s^4x^{-1}y^{-1}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}.$$

Taking the trace we obtain

$$\begin{aligned}\langle\partial_\mu\phi|g|\partial_\mu\phi\rangle &= s^4(xy+xy^{-1}+x^{-1}y+x^{-1}y^{-1}) \\ &= s^4(x+x^{-1})(y+y^{-1}) \\ &= s^4\chi_{\frac{1}{2}}(x^2)\chi_{\frac{1}{2}}(y^2).\end{aligned} \tag{C.45}$$

Carrying on this way to higher descendents we will learn that

$$\mathrm{Tr}(g^2) = \mathrm{Tr}(e^{2tD+2i\theta_L J_{3,L}+2i\theta_R J_{3,R}}) = \sum_i \langle i|e^{2tD+2i\theta_L J_{3,L}+2i\theta_R J_{3,R}}|i\rangle = \chi_V(s^2,x^2,y^2)$$

Thus, the character for the representation obtained by taking the product of two copies of the representation that the free scalar field belongs to is

$$\chi_{Sym(V^{\otimes 2})} = \frac{1}{2}\Big((\chi_V(s,x,y))^2 + \chi_V(s^2,x^2,y^2)\Big) \tag{C.46}$$

Now how do we compute the LHS of (C.46). We begin by defining

$$P(s,x,y) = \frac{1}{(1-sx^{1/2}y^{1/2})(1-sx^{1/2}y^{-1/2})(1-sx^{-1/2}y^{1/2})(1-sx^{-1/2}y^{-1/2})} \tag{C.47}$$

By expanding $P(s,x,y)$, we find that

$$P(s,x,y) = \sum_{p,q=0}^{\infty} s^{2p+q} \chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y). \tag{C.48}$$

Using the identity

$$\frac{1}{1-t^2} = \frac{1}{1-t}\frac{1}{1+t}, \tag{C.49}$$

for each of the four factors in $P(s,x,y)$, we find

$$\begin{aligned} P(s^2,x^2,y^2) &= P(s,x,y)P(-s,x,y) && \text{(C.50)}\\ &= \sum_{p,q=0}^{\infty} (-s)^{2p+q} \chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) P(s,x,y). && \text{(C.51)} \end{aligned}$$

Thus,

$$\begin{aligned} \chi_V(s^2,x^2,y^2) &= s^2(1-s^4)P(s^2,x^2,y^2)\\ &= s^2(1-s^4)P(s,x,y)P(-s,x,y)\\ &= s^2(1-s^4)P(s,x,y)\sum_{p,q=0}^{\infty}(-s)^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\\ &= s^2(1+s^2)P(s,x,y)\sum_{q=0}^{\infty}(-s)^{q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\\ &= s^2P(s,x,y) - \sum_{q=0}^{\infty}(-1)^q\left[s^{3+q}\chi_{\frac{q+1}{2}}(x)\chi_{\frac{q+1}{2}}(y) - s^{4+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\right]P(s,x,y)\\ &= s^2P(s,x,y) - \sum_{d=3}^{\infty}(-1)^{d+1}\left[s^{d}\chi_{\frac{d-2}{2}}(x)\chi_{\frac{d-2}{2}}(y) - s^{d+1}\chi_{\frac{d-3}{2}}(x)\chi_{\frac{d-3}{2}}(y)\right]P(s,x,y)\\ &. \end{aligned} \tag{C.52}$$

Note also that

$$
\begin{aligned}
(\chi_V(s,x,y))^2 =&\chi_{[1,0,0]} \times \chi_{[1,0,0]} \\
=&\sum_{q=0}^{\infty} s^{q+1}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \times s(1-s^2)P(s,x,y) \\
=&s^2 P(s,x,y) + \sum_{q=0}^{\infty}\left[s^{3+q}\chi_{\frac{q+1}{2}}(x)\chi_{\frac{q+1}{2}}(y) - s^{4+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\right]P(s,x,y) \\
=&s^2 P(s,x,y) + \sum_{d=3}^{\infty}\left[s^{d}\chi_{\frac{d-2}{2}}(x)\chi_{\frac{d-2}{2}}(y) - s^{d+1}\chi_{\frac{d-3}{2}}(x)\chi_{\frac{d-3}{2}}(y)\right]P(s,x,y)
\end{aligned}
\tag{C.53}
$$

Thus,

$$
\begin{aligned}
\chi_{Sym(V^{\otimes 2})} =&\frac{1}{2}\left((\chi_V(s,x,y))^2 + \chi_V(s^2,x^2,y^2)\right) \\
=&s^2 P(s,x,y) + \sum_{d=1}^{\infty}\left[s^{2d+2}\chi_{\frac{2d}{2}}(x)\chi_{\frac{2d}{2}}(y) - s^{2d+3}\chi_{\frac{2d-1}{2}}(x)\chi_{\frac{2d-1}{2}}(y)\right]P(s,x,y)
\end{aligned}
\tag{C.54}
$$

Thus, we have

$$
Sym(D_{[100]} \otimes D_{[100]}) = \mathcal{A}_{[200]} + \sum_{k_1=1} D_{[2k_1+2,\frac{2k_1}{2},\frac{2k_1}{2}]} \tag{C.55}
$$

The term $\mathcal{A}_{[200]}$ is the representation for a spinless scalar of dimension 2 - the corresponding primary is ϕ^2. This rep has no null states. The terms $D_{[2k_1+2,\frac{2k_1}{2},\frac{2k_1}{2}]}$ are conserved currents. These reps have null states.

In the work [35] solving the conservation equation

$$
\partial^{\mu} D_{\mu}\mathcal{O}_s(z,x) = 0.
$$

they gave the result

$$
\begin{aligned}
&\alpha^s C_s^{\gamma}\left(\frac{\beta}{\alpha}\right) = \mathcal{O}_s \qquad\qquad (\text{C.56}) \\
=&\frac{\sqrt{\pi}\Gamma(d/2+s-1)\Gamma(d+s-3)}{2^{d-4}\Gamma(\frac{d-3}{2})}\sum_{k=0}^{s}\frac{(-1)^k(z\cdot\partial_1)^{s-k}(z\cdot\partial_2)^k}{k!(s-k)!\Gamma(k+d/2-1)\Gamma(s-k+d/2-1)},
\end{aligned}
$$

where $\mathcal{O}_s$ is a conserved spin current with spin s and dimension $s+d-2$ and, the function $C_s^\gamma(\frac{\beta}{\alpha})$ is a Gegenbauer polynomial. In 4 dimension

$$\begin{aligned}\mathcal{O}_s =&(\Gamma(s+1))^2 \sum_{k=0}^{s} \frac{(-1)^k (z\cdot\partial)^{s-k}\phi(z\cdot\partial)^k\phi}{k!(s-k)\Gamma(k+1)\Gamma(s-k+1)} \\ =&(s!)^2 \sum_{k=0}^{s} \frac{(-1)^k (z\cdot\partial)^{s-k}\phi(z\cdot\partial)^k\phi}{(k!(s-k)!)^2}.\end{aligned} \tag{C.57}$$

The term $A_{[2,0,0]}$ is the representation for the term ϕ^2, which is a primary. Each $\mathcal{O}_s$ from above is in representation $D_{[s+2,\frac{s}{2},\frac{s}{2}]}$. If s is odd, $\mathcal{O}_s = 0$ so we have a primary for each representation on the right hand side of ((C.55)).

C.4 Character For Product of Many Scalar Field Operators

The character relevant for a product of three fields is

$$\chi_{Sym(V^{\otimes 3})} = \frac{1}{6}\Big((\chi_V(s,x,y))^3 + 3\chi_V(s,x,y)\chi_V(s^2,x^2,y^2) + 2\chi_V(s^3,x^3,y^3)\Big) \tag{C.58}$$

The character relevant for a product of four fields is

$$\begin{aligned}\chi_{Sym(V^{\otimes 4})} =&\frac{1}{24}\Big((\chi_V(s,x,y))^4 + 6(\chi_V(s,x,y))^2\chi_V(s^2,x^2,y^2) \\ &+ 8\chi_V(s^3,x^3,y^3)\chi_V(s,x,y) + 3(\chi_V(s^2,x^2,y^2))^2 \\ &+ 6\chi_V(s^4,x^4,y^4)\Big)\end{aligned} \tag{C.59}$$

The character relevant for a product of five fields is

$$\begin{aligned}\chi_{Sym(V^{\otimes 5})} =&\frac{1}{120}\Big((\chi_V(s,x,y))^5 + 10(\chi_V(s,x,y))^3\chi_V(s^2,x^2,y^2) \\ &+ 20\chi_V(s^3,x^3,y^3)(\chi_V(s,x,y))^2 + 30\chi_V(s^4,x^4,y^4)\chi_V(s,x,y) \\ &+ 15(\chi_V(s^2,x^2,y^2))^2\chi_V(s,x,y) + 20\chi_V(s^2,x^2,y^2)\chi_V(s^3,x^3,y^3) \\ &+ 24\chi_V(s^5,x^5,y^5)\Big)\end{aligned} \tag{C.60}$$

Using the above characters, we can compute the products

$$Sym(D_{[100]} \otimes D_{[100]} \otimes D_{[100]}) \tag{C.61}$$

which correspond the product of 3 scalar operators, $\phi \otimes \phi \otimes \phi = \phi^3$ together with the descendents obtained from acting with the spacetime derivatives ∂_μ. We also look at products

$$Sym(D_{[100]} \otimes D_{[100]} \otimes D_{[100]} \otimes D_{[100]}) \tag{C.62}$$

which correspond to the scalar operator product $\phi \otimes \phi \otimes \phi \otimes \phi = \phi^4$. And also

$$Sym(D_{[100]} \otimes D_{[100]} \otimes D_{[100]} \otimes D_{[100]} \otimes D_{[100]}) \tag{C.63}$$

To be able to calculate these products we first revert back to the idea of a characters in quantum mechanics. Consider the character of spin j

$$\begin{aligned} \chi_j(x) &= x^j + x^{j-1} + \cdots + x^{-j+1} + x^{-j} \\ &= \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \end{aligned} \tag{C.64}$$

Using this we find

$$\begin{aligned} \chi_{\frac{k}{2}}(x^n) &= x^{\frac{nk}{2}} + x^{\frac{n(k-2)}{2}} + \cdots + x^{-\frac{n(k-2)}{2}} + x^{-\frac{nk}{2}} \\ &= \left(\frac{x^{\frac{nk}{2}} + x^{\frac{n(k-2)}{2}} + \cdots + x^{-\frac{n(k-2)}{2}} + x^{-\frac{nk}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \right) (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \\ &= \sum_{l=0,1,\ldots}^{\lfloor k/2 \rfloor} \chi_{\frac{kn}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor (k-1)/2 \rfloor} \chi_{\frac{kn}{2}-nl-1}(x) \end{aligned} \tag{C.65}$$

To obtain the last line, multiply the numerator out and collect terms.

Using the formulas for $\chi_{\frac{q}{2}}(x^n)$ that we have just derived, we find

$$\begin{aligned}
P(s^n, x^n, y^n) &= \sum_{p,q=0}^{\infty} s^{2np+nq} \chi_{\frac{q}{2}}(x^n) \chi_{\frac{q}{2}}(y^n) \qquad \text{(C.66)} \\
&= \frac{1}{1-s^{2n}} \sum_{q=0}^{\infty} s^{nq} \chi_{\frac{q}{2}}(x^n) \chi_{\frac{q}{2}}(y^n) \\
&= \frac{1}{1-s^{2n}} \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(x) \right] \\
&\quad \times \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(y) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(y) \right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\chi_{V^+}(s^n, x^n, y^n) &= P(s^n, x^n, y^n) s^n (1-s^{2n}) \qquad \text{(C.67)} \\
&= s^n \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(x) \right] \\
&\quad \times \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(y) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(y) \right]
\end{aligned}$$

We also need an identity which rewrites $\chi_{V^+}(s^n, x^n, y^n)$ as $SU(2)$ characters multiplied by $P(s, x, y)$; these can very easily be translated into $\mathcal{A}_{[\cdot,\cdot,\cdot]}$s. Towards this end, note that

$$1 = P(s,x,y)(1 - sx^{1/2}y^{1/2})(1 - sx^{1/2}y^{-1/2})(1 - sx^{-1/2}y^{1/2})(1 - sx^{-1/2}y^{-1/2}) \qquad \text{(C.68)}$$

$$= P(s,x,y)\left[1 + s^4 - s(1+s^2)\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y) + s^2(\chi_1(x) + \chi_1(y))\right]$$

A straight forward computation now gives

$$\chi_{V^+}(s^n, x^n, y^n) = P(s^n, x^n, y^n) s^n (1 - s^{2n}) \tag{C.69}$$

$$= s^n \Bigg[(1+s^4) \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_1}(x) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_1-1}(x) \right]$$

$$\times \left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-1}(y) \right]$$

$$- s(1+s^2) \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_1}(x) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_1-1}(x) + \sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_1}(x) \right.$$

$$\left. - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_1-1}(x) \right] \times \left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_2}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_2-1}(y) \right.$$

$$\left. + \sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_2}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_2-1}(y) \right]$$

$$+ s^2 \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_1}(x) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_1-1}(x) \right]$$

$$\times \left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2+1}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2}(y) + \sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2}(y) \right.$$

$$\left. - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-1}(y) + \sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-1}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-2}(y) \right]$$

$$+ s^2 \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_1}(y) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_1-1}(y) \right]$$

$$\times \left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2+1}(x) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2}(x) + \sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2}(x) \right.$$

$$\left. - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-1}(x) + \sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-1}(x) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-2}(x) \right] \Bigg] P(s,x,y)$$

Return to

$$\chi_{Sym(V^{\otimes 3})} = \frac{1}{6} \left((\chi_{V_+}(s,x,y))^3 + 3\chi_{V_+}(s,x,y)\chi_{V_+}(s^2,x^2,y^2) + 2\chi_{V_+}(s^3,x^3,y^3) \right) \tag{C.70}$$

We know the decomposition of $(\chi_{V_+}(s,x,y))^3$ into irreducible characters - all we need for this is the $SU(2)$ character product rule. For $\chi_{V_+}(s,x,y)\chi_{V_+}(s^2,x^2,y^2)$ use (C.67) to evaluate $\chi_{V_+}(s^2,x^2,y^2)$ and use the known formula for $\chi_{V_+}(s,x,y)$. For $\chi_{V_+}(s^3,x^3,y^3)$ use (C.69). The result is (each term in square brackets collects all reps of a given dimension; I have tried to indicate the origin of each term by keeping the coefficients 2 and 3 that appear in (C.70))

$$\begin{aligned}
\chi_{Sym(V^{\otimes 3})} = \frac{1}{6}\Bigg[& \Big[\mathcal{A}_{[3,0,0]} + 3\times\mathcal{A}_{[3,0,0]} + 2\times\mathcal{A}_{[3,0,0]}\Big] \qquad \text{(C.71)}\\
& + \Big[2\mathcal{A}_{[4,\frac{1}{2},\frac{1}{2}]} + 3\times 0 + 2\times(-\mathcal{A}_{[4,\frac{1}{2},\frac{1}{2}]})\Big]\\
& + \Big[3\mathcal{A}_{[5,1,1]} + \mathcal{A}_{[5,0,1]} + \mathcal{A}_{[5,1,0]} + 3\times(-\mathcal{A}_{[5,0,1]} - \mathcal{A}_{[5,1,0]} + \mathcal{A}_{[5,1,1]}) + 2(\mathcal{A}_{[5,0,1]} + \mathcal{A}_{[5,1,0]})\Big]\\
& + \Big[4\mathcal{A}_{[6,\frac{3}{2},\frac{3}{2}]} + 2\mathcal{A}_{[6,\frac{1}{2},\frac{3}{2}]} + 2\mathcal{A}_{[6,\frac{3}{2},\frac{1}{2}]} + 3\times 0 + 2\times(\mathcal{A}_{[6,\frac{3}{2},\frac{3}{2}]} - \mathcal{A}_{[6,\frac{3}{2},\frac{1}{2}]} - \mathcal{A}_{[6,\frac{1}{2},\frac{3}{2}]})\Big]\\
& + \Big[5\mathcal{A}_{[7,2,2]} + \mathcal{A}_{[7,0,2]} + \mathcal{A}_{[7,2,0]} + 3\mathcal{A}_{[7,1,2]} + 3\mathcal{A}_{[7,2,1]}\\
& + 3\times(\mathcal{A}_{[7,2,2]} + \mathcal{A}_{[7,0,2]} + \mathcal{A}_{[7,2,0]} - \mathcal{A}_{[7,2,1]} - \mathcal{A}_{[7,1,2]}) + 2\times(-\mathcal{A}_{[7,2,2]} + \mathcal{A}_{[7,2,0]} + \mathcal{A}_{[7,0,2]})\Big]\\
& + \Big[6\mathcal{A}_{[8,\frac{5}{2},\frac{5}{2}]} + 4\mathcal{A}_{[8,\frac{3}{2},\frac{5}{2}]} + 4\mathcal{A}_{[8,\frac{5}{2},\frac{3}{2}]} + 2\mathcal{A}_{[8,\frac{1}{2},\frac{5}{2}]} + 2\mathcal{A}_{[8,\frac{5}{2},\frac{1}{2}]}\\
& + 3\times 0 + 2\times(\mathcal{A}_{[8,\frac{5}{2},\frac{3}{2}]} - \mathcal{A}_{[8,\frac{1}{2},\frac{5}{2}]} - \mathcal{A}_{[8,\frac{5}{2},\frac{1}{2}]} + \mathcal{A}_{[8,\frac{3}{2},\frac{5}{2}]})\Big]\\
& + [7\mathcal{A}_{[9,3,3]} + 5\mathcal{A}_{[9,2,3]} + 5\mathcal{A}_{[9,3,2]} + 3\mathcal{A}_{[9,1,3]} + 3\mathcal{A}_{[9,3,1]} + \mathcal{A}_{[9,0,3]} + \mathcal{A}_{[9,3,0]}\\
& + 3\times(\mathcal{A}_{[9,1,3]} + \mathcal{A}_{[9,3,1]} - \mathcal{A}_{[9,0,3]} - \mathcal{A}_{[9,2,3]} - \mathcal{A}_{[9,3,0]} - \mathcal{A}_{[9,3,2]} + \mathcal{A}_{[9,3,3]})\\
& + 2\times(\mathcal{A}_{[9,3,3]} + \mathcal{A}_{[9,3,0]} - \mathcal{A}_{[9,3,2]} + \mathcal{A}_{[9,0,3]} - \mathcal{A}_{[9,2,3]})]\\
& + [8\mathcal{A}_{[10,\frac{7}{2},\frac{7}{2}]} + 6\mathcal{A}_{[10,\frac{5}{2},\frac{7}{2}]} + 6\mathcal{A}_{[10,\frac{7}{2},\frac{5}{2}]} + 4\mathcal{A}_{[10,\frac{7}{2},\frac{3}{2}]} + 4\mathcal{A}_{[10,\frac{3}{2},\frac{7}{2}]} + 2\mathcal{A}_{[10,\frac{7}{2},\frac{1}{2}]} + 2\mathcal{A}_{[10,\frac{1}{2},\frac{7}{2}]}\\
& + 3\times 0 + 2\times(\mathcal{A}_{[10,\frac{7}{2},\frac{3}{2}]} + \mathcal{A}_{[10,\frac{3}{2},\frac{7}{2}]} - \mathcal{A}_{[10,\frac{7}{2},\frac{7}{2}]} - \mathcal{A}_{[10,\frac{7}{2},\frac{1}{2}]} - \mathcal{A}_{[10,\frac{1}{2},\frac{7}{2}]})]\\
& + \ldots\\
= {} & \mathcal{A}_{[3,0,0]} + \mathcal{A}_{[5,1,1]} + \mathcal{A}_{[6,\frac{3}{2},\frac{3}{2}]} + \mathcal{A}_{[7,2,2]} + \mathcal{A}_{[7,0,2]} + \mathcal{A}_{[7,2,0]}\\
& + \mathcal{A}_{[8,\frac{5}{2},\frac{5}{2}]} + \mathcal{A}_{[8,\frac{3}{2},\frac{5}{2}]} + \mathcal{A}_{[8,\frac{5}{2},\frac{3}{2}]} + 2\mathcal{A}_{[9,3,3]} + \mathcal{A}_{[9,1,3]} + \mathcal{A}_{[9,3,1]}\\
& + \mathcal{A}_{[10,\frac{7}{2},\frac{7}{2}]} + \mathcal{A}_{[10,\frac{7}{2},\frac{5}{2}]} + \mathcal{A}_{[10,\frac{5}{2},\frac{7}{2}]} + \mathcal{A}_{[10,\frac{7}{2},\frac{3}{2}]} + \mathcal{A}_{[10,\frac{3}{2},\frac{7}{2}]} + \ldots
\end{aligned}$$

