

Information-Driven Pricing Kernel Models

by

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A thesis submitted for the degree of
Doctor of Philosophy

2013

To Ba and Bapuji

Abstract

This thesis presents a range of related pricing kernel models that are driven by incomplete information about a series of future unknowns. These unknowns may, for instance, represent fundamental macroeconomic, political or social random variables that are revealed at future times. They may also represent latent or hidden factors that are revealed asymptotically. We adopt the information-based approach of Brody, Hughston and Macrina (BHM) to model the information processes associated with the random variables. The market filtration is generated collectively by these information processes. By directly modelling the pricing kernel, we generate information-sensitive arbitrage-free models for the term structure of interest rates, the excess rate of return required by investors, and security prices. The pricing kernel is modelled by a supermartingale to ensure that nominal interest rates remain non-negative. To begin with, we primarily investigate finite-time pricing kernel models that are sensitive to Brownian bridge information. The BHM framework for the pricing of credit-risky instruments is extended to a stochastic interest rate setting. In addition, we construct recovery models, which take into consideration information about, for example, the state of the economy at the time of default. We examine various explicit examples of analytically tractable information-driven pricing kernel models. We develop a model that shares many of the features of the rational lognormal model, and investigate examples of heat kernel models. It is shown that these models may result in discount bonds and interest rates being bounded by deterministic functions. In certain situations, incoming information about random variables may exhibit jumps. To this end, we construct a more general class of finite-time pricing kernel models that are driven by Lévy random bridges. Finally, we model the aggregate impact of uncertainties on a financial market by randomised mixtures of Lévy and Markov processes respectively. It is assumed that market participants have incomplete information about the underlying random mixture. We apply results from non-linear filtering theory and construct Flesaker-Hughston models and infinite-time heat kernel models based on these randomised mixtures.

Declaration

I declare that this thesis is my own work, except where otherwise indicated. Due acknowledgment has been made in the text to all other material used. This thesis is being submitted for the Degree of Doctor of Philosophy to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other university.

Signature

Date

Acknowledgments

I'm grateful to my advisor, Andrea Macrina, for the effort he has put into supervising me from abroad. I want to thank him for giving me the opportunity to work with him on interesting research, for his guidance and constant encouragement and, for being both a mentor and a friend.

I wish to thank Ebrahim Momoniat, Head of School of Computational and Applied Mathematics, for his support for this project. I'm deeply grateful to David Taylor, from whom I have learned so much over the years. I'm thankful for his enduring and generous support for my postgraduate studies. I would also like to express my gratitude to Raouf Ghomrasni for helpful comments on part of this work and for suggesting useful references. Further thanks are due to David Sherwell and members of the Programme in Advanced Mathematics of Finance.

Part of this work was developed when I was a visiting research student in the Department of Mathematics, King's College London (1 October 2010 – 30 September 2011). I owe thanks to members of the Financial Mathematics group, especially D. Brigo and C. Buescu, for valuable suggestions. I'm grateful to D. C. Brody, H. Geman, L. P. Hughston, W. T. Shaw, and members of the London Graduate School in Mathematical Finance for helpful remarks. Further thanks are due to members of the Department of Mathematics, University College London, for their hospitality.

Some of this research was carried out during visits to Japan. I would like to thank the staff and students at Department of Mathematical Sciences, Ritsumeikan University; Institute of Economic Research, Kyoto University; and Graduate School of International Corporate Strategy, Hitotsubashi University, for their hospitality. I am especially grateful to J. Akahori, T. Honda, A. Kohatsu-Higa, R. Miura, H. Nakagawa, K. Ohashi and J. Sekine for stimulating discussions.

The financial support of the Programme in Advanced Mathematics of Finance; School of Computational and Applied Mathematics; National Research Foundation of South Africa (NRF)¹; and the Commonwealth Scholarship Commission in the United Kingdom, is gratefully acknowledged.

¹Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the NRF.

Special thanks are due to Ed Hoyle for providing *Mathematica* code for the simulations of the Lévy random bridges used in Chapter 6. I am very grateful to Olivier Menoukeu Pamen, for answering many questions and for commenting on this thesis. Further thanks are due to Maria Alessandra Crisafi (for insightful discussion and helpful comments related to Chapter 7), Junhuan Zhang (for sharing ideas at KCL) and Nien-Lin Liu and Yuri Imamura (for interesting research discussions and for teaching me a great deal about Japanese culture).

Finally, immeasurable thanks are due to my entire family. I'm especially grateful to my parents, Anant and Nisha, for their love and wholehearted support; to my aunts, Bhavna and Vasanti, and uncle Rasu, for their constant encouragement; and to my sister Sonali, for being both my most exacting critic and my best friend.

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Chapter 1

Introduction

Information is an important commodity in financial markets as it shapes investors' perceptions and decisions. In this thesis, we take the view that information is a key determinant of and driving force behind the term structure of interest rates and the prices of financial securities. We assume that there are numerous underlying fundamental factors, the values of which are not directly observable and will only be revealed at future times. Examples include (but are not limited to) macroeconomic, political, social and demographic random variables. We posit that incoming news about these factors influences the dynamics of interest rates, the future cash flows (and hence, prices) of securities and the excess rate of return demanded by investors for taking on risk. Information about latent or hidden factors that are revealed asymptotically may also account for fluctuations in price levels and the term structure of interest rates. Security prices may, in addition, be affected by information about idiosyncratic random variables. The information available in markets is seldom completely reliable since, as it circulates, it is corrupted by noise and speculation (much like a radio signal is disturbed by static during transmission). Investors make financial decisions on the basis of this partial or incomplete information and over time acquire knowledge, and update their expectations with the arrival of new information. We are primarily concerned with the problem of pricing in the presence of incomplete information.

According to Cochrane [27], in order to price any security, we have to account for (i) the delay and (ii) the risk of its future cash flows. Here, it is shown that the value of any security is given by the expected value of its discounted future cash flows, where a single pricing kernel (state-price density or stochastic discount factor) is used for all types of securities. This results in a consistent approach to pricing. Macrina [68] notes that a pricing kernel captures both investor impatience and attitudes towards risk. Therefore, by specifying a model for a pricing kernel, we implicitly model the term structure of interest rates and the excess rate of return demanded by market participants for investing in risky assets. We are of

the opinion that both investor impatience and attitudes towards risk are influenced by evolving information about fundamental factors that is available in the market.

A significant assumption and inherent feature of the models constructed in this thesis is that nominal interest rates are non-negative. We recall that a nominal interest rate reflects (i) the revenue earned from lending money, (ii) the cost of borrowing money, (iii) the opportunity cost of holding cash, and (iv) time preference (see Eyster [33]). Under normal economic conditions, we expect borrowers to make interest payments to lenders; that is, typically interest rates are bounded from below by zero.

A well-known no-arbitrage argument for why nominal interest rates should not be negative is the “money-under-the-mattress” argument: We assume that market participants can hold cash¹. Investors would rather hold onto cash than earn a negative return on investments. Alternatively, by borrowing at a negative interest rate and holding cash until the repayment date, borrowers can realize a risk-free profit from a zero initial investment. Therefore, in order to prevent such arbitrage opportunities, nominal interest rates should be non-negative.

Keister [64] notes that financial markets are for the most part designed to function under non-negative interest rates and may experience considerable disruptions under negative interest rates. There are numerous criticisms of negative interest rates on the grounds that financial motives and behavior become counterintuitive. For instance, lenders have to make interest payments to borrowers. Garbade & McAndrews [42] note that “socially unproductive practices” may be encouraged, and since spending, as opposed to saving for the future, is rewarded, we may see individuals and governments becoming excessively extravagant. Moreover, so-called “interest avoidance strategies” could emerge where market participants prefer to make payments quickly and receive payments in forms that can be collected slowly ([42]). In addition, Butler [24] notes that extended periods of negative interest rates could ultimately lead to an increasingly cash-based society and possible bank runs.

A drawback of several well-known models for the term structure of interest rates is that they allow nominal interest rates and bond yields to become negative. Examples are the Gaussian models, such as the Merton, Vasicek, Ho-Lee and Hull-White models, which have the property that the short rate is normally distributed (and discount bond prices are lognormally distributed). A number of term structure models have been developed over the years to remedy this shortcoming. Examples include the Dothan, Brennan-Schwartz, Cox-Ingersoll-Ross, Longstaff, Black-Derman-Toy, Miltersen-Sandmann-Sondermann, Black-Karasinski and Rendleman-Bartter models. In addition, “shadow-rate models” (see Black [11] and Gorovoi & Linetsky [45]) have also been developed, where

¹We shall ignore the costs of hoarding cash.

the interest rate is modelled as an option to ensure non-negativity. In this thesis, we shall use a pricing kernel approach to construct non-negative interest rate models. A striking feature of pricing kernel models is the ease with which non-negativity of interest rates can be ensured: we simply require that the pricing kernel is a positive supermartingale.

The primary objective of this thesis is to demonstrate how a range of pricing kernel models which are sensitive to noisy macroeconomic information may be constructed. These information-sensitive pricing kernel models describe the evolution of interest rates and allow for consistent, arbitrage-free pricing across asset classes.

This thesis is organized as follows. In Chapter 2, we introduce the reader to the theoretical background. There are three pillars upon which the material in this thesis is based: (i) pricing kernel modelling, (ii) stochastic filtering theory, and (iii) information and filtration modelling. To begin with, we define the pricing kernel and discuss its significance in the context of interest rate modelling. We summarize some important representations of pricing kernels which have appeared in the literature over the past two decades, and explain the interrelationships between them. We then provide an introduction to classical filtering theory. We describe the filtering problem and provide the mathematical results required to treat such a problem. Next, we describe an “information-based” approach to asset pricing: we consider the framework by Brody et al. [19, 21] and Macrina [68] for asset pricing under incomplete information. Instead of assuming that asset prices are adapted to a “pre-specified filtration”, the information that is available to market participants, i.e. the market filtration, is modelled explicitly. The presence of incomplete information leads to a filtering problem. We make explicit the connections with standard filtering theory. Furthermore, since we consider asset pricing in the absence of arbitrage, all pricing results can be equivalently formulated in terms of a pricing kernel. Thus, Section 2.3 brings together the themes of pricing, filtering and filtration modelling. It also gives us the opportunity to establish the notation that we will use hereafter.

In Chapter 3, we begin by describing an approach by Hughston & Macrina [55] where pricing kernel models which are sensitive to Brownian bridge information are constructed over a finite time interval. The idea is as follows: There is a set of fixed future dates at which the values of independent fundamental macroeconomic random variables will be successively revealed. Prior to these dates, the economic factors are not directly observable and investors possess incomplete information about them. The information is modelled by independent Brownian bridge information processes as in [19, 21, 68], and the market filtration is generated by these information processes. The pricing kernel is modelled by the product of a martingale (driven by the information processes) and a function of the information

processes and time. The choice of the function is a modelling input. The martingale is chosen such that it induces a change of measure to an auxiliary measure under which the Brownian bridge information processes have the law of Brownian bridges. The introduction of the auxiliary measure proves to be convenient for calculation purposes since, informally speaking, the signal component is removed from each information process under the new measure. This technique will be used extensively in this thesis and will be adapted later in this chapter and in Chapter 6. In order to ensure non-negativity of the short rate, it is shown in [55] that the input function must satisfy a partial differential inequality (hereafter PDI). We are interested in characterizing the class of functions that satisfy this PDI as such functions generate appropriate models for nominal interest rates.

To this end, we first extend the above results to an infinite time setup. Here, we take the set of random variables to be independent factors that are revealed asymptotically. We assume that each information process is modelled by a Brownian motion with a random drift (see Brody et al. [22]). We use a similar approach to obtain results which are the Brownian counterparts of the Brownian bridge results. In particular, the martingale in the pricing kernel model is now chosen such that it induces a change of measure to an auxiliary measure under which the Brownian information processes have the law of Brownian motion. Here, the input function must also satisfy a PDI to ensure non-negativity of the short rate.

We show in both settings that if the function of the information processes and time is a supermartingale under the respective auxiliary measure, then, provided the function is differentiable in time and twice differentiable in the space variables, it satisfies the appropriate PDI. Next, we use the theory of space-time superharmonic functions for Brownian motion and Brownian bridges in order to characterize solutions to the respective PDIs. In the final part of this chapter, we recall the weighted heat kernel approaches described in Chapter 2. The approaches of Akahori et al. [2] and Akahori & Macrina [3] provide useful methods for constructing processes which are supermartingales under the respective auxiliary measures, and for which the underlying functions satisfy the PDIs.

In Chapter 4 we generalize the BHM framework for the pricing of credit-risky assets to a stochastic interest rate setting, by using the pricing kernel approach of Hughston & Macrina [55]. The material in this chapter is based on Macrina & Parbhoo [70]. Additional details are, however, discussed in Sections 4.3, 4.4 and 4.6. The basic setup is as follows: We consider two fixed future dates at which the value of two independent random variables are successively revealed. The first random variable represents an idiosyncratic factor that is associated with a debt issuer that is used to model the cash flows of the securities, while the second random variable is a fundamental factor (macroeconomic, or otherwise). Once again, we assume that investors possess incomplete information about these

random variables, and the market filtration is generated by the associated information processes. To begin with, we consider a credit-risky discount bond with a payoff that is a function of the credit-related idiosyncratic random variable and the information available at bond maturity about the fundamental factor. The change-of-measure technique discussed in Chapter 3 is used to derive the bond price process. We also calculate the yield spread which is a measure of the excess return that a credit-risky bond provides over that of a discount bond, and consider a pricing kernel model which may be sensitive to information about default. Next, a number of models for random recovery are constructed. In the event of default of a credit-risky bond, the amount recovered may, for instance, depend on factors specific to the firm issuing the bond, and on available information about the future state of the economy. Furthermore, we provide additional insights on how empirical evidence on corporate bonds may help us to construct models for creditor recovery.

A pricing formula for a European call option written on a binary bond is derived. In addition, we consider a multi-dimensional setting and examine the pricing of credit-inflation hybrid instruments and credit-risky bonds traded in a foreign currency. In both cases, we make use of the foreign exchange analogy. We also extend the framework to price credit-risky coupon-bearing bonds for which default can occur at any of the coupon dates. The models considered so far have all been based on Brownian bridge information processes. To end this chapter, we construct a pricing kernel model that also reacts to the cumulative debt of a country over a finite period of time. We assume that the debt is increasing over that period of time and use a gamma bridge accumulation process (see Brody et al. [20]) to model the aggregate debt. In doing so, we incorporate jumps into the model.

Chapter 5 is concerned with the construction and analysis of several explicit finite-time information-sensitive pricing kernel models driven by Brownian bridge information. These models are shown to satisfy the PDI for non-negativity of interest rates in Chapter 3. To begin with, a novel information-driven model – which shares many properties with the rational lognormal model by Flesaker & Hughston [36] – is developed. We show that discount bond prices and interest rates are bounded by deterministic functions. The boundedness feature also has implications for the pricing of other types of securities. We demonstrate this by discussing the pricing of bond options and swaptions. The model has the attractive feature of producing analytical formulae for the prices of such instruments. In addition, we compare the constructed model with the rational lognormal model and describe an interesting link with the space-time superharmonic functions considered in Chapter 3.

Next, we consider two explicit examples of models based on Brownian bridge

information which have been constructed by Akahori & Macrina [3]: a quadratic model and an exponential-quadratic model. We show that the discount bond prices and interest rates are also bounded in these models. Simulations are used to further illustrate this. In recent work, Macrina [69] has extended the heat kernel models in [3], to produce models which have additional flexibility for calibration. We show that this approach can be used, for example, to generate models based on Brownian bridge information. In particular, we examine a class of analytical models considered in [69], which we dub the “(bA)” models, and provide sufficient conditions for such models to produce bounded discount bond prices and interest rates.

To end this chapter, we recall that real interest rates may become negative. We construct one possible example of an information-sensitive real pricing kernel model, which coupled with a suitable model for the nominal pricing kernel, results in a model for inflation which is impacted by the evolution of macroeconomic information.

Chapter 6 closely follows the ideas in Section 3.1; however, we construct pricing kernel models based on Lévy random bridge (LRB) information. In doing so, we generalize the Brownian bridge information setup to include the situation where the information process jumps. We provide background on Lévy processes, Lévy bridges and Lévy random bridges from Hoyle [52] and Hoyle et al. [53]. We show the relationship between the results in a Brownian random bridge setup with the previously derived expressions in the Hughston-Macrina framework. In addition, a simple example of a model based on a $1/2$ -stable random bridge is constructed. In principle, we can consider security pricing (along similar lines to Chapter 4) with LRB information. To avoid too much repetition, we only consider the valuation of (i) a discount bond option and (ii) a credit-risky bond.

In Chapter 7, we construct models which differ considerably in style from those considered thus far. Here, we build infinite-time models driven by randomised mixtures of stochastic processes. The material in this chapter is based on Macrina & Parbhoo [71]; however, further findings are discussed in Section 7.2. We take the view that there are a number of sources of uncertainty that influence financial markets over time e.g. economic, geopolitical and demographic factors. We can think of these factors as bringing about moves in financial markets. We model the impact of each factor by a stochastic process, and we model the combined impact of these uncertainties by a randomised mixture of these stochastic processes. Even though the factors jointly influence markets, the impact of a particular factor may be felt to a greater (lesser) extent at any time. To begin with, we assume that these stochastic processes are Lévy processes and we define a randomised Esscher martingale. We provide additional background on the use of the standard Esscher transform as a change-of-measure, and we describe how a randomised Esscher

martingale may also be used to induce a change of measure.

It is assumed that market participants cannot directly observe the precise underlying mixture of Lévy processes and possess genuine information about the mixture that is distorted by Brownian noise. Thus, we have a stochastic filtering problem at hand. We can compute the optimal filter of the randomised Esscher martingale family, which we call the filtered Esscher martingale family. By using results from filtering theory described in Chapter 2, we can express the filtered Esscher martingale family in terms of a conditional density process. Some explicit examples based on different Lévy processes are provided. We may also want to model an information process with discontinuous noise: as an example, we construct a gamma information process.

Having specified the building blocks, we now construct Flesaker-Hughston pricing kernel models that are based on the filtered Esscher martingales. We construct examples of models based on a Brownian information process and (i) a Brownian motion, (ii) a gamma process and (iii) a variance-gamma process, respectively. In the Brownian case, we derive the dynamics and simulate sample paths for the bond price process and interest rate. In the gamma case, we perform a detailed analysis of the sensitivity of the model parameters. We observe that the choice of the random mixing function significantly influences the model dynamics. To further demonstrate this, we construct a so-called “chameleon” random mixer which changes its form at a random time. Next, we simulate the associated model-generated discount bond curves and yield curves in order to determine which types of yield curve movements the models can produce. We also consider the pricing of a European call option on a discount bond and simulate the option price surfaces.

The constructed class of pricing kernel models are extended by applying the weighted heat kernel approach of Akahori et al. [2], to generate models that are driven by randomised mixtures of Markov processes. Here, we are able to build models using Markov processes which may have dependent increments. For the sake of an example, we consider a quadratic model driven by an Ornstein-Uhlenbeck process and a Brownian information process. In the final part of this chapter, we attempt to classify the considered randomised mixture models with non-negative interest rates, and we illustrate the relationships between these classes.

In short, this thesis makes a distinct contribution to the fields of pricing kernel theory, stochastic filtering theory, and information-based asset pricing. Chapters 3, 5 and 6 are partially expository, but also serve to extend the ideas of Hughston & Macrina [55], Akahori & Macrina [3], Macrina [69] and Hoyle et al. [53] among others, in new directions. In particular, we provide original analysis on the PDI for non-negative interest rates; we construct explicit examples of information-based

term structure models and provide novel numerical analysis; and we examine interest rate modelling for the more generalized class of Lévy random bridges. The most significant original contributions of this thesis appear in Chapter 4, where we price a range of credit-risky instruments in a stochastic interest rate setting, and in Chapter 7, where we develop pricing kernel models driven by a mixture of stochastic processes, where a random parameter is the source of the mixture.