In a similar way we find

$$\begin{aligned}
\chi_{Sym(V^{\otimes 4})} = {} & \mathcal{A}_{[4,0,0]} + \mathcal{A}_{[6,1,1]} + \mathcal{A}_{[7,\frac{3}{2},\frac{3}{2}]} + \mathcal{A}_{[8,0,0]} + \mathcal{A}_{[8,0,2]} + \mathcal{A}_{[8,2,0]} + \mathcal{A}_{[8,1,1]} + 2\mathcal{A}_{[8,2,2]} \qquad \text{(C.72)}\\
& + \mathcal{A}_{[9,\frac{3}{2},\frac{1}{2}]} + \mathcal{A}_{[9,\frac{5}{2},\frac{1}{2}]} + \mathcal{A}_{[9,\frac{1}{2},\frac{3}{2}]} + \mathcal{A}_{[9,\frac{5}{2},\frac{3}{2}]} + \mathcal{A}_{[9,\frac{1}{2},\frac{5}{2}]} + \mathcal{A}_{[9,\frac{3}{2},\frac{5}{2}]} + \mathcal{A}_{[9,\frac{5}{2},\frac{5}{2}]}\\
& \mathcal{A}_{[10,0,0]} + 2\mathcal{A}_{[10,1,1]} + \mathcal{A}_{[10,2,1]} + 2\mathcal{A}_{[10,3,1]} + \mathcal{A}_{[10,1,2]} + 2\mathcal{A}_{[10,2,2]} + \mathcal{A}_{[10,3,2]}\\
& + 2\mathcal{A}_{[10,1,3]} + \mathcal{A}_{[10,2,3]} + 3\mathcal{A}_{[10,3,3]} + \ldots
\end{aligned}$$

$$\chi_{Sym(V^{\otimes 5})} = \mathcal{A}_{[5,0,0]} + \mathcal{A}_{[7,1,1]} + \mathcal{A}_{[8,\frac{3}{2},\frac{3}{2}]} + \mathcal{A}_{[9,0,0]} + \mathcal{A}_{[9,1,1]} + \mathcal{A}_{[9,2,0]} + \mathcal{A}_{[9,0,2]} + 2\mathcal{A}_{[9,2,2]} \tag{C.73}$$

$$+ \mathcal{A}_{[10,\frac{1}{2},\frac{1}{2}]} + \mathcal{A}_{[10,\frac{3}{2},\frac{1}{2}]} + \mathcal{A}_{[10,\frac{1}{2},\frac{3}{2}]} + \mathcal{A}_{[10,\frac{3}{2},\frac{3}{2}]} + \mathcal{A}_{[10,\frac{1}{2},\frac{5}{2}]} + \mathcal{A}_{[10,\frac{5}{2},\frac{1}{2}]} + \mathcal{A}_{[10,\frac{5}{2},\frac{3}{2}]}$$
$$+ \mathcal{A}_{[10,\frac{3}{2},\frac{5}{2}]} + 2\mathcal{A}_{[10,\frac{5}{2},\frac{5}{2}]} + \ldots$$

C.5 Generating Functions for Primary Operators

Here we will derive the generating function for the primary operators by first showing the basic idea with the matrix

$$M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \tag{C.74}$$

From here we can verify that

$$\chi_{\square}(M) = a + b \tag{C.75}$$

$$\chi_{\square\square}(M) = a^2 + ab + b^2 \tag{C.76}$$

$$\chi_{\square\square\square}(M) = a^3 + a^2 b + ab^2 + b^3 \tag{C.77}$$

$$\chi_{\square\square\square\square}(M) = a^4 + a^3 b + a^2 b^2 + ab^3 + b^4 \tag{C.78}$$

We can continue this way to higher order terms. We introduce a formula which can easily compute the characters χ_n. This formula contains a parameter t^n which collects terms belonging to the character χ_n,

$$\frac{1}{\det(1 - tM)} = \frac{1}{(1-ta)(1-tb)} \tag{C.79}$$
$$= \sum_n t^n \chi_{(n)}(M) \tag{C.80}$$

where M is a matrix, a and b are the eigenvalues of M. We verify this formula by simply expanding the RHS as follows

$$
\begin{aligned}
\frac{1}{(1-ta)(1-tb)} =&(1+ta+t^2a^2+t^3a^3+\cdots)(1+tb+t^2b^2+t^3b^3+\cdots) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(C.81)} \\
=&1+(a+b)t+(a^2+ab+b^2)t^2+(a^3+a^2b+ab^2+b^3)t^3+\cdots \\
=&1+\chi_1 t+\chi_2 t^2+\chi_3 t^3+\cdots \\
=&\sum_n t^n \chi_{(n)}(M)
\end{aligned}
$$

We will now motivate how the term

$$\frac{1}{\det(1-tM)}$$

in (C.81) comes from considering the Guassian integral

$$(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n} = \frac{1}{\pi^N}\int\prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_k\bar{z}^k}\frac{1}{n!}z_{i_1}\cdots z_{i_n}\bar{z}^{j_1}\cdots\bar{z}^{j_n} \tag{C.82}$$

To evaluate the integral above, we study the generating function below

$$I=\frac{1}{\pi^N}\int\prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_k\bar{z}^k+\sum_k(\bar{j}^k z_k+j_k\bar{z}^k)} \tag{C.83}$$

We can evaluate this integral by completing the square. After that we obtain I_n in (C.82) by taking derivatives of I and setting $j_i=\bar{j}^i=0$. This is carried out as follows:
Start with the I integral,

$$I=\frac{1}{\pi^N}\int\prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_k\bar{z}^k+\sum_k(\bar{j}^k z_k+j_k\bar{z}^k)} \tag{C.84}$$

We shift $z\to z+j$ and $\bar{z}\to\bar{z}+\bar{j}$ and end up with

$$I=\frac{1}{\pi^N}\int dz_i d\bar{z}_i e^{-\sum z_k\bar{z}^k+\sum\bar{j}^k j_k}. \tag{C.85}$$

Performing the Guassian integral, we are left with

$$I=e^{\sum_k\bar{j}^k j_k}. \tag{C.86}$$

Applying the derivatives with respect to $\bar{j}$ and j as follows

$$\frac{\delta}{\delta j_{j_1}}\cdots\frac{\delta}{\delta j_{i_n}}\frac{\delta}{\delta\bar{j}^{i_1}}\frac{\delta}{\delta\bar{j}^{i_2}}\cdots\frac{\delta}{\delta\bar{j}^{j_n}}(I)=\frac{\delta}{\delta j_{j_1}}\cdots\frac{\delta}{\delta j_{i_n}}\frac{\delta}{\delta\bar{j}^{i_1}}\frac{\delta}{\delta\bar{j}^{i_2}}\cdots\frac{\delta}{\delta\bar{j}^{j_n}}e^{\sum_k\bar{j}^k j_k} \tag{C.87}$$

$$\begin{aligned}(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n}=&\frac{1}{n!}\frac{\delta}{\delta j_{j_1}}\cdots\frac{\delta}{\delta j_{i_n}}\frac{\delta}{\delta\bar{j}^{i_1}}\frac{\delta}{\delta\bar{j}^{i_2}}\cdots\frac{\delta}{\delta\bar{j}^{j_n}}(\bar{j}^{k_1}\cdots\bar{j}^{k_n}j_{k_1}\cdots j_{k_n})\\ =&\frac{1}{n!}\sum_{\sigma\epsilon S_n}\delta^{j_1}_{i_{\sigma(1)}}\cdots\delta^{j_1}_{i_{\sigma(n)}},\end{aligned}$$

where S_n is the symmetric group and σ is an element of a group. This shows that

$$(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n}=\frac{1}{n!}\sum_{\sigma\in S_n}\delta^{j_1}_{i_{\sigma(1)}}\delta^{j_2}_{i_{\sigma(2)}}\cdots\delta^{j_n}_{i_{\sigma(n)}} \tag{C.88}$$

From here we will argue that

$$M^{i_1}_{j_1}M^{i_2}_{j_2}\cdots M^{i_n}_{j_n}(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n}=\chi_R(M) \tag{C.89}$$

where R is a Young diagram with one row of n boxes. We argue this as follows,

$$\begin{aligned}(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n}M^{i_1}_{j_1}M^{i_2}_{j_2}\cdots M^{i_n}_{j_n}=&\frac{1}{n!}\sum_{\sigma\epsilon S_n}\delta^{j_1}_{i_{\sigma(1)}}\cdots\delta^{j_1}_{i_{\sigma(n)}}M^{i_1}_{j_1}M^{i_2}_{j_2}\cdots M^{i_n}_{j_n}\\ =&\frac{1}{n!}\sum_{\sigma\epsilon S_n}M^{i_1}_{i_{\sigma(1)}}\cdots M^{i_n}_{i_{\sigma(n)}}\\ =&\frac{1}{n!}\sum_{\sigma\epsilon S_n}\mathrm{Tr}(\sigma M^{\otimes n})\\ =&\chi_R(M).\end{aligned} \tag{C.90}$$

In the above R is a Young diagram with a single row of n boxes.

Now, consider the integral

$$Z=\frac{1}{\pi^N}\int\prod_{i=1}^N dz_k d\bar{z}_k e^{-\sum_{i,j}z_i O^i_j\bar{z}^j}=\frac{1}{\pi^N}\int\prod_{i=1}^N dz_k d\bar{z}_k e^{-\sum_{i,j}z_i(\delta^i_j-tM^i_j)\bar{z}^j} \tag{C.91}$$

We know that this integral evaluates to

$$\frac{1}{\pi^N}\int\prod_{i=1}^{N}dz_k d\bar{z}_k e^{-\sum_{i,j} z_i O^i_j \bar{z}^j} = \frac{1}{\det O} \tag{C.92}$$

Thus, we have

$$Z = \frac{1}{\det(1-tM)} \tag{C.93}$$

Now compute Z by expanding in t

$$\begin{aligned} Z =& \frac{1}{\pi^N}\int\prod_{i=1}^{N}dx_i dy_i e^{-\sum_i z_i \bar{z}^i}\sum_{n=0}^{\infty}t^n M^{i_1}_{j_1}M^{i_2}_{j_2}\cdots M^{i_n}_{j_n}z_{i_1}\cdots z_{i_n}\bar{z}^{j_1}\cdots\bar{z}^{j_n} \\ =& \sum_n t^n \chi_{(n)}(M) \end{aligned} \tag{C.94}$$

Comparing (C.93) and (C.94) now proves (C.79). For our character problem, we are interested in the case that

$$M = s^D x^{J_{3,L}} y^{J_{3,R}} \tag{C.95}$$

Looking at (C.6), we can see that

$$\frac{1}{\det(1-tM)} = \prod_{q=0}^{\infty}\prod_{a=-\frac{q}{2}}^{\frac{q}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}\frac{1}{1-ts^{q+1}x^a y^b} \tag{C.96}$$

The remaining task is to decompose this into $SO(4,2)$ representations, then we will be able to generate the spectrum of primary operators for n copies of the scalar field.

C.5.1 Partition functions

Consider a partition function that is a sum of SU(2 characters [60]

$$Z(x) = \mathrm{Tr}\Big(x^{J_3}\Big) = \sum_j N_j \chi_j(x) = Z_0 + \sum_{k=1}^{\infty} Z_k(x^k + x^{-k}) \tag{C.97}$$

Since all characters are invariant under the transformation $x \to \frac{1}{x}$ $Z(x)$ must also be invariant under this transformation. The last equality manifests that symmetry. N_j counts the number of times the spin j rep appears. To determine N_j we make use of the orthogonality of characters.

If we set $x = e^{i\theta}$, the results in (C.1) already imply that

$$\chi_j(x) = \frac{\sin\Big((j+\frac{1}{2})\theta\Big)}{\sin\Big(\frac{\theta}{2}\Big)}. \tag{C.98}$$

We can verify character orthogonality

$$\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)\chi_k(x) = \delta_{jk} \tag{C.99}$$

as follows

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)\chi_k(x) =& \int_0^{4\pi} \sin\frac{\sin^2\frac{\theta}{2}}{2\pi}\Big(\frac{\sin((j+1/2)\theta)}{\sin\frac{\theta}{2}}\Big)\Big(\frac{\sin((k+1/2)\theta)}{\sin\frac{\theta}{2}}\Big) \qquad \text{(C.100)}\\
=& \int_0^{4\pi} d\theta \frac{\sin((j+1/2)\theta)}{2\pi}\sin((k+1/2)\theta)\\
=& \int_0^{4\pi} \frac{d\theta}{2\pi}\Big(\frac{e^{i(j+\frac{1}{2})\theta} - e^{-i(j+\frac{1}{2})\theta}}{2i}\Big)\Big(\frac{e^{i(k+\frac{1}{2})\theta} - e^{-i(k+\frac{1}{2})\theta}}{2i}\Big)\\
=& -\int_0^{4\pi} \frac{d\theta}{8\pi}\Big(e^{i(j+k+1)\theta} - e^{i(j-k)\theta} - e^{i(k-j)\theta} + e^{-(j+k+1)\theta}\Big)\\
=& -\frac{1}{8\pi}\Big(4\pi\delta_{j+1,-k} - 4\pi\delta_{j,k} - 4\pi\delta_{j,k} + 4\pi\delta_{-j-1,k}\Big)\\
=& -\frac{1}{8\pi}\Big(-4\pi\delta_{j,k} - 4\pi\delta_{j,k}\Big)\\
=&\delta_{j,k},
\end{aligned}$$

We can also show that

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)(x^k + x^{-k}) =&\delta_{j,k} - \delta_{j+1,k} \qquad \text{(C.101)}\\
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x) =&\delta_{j,0}
\end{aligned}$$

as follows:

$$\int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi} \chi_j(x)(x^k + x^{-k}) = \int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi} \frac{\sin((j+1/2)\theta)}{\sin\left(\frac{\theta}{2}\right)} (e^{ik\theta} + e^{-ik\theta}) \tag{C.102}$$

$$= \int_0^{4\pi} \frac{d\theta}{2\pi} \sin\frac{\theta}{2} \sin\left((j + \frac{1}{2})\theta\right) e^{ik\theta} + \int_0^{4\pi} \frac{d\theta}{2\pi} \sin\frac{\theta}{2} \sin\left((j+\frac{1}{2})\theta\right) e^{-ik\theta}$$

$$= \int_0^{4\pi} \frac{d\theta}{2\pi} \left(\frac{e^{-i\theta/2} - e^{-i\theta/2}}{2i}\right)\left(\frac{e^{i(j+1/2)\theta} - e^{-i(j+1/2)\theta}}{2i}\right) e^{ik\theta}$$

$$+ \int_0^{4\pi} \frac{d\theta}{2\pi} \left(\frac{e^{-i\theta/2} - e^{-i\theta/2}}{2i}\right)\left(\frac{e^{i(j+1/2)\theta} - e^{-i(j+1/2)\theta}}{2i}\right) e^{-ik\theta}.$$

Multiply out the exponentials to obtain

$$\int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi} \chi_j(x)(x^k + x^{-k}) = -\int_0^{4\pi} \frac{d\theta}{8\pi} \left(e^{i(j+k+1)\theta} - e^{i(k-j)\theta} - e^{i(k+j)\theta} + e^{-i(j-k+1)\theta}\right) \tag{C.103}$$

$$- \int_0^{4\pi} \frac{d\theta}{8\pi} \left(e^{i(j-k+1)\theta} - e^{-i(k+j)\theta} - e^{i(k-j)\theta} + e^{-i(j+k+1)\theta}\right)$$

$$= - \frac{1}{8\pi}\left(4\pi\delta_{j+1,-k} - 4\pi\delta_{j,k} - 4\pi\delta_{j,-k} + 4\pi\delta_{j+1,k}\right)$$

$$- \frac{1}{8\pi}\left(4\pi\delta_{j+1,k} - 4\pi\delta_{j,-k} - 4\pi\delta_{j,k} + 4\pi\delta_{j+k,-k}\right)$$

$$= \delta_{j,k} - \delta_{j+1,k},$$

Finally,

$$\int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi} \chi_j(x) = \int_0^{4\pi} \frac{d\theta}{2\pi} \sin\frac{\theta}{2} \sin((j+1/2)\theta) \tag{C.104}$$

$$= \frac{1}{2}\int_0^{4\pi} \frac{d\theta}{2\pi}\left(\cos(j\theta) - \cos((j+1)\theta)\right)$$

$$= \delta_{j,0}.$$

Using the orthogonality of the characters we can see that

$$\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi} \chi_j(x) Z(x) = \int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi} \chi_j(x) \sum_k N_k \chi_k(x)$$

$$= N_j \tag{C.105}$$

We also have

$$
\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)Z(x) &= \int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)\left(Z_0+\sum_{k=1}^{\infty}Z_k(x^k+x^{-k})\right) \\
&= Z_j - Z_{j+1} \qquad (j>0) \\
&= Z_0 \qquad (j=0)
\end{aligned}
\tag{C.106}
$$

If we define the generating function

$$
G(x)=\sum_j x^j N_j
$$

and the "regular part of a function" as

$$
\left[\sum_n a_n x^n\right]_{\geq} = \sum_{n=0}^{\infty} a_n x^n
$$

we remove the negative powers of x by writing $Z(x)$ as

$$
G(x)=\left[(1-\frac{1}{x})Z(x)\right]_{\geq} \tag{C.107}
$$

where the multiplication with the factor $(-\frac{1}{x})$ removes negative powers.

Now, considering the $SO(4,2)$ group. In our case, the partition function $Z_n(s,x,y)$ is given by $\chi_{Sym(V^{\otimes n})}(s,x,y)$, where we have

$$
Z_n(s,x,y)=\chi_{Sym(V^{\otimes n})}(s,x,y)=\sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]}\chi_{[\Delta,j_1,j_2]}(s,x,y) \tag{C.108}
$$

We will restrict to the case that $n\geq 3$. In this case, we know that the only characters $\chi_{[\Delta,j_1,j_2]}(s,x,y)$ which contribute do not saturate the unitarity bound and hence do not have any null states. In this case we have

$$
\chi_{[\Delta,j_1,j_2]}(s,x,y)=\frac{s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})} \tag{C.109}
$$

Thus,

$$Z_n(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}}) = \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y)$$
.
$$\tag{C.110}$$

The right hand side of this last equation is a sum of (products of) $SU(2)$ characters, so we can treat this using our $SU(2)$ method derived above. To remove terms with negative powers we multiply both the expansions above with the factors $(1-\frac{1}{x})(1-\frac{1}{y})$. This results in

$$G_n(s,x,y) = \left[(1-\frac{1}{x})(1-\frac{1}{y})Z_n(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\right]_{\geq} \tag{C.111}$$

$$= \sum_{n,j_1,j_2} N_{[n,j_1,j_2]} s^n x^{j_1} y^{j_2} \tag{C.112}$$

where $G_n(s,x,y)$ is the generating function for primary operators. Recall that

$$\prod_{q=0}^{\infty}\prod_{a=-\frac{q}{2}}^{\frac{q}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{1-ts^{q+1}x^a y^b} = \sum_{n=0}^{\infty} t^n Z_n(s,x,y).$$

Expansion of $G_n(s,x,y)$ for $n=3,4,5$ we find

$$\begin{aligned} G_3(s,x,y) =& s^3 + s^5 xy + s^6 x^{\frac{3}{2}} y^{\frac{3}{2}} + s^7 x^2 y^2 + s^7 x^2 + s^7 y^2 + s^8 x^{\frac{5}{2}} y^{\frac{5}{2}} \\ &+ s^8 x^{\frac{3}{2}} y^{\frac{5}{2}} + s^8 x^{\frac{5}{2}} y^{\frac{3}{2}} + 2s^9 x^3 y^3 + s^9 xy^3 + s^9 x^3 y + s^{10} x^{\frac{7}{2}} y^{\frac{7}{2}} \\ &+ s^{10} x^{\frac{7}{2}} y^{\frac{5}{2}} + s^{10} x^{\frac{5}{2}} y^{\frac{7}{2}} + s^{10} x^{\frac{7}{2}} y^{\frac{3}{2}} + s^{10} x^{\frac{3}{2}} y^{\frac{7}{2}} + ... \end{aligned} \tag{C.113}$$

$$\begin{aligned} G_4(s,x,y) =& s^4 + s^6 xy + s^7 x^{\frac{3}{2}} y^{\frac{3}{2}} + 2s^8 x^2 y^2 + s^8 x^2 + s^8 xy + s^8 y^2 \\ &+ s^9 x^{\frac{5}{2}} y^{\frac{5}{2}} + s^9 x^{\frac{5}{2}} y^{\frac{3}{2}} + s^9 x^{\frac{5}{2}} y^{\frac{1}{2}} + s^9 x^{\frac{3}{2}} y^{\frac{5}{2}} + s^9 x^{\frac{3}{2}} y^{\frac{1}{2}} + s^9 x^{\frac{1}{2}} y^{\frac{5}{2}} + \\ &+ s^9 x^{\frac{1}{2}} y^{\frac{3}{2}} + ... \end{aligned} \tag{C.114}$$

$$
\begin{aligned}
G_5(s,x,y) = s^5 + s^7xy + s^8x^{\frac{3}{2}}y^{\frac{3}{2}} + 2s^9x^2y^2 + s^9x^2 + s^9y^2 + s^9xy \\
+ s^{10}x^{\frac{3}{2}}y^{\frac{3}{2}} + s^{10}x^{\frac{3}{2}}y^{\frac{5}{2}} + s^{10}x^{\frac{3}{2}}y^{\frac{1}{2}} + s^{10}x^{\frac{5}{2}}y^{\frac{3}{2}} + 2s^{10}x^{\frac{5}{2}}y^{\frac{5}{2}} \\
+ s^{10}x^{\frac{5}{2}}y^{\frac{1}{2}} + s^{10}x^{\frac{1}{2}}y^{\frac{3}{2}} + s^{10}x^{\frac{1}{2}}y^{\frac{5}{2}} + s^{10}x^{\frac{1}{2}}y^{\frac{1}{2}} + ...
\end{aligned}
\tag{C.115}
$$

C.6 Generating Function for free fermion

When considering the generating function for the free fermions, we should bare in mind that the fields are Grassman fields, they anticommute. Consequently, instead of taking the symmetric product of representations, we should be taking the antisymmetric product. Consider the matrix

$$
M = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{C.116}
$$

The relevant Schur polynomials are

$$
\chi_{\tiny\yng(1)}(M) = a + b + c
$$

$$
\chi_{\tiny\yng(1,1)}(M) = ab + ac + bc
$$

$$
\chi_{\tiny\yng(1,1,1)}(M) = abc
$$

This gives the formula

$$
\det(1 + tM) = (1 + ta)(1 + tb)(1 + tc) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M) \tag{C.117}
$$

We can prove the formula quoted above in (C.117) by starting from the integral

$$
\int \prod_{i=1}^{N} d\psi_i d\bar{\psi}^i e^{-\sum_{i,j} \psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j}
$$

which runs over the Grassman variables $\psi_i, \bar{\psi}^i$. Consider the integral

$$\int \prod_{i=1}^{N} d\psi_i d\psi^i e^{-\sum_{i,j=1}^{N} \psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j} \tag{C.118}$$

We will evaluate this integral with $N = 3$ and comment on the general case. We convert the summation in the exponential to a matrix multiplication,

$$\begin{aligned}\sum_{i,j=1}^{3} \psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j &= \begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix} \left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + t \begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix} \right) \begin{bmatrix}\bar{\psi}_1\\ \bar{\psi}_2\\ \bar{\psi}_3\end{bmatrix} \\ &= \psi(\mathbf{1} + tM)\bar{\psi}.\end{aligned} \tag{C.119}$$

The index i in the products run from 1 to 3. We expand the exponential as follows,

$$\begin{aligned}\int \prod_{i=1}^{N} d\psi_i d\psi^i e^{-\sum_{i,j=1}^{N} \psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j} &= \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \Big(1 - \psi(\mathbf{1} + tM)\bar{\psi} \\ &+ \frac{1}{2!}(\psi(\mathbf{1} + tM)\bar{\psi})^2 - \frac{1}{3!}(\psi(\mathbf{1} + tM)\bar{\psi})^3 + \cdots \Big).\end{aligned} \tag{C.120}$$

Only the fourth term yields a non-zero integral, therefore

$$\begin{aligned}\int \prod_{i=1}^{N} d\psi_i d\psi^i e^{-\sum_{i,j=1}^{N} \psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j} &= -\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \frac{1}{3!}(\psi(\mathbf{1} + tM)\bar{\psi})^3 \\ &= -\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \frac{1}{3!}\Big(\psi_1(1+ta)\bar{\psi}_1 + \psi_2(1+tb)\bar{\psi}_2 + \psi_3(1+tb)\bar{\psi}_3\Big)^3 \\ &= -\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \Big((1+ta)(1+tb)(1+tc)\Big)\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2\psi_3\bar{\psi}_3 \\ &= (1+ta)(1+tb)(1+tc) \\ &= 1 + (a+b+c)t + (ab+ac+bc)t^2 + (abc)t^3 \\ &= \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M) \\ &= \det(1 + tM)\end{aligned}$$

In general, only the term of the form $-\frac{1}{N!}(\psi(\mathbf{1} + tM)\bar{\psi})^N$ would contribute.

For the case that $M = s^D x^{J_{3,L}} y^{J_{3,R}}$ we have

$$\det(1+tM) = \prod_{t=0}^{\infty} \prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} (1 + ts^{\frac{3}{2}+q} x^a y^b).$$

From [14] the character of a left handed Weyl spinor is

$$\chi_L(s,x,y) = s^{\frac{3}{2}}(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y))P(s,x,y)$$

This formula can be simplified into

$$\chi_L(s,x,y) = s^{\frac{3}{2}} \sum_{q=0}^{\infty} s^q \chi_{\frac{q+1}{2}}(x)\chi_{\frac{q}{2}}(y).$$

The simplification is carried out as follows:

$$s^{3/2}(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y))P(s,x,y) = s^{3/2}(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y)) \sum_{p,q=0}^{\infty} s^{2p+q} \chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(x) \tag{C.121}$$

Using the formula

$$\chi_{j\otimes j'}(x) = \chi_j(x)\chi_{j'}(x) = \sum_{k=|j-j'|}^{j+j'} \chi_k(x), \tag{C.122}$$

we obtain

$$\chi_L(s,x,y) = s^{3/2} \sum_{p,q=0}^{\infty} s^{2p+q} \Bigg(\sum_{k=|\frac{q}{2}-\frac{1}{2}|}^{\frac{q}{2}+\frac{1}{2}} \chi_k(x)\chi_{\frac{q}{2}}(y) - s \sum_{k=|\frac{q}{2}-\frac{1}{2}|}^{\frac{q}{2}+\frac{1}{2}} \chi_{\frac{q}{2}}(x)\chi_k(y) \Bigg) \tag{C.123}$$

$$= s^{\frac{3}{2}} \sum_{p,q=0}^{\infty} s^{2p+q} \Bigg(\chi_{\frac{q}{2}-\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y) + \chi_{\frac{q}{2}+\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y) - s\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}-\frac{1}{2}}(y) - s\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}+\frac{1}{2}}(y) \Bigg).$$

If we shift $q \to 1$ on the first and third term, the first term then cancels with the last term and, the second and third terms simplify into

$$\chi_L(s,x,y) = s^{3/2}(1-s^2) \sum_{p,q=0}^{\infty} s^{2p+q} \chi_{\frac{q}{2}+\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y). \tag{C.124}$$

Summing over the p terms we obtain

$$\chi_L(s,x,y) = \sum_{q=0}^{\infty} s^{\frac{3}{2}+q}\chi_{\frac{q}{2}+\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y) \tag{C.125}$$

The generating function for primary operators in the free fermion CFT is given by

$$G_n(s,x,y) = \left[(1-\frac{1}{x})(1-\frac{1}{y})Z_n(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\right]_{\geq}$$

where now $Z_n(s,x,y)$ is defined by

$$\det(1+tM) = \prod_{t=0}^{\infty}\prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}(1+ts^{\frac{3}{2}+q}x^a y^b) = \sum_{n=0}^{\infty} t^n Z_n(s,x,y)$$

An expansion of $G_n(s,x,y)$ for $n=3$ yields

$$G_3(s,x,y) = s^{\frac{11}{2}}x\sqrt{y} + s^{\frac{13}{2}}x^{\frac{5}{2}} + s^{\frac{15}{2}}y^{\frac{3}{2}} + s^{\frac{15}{2}}x^3y^{\frac{3}{2}} + ...$$

C.7 Generating function of $O(N)$ vector model characters

To count the primaries in the $O(N)$ vector model, we needed explicit expressions for the characters of $V_+^{\otimes}$ projected to the trivial of $S_n[S_2]$. Here we will derive

$$\mathcal{Z}(t,Q) = \sum_{n=0}^{\infty} t^n \chi_{\mathcal{H}_n}(Q) \tag{C.126}$$

The generating function of characters for $\mathcal{H}_n$, the $S_n[S_2]$ invariant subspace of $V^{\otimes n}$, is

$$
\begin{aligned}
\mathcal{Z}(t,Q) &= \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} \sum_{\sigma \in S_n[S_2]} \mathrm{tr}_{V^{\otimes 2n}}(\sigma Q^{\otimes 2n}) \\
&= \sum_{n=0}^{\infty} t^n \sum_{p \vdash 2n} \mathcal{Z}_{\vec{p}}^{S_n[S_2]} \prod_i (\mathrm{tr} Q^i)^{p_i} \\
&= \sum_{n=0}^{\infty} t^n \sum_{p \vdash 2n} \mathcal{Z}_{\vec{p}}^{S_n[S_2]} \prod_i (\sum_a q_a^i)^{p_i}
\end{aligned} \tag{C.127}
$$

where $\mathcal{Z}_{\vec{p}}^{S_n[S_2]}$ is the number of permutations in $S_n[S_2]$ with cycle structure $\vec{p}$, divided by the order of $S_n[S_2]$. The cycle polynomials are

$$
\mathcal{Z}^{S_n[S_2]}(\vec{x}) = \sum_{p \vdash 2n} \mathcal{Z}_{\vec{p}}^{S_n[S_2]} \prod_i x_i^{p_i} \tag{C.128}
$$

The generating function of the cycle polynomial is given by

$$
\begin{aligned}
\mathcal{Z}(t,\vec{x}) &= \sum_{n=0}^{\infty} t^n \mathcal{Z}^{S_n[S_2]}(\vec{x}) \\
&= e^{\sum_{i=1}^{\infty} \frac{t^i}{2i}(x_{2i}+x_i^2)}
\end{aligned} \tag{C.129}
$$

Comparing (C.127) and (C.129) we see that

$$
\begin{aligned}
\mathcal{Z}(t,Q) &= \mathcal{Z}(t, x_i \to \sum_a q_a^i) \\
&= e^{\sum_{i=1}^{\infty} \frac{t^i}{2i}((\sum_a q_a^i)^2 + \sum_a q_a^{2i})} \\
&= e^{\sum_{i=1}^{\infty} \frac{t^i}{2i}(\sum_a \sum_b q_a^i q_b^i + \sum_a q_a^{2i})} \\
&= e^{\sum_{a,b} \sum_{i=1}^{\infty} \frac{t^i q_a^i q_b^i}{2i} + \sum_a \sum_i \frac{t^i}{2i} q_a^{2i}} \\
&= e^{-\frac{1}{2}\sum_{a,b} \log(1-tq_a q_b) - \frac{1}{2}\sum_a \log\left(1-tq_a^2\right)} \\
&= \prod_a \frac{1}{\sqrt{1-q_a^2}} \prod_{a,b} \frac{1}{\sqrt{(1-tq_a q_b)}} \\
&= \prod_a \frac{1}{\sqrt{1-q_a^2}} \prod_{a<b} \frac{1}{\sqrt{(1-tq_a q_b)}}.
\end{aligned} \tag{C.130}
$$

Appendix D

Counting Primaries from State Counting

In this appendix we will explain how (5.85) was derived. Consider the equation

$$\left[Z_3(s,x,y)(1-\frac{1}{y})\right]_{\geq} = \frac{1}{2\pi i}\oint \frac{(1-\frac{1}{z^2})Z_3(s,x,z^2)}{z-\sqrt{y}}. \tag{D.1}$$

This equation can be represented in a more simple form as follows,

$$[(1-\frac{1}{a})Z(a)]_{a\geq 0} = \frac{1}{2\pi}\oint \frac{(1-\frac{1}{z^2})}{z-\sqrt{a}}Z(z^2). \tag{D.2}$$

The LHS of this equation is obtained by performing a Laurent expansion which is truncated by removing terms with negative powers of a. The RHS is equal to the LHS since we can consider a Cauchy integral

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{z^n}{z-a}dz \tag{D.3}$$

where n is an integer and $\mathcal{C}$ is a contour around a circle with radius $|a|$. From Cauchy integrals we know that for $n < 0$ the integral is zero and, for $n \geq 0$ the results is a^n. Therefore we can conclude that

$$[Z(a)]_{a\geq 0} = \oint \frac{Z(z)}{z-a}dz. \tag{D.4}$$

If $Z(z)$ is a Laurent expansion

$$Z(z) = \sum_{-\infty}^{\infty} c_n z^n. \tag{D.5}$$

This integral

$$\oint \frac{Z(z)}{z-a} dz \tag{D.6}$$

will only keep terms with positive powers. This analysis only works for function $Z(z)$ with positive integer powers. In our case where we are dealing with the $SU(2)$ partition function $Z(z)$, half integer powers arise. We avoid this problem by taking $a \to a^2$ before we compute the contour integral and then we take $a \to \sqrt{a}$ afterwards. Mathematically it is implemented as follows

$$[Z(a)]_{a\geq 0} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{Z(z^2)}{z-\sqrt{z}}. \tag{D.7}$$

Therefore

$$[(1-\frac{1}{a})]_{\geq 0} = \frac{1}{2\pi i} \oint \frac{(1-\frac{1}{z^2})Z(z^2)}{z-\sqrt{a}}. \tag{D.8}$$

In our case we have

$$\left[Z_3(s,x,y)(1-\frac{1}{y})\right]_{\geq 0} = \oint dz \frac{(1-\frac{1}{z^2})Z_3(s,x,y)}{z-\sqrt{a}}, \tag{D.9}$$

which performs Laurent expansion of $Z(s,x,y)(1-\frac{1}{y})$ and keeps only terms with positive powers of y.

Appendix E

Constructing Primary Operators

In this appendix we will describe the construction of some non-trivial primaries.

E.1 Higher Spin Operators

To construct higher spin primary operators for a n scalar fields, the operators must obey the condition

$$
\begin{aligned}
K_\mu \mathcal{O} =& \sum_i^n \frac{\partial}{\partial x_i^\mu} \mathcal{O} \\
=& 0,
\end{aligned}
\tag{E.1}
$$

where K_μ is the generator of special conformal transformations. The condition above shows that the polynomial should be translational invariant. From these operators we need to remove null states set by the equation of motion,

$$
\sum_\mu \frac{\partial}{\partial x_i^\mu} \frac{\partial}{\partial x_i^\mu} \mathcal{O} = 0. \tag{E.2}
$$

This shows that $\mathcal{O}$ is a harmonic polynomial. We can move into the complex plane and write the x_μ as follows

$$
z_i = x_i^1 + i x_i^2, \qquad w_i = x_i^3 + i x_i^4. \tag{E.3}
$$

The translational invariants we can use to build the polynomial are

$$\frac{z_1 - z_2}{\sqrt{2}}, \quad \frac{z_1 + z_2 - 2z_3}{\sqrt{6}}, \quad \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}}, \tag{E.4}$$

which are in the hook representation of S_4. Any polynomial built using these invariants are harmonic since

$$\begin{aligned}\sum_{\mu} \frac{\partial}{\partial x_i^{\mu}} \frac{\partial}{\partial x_i^{\mu}} \mathcal{O}(z_1, z_2, z_3, z_4) =& 4\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_i} + \frac{\partial^2}{\partial w_i \partial \bar{w}_i}\right)\mathcal{O}(z_1, z_2, z_3, z_4) \\ =& 0.\end{aligned} \tag{E.5}$$

The polynomial we want to build is constructed from scalar fields, so we have to enforce bosonic statistics. We achieve this by acting with a projector

$$P_R = \frac{d_R}{4!} \sum_{\sigma \epsilon S_4} \chi_R(\sigma) \cdot \sigma \tag{E.6}$$

where $R = \yng(4)$. In the representation space this projector is given as

$$P_R = \frac{d_R}{4!} \sum_{\sigma \epsilon S_4} \chi_R(\sigma) \cdot \Gamma_S(\sigma), \tag{E.7}$$

where

$$\begin{aligned} S =& \underbrace{\yng(4,1) \otimes \cdots \otimes \yng(4,1)}_{\text{k times}} \\ =& \yng(4,1)^{\otimes k}, \end{aligned} \tag{E.8}$$

and $\Gamma_S(\sigma)$ is the matrix representation of group element σ and is given as

$$\begin{aligned} \Gamma_S(\sigma) =& \underbrace{\Gamma_{\yng(4,1)}(\sigma) \otimes \cdots \otimes \Gamma_{\yng(4,1)}(\sigma)}_{\text{k times}} \\ =& \left(\Gamma_{\yng(4,1)}(\sigma)\right)^{\otimes k}. \end{aligned} \tag{E.9}$$

We let the projector P_R act on the polynomial,

$$\underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{\text{k times}} = \underbrace{\begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \\ \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \\ \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \end{bmatrix}}_{\text{k times}} \tag{E.10}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} = \begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \\ \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \end{bmatrix}^{\otimes k},$$

where

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \\ \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \end{bmatrix}. \tag{E.11}$$

The polynomials e_1, e_2 and e_3 are harmonic and correspond to the Young-Yamanouchi states

$$e_1 \leftrightarrow \ytableaushort{134,2} \qquad e_2 \leftrightarrow \ytableaushort{124,3} \qquad e_3 \leftrightarrow \ytableaushort{123,4} \qquad ,$$

and further more the monomial z_1 corresponds to the Young diagram box labelled 1 on each Young-Yamanouchi state, z_2 corresponds to the box labelled 2, z_3 corresponds to the box labelled 3 and z_4 to the box labeled 4. This irreducible representation has dimension $d_{\ydiagram{3,1}} = 3$, and the basis for this representation are given by the Young-Yamanouchi states,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{E.12}$$