Chapter 2

Pricing, filtering, and filtration modelling

To begin with, we provide some background on pricing kernels, filtering theory and security pricing under incomplete information. We do not attempt an exhaustive survey of the vast literature on these topics. Rather, our intent is to provide a summary of fundamental results which form the foundation of the material in subsequent chapters.

2.1 Pricing kernel models

2.1.1 Background

We model uncertainty by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the real probability measure and $\{\mathcal{F}_t\}_{t \geq 0}$ is the market filtration representing the flow of information in a financial market. By a filtration, we mean a family of sub-sigma-fields of $\mathcal{F}$ that is increasing; that is, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. It is assumed that the filtered probability space satisfies the “usual conditions”:

- (i) $\mathcal{F}$ is complete, i.e. $A \subset B$, $B \in \mathcal{F}$ and $\mathbb{P}(B) = 0$ implies that $A \in \mathcal{F}$ and $\mathbb{P}(A) = 0$;
- (ii) $\mathcal{F}_0$ contains all the $\mathbb{P}$ -null sets;
- (iii) $\{\mathcal{F}_t\}$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{u > t} \mathcal{F}_u$ for all $t \geq 0$ (see Hunt & Kennedy [58]).

We shall assume the absence of arbitrage; however we do not assume market completeness. In what follows, all considered stochastic processes are assumed to be càdlàg (right continuous with left limits). In the rest of Section 2.1, we shall write $\mathbb{E}[\cdot]$ for $\mathbb{E}^{\mathbb{P}}[\cdot]$.

The following definition is taken from Axiom A2 in Hughston & Mina [56]:

Definition 2.1.1. Let $\{S_t\}_{t \geq 0}$ be the price process of an asset with cumulative dividend process $\{\Delta_t\}_{t \geq 0}$. We assume that there exists a process $\{\pi_t\}_{t \geq 0}$ satisfying $\pi_t > 0$ almost surely for all $t \geq 0$ such that the “deflated total value process” $\{\bar{S}_t\}_{t \geq 0}$ defined by

$$\bar{S}_t = \pi_t S_t + \int_0^t \pi_u d\Delta_u \quad (2.1.1)$$

is an $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale. We call the process $\{\pi_t\}$ a **pricing kernel**.

Remark 2.1.1. Let $\{S_t\}$ be the price process of a non-dividend-paying asset. We assume that there exists a pricing kernel $\{\pi_t\}_{t \geq 0}$ satisfying $\pi_t > 0$ almost surely for all $t \geq 0$ such that the deflated total value process $\{\bar{S}_t\}_{t \geq 0}$ defined by

$$\bar{S}_t = \pi_t S_t \quad (2.1.2)$$

is an $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale.

Typically, in mathematical finance, pricing results are formulated in terms of a so-called “numeraire pair” (see [58]). A numeraire pair $(\{N_t\}, \mathbb{N})$ consists of a strictly positive numeraire (or unit of account) $\{N_t\}$, and a martingale measure $\mathbb{N}$ under which a price process normalized by $\{N_t\}$ is a martingale. It can be shown that the existence of a numeraire pair implies the absence of arbitrage in a continuous-time setting. Furthermore, in a complete market, given a numeraire, there exists a unique martingale measure. Hunt & Kennedy [58] note that a pricing kernel is closely related to a numeraire pair: It is proved that

- (i) an economy admits a numeraire pair if, and only if, there exists a pricing kernel;
- (ii) a price process normalized by $\{N_t\}$ is a $\mathbb{N}$ -martingale for all numeraire pairs $(\{N_t\}, \mathbb{N})$ if, and only if, the price process deflated by $\{\pi_t\}$ is a $\mathbb{P}$ -martingale for all pricing kernels $\{\pi_t\}$; and
- (iii) any two numeraire pairs, $(\{N_t\}, \mathbb{N}_1)$ and $(\{N_t\}, \mathbb{N}_2)$, with common numeraire $\{N_t\}$ agree if, and only if, any two pricing kernels, $\{\pi_t^{(1)}\}$ and $\{\pi_t^{(2)}\}$ agree.

It follows that if a security delivers a single random cash flow H_T at time T , then its price at $0 \leq t < T$ is given by the following pricing kernel valuation formula:

$$H_t = \frac{1}{\pi_t} \mathbb{E}[\pi_T H_T \mid \mathcal{F}_t]. \quad (2.1.3)$$

Since a T -maturity discount bond $\{P_{tT}\}_{0 \leq t \leq T}$ delivers a cash flow of $H_T = 1$ at time T , its value at $0 \leq t \leq T$ is

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}[\pi_T \mid \mathcal{F}_t]. \quad (2.1.4)$$

Clearly $P_{tT} > 0$ and $P_{tt} = 1$ for all $0 \leq t \leq T$. Using equation (2.1.4) as a starting point for term structure modelling has an important advantage: non-negativity of interest rates can be guaranteed by imposing conditions directly on the pricing kernel.

Definition 2.1.2. A *positive (resp. non-negative) interest rate model* is a term structure model for which

- (i) the prices of zero-coupon bonds $\{P_{tT}\}$ satisfy $0 < P_{tT} \leq 1$;
- (ii) P_{tT} are decreasing (resp. non-increasing) in the maturity index $T < \infty$.

A sufficient condition for non-negativity of interest rates is that the pricing kernel $\{\pi_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -supermartingale, i.e. $\mathbb{E}[\pi_u] < \infty$ and

$$\mathbb{E}[\pi_v | \mathcal{F}_u] \leq \pi_u \quad \text{for } 0 \leq u \leq v. \quad (2.1.5)$$

The supermartingale property (2.1.5) implies that $P_{tT} \leq 1$ for $0 \leq t \leq T$, and positivity of the discount bond price follows from the positivity of the pricing kernel. Furthermore, for $T_1 \leq T_2$, discount bond prices satisfy

$$P_{tT_2} = \frac{1}{\pi_t} \mathbb{E}[\pi_{T_2} | \mathcal{F}_t] = \frac{1}{\pi_t} \mathbb{E}[\mathbb{E}[\pi_{T_2} | \mathcal{F}_{T_1}] | \mathcal{F}_t] \leq \frac{1}{\pi_t} \mathbb{E}[\pi_{T_1} | \mathcal{F}_t] = P_{tT_1}. \quad (2.1.6)$$

This follows from the tower rule and the fact that the conditional expectation preserves ordering almost surely, i.e. if $Y \leq X$ a.s. and $\mathbb{E}[|X|], \mathbb{E}[|Y|] < \infty$, then

$$\mathbb{E}[Y | \mathcal{B}] \leq \mathbb{E}[X | \mathcal{B}] \quad \text{a.s.} \quad (2.1.7)$$

where $\mathcal{B}$ is a sigma-field (see Roussas [90]).

The idea of using the pricing kernel for interest rate modelling originated from Constantinides [28]. Thereafter, a number of related approaches to pricing kernel modelling have appeared in the literature, most notably: the characterization of HJM models with positive interest rates by Flesaker & Hughston [36]; the “potential approach” to term structure modelling by Rogers [88]; and the chaotic approach to interest rate modelling by Hughston & Rafailidis [57] and Brody & Hughston [17]. Rutkowski [91], Glasserman & Jin [44] and Hunt & Kennedy [58] show some interesting connections between these approaches. More recently, Akahori et al. [2] and Akahori & Macrina [3] have introduced weighted heat kernel pricing kernel models. We now provide a summary of several significant representations of the pricing kernel. We use insights gained from Parbhoo [82] and the references therein.

2.1.2 Representations of pricing kernels

Radon-Nikodym derivative representation

We begin by considering a Brownian setup, in which the market filtration $\{\mathcal{F}_t\}$ is generated by a Brownian motion $\{W_t\}_{t \geq 0}$. Let $\{n_t\}_{t \geq 0}$ be the strictly positive money market account given by

$$n_t = \exp \left(\int_0^t r_s ds \right) \quad (2.1.8)$$

where $\{r_t\}_{t \geq 0}$ is the short rate. We assume that underlying assets and derivatives are priced in the market by using a fixed equivalent martingale measure $\mathbb{Q}$, with $\{n_t\}$ as the numeraire. Then, if an asset delivers a single random cash flow H_T at time T , its value at $0 \leq t < T$ is given by the usual discounted expectation:

$$S_t = n_t \mathbb{E}^{\mathbb{Q}} \left[\frac{H_T}{n_T} \middle| \mathcal{F}_t \right]. \quad (2.1.9)$$

Since the measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, there exists a strictly positive change-of-measure martingale $Z_t = d\mathbb{Q}/d\mathbb{P} |_{\mathcal{F}_t}$ with $Z_0 = 1$. By Bayes' formula, we can write

$$S_t = \frac{n_t}{Z_t} \mathbb{E} \left[\frac{Z_T}{n_T} H_T \middle| \mathcal{F}_t \right]. \quad (2.1.10)$$

By comparing (2.1.3) with (2.1.10), we see that the corresponding pricing kernel is given by

$$\pi_t = \frac{Z_t}{n_t} \quad (2.1.11)$$

(see Björk [10]). Rewriting (2.1.11), we see that the pricing kernel is of the form

$$\pi_t = \exp \left[- \int_0^t \left(r_s + \frac{1}{2} \lambda_s^2 \right) ds - \int_0^t \lambda_s dW_s \right], \quad (2.1.12)$$

where $\{\lambda_t\}_{t \geq 0}$ denotes the market price of risk, and $\{\pi_t\}$ is governed by the dynamics

$$d\pi_t = -r_t \pi_t dt - \lambda_t \pi_t dW_t. \quad (2.1.13)$$

It is evident that positivity of the short rate is equivalent to the condition that $\{\pi_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -supermartingale. Equation (2.1.12) shows that the specification of a model for the pricing kernel is equivalent to choosing a model for the short rate and the market risk premium. In a more general Lévy setup, choosing a model for the pricing kernel is equivalent to specifying a model for the short rate and the excess rate of return.

Potentials

In an infinite-time setting, in addition to the requirement that nominal interest rates are non-negative, we may insist that the value of a discount bond should vanish in the limit of infinite maturity. Then, we require that $\{\pi_t\}$ is a positive supermartingale which satisfies:

$$\lim_{t \rightarrow \infty} \mathbb{E}[\pi_t] = 0. \quad (2.1.14)$$

We can characterize processes which have this property by recalling the following definition from Meyer [74, 75]:

Definition 2.1.3. *Let $\{X_t\}_{t \geq 0}$ be a right-continuous supermartingale. We say that $\{X_t\}$ is a **potential** if the random variables X_t are positive, and if*

$$\lim_{t \rightarrow \infty} \mathbb{E}[X_t] = 0. \quad (2.1.15)$$

Every potential constitutes an admissible pricing kernel (see e.g. Björk [10] and Andruszkiewicz & Brody [4]). Therefore, we can utilize the theory of potentials to construct pricing kernel models. To this end, we shall use the classic representation of “class (D) potentials” in [74, 75]. First, we need to define what we mean by a class (D) process.

Definition 2.1.4. *Let $\{X_t\}_{t \geq 0}$ be a right-continuous stochastic process that is adapted to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The process $\{X_t\}_{t \geq 0}$ is said to belong to the **class (D)** if the random variables X_τ are uniformly integrable, where τ is any finite $\{\mathcal{F}_t\}$ -stopping time.*

In what follows, we suppose that the process $\{A_t\}_{t \geq 0}$ has positive right-continuous non-decreasing paths, such that $A_0 = 0$ almost surely. In addition, we let $\{A_t\}$ be integrable; that is, $\mathbb{E}[A_\infty] < \infty$, where $A_\infty := \lim_{t \rightarrow \infty} A_t$. We assume that $\{A_t\}$ is adapted to the market filtration $\{\mathcal{F}_t\}$.

Proposition 2.1.1. *Let $\{\zeta_t\}_{t \geq 0}$ be a right-continuous version of the process $\{\mathbb{E}[A_\infty | \mathcal{F}_t] - A_t\}$. Then $\{\zeta_t\}$ is a class (D) potential.*

Proof. This proof follows [75]. Let $\{M_t\}_{t \geq 0}$ be a right continuous version of the martingale $\mathbb{E}[A_\infty | \mathcal{F}_t]$. Then, since $\{A_t\}$ is a right-continuous submartingale, $\{M_t - A_t\}$ is a right-continuous supermartingale. Furthermore,

$$\lim_{t \rightarrow \infty} \mathbb{E}[M_t - A_t] = \lim_{t \rightarrow \infty} \mathbb{E}[A_\infty] - \lim_{t \rightarrow \infty} \mathbb{E}[A_t] = 0, \quad (2.1.16)$$

and $M_t - A_t \geq 0$ for each $t \geq 0$. Let V denote the set of all $\{\mathcal{F}_t\}$ -stopping times. Then, for $\tau \in V$ the random variables M_τ are uniformly integrable. The random variables A_τ are also uniformly integrable since they are dominated by A_∞ . Thus, any right-continuous version, $\{\zeta_t\}$, is a class (D) potential. $\square$

Meyer [74] refers to the process $\{\zeta_t\}_{t \geq 0}$ as the *potential generated by $\{A_t\}$* . Moreover, the following stronger result can be proved.

Theorem 2.1.1. *A potential $\{X_t\}_{t \geq 0}$ belongs to the class (D) if, and only if, it is generated by a process $\{A_t\}$.*

Proof. The sufficiency is evident from Proposition 2.1.1. The proof of the necessity is less direct and can be found in [74]. $\square$

Hence, to construct a pricing kernel model which produces discount bond prices with the above-mentioned economically desirable properties, it is enough to choose an appropriate process $\{A_t\}$, and to model the pricing kernel by

$$\pi_t = \mathbb{E}[A_\infty | \mathcal{F}_t] - A_t. \quad (2.1.17)$$

Equation (2.1.17) is significant because, in many of the pricing kernel frameworks we shall consider hereafter, the pricing kernel admits this particular representation.

The Flesaker-Hughston framework

The “positive interest” framework was constructed by Flesaker & Hughston [36] and developed further in Flesaker & Hughston [37, 38, 39]. These models form the subclass of the models by Heath et al. [49], for which interest rates are positive. Even though the original formulation of the Flesaker-Hughston framework was not directly in terms of pricing kernels, there is a very close relationship between the pricing kernel approach and Flesaker-Hughston models.

In the Flesaker-Hughston framework, the pricing kernel is modelled by

$$\pi_t = \int_t^\infty \rho(u) m_{tu} du. \quad (2.1.18)$$

Here, we choose a positive function

$$\rho(t) = -\partial_t P_{0t} \quad (2.1.19)$$

to match the initial term structure, and $\{m_{tu}\}_{0 \leq t \leq u < \infty}$ is a family of positive unit-initialized martingales; that is,

- (i) $m_{tu} > 0$ for $0 \leq t \leq u < \infty$;
- (ii) $m_{0u} = 1$ for $0 \leq u < \infty$;
- (iii) $\mathbb{E}[m_{tu} | \mathcal{F}_s] = m_{su}$ for $0 \leq s \leq t \leq u < \infty$,

where $\{\mathcal{F}_t\}$ is the market filtration. By substituting (2.1.18) into the bond pricing formula (2.1.4), and by using Fubini's theorem and the martingale property, it follows that the price of a discount bond with maturity T is given by

$$P_{tT} = \frac{\int_T^\infty \rho(u) m_{tu} du}{\int_t^\infty \rho(u) m_{tu} du}. \quad (2.1.20)$$

Moreover, since the short rate is given by $r_t = -\partial_T \ln(P_{tT})|_{T=t}$, the expression for the short rate is

$$r_t = \frac{\rho(t) m_{tt}}{\int_t^\infty \rho(u) m_{tu} du}. \quad (2.1.21)$$

Here, interest rates are clearly positive by construction. Therefore, in order to model the interest rate system using the Flesaker-Hughston framework, we need to give the initial term structure and an explicit positive martingale family $\{m_{tu}\}$ as inputs. The relationship between Flesaker-Hughston models and class (D) potentials is made explicit by Glasserman & Jin [44] and Hunt & Kennedy [58], among others.

Proposition 2.1.2. *Let $\{m_{tu}\}_{0 \leq u < \infty}$ be a family of positive martingales and let $\rho(t)$ be given by (2.1.19). Then the pricing kernel given by (2.1.18) has the representation (2.1.17), where*

$$A_t = \int_0^t \rho(u) m_{uu} du, \quad (2.1.22)$$

That is, (2.1.18) is the potential generated by (2.1.22) and is, thus, a potential of class (D).

Proof. Firstly, we note that $\{A_t\}$ is a positive, continuous, increasing and integrable process with $A_0 = 0$. The potential generated by (2.1.22) is given by

$$\zeta_t = \mathbb{E}[A_\infty | \mathcal{F}_t] - A_t = \mathbb{E}[A_\infty - A_t | \mathcal{F}_t]. \quad (2.1.23)$$

By Fubini's theorem and the martingale property, we can simplify this expression to obtain

$$\zeta_t = \int_t^\infty \rho(u) m_{tu} du. \quad (2.1.24)$$

That $\{\zeta_t\}$ is a class (D) potential follows from Theorem 2.1.1. Therefore, (2.1.18) is a class (D) potential. $\square$

Next, a partial converse to this result is stated (see, for instance [10, 44, 58]).

Lemma 2.1.2. *Given a continuous, increasing, integrable process $\{A_t\}$ of the form*

$$A_t = \int_0^t a_u du, \quad (2.1.25)$$

where $\{a_t\}_{t \geq 0}$ is a positive process and $A_0 = 0$, let the pricing kernel be given by (2.1.17). Then there exists a positive deterministic function $\rho(u)$ and a positive family of unit-initialized martingales $\{m_{tu}\}_{0 \leq t \leq u < \infty}$, defined by

$$\rho(u) = \frac{\mathbb{E}[a_u]}{\pi_0} \quad m_{tu} = \frac{\mathbb{E}[a_u | \mathcal{F}_t]}{\mathbb{E}[a_u]} \quad (2.1.26)$$

such that the pricing kernel has a Flesaker-Hughston representation:

$$\pi_t = \pi_0 \int_t^\infty \rho(u) m_{tu} du, \quad (2.1.27)$$

where π_0 is a constant scaling factor.

Proof. For $0 \leq t \leq T < \infty$, we define $N_{tT} = \mathbb{E}[\pi_T | \mathcal{F}_t]$. It follows that $\partial_T N_{tT} = -\mathbb{E}[a_T | \mathcal{F}_t]$. We can write

$$\rho(u) = \frac{\mathbb{E}[a_u]}{\pi_0}. \quad (2.1.28)$$

We set

$$m_{tu} = \frac{\mathbb{E}[a_u | \mathcal{F}_t]}{\mathbb{E}[a_u]}. \quad (2.1.29)$$

It can be proved that $\{m_{tT}\}$ is a positive unit-initialized martingale. Then we can write

$$N_{tT} = \pi_0 \int_T^\infty \rho(u) m_{tu} du. \quad (2.1.30)$$

Equation (2.1.30) follows since $N_{tt} = \pi_t$. $\square$

It follows that Flesaker-Hughston models are precisely the class of pricing kernels that are class (D) potentials generated by positive increasing, integrable processes $\{A_t\}$ of the form (2.1.25) (see [58]). Therefore, to model such class (D) potentials, it suffices to specify a family of positive martingales.