We obtain the matrix representation $\Gamma_S(\sigma)$ by acting with permutation group elements on the Young-Yamanouchi states (polynomials). Consider the action of the permutation group element (23) on the Young-Yamanouchi state e_1,

$$(23)e_1 = (23)\left(\frac{z_1 - z_2}{\sqrt{2}}\right) \tag{E.13}$$

this action exchanges the position of z_2 and z_3, therefore

$$\begin{aligned}(23)e_1 =&(23)\left(\frac{z_1 - z_2}{\sqrt{2}}\right) \\ =&\frac{z_1 - z_3}{\sqrt{2}} \\ =&\frac{1}{2}\left(\frac{z_1 - z_2}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{z_1 + z_2 - 2z_3}{\sqrt{6}}\right) \\ =&\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2.\end{aligned} \tag{E.14}$$

The action of the group element (23) on e_2 is

$$\begin{aligned}(23)e_2 =&(23)\frac{(z_1 + z_2 - 2z_3)}{\sqrt{6}} \\ =&\frac{z_1 + z_3 - 2z_2}{\sqrt{6}} \\ =&\sqrt{\frac{3}{2}}\left(\frac{z_1 - z_2}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{z_1 + z_2 - 2z_3}{\sqrt{6}}\right) \\ =&\sqrt{\frac{3}{2}}e_1 - \frac{1}{2}e_2.\end{aligned} \tag{E.15}$$

Lastly the action of the permutation group element (23) on e_3 is

$$\begin{aligned}(23)e_3 =&(12)\left(\frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}}\right) \\ =&\frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \\ =&e_3.\end{aligned} \tag{E.16}$$

Therefore the matrix representation for the group element (23) is

$$\Gamma_Q((23)) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{E.17}$$

One finds the matrix representation $\Gamma_Q((12))$ by following the same procedure as above. We act with the group element (12) on the states e_1, e_2 and e_3, this time the group element (12) interchanges the position of z_1 and z_2. We find the matrix representation

$$\Gamma_Q((12)) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{E.18}$$

The matrix representation for the group element (34) is obtained by interchanging the positions of z_3 and z_4 on the states e_1, e_2 and e_3. The result is

$$\Gamma_Q((34)) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{8}}{3} \\ 0 & \frac{\sqrt{8}}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{E.19}$$

The identity is

$$\Gamma_Q((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{E.20}$$

We can find the matrix representation of the remaining group elements of S_4 by multiplying the matrices above. When $k = 2$ we have the projector acting on the polynomial as follows

$$P_{R^2} = \frac{d_{\square\square\square\square}}{4!} \sum_{\sigma \epsilon S_4} \chi_{\square\square\square\square}(\sigma) \Gamma(\sigma) \otimes \Gamma(\sigma) \cdot \begin{bmatrix} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \end{bmatrix}^{\otimes 2}, \tag{E.21}$$

where $d_R = 1$ and $\chi_R(\sigma) = 1$. Using mathematica we find the following result for P_{R^2}

$$P_{R^2} = \frac{d_{\square\square\square\square}}{4!} \sum_{\sigma \epsilon S_4} \chi_{\square\square\square\square}(\sigma)\Gamma(\sigma) \otimes \Gamma(\sigma) \cdot \begin{bmatrix} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \end{bmatrix}^{\otimes 2} \tag{E.22}$$

$$= \begin{bmatrix} \frac{1}{12}P \\ 0 \\ 0 \\ 0 \\ \frac{1}{12}P \\ 0 \\ 0 \\ 0 \\ \frac{1}{12}P \end{bmatrix}$$

where

$$\begin{aligned} P =&(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_1 - z_4)^2 + (z_2 - z_3)^2 + (z_2 - z_4)^2 + (z_3 - z_4)^2 \\ =&\sum_{i<j}^{4} (z_i - z_j)^2, \end{aligned} \tag{E.23}$$

and $i, j = 1, \cdots 4$. The results for P_{R^3} are

$$P_{R^3} = \frac{1}{24} = \sum_{\sigma \epsilon S_4} \chi_{\square\square\square\square}(\sigma)\Gamma(\sigma) \otimes \Gamma(\sigma) \otimes \Gamma(\sigma) \cdot \begin{bmatrix} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \end{bmatrix}^{\otimes 3} \tag{E.24}$$

$$\begin{aligned} =&P_3\Big(0, \frac{1}{4\sqrt{6}}, \frac{1}{8\sqrt{3}}, \frac{1}{4\sqrt{6}}, 0, 0, \frac{1}{8\sqrt{3}}, 0, 0, \frac{1}{4\sqrt{6}}, 0, 0, 0, -\frac{1}{4\sqrt{6}}, \frac{1}{8\sqrt{3}}, 0, \frac{1}{8\sqrt{3}}, \\ &0, \frac{1}{8\sqrt{3}}, 0, 0, 0, \frac{1}{8\sqrt{3}}, 0, 0, 0, -\frac{1}{4\sqrt{6}}\Big), \end{aligned}$$

where

$$
\begin{aligned}
P_3 =&(z_4 + z_2 - 2z_3)(z_4 + z_3 - 2z_2)(z_2 + z_3 - 2z_4) \\
&+ (z_1 + z_4 - 2z_3)(z_1 + z_3 - 2z_4)(z_4 + z_3 - 2z_1) \\
&+ (z_1 + z_2 - 2z_4)(z_1 + z_4 - 2z_2)(z_2 + z_4 - 2z_1) \\
&+ (z_1 + z_2 - 2z_3)(z_1 + z_3 - 2z_2)(z_2 + z_3 - 2z_1) \\
=& \sum_{i<j<k}^{4} (z_i + z_j - z_k)(z_j + z_k - z_i)(z_k + z_i - z_j).
\end{aligned}
\tag{E.25}
$$

The construction for both P_{R^2} and P_{R^3} give one solution which shows that there is only one invariant symmetric subspace. This can also be seen from the eigenvalue spectrum of both P_{R^2} and P_{R^3} projectors, where their spectrum contains a single eigenvalue of 1 and the rest of the eigenvalues are 0. This means there is only one possible way to write a symmetric invariant polynomial.

For the case of P_{R^4}, the eigenvalue spectrum of the projector contains 4 eigenvalues of 1 and the rest are 0. And the results of the P_{R^4} projector contain 4 different solutions, which implies that there are 4 possible ways to write a symmetric invariant polynomial. This way of counting is related to the number of times the representation r ($r = \yng(4)$) appears in a tensor product $\yng(3,1)^{\otimes k}$.

We know that the matrices with the same cycle structure have the same trace and the character of tensor product is equal to the product of the characters of each representation. Therefore

$$
\begin{gathered}
\chi_{\yng(3,1)}((1)) = 3, \quad \chi_{\yng(3,1)}((..)) = 1 \quad \text{for 2 cycles}, \quad \chi_{\yng(3,1)}((...)) = 0 \quad \text{for 3 cycles}, \\
\chi_{\yng(3,1)}((..)(..)) = -1 \quad \text{for 2 2 cycles}, \quad \chi_{\yng(3,1)}((....)) = 1 \quad \text{for 4 cycles},
\end{gathered}
\tag{E.26}
$$

and the character of the tensor product is

$$
\begin{gathered}
\chi_{\yng(3,1)^{\otimes 2}}((1)) = 3^2, \quad \chi_{\yng(3,1)^{\otimes 2}}((..)) = 1 \quad \text{for 2 cycles}, \quad \chi_{\yng(3,1)^{\otimes 2}}((...)) = 0 \quad \text{for 3 cycles}, \\
\chi_{\yng(3,1)^{\otimes 2}}((..)(..)) = (-1)^2 \quad \text{for 2 2 cycles}, \quad \chi_{\yng(3,1)^{\otimes 2}}((....)) = 1 \quad \text{for 4 cycles}.
\end{gathered}
\tag{E.27}
$$

The formula for the number of times $n_r(k)$ the represenatation r ($r = \ydiagram{4}$) appears in a tensor product $S = \ydiagram{3,1}^{\otimes 2}$ is given as

$$
\begin{aligned}
n_r =& \frac{1}{24} \sum_{\sigma \epsilon S_4} \chi_r(\sigma) \chi_S(\sigma) \\
=& \frac{1}{24} \sum_{\sigma \epsilon S_4} \chi_{\ydiagram{4}}(\sigma) \chi_{\ydiagram{3,1}^{\otimes 2}}(\sigma) \\
=& \frac{1}{24} \Big(\chi_{\ydiagram{4}}((1)) \chi_{\ydiagram{3,1}^{\otimes 2}}((1)) + 6\chi_{\ydiagram{4}}((..)) \chi_{\ydiagram{3,1}^{\otimes 2}}((..)) + 8\chi_{\ydiagram{4}}((...)) \chi_{\ydiagram{3,1}^{\otimes 2}}((...)) \\
& 3\chi_{\ydiagram{4}}((..)(..)) \chi_{\ydiagram{3,1}^{\otimes 2}}((..)(..)) + 6\chi_{\ydiagram{4}}((....)) \chi_{\ydiagram{3,1}^{\otimes 2}}((....)) \Big) \\
=& \frac{1}{24} \Big((1)(3)^2 + 6(1)(1)^2 + 3(1)(0) + 3(1)(-1)^2 + 6(1)(-1)^2 \Big) \\
=& 1
\end{aligned}
\tag{E.28}
$$

and for $S = \ydiagram{3,1}^{\otimes 3}$ we have

$$
\begin{aligned}
n_r =& \frac{1}{24} \sum_{\sigma \epsilon S_4} \chi_{\ydiagram{4}}(\sigma) \chi_{\ydiagram{3,1}^{\otimes 3}}(\sigma) \\
=& \frac{1}{24} (1(3)^3 + 6(1)(1)^3 + 8(1)(0) + 3(1)(-1)^3 + 6(1)(-1)^3 \Big) \\
=& 1,
\end{aligned}
$$

and for $S = \ydiagram{3,1}^{\otimes 4}$ we obtain

$$
\begin{aligned}
n_r =& \frac{1}{24} \sum_{\sigma \epsilon S_4} \chi_{\ydiagram{4}}(\sigma) \chi_{\ydiagram{3,1}^{\otimes 4}}(\sigma) \\
=& 4.
\end{aligned}
\tag{E.29}
$$

In general $n_r(k)$ is

$$
n_{\ydiagram{4}}(k) = \frac{3^k + 6(1)^k + 9(-1)^k}{24} \qquad k > 1. \tag{E.30}
$$

E.1.1 Primary Operators built with z and w operators

Note that the operators $z, \bar{z}, w$ and $\bar{w}$ have the spin or charge assignments (j_L, j_R) given as

$$\begin{aligned} z &\leftrightarrow (\frac{1}{2}, \frac{1}{2}) \\ \bar{z} &\leftrightarrow (-\frac{1}{2}, -\frac{1}{2}) \\ w &\leftrightarrow (\frac{1}{2}, -\frac{1}{2}) \\ \bar{w} &\leftrightarrow (-\frac{1}{2}, \frac{1}{2}). \end{aligned}$$

Primary operators built above using the projector P_R and the translation invariants given above have spin or charge assignments

$$(j_L^{\frac{k}{2}}, j_R^{\frac{k}{2}}).$$

We can also build another set of primary operators from using the monomials

$$z_i = x_i^1 + ix_i^2, \qquad w_i = x_i^3 + ix_i^4. \tag{E.31}$$

The translational invariant states we use to build these primary operators are

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \\ \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \end{bmatrix}, \tag{E.32}$$

from the complex z subspace and, for the w complex subspace the translational invariants monomials are

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{w_1 - w_2}{\sqrt{2}} \\ \frac{w_1 + w_2 - 2w_3}{\sqrt{6}} \\ \frac{w_1 + w_2 + w_3 - 3w_4}{\sqrt{12}} \end{bmatrix}. \tag{E.33}$$

We combine these by forming a tensor product as follows

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^{\otimes l} = \begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \\ \frac{z_1 + z_2 + z_3 - 3z_4}{\sqrt{12}} \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} \frac{w_1 - w_2}{\sqrt{2}} \\ \frac{w_1 + w_2 - 2w_3}{\sqrt{6}} \\ \frac{w_1 + w_2 + w_3 - 3w_4}{\sqrt{12}} \end{bmatrix}^{\otimes l}. \tag{E.34}$$

We project these states onto a symmetric subspace by acting with a projector $P_{R_{zw}}$, where we have projector $\Gamma_{S_z}(\sigma)$ acting on the state

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{E.35}$$

and, projector Γ_{S_w} acting on the state

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \tag{E.36}$$

as follows

$$\begin{aligned} P_{R_{zw}} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^{\otimes l} &= \frac{d_{\Box\Box\Box\Box}}{4!} \sum_{\sigma \epsilon S_4} \chi_{\Box\Box\Box\Box}(\sigma) (\Gamma_{S_z}(\sigma))^{\otimes k} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} \otimes (\Gamma_{S_w}(\sigma))^{\otimes l} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^{\otimes l} \qquad \text{(E.37)} \\ &= \frac{1}{4!} \sum_{\sigma \epsilon S_4} (\Gamma_{S_z}(\sigma))^{\otimes k} \otimes (\Gamma_{S_w}(\sigma))^{\otimes l} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^{\otimes l} . \end{aligned}$$

The primary operators built using z and w co-ordinates have the spin or charge assignments

$$\left(j_L^{\frac{l+k}{2}}, j_R^{\frac{l-k}{2}} \right). \tag{E.38}$$

We can build another set of primary operators using z and $\bar{w}$ co-ordinates in the same way,

$$P_{R_{z\bar{w}}} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \end{bmatrix}^{\otimes l} = \frac{1}{4!} \sum_{\sigma \epsilon S_4} (\Gamma_{S_z}(\sigma))^{\otimes k} \otimes (\Gamma_{S_{\bar{w}}}(\sigma))^{\otimes l} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \end{bmatrix}^{\otimes l} . \tag{E.39}$$

These operators will have the charge assignments

$$\left(j_L^{\frac{l-k}{2}}, j_R^{\frac{l+k}{2}}\right). \tag{E.40}$$

The number of times $N_{r=\yng(4)}(k,l)$ the representation $r = \yng(4)$ appears in the tensor product $\yng(3,1)^{\otimes(k+l)}$ is

$$N_r(k,l) = \frac{3^{k+l} + 6(1)^{k+l} + 9(-1)^{k+l}}{24} \qquad k,l > 1. \tag{E.41}$$

Using (E.37), $k = 1$ and $l = 1$ we obtain that

$$P_{Rzw} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathcal{O}^{(k=1,l=1)}(z,w)\left(\frac{1}{12}, 0, 0, 0, \frac{1}{12}, 0, 0, 0, \frac{1}{12}\right), \tag{E.42}$$

where $\mathcal{O}^{(k=1,l=1)}(z,w)$ is

$$\begin{aligned} \mathcal{O}^{(1,1)}(z,w) =& (z_1 - z_2)(w_1 - w_2) + (z_1 - z_3)(w_1 - w_3) + (z_1 - z_4)(w_1 - w_4) \\ &+ (z_2 - z_3)(w_2 - w_3) + (z_2 - z_4)(w_2 - w_4) + (z_3 - z_4)(w_3 - w_4) \\ =& \sum_{i<j}^{4} (z_i - z_j)(w_i - w_j). \end{aligned} \tag{E.43}$$

The projector P_{Rzw} maps the translational invariant state

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{E.44}$$

into a symmetric subspace $\yng(4)$, where the polynomial is made up of the invariants $(z_i - z_j)$ and $(w_i - w_j)$. These invariants combine into a product $(z_i - z_j)(w_i - w_j)$, and all the possible combinations of i, j of this product are summed with the restriction that $i > j$. Hence

$$\mathcal{O}^{(1,1)}(z,w) = \sum_{i<j}^{4} (z_i - z_j)(w_i - w_j). \tag{E.45}$$

When $k = 0$ and $l = 2$ we find

$$P_{Rzw} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^{\otimes 2} = \mathcal{O}^{(k=0,l=2}(w)\left(\frac{1}{12}, 0, 0, 0, \frac{1}{12}, 0, 0, 0, \frac{1}{12}\right), \tag{E.46}$$

where $\mathcal{O}^{(k=0,l=2)}(w)$ is

$$\begin{aligned}\mathcal{O}^{(0,2)}(w) =&(w_1 - w_2)^2 + (w_1 - w_3)^2 + (w_1 - w_4)^2 + (w_2 - w_3)^2 + (w_2 - w_4)^2 \\ &+ (w_3 - w_4)^2 \\ =&\sum_{i<j}^{4}(w_i - w_j)^2.\end{aligned} \tag{E.47}$$

Here the operator P_{Rzw} maps the translational invariant polynomial

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^{\otimes 2}$$

into a symmetric subspace where the polynomial is made up of the invariants $(w_i - w_j)^2$ and, all the possible combinations of $(w_i - w_j)^2$ are summed with a restriction that $i < j$.

When $k = 2$ and $l = 1$ we obtain

$$\begin{aligned}P_{Rzw} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^{\otimes 2} \otimes \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} =&\mathcal{O}^{(2,1)}(z, w)\Bigg(0, -\frac{1}{72\sqrt{6}}, -\frac{1}{144\sqrt{3}}, \frac{1}{72\sqrt{6}}, 0, 0, -\frac{1}{144\sqrt{3}}, \\ &0, 0, -\frac{1}{72\sqrt{6}}, 0, 0, 0, \frac{1}{72\sqrt{6}}, -\frac{1}{144\sqrt{3}}, 0, -\frac{1}{144\sqrt{3}}, 0, \\ &-\frac{1}{144\sqrt{3}}, 0, 0, 0, -\frac{1}{144\sqrt{3}}, 0, 0, 0, \frac{1}{72\sqrt{3}}\Bigg),\end{aligned} \tag{E.48}$$

where $\mathcal{O}^{(2,1)}(z, w)$ is

$$
\begin{aligned}
\mathcal{O}^{(2,1)}(z,w) =&(z_1+z_4-2z_3)(z_1+z_3-2z_4)(w_4+w_3-2w_1) \\
&+(z_1+z_4-2z_3)(w_1+w_3-2w_4)(z_4+z_3-2z_1) \\
&+(w_1+w_4-2w_3)(z_1+z_3-2z_4)(z_4+z_3-2z_1) \\
&+(w_1+w_2-2w_4)(z_1+z_4-2z_2)(z_2+z_4-2z_1) \\
&+(z_1+z_2-2z_4)(w_1+w_4-2w_2)(z_2+z_4-2z_1) \\
&+(z_1+z_2-2z_4)(z_1+z_4-2z_2)(w_2+w_4-2w_1) \\
&+(w_1+w_2-2w_3)(z_1+z_3-2z_2)(z_2+z_3-2z_1) \\
&+(z_1+z_2-2z_3)(w_1+w_3-2w_2)(z_2+z_3-2z_1) \\
&+(z_1+z_2-2z_3)(z_1+z_3-2z_2)(w_2+w_3-2w_1) \\
&+(w_4+w_2-2w_3)(z_4+z_3-2z_2)(z_2+z_3-2z_4) \\
&+(z_4+z_2-2z_3)(w_4+w_3-2w_2)(z_2+z_3-2z_4) \\
&+(z_4+z_2-2z_3)(z_4+z_3-2z_2)(w_2+w_3-2w_4) \\
=&\sum_{i<j<k}^{4}\Big[(z_i+z_j-z_k)(z_j+z_k-z_i)(w_k+w_i-w_j) \\
&+(z_i+z_j-z_k)(w_j+w_k-w_i)(z_k+z_i-z_j) \\
&+(w_i+w_j-w_k)(z_j+z_k-z_i)(z_k+z_i-z_j)\Big].
\end{aligned}
\tag{E.49}
$$