Conditional variance representation

Another closely related method for modelling pricing kernels is the so-called “chaotic approach to interest rate modelling” by Hughston & Rafailidis [57]. Other significant contributions to this line of research include the work of Brody & Hughston [17], and more recently, Grasselli & Tsujimoto [47] and Hughston & Mina [56]. Let L^2 denote the space of square-integrable functions. We assume that the market filtration $\{\mathcal{F}_t\}$ is generated by a single Brownian motion $\{W_t\}$. In [17] the pricing kernel is modelled by the conditional variance of a square-integrable random variable X_∞ , i.e.

$$\pi_t = \mathbb{E}^\mathbb{P}[X_\infty^2 | \mathcal{F}_t] - (\mathbb{E}^\mathbb{P}[X_\infty | \mathcal{F}_t])^2. \quad (2.1.31)$$

A useful way of modelling the square-integrable random variable X_∞ is by means of the Wiener chaos expansion. For compactness, we do not discuss the theory behind this expansion; the reader is referred to Wiener [98], Itô [60], Øksendal [79] and Grasselli & Hurd [46] for further details. We can express X_∞ by the following series of iterated Itô integrals:

$$\begin{aligned} X_\infty = & \int_0^\infty \phi_1(s_1) dW_{s_1} + \int_0^\infty \int_0^{s_1} \phi_2(s_1, s_2) dW_{s_2} dW_{s_1} \\ & + \int_0^\infty \int_0^{s_1} \int_0^{s_2} \phi_3(s_1, s_2, s_3) dW_{s_3} dW_{s_2} dW_{s_1} + \dots \end{aligned} \quad (2.1.32)$$

where each integrand $\phi_i(\cdot) \in L^2(\mathbb{R}_+^i)$ is a deterministic function of i variables. The inputs of “chaotic” pricing kernel models are these so-called *chaos coefficients*. Equation (2.1.31) can be substituted into the bond price formula, where X_∞ is given by (2.1.32), in order to obtain an expression for the discount bond price. An interest rate model that only contains terms up to order n in the expansion of X_∞ is called an n th-order chaos model. First chaos models are examples of completely deterministic interest rate models since both the pricing kernel and the zero-coupon bond system are deterministic. The second chaos models are the simplest models that introduce stochasticity (see [17, 57]). It is worth noting that equation (2.1.31) can also be expressed in the form of a class (D) potential (2.1.17) (see [46, 47]), and thus generates Flesaker-Hughston models.

Rogers’ explicit models

Rogers [88, 89] builds explicit families of interest rate models by using Markov processes to construct potentials. In what follows, we give a brief outline of the approach taken. Let $\{X_t\}_{t \geq 0}$ be a Markov process on a state space $\mathbb{X}$. Let $g : \mathbb{X} \rightarrow \mathbb{R}_+$ be a measurable bounded function and let α be a positive constant. Let R_α denote the resolvent operator of the Markov process; that is, the Laplace transform of the transition function:

$$R_\alpha g(x) = \mathbb{E} \left[\int_0^\infty e^{-\alpha s} g(X_s) ds \mid X_0 = x \right]. \quad (2.1.33)$$

In [88] it is noted that the requirement that $g(x)$ is bounded is a sufficient (but not necessary) condition for $R_\alpha g(x)$ to be well-defined. The pricing kernel is modelled by

$$\pi_t = e^{-\alpha t} R_\alpha g(X_t) = \int_0^\infty e^{-\alpha(t+s)} \mathbb{E}[g(X_{t+s}) \mid X_t] ds. \quad (2.1.34)$$

Different choices of $g(x)$, α and $\{X_t\}$ produce a variety of examples of term structure models. Explicit constructions of models based on Markov diffusions can

be found in [88], whereas [89] and related work cited therein, investigate models based on finite-state Markov chains.

Let $\{\mathcal{F}_t\}$ be the natural filtration of the process $\{X_t\}$. Then, it can be shown that the pricing kernel (2.1.34) can be written in the form of equation (2.1.17). That is, (2.1.34) is a potential generated by

$$A_t = \int_0^t e^{-\alpha s} g(X_s) ds, \quad (2.1.35)$$

and thus, a potential of class (D); see [89]. It follows that these pricing kernel models belong to the Flesaker-Hughston class. In particular, this approach produces explicit examples of Flesaker-Hughston models that are driven by Markov processes. The relationship between the two approaches is seen more directly by noting that (2.1.34) can be expressed as (2.1.27) with

$$\rho(u) = \frac{e^{-\alpha u} \mathbb{E}[g(X_u)]}{\pi_0} \quad \text{and} \quad m_{tu} = \frac{\mathbb{E}[g(X_u) | \mathcal{F}_t]}{\mathbb{E}[g(X_u)]}. \quad (2.1.36)$$

A similar result can be found in Björk [10].

Weighted heat kernel models

More recently, Akahori et al. [2] have introduced a *weighted heat kernel* approach as a technique for generating supermartingales from time-homogeneous Markov processes. We now summarize the main elements of this approach. Let $\{X_t\}_{t \geq 0}$ be a time-homogeneous Markov process defined on a Polish¹ state space $\mathbb{S}$. Let $\mathcal{U} = \{(u, t) \in (0, \infty) \times [0, \infty)\}$.

Definition 2.1.5. *A measurable function $p : \mathcal{U} \times \mathbb{S} \rightarrow \mathbb{R}$ is known as a **propagator** if it satisfies*

$$\mathbb{E}[p(u, t, X_t) | X_s = x] = p(u + t - s, s, x) \quad (2.1.37)$$

for $(u, t) \in \mathcal{U}$, $0 \leq s \leq t$ and $x \in \mathbb{S}$.

In [2], the following useful example of a propagator is provided:

$$p(u, t, X_t) = \mathbb{E}[g(t + u, X_{t+u}) | X_t] \quad (2.1.38)$$

where it is sufficient to choose $g : \mathbb{S} \rightarrow \mathbb{R}_+$ to be a measurable, bounded function.²

¹A Polish space is a metric space that is complete (every Cauchy sequence is convergent) and separable (it contains a countable dense subset), see e.g. Serfozo [93].

²Once a Markov process $\{X_t\}$ has been selected, we may be able to choose $g(t, x)$ to be a measurable, integrable function.

Furthermore, we define a weight function by a measurable function $w : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}_+$ which satisfies

$$w(t, v - s) \leq w(t - s, v) \quad (2.1.39)$$

for arbitrary $t, v \in \mathbb{R}_0^+$ and $s \leq t \wedge v$. For $p : \mathcal{U} \times \mathbb{S} \rightarrow \mathbb{R}_+$, it can be shown that the process

$$\nu_t = \int_0^\infty w(t, u) p(u, t, X_t) du \quad (2.1.40)$$

is a positive supermartingale. If we let the propagator be given by (2.1.38), then we can model the pricing kernel by

$$\pi_t = \int_0^\infty w(t, u) \mathbb{E}[g(t + u, X_{t+u}) | X_t] du. \quad (2.1.41)$$

We are able to generate non-negative interest rate models by specifying a time-homogeneous Markov process $\{X_t\}$ and by choosing appropriate functions $w(t, u)$ and $g(t, x)$. By substituting (2.1.41) into (2.1.4) we obtain the following discount bond price formula:

$$P_{tT} = \frac{\int_{T-t}^\infty w(T, u - T + t) \mathbb{E}[g(t + u, X_{t+u}) | X_t] du}{\int_0^\infty w(t, u) \mathbb{E}[g(t + u, X_{t+u}) | X_t] du}. \quad (2.1.42)$$

The weighted heat kernel approach can be seen as a generalization of Rogers' approach when the underlying Markov process is time-homogeneous. In particular, from equations (2.1.34) and (2.1.41), we see that Rogers' models can be recovered in this setup by considering a function $g(x)$ with no time dependence, and by choosing the weight function to be

$$w(t, u) = \exp[-\alpha(t + u)], \quad (2.1.43)$$

where α is a positive constant. We have shown that Rogers' models belong to the Flesaker-Hughston class, and are class (D) potentials. It is natural to ask if such a relationship exists for weighted heat kernel models. We show this explicitly in the following proposition.

Proposition 2.1.3. *Let $\{X_t\}$ be a time-homogeneous Markov process and let $\{\mathcal{F}_t\}$ be the filtration generated by $\{X_t\}$. Let $g : \mathbb{S} \rightarrow \mathbb{R}$ be a measurable, bounded function, and let the weight function be given by*

$$w(t, u) = \bar{w}(t + u) \quad (2.1.44)$$

where $\bar{w} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded non-increasing function. We assume that

$$\int_0^\infty \bar{w}(t + u) \mathbb{E}[g(t + u, X_{t+u})] du < \infty. \quad (2.1.45)$$

Then the pricing kernel is given by

$$\pi_t = \int_0^\infty \bar{w}(t+u) \mathbb{E}[g(t+u, X_{t+u}) | X_t] du. \quad (2.1.46)$$

A sufficient condition for (2.1.46) to be a potential is that $\bar{w}(s) \rightarrow 0$ in the limit as $s \rightarrow \infty$. In this case, the pricing kernel (2.1.46) is a potential generated by

$$A_t = \int_0^t \bar{w}(u) g(u, X_u) du; \quad (2.1.47)$$

that is, a potential of class (D). Moreover, (2.1.46) can be expressed by (2.1.27) with

$$\rho(u) = \frac{\bar{w}(u) \mathbb{E}[g(u, X_u)]}{\pi_0} \quad \text{and} \quad m_{tu} = \frac{\mathbb{E}[g(u, X_u) | \mathcal{F}_t]}{\mathbb{E}[g(u, X_u)]}. \quad (2.1.48)$$

The constant π_0 is a scaling factor. Therefore, such weighted heat kernel models belong to the Flesaker-Hughston class of models.

For interest rate modelling with a fixed finite time horizon $0 < U < \infty$, it is sufficient that the pricing kernel is a positive supermartingale over the interval $[0, U)$ for interest rates to remain non-negative. To this end, the weighted heat kernel approach of Akahori et al. [2] has been adapted by Akahori & Macrina [3] to generate supermartingales driven by time-inhomogeneous Markov processes. The ideas are similar to those considered above. However, now we need to

- (i) specify a time-inhomogeneous Markov process $\{L_{tU}\}_{0 \leq t \leq U}$ taking values in a Polish space $\mathbb{V}$;
- (ii) choose a measurable, integrable function $G : [0, U) \rightarrow \mathbb{R}_+$; and
- (iii) specify $w : [0, U] \times [0, U] \rightarrow \mathbb{R}_+$ satisfying (2.1.39).

Then, for $0 \leq t < U$, we can model the pricing kernel by

$$\pi_t = \int_0^{U-t} w(t, u) \mathbb{E}[G(u+t, L_{u+t, U}) | L_{tU}] du. \quad (2.1.49)$$

It is shown in [3] that (2.1.49) is a positive supermartingale. By substituting (2.1.49) into the bond price formula (2.1.4), we see that the price of a discount bond with maturity T is given by

$$P_{tT} = \frac{\int_{T-t}^{U-t} w(T, u-T+t) \mathbb{E}[G(t+u, L_{t+u, U}) | L_{tU}] du}{\int_0^{U-t} w(t, u) \mathbb{E}[G(t+u, L_{t+u, U}) | L_{tU}] du}. \quad (2.1.50)$$

2.2 Filtering theory

The main objective of stochastic filtering theory is to estimate a “signal” which cannot be observed directly based on observations of an associated process. Many significant advances in the theory of filtering were made during the 1960’s and 1970’s: some important contributions include the works of Kushner [65, 66], Kallianpur & Striebel [62, 63], Zakai [102] and Fujisaki et al. [41]. There is a vast body of literature on the subject and a variety of applications in fields ranging from signal processing and navigation to biology, stochastic control and mathematical finance (see van Handel [95]). This section serves as an overview of non-linear filtering in continuous time. The discussion which follows is largely based on Chapter 5 of Xiong [99], Chapter 3 of Bain & Crisan [7], and Crisan [31].

2.2.1 The filtering problem

Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{P})$ be a filtered probability space. We shall take $\{\mathcal{G}_t\}$ to be

$$\mathcal{G}_t = \sigma(X_0, \{W_s\}_{0 \leq s \leq t}, \{B_s\}_{0 \leq s \leq t}), \quad (2.2.1)$$

where X_0 is a random variable, $\{W_t\}$ is a p -dimensional Brownian motion, and $\{B_t\}$ is an m -dimensional Brownian motion. We assume that X_0 , $\{W_t\}$ and $\{B_t\}$ are all independent of each other. For the remainder of Section 2.2, we shall take $\mathbb{P}$ to be an arbitrary measure without any particular interpretation, so as to remain as general as possible. We shall use the notation $\mathbb{E}[\cdot]$ for $\mathbb{E}^{\mathbb{P}}[\cdot]$. To begin with, we consider two processes:

We suppose that $\{X_t\}$ is the following d -dimensional diffusion driven by the p -dimensional $(\{\mathcal{G}_t\}, \mathbb{P})$ -Brownian motion $\{W_t\}$, where X_0 is a random variable:

$$X_t = X_0 + \int_0^t c(s, X_s) ds + \int_0^t \varsigma(s, X_s) dW_s. \quad (2.2.2)$$

We assume that $c : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\varsigma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times p}$ satisfy the global Lipschitz and linear growth conditions. That is, there exists a constant $K > 0$ such that for all $x, y \in \mathbb{R}^d$, we have

$$\|c(t, x) - c(t, y)\| + \|\varsigma(t, x) - \varsigma(t, y)\| \leq K\|x - y\|, \quad (2.2.3)$$

$$\|c(t, x)\|^2 + \|\varsigma(t, x)\|^2 \leq K^2(1 + \|x\|^2). \quad (2.2.4)$$

We recall that for $z \in \mathbb{R}^d$, the Euclidean norm is defined by

$$\|z\| = \left(\sum_{i=1}^d z_i^2 \right)^{1/2}, \quad (2.2.5)$$

and for a matrix $A \in \mathbb{R}^{d \times p}$, we have

$$\|A\| = \left(\sum_{i=1}^d \sum_{j=1}^p A_{ij}^2 \right)^{1/2}. \quad (2.2.6)$$

Under these conditions, equation (2.2.2) has a unique solution (see Øksendal [80]). We call $\{X_t\}$, the *signal process*.

Next, we let $\ell : \mathbb{R}_+ \times \mathbb{S} \rightarrow \mathbb{R}^m$ be a measurable function. Then the stochastic process $\{Y_t\}_{t \geq 0}$ defined by

$$Y_t = \int_0^t \ell(s, X_s) ds + B_t \quad (2.2.7)$$

is called the *observation process*. The $(\{\mathcal{G}_t\}, \mathbb{P})$ -Brownian motion $\{B_t\}$ is independent of the signal process $\{X_t\}$. We also impose global Lipschitz and linear growth conditions on $\ell(t, x)$, that is, we assume that there exists a constant $M > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\|\ell(t, x) - \ell(t, y)\| \leq M\|x - y\|, \quad (2.2.8)$$

$$\|\ell(t, x)\|^2 \leq M^2(1 + \|x\|^2). \quad (2.2.9)$$

In addition, we assume that

$$\mathbb{P} \left[\int_0^t \|\ell(s, X_s)\| ds < \infty \right] = 1. \quad (2.2.10)$$

Bain & Crisan [7] note that condition (2.2.10) ensures that the Riemann integral in (2.2.7) exists almost surely. We denote by $\{\mathcal{F}_t\}$ the filtration generated by the observation process, i.e.

$$\mathcal{F}_t = \sigma(\{Y_s\}_{0 \leq s \leq t}). \quad (2.2.11)$$

Since $\ell(t, x)$ is a measurable function and $\{B_t\}$ is adapted to $\{\mathcal{G}_t\}$, it follows that $\{Y_t\}$ is adapted to $\{\mathcal{G}_t\}$. Therefore, $\mathcal{F}_t \subset \mathcal{G}_t$. The filtration $\{\mathcal{F}_t\}$ can be seen as the information available from which one estimates the signal. In this setup, there is no information available at time $t = 0$ since $Y_0 = 0$.

Since we wish to estimate X_t based on the information we have gathered from observing the process $\{Y_s\}$ over $[0, t]$, a natural question which arises is: *What is the best estimate?* It turns out that the estimate for X_t which has the minimum quadratic error is given by

$$\hat{X}_t = \mathbb{E}[X_t | \mathcal{F}_t] \quad (2.2.12)$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}$. The proof of this result is given by Lemma A.1.1 in Appendix A with $\xi = X_t$ and $\mathcal{F} = \mathcal{F}_t$. In the case of a function of the signal $g(X_t)$, Xiong [99] notes that

$g(\mathbb{E}[X_t | \mathcal{F}_t])$ is not the minimum quadratic error estimate if $g(x)$ is a non-linear function. By Lemma A.1.1 with $\xi = g(X_t)$ and $\mathcal{F} = \mathcal{F}_t$, we deduce that

$$\widehat{g(X_t)} = \mathbb{E}[g(X_t) | \mathcal{F}_t] \quad (2.2.13)$$

is the estimate of $g(X_t)$ with the minimum quadratic error. With this in mind, we are now in a position to formally define the filtering problem. To this end, we refer to [7]. Let $C_b(\mathbb{R}^d)$ denote the class of bounded continuous functions on $\mathbb{R}^d$.

Definition 2.2.1. *The **filtering problem** consists of establishing the conditional distribution Θ_t of the signal X_t , given the information accumulated from observing $\{Y_s\}$ over $[0, t]$, contained in $\{\mathcal{F}_t\}$. For $\varphi \in C_b(\mathbb{R}^d)$, we need to determine*

$$\Theta_t(\varphi) := \mathbb{E}[\varphi(X_t) | \mathcal{F}_t] = \int_{\mathbb{R}^d} \varphi(x) \Theta_t(dx). \quad (2.2.14)$$

The conditional distribution is the best estimate of the signal process based on the observation filtration. It is also known as the *optimal filter*. We can take one of two possible approaches in order to derive the filtering equations: The first method was largely developed by Fujisaki et al. [41] and is known as the *innovation approach* (martingale approach). The central idea here is that a square-integrable martingale can be represented as a stochastic integral with respect to an “innovation process”. This is used to derive the Fujisaki-Kallianpur-Kunita (FKK) filtering equation for $\{\Theta_t\}$. The mathematical details can be found in [41], and Kallianpur [61], among other filtering texts. We shall use another method called the *change-of-measure approach* (reference measure approach). This method involves constructing a new measure $\mathbb{M}$ under which $\{Y_t\}$ is a $\{\mathcal{G}_t\}$ -adapted Brownian motion independent of $\{X_t\}$. A generalized Bayes’ formula is derived for $\{\Theta_t\}$. This formula can be expressed in terms of an associated “unnormalised” conditional distribution $\{\theta_t\}$, and we can obtain an equation for the dynamics of $\{\theta_t\}$. These results lead to a stochastic differential equation for the optimal filter by Itô’s formula.