Here the projector produces a symmetric polynomial $\mathcal{O}^{(2,1)}(z,w)$ which is composed of invariants $(z_i+z_j-2z_k)$ and $(w_i+w_j-2w_k)$. The polynomial $\mathcal{O}^{(2,1)}$ contains a sum of all the possible distinct products of

$$(z_i+z_j-2z_k)(z_j+z_k-z_i)(w_k+w_i-w_j).$$

E.1.2 Primary operators for 3 scalar fields

Consider the projection operator for 3 scalar fields

$$P_{Rzw} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{\otimes l} = \frac{1}{6} \sum_{\sigma \epsilon S_3} \chi_{\yng(3)}(\sigma)(\Gamma_{S_z}(\sigma))^{\otimes k} \otimes (\Gamma_{S_w}(\sigma))^{\otimes l} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^{\otimes k} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{\otimes l}, \tag{E.50}$$

where

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{z_1 - z_2}{\sqrt{2}} \\ \frac{z_1 + z_2 - 2z_3}{\sqrt{6}} \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{w_1 - w_2}{\sqrt{2}} \\ \frac{w_1 + w_2 - 2w_3}{\sqrt{6}} \end{bmatrix},$$

and

$$(\Gamma_{S_z}(\sigma))^{\otimes k} = \underbrace{\Gamma_{\yng(2,1)}(\sigma) \otimes \cdots \otimes \Gamma_{\yng(2,1)}(\sigma)}_{\text{k times}}. \tag{E.51}$$

From equation (E.50) above the representation $\yng(3)$ appears $N_{\yng(3)}(k,l)$ times,

$$N_{\yng(3)}(k,l) = \frac{2^{k+1} + 2(-1)^{k+l}}{6}, \tag{E.52}$$

in the tensor product $\yng(2,1)^{\otimes(k+l)}$. This integer counts the number of different solutions equation (E.50) can have. When $k = 3$ and $l = 1$, this equation will have 3 solutions since

$$\begin{aligned} N_{\yng(3)}(k,l) =& \frac{2^{k+l} + 2(-1)^{k+l}}{6} \\ =& \frac{2^{3+1} + 2(-1)^{3+1}}{6} \\ =& 3. \end{aligned} \tag{E.53}$$

Which means there are 3 different subspaces carrying the representation $\yng(3)$. To obtain a single solution we act with the projector $P_{\yng(2,2)}$ on equation (E.50), given as

$$P_{\yng(2,2)} = \frac{d_{\yng(2,2)}}{24} \sum_{\sigma \epsilon S_4} \chi_{\yng(2,2)}(\sigma)\Gamma(\sigma), \tag{E.54}$$

where $d_{\boxplus} = 2$ and

$$\Gamma((12)) = P_{4\times 4} \otimes \mathbb{1}_{2\times 2} \otimes \mathbb{1}_{2\times 2} \qquad \Gamma((23)) = \mathbb{1}_{2\times 2} \otimes P_{4\times 4} \otimes \mathbb{1}_{2\times 2} \tag{E.55}$$

$$\Gamma((34)) = \mathbb{1}_{2\times 2} \otimes \mathbb{1}_{2\times 2} \otimes P_{4\times 4} \tag{E.56}$$

and

$$P_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{E.57}$$

The matrix representation $\Gamma_{\boxplus}(\sigma)$ corresponding to the character $\chi_{\boxplus}(\sigma)$ is

$$\Gamma((12)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \Gamma((23)) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad \Gamma((34)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{E.58}$$

Therefore

$$\chi_{\boxplus}((\cdot\cdot)) = 0, \qquad \chi_{\boxplus}((\cdot\cdot)(\cdot\cdot)) = 2, \qquad \chi_{\boxplus}((\cdot\cdot\cdot)) = -1 \qquad \chi_{\boxplus}((\cdot\cdot\cdot\cdot)) = 0. \tag{E.59}$$

The projection operator $P_{\boxplus} \cdot P_{Rzw}$ has eigenvalues

$$\text{Eigenvalues}[P_{\boxplus} \cdot P_{Rzw}] = \Big(1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big), \tag{E.60}$$

and it projects the state

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^{\otimes 3} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{\otimes 1}.$$

This results in the formula

$$P_{\boxplus} \cdot P_{Rzw} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^{\otimes 3} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{\otimes 1} = P_{\boxplus} \cdot \left[\frac{1}{6} \sum_{\sigma \epsilon S_3} \chi_{\square\square\square}(\sigma) (\Gamma_{S_z}(\sigma))^{\otimes 3} \otimes (\Gamma_{S_w}(\sigma))^{\otimes 1} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^{\otimes 3} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{\otimes 1} \right] \tag{E.61}$$

Appendix F

Testing the Primary Condition

F.1 Primary Operators

In this appendix we will illustrate that the many-body polynomials we have constructed are indeed primary operators once we transform back to the CFT language. We will make use of the conformal group algebra

$$\begin{aligned}
&[K_\mu, P_\nu] = 2M_{\mu\nu} - 2D\delta_{\mu\nu} \\
&[D, P_\mu] = P_\mu \\
&[D, K_\mu] = -K_\mu \\
&[M_{\mu\nu}, K_\alpha] = \delta_{\nu\alpha}K_\mu - \delta_{\mu\alpha}K_\nu \\
&[M_{\mu\nu}, K_\alpha] = \delta_{\nu\alpha}P_\mu - \delta_{\mu\alpha}P_\nu
\end{aligned} \tag{F.1}$$

Consider the polynomial

$$\Psi = (w^{(3)}(\bar{z}^{(2)} - \bar{z}^{(1)}) + w^{(2)}(\bar{z}^{(1)} - \bar{z}^{(3)}) + w^{(1)}(\bar{z}^{(3)} - \bar{z}^{(2)}))^{2n}, \tag{F.2}$$

A simple case we will take is $n = 2$. We will make the translation

$$z_1^k w_1^l z_2^q w_2^r z_3^t w_3^v \;\leftrightarrow\; \left(\frac{P_z^k P_w^l}{k!l!}\phi\right)\left(\frac{P_z^q P_w^r}{q!r!}\phi\right)\left(\frac{P_z^t P_w^v}{t!v!}\phi\right) \tag{F.3}$$

to obtain

$$\Psi = \frac{1}{4}\phi P_{\bar{z}}^2\phi P_w^2\phi - \frac{1}{2}P_{\bar{z}}^2\phi(P_w\phi)^2 - \phi(P_{\bar{z}}P_w\phi)^2 - \frac{1}{2}(P_{\bar{z}}\phi)^2 P_w^2\phi + 2(P_{\bar{z}}P_w\phi)(P_w\phi)(P_{\bar{z}}\phi) \tag{F.4}$$

where $P_{\bar{z}} = P_1 - iP_2$ and $P_w = P_3 + iP_4$. Acting with the special conformal operator K_μ on Ψ, the operator Ψ should be annihilated if it is a primary operator. Action of K_μ on Ψ gives

$$\begin{aligned}
K_\mu\psi =& \frac{1}{4}\phi K_\mu P_{\bar{z}}^2\phi P_w^2\phi + \frac{1}{4}\phi P_{\bar{z}}^2\phi K_\mu P_w^2\phi && \text{(F.5)}\\
&- \frac{1}{2}K_\mu P_{\bar{z}}^2\phi P_w\phi P_w\phi - \frac{1}{2}P_{\bar{z}}^2\phi K_\mu P_w\phi P_w\phi - \frac{1}{2}P_{\bar{z}}^2\phi P_w\phi K_\mu P_w\phi \\
&- \phi K_\mu P_{\bar{z}}P_w\phi P_{\bar{z}}P_w\phi - \phi P_{\bar{z}}P_w\phi K_\mu P_{\bar{z}}P_w\phi && \text{(F.6)}\\
&- \frac{1}{2}K_\mu P_{\bar{z}}\phi P_{\bar{z}}\phi P_w^2\phi - \frac{1}{2}P_{\bar{z}}\phi K_\mu P_{\bar{z}}\phi P_w^2\phi - \frac{1}{2}P_{\bar{z}}\phi P_{\bar{z}}\phi K_\mu P_w^2\phi \\
&+ 2K_\mu P_{\bar{z}}P_w\phi P_w\phi P_{\bar{z}}\phi + 2P_{\bar{z}}P_w\phi K_\mu P_w\phi P_{\bar{z}}\phi + 2P_{\bar{z}}P_w\phi P_w\phi K_\mu P_{\bar{z}}\phi
\end{aligned}$$

First consider the term

$$\begin{aligned}
K_\mu P_{\bar{z}}^m\phi =& \sum_{r=0}^{m-1} P_{\bar{z}}^r[K_\mu, P_{\bar{z}}]P_{\bar{z}}^{m-1-r}\phi && \text{(F.7)}\\
=& \sum_{r=0}^{m-1} P_{\bar{z}}^r[K_\mu, P_1 - iP_2]P_{\bar{z}}^{m-1-r}\phi \\
=& \sum_{r=0}^{m-1} P_{\bar{z}}^r\Big(2M_{\mu1} - 2D\delta_{\mu1} - 2iM_{\mu2} + 2iD\delta_{\mu2}\Big)P_{\bar{z}}^{m-1-r}\phi \\
=& \sum_{r=0}^{m-1} P_{\bar{z}}^r\Big(- 2(\delta_{\mu1} - i\delta_{\mu2})D + (M_{\mu1} - iM_{\mu2}\Big)P_{\bar{z}}^{m-1-r}\phi. && \text{(F.8)}
\end{aligned}$$

Using the formula

$$
\begin{aligned}
2\sum_{r=0}^{m-1} P_{\bar{z}}^{r} M_{\mu 1} P_{\bar{z}}^{m-1-r}\phi = &2\sum_{r=0}^{m-1}\sum_{s=0}^{m-r-2} P_{\bar{z}}^{r} P_{\bar{z}}^{s}[M_{\mu 1}, P_{\bar{z}}] P_{\bar{z}}^{m-r-s-2}\phi \\
=&2\sum_{r=0}^{m-1}\sum_{s=0}^{m-r-2} P_{\bar{z}}^{r} P_{\bar{z}}^{s}[M_{\mu 1}, P_1 - iP_2] P_{\bar{z}}^{m-r-s-2}\phi \\
=&2\sum_{r=0}^{m-1}\sum_{s=0}^{m-r-2} P_{\bar{z}}^{r} P_{\bar{z}}^{s}\Big(\delta_{11}P_\mu - \delta_{\mu 1}P_1 - i\delta_{12}P_\mu + i\delta_{\mu 2}P_1\Big) P_{\bar{z}}^{m-r-s-2}\phi \\
=&\sum_{r=0}^{m-1}\sum_{s=0}^{m-r-2}\Big(P_\mu - \delta_{\mu 1}P_1 + i\delta_{\mu 2}P_1\Big) P_{\bar{z}}^{m-1}\phi \\
=&-2(\delta_{\mu 1} - i\delta_{\mu 2})\sum_{r=0}^{m-1}(m-r-1)P_1 P_{\bar{z}}^{m-2}\phi + 2\sum_{r=0}^{m-1}(m-r-1)P_\mu P_{\bar{z}}^{m-2}\phi \\
=&-2(\delta_{\mu 1} - i\delta_{\mu 2})\Big(m^2 - \frac{m(m-1)}{2} - m\Big)P_1 P_{\bar{z}}^{m-2}\phi \\
&+2\Big(m^2 - \frac{m(m-1)}{2} - m\Big)P_\mu P_{\bar{z}}^{m-2}\phi,
\end{aligned}
\tag{F.9}
$$

it follows that

$$
\begin{aligned}
2\sum_{r=0}^{m-1} P_{\bar{z}}^{r} M_{\mu 1} P_{\bar{z}}^{m-1-r}\phi = &2i(\delta_{\mu 1} - i\delta_{\mu 2})\Big(m^2 - \frac{m(m-1)}{2} - m\Big)P_2 P_{\bar{z}}^{m-2}\phi \\
&-2\Big(m^2 - \frac{m(m-1)}{2} - m\Big)P_\mu P_{\bar{z}}^{m-2}\phi.
\end{aligned}
\tag{F.10}
$$

Note that in the computation above we have used the fact that $M_{\mu\nu}\phi = 0$. We also know that

$$
DP_{\bar{z}}\phi^{m-r-1}\phi = (m-r)P_{\bar{z}}^{m-r-1}\phi. \tag{F.11}
$$

Plugging (F.9), (F.10), (F.11) back into (F.7) into we obtain

$$
K_\mu P_{\bar{z}}^{m}\phi = -2m^2(\delta_{\mu 1} - i\delta_{\mu 2})P_{\bar{z}}^{m-1}\phi. \tag{F.12}
$$

For $m = 2$

$$K_\mu P_{\bar{z}}^2\phi = -8(\delta_{\mu 1} - i\delta_{\mu 2})P_{\bar{z}}\phi \tag{F.13}$$

Using the same idea to treat the term $K_\mu P_{\bar{z}}P_w\phi$ we find

$$K_\mu P_{\bar{z}}P_w\phi = -4(\delta_{\mu 1} - i\delta_{\mu 2})P_w\phi - 4(\delta_{\mu 3} + i\delta_{\mu 4})P_{\bar{z}}\phi. \tag{F.14}$$

Denoting $A = -(\delta_{\mu 1} - i\delta_{\mu 2})$ and $B = -(\delta_{\mu 3} + i\delta_{\mu 4})$ and plugging back the above results back in (F.5) we find

$$K_\mu\Psi = 0. \tag{F.15}$$

This shows that the operator Ψ is indeed a primary operator.

Lets consider another set of primary operators that are constructed from taking a determinant of the coordinates variables. Consider the operator

$$Q = \det\begin{bmatrix} z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ (z^{(1)})^2 & (z^{(2)})^2 & (z^{(3)})^2 & (z^{(4)})^2 \\ \bar{z}^{(1)} & \bar{z}^{(2)} & \bar{z}^{(3)} & \bar{z}^{(4)} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{F.16}$$

,

For odd powers of Q acting on the scalar field $\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\big|_{z_1\cdots z_4=z}$ yield zero operators and even powers of Q give non-zero primary operators. Consider Q acting on $\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\big|_{z_1\cdots z_4=z}$,

$$
\begin{aligned}
Q[\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)]\Big|_{z_1\cdots z_4=z} =&\Big(z^{(1)}(z^{(2)})^2\bar{z}^{(3)} - z^{(1)}(z^{(2)})^2\bar{z}^{(4)} - z^{(1)}(z^{(3)})^2\bar{z}^{(2)} \\
&+ z^{(1)}(z^{(3)})^2\bar{z}^{(4)} + z^{(1)}(z^{(4)})^2\bar{z}^{(2)} - z^{(1)}(z^{(4)})^2\bar{z}^{(3)} \\
&- z^{(2)}(z^{(1)})^2\bar{z}^{(3)} + z^{(2)}(z^{(1)})^2\bar{z}^{(4)} + z^{(2)}(z^{(3)})^2\bar{z}^{(1)} \\
&- z^{(2)}(z^{(3)})^2\bar{z}^{(4)} - z^{(2)}(z^{(4)})^2\bar{z}^{(1)} + z^{(2)}(z^{(4)})^2\bar{z}^{(3)} \\
&+ z^{(3)}(z^{(1)})^2\bar{z}^{(2)} - z^{(3)}(z^{(1)})^2\bar{z}^{(4)} - z^{(3)}(z^{(2)})^2\bar{z}^{(1)} \\
&+ z^{(3)}(z^{(2)})^2\bar{z}^{(4)} + z^{(3)}(z^{(4)})^2\bar{z}^{(1)} - z^{(3)}(z^{(4)})^2\bar{z}^{(2)} \\
&- z^{(4)}(z^{(1)})^2\bar{z}^{(2)} + z^{(4)}(z^{(1)})^2\bar{z}^{(3)} + z^{(4)}(z^{(2)})^2\bar{z}^{(1)} \\
&- z^{(4)}(z^{(2)})^2\bar{z}^{(3)} - z^{(4)}(z^{(3)})^2\bar{z}^{(1)} + z^{(4)}(z^{(3)})^2\bar{z}^{(2)}\Big) \\
&\times [\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)]\Big|_{z_1\cdots z_4=z}.
\end{aligned}
\tag{F.17}
$$

Making the transformation $x^k \to \frac{P^k}{k!}$ we obtain

$$
\begin{aligned}
Q[\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)]\Big|_{z_1\cdots z_4=z} =&P_z\phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi\phi - P_z\phi\frac{P_z^2}{2!}\phi\phi P_{\bar{z}}\phi - P_z\phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi\phi \\
&+ P_z\phi\phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi + P_z\phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi\phi - P_z\phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi\phi \\
&- \frac{P_z^2}{2!}\phi P_z\phi P_{\bar{z}}\phi\phi + \frac{P_z^2}{2!}\phi P_z\phi\phi P_{\bar{z}}\phi + \bar{z}\phi P_z\phi\frac{P_z^2}{2!}\phi\phi \\
&- \phi P_z\phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi - P_{\bar{z}}\phi P_z\phi\phi\frac{P_z^2}{2!}\phi + \phi P_z\phi P_{\bar{z}}\phi\frac{P_z^2}{2!}\phi \\
&+ \frac{P_z^2}{2!}\phi P_{\bar{z}}\phi P_z\phi\phi - \frac{P_z^2}{2!}\phi\phi P_z\phi P_{\bar{z}}\phi - P_{\bar{z}}\phi\frac{P_z^2}{2!}P_z\phi\phi \\
&+ \phi\frac{P_z^2}{2!}\phi P_z\phi P_{\bar{z}}\phi + \phi P_{\bar{z}}\phi P_z\phi\frac{P_z^2}{2!}\phi - \phi P_{\bar{z}}\phi P_z\phi\frac{P_z}{2!}\phi \\
&- \frac{P_z^2}{2!}\phi P_{\bar{z}}\phi\phi P_z\phi + \frac{P_z^2}{2!}\phi\phi P_{\bar{z}}\phi P_z\phi + P_{\bar{z}}\phi\frac{P_z^2}{2!}\phi\phi P_z\phi \\
&- \phi\frac{P_z^2}{2!}\phi P_{\bar{z}}\phi P_z\phi - P_{\bar{z}}\phi\phi\frac{P_z^2}{2!}\phi P_z\phi + \phi P_{\bar{z}}\phi\frac{P_z^2}{2!}\phi P_z\phi \\
=&0
\end{aligned}
\tag{F.18}
$$

For even powers of Q we get non-zero results. For Q^2 we get

$$Q^2 = \det\begin{bmatrix} z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ (z^{(1)})^2 & (z^{(2)})^2 & (z^{(3)})^2 & (z^{(4)})^2 \\ \bar{z}^{(1)} & \bar{z}^{(2)} & \bar{z}^{(3)} & \bar{z}^{(4)} \\ 1 & 1 & 1 & 1 \end{bmatrix} \times \det\begin{bmatrix} z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ (z^{(1)})^2 & (z^{(2)})^2 & (z^{(3)})^2 & (z^{(4)})^2 \\ \bar{z}^{(1)} & \bar{z}^{(2)} & \bar{z}^{(3)} & \bar{z}^{(4)} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{F.19}$$

Note that when the derivatives act on the fields, the cross terms are equal and we add them together. The non-cross terms are also equal and we add these together too. Therefore we have

$$Q^2 = 4(z^{(1)})^2 \begin{vmatrix} (z_2)^2 & (z_3)^2 & (z_4)^2 \\ \bar{z}_2 & \bar{z}_3 & \bar{z}_4 \\ 1 & 1 & 1 \end{vmatrix}^2 - 12 z_1 z_2 \begin{vmatrix} (z_2)^2 & (z_3)^2 & (z_4)^2 \\ \bar{z}_2 & \bar{z}_3 & \bar{z}_4 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} (z_1)^2 & (z_3)^2 & (z_4)^2 \\ \bar{z}_1 & \bar{z}_3 & \bar{z}_4 \\ 1 & 1 & 1 \end{vmatrix}, \tag{F.20}$$

and

$$\begin{aligned} Q^2[\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)]\Big|_{z_1,\cdots,z_4=z} &= 4z_1^2\Big((z_2)^2(\bar{z}_3-\bar{z}_4)-(z_3)^2(\bar{z}_2-\bar{z}_4)+(z_4)^2(\bar{z}_2-\bar{z}_3)\Big)^2 \qquad \text{(F.21)} \\ &- 12 z_2 z_1\Big((z_2)^2(\bar{z}_3-\bar{z}_4)-(z_3)^2(\bar{z}_2-\bar{z}_4)+(z_4)^2(\bar{z}_2-\bar{z}_3)\Big)^2 \\ &\times \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\Big|_{z_1,\cdots,z_4=z}, \end{aligned}$$

these becomes at $z_1, \cdots, z_4 = z$

$$
\begin{aligned}
Q^2\phi^4(z) =& -\frac{1}{2}P_z^2\phi P_z^4\phi P_{\bar{z}}\phi P_{\bar{z}}\phi - 2P_z^2\phi P_z^2\phi P_z^2 P_{\bar{z}}\phi P_{\bar{z}}\phi - P_z^2\phi P_z^2 P_{\bar{z}}\phi P_z^2 P_{\bar{z}}\phi\phi \qquad \text{(F.22)}\\
& -\frac{3}{2}P_z^2\phi P_{\bar{z}}^2\phi P_z^2\phi P^2\phi + 4P_z^2\phi P_{\bar{z}}\phi P_z^2 P_{\bar{z}}\phi P_z^2\phi - 2P_z^2\phi P_z^2 P_{\bar{z}}\phi P^2 P_{\bar{z}}\phi\phi\\
& +\frac{1}{4}P_z^2\phi P_z^4\phi P_{\bar{z}}^2\phi\phi - \frac{1}{3}P_z^3\phi P_z^3\phi P_{\bar{z}}^2\phi\phi + \frac{2}{3}P_z^3\phi P_z^3\phi P_{\bar{z}}\phi P_{\bar{z}}\phi\\
& - P_z P_{\bar{z}}\phi P_z P_{\bar{z}}\phi P_z^4\phi\phi + 2P_z\phi P_z P_{\bar{z}}\phi P_z^4\phi P_{\bar{z}}\phi - \frac{1}{2}P_z\phi P_z\phi P_z^4\phi P_{\bar{z}}^2\phi\\
& + 6P_z P_{\bar{z}}\phi P_z P_{\bar{z}}\phi P_z^2\phi P_z^2\phi - 12P_z\phi P_z P_{\bar{z}}\phi P^2 P_{\bar{z}}\phi P_z^2\phi\\
& + 6P_z\phi P_z\phi P_z^2 P_{\bar{z}}\phi P_z^2 P_{\bar{z}}\phi + 4P_z P_{\bar{z}}\phi P_z^3\phi P_z^2 P_{\bar{z}}\phi\phi - 4P_z\phi P_z^3\phi P_z^2 P_{\bar{z}}\phi P_{\bar{z}}\phi\\
& - 4P_z P_{\bar{z}}\phi P_z^3\phi P_z^2\phi P_{\bar{z}}\phi + \frac{1}{3}P_z\phi P^3\phi P_z^2\phi P_{\bar{z}}^2\phi.
\end{aligned}
$$

We can verify that $Q^2\phi^4(z)$ is a Primary operator by acting with a special conformal operator K_μ. We will first act on each term on (F.22) by K_μ then add up the terms together. Let $A = \delta_{1\mu} - i\delta_{2\mu}$ and $B = \delta_{3\mu} + i\delta_{4\mu}$. Then

$$
\begin{aligned}
K_\mu(P_z^2\phi P_z^4\phi P_{\bar{z}}\phi P_{\bar{z}}\phi) =& 8BP_z\phi P_z^4\phi P_{\bar{z}}\phi P_{\bar{z}}\phi + 32BP_z^2\phi P_z^3\phi P_{\bar{z}}\phi P_{\bar{z}}\phi + 4AP_z^2\phi P_z^4\phi P_{\bar{z}}\phi\phi \qquad \text{(F.23)}\\
=& O_1
\end{aligned}
$$

$$
\begin{aligned}
K_\mu(P_z^2\phi P_z^2\phi P_z^2 P_{\bar{z}}\phi P_{\bar{z}}\phi) =& 16BP_z\phi P_z^2\phi P_z^2 P_{\bar{z}}\phi P_{\bar{z}}\phi + 8BP_z^2\phi P_z^2\phi P_z P_{\bar{z}}\phi P_{\bar{z}}\phi \qquad \text{(F.24)}\\
& 2AP_z^2\phi P_z^2\phi P_z^2\phi P_{\bar{z}}\phi + 2AP_z^2\phi P_z^2\phi P_z^2 P_{\bar{z}}\phi\phi\\
=& O_2,
\end{aligned}
$$

$$
\begin{aligned}
K_\mu(P_z^2\phi P_z^2 P_{\bar{z}}\phi P_z^2 P_{\bar{z}}\phi\phi) =& 8BP_z\phi P_z^2 P_{\bar{z}}\phi P_z^2 P_{\bar{z}}\phi\phi + 16BP_z^2\phi P_z P_{\bar{z}}\phi P_z^2 P_{\bar{z}}\phi\phi \qquad \text{(F.25)}\\
& + 4AP_z^2\phi P_z^2 P_{\bar{z}}\phi P_z^2\phi\phi\\
=& O_3,
\end{aligned}
$$

$$
\begin{aligned}
K_\mu(P_z^2\phi P_{\bar{z}}^2\phi P_z^2\phi P^2\phi) =& 24BP_z\phi P_{\bar{z}}^2\phi P_z^2\phi P^2\phi + 8AP_z^2\phi P_{\bar{z}}\phi P_z^2\phi P^2\phi \qquad \text{(F.26)}\\
=& O_4,
\end{aligned}
$$

$$
\begin{aligned}
K_\mu(P_z^2\phi P_{\bar z}\phi P_z^2 P_{\bar z}\phi P_z^2\phi) =&16BP_z\phi P_{\bar z}\phi P_z^2 P_{\bar z}\phi P_z^2\phi + 2AP_z^2\phi P_z^2 P_{\bar z}\phi P_z^2\phi\phi \\
&+ BP_z^2\phi P_{\bar z}\phi P_z P_{\bar z}\phi P_z^2\phi + 2AP_z^2\phi P_{\bar z}\phi P_z^2\phi P_z^2\phi \\
=&O_5,
\end{aligned} \tag{F.27}
$$

$$
\begin{aligned}
K_\mu(P_z^2\phi P_z^2 P_{\bar z}\phi P^2 P_{\bar z}\phi\phi) =&8BP_z\phi P_z^2 P_{\bar z}\phi P^2 P_{\bar z}\phi\phi + 16BP_z^2\phi P_z P_{\bar z}\phi P^2 P_{\bar z}\phi\phi \\
&+ 4AP_z^2\phi P_z^2\phi P^2 P_{\bar z}\phi\phi \\
=&O_6,
\end{aligned} \tag{F.28}
$$

$$
\begin{aligned}
K_\mu(P_z^2\phi P_z^4\phi P_{\bar z}^2\phi\phi) =&8BP_z\phi P_z^4\phi P_{\bar z}^2\phi\phi + 32BP_z^2\phi P_z^3\phi P_{\bar z}^2\phi\phi \\
&+ 8AP_z^2\phi P_z^4\phi P_{\bar z}\phi\phi \\
=&O_7,
\end{aligned} \tag{F.29}
$$

$$
\begin{aligned}
K_\mu(P_z^3\phi P_z^3\phi P_{\bar z}^2\phi\phi) =&36BP_z^2\phi P_z^3\phi P_{\bar z}^2\phi\phi + 8AP_z^3\phi P_z^3\phi P_{\bar z}\phi\phi \\
=&O_8,
\end{aligned} \tag{F.30}
$$

$$
\begin{aligned}
K_\mu(P_z^3\phi P_z^3\phi P_{\bar z}\phi P_{\bar z}\phi) =&36BP_z^2\phi P_z^3\phi P_{\bar z}\phi P_{\bar z}\phi + 4AP_z^3\phi P_z^3\phi P_{\bar z}\phi\phi \\
=&O_9,
\end{aligned} \tag{F.31}
$$

$$
\begin{aligned}
K_\mu(P_z P_{\bar z}\phi P_z P_{\bar z}\phi P_z^4\phi\phi) =&4BP_{\bar z}\phi P_z P_{\bar z}\phi P_z^4\phi\phi + 4AP_z\phi P_z P_{\bar z}\phi P_z^4\phi\phi \\
&+ 32BP_z P_{\bar z}\phi P_z P_{\bar z}\phi P_z^3\phi\phi \\
=&O_{10},
\end{aligned} \tag{F.32}
$$

$$
\begin{aligned}
K_\mu(P_z\phi P_z P_{\bar z}\phi P_z^4\phi P_{\bar z}\phi) =&2B\phi P_z P_{\bar z}\phi P_z^4\phi P_{\bar z}\phi + 2BP_z\phi P_{\bar z}\phi P_z^4\phi P_{\bar z}\phi \\
&+ 2AP_z\phi P_z\phi P_z^4\phi P_{\bar z}\phi + 32BP_z\phi P_z P_{\bar z}\phi P_z^3\phi P_{\bar z}\phi \\
&+ 2AP_z\phi P_z P_{\bar z}\phi P_z^4\phi\phi \\
=&O_{11},
\end{aligned} \tag{F.33}
$$

$$
\begin{aligned}
K_\mu(P_z\phi P_z\phi P_z^4\phi P_{\bar{z}}^2\phi) =& 4B\phi P_z\phi P_z^4\phi P_{\bar{z}}^2\phi + 32BP_z\phi P_z\phi P_z^3\phi P_{\bar{z}}^2\phi \\
&+ 8AP_z\phi P_z\phi P_z^4\phi P_{\bar{z}}\phi \\
=& O_{12},
\end{aligned} \tag{F.34}
$$

$$
\begin{aligned}
K_\mu(P_zP_{\bar{z}}\phi P_zP_{\bar{z}}\phi P_z^2\phi P_z^2\phi) =& 4BP_{\bar{z}}\phi P_zP_{\bar{z}}\phi P_z^2\phi P_z^2\phi + 4AP_z\phi P_zP_{\bar{z}}\phi P_z^2\phi P_z^2\phi \\
&+ 16BP_zP_{\bar{z}}\phi P_zP_{\bar{z}}\phi P_z\phi P_z^2\phi \\
=& O_{13},
\end{aligned} \tag{F.35}
$$

$$
\begin{aligned}
K_\mu(P_z\phi P_zP_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi P_z^2\phi) =& 2B\phi P_zP_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi P_z^2\phi + 2BP_z\phi P_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi P_z^2\phi \\
&+ 2AP_z\phi P_z\phi P_z^2P_{\bar{z}}\phi P_z^2\phi + 8BP_z\phi P_zP_{\bar{z}}\phi P_zP_{\bar{z}}\phi P_z^2\phi \\
&+ 2AP_z\phi P_zP_{\bar{z}}\phi P_z^2\phi P_z^2\phi + 8BP_z\phi P_zP_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi P_z\phi \\
=& O_{14},
\end{aligned} \tag{F.36}
$$

$$
\begin{aligned}
K_\mu(P_z\phi P_z\phi P_z^2P_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi) =& 4B\phi P_z\phi P_z^2P_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi + 16BP_z\phi P_z\phi P_zP_{\bar{z}}\phi P_z^2P_{\bar{z}}\phi \\
&+ 4AP_z\phi P_z\phi P_z^2\phi P_z^2P_{\bar{z}}\phi \\
=& O_{15},
\end{aligned} \tag{F.37}
$$

$$
\begin{aligned}
K_\mu(P_zP_{\bar{z}}\phi P_z^3\phi P_z^2P_{\bar{z}}\phi\phi) =& 2BP_{\bar{z}}\phi P_z^3\phi P_z^2P_{\bar{z}}\phi\phi + 2AP_z\phi P_z^3\phi P_z^2P_{\bar{z}}\phi\phi \\
&+ 18BP_zP_{\bar{z}}\phi P_z^2\phi P_z^2P_{\bar{z}}\phi\phi + 8BP_zP_{\bar{z}}\phi P_z^3\phi P_zP_{\bar{z}}\phi\phi \\
&+ 2AP_zP_{\bar{z}}\phi P_z^3\phi P_z^2\phi\phi \\
=& O_{16},
\end{aligned} \tag{F.38}
$$

$$
\begin{aligned}
K_\mu(P_z\phi P_z^3\phi P_z^2P_{\bar{z}}\phi P_{\bar{z}}\phi) =& 2B\phi P_z^3\phi P_z^2P_{\bar{z}}\phi P_{\bar{z}}\phi + 18BP_z\phi P_z^2\phi P_z^2P_{\bar{z}}\phi P_{\bar{z}}\phi \\
&+ 8BP_z\phi P_z^3\phi P_zP_{\bar{z}}\phi P_{\bar{z}}\phi + 2AP_z\phi P_z^3\phi P_z^2\phi P_{\bar{z}}\phi \\
&+ 2AP_z\phi P_z^3\phi P_z^2P_{\bar{z}}\phi\phi \\
=& O_{17},
\end{aligned} \tag{F.39}
$$

$$K_\mu(P_zP_{\bar z}\phi P_z^3\phi P_z^2\phi P_{\bar z}\phi) = 2BP_{\bar z}\phi P_z^3\phi P_z^2\phi P_{\bar z}\phi + 2AP_z\phi P_z^3\phi P_z^2\phi P_{\bar z}\phi \\ + 18BP_zP_{\bar z}\phi P_z^2\phi P_z^2\phi P_{\bar z}\phi + 8BP_zP_{\bar z}\phi P_z^3\phi P_z\phi P_{\bar z}\phi \\ + 2AP_zP_{\bar z}\phi P_z^3\phi P_z^2\phi\phi \\ = O_{18}, \tag{F.40}$$

$$K_\mu(P_z\phi P^3\phi P_z^2\phi P_{\bar z}^2\phi) = 2B\phi P^3\phi P_z^2\phi P_{\bar z}^2\phi + 18BP_z\phi P^2\phi P_z^2\phi P_{\bar z}^2\phi \\ 8BP_z\phi P^3\phi P_z\phi P_{\bar z}^2\phi + 8AP_z\phi P^3\phi P_z^2\phi P_{\bar z}\phi \\ = O_{19}. \tag{F.41}$$

Summing these terms we find

$$K_\mu(Q^2\phi^4(z)) = -\frac{1}{2}O_1 - 2O_2 - O_3 - \frac{3}{2}O_4 + 4O_5 - 2O_6 + \frac{1}{4}O_7 - \frac{1}{3}O_8 + \frac{2}{3}O_9 \\ - O_{10} + 2O_{11} - \frac{1}{2}O_{12} + 6O_{13} - 12O_{14} - 6O_{15} + 4O_{16} - 4O_{17} \\ - 4O_{18} + \frac{1}{3}O_{19} \\ = 0. \tag{F.42}$$

This shows that $(Q^2\phi^4(z))$ is a primary.

F.2 Transformation between coordinates variables and spacetime derivarives

We can translate polynomials back to momentum operators by using the following transformation

$$P_{\bar z}^k = a_k \bar z^k \tag{F.43}$$

where $\bar z = x_1 - ix_2$ and $P_{\bar z} = P_1 - iP_2$. We determine a_k as follows,

$$P_{\bar z}^{k+1} = a_{k+1}\bar z^{k+1} \tag{F.44}$$

and

$$\begin{aligned} a_{k+1}\bar{z}^{k+1} =& P_{\bar{z}}(a_k\bar{z}^k) \\ =& a_k(P_1 - iP_2)\bar{z}^k \end{aligned} \tag{F.45}$$

using the expression $P_\mu = x^2\partial_\mu - 2x_\mu x \cdot \partial - 2x_\mu$ for the momentum operator. We can further simplify the expression above,

$$\begin{aligned} a_k(P_1 - iP_2)\bar{z}^k =& a_k(kx^2 - 2kx_1(x_1 - ix_2) - 2x_1\bar{z})\bar{z}^{k-1} \\ & - ia_k(-ikx^2 + 2ikx_2(x_1 - ix_2) - 2x_2\bar{z})\bar{z}^{k-1} \\ =& -2a_k(k+1)\bar{z}^{k+1}. \end{aligned} \tag{F.46}$$

Thus

$$a_{k+1} = -2(k+1)a_k, \tag{F.47}$$

which implies that

$$a_k = (-1)^k 2^k k!. \tag{F.48}$$

Similar reasoning shows that

$$P_w^k \to b_k w^k \tag{F.49}$$

$$b_k \to (-1)^k 2^k k!, \tag{F.50}$$

where $w = x_3 + ix_4$ and $P_w = P_3 + iP_4$. Therefore the transformation of a $\bar{z}$ and w polynomial is as follows,

$$\bar{z}^k \to \frac{P_{\bar{z}}^k}{(-1)^k 2^k k!} \qquad w^l \to \frac{P_w^l}{(-1)^l 2^l l!}. \tag{F.51}$$

Appendix G

Formulas for Fermions

Consider the character generating function for the fermions

$$F(t,s,x,y) = \prod_{q=0}^{\infty}(1 + ts^{q+\frac{3}{2}}x^{\frac{q+1}{2}}y^{\frac{q}{2}}). \tag{G.1}$$

Before making an expansion we will drop the terms $x^{\frac{q+1}{2}}$ and $y^{\frac{q}{2}}$ since it is easy to reinsert these variables after expansion. Therefore

$$F(t,s) = \prod_{q=0}^{\infty}(1 + ts^{q+\frac{3}{2}}). \tag{G.2}$$

The derivative with respect to the parameter t yield

$$\begin{aligned}\frac{\partial F}{\partial t} =& \sum_{a=0}^{\infty}\frac{s^{\frac{3}{2}+a}}{(1+ts^{\frac{3}{2}+a})}\prod_{q=0}^{\infty}(1+ts^{q+\frac{3}{2}}) \\ =& f_1 F\end{aligned} \tag{G.3}$$

where

$$f_k(t,s) = (-1)^{k-1}(k-1)!\sum_a \frac{s^{k(a+\frac{3}{2})}}{(1+ts^{(\frac{3}{2}+a)})^k} \quad \text{and} \quad f_1(t,s) = \sum_a \frac{s^{a+\frac{3}{2}}}{1+ts^{a+\frac{3}{2}}}, \tag{G.4}$$

The 2^{nd}, 3^{rd} and 4^{th} t-derivatives of F give us

$$\frac{\partial^2 F}{\partial t^2} = \Big(- \sum_a \frac{s^{2(\frac{3}{2}+a)}}{(1+ts^{\frac{3}{2}+a})^2} + \sum_{a,b} \frac{s^{(\frac{3}{2}+a)} s^{(\frac{3}{2}+b)}}{(1+ts^{\frac{3}{2}+a})(1+ts^{\frac{3}{2}+b})} \Big) \prod_{q=0}^{\infty} (1+ts^{\frac{3}{2}+q}) \tag{G.5}$$

$$= (f_1^2 + f_2) F,$$

$$\begin{aligned} \frac{\partial^3 F}{\partial t^3} =& \Big(2 \sum_a \frac{s^{3(\frac{3}{2}+a)}}{(1+ts^{\frac{3}{2}+a})^3} - 3 \sum_{a,b} \frac{s^{2(\frac{3}{2}+a)} s^{(\frac{3}{2}+b)}}{(1+ts^{\frac{3}{2}+a})^2(1+ts^{\frac{3}{2}+b})} \\ &+ \sum_{a,b,c} \frac{s^{(\frac{3}{2}+a)} s^{(\frac{3}{2}+b)} s^{(\frac{3}{2}+c)}}{(1+ts^{\frac{3}{2}+a})(1+ts^{\frac{3}{2}+b})(1+ts^{\frac{3}{2}+c})} \Big) \prod_{q=0}^{\infty} (1+ts^{\frac{3}{2}+q}) \\ =& (f_3 + 3f_2 f_1 + f_1^3) F, \end{aligned} \tag{G.6}$$

and lastly

$$\frac{\partial^4 F}{\partial t^4} = (f_4 + 4f_3 f_1 + 3f_2 f_2 + 6f_2 f_1 f_1 + f_1^4) F. \tag{G.7}$$