2.2.2 The change-of-measure approach

In order to calculate equation (2.2.14), we construct a new probability measure on Ω under which the observation process $\{Y_t\}$ is a Brownian motion. Let $\{\mathcal{E}_t\}_{t \geq 0}$ be the process defined by

$$\mathcal{E}_t = \exp \left(- \int_0^t \ell(s, X_s)^\top dB_s - \frac{1}{2} \int_0^t \|\ell(s, X_s)\|^2 ds \right). \quad (2.2.15)$$

Under the Novikov condition

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^t \|\ell(s, X_s)\|^2 ds \right) \right] < \infty, \quad (2.2.16)$$

the process $\{\mathcal{E}_t\}$ is a $(\{\mathcal{G}_t\}, \mathbb{P})$ -martingale. However, Bain & Crisan [7] note that it is difficult to verify (2.2.16). Instead, in [7] it is proved that if the function $\ell(t, x)$ satisfies

$$\mathbb{E} \left[\int_0^t \|\ell(s, X_s)\|^2 ds \right] < \infty, \quad (2.2.17)$$

$$\mathbb{E} \left[\int_0^t \mathcal{E}_s \|\ell(s, X_s)\|^2 ds \right] < \infty \quad (2.2.18)$$

for all $t > 0$, then $\{\mathcal{E}_t\}$ is a $(\{\mathcal{G}_t\}, \mathbb{P})$ -martingale. Since $\mathcal{E}_t > 0$ and $\mathcal{E}_0 = 1$, we can now introduce a probability measure $\mathbb{M}$ on Ω . We define its Radon-Nikodym derivative with respect to $\mathbb{P}$ by

$$\frac{d\mathbb{M}}{d\mathbb{P}} \Big|_{\mathcal{G}_t} = \mathcal{E}_t. \quad (2.2.19)$$

Proposition 2.2.1. *If conditions (2.2.17) and (2.2.18) hold, then the observation process $\{Y_t\}$ is a $\{\mathcal{G}_t\}$ -adapted Brownian motion independent of $\{X_t\}$ under $\mathbb{M}$. Furthermore, the distribution of the signal process $\{X_t\}$ is the same under both $\mathbb{M}$ and $\mathbb{P}$.*

Proof. We follow [7] for the proof. By Girsanov's theorem

$$Y_t = \int_0^t \ell(s, X_s) ds + B_t \quad (2.2.20)$$

is a m -dimensional $(\{\mathcal{G}_t\}, \mathbb{M})$ -Brownian motion. For a bounded measurable function f defined on the product of the path spaces for the pair process (X, Y) , it can be shown that

$$\mathbb{E}[f(X, Y)\mathcal{E}_t] = \mathbb{E}[f(X, B)]. \quad (2.2.21)$$

Here, the processes $\{X_s\}$, $\{Y_s\}$ are regarded up to time t . Therefore, we have

$$\mathbb{E}^{\mathbb{M}}[f(X, Y)] = \mathbb{E}[f(X, Y)\mathcal{E}_t] = \mathbb{E}[f(X, B)]. \quad (2.2.22)$$

This means that $\{X_s\}$ and $\{Y_s\}$ have the same joint distribution under $\mathbb{M}$ as $\{X_s\}$ and $\{B_s\}$ under $\mathbb{P}$. Therefore, $\{X_s\}$ and $\{Y_s\}$ are $\mathbb{M}$ -independent since $\{X_s\}$ and $\{B_s\}$ are $\mathbb{P}$ -independent. Furthermore, suppose that $f(X, Y) = f(X)$. Then, from (2.2.22)

$$\mathbb{E}^{\mathbb{M}}[f(X)] = \mathbb{E}[f(X)\mathcal{E}_t] = \mathbb{E}[f(X)]. \quad (2.2.23)$$

Thus $\{X_t\}$ has the same law under $\mathbb{M}$ and $\mathbb{P}$. $\square$

Next, we define the process $\{\Lambda_t\}_{t \geq 0}$ by $\Lambda_t = \mathcal{E}_t^{-1}$ for $t \geq 0$. Under $\mathbb{M}$, $\{\Lambda_t\}$ satisfies the following stochastic differential equation:

$$d\Lambda_t = \Lambda_t \ell(t, X_t)^\top dY_t \quad (2.2.24)$$

and

$$\Lambda_t = \exp \left(\int_0^t \ell(s, X_s)^\top dY_s - \frac{1}{2} \int_0^t \|\ell(s, X_s)\|^2 ds \right) \quad (2.2.25)$$

where we denote the transpose of a vector u by $u^\top$. Since $\mathbb{E}^{\mathbb{M}}[\Lambda_t] = \mathbb{E}[\Lambda_t \mathcal{E}_t] = 1$, $\{\Lambda_t\}$ is a $\{\mathcal{G}_t\}$ -adapted martingale under $\mathbb{M}$ and

$$\left. \frac{d\mathbb{P}}{d\mathbb{M}} \right|_{\mathcal{G}_t} = \Lambda_t \quad (2.2.26)$$

for $t \geq 0$.

The next theorem provides a type of Bayes' formula for filtering which enables us to calculate the conditional expectation (2.2.14) by using the probability measure $\mathbb{M}$. This result was shown by Kallianpur & Striebel [62, 63] and is known as the Kallianpur-Striebel formula.

Theorem 2.2.1. (Kallianpur-Striebel formula) *Assume that (2.2.17) and (2.2.18) hold. For $\varphi \in C_b^2(\mathbb{R}^d)$ and $t \geq 0$, the optimal filter is given by*

$$\Theta_t(\varphi) = \frac{\mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t) | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{M}}[\Lambda_t | \mathcal{F}_t]} \quad \mathbb{M} - a.s. \quad (2.2.27)$$

Proof. We follow [7] for the proof of this result. We need to prove that

$$\Theta_t(\varphi) \mathbb{E}^{\mathbb{M}}[\Lambda_t | \mathcal{F}_t] = \mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t) | \mathcal{F}_t] \quad \mathbb{M} - a.s. \quad (2.2.28)$$

Let b be an arbitrary $\{\mathcal{F}_t\}$ -measurable bounded random variable. Then, since $\Theta_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{F}_t]$ and using the tower property of conditional expectation, we can write

$$\mathbb{E}[\Theta_t(\varphi)b] = \mathbb{E}[\varphi(X_t)b]. \quad (2.2.29)$$

Next, we perform a change of measure to $\mathbb{M}$:

$$\mathbb{E}^{\mathbb{M}}[\Lambda_t \Theta_t(\varphi)b] = \mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t)b]. \quad (2.2.30)$$

By the tower rule of conditional expectation, we have

$$\mathbb{E}^{\mathbb{M}} [\mathbb{E}^{\mathbb{M}}[\Lambda_t \Theta_t(\varphi)b | \mathcal{F}_t]] = \mathbb{E}^{\mathbb{M}} [\mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t)b | \mathcal{F}_t]] . \quad (2.2.31)$$

We use the fact that b and $\Theta_t(\varphi)$ are adapted to $\{\mathcal{F}_t\}$ to write

$$\mathbb{E}^{\mathbb{M}} [b \Theta_t(\varphi) \mathbb{E}^{\mathbb{M}}[\Lambda_t | \mathcal{F}_t]] = \mathbb{E}^{\mathbb{M}} [b \mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t) | \mathcal{F}_t]] . \quad (2.2.32)$$

Thus, we have proved formula (2.2.27). $\square$

Remark 2.2.1. Bain & Crisan [7] make the following observations. Firstly, since $\mathbb{M}$ and $\mathbb{P}$ are equivalent measures, we can also write

$$\Theta_t(\varphi) = \frac{\mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t) | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{M}}[\Lambda_t | \mathcal{F}_t]} \quad \mathbb{P} - a.s. \quad (2.2.33)$$

Furthermore, the Kallianpur-Striebel formula holds for any Borel-measurable φ , not necessarily bounded, such that $\mathbb{E}[|\varphi(X_t)|] < \infty$.

Definition 2.2.2. For $t \geq 0$, the *unnormalized conditional distribution* $\{\theta_t\}_{t \geq 0}$ is a $\{\mathcal{F}_t\}$ -adapted process given by

$$\theta_t(\varphi) = \mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X_t) | \mathcal{F}_t]. \quad (2.2.34)$$

Thus, for $t \geq 0$, we can write the Kallianpur-Striebel formula in terms of the unnormalized conditional distribution:

$$\Theta_t(\varphi) = \frac{\theta_t(\varphi)}{\theta_t(1)} \quad \mathbb{M}, \mathbb{P} - a.s. \quad (2.2.35)$$

The denominator of (2.2.35) can be seen as a normalizing factor (see [7]). Thus, $\{\theta_t\}$ is considered to be an “unnormalized” conditional distribution.

2.2.3 Fundamental filtering equations

In what follows, we shall assume that for all $t \geq 0$, conditions (2.2.17) and (2.2.18) hold and that

$$\mathbb{M} \left[\int_0^t [\theta_s(|\ell|)]^2 ds < \infty \right] = 1 \quad (2.2.36)$$

(see [7]). We now derive an equation for the dynamics of $\{\theta_t\}$. This equation is known as the Zakai equation. We recall that the infinitesimal generator $\mathcal{L}_t$ associated with the signal process in (2.2.2) is a second-order differential operator given by

$$\begin{aligned} \mathcal{L}_t g &= \sum_{i=1}^d c_i(t, x) \partial_{x_i} g + \frac{1}{2} \sum_{i,j=1}^d \sum_{k=1}^p \varsigma_{ik}(t, x) \varsigma_{jk}(t, x) \partial_{x_i x_j} g \\ &= \sum_{i=1}^d c_i(t, x) \partial_{x_i} g + \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \partial_{x_i x_j} g \end{aligned} \quad (2.2.37)$$

where

$$a_{ij}(t, x) = \sum_{k=1}^p \varsigma_{ik}(t, x) \varsigma_{jk}(t, x) = (\varsigma(t, x) \varsigma(t, x)^\top)_{ij}, \quad (2.2.38)$$

and $g(x) \in \mathcal{D}(\mathcal{L}_t)$. We denote by $\mathcal{D}(\mathcal{L}_t)$ the domain of the infinitesimal generator. If $g(x) \in C_b^2(\mathbb{R}^d)$, then $g(x) \in \mathcal{D}(\mathcal{L}_t)$. Furthermore, we assume that $1 \in \mathcal{D}(\mathcal{L}_t)$ and $\mathcal{L}_t 1 = 0$.

Theorem 2.2.2. (Zakai equation) *The unnormalized conditional distribution $\{\theta_t\}$ satisfies the following equation:*

$$\theta_t(\varphi) = \theta_0(\varphi) + \int_0^t \theta_s(\mathcal{L}_s \varphi) ds + \int_0^t \theta_s(\varphi \ell^\top) dY_s, \quad (2.2.39)$$

for $\varphi \in \mathcal{D}(\mathcal{L}_t)$.

Proof. We follow van Handel [96] for this proof. By Itô's formula, we can show that

$$\begin{aligned} \Lambda_t \varphi(X_t) &= \varphi(X_0) + \int_0^t \Lambda_s \mathcal{L}_s \varphi(X_s) ds + \int_0^t \Lambda_s \varphi(X_s) \ell^\top(s, X_s) dY_s \\ &\quad + \int_0^t \Lambda_s \nabla_x^\top \varphi(X_s) \zeta(s, X_s) dW_s, \end{aligned} \quad (2.2.40)$$

where ∇_x denotes the gradient. We can now compute the conditional expectation of equation (2.2.40). All the integrands are square integrable; therefore, we can apply Lemma A.2.1. This result and its proof are provided in Section A.2 of Appendix A. We obtain

$$\begin{aligned} \mathbb{E}^\mathbb{M}[\Lambda_t \varphi(X_t) | \mathcal{F}_t] &= \mathbb{E}^\mathbb{M}[\varphi(X_0) | \mathcal{F}_t] + \int_0^t \mathbb{E}^\mathbb{M}[\Lambda_s \mathcal{L}_s \varphi(X_s) | \mathcal{F}_s] ds \\ &\quad + \int_0^t \mathbb{E}^\mathbb{M} \left[\Lambda_s \varphi(X_s) \ell^\top(s, X_s) \middle| \mathcal{F}_s \right] dY_s. \end{aligned} \quad (2.2.41)$$

We recall from Proposition 2.2.1 that X_t is independent of Y_t (and consequently, of $\mathcal{F}_t$) under $\mathbb{M}$. Therefore,

$$\mathbb{E}^\mathbb{M}[\varphi(X_0) | \mathcal{F}_t] = \mathbb{E}^\mathbb{M}[\varphi(X_0)] = \mathbb{E}[\varphi(X_0)]. \quad (2.2.42)$$

Thus, the result is proved. $\square$

In order to solve the filtering problem, an equation for the evolution of the normalised conditional distribution $\{\Theta_t\}$ is required. This equation is called the Kushner-Stratonovich or FKK equation.

Theorem 2.2.3. (Kushner-Stratonovich/FKK equation) *The conditional distribution $\{\Theta_t\}$, satisfies the following equation:*

$$\Theta_t(\varphi) = \Theta_0(\varphi) + \int_0^t \Theta_s(\mathcal{L}_s \varphi) ds + \int_0^t [\Theta_s(\varphi \ell^\top) - \Theta_s(\varphi) \Theta_s(\ell^\top)] (dY_s - \Theta_s(\ell) ds), \quad (2.2.43)$$

for $\varphi \in \mathcal{D}(\mathcal{L}_t)$.

Proof. We recall from (2.2.39) that

$$d\theta_t(\varphi) = \theta_t(\mathcal{L}_t\varphi)dt + \theta_t(\varphi\ell^\top)dY_t, \quad (2.2.44)$$

and use the fact that $\mathcal{L}_t 1 = 0$. We apply Itô's quotient formula to equation (2.2.35) to obtain

$$\begin{aligned} d\Theta_t(\varphi) = & \left[\frac{\theta_t(\mathcal{L}_t\varphi)}{\theta_t(1)} - \left\{ \frac{\theta_t(\varphi\ell^\top)}{\theta_t(1)} - \Theta_t(\varphi)\frac{\theta_t(\ell^\top)}{\theta_t(1)} \right\} \frac{\theta_t(\ell)}{\theta_t(1)} \right] dt \\ & + \left[\frac{\theta_t(\varphi\ell^\top)}{\theta_t(1)} - \Theta_t(\varphi)\frac{\theta_t(\ell^\top)}{\theta_t(1)} \right] dY_t \end{aligned} \quad (2.2.45)$$

This simplifies to

$$d\Theta_t(\varphi) = \Theta_t(\mathcal{L}_t\varphi)dt + [\Theta_t(\varphi\ell^\top) - \Theta_t(\varphi)\Theta_t(\ell^\top)](dY_t - \Theta_t(\ell)dt). \quad (2.2.46)$$

□

We are now in a position to introduce the following important concept in filtering theory.

Definition 2.2.3. *The stochastic process $\{v_t\}_{t \geq 0}$ defined by*

$$v_t = Y_t - \int_0^t \Theta_s(\ell) ds, \quad (2.2.47)$$

*is known as the **innovation process**.*

Under the assumption (2.2.17), the innovation process is well-defined.

Proposition 2.2.2. *The process $\{v_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion.*

Proof. The process $\{v_t\}$ is adapted to $\{\mathcal{F}_t\}$ since $\{Y_t\}$ and $\int_0^t \Theta_s(\ell) ds$ are $\{\mathcal{F}_t\}$ -adapted. According to Lévy's characterization of Brownian motion, a stochastic process is a Brownian motion if, and only if, it is a continuous local martingale starting at 0, with quadratic variation given by t (see, for instance, Protter [85]). Firstly, we observe that $\{v_t\}$ is integrable since (2.2.17) is assumed. For $s \leq t$,

$$\begin{aligned} \mathbb{E}[v_t | \mathcal{F}_s] &= \mathbb{E}\left[Y_t - \int_0^t \Theta_r(\ell) dr \middle| \mathcal{F}_s\right] \\ &= \mathbb{E}\left[B_t - B_s + \int_s^t [\ell(r, X_r) - \Theta_r(\ell)] dr \middle| \mathcal{F}_s\right] + v_s. \end{aligned} \quad (2.2.48)$$

Here, we have substituted in (2.2.7) for $\{Y_t\}$. By the independent increments of Brownian motion, and since $B_t - B_s$ is independent of $\{X_r\}_{r \leq s}$, we have that $B_t - B_s$ is independent of $\mathcal{F}_s$. Thus, we obtain

$$\begin{aligned} \mathbb{E}[v_t | \mathcal{F}_s] &= \int_s^t \mathbb{E}[\ell(r, X_r) - \Theta_r(\ell) | \mathcal{F}_s] dr + v_s \\ &= \int_s^t \mathbb{E}[\ell(r, X_r) - \mathbb{E}[\ell(r, X_r) | \mathcal{F}_r] | \mathcal{F}_s] dr + v_s \\ &= v_s \end{aligned} \quad (2.2.49)$$

by using equation (2.2.14) and the tower property of conditional expectation. It can be shown that $dv_t dv_t = dt$. Therefore, the result follows by Lévy's characterization of Brownian motion. $\square$

We can rewrite the Kushner-Stratonovich equation in terms of the innovations process:

$$\Theta_t(\varphi) = \Theta_0(\varphi) + \int_0^t \Theta_s(\mathcal{L}_s \varphi) ds + \int_0^t [\Theta_s(\varphi \ell^\top) - \Theta_s(\varphi) \Theta_s(\ell^\top)] dv_s. \quad (2.2.50)$$

We now assume that the conditional distribution Θ_t has a density $f_t(x)$, such that

$$\Theta_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{F}_t] = \int_{\mathbb{R}^d} \varphi(x) f_t(x) dx. \quad (2.2.51)$$

Theorem 2.2.4. (Kushner Theorem) *Let Θ_t have a density $f_t(x)$. Then equation (2.2.50) leads to following equation for the conditional density*

$$df_t(x) = \mathcal{L}_t^* f_t(x) dt + f_t(x) [\ell^\top(t, x) - \Theta_t(\ell^\top)] dv_t \quad (2.2.52)$$

with initial condition $f_0(x) \in L^2(\mathbb{R}^d)$, where $\mathcal{L}_t^*$ is the formal adjoint operator of $\mathcal{L}_t$ given by

$$\mathcal{L}_t^* \phi = \frac{1}{2} \sum_{i,j=1}^d \partial_{y_i y_j} [a_{ij}(t, y) \phi] - \sum_{i=1}^d \partial_{y_i} [c_i(t, y) \phi], \quad (2.2.53)$$

where $a_{ij}(t, y)$ is given by (2.2.38).

Proof. We follow Yazigi [101] for the derivation of this result. We start by writing (2.2.50) in terms of the conditional density:

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) f_t(x) dx &= \int_{\mathbb{R}^d} \varphi(x) f_0(x) dx + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_s \varphi(x) f_s(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} f_s(x) [\varphi(x) \ell^\top(s, x) - \varphi(x) \Theta_s(\ell^\top)] dx dv_s. \end{aligned} \quad (2.2.54)$$

We use Fubini's theorem and the fact that $\langle \mathcal{L}_t u, v \rangle = \langle u, \mathcal{L}_t^* v \rangle$,³ to write

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) f_t(x) dx &= \int_{\mathbb{R}^d} \varphi(x) f_0(x) dx + \int_{\mathbb{R}^d} \varphi(x) \int_0^t \mathcal{L}_s^* f_s(x) ds dx \\ &\quad + \int_{\mathbb{R}^d} \varphi(x) \int_0^t f_s(x) [\ell^\top(s, x) - \Theta_s(\ell^\top)] dv_s dx, \end{aligned} \quad (2.2.55)$$

Therefore, we obtain

$$f_t(x) = f_0(x) + \int_0^t \mathcal{L}_s^* f_s(x) ds + \int_0^t f_s(x) [\ell^\top(s, x) - \Theta_s(\ell^\top)] dv_s. \quad (2.2.56)$$

□

van Handel [96] notes that this is a non-linear partial integro-differential equation. Suppose that we also assume the existence of a conditional density $q_t(x)$ for the unnormalised conditional distribution of X_t given $\mathcal{F}_t$, i.e.

$$\theta_t(\varphi) = \int_{\mathbb{R}^d} \varphi(x) q_t(x) dx. \quad (2.2.57)$$

Then by writing the Zakai equation in terms of the density $q_t(x)$ and by following similar arguments to those in the proof of Theorem 2.2.4, we can show that

$$dq_t(x) = \mathcal{L}_t^* q_t(x) dt + q_t(x) \ell(t, x)^\top dY_t. \quad (2.2.58)$$

This equation is a linear stochastic partial differential equation.

Proposition 2.2.3. *Let the unconditional distribution θ_t have a density $q_t(x)$. Then, the density $f_t(x)$ of the conditional distribution Θ_t is given by*

$$f_t(x) = \frac{q_t(x)}{\int_{\mathbb{R}^d} q_t(y) dy}. \quad (2.2.59)$$

Proof. We substitute (2.2.57) into formula (2.2.35) to obtain

$$\int_{\mathbb{R}^d} \varphi(x) f_t(x) dx = \frac{\int_{\mathbb{R}^d} \varphi(x) q_t(x) dx}{\int_{\mathbb{R}^d} q_t(y) dy}. \quad (2.2.60)$$

The result follows by differentiation. □

The reader should refer to [101] and the references therein, for details relating to the existence and uniqueness of the conditional density.

³Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product.

2.2.4 A special case

So far, we have presented the general setup for stochastic filtering in which the signal is modelled by a process $\{X_t\}$. In this thesis, we are interested in a special type of filtering problem; that is, the estimation of a signal random variable using information acquired from the observation process. To obtain the required results, we can use the wider framework where c is taken to be a d -dimensional null vector and ς is a $d \times p$ zero matrix. Then from (2.2.2), the signal is the random variable X_0 . We recall that in the general case the filtration $\{\mathcal{G}_t\}$ is given by equation (2.2.1), and can be thought of as the filtration generated by the signal $\{X_t\}$ and the independent m -dimensional noise $\{B_t\}$. Therefore, in the current setup we have

$$\mathcal{G}_t = \sigma(X_0, \{B_s\}_{s \leq t}). \quad (2.2.61)$$

In order to keep notation simple, we shall denote the signal random variable by X . Thus, $X : \Omega \rightarrow \mathbb{R}^d$ is $\mathcal{G}_t$ -measurable for all $t \geq 0$, and $\{B_t\}$ is a $(\{\mathcal{G}_t\}, \mathbb{P})$ -Brownian motion independent of X . We model the observation process by

$$Y_t = \int_0^t \ell(s, X) ds + B_t. \quad (2.2.62)$$

The observation filtration is given by $\mathcal{F}_t = \sigma(\{Y_s\}_{s \leq t})$ with $\mathcal{F}_t \subset \mathcal{G}_t$. In this case, the filtering problem is to determine

$$\Theta_t(\varphi) = \mathbb{E}[\varphi(X) | \mathcal{F}_t] = \int_{\mathbb{R}^d} \varphi(x) \Theta_t(dx). \quad (2.2.63)$$

To this end, we can use the Kallianpur-Striebel formula to obtain

$$\Theta_t(\varphi) = \frac{\mathbb{E}^{\mathbb{M}}[\Lambda_t \varphi(X) | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{M}}[\Lambda_t | \mathcal{F}_t]} \quad \mathbb{M}, \mathbb{P} - a.s. \quad (2.2.64)$$

where

$$\Lambda_t = \exp \left(\int_0^t \ell(s, X)^\top dY_s - \frac{1}{2} \int_0^t \|\ell(s, X)\|^2 ds \right). \quad (2.2.65)$$

Here, the Zakai equation for the unnormalized conditional distribution reduces to

$$\theta_t(\varphi) = \theta_0(\varphi) + \int_0^t \theta_s(\varphi \ell^\top) dY_s, \quad (2.2.66)$$

and the Kushner-Stratonovich equation is

$$\Theta_t(\varphi) = \Theta_0(\varphi) + \int_0^t [\Theta_s(\varphi \ell^\top) - \Theta_s(\varphi) \Theta_s(\ell^\top)] dv_s, \quad (2.2.67)$$

where the innovation process $\{v_t\}$ is given by

$$v_t = Y_t - \int_0^t \Theta_s(\ell) ds. \quad (2.2.68)$$

If we suppose that θ_t , the unconditional distribution of X given $\mathcal{F}_t$, possesses a density $q_t(x)$, then it can be shown that the dynamics of this density are given by

$$dq_t(x) = q_t(x)\ell(t, x)^\top dY_t. \quad (2.2.69)$$

The solution to this equation is

$$q_t(x) = q_0(x) \exp \left(\int_0^t \ell(s, x)^\top dY_s - \frac{1}{2} \int_0^t \|\ell(s, x)\|^2 ds \right). \quad (2.2.70)$$

By using equation (2.2.59), we obtain the following density for Θ_t :

$$f_t(x) = \frac{f_0(x) \exp \left(\int_0^t \ell(s, x)^\top dY_s - \frac{1}{2} \int_0^t \|\ell(s, x)\|^2 ds \right)}{\int_{\mathbb{R}^d} f_0(y) \exp \left(\int_0^t \ell(s, y)^\top dY_s - \frac{1}{2} \int_0^t \|\ell(s, y)\|^2 ds \right) dy}, \quad (2.2.71)$$

where $f_0(x) = q_0(x)$ is the initial density of X . Moreover, the conditional density $f_t(x)$ satisfies

$$df_t(x) = f_t(x) [\ell^\top(t, x) - \Theta_t(\ell^\top)] dv_t. \quad (2.2.72)$$

In Chapter 7, we shall consider a filtering problem of this type.

2.3 An information-based approach

2.3.1 The BHM framework

We begin by describing some of the underlying ideas of this approach. The following synopsis is largely based on the work of Brody et al. [19, 21] and Macrina [68]. We model uncertainty by specifying a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on which the filtration $\{\mathcal{F}_t\}$, will be explicitly constructed. The filtration $\{\mathcal{F}_t\}$ is taken to be the market filtration. The prices of all assets are adapted to $\{\mathcal{F}_t\}$. The market is assumed to be arbitrage-free and incomplete, and the existence of an established pricing kernel is assumed. This guarantees the existence of an associated equivalent martingale measure $\mathbb{Q}$ (see [68]). Moreover, for convenience, it is assumed that interest rates are deterministic. Thus for $t \leq T$, the price of a discount bond is given by

$$P_{tT} = \frac{n_t}{n_T} = \exp \left(- \int_t^T r_s ds \right) \quad (2.3.1)$$

where $\{n_t\}$ is the money market account process and $\{r_t\}$ is the deterministic short rate.

Let us consider an asset $\{S_t\}$ which generates a series of random cash flows D_k at fixed future dates T_k ($k = 1, \dots, n$). Then the value of such an asset at

some time $t < T_1$ is given by the usual discounted risk-neutral expectation⁴:

$$S_t = \sum_{k=1}^n P_{tT_k} \mathbb{E}^{\mathbb{Q}}[D_k | \mathcal{F}_t] \quad (2.3.2)$$

For each $k = 1, \dots, n$, we assume that the cashflow D_k can be expressed as a non-negative function $F_k(x_1, \dots, x_k)$ of a set of independent market factors $X_{T_1}, \dots, X_{T_k}$, i.e.

$$D_k = F_k(X_{T_1}, X_{T_2}, \dots, X_{T_k}). \quad (2.3.3)$$

For each $t \leq T_k$, we assume that all the information available to market participants about X_{T_k} is contained in an information process. This information process is composed of two parts: a *signal* term, which contains genuine information about the market factor, and a *noise* component which corrupts or distorts the signal. The signal and noise terms are assumed to be independent, that is, the noise contains no useful information about the value of the cash flow. It is assumed that the information process takes the form

$$I_{tT_k} = \sigma_{T_k} X_{T_k} t + \beta_{tT_k}, \quad (2.3.4)$$

where $\{\beta_{tT_k}\}$ represents a standard Brownian bridge over the interval $[0, T_k]$. The true value of X_{T_k} is disclosed at time T_k . Therefore, each market factor X_{T_k} is $\mathcal{F}_{T_k}$ -measurable. The parameter σ_{T_k} can be interpreted as the constant information flow rate associated with the factor X_{T_k} . Higher values of σ_{T_k} correspond to greater transparency concerning X_{T_k} ; while low σ_{T_k} implies that market participants have little knowledge about the value of X_{T_k} until very close to time T_k . We assume that the information processes $\{I_{tT_k}\}$ collectively generate the market filtration, i.e.

$$\mathcal{F}_t := \sigma(\{I_{sT_1}\}_{0 \leq s \leq t}, \dots, \{I_{sT_n}\}_{0 \leq s \leq t}). \quad (2.3.5)$$

The information process $\{I_{tT_k}\}$ is $\{\mathcal{F}_t\}$ -adapted. However, the Brownian bridge is not $\{\mathcal{F}_t\}$ -adapted. If this were the case, market participants would be able to infer the signal based on the information process and the noise.

For convenience, we now consider the simplest model arising in the BHM framework; that is, an asset which generates a single random cashflow

$$D_T = X_T \quad (2.3.6)$$

at time T . In this case, we can regard the cash flow D_T as being the relevant market factor⁵ ([68]). Examples of such an asset include a credit-risky discount

⁴A more general expression for the value of the asset at any time $t \geq 0$ is given in [21], under the assumption that the asset price goes ex-dividend once a dividend is paid.

⁵Thus, here we shall assume that X_T takes positive values.

bond maturing at time T , or an asset which pays a single dividend at time T . We assume that the information about the cash flow is contained in the process

$$I_{tT} = \sigma t D_T + \beta_{tT}, \quad (2.3.7)$$

and that the *a priori* probability distribution of D_T is known. Here, the random variable D_T is taken to be a continuous random variable with *a priori* probability density $p(x)$.⁶ Let $\{\mathcal{F}_t\}$ be the filtration generated by $\{I_{tT}\}$. It is proved in, for example, [19, 21, 68] that the information process $\{I_{tT}\}$ satisfies the Markov property with respect to $\{\mathcal{F}_t\}$.

In order to compute the conditional probability density $\{\pi_t(x)\}$ of D_T given the information I_{tT} , i.e.

$$\pi_t(x) = \frac{d}{dx} \mathbb{Q}[D_T \leq x \mid I_{tT}], \quad (2.3.8)$$

we apply Bayes' formula and use the fact that β_{tT} is a Gaussian random variable with mean 0 and variance $t(T-t)/T$. Let $\rho(I_{tT} \mid D_T = x)$ denote the conditional density function for I_{tT} given that $D_T = x$. Then,

$$\begin{aligned} \pi_t(x) &= \frac{p(x)\rho(I_{tT} \mid D_T = x)}{\int_0^\infty p(y)\rho(I_{tT} \mid D_T = y) dy} \\ &= \frac{p(x) \exp\left[\frac{T}{T-t}(\sigma x I_{tT} - \frac{1}{2}\sigma^2 x^2 t)\right]}{\int_0^\infty p(y) \exp\left[\frac{T}{T-t}(\sigma y I_{tT} - \frac{1}{2}\sigma^2 y^2 t)\right] dy}. \end{aligned} \quad (2.3.9)$$

It follows that the asset price $\{S_t\}$ for $t < T$ is given by

$$\begin{aligned} S_t &= P_{tT} \mathbb{E}^\mathbb{Q}[D_T] \\ &= P_{tT} \int_0^\infty x \pi_t(x) dx \end{aligned} \quad (2.3.10)$$

$$= P_{tT} \frac{\int_0^\infty x p(x) \exp\left[\frac{T}{T-t}(\sigma x I_{tT} - \frac{1}{2}\sigma^2 x^2 t)\right] dx}{\int_0^\infty p(y) \exp\left[\frac{T}{T-t}(\sigma y I_{tT} - \frac{1}{2}\sigma^2 y^2 t)\right] dy}. \quad (2.3.11)$$

We are now in a position to examine the dynamics of $\{S_t\}$. In [21, 68], the following results are obtained:

Proposition 2.3.1. *The stochastic differential equation associated with the asset price process $\{S_t\}$ is given by*

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t, \quad (2.3.12)$$

where $r_t = -\partial_T \ln(P_{tT})|_{T=t}$. The absolute price volatility $\{\Gamma_{tT}\}$ is given by

$$\Gamma_{tT} = P_{tT} \frac{\sigma T}{T-t} V_t, \quad (2.3.13)$$

⁶If D_T is assumed to be a discrete random variable, analogous results hold.

where V_t is the conditional variance of D_T ; that is

$$V_t = \int_0^\infty x^2 \pi_t(x) dx - \left(\int_0^\infty x \pi_t(x) dx \right)^2. \quad (2.3.14)$$

The process $\{W_t\}_{0 \leq t < T}$ is of the form

$$W_t = I_{tT} + \int_0^t \frac{I_{sT}}{T-s} ds - \sigma T \int_0^t \frac{1}{T-s} \mathbb{E}^\mathbb{Q}[D_T | I_{sT}] ds. \quad (2.3.15)$$

Proof. First we apply Itô's formula to (2.3.9) to compute the dynamics of the conditional density. After simplification, we can write

$$d\pi_t(x) = \frac{\sigma T}{T-t} (x - \mathbb{E}^\mathbb{Q}[D_T | I_{tT}]) \pi_t(x) dW_t \quad (2.3.16)$$

where $\{W_t\}$ is given by (2.3.15). It follows from (2.3.10) that

$$dS_t = r_t S_t dt + P_{tT} \int_0^\infty x d\pi_t(x) dx. \quad (2.3.17)$$

After substituting in (2.3.16), this expression reduces to (2.3.12). $\square$

The process $\{W_t\}$ that turns up in the conditional density dynamics (and hence, in the asset price dynamics) is a $\mathbb{Q}$ -Brownian motion, adapted to the filtration generated by the information process. This is shown by proving that $\{W_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{Q})$ -martingale, and by observing that $dW_t dW_t = dt$. The result follows by Lévy's characterization of Brownian motion (see [21, 68] for details).

For comparison's sake, it is worth pointing out that the starting point in much of the mathematical finance literature is to specify a stochastic differential equation driven by a Brownian motion for the asset price dynamics. In the BHM framework, however, no prior assumptions are made about the asset price dynamics. Instead, the dynamics are deduced and the existence of a Brownian driver $\{W_t\}$ is shown ([68]). This process is referred to as the “innovation process”. We remark that in classical filtering theory, the innovation process is a Brownian motion which emerges in the Kushner-Stratonovich/FKK equation for the conditional distribution and in the Kushner equation (2.2.52) for the conditional density. More will be said on the explicit connections between information-based asset pricing and filtering in the next section.

The abovementioned information-based models have been used by Brody et al. [19] for the pricing of credit-risky securities, and by Brody et al. [21] for the pricing of assets with a range of dividend structures. More recently, Brody et al. [15] have used such models for the pricing of assets in a market with asymmetric information. It is worth mentioning that the model (2.3.7) is not the only type

of information model that one can consider. In Brody et al. [22] the following information model is constructed:

$$I_t = \sigma t X + B_t. \quad (2.3.18)$$

Here $\{B_t\}$ is a Brownian motion independent of the random variable X . The market filtration $\{\mathcal{F}_t\}$ is generated by the process $\{I_t\}$ and X is an $\mathcal{F}_\infty$ -measurable random variable. This information process also has the Markov property with respect to its own filtration ([22]). Brody et al. [20] have used gamma bridge cumulative gains processes for the pricing of insurance and reinsurance products. The Brownian information and Brownian bridge information models have the property that they are additive models; that is, the information is given by the sum of a signal term and the noise term. In [20], multiplicative models are considered, i.e. the cumulative gains process is modelled as the product of the signal and the noise components. Further developments to the theory have been made by Hoyle [52] and Hoyle et al. [53]. In these works, an entire class of information processes called Lévy random bridges has been constructed. We shall consider such information processes more closely in Chapter 6.

2.3.2 Connections with filtering theory

We now establish the explicit mathematical links between the BHM information-based approach and filtering theory. To this end, we return to the setup and notation used in Section 2.2.4, where $\mathbb{P}$ is an arbitrary probability measure. For expectations with respect to the measure $\mathbb{P}$, we shall use the notation $\mathbb{E}[\cdot]$ to denote $\mathbb{E}^\mathbb{P}[\cdot]$. For convenience, we shall restrict our attention to the one-dimensional case in Section 2.2.4, i.e. we assume that $d = 1$ and $m = 1$.

Brownian bridge information

Let us suppose that

$$\ell(t, X) = \frac{\sigma T X}{T - t}. \quad (2.3.19)$$

Then from (2.2.62), we see that the observation process is given by

$$Y_t = \sigma T X \int_0^t \frac{1}{T - s} ds + B_t. \quad (2.3.20)$$

We recall from Section 2.2.4 that X and $\{B_t\}$ are adapted to the filtration $\{\mathcal{G}_t\}$. We define $\{\mathcal{F}_t\}$ to be the filtration generated by the observation process $\{Y_t\}$. We note that $\mathcal{F}_t \subset \mathcal{G}_t$.

Proposition 2.3.2. *Let $f_0(x)$ denote the a priori density of the random variable X . The conditional density $f_t(x)$ of X is given by*

$$f_t(x) = \frac{f_0(x) \exp \left[\frac{T}{T-t} (\sigma x I_{tT} - \frac{1}{2} \sigma^2 x^2 t) \right]}{\int_{\mathbb{R}} f_0(y) \exp \left[\frac{T}{T-t} (\sigma y I_{tT} - \frac{1}{2} \sigma^2 y^2 t) \right] dy} \quad (2.3.21)$$

where the process $\{I_{tT}\}_{t \leq T}$ is defined by

$$I_{tT} = \sigma X t + \beta_{tT}, \quad (2.3.22)$$

and $\{\beta_{tT}\}_{t \leq T}$ is a $(\{\mathcal{G}_t\}, \mathbb{P})$ -Brownian bridge defined by

$$\beta_{tT} = (T-t) \int_0^t \frac{1}{T-s} dB_s. \quad (2.3.23)$$

Proof. We recall that the expression for the conditional density is given by equation (2.2.71). First, we need to compute the exponent which appears in this formula:

$$\int_0^t \ell(s, x) dY_s - \frac{1}{2} \int_0^t \ell^2(s, x) ds. \quad (2.3.24)$$

From (2.3.19) and (2.3.20) it follows that

$$\begin{aligned} \int_0^t \ell(s, x) dY_s - \frac{1}{2} \int_0^t \ell^2(s, x) ds \\ = \int_0^t \frac{\sigma T x}{T-s} \left(\frac{\sigma T X}{T-s} ds + dB_s \right) - \frac{1}{2} \int_0^t \frac{\sigma^2 T^2 x^2}{(T-s)^2} ds \end{aligned} \quad (2.3.25)$$

This can be written in the form

$$\begin{aligned} \frac{\sigma T x}{T-t} \left[(T-t) \int_0^t \frac{1}{T-s} dB_s + \sigma T X (T-t) \int_0^t \frac{1}{(T-s)^2} ds \right] \\ - \frac{1}{2} \sigma^2 T^2 x^2 \int_0^t \frac{1}{(T-s)^2} ds. \end{aligned} \quad (2.3.26)$$

It turns out that the first integral produces a $(\mathcal{G}_t, \mathbb{P})$ -Brownian bridge process $\{\beta_{tT}\}$ over $[0, T]$; that is

$$\beta_{tT} = (T-t) \int_0^t \frac{1}{T-s} dB_s. \quad (2.3.27)$$

Moreover, the deterministic integral simplifies to

$$\int_0^t \frac{1}{(T-s)^2} ds = \frac{t}{T(T-t)}. \quad (2.3.28)$$

Let us define the process $\{I_{tT}\}$ by (2.3.22). Then, by putting these results together and simplifying, we obtain

$$\int_0^t \ell(s, x) dY_s - \frac{1}{2} \int_0^t \ell^2(s, x) ds = \frac{T}{T-t} \left(\sigma x I_{tT} - \frac{1}{2} \sigma^2 x^2 t \right). \quad (2.3.29)$$

Therefore, the conditional density (2.2.71) can be written in the form of (2.3.21). $\square$

Remark 2.3.1. We have followed a calculation by Filipović et al. [35] in the above proof. The study [35] is concerned with the development of conditional density models for asset pricing. Here, a dynamical equation (referred to as the “master” equation) is derived for the conditional density. We remark that the master equation corresponds exactly with the Kushner equation (2.2.72) in Section 2.2.4. It is, indeed, noted in [35] that the construction of conditional density models admits an interpretation as a kind of a filtering problem.

Proposition 2.3.3. *The innovation process $\{v_t\}_{0 \leq t < T}$ can be written as*

$$v_t = I_{tT} + \int_0^t \frac{I_{sT}}{T-s} ds - \sigma T \int_0^t \frac{1}{T-s} \mathbb{E}[X | I_{sT}] ds \quad (2.3.30)$$

where $\{I_{tT}\}$ is given by (2.3.22).