In general

$$\frac{\partial^n}{\partial t^n} F(t,s) = \sum_{n_1,\cdots,n_q} \sum_{k_1,\cdots,k_q} \frac{(n_1 k_1 + \cdots + n_q k_q)!}{n_1! \cdots n_q! (k_1!)^{n_1} \cdots (k_q!)^{n_q}} f_{k1}^{n_1} \cdots f_{k_q}^{n_q} \delta_{n, n_1 k_1 + \cdots + n_q k_q} F(t,s). \tag{G.8}$$

We can see that $F(0,s) = 1$ and

$$\begin{aligned} f_k(0,s) =& (-1)^{k-1} (k-1)! \sum_{a=0}^{\infty} s^{k(\frac{3}{2}+a)} \\ =& (-1)^{k-1} (k-1)! \frac{s^{\frac{3}{2}k}}{(1-s^k)}. \end{aligned} \tag{G.9}$$

Plugging the simplification of $f_k(0,s)$ into equation (G.8) we obtain

$$
\begin{aligned}
\left.\frac{\partial^n F}{\partial t^n}\right|_{t=0} =& \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q} \frac{(n_1k_1+\cdots+n_qk_q)!}{n_1!\cdots n_q!(k_1)^{n_1}\cdots(k_q)^{n_q}}\left(\frac{s^{\frac{3}{2}k_1}}{1-s^{k_1}}\right)^{n_1}\cdots\left(\frac{s^{\frac{3}{2}k_q}}{1-s^{k_q}}\right)^{n_q}\times \\
& (-1)^{k_1-1}\times\cdots\times(-1)^{k_q-1}\delta_{n,n_1k_1+\cdots+n_qk_q} \\
=& \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q} \frac{n!s^{\frac{3}{2}n}}{n_1!\cdots n_q!(k_1)^{n_1}\cdots(k_q)^{n_q}}\left(\frac{s^{\frac{3}{2}k_1}}{1-s^{k_1}}\right)^{n_1}\cdots\left(\frac{s^{\frac{3}{2}k_q}}{1-s^{k_q}}\right)^{n_q}\times \\
& (-1)^{k_1-1}\times\cdots\times(-1)^{k_q-1}\delta_{n,n_1k_1+\cdots+n_qk_q}.
\end{aligned} \tag{G.10}
$$

In the end we have

$$
\begin{aligned}
\frac{1}{n!}\frac{\partial^n F}{\partial t^n} =& \chi_n(s,x,y) \\
=& s^{\frac{n}{2}}\,\mathrm{Tr}(P_{[1^n]}s^{L_0}).
\end{aligned} \tag{G.11}
$$

Using the general formula in (G.10) for finite n cases we obtain

$$
\begin{aligned}
\left.\frac{1}{2!}\frac{\partial^2}{\partial t^2}F(t,s)\right|_{t=0} =& \frac{s^3}{(1-s)(1-s^2)} - \frac{s^3}{(1-s)^2} \\
=& \frac{s^4}{(1-s)(1-s^2)} \\
=& s\mathrm{Tr}(P_{[1^2]}s^{L_0}),
\end{aligned} \tag{G.12}
$$

$$
\begin{aligned}
\left.\frac{1}{3!}\frac{\partial^3}{\partial t^3}F(t,s)\right|_{t=0} =& \frac{2s^{\frac{9}{2}}}{3!(1-s^3)} - \frac{3s^{\frac{9}{2}}}{3!(1-s^2)(1-s)} + \frac{s^{\frac{9}{2}}}{3!(1-s)^3} \\
=& \frac{s^{\frac{15}{2}}}{(1-s)(1-s^2)(1-s^3)} \\
=& s^{\frac{3}{2}}\,\mathrm{Tr}(P_{[1^3]}s^{L_0}),
\end{aligned} \tag{G.13}
$$

and

$$\frac{1}{4!}\frac{\partial^4}{\partial t^4}F(t,s)\bigg|_{t=0} = -\frac{6s^6}{4!(1-s^4)} + \frac{8s^6}{4!(1-s^3)(1-s)} + \frac{3s^6}{4!(1-s^2)(1-s^2)} - \frac{6s^6}{4!(1-s^2)(1-s)^2} \tag{G.14}$$

$$\begin{aligned} &+ \frac{s^6}{4!(1-s)^4} \\ =&\frac{s^{12}}{(1-s)(1-s^2)(1-s^3)(1-s^4)} \\ =&s^2\mathrm{Tr}(P_{[1^4]}s^{L_0}). \end{aligned}$$

From the cases above we can infer that in general we have

$$\begin{aligned} \frac{1}{n!}\frac{\partial^n F}{\partial t^n} =&s^{\frac{n}{2}}\mathrm{Tr}(P_{[1^n]}s^{L_0}) \\ =&\prod_{i=1}^{n}\frac{s^{\frac{n}{2}(n+2)}}{(1-s^i)}. \end{aligned} \tag{G.15}$$

Reinstating x and y we have

$$F(t,s,x,y) = \prod_{q=0}^{\infty}(1+ts^{q+\frac{3}{2}}x^{\frac{1}{2}(q+1)}y^{\frac{q}{2}}) \tag{G.16}$$

and

$$\begin{aligned} \frac{\partial F(t,s,x,y)}{\partial t} &= \sum_{a=0}^{\infty}\frac{s^{a+\frac{3}{2}}x^{\frac{1}{2}(a+1)}y^{\frac{a}{2}}}{(1+ts^{a+\frac{3}{2}}x^{\frac{1}{2}(a+1)}y^{\frac{a}{2}})}F(t,s,x,y) \\ f_k(t,s,x,y) &= (-1)^{k-1}(k-1)!\sum_{a=0}^{\infty}\frac{s^{k(a+\frac{3}{2})}x^{\frac{k}{2}(a+1)}y^{\frac{ka}{2}}}{(1+ts^{a+\frac{3}{2}}x^{\frac{1}{2}(a+1)}y^{\frac{a}{2}})^k}. \end{aligned} \tag{G.17}$$

Therefore

$$f_k(t,s,x,y) = (-1)^{k-1}(k-1)!\frac{s^{\frac{3}{2}k}x^{\frac{k}{2}}}{(1-s^kx^{\frac{k}{2}}y^{\frac{k}{2}})}, \tag{G.18}$$

and hence

$$\frac{1}{n!}\frac{\partial F}{\partial t}\Big|_{t=0} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{s^{\frac{3}{2}n}x^{\frac{n}{2}}}{n_1!\cdots n_q!(k_1)^{n_1}\cdots(k_q)^{n_q}}\Big(\frac{1}{1-s^{k_1}x^{\frac{k_1}{2}}y^{\frac{k_1}{2}}}\Big)^{n_1}\cdots\Big(\frac{1}{1-s^{k_q}x^{\frac{k_q}{2}}y^{\frac{k_q}{2}}}\Big)^{n_q}\times \tag{G.19}$$

$$(-1)^{k_1-1}\times\cdots\times(-1)^{k_q-1}\delta_{n,n_1k_1+\cdots+n_qk_q}$$

Using the equation (G.19), we find

$$\frac{1}{2!}\frac{\partial F}{\partial t^2}\Big|_{t=0} = \frac{s^4x^{\frac{3}{2}}y^{\frac{1}{2}}}{(1-s^2xy)(1-sx^{\frac{1}{2}}y^{\frac{1}{2}})}, \tag{G.20}$$

$$\frac{1}{3!}\frac{\partial F}{\partial t^3}\Big|_{t=0} = \frac{s^{\frac{15}{2}}x^3y^{\frac{3}{2}}}{(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^2xy)(1-sx^{\frac{1}{2}}y^{\frac{1}{2}})}, \tag{G.21}$$

and

$$\frac{1}{4!}\frac{\partial F}{\partial t^4}\Big|_{t=0} = \frac{s^{12}x^5y^3}{(1-s^4x^2y^2)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^2xy)(1-sx^{\frac{1}{2}}y^{\frac{1}{2}})}. \tag{G.22}$$

In general

$$\frac{1}{n!}\frac{\partial^n F}{\partial t^n}\Big|_{t=0} = x^{\frac{n(n+1)}{4}}y^{\frac{n(n-1)}{4}}s^{\frac{n}{2}(n+2)}\prod_{i=1}^{n}\frac{1}{(1-s^ix^{\frac{i}{2}}y^{\frac{i}{2}})}. \tag{G.23}$$

The generating function $G_n^{z,w}(s,x,y)$ for the extremal primary operators with dimension and charges respectively,

$$\Delta = n+q \qquad J=(J_3^L,J_3^R)=(\frac{1}{2}+q,\frac{q}{2}-m), \tag{G.24}$$

is given as

$$
\begin{aligned}
G_n^{z,w}(s,x,y) =&(1-\sqrt{xy})\frac{1}{n!}\frac{\partial^n F}{\partial t^n}\bigg|_{t=0} \\
=&(1-s\sqrt{xy})\chi_n(s,x,y) \\
=&Z_n(s,x,y),
\end{aligned}
\tag{G.25}
$$

where $Z_n = (1-s\sqrt{xy})\chi_n(s,x,y)$. From the numerical computation, the expansion of $G_3(s,x,y)$ is

$$
G_3(s,x,y) = s^{\frac{15}{2}}x^3y^{\frac{3}{2}} + s^{\frac{19}{2}}x^4y^{\frac{5}{2}} + s^{\frac{21}{2}}x^{\frac{9}{2}}y^3 + s^{\frac{23}{2}}x^5y^{\frac{7}{2}} + s^{\frac{25}{2}}x^{\frac{11}{2}}y^4 + \cdots .
\tag{G.26}
$$

From here we will consider another set of extremal operators. We consider the set of extremal operators with dimension $\Delta = n+q$ and $J_3^L = \frac{3}{2}+q$ maximum spin. We begin by studying the character generating function

$$
F_2(t,s,x,y) = \prod_{q=0}^{\infty}\prod_{m=0}^{q}(1+ts^{q+\frac{3}{2}}x^{\frac{q+1}{2}}y^{m-\frac{q}{2}}).
\tag{G.27}
$$

The derivatives with respect to parameter t yields

$$
\begin{aligned}
\frac{\partial}{\partial t}F_2(t,s,x,y) =&\sum_{a=0}^{\infty}\sum_{m=0}^{a}\frac{s^{a+\frac{3}{2}}x^{\frac{a+1}{2}}y^{m-\frac{a}{2}}}{(1+ts^{a+\frac{3}{2}}x^{\frac{a+1}{2}}y^{m-\frac{a}{2}})} \\
=&f_1(t,s,x,y)F_2(t,s,x,y),
\end{aligned}
\tag{G.28}
$$

where

$$
f_k(t,s,x,y) = \frac{\partial^{k-1}}{\partial t^{k-1}}F_1(t,s,x,y) = (-1)^{k-1}(k-1)!\sum_{a=0}^{\infty}\sum_{m=0}^{a}\frac{s^{k(a+\frac{3}{2})}x^{k(\frac{a+1}{2})}y^{k(m-\frac{a}{2})}}{(1+ts^{a+\frac{3}{2}}x^{\frac{a+1}{2}}y^{m-\frac{a}{2}})^k}.
\tag{G.29}
$$

We know that $F_2(t,s,x,y) = 1$ and

$$
\begin{aligned}
f_k(0,s,x,y) =&(-1)^{k-1}(k-1)!\sum_{a=0}^{\infty}\sum_{m=0}^{a} s^{k(a+\frac{3}{2})}x^{k(\frac{a+1}{2})}y^{k(m-\frac{a}{2})} \\
=&(-1)^{k-1}(k-1)!\frac{s^{\frac{3}{2}k}x^{\frac{1}{2}k}}{(1-s^k x^{\frac{k}{2}}y^{\frac{k}{2}})(1-s^k x^{\frac{k}{2}}y^{-\frac{k}{2}})}.
\end{aligned} \tag{G.30}
$$

Thus

$$
\begin{aligned}
\frac{1}{n!}\frac{\partial^n}{\partial t^n}F_2(t,s,x,y)\Big|_{t=0} =& \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{n!x^{\frac{1}{2}n}s^{\frac{3}{2}n}}{n_1!\cdots n_q!(k_1)^{n_1}\cdots(k_q)^{n_q}}\left(\frac{1}{1-s^{k_1}x^{k_1}y^{k_1/2}}\right)^{n_1}\times \\
&\cdots\left(\frac{1}{1-s^{k_q}x^{k_q}y^{-k_q/2}}\right)^{n_q}(-1)^{k_1-1}\times\cdots\times(-1)^{k_q-1} \\
&\times\delta_{n,n_1k_1+\cdots+n_qk_q} \\
=&\chi_n(s,x,y).
\end{aligned} \tag{G.31}
$$

The generating function for the primaries with dimension $\Delta = n+q$ and maximum spin $J_3^L = \frac{3}{2}+q$ is given as

$$
G_n^{z,w}(s,x,y) = \left[\left(1-\frac{1}{y}\right)Z_n^{z,w}(s,x,y)\right]_{\geq 0}, \tag{G.32}
$$

where $Z_n(s,x,y)$ is given as

$$
Z_n(s,x,y) = \left(1-s\sqrt{xy}\right)\left(1-s\sqrt{\frac{x}{y}}\right)\chi_n(s,x,y). \tag{G.33}
$$

The expansion of $G_3^{z,w}(s,x,y)$ is

$$
\begin{aligned}
G_3^{z,w}(s,x,y) =&s^{\frac{13}{2}}x^{\frac{5}{2}}+s^{\frac{15}{2}}x^3y^{\frac{3}{2}}+s^{\frac{17}{2}}x^{\frac{7}{2}}y+s^{\frac{19}{2}}x^4y^{\frac{3}{2}}+s^{\frac{19}{2}}x^4y^{\frac{5}{2}}+s^{\frac{21}{2}}x^{\frac{9}{2}} \\
&+s^{\frac{21}{2}}x^{\frac{9}{2}}y^2+s^{\frac{21}{2}}x^{\frac{9}{2}}y^3+\cdots.
\end{aligned} \tag{G.34}
$$

G.1 Fermion Projectors

Using the formula (G.31), for $n = 2$ we find that

$$\chi_2(s,x,y) = s^3 x\Big(\frac{s\sqrt{xy} + s\sqrt{\frac{x}{y}}}{(1-s^2xy)(1-s^2\frac{x}{y})(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})}\Big), \tag{G.35}$$

and consequently

$$Z_2(s,x,y) = s^3 x\Big(\frac{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(s\sqrt{xy} + s\sqrt{\frac{x}{y}})}{(1-s^2xy)(1-s^2\frac{x}{y})(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})}\Big). \tag{G.36}$$

The generating functions used to construct $Z_2(s,x,y)$ are

$$\begin{aligned} Z_2(s,x,y) =& s^3 x\Big(Z_{SH}(q_1, \begin{smallmatrix}\square\\\square\end{smallmatrix})Z_{SH}(q_2, \square\square) + Z_{SH}(q_1, \square\square)Z_{SH}(q_2, \begin{smallmatrix}\square\\\square\end{smallmatrix})\Big) \qquad \text{(G.37)}\\ =& \frac{s^3 x(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(s\sqrt{xy} + s\sqrt{\frac{x}{y}})}{(1-s^2xy)(1-s^2\frac{x}{y})(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})}, \end{aligned}$$

where $q_1 = s\sqrt{xy}$, $q_2 = s\sqrt{\frac{x}{y}}$ and

$$Z_{SH}(q,\Lambda) = (1-q)q^{\sum_i \frac{c_i(c_i-1)}{2}} \prod_b \frac{1}{1-q^{h_b}}. \tag{G.38}$$

For $n = 3$ we obtain

$$\begin{aligned} \chi_3(s,x,y) =& s^{\frac{9}{2}} x^{\frac{3}{2}} \Big(\frac{q_1^3 + q_2^3 + q_1^3(1+q_2)q_2 + q_2^3(1+q_1)q_1}{(1-q_1)(1-q_2)(1-q_1^2)(1-q_2^2)(1-q_1^3)(1-q_2^3)}\Big) \qquad \text{(G.39)}\\ Z_3(s,x,y) =& s^{\frac{9}{2}} x^{\frac{3}{2}}(1-q_1)(1-q_2)\Big(\frac{q_1^3 + q_2^3 + q_1^3(1+q_2)q_2 + q_2^3(1+q_1)q_1}{(1-q_1)(1-q_2)(1-q_1^2)(1-q_2^2)(1-q_1^3)(1-q_2^3)}\Big) \end{aligned}$$

We can construct $Z_3(s,x,y)$ using the generating functions below

$$
\begin{aligned}
Z_3(s,x,y) =& s^{\frac{9}{2}} x^{\frac{3}{2}} \Big(Z_{SH}(q_1, \ydiagram{1,1,1}) Z_{SH}(q_2, \ydiagram{3}) + Z_{SH}(q_1, \ydiagram{2,1}) Z_{SH}(q_2, \ydiagram{2,1}) + Z_{SH}(q_1, \ydiagram{3}) Z_{SH}(q_2, \ydiagram{1,1,1}) \Big) \\
=& s^{\frac{9}{2}} x^{\frac{3}{2}} (1-q_1)(1-q_2) \Big(\frac{q_1^3 + q_2^3 + q_1^3(1+q_2)q_2 + q_2^3(1+q_1)q_1}{(1-q_1)(1-q_2)(1-q_1^2)(1-q_2^2)(1-q_1^3)(1-q_2^3)} \Big).
\end{aligned}
\tag{G.40}
$$

Performing a power series expansion of $Z_3(s,x,y)$ we find

$$
\begin{aligned}
Z_3(s,x,y) =& (q_2 + q^3 + q_2^5 + \cdots) + q_1(1 + q_2^2 + q_2^4 + q_2^6 + \cdots) + q_1^2(q_2 + q_2^3 + q_2^5 + \cdots) \\
& + q_1^3(1 + q_2^2 + q_2^4 + \cdots) + q_1^4(q_2 + q_2^3 + q_2^5 + \cdots) + q_1^5(1 + q_2^3 + q_2^5 + \cdots) + \cdots \\
=& \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (q_2^{2n+1} q_1^{2m} + q_2^{2n} q_1^{2m+1}) \\
=& \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Bigg(\Big(s\sqrt{\frac{x}{y}} \Big)^{2n+1} (s\sqrt{xy})^{2m} + \Big(s\sqrt{\frac{x}{y}} \Big)^{2n} (s\sqrt{xy})^{2m+1} \Bigg) \\
=& s x^{\frac{1}{2}} y^{\frac{1}{2}} + s x^{\frac{1}{2}} y^{-\frac{1}{2}} + s^3 s x^{\frac{3}{2}} y^{-\frac{3}{2}} + s^3 x^{\frac{3}{2}} y^{-\frac{1}{2}} + s^3 s x^{\frac{3}{2}} y^{\frac{1}{2}} + s^3 s x^{\frac{3}{2}} y^{\frac{3}{2}} + s^5 x^{\frac{5}{2}} y^{-\frac{5}{2}} \\
& + s^5 x^{\frac{5}{2}} y^{-\frac{3}{2}} + s^3 x^{\frac{5}{2}} y^{-\frac{1}{2}} + s^5 x^{\frac{5}{2}} y^{\frac{1}{2}} + s^5 x^{\frac{5}{2}} y^{\frac{3}{2}} + s^5 x^{\frac{5}{2}} y^{\frac{5}{2}} + \cdots .
\end{aligned}
\tag{G.41}
$$

From the last line of (G.41) we observe that $Z_3(s,x,y)$ contains primary operators with negative spin (momentum) powers. To eliminate these primary operators, we have to compute

$$
G_3(s,x,y) = \Big[\Big(1 - \frac{1}{y} \Big) Z_2(s,x,y) \Big]_{\geq 0}. \tag{G.42}
$$

We also note that the primary operators generated by $Z_3(s,x,y)$ at each dimension s^d, is a series of primary operators with an extremum j_L spin (x^{j_L}). This series is of a particular form

$$
s^{2j_L} \sum_{m=-j_L}^{j_L} x^{j_L} y^{j_L - 2m}, \tag{G.43}
$$

where j_L is the maximum left spin.