Proof. From equations (2.2.68) and (2.3.19), we have that

$$v_t = Y_t - \sigma T \int_0^t \frac{1}{T-s} \mathbb{E}[X | I_{sT}] ds. \quad (2.3.31)$$

It remains to show that

$$Y_t = I_{tT} + \int_0^t \frac{I_{sT}}{T-s} ds. \quad (2.3.32)$$

We can write

$$dI_{tT} + \frac{I_{tT}}{T-t} dt = \sigma X dt + d\beta_{tT} + \frac{\sigma t X}{T-t} dt + \frac{\beta_{tT}}{T-t} dt. \quad (2.3.33)$$

From (2.3.27), it follows that

$$d\beta_{tT} = - \left(\int_0^t \frac{1}{T-s} dB_s \right) dt + dB_t. \quad (2.3.34)$$

Therefore, (2.3.33) simplifies to

$$\begin{aligned} dI_{tT} + \frac{I_{tT}}{T-t} dt &= \left(\sigma X + \frac{\sigma X t}{T-t} \right) dt + dB_t \\ &= \ell(t, X) dt + dB_t. \end{aligned} \quad (2.3.35)$$

The required result is proved. $\square$

By (2.2.72), we see that the conditional density $f_t(x)$ satisfies

$$df_t(x) = \frac{\sigma T}{T-t} (x - \mathbb{E}[X | \mathcal{F}_t]) f_t(x) dv_t. \quad (2.3.36)$$

We recall that the connection between the observation process $\{Y_s\}$ and the process $\{I_{tT}\}$ is given by (2.3.32), or equivalently, by the following relation from [35]:

$$I_{tT} = (T-t) \int_0^t \frac{1}{T-s} dY_s. \quad (2.3.37)$$

Thus, we can also think of the observation filtration $\{\mathcal{F}_t\}$ in the following terms:

$$\mathcal{F}_t = \sigma(\{I_{sT}\}_{0 \leq s \leq T}). \quad (2.3.38)$$

Here, $\{I_{tT}\}_{0 \leq t \leq T}$ is an information process about the random variable X which is revealed at time T . Therefore, we see that this filtering problem corresponds exactly to the Brownian bridge information model of Brody et al. [19, 21] and Macrina [68] where $\{\mathcal{F}_t\}$ is the market filtration, and the arbitrary measure $\mathbb{P}$ is taken to be the preferred risk-neutral measure $\mathbb{Q}$. We shall consider this type of information model in much of this thesis.

Brownian information

Let us now assume that

$$\ell(s, X) = \sigma X. \quad (2.3.39)$$

Then the observation process is given by

$$Y_t = \sigma X t + B_t. \quad (2.3.40)$$

Let $\{\mathcal{F}_t\}$ be the filtration generated by $\{Y_t\}$, with $\mathcal{F}_t \subset \mathcal{G}_t$. Then from equation (2.2.71), we can express the conditional density $f_t(x)$ by

$$f_t(x) = \frac{\exp(\sigma x Y_t - \frac{1}{2} \sigma^2 x^2 t)}{\int_{\mathbb{R}} \exp(\sigma y Y_t - \frac{1}{2} \sigma^2 y^2 t) dy}. \quad (2.3.41)$$

Moreover, the innovations process is

$$v_t = Y_t - \sigma \int_0^t \mathbb{E}[X | \mathcal{F}_s] ds. \quad (2.3.42)$$

The dynamical equation for the conditional density is given by

$$df_t(x) = \sigma (x - \mathbb{E}[X | \mathcal{F}_t]) f_t(x) dv_t. \quad (2.3.43)$$

In this case, the observation process corresponds exactly with the information process considered, for instance, in Brody et al. [22] and Brody et al. [23], where the market filtration is given by the observation filtration $\{\mathcal{F}_t\}$. We shall consider such information models in Chapter 3 and in Chapter 7.

2.3.3 A pricing kernel formulation

In the information-based approach, a preferred equivalent martingale measure $\mathbb{Q}$ is used from the outset for the pricing of risky assets and derivative securities. In fact, the real probability measure does not enter into the discussion (see Brody et al. [21]). We now change tack to set up the theory using a pricing kernel approach. We model the financial market by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the filtration $\{\mathcal{F}_t\}$, will be explicitly constructed. Here, $\mathbb{P}$ is the real probability measure and $\{\mathcal{F}_t\}$ is the market filtration. We shall assume the absence of arbitrage and the existence of an established pricing kernel $\{\pi_t\}$. Market completeness is not assumed. In what follows, $\mathbb{E}[\cdot]$ is used for $\mathbb{E}^{\mathbb{P}}[\cdot]$.

In this setup, the price at time $t \leq T$ of an asset which generates a single random cash flow $D_T = X_T$ at time T , is

$$S_t = \frac{1}{\pi_t} \mathbb{E}[\pi_T D_T | \mathcal{F}_t] \quad (2.3.44)$$

where $\{\pi_t\}$ is the pricing kernel. In Section 2.1, it has been noted that the pricing kernel can be expressed by $\pi_t = Z_t/n_t$, where $\mathbb{Q}$ is the associated risk neutral measure, $\{n_t\}$ is the money market account numeraire, and $Z_t = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t}$ is the change-of-measure martingale. In a deterministic interest rate setting, the pricing formula reduces to

$$S_t = \frac{n_t}{Z_t} \mathbb{E} \left[\frac{Z_T}{n_T} D_T \middle| \mathcal{F}_t \right] = \frac{n_t}{n_T} \frac{\mathbb{E}[Z_T D_T | \mathcal{F}_t]}{Z_t} = P_{tT} \mathbb{E}^{\mathbb{Q}}[D_T | \mathcal{F}_t] \quad (2.3.45)$$

from (2.3.1) and Bayes' formula. This is exactly the pricing formula used in the BHM framework. In this thesis, however, we are interested in modelling the more general stochastic interest rate environment. To this end, we shall work with the real measure $\mathbb{P}$ and build models for the pricing kernel $\{\pi_t\}$. In the spirit of information-based asset pricing, we shall model market information by way of information processes and construct the market filtration $\{\mathcal{F}_t\}$.

Chapter 3

Information-sensitive pricing kernels

In this chapter, we provide a detailed discussion of the pricing kernel approach of Hughston & Macrina [55], where an information-sensitive pricing kernel that exists up to a finite time horizon is constructed. We show that the proposed framework is flexible enough to generate infinite-time pricing kernel models. In both settings, the pricing kernel is modelled by the product of a martingale and a suitable function of time and the information processes. Since we are interested in the construction of non-negative interest rate models, we also attempt to address the following questions: *Which classes of functions satisfy (i) a PDI derived in [55] and, (ii) the corresponding PDI in the infinite time setting, for non-negativity of the short rate?* Next, we show that the weighted heat kernel method proposed by Akahori & Macrina [3] in a finite-time setting, and by Akahori et al. [2] in an infinite-time setting, can be used to generate suitable information-driven pricing kernel models for which the respective PDIs are satisfied.

3.1 Finite-horizon pricing kernel models

To begin with, we describe the approach used by Hughston & Macrina [55] for the pricing of fixed-income instruments in the general multi-factor setting. To model uncertainty in the financial market, we consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\mathbb{P}$ denotes the real probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the market filtration which models the evolution of information over time. In the rest of this thesis, we denote by $\mathbb{E}[\cdot]$, the expectation under the real measure $\mathbb{P}$. Next, we introduce a set of fixed dates $U_1 \leq U_2 \leq \dots \leq U_n$. To each date U_k ($k = 1, \dots, n$), we attach an independent random variable X_{U_k} . These random variables may be either discrete or continuous with a given *a priori* distribution. In what follows, we assume that each X_{U_k} is a continuous random variable that

takes values in $\mathbb{R}$, with probability density $p_k(x)$. The random variables are taken to be macroeconomic factors, the values of which are successively revealed. With each macroeconomic factor X_{U_k} , we associate an independent Markov process $\{I_{tU_k}\}_{0 \leq t \leq U_k}$. This process models noisy information that is available to the market participants about X_{U_k} . Moreover, we assume that the market filtration $\{\mathcal{F}_t\}$ is generated jointly by all of these information processes, that is,

$$\mathcal{F}_t = \sigma(\{I_{sU_1}\}_{0 \leq s \leq t}, \dots, \{I_{sU_n}\}_{0 \leq s \leq t}). \quad (3.1.1)$$

The absence of arbitrage in the economy is characterized by the existence of a pricing kernel $\{\pi_t\}$. The pricing kernel is a strictly positive process and is adapted to the market filtration. We shall be concerned with a pricing kernel $\{\pi_t\}$ of the form

$$\pi_t := F(t, I_{tU_1}, \dots, I_{tU_n}) \quad (3.1.2)$$

for some positive function $F(t, x_1, \dots, x_n)$. Such a pricing kernel is driven by information, owing to its dependence on the random variables $I_{tU_1}, \dots, I_{tU_n}$. Discount bond prices are determined for $0 \leq t \leq T$, by the following relation:

$$P_{tT} = \frac{\mathbb{E}[\pi_T | \mathcal{F}_t]}{\pi_t}. \quad (3.1.3)$$

We can make use of the Markov property of $\{I_{tU_k}\}$ to write

$$P_{tT} = \frac{\mathbb{E}[F(T, I_{TU_1}, \dots, I_{TU_n}) | I_{tU_1}, \dots, I_{tU_n}]}{F(t, I_{tU_1}, \dots, I_{tU_n})} \quad (3.1.4)$$

for $0 \leq t \leq T < U_1 \leq \dots \leq U_n$. If these expectations can be computed explicitly, we have an analytically tractable model. In general, we can infer the short rate, up to a sign, from the drift of the pricing kernel. Therefore equation (3.1.2) generates arbitrage-free term structure models in which the short rate and the prices of discount bonds fluctuate over time as a result of the disclosure of information to market participants about relevant factors affecting the economy.

We now model noisy information about each factor X_{U_k} by a Brownian bridge information process $\{I_{tU_k}\}_{0 \leq t \leq U_k}$ ($k = 1, \dots, n$) given by

$$I_{tU_k} = \sigma_k t X_{U_k} + \beta_{tU_k} \quad (3.1.5)$$

(see Brody et al. [19, 21] and Macrina [68]). We recall that σ_k represents the rate at which genuine information about X_{U_k} is revealed in the market as time progresses. Each process $\{\beta_{tU_k}\}_{0 \leq t \leq U_k}$ is a Brownian bridge that is independent of X_{U_k} . This term represents Gaussian noise which perturbs the signal X_{U_k} and vanishes at time U_k . The signal and noise are not observable at times $t < U_k$; market participants observe only the sum of these components. It is shown in

e.g. [68] that $\{I_{tU_k}\}$ is a Markov process. Hughston & Macrina [55] construct the following multi-factor pricing kernel model

$$\pi_t = M_t^{(1)} \cdots M_t^{(n)} f(t, I_{tU_1}, \dots, I_{tU_n}) \quad (3.1.6)$$

where $f(t, x_1, \dots, x_n)$ is a positive function and, each $\{M_t^{(k)}\}_{0 \leq t < U_k}$ ($k = 1, \dots, n$) is a martingale defined by

$$M_t^{(k)} = \left(\int_{-\infty}^{\infty} e^{\frac{U_k}{U_k-t} (\sigma_k x I_{tU_k} - \frac{1}{2} \sigma_k^2 x^2 t)} p_k(x) dx \right)^{-1}, \quad (3.1.7)$$

with dynamics given by

$$\frac{dM_t^{(k)}}{M_t^{(k)}} = -\frac{\sigma_k U_k}{U_k - t} \mathbb{E}[X_{U_k} | I_{tU_k}] dW_t^{(k)}. \quad (3.1.8)$$

Each $\{W_t^{(k)}\}_{0 \leq t < U_k}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by

$$W_t^{(k)} = I_{tU_k} + \int_0^t \frac{I_{sU_k}}{U_k - s} ds - \int_0^t \frac{\sigma_k U_k}{U_k - s} \mathbb{E}[X_{U_k} | I_{sU_k}] ds, \quad (3.1.9)$$

where

$$dW_t^{(i)} dW_t^{(j)} = \begin{cases} dt & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

by the independence of the information processes. We note that for each k , $\{M_t^{(k)}\}$ is a martingale which induces a change-of-measure from $\mathbb{P}$ to a measure $\mathbb{B}^k$, under which I_{tU_k} has the law of a Brownian bridge. We show this explicitly by rewriting (3.1.9) in the following form:

$$dI_{tU_k} = dW_t^{(k)} + \left(-\frac{I_{tU_k}}{U_k - t} + \frac{\sigma_k U_k}{U_k - t} \mathbb{E}[X_{U_k} | I_{tU_k}] \right) dt. \quad (3.1.10)$$

Then by Girsanov's theorem, it follows that

$$b_t^{(k)} = W_t^{(k)} + \int_0^t \frac{\sigma_k U_k}{U_k - s} \mathbb{E}[X_{U_k} | I_{sU_k}] ds \quad (3.1.11)$$

is a $(\{\mathcal{F}_t\}, \mathbb{B}^k)$ -Brownian motion. Thus we can write

$$dI_{tU_k} = db_t^{(k)} - \frac{I_{tU_k}}{U_k - t} dt. \quad (3.1.12)$$

These are exactly the dynamics of a $\mathbb{B}^k$ standard Brownian bridge with termination time U_k .

Proposition 3.1.1. *The process $\{M_t\}_{0 \leq t < U_1}$ defined by*

$$M_t := M_t^{(1)} \cdots M_t^{(n)} \quad (3.1.13)$$

is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale.

Proof. By the Markov property and the independence of the information processes, we have for $s \leq t < U_1$, that

$$\begin{aligned} \mathbb{E}[M_t | \mathcal{F}_s] &= \mathbb{E}[M_t^{(1)} \cdots M_t^{(n)} | I_{sU_1}, \dots, I_{sU_n}] \\ &= \mathbb{E}[M_t^{(1)} | I_{sU_1}] \cdots \mathbb{E}[M_t^{(n)} | I_{sU_n}] \\ &= M_s^{(1)} \cdots M_s^{(n)} \\ &= M_s. \end{aligned} \quad (3.1.14)$$

Furthermore, $\mathbb{E}[|M_t|] < \infty$ for $t < U_1$. $\square$

We observe that $M_t > 0$ for $0 \leq t < U_1$ and satisfies $\mathbb{E}[M_t] = 1$. Therefore, the martingale $\{M_t\}_{0 \leq t < U_1}$ can be used to induce a change of measure, which we denote by $\mathbb{B}$.

Proposition 3.1.2. *For each $k = 1, \dots, n$, $\{b_t^{(k)}\}_{0 \leq t < U_1}$ is a $(\{\mathcal{F}_t\}, \mathbb{B})$ -Brownian motion.*

Proof. By a measure change, we see that for $s \leq t < U_1$,

$$\begin{aligned} \mathbb{E}^{\mathbb{B}}[b_t^{(k)} | \mathcal{F}_s] &= \mathbb{E}^{\mathbb{B}}[b_t^{(k)} | I_{sU_1}, \dots, I_{sU_n}] \\ &= \frac{1}{M_s} \mathbb{E}[M_t b_t^{(k)} | I_{sU_1}, \dots, I_{sU_n}] \\ &= \frac{1}{M_s} \mathbb{E}[M_t^{(k)} b_t^{(k)} | I_{sU_k}] \prod_{i=1}^{k-1} \mathbb{E}[M_t^{(i)} | I_{sU_i}] \prod_{j=k+1}^n \mathbb{E}[M_t^{(j)} | I_{sU_j}] \\ &= \frac{1}{M_s^{(k)}} \mathbb{E}[M_t^{(k)} b_t^{(k)} | I_{sU_k}] \\ &= \mathbb{E}^{\mathbb{B}^k}[b_t^{(k)} | I_{sU_k}] \\ &= b_s^{(k)}. \end{aligned} \quad (3.1.15)$$

Thus $\{b_t^{(k)}\}$ is a continuous $(\{\mathcal{F}_t\}, \mathbb{B})$ -martingale with $b_0^{(k)} = 0$. In addition, $db_t^{(k)} db_t^{(k)} = dt$. Therefore, the result follows by Lévy's characterization of Brownian motion. $\square$

Proposition 3.1.3. *Under $\mathbb{B}$, each information process has, over the interval $[0, U_1)$, the distribution of a Brownian bridge on the interval from 0 to the horizon of the information process.*

Proof. This is evident from (3.1.12) and Proposition 3.1.2. We can also derive expressions for the mean and covariance function for the information process under $\mathbb{B}$. We obtain precisely the expressions for the mean and covariance of a standard Brownian bridge with termination time U_k . That is to say, by a change of measure and the independence of the information processes, we see that for $k = 1, \dots, n$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{B}}[I_{tU_k}] &= \mathbb{E}[M_t I_{tU_k}] \\
&= \mathbb{E}[M_t^{(1)} \dots M_t^{(n)} I_{tU_k}] \\
&= \mathbb{E}[M_t^{(k)} I_{tU_k}] \mathbb{E}[M_t^{(1)} \dots M_t^{(k-1)} M_t^{(k+1)} \dots M_t^{(n)}] \\
&= \mathbb{E}^{\mathbb{B}^k}[I_{tU_k}] \\
&= 0,
\end{aligned} \tag{3.1.16}$$

and for $s \leq t < U_1$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{B}}[I_{sU_k} I_{tU_k}] &= \mathbb{E}[M_t I_{sU_k} I_{tU_k}] \\
&= \mathbb{E}[M_t^{(1)} \dots M_t^{(n)} I_{sU_k} I_{tU_k}] \\
&= \mathbb{E}[M_t^{(k)} I_{sU_k} I_{tU_k}] \mathbb{E}[M_t^{(1)} \dots M_t^{(k-1)} M_t^{(k+1)} \dots M_t^{(n)}] \\
&= \mathbb{E}^{\mathbb{B}^k}[I_{sU_k} I_{tU_k}] \\
&= \frac{s(U_k - t)}{U_k}.
\end{aligned} \tag{3.1.17}$$

□

We now return to bond pricing and consider the price of a discount bond with maturity $T < U_1$. In the current setup we have

$$P_{tT} = \frac{\mathbb{E}[M_T f(T, I_{TU_1}, \dots, I_{TU_n}) | I_{tU_1}, \dots, I_{tU_n}]}{M_t f(t, I_{tU_1}, \dots, I_{tU_n})}. \tag{3.1.18}$$

By applying Bayes' formula, we obtain

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}[f(T, I_{TU_1}, \dots, I_{TU_n}) | I_{tU_1}, \dots, I_{tU_n}]}{f(t, I_{tU_1}, \dots, I_{tU_n})}. \tag{3.1.19}$$

In order to simplify the expression for the discount bond, Hughston & Macrina [55] define a collection of random variables $\{Y_{tT}^{(k)}\}_{k=1, \dots, n}$ by

$$Y_{tT}^{(k)} = I_{TU_k} - \frac{U_k - T}{U_k - t} I_{tU_k}. \tag{3.1.20}$$

It can be shown that under the measure $\mathbb{B}$, each $Y_{tT}^{(k)}$ ($k = 1, \dots, n$) is a Gaussian random variable with zero mean and standard deviation given by

$$\nu_{tT}^{(k)} = \sqrt{\frac{(T - t)(U_k - T)}{U_k - t}}. \tag{3.1.21}$$

It can be verified that $Y_{tT}^{(k)}$ is independent of I_{tU_k} under $\mathbb{B}$ by showing that $\text{Cov}^{\mathbb{B}}[Y_{tT}^{(k)} I_{tU_k}] = 0$. Next, we introduce a set of Gaussian random variables Y_k , with zero mean and unit variance; this allows us to write $Y_{tT}^{(k)} = \nu_{tT}^{(k)} Y_k$. Since each I_{tU_k} is $\mathcal{F}_t$ -measurable and Y_k is independent of I_{tU_k} , we can express the price of a sovereign bond by the following Gaussian integral:

$$P_{tT} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{f\left(T, \nu_{tT}^{(1)} y_1 + \frac{U_1-T}{U_1-t} I_{tU_1}, \dots, \nu_{tT}^{(n)} y_n + \frac{U_n-T}{U_n-t} I_{tU_n}\right)}{f(t, I_{tU_1}, \dots, I_{tU_n})} \times \frac{1}{(\sqrt{2\pi})^n} \exp\left[-\frac{1}{2}(y_1^2 + \dots + y_n^2)\right] dy_1 \dots dy_n. \quad (3.1.22)$$

We can construct a range of interest rate models in this framework by specifying the function $f(t, x_1, \dots, x_n)$. Let $C^{1,2}$ denote the space of functions for which continuous derivatives up to the first order in time and second order in the spatial variables exist. We use subscript notation to denote partial derivatives; that is to say $\partial_x u = \partial u / \partial x$ and $\partial_{xx} u = \partial^2 u / \partial x^2$. For $f(t, x_1, \dots, x_n) \in C^{1,2}([0, U_1] \times \mathbb{R}^n)$, it is shown in [55] that the dynamics of the pricing kernel are given by

$$\begin{aligned} \frac{d\pi_t}{\pi_t} = & \frac{1}{f(t, I_{tU_1}, \dots, I_{tU_n})} \left\{ \left[\partial_t f + \sum_{k=1}^n \left(\frac{1}{2} \partial_{x_k x_k} f - \frac{I_{tU_k}}{U_k - t} \partial_{x_k} f \right) \right] dt \right. \\ & \left. + \sum_{k=1}^n \left(\partial_{x_k} f - \frac{\sigma_k U_k}{U_k - t} \mathbb{E}[X_{U_k} | I_{tU_k}] f(t, I_{tU_1}, \dots, I_{tU_n}) \right) dW_t^{(k)} \right\}. \end{aligned} \quad (3.1.23)$$

We recall that the drift of the pricing kernel determines the short rate, that is,

$$r_t = \frac{1}{f(t, I_{tU_1}, \dots, I_{tU_n})} \left[\sum_{k=1}^n \left(\frac{I_{tU_k}}{U_k - t} \partial_{x_k} f - \frac{1}{2} \partial_{x_k x_k} f \right) - \partial_t f \right]. \quad (3.1.24)$$

The volatility term associated with $W_t^{(k)}$ determines the k^{th} component of the market price of risk:

$$\lambda_t^k = \frac{1}{f(t, I_{tU_1}, \dots, I_{tU_n})} \left(\frac{\sigma_k U_k}{U_k - t} \mathbb{E}[X_{U_k} | I_{tU_k}] f(t, I_{tU_1}, \dots, I_{tU_n}) - \partial_{x_k} f \right). \quad (3.1.25)$$

For non-negativity of interest rates, the function $f(t, x_1, \dots, x_n)$ must satisfy the following inequality:

$$\sum_{k=1}^n \left(\frac{x_k}{U_k - t} \partial_{x_k} f - \frac{1}{2} \partial_{x_k x_k} f \right) - \partial_t f \geq 0. \quad (3.1.26)$$

While in [55] the PDI is strict, we are interested in classifying those functions $f(t, x_1, \dots, x_n)$ for which negative interest rates are excluded.

Remark 3.1.1. For convenience, in much of the discussion in Sections 3.4 and 3.5 and in later chapters, we consider a one-dimensional model in which a random variable X_U is revealed at time U . Here, we model noisy information in the market about X_U by $I_{tU} = \sigma t X_U + \beta_{tU}$. The pricing kernel is taken to be of the form $\pi_t = M_t f(t, I_{tU})$, where $\{M_t\}_{0 \leq t < U}$ is the density martingale associated with a change of measure from $\mathbb{P}$ to the bridge measure $\mathbb{B}$ under which the information process has the law of a Brownian bridge; that is

$$M_t = \left(\int_{-\infty}^{\infty} e^{\frac{U}{U-t}(\sigma x I_{tU} - \frac{1}{2}\sigma^2 x^2 t)} p(x) dx \right)^{-1}. \quad (3.1.27)$$

In this case, the price of a discount bond is given by the following Gaussian integral:

$$P_{tT} = \frac{1}{f(t, I_{tU})} \int_{-\infty}^{\infty} f\left(T, \nu_{tT}y + \frac{U-T}{U-t}I_{tU}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy. \quad (3.1.28)$$

In order to guarantee non-negativity of the short rate, the function $f(t, x)$ must satisfy

$$\frac{x}{U-t} \partial_x f - \frac{1}{2} \partial_{xx} f - \partial_t f \geq 0. \quad (3.1.29)$$

3.2 Infinite-time pricing kernel models

The approach in [55] can be extended to an infinite time setup. We now assume that $U_1 = U_2 = \dots = U_n = \infty$ and give Brownian analogues of the Brownian bridge-based results in Section 3.1. For each $k = 1, \dots, n$, we consider the following information process

$$I_t^{(k)} = \sigma_k t X^{(k)} + B_t^{(k)} \quad (3.2.1)$$

where $\{B_t^{(k)}\}_{t \geq 0}$ is a Brownian motion independent of $X^{(k)}$. Here, each $X^{(k)}$ is an $\mathcal{F}_\infty$ -measurable random variable. The underlying variables $X^{(k)}$ can be viewed as being latent or hidden factors. The resulting pricing kernel models have a mechanism for learning about such factors. We remark that the signal term in (3.2.1) is linear in the time variable t , while $B_t^{(k)} \stackrel{d}{=} Y\sqrt{t}$ where Y is a standard normal random variable. Thus, the growth of the noise term is of the order of $\sqrt{t}$. This means that observations of $\{I_t^{(k)}\}$ reveal $X^{(k)}$ asymptotically, as the signal dominates the noise. This is also noted, for instance, by Brody et al. [22] and Brody et al. [23] where such information processes are used in a different context. It can be shown that for each k , $\{I_t^{(k)}\}$ is a Markov process (see [22]). We now assume that the market filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is given by

$$\mathcal{F}_t = \sigma\left(\{I_s^{(1)}\}_{0 \leq s \leq t}, \dots, \{I_s^{(n)}\}_{0 \leq s \leq t}\right). \quad (3.2.2)$$

We model the pricing kernel by

$$\pi_t = m_t^{(1)} \cdots m_t^{(n)} g(t, I_t^{(1)}, \dots, I_t^{(n)}) \quad (3.2.3)$$

where $g(t, x_1, \dots, x_n)$ is a positive function and, for $k = 1, \dots, n$, $\{m_t^{(k)}\}_{0 \leq t < \infty}$ are martingales with dynamics given by

$$\frac{dm_t^{(k)}}{m_t^{(k)}} = -\sigma_k \mathbb{E}[X^{(k)} | I_t^{(k)}] dw_t^{(k)}, \quad (3.2.4)$$

where for each k , $\{w_t^{(k)}\}_{0 \leq t < \infty}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by

$$w_t^{(k)} = I_t^{(k)} - \int_0^t \sigma_k \mathbb{E}[X^{(k)} | I_s^{(k)}] ds. \quad (3.2.5)$$

Since the information processes are independent, it follows that

$$dw_t^{(i)} dw_t^{(j)} = \begin{cases} dt & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For each k , $\{m_t^{(k)}\}$ is a martingale which induces a change-of-measure from $\mathbb{P}$ to a measure $\mathbb{Z}^k$, under which $I_t^{(k)}$ has the law of a Brownian motion.

Proposition 3.2.1. *The process $\{m_t\}_{0 \leq t < \infty}$ defined by*

$$m_t := m_t^{(1)} \cdots m_t^{(n)} \quad (3.2.6)$$

is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale.

Proof. The proof follows that of Proposition 3.1.1. □

Since $m_t > 0$ for $t \geq 0$ and $\mathbb{E}[m_t] = 1$, the martingale $\{m_t\}$ can be used to induce a change of measure, which we denote by $\mathbb{Z}$.

Proposition 3.2.2. *Under $\mathbb{Z}$, each information process $\{I_t^{(k)}\}$ ($k = 1, \dots, n$) has the distribution of a Brownian motion.*

Proof. The proof is similar to that of Proposition 3.1.2. In particular, we can show that for each k , $\{I_t^{(k)}\}$ is a continuous $(\{\mathcal{F}_t\}, \mathbb{Z})$ -martingale starting at 0, for which $dI_t^{(k)} dI_t^{(k)} = dt$. The result follows by Lévy's characterization of Brownian motion. □

In this setting, the price at $t \leq T$ of a discount bond with maturity $T < \infty$ is given by

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{Z}} \left[g(T, I_T^{(1)}, \dots, I_T^{(n)}) \mid I_t^{(1)}, \dots, I_t^{(n)} \right]}{g(t, I_t^{(1)}, \dots, I_t^{(n)})}. \quad (3.2.7)$$

In order to simplify this expression, we follow [55] and introduce a collection of random variables $\{Z_{tT}^{(k)}\}_{k=1,\dots,n}$ where

$$Z_{tT}^{(k)} = I_T^{(k)} - I_t^{(k)}. \quad (3.2.8)$$

Each $Z_{tT}^{(k)}$ ($k = 1, \dots, n$) is a Gaussian random variable with zero mean and standard deviation given by $\sqrt{T-t}$ under the measure $\mathbb{Z}$. Since

$$\text{Cov}^{\mathbb{Z}}[Z_{tT}^{(k)} I_t^{(k)}] = \mathbb{E}^{\mathbb{Z}}[Z_{tT}^{(k)} I_t^{(k)}] = \mathbb{E}^{\mathbb{Z}}[(I_T^{(k)} - I_t^{(k)}) I_t^{(k)}] = 0, \quad (3.2.9)$$

by the independent increments of $\{I_t^{(k)}\}$ under $\mathbb{Z}$, it follows that $Z_{tT}^{(k)}$ is independent of $I_t^{(k)}$ under $\mathbb{Z}$. Therefore, we can write

$$\begin{aligned} P_{tT} = & \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{g\left(T, \sqrt{T-t} z_1 + I_t^{(1)}, \dots, \sqrt{T-t} z_n + I_t^{(n)}\right)}{g(t, I_t^{(1)}, \dots, I_t^{(n)})} \\ & \times \frac{1}{(\sqrt{2\pi})^n} \exp\left[-\frac{1}{2}(z_1^2 + \dots + z_n^2)\right] dz_1 \dots dz_n. \end{aligned} \quad (3.2.10)$$

By the Itô formula, the dynamics of the pricing kernel are given by

$$\begin{aligned} \frac{d\pi_t}{\pi_t} = & \frac{1}{g(t, I_t^{(1)}, \dots, I_t^{(n)})} \left[\partial_t g + \frac{1}{2} \sum_{k=1}^n \partial_{x_k x_k} g \right] dt \\ & + \frac{1}{g(t, I_t^{(1)}, \dots, I_t^{(n)})} \sum_{k=1}^n \left(\partial_{x_k} g - \sigma_k \mathbb{E}[X^{(k)} | I_t^{(k)}] g(t, I_t^{(1)}, \dots, I_t^{(n)}) \right) dw_t^{(k)} \end{aligned} \quad (3.2.11)$$

for $g(t, x_1, \dots, x_n) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, where $\partial_t g$ is the continuous first derivative of $g(t, x_1, \dots, x_n)$ in t , and $\partial_{x_k} g$ and $\partial_{x_k x_k} g$ are continuous first and second derivatives of $g(t, x_1, \dots, x_n)$ in x_k , respectively. We obtain the following expressions for the short rate and the k^{th} component of the market price of risk:

$$r_t = -\frac{1}{g(t, I_t^{(1)}, \dots, I_t^{(n)})} \left[\partial_t g + \frac{1}{2} \sum_{k=1}^n \partial_{x_k x_k} g \right], \quad (3.2.12)$$

$$\lambda_t^k = \frac{1}{g(t, I_t^{(1)}, \dots, I_t^{(n)})} \left(\sigma_k \mathbb{E}[X^{(k)} | I_t^{(k)}] g(t, I_t^{(1)}, \dots, I_t^{(n)}) - \partial_{x_k} g \right). \quad (3.2.13)$$

The function $g(t, x_1, \dots, x_n)$ must satisfy the following PDI for non-negativity of the short rate:

$$-\frac{1}{2} \sum_{k=1}^n \partial_{x_k x_k} g - \partial_t g \geq 0. \quad (3.2.14)$$

Remark 3.2.1. In what follows, for simplicity, we shall consider a one-dimensional model. We model the pricing kernel by $\pi_t = m_t g(t, I_t)$, where $g(t, x)$ is a positive function and $\{m_t\}_{t \geq 0}$ is the change-of-measure martingale from $\mathbb{P}$ to the measure $\mathbb{Z}$, under which the information process $\{I_t\}$ has the law of a Brownian motion. In this case, the price of a discount bond with maturity T is given by

$$P_{tT} = \frac{1}{g(t, I_t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} g(T, \sqrt{T-t}z + I_t) dz \quad (3.2.15)$$

and the associated PDI reduces to

$$-\frac{1}{2} \partial_{xx} g - \partial_t g \geq 0. \quad (3.2.16)$$

3.3 Functions that satisfy the PDIs

Proposition 3.3.1. *Let $\{\mathcal{F}_t\}$ be the filtration generated by the information processes $\{I_{tU_1}\}, \dots, \{I_{tU_n}\}$, and let the pricing kernel be modelled by*

$$\pi_t = M_t f(t, I_{tU_1}, \dots, I_{tU_n}). \quad (3.3.1)$$

Here, $\{M_t\}_{0 \leq t < U_1}$ is given by (3.1.13) and defines a bridge measure $\mathbb{B}$ such that Proposition 3.1.3 holds. Let $f(t, x_1, \dots, x_n) \in C^{1,2}([0, U_1] \times \mathbb{R}^n)$ be a positive function. If the process $\{f(t, I_{tU_1}, \dots, I_{tU_n})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ supermartingale for $0 \leq t < U_1$, then $f(t, x_1, \dots, x_n)$ satisfies the inequality (3.1.26).

Proof. A change of measure argument shows that if $\{f(t, I_{tU_1}, \dots, I_{tU_n})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale, the pricing kernel $\{\pi_t\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{P})$ -supermartingale. Thus, the short rate is guaranteed to be non-negative. For $f(t, x_1, \dots, x_n) \in C^{1,2}([0, U_1] \times \mathbb{R}^n)$, the function $f(t, x_1, \dots, x_n)$ must satisfy (3.1.26); see Macrina & Parbhoo [70]. $\square$

Furthermore, we obtain the following result in the infinite-time setting along the same lines:

Proposition 3.3.2. *Let $\{\mathcal{F}_t\}$ be the filtration generated by the information processes $\{I_t^{(1)}\}, \dots, \{I_t^{(n)}\}$, and let the pricing kernel be modelled by*

$$\pi_t = m_t g(t, I_t^{(1)}, \dots, I_t^{(n)}). \quad (3.3.2)$$

Here, $\{m_t\}_{t \geq 0}$ is given by (3.2.6) and defines a measure $\mathbb{Z}$ such that Proposition 3.2.2 holds. Let $g(t, x_1, \dots, x_n) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ be a positive function. If the process $\{g(t, I_t^{(1)}, \dots, I_t^{(n)})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{Z})$ -supermartingale for $t \geq 0$, then $g(t, x_1, \dots, x_n)$ satisfies the inequality (3.2.14).

3.4 Space-time harmonic and superharmonic functions

For convenience, we focus henceforth on the one-dimensional PDIs given by (3.1.29) and (3.2.16). However, the ideas presented in this section also hold in the general multi-dimensional setting. Our objective here is to characterize functions which satisfy these PDIs. To this end, we recall the following results from Hirsch et al. [51] and Profeta et al. [84].

Definition 3.4.1. *We say that a function $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a **space-time harmonic function for Brownian motion** if $\{h(t, Z_t)\}_{t \geq 0}$ is an $\{\mathcal{H}_t\}_{t \geq 0}$ martingale where $\{Z_t\}_{t \geq 0}$ is a standard Brownian motion and $\{\mathcal{H}_t\}_{t \geq 0}$ denotes its natural filtration.*

Remark 3.4.1. Let $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a $C^{1,2}$ function such that

$$\int_0^t \mathbb{E} [(\partial_x h(s, Z_s))^2] ds < \infty \quad (3.4.1)$$

for all $t \geq 0$. Then $h(t, x)$ is a space-time harmonic function if, and only if, for all $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\partial_t h + \frac{1}{2} \partial_{xx} h = 0. \quad (3.4.2)$$

Example 3.4.1. For $\lambda \in \mathbb{R}$, the function

$$h(t, x) = \exp \left(\lambda x - \frac{1}{2} \lambda^2 t \right) \quad (3.4.3)$$

is a space-time harmonic function.

A well-known theorem by Widder [97] provides a representation for all positive space-time harmonic functions (see, for instance, [51] and [84]). For completeness we include a statement of this theorem.¹

Theorem 3.4.1. (Widder's theorem) *Every positive space-time harmonic function $h(t, x)$ can be written in the form*

$$h(t, x) = \int_{-\infty}^{\infty} \exp \left(\lambda x - \frac{1}{2} \lambda^2 t \right) \mu(d\lambda) \quad (3.4.4)$$

where $\lambda \in \mathbb{R}$ and μ is a positive measure of finite total mass.

¹Some applications of Widder's theorem to finance can be found in, e.g. Berrier et al. [8] and Musiela & Zariphopoulou [76], where optimal portfolio choice based on the forward performance criterion is considered. Here, Widder's theorem is used to construct solutions to PDEs for the local risk tolerance and differential input functions.

Next, we recall that there is a well-known correspondence between harmonic and superharmonic functions and martingales and supermartingales, respectively. In particular, a function $u(x)$ is said to be harmonic (resp. superharmonic) if $\{u(Z_t)\}_{t \geq 0}$ is a martingale (resp. supermartingale), where $\{Z_t\}$ is a Brownian motion. A twice differentiable function $u(x)$ is a harmonic (resp. superharmonic) function if, and only if,

$$\frac{1}{2}\partial_{xx}u = 0 \quad \left(\text{resp. } \frac{1}{2}\partial_{xx}u \leq 0 \right). \quad (3.4.5)$$

With these relationships in mind, we can now extend the above-mentioned considerations to space-time superharmonic functions.

Definition 3.4.2. We say that a function $\ell : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a **space-time superharmonic function for Brownian motion** if $\{\ell(t, Z_t)\}_{t \geq 0}$ is an $\{\mathcal{H}_t\}_{t \geq 0}$ supermartingale where $\{Z_t\}_{t \geq 0}$ is a standard Brownian motion and $\{\mathcal{H}_t\}_{t \geq 0}$ denotes its natural filtration.

Remark 3.4.2. Let $\ell : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a $C^{1,2}$ function such that

$$\int_0^t \mathbb{E} [(\partial_x \ell(s, Z_s))^2] ds < \infty \quad (3.4.6)$$

for all $t \geq 0$. Then $\ell(t, x)$ is a space-time superharmonic function if, and only if, for all $(t, x) \in (0, \infty) \times \mathbb{R}$,

$$\partial_t \ell + \frac{1}{2}\partial_{xx}\ell \leq 0. \quad (3.4.7)$$

Space-time harmonic functions may be used to construct space-time superharmonic functions as in the following example.

Example 3.4.2. Let $\gamma_1(t)$ and $\gamma_2(t)$ be positive non-increasing deterministic functions. For $\lambda \in \mathbb{R}$, the function

$$\ell(t, x) = \gamma_1(t) \exp \left(\lambda x - \frac{1}{2}\lambda^2 t \right) + \gamma_2(t) \quad (3.4.8)$$

is a space-time superharmonic function.

To put these definitions and statements into perspective, we return to the infinite time pricing kernel model considered in Remark 3.2.1.

Proposition 3.4.1. Let the pricing kernel be modelled as in Remark 3.2.1. Let $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a positive $C^{1,2}$ space-time superharmonic function for Brownian motion. Then $g(t, x)$ satisfies the PDI (3.2.16), and the resulting short rate model is non-negative for $t \geq 0$.

We see that if $g(t, x)$ is chosen such that it is a positive $C^{1,2}$ space-time superharmonic function, then $\{g(t, I_t)\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{Z})$ -supermartingale since the information process $\{I_t\}$ has the law of a Brownian motion under the measure $\mathbb{Z}$. Thus, the above result is consistent with Proposition 3.3.2, and shows which functions are suitable for constructing models for nominal interest rates in an infinite time setting. Hirsch et al. [51] remark that the notion of a space-time harmonic function with respect to Brownian motion is well understood and useful. This motivates the authors of [51] to define similar functions for other processes, such as Brownian and Lévy sheets. In the same vein, we consider space-time harmonic functions for Brownian bridges. We refer to Profeta et al. [84], where such functions appear in the discussion of past-future harmonic functions.

Definition 3.4.3. For any given $V > 0$ and $y \in \mathbb{R}$, let the process $\{B_{tV}\}_{0 \leq t \leq V}$ be a Brownian bridge of length V such that $B_{VV} = y$ a.s. We say that a function $H : [0, V) \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a **space-time harmonic function for the Brownian bridge** $\{B_{tV}\}$ if $\{H(t, B_{tV})\}_{0 \leq t < V}$ is an $\{\mathcal{H}_t\}_{0 \leq t < V}$ martingale where $\{\mathcal{H}_t\}$ denotes the natural filtration of $\{B_{tV}\}$.