For $n = 4$ we find

$$\chi_4(s,x,y) = \frac{s^6 x^2 Q}{(1-q_1^4)(1-q_2^4)(1-q_1^3)(1-q_2^3)(1-q_1^2)(1-q_2^2)(1-q_1)(1-q_2)} \tag{G.44}$$

$$Z_4(s,x,y) = s^6 x^2 \frac{(1-q_1)(1-q_2)Q}{(1-q_1^4)(1-q_2^4)(1-q_1^3)(1-q_2^3)(1-q_1^2)(1-q_2^2)(1-q_1)(1-q_2)}$$

where

$$\begin{aligned} Q =& q_1^6 + q_2^6 + q_1^5 q_2(1 + q_2 + q_2^2) + q_1 q_2^3(1 + q_2 + q_2^2) + q_1^4 q_2(1 + 2q_2 + q_2^2 + q_2^3) \\ &+ q_1^2 q_2^2(1 + q_2 + 2q_2^2 + q_2^3) + q_2^3 q_1(1 + q_2 + 2q_2 + q_2^3 + q_2^4)). \end{aligned} \tag{G.45}$$

We construct $Z_4(s,x,y)$ using the projectors

$$\begin{aligned} Z_4(s,x,y) =& s^6 x^2 \Big(Z_{SH}(q_1, \ydiagram{1,1,1,1}) Z_{SH}(q_2, \ydiagram{4}) + Z_{SH}(q_1, \ydiagram{2,1,1}) Z_{SH}(q_2, \ydiagram{3,1}) \\ &+ Z_{SH}(q_1, \ydiagram{2,2}) Z_{SH}(q_2, \ydiagram{2,2}) + Z_{SH}(q_1, \ydiagram{3,1}) Z_{SH}(q_2, \ydiagram{2,1,1}) \\ &+ Z_{SH}(q_1, \ydiagram{4}) Z_{SH}(q_2, \ydiagram{1,1,1,1}) \Big), \end{aligned} \tag{G.46}$$

to find

$$Z_4(s,x,y) = s^6 x^2 \frac{(1-q_1)(1-q_2)Q}{(1-q_1^4)(1-q_2^4)(1-q_1^3)(1-q_2^3)(1-q_1^2)(1-q_2^2)(1-q_1)(1-q_2)}, \tag{G.47}$$

where Q is the same as in (G.45). One more last check for $n = 5$ before we give a generic formula for counting. From Taylor expansion we find $\chi_5(s,x,y)$

and $Z_5(s,x,y)$ to be

$$\chi_5(s,x,y) = \frac{s^{\frac{15}{2}} x^{\frac{5}{2}} P}{(1-q_1^5)(1-q_2^5)(1-q_1^4)(1-q_2^4)(1-q_1^3)(1-q_2^3)(1-q_1^2)(1-q_2^2)(1-q_1)(1-q_2)} \tag{G.48}$$

$$Z_5(s,x,y) = \frac{s^{\frac{15}{2}} x^{\frac{5}{2}} (1-q_1)(1-q_2) P}{(1-q_1^5)(1-q_2^5)(1-q_1^4)(1-q_2^4)(1-q_1^3)(1-q_2^3)(1-q_1^2)(1-q_2^2)(1-q_1)(1-q_2)},$$

where P is

$$\begin{aligned} P =& q_1^{10} + q_2^{10} + q_1^9 q_2 (1 + q_2 + q_2^2 + q_2^3) + q_1 q_2^6 (1 + q_2 + q_2^2 + q_2^3) \\ &+ q_1^8 q_2 (1 + 2q_2 + 2q_2^2 + 2q_2^3 + q_2^4 + q_2^5) + q_1^2 q_2^4 (1 + q_2 + 2q_2^2 + 2q_2^3 + 2q_2^4 + q_2^5) \\ &+ q_1^7 q_2 (1 + 2q_2 + 3q_2^2 + 3q_2^3 + 3q_2^4 + 2q_2^5 + q_2^6) + q_1^3 q_2^3 (1 + 2q_2 + 3q_2^2 + 3q_2^3 + 3q_2^4 + 2q_2^5 + q_2^6) \\ &+ q_1^5 q_2^2 (1 + 3q_2 + 4q_2^2 + 6q_2^3 + 4q_2^4 + 3q_2^5 + q_2^6) + q_1^6 q_2 (1 + 2q_2 + 3q_2^2 + 4q_2^3 + 4q_2^4 + 3q_2^5 + 2q_2^6 + q_2^7) \\ &+ q_1^4 q_2^2 (1 + 2q_2 + 3q_2^2 + 4q_2^3 + 3q_2^5 + 2q_2^6 + q_2^7). \end{aligned} \tag{G.49}$$

Using the generating functions we construct $Z_5(s,x,y)$ as follows

$$\begin{aligned} Z_5(s,x,y) =& s^{\frac{15}{2}} x^{\frac{5}{2}} \Big(Z_{SH}(q_1, \ydiagram{1,1,1,1,1}) Z_{SH}(q_2, \ydiagram{5}) + Z_{SH}(q_1, \ydiagram{2,1,1,1}) Z_{SH}(q_2, \ydiagram{4,1}) \\ &+ Z_{SH}(q_1, \ydiagram{2,2,1}) Z_{SH}(q_2, \ydiagram{3,2}) + Z_{SH}(q_1, \ydiagram{3,1,1}) Z_{SH}(q_2, \ydiagram{3,1,1}) + Z_{SH}(q_1, \ydiagram{3,2}) Z_{SH}(q_2, \ydiagram{2,2,1}) \\ &+ Z_{SH}(q_1, \ydiagram{4,1}) Z_{SH}(q_2, \ydiagram{2,1,1,1}) + Z_{SH}(q_1, \ydiagram{5}) Z_{SH}(q_2, \ydiagram{1,1,1,1,1}) \Big) \\ =& \frac{s^{\frac{15}{2}} x^{\frac{5}{2}} (1-q_1)(1-q_2) P}{(1-q_1^5)(1-q_2^5)(1-q_1^4)(1-q_2^4)(1-q_1^3)(1-q_2^3)(1-q_1^2)(1-q_2^2)(1-q_1)(1-q_2)}. \end{aligned} \tag{G.50}$$

The general formula for counting primaries constructed using n fermions is

$$Z_n(s,x,y) = s^{\frac{3}{2}n} x^{\frac{n}{2}} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda) Z_{SH}(q_2, \Lambda^T). \tag{G.51}$$

Note that the coefficient $s^{\frac{3}{2}n} x^{\frac{n}{2}}$ originates from the dimension and spin of n

Weyl spinors. A left handed Weyl spinor is in the following representation

$$\left|\Delta, J_L^3, j_R^3\right\rangle = \left|\frac{3}{2}, \frac{1}{2}, 0\right\rangle \tag{G.52}$$

.

The product of n left handed Weyl spinors is an $s^{\frac{3}{2}n}x^{\frac{n}{2}}$ irrep. As we can see from the example of the $Z_n(s, x, y)$ generating functions, we pair $Z_{SH}(q_1, \Lambda)$ with $Z_{SH}(q_2, \Lambda^T)$. Note that by the transpose of Young diagram we mean,

$$\ydiagram{1,1,1,1}^{\,T} = \ydiagram{4} \qquad \ydiagram{2,1,1}^{\,T} = \ydiagram{3,1} \qquad \ydiagram{2,1}^{\,T} = \ydiagram{2,1}. \tag{G.53}$$

G.1.1 How $Z_n(s, x, y)$ Fermion Generating Functions Work

Consider the $Z_3(s, x, y)$ in (G.40). The product of generating functions for Z_3 are

$$Z_{SH}(q_1, \ydiagram{1,1,1})Z_{SH}(q_2, \ydiagram{3}), \qquad Z_{SH}(q_1, \ydiagram{2,1})Z_{SH}(q_2, \ydiagram{2,1}), \qquad Z_{SH}(q_1, \ydiagram{3})Z_{SH}(q_2, \ydiagram{1,1,1}). \tag{G.54}$$

These products are actually counting for a tensor product of Young diagram representation space. The tensor product of these Young diagrams yields the following subspaces

$$\begin{aligned} &\ydiagram{1,1,1} \otimes \ydiagram{3} = \ydiagram{1,1,1} \\ &\ydiagram{2,1} \otimes \ydiagram{2,1} = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1} \\ &\ydiagram{3} \otimes \ydiagram{1,1,1} = \ydiagram{1,1,1}. \end{aligned} \tag{G.55}$$

We see that from the tensor product of Young diagrams above, every sum of the representation subspaces obtained on the left hand side contains an antisymmetric subspace representation,

$$\ydiagram{1,1,1}. \tag{G.56}$$

Further, the antisymmetric representation always occurs with multiplicity 1. Thus the product of these generating functions count the antisymmeytric subspace representation, which is the kind of representation we want to project to, since we are dealing with Fermions.

The product of generating functions contained in $Z_4(s, x, y)$ in (G.46) have a tensor product that gives the following sum of subspaces

$$
\begin{aligned}
&\ydiagram{1,1,1,1} \otimes \ydiagram{4} = \ydiagram{1,1,1,1} \\
&\ydiagram{2,1,1} \otimes \ydiagram{3,1} = \ydiagram{1,1,1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \\
&\ydiagram{3,1} \otimes \ydiagram{2,1,1} = \ydiagram{1,1,1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \\
&\ydiagram{2,2} \otimes \ydiagram{2,2} = \ydiagram{4} \oplus \ydiagram{2,2} \oplus \ydiagram{1,1,1,1} \\
&\ydiagram{4} \otimes \ydiagram{1,1,1,1} = \ydiagram{1,1,1,1}.
\end{aligned}
\tag{G.57}
$$

One can check whether the sum of Young diagrams appearing on the RHS are correct by computing the sum of the dimensions of Young diagrams on the RHS and, compare the answer with the dimension on the LHS. We can see that every tensor product contains an antisymmetric representation. Which means that these tensor products do contain the antisymmetric subspace. In a nutshell a tensor product of a Young diagram with its transposed Young diagram will contain an antisymmetric representation with multiplicity 1.

Appendix H

Total Spin Operators

In this appendix we will compute the total spin of the polynomials we constructed in chapter 5. The total spin is given by

$$J \cdot J\mathcal{O}(z,w) = s(s+1)\mathcal{O}(z,w), \tag{H.1}$$

where s is the spin of the polynomial and $J \cdot J$ is the total spin operator given as

$$J^R \cdot J^R = (J_1^R)^2 + (J_2^R)^2 + (J_3^R)^2, \tag{H.2}$$

or

$$J^L \cdot J^L = (J_1^L)^2 + (J_2^L)^2 + (J_3^L)^2. \tag{H.3}$$

The $SO(4)$ generators are given as

$$J_3^R = z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} + w_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial \bar{w}_i}$$

$$J_+^R = w_i \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial \bar{w}_i} \qquad J_-^R = \bar{z}_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial z_i}$$

$$J_3^L = z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} - w_i \frac{\partial}{\partial w_i} + \bar{w}_i \frac{\partial}{\partial \bar{w}_i}$$

$$J_+^L = \bar{w}_i \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial w_i} \qquad J_-^L = \bar{z}_i \frac{\partial}{\partial \bar{w}_i} - w_i \frac{\partial}{\partial z_i}.$$

Using the $SO(4)$ generators given above we can find the generators J_1^L, J_2^L,

J_1^R and J_2^R, since

$$J_+^R = \frac{1}{2}(J_1^R + iJ_2^R) \quad \text{and} \quad J_-^R = \frac{1}{2}(J_1^R - iJ_2^R) \tag{H.4}$$
$$J_+^L = \frac{1}{2}(J_1^L + iJ_2^L) \quad \text{and} \quad J_-^L = \frac{1}{2}(J_1^L - iJ_2^L).$$

Hence we find the generators are

$$J_1^R = \bar{z}_i \frac{\partial}{\partial w_i} - z_i \frac{\partial}{\partial \bar{w}_i} + w_i \frac{\partial}{\partial \bar{z}_i} - \bar{w}_i \frac{\partial}{\partial z_i} \tag{H.5}$$

$$J_2^R = i\left(z_i \frac{\partial}{\partial \bar{w}_i} + \bar{z}_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial z_i} - w_i \frac{\partial}{\partial \bar{z}_i} \right) \tag{H.6}$$

$$J_1^L = \bar{w}_i \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial w_i} + \bar{z}_i \frac{\partial}{\partial \bar{w}_i} - w_i \frac{\partial}{\partial z_i} \tag{H.7}$$

$$J_2^L = i\left(z_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial \bar{z}_i} + \bar{z}_i \frac{\partial}{\partial \bar{w}_i} - w_i \frac{\partial}{\partial z_i} \right). \tag{H.8}$$

The total spin operators become

$$J^R \cdot J^R = z_i z_j \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} + 2 z_i w_j \frac{\partial}{\partial z_i} \frac{\partial}{w_j} + w_i w_j \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j}, \tag{H.9}$$

and

$$J^L \cdot J^L = z_i z_j \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} - 2 z_i w_j \frac{\partial}{\partial z_i} \frac{\partial}{w_j} + 4 w_i z_j \frac{\partial}{\partial z_i} \frac{\partial}{\partial w_j} + w_i w_j \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j}. \tag{H.10}$$

Now we can compute the spin of the polynomial in $\mathcal{O}_{\phi\phi\phi}$ constructed from 3 scalar fields,

$$
\begin{aligned}
J^R \cdot J^R \mathcal{O}_{\phi\phi\phi}(z,w) =& J^R \cdot J^R (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)\times \\
& (w_3(z_1 - z_2) + w_1(z_2 - z_3) + w_2(z_3 - z_1)) \\
=& 20\mathcal{O}^{(4,1)}_{\phi\phi\phi}(z,w) \\
=& 4(4+1)\mathcal{O}^{(4,1)}_{\phi\phi\phi}(z,w),
\end{aligned}
\tag{H.11}
$$

therefore the operator $\mathcal{O}_{\phi\phi\phi}(z,w)$ has right spin $s = 4$. The left hand spin is computed as follows

$$
\begin{aligned}
J^L \cdot J^L \mathcal{O}_{\phi\phi\phi}(z,w) =& J^L \cdot J^L (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)\times \\
& (w_3(z_1 - z_2) + w_1(z_2 - z_3) + w_2(z_3 - z_1)) \\
=& 30\mathcal{O}^{(4,1)}_{\phi\phi\phi}(z,w) \\
=& 5(5+1)\mathcal{O}^{(4,1)}_{\phi\phi\phi}(z,w),
\end{aligned}
\tag{H.12}
$$

so that $s = 5$ for the left hand spin.

Appendix I

The Hilbert Series for $Z_3(s,x,y)$

Here we consider the Hilbert series $Z_3(s,x,y)$ for counting of extremal primaries built using 3 scalar fields. This Hilbert series has a non-trivial numerator

$$Z_3^{z,w} = \frac{s^3(1 - s^5x^{\frac{5}{2}}(\sqrt{y} + \frac{1}{\sqrt{y}}) - s^6x^3(\frac{1}{y} + 1 + y) - s^{14}x^7 + s^8x^4(y + 1 + \frac{1}{y}) + s^9x^{\frac{9}{2}}(\sqrt{y} + \frac{1}{\sqrt{y}})}{(1 - s^xy)(1 - s^2x)(1 - s^{\frac{x}{y}})(1 - s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1 - s^3x^{\frac{3}{2}}y^{\frac{1}{2}})(1 - s^3x^{\frac{3}{2}}y^{\frac{-1}{2}})(1 - s^3x^{\frac{3}{2}}y^{\frac{-3}{2}})} \tag{I.1}$$

Our goal is to explain how the numerator of $Z_3(s,x,y)$ encodes relations between the generators of the ring as well as relations between those relations.

From the denominator of the Hilbert series, we have 7 generators. We can easily identify them as follows

$$G_1 = (z_{12})^2 + (z_{13})^2 + (z_{23})^2 \leftrightarrow s^2xy$$

$$G_2 = z_{12}w_{12} + z_{13}w_{13} + z_{23}w_{23} \leftrightarrow x^2x$$

$$G_3 = (w_{12})^2 + (w_{13})^2 + (w_{23})^2 \leftrightarrow s^2xy^{-1}$$

$$G_4 = (z_{13} + z_{23})(z_{31} + z_{21})(z_{12} + z_{32}) \leftrightarrow s^3x^{\frac{3}{2}}y^{\frac{3}{2}}$$

$$
\begin{aligned}
G_5 =& (w_{13}+w_{23})(z_{31}+z_{21})(z_{12}+z_{32}) \\
&+ (z_{13}+z_{23})(w_{31}+w_{21})(z_{12}+z_{32}) \\
&+ (z_{13}+z_{23})(z_{31}+z_{12})(w_{12}+w_{32}) \leftrightarrow s^3x^{\frac{3}{2}}y^{\frac{1}{2}}
\end{aligned} \tag{I.2}
$$

$$
\begin{aligned}
G_6 =& (w_{13}+w_{23})(w_{31}+w_{21})(z_{12}+z_{32}) \\
&+ (z_{13}+z_{23})(w_{31}+w_{21})(w_{12}+w_{32}) \\
&+ (w_{13}+w_{23})(z_{31}+z_{12})(w_{12}+w_{32}) \leftrightarrow s^3x^{\frac{3}{2}}y^{\frac{-3}{2}}
\end{aligned} \tag{I.3}
$$

From the numerator of the Hilbert series, the terms with a negative sign should correspond to relations between the generators of the degree given by the monomial. From $-s^5x^{\frac{5}{2}}(\sqrt{y}+\frac{1}{\sqrt{y}}) - s^6x^3(\frac{1}{y}+1+y) - s^{14}x^7$ we have 6 relations. They are

$$\chi_1 = 3G_3G_4 - 2G_2G_5 + G_1G_6 = 0 \leftrightarrow s^5x^{\frac{5}{2}}\sqrt{y}$$

$$\chi_2 = G_3G_5 - 2G_2G_6 + 3G_1G_7 = 0 \leftrightarrow s^5x^{\frac{5}{2}}y^{\frac{1}{2}}$$

$$\chi_3 = 4G_1G_2^2 - G_1^2G_3 = 0 \leftrightarrow s^6x^3y$$

$$\chi_4 = 4G_2^3 - G_1G_2G_3 = 0 \leftrightarrow s^6x^3$$

$$\chi_5 = 4G_2^2G_3 - G_1G_3^2 = 0 \leftrightarrow s^6x^3y^{-1}$$

$$\chi_6 = G_2^7 - G_1G_2^5G_3 + \frac{1}{9}G_2^4G_5G_6 - G_2^4G_4G_7 = 0 \leftrightarrow s^{14}x^7$$

Again from the numerator of the Hilbert series, the terms with positive sign should corresponds to relations between the relations, again of the degree given by the monomial. From $s^8x^4(y+1+\frac{1}{y}) + s^9x^{\frac{9}{2}}(\sqrt{y}+\frac{1}{\sqrt{y}})$ we have 5 relations among the relations. They are

$$
\begin{aligned}
4\chi_5G_2 + \chi_4G_3 &= 0 \leftrightarrow s^8x^4y^{-1} \\
\chi_5G_1 - \chi_3G_3 &= 0 \leftrightarrow s^8x^4
\end{aligned} \tag{I.4}
$$

$$\chi_4 G_1 + 4\chi_3 G_2 = 0 \leftrightarrow s^8 x^4 y \tag{I.5}$$
$$\chi_2 G_2^2 - \frac{1}{4}\chi_2 G_1 G_3 - \chi_5 G_5 - \frac{1}{2}\chi_4 G_6 - 3\chi_3 G_7 = 0 \leftrightarrow s^9 x^{\frac{9}{2}} y^{\frac{-1}{2}}$$
$$-4\chi_1 G_2^2 + \chi_1 G_1 G_3 + 12\chi_5 G_4 + 2\chi_4 G_5 + 4\chi_3 G_6 = 0 \leftrightarrow s^9 x^{\frac{9}{2}} y^{\frac{1}{2}}$$

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