Remark 3.4.3. Let $H : [0, V) \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a $C^{1,2}$ function such that

$$\int_0^t \mathbb{E} [(\partial_x H(s, B_{sV}))^2] ds < \infty \quad (3.4.9)$$

for all $t < V$. Then $H(t, x)$ is a space-time harmonic function for the Brownian bridge with length V and terminal value y if, and only if, for all $(t, x) \in (0, V) \times \mathbb{R}$,

$$\partial_t H + \frac{y-x}{V-t} \partial_x H + \frac{1}{2} \partial_{xx} H = 0 \quad (3.4.10)$$

for $x, y \in \mathbb{R}$.

Example 3.4.3. For $0 \leq t < U$,

$$H(t, x) = \sqrt{U-t} \exp \left[\frac{x^2}{2(U-t)} \right] \quad (3.4.11)$$

is a space-time harmonic function for a Brownian bridge with length U and terminal value 0.

We can use these ideas to define space-time superharmonic functions.

Definition 3.4.4. For any given V and y , let the process $\{B_{tV}\}_{0 \leq t \leq V}$ be a Brownian bridge of length V such that $B_{VV} = y$ a.s. We say that a function $L : [0, V) \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a **space-time superharmonic function for the Brownian bridge** $\{B_{tV}\}$ if $\{L(t, B_{tV})\}_{0 \leq t < V}$ is an $\{\mathcal{H}_t\}_{0 \leq t < V}$ supermartingale where $\{\mathcal{H}_t\}$ denotes the natural filtration of $\{B_{tV}\}$.

Remark 3.4.4. Let $L : [0, V) \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a $C^{1,2}$ function such that

$$\int_0^t \mathbb{E} [(\partial_x L(s, B_{sV}))^2] ds < \infty \quad (3.4.12)$$

for all $t < V$. Then $L(t, x)$ is a space-time superharmonic function for the Brownian bridge with length V and terminal value y if, and only if, for all $(t, x) \in (0, V) \times \mathbb{R}$,

$$\partial_t L + \frac{y-x}{V-t} \partial_x L + \frac{1}{2} \partial_{xx} L \leq 0 \quad (3.4.13)$$

for $x, y \in \mathbb{R}$.

For the finite-time one-dimensional pricing kernel model considered in Remark 3.1.1, we can conclude the following:

Proposition 3.4.2. *Let the pricing kernel be modelled as in Remark 3.1.1. Let $f : [0, V) \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a positive $C^{1,2}$ space-time superharmonic function for a standard Brownian bridge with length U . Then $f(t, x)$ satisfies (3.1.29), and the resulting short rate model is non-negative for $0 \leq t < U$.*

We recall that the information process $\{I_{tU}\}$ has the law of a standard Brownian bridge with length U , under $\mathbb{B}$. If $f(t, x)$ is chosen to be a positive $C^{1,2}$ space-time superharmonic function for a standard Brownian bridge with length U , ending at 0, then $\{f(t, I_{tU})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale. Therefore, this result is in agreement with Proposition 3.3.1 and determines the type of functions we can consider for interest rate modelling in the finite-time setup.

We conclude this section with the following useful observation from [84]. Let $\mathbf{H}_B$ denote the set of space-time harmonic functions for Brownian motion and let $\mathbf{H}_{br,V}^{0 \rightarrow y}$ denote the set of space-time harmonic functions for the Brownian bridge of length V ending at y . Let $h(u, z) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, be a $C^{1,2}$ function and, let $H(t, x) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, be a $C^{1,2}$ function defined by

$$H(t, x) := h\left(\frac{t}{V-t}, \frac{xV-yt}{(V-t)\sqrt{V}}\right). \quad (3.4.14)$$

It can be shown that the technical condition

$$\int_0^t \mathbb{E} [(\partial_x H(s, B_{sV}))^2] ds < \infty \quad (3.4.15)$$

holds for all $0 \leq t < V$, where $\{B_{sV}\}$ is a Brownian bridge with length V and terminal value y , if, and only if,

$$\int_0^{t'} \mathbb{E} [(\partial_z h(u, Z_u))^2] du < \infty \quad (3.4.16)$$

is true for all $t' \geq 0$, where $t' = t/(V - t)$ and $\{Z_u\}$ is a Brownian motion. Then, since

$$\partial_t H + \frac{y-x}{V-t} \partial_x H + \frac{1}{2} \partial_{xx} H = \frac{V}{(V-t)^2} \left(\partial_u h + \frac{1}{2} \partial_{zz} h \right), \quad (3.4.17)$$

it follows that

$$\partial_t H + \frac{y-x}{V-t} \partial_x H + \frac{1}{2} \partial_{xx} H = 0 \quad (3.4.18)$$

if, and only if,

$$\partial_u h + \frac{1}{2} \partial_{zz} h = 0. \quad (3.4.19)$$

Therefore, $H(t, x)$ is a space-time harmonic function for the Brownian bridge $\{B_{tV}\}$ with length V and terminal value y if, and only if, $h(u, z)$ is a space-time harmonic function for Brownian motion. In this way, Profeta et al. [84] show the existence of a bijective correspondence between $\mathbf{H}_B$ and $\mathbf{H}_{br,V}^{0 \rightarrow y}$.

Next, we let $\mathbf{L}_B$ denote the set of space-time superharmonic functions for Brownian motion and let $\mathbf{L}_{br,V}^{0 \rightarrow y}$ denote the set of space-time superharmonic functions for the Brownian bridge of length V ending at y . Let $\ell(u, z) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a $C^{1,2}$ function. We use the above change of variables to define the following $C^{1,2}$ function

$$L(t, x) := \ell \left(\frac{t}{V-t}, \frac{xV-yt}{(V-t)\sqrt{V}} \right). \quad (3.4.20)$$

Under similar technical conditions and by analogous arguments, we can conclude that since

$$\partial_t L + \frac{y-x}{V-t} \partial_x L + \frac{1}{2} \partial_{xx} L = \frac{V}{(V-t)^2} \left(\partial_u \ell + \frac{1}{2} \partial_{zz} \ell \right), \quad (3.4.21)$$

$L(t, x)$ is a space-time superharmonic function for the Brownian bridge $\{B_{tV}\}$ with length V and terminal value y if, and only if, $\ell(u, z)$ is a space-time superharmonic function for Brownian motion. In so doing, we show that there is also a relationship between the sets $\mathbf{L}_B$ and $\mathbf{L}_{br,V}^{0 \rightarrow y}$. The relations (3.4.14) and (3.4.20) are useful as they give us a change of variable which enables us to construct Brownian bridge-based analogues of infinite time Brownian pricing kernel models for which nominal interest rates are non-negative. We shall demonstrate this more explicitly in Chapter 5.

3.5 Weighted heat kernel models and PDIs

The weighted heat kernel approach of Akahori & Macrina [3] is a convenient way to generate pricing kernels driven by time-inhomogeneous Markov processes. In [3], it is shown that the constructed pricing kernels can be used to generate

information-based interest rate models, if time-inhomogeneous Markov processes are used to model noisy information about macroeconomic factors. In particular, when the underlying driver is a Brownian bridge information process, the pricing kernels constructed by [3] can be plugged into the Hughston-Macrina framework to generate suitable models for nominal interest rates. To show this, we recall that the information process $\{I_{tU}\}$ has the law of a Brownian bridge under $\mathbb{B}$. The Brownian bridge is an example of a time-inhomogeneous Markov process with respect to its own filtration. This is evident since, for $0 \leq t_0 \leq t_1 < U$, the transition density of a Brownian bridge is given by

$$\varrho(t_0, t_1; x, y) = \frac{1}{\sqrt{\frac{2\pi(U-t_1)(t_1-t_0)}{U-t_0}}} \exp \left[-\frac{\left(y - \frac{U-t_1}{U-t_0}x\right)^2}{\frac{2(U-t_1)(t_1-t_0)}{U-t_0}} \right], \quad (3.5.1)$$

and $\varrho(t_0, t_1; x, y)$ does not only depend on the time variables t_0 and t_1 through their difference $t_1 - t_0$.

Let $w : [0, U] \times [0, U] \rightarrow \mathbb{R}_+$ be a weight function that satisfies

$$w(t, u - s) \leq w(t - s, u) \quad (3.5.2)$$

for $U > 0$ and $s \leq t \wedge u$. Then, Akahori & Macrina [3] show that for a positive integrable function $F(t, x)$, the process $\{f(t, I_{tU})\}$ given by

$$f(t, I_{tU}) = \int_0^{U-t} w(t, u) \mathbb{E}^{\mathbb{B}}[F(t+u, I_{t+u,U}) | I_{tU}] du \quad (3.5.3)$$

is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale for $t < U$.

The proof of this result goes as follows. We define the process $\{p(t, u, I_{tU})\}$ by

$$p(t, u, I_{tU}) = \mathbb{E}^{\mathbb{B}}[F(t+u, I_{t+u,U}) | I_{tU}], \quad (3.5.4)$$

where $0 \leq u \leq U - t$. Then for $0 \leq s \leq t < U$ we have

$$\begin{aligned} \mathbb{E}^{\mathbb{B}}[f(t, I_{tU}) | I_{sU}] &= \int_0^{U-t} w(t, u) \mathbb{E}^{\mathbb{B}}[p(t, u, I_{tU}) | I_{sU}] du \\ &= \int_0^{U-t} w(t, u) p(s, u + t - s, I_{sU}) du \\ &= \int_{t-s}^{U-s} w(t, v - t + s) p(s, v, I_{sU}) dv. \end{aligned} \quad (3.5.5)$$

Here we have made use of the tower rule of conditional expectation and the Markov property of $\{I_{tU}\}$. Next, by (3.5.2) we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{B}}[f(t, I_{tU}) | I_{sU}] &\leq \int_{t-s}^{U-s} w(t - (t - s), v) p(s, v, I_{sU}) dv \\ &\leq \int_0^{U-s} w(s, v) p(s, v, I_{sU}) dv = f(s, I_{sU}). \end{aligned} \quad (3.5.6)$$

Furthermore,

$$\mathbb{E}^{\mathbb{B}}[f(t, I_{tU})] = \int_0^{U-t} w(t, u) \mathbb{E}^{\mathbb{B}}[F(t+u, I_{t+u,U})] du < \infty. \quad (3.5.7)$$

Thus, $\{f(t, I_{tU})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale, where $\{\mathcal{F}_t\}$ is generated by $\{I_{tU}\}$. It follows from Proposition 3.3.1 that if the underlying function $f(t, x) \in C^{1,2}$, then $f(t, x)$ satisfies inequality (3.1.29).

Akahori et al. [2] consider weighted heat kernel models for pricing kernels that are driven by time-homogeneous Markov processes. The constructed models are infinite-time models. Brownian motion is an example of a time homogeneous Markov process since its transition density is given by

$$\varrho(t_0, t_1; x, y) = \frac{1}{\sqrt{2\pi(t_1 - t_0)}} \exp \left[-\frac{(y - x)^2}{2(t_1 - t_0)} \right], \quad (3.5.8)$$

for $0 \leq t_0 \leq t_1$, and dependence on the time variables t_0 and t_1 in (3.5.8) appears only through the difference $t_1 - t_0$. Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a weight function that satisfies (3.5.2) for arbitrary $t, u \in \mathbb{R}_+$ and $s \leq t \wedge u$ and let $G(t, x)$ be a positive integrable function. We recall that $\{I_t\}$ has the distribution of a Brownian motion under the measure $\mathbb{Z}$. It follows that the process $\{g(t, I_t)\}$ defined by

$$g(t, I_t) = \int_0^\infty w(t, s) \mathbb{E}^{\mathbb{Z}}[G(t+s, I_{t+s}) | I_t] ds \quad (3.5.9)$$

is a positive $(\{\mathcal{F}_t\}, \mathbb{Z})$ -supermartingale, where the filtration $\{\mathcal{F}_t\}$ is generated by $\{I_t\}$. By Proposition 3.3.2, we conclude that if the underlying function $g(t, x) \in C^{1,2}$, then $g(t, x)$ satisfies inequality (3.2.16).

We remark that we have focused on one-dimensional models in the preceding discussion, however, the same ideas can be used in a multi-dimensional setting.

Chapter 4

Security pricing with information-sensitive interest rates

In the class of information-based credit-risk models by Brody et al. [19], for simplicity, it is assumed that interest rates are deterministic. The first extension of these models that includes stochastic interest rates was proposed by Rutkowski & Yu [92]. Here, the authors consider the pricing of a credit-risky bond with a future random payoff at maturity. A noisy Brownian bridge information process is associated with the random cash-flow. It is assumed that the market filtration is generated jointly by the information process (or equivalently, the corresponding innovations process), and by an independent Brownian motion which drives the interest rate process.

In this chapter, we develop security pricing models with information-sensitive stochastic discount factors. We use the approach of Hughston & Macrina [55] discussed in Section 3.1, and extend results from the information-based approach to asset pricing presented, e.g. in Brody et al. [19, 21]. This chapter contains material which appears in Macrina & Parbhoo [70]; however further insights are offered in Sections 4.3, 4.4 and 4.6. It is worth mentioning that while we develop security pricing models using the Brownian bridge information setup in Section 3.1, similar results may also be obtained using the ideas in Section 3.2.

4.1 Credit-risky discount bonds

We define the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ where $\mathbb{P}$ is the real probability measure. We introduce two dates T and U , where $T < U$, and we attach two independent factors X_T and X_U to these dates respectively. The random variable

X_U is assumed to be continuous¹ and is taken to be a macroeconomic factor. For example, X_U could be the GDP level of the economy at time U . Other examples of macroeconomic factors include: the debt level, the debt to GDP ratio, the budget deficit, the unemployment level or the balance of trade of the economy at time U . We remark that, while it is assumed in [55] and [70] that X_U is a macroeconomic factor, the proposed frameworks lend themselves to more flexibility. For instance, we could let X_U model uncertainty about the political landscape such as the outcome of an election, or the level of civil unrest in a region at time U . Alternatively, the random variable may be associated with demographic uncertainty at time U , such as the size of the retired population in the economy, and so on. In fact, the random variable X_U does not necessarily have to represent domestic uncertainty. The prices of securities in the economy may be greatly influenced by information about an external or foreign factor, the value of which will be revealed at a future time U . Recent evidence of this is that bond prices and yields in a number of European countries have been very sensitive to noisy news of a potential “Grexit”², particularly between May and June of 2012 in the run-up to the Greek national election.

Next, for convenience, we assume that X_T is a discrete random variable that takes the values

$$X_T = x_i \quad (4.1.1)$$

($i = 0, 1, \dots, n$) with *a priori* probabilities

$$\mathbb{P}[X_T = x_i] = p_i. \quad (4.1.2)$$

We take X_T to be the random variable by which the future payoff of a credit-risky bond issued by a firm is modelled. With the two X -factors, we associate the independent information processes $\{I_{tT}\}_{0 \leq t \leq T}$ and $\{I_{tU}\}_{0 \leq t \leq U}$ given by

$$I_{tU} = \sigma_1 t X_U + \beta_{tU}, \quad I_{tT} = \sigma_2 t X_T + \beta_{tT}. \quad (4.1.3)$$

We assume that the market filtration $\{\mathcal{F}_t\}$ is generated by both information processes $\{I_{tT}\}$ and $\{I_{tU}\}$. Then, the price at time $t \leq T$ of a credit-risky discount bond with maturity $T < U$, and payoff $H_T \in [0, 1]$ is given by

$$B_{tT} = \frac{\mathbb{E}[\pi_T H_T | \mathcal{F}_t]}{\pi_t}, \quad (4.1.4)$$

where $\{\pi_t\}_{0 \leq t < U}$ is the pricing kernel. We apply the pricing kernel model proposed in [55], that is,

$$\pi_t = M_t f(t, I_{tU}), \quad (4.1.5)$$

¹However, the subsequent results hold for an arbitrary probability distribution of X_U .

²A now common term first used by Citigroup analysts Willem Buiter and Ebrahim Rahbari to describe a Greek exit from the Eurozone (see *The Economist*, 7 February 2012).

where $\{M_t\}_{0 \leq t < U}$ is a positive martingale that satisfies

$$dM_t = -\frac{\sigma_1 U}{U-t} \mathbb{E}[X_U | I_{tU}] M_t dW_t, \quad (4.1.6)$$

and where $\{W_t\}_{0 \leq t < U}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by

$$W_t = I_{tU} + \int_0^t \frac{I_{sU}}{U-s} ds - \int_0^t \frac{\sigma_1 U}{U-s} \mathbb{E}[X_U | I_{sU}] ds. \quad (4.1.7)$$

Since the pricing kernel depends on $\{I_{tU}\}$, interest rates and the market price of risk will fluctuate as noisy information spreads through the market about the likely value of the fundamental risk factor X_U at time U .

Proposition 4.1.1. *Let $\{\pi_t\}$ be given by (4.1.5). Let the payoff H_T of a T -maturity credit-risky bond be a function of X_T and the information about X_U at time T , that is,*

$$H_T = H(X_T, I_{TU}), \quad (4.1.8)$$

where $H : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$. Then the price at time $t \leq T$ of the credit-risky bond is given by

$$\begin{aligned} B_{tT} = \frac{1}{f(t, I_{tU})} \sum_{i=0}^n \pi_{it} \int_{-\infty}^{\infty} f\left(T, \nu_{tTy} + \frac{U-T}{U-t} I_{tU}\right) H\left(x_i, \nu_{tTy} + \frac{U-T}{U-t} I_{tU}\right) \\ \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy, \end{aligned} \quad (4.1.9)$$

where $\{\pi_{it}\}$ is the conditional probability that $X_T = x_i$:

$$\pi_{it} = \mathbb{P}[X_T = x_i | I_{tT}] = \frac{p_i \exp\left[\frac{T}{T-t} (\sigma_2 x_i I_{tT} - \frac{1}{2} \sigma_2^2 x_i^2 t)\right]}{\sum_{i=0}^n p_i \exp\left[\frac{T}{T-t} (\sigma_2 x_i I_{tT} - \frac{1}{2} \sigma_2^2 x_i^2 t)\right]}. \quad (4.1.10)$$

Proof. From equations (4.1.4), (4.1.5) and (4.1.8), we have

$$B_{tT} = \frac{\mathbb{E}[M_T f(T, I_{TU}) H(X_T, I_{TU}) | \mathcal{F}_t]}{M_t f(t, I_{tU})} \quad (4.1.11)$$

for $t \leq T$. By the tower rule of conditional expectation, we can write

$$B_{tT} = \frac{\mathbb{E}[\mathbb{E}[M_T f(T, I_{TU}) H(X_T, I_{TU}) | \sigma(\mathcal{F}_t, X_T)] | \mathcal{F}_t]}{M_t f(t, I_{tU})}. \quad (4.1.12)$$

We recall that the information processes are independent Markov processes. Therefore, the price of the credit-risky discount bond can be expressed by

$$B_{tT} = \frac{\mathbb{E}[\mathbb{E}[M_T f(T, I_{TU}) H(X_T, I_{TU}) | I_{tU}, I_{tT}, X_T] | I_{tU}, I_{tT}]}{M_t f(t, I_{tU})}. \quad (4.1.13)$$

By applying Bayes' formula to the inner expectation, we can perform a change of measure to the bridge measure $\mathbb{B}$:

$$B_{tT} = \frac{1}{f(t, I_{tU})} \mathbb{E} \left[\mathbb{E}^{\mathbb{B}} [f(T, I_{TU}) H(X_T, I_{TU}) \mid I_{tU}, I_{tT}, X_T] \mid I_{tU}, I_{tT} \right]. \quad (4.1.14)$$

We define a random variable Y_{tT} by

$$Y_{tT} = I_{TU} - \frac{U-T}{U-t} I_{tU}. \quad (4.1.15)$$

We recall that $\{I_{tU}\}$ has the law of a Brownian bridge under $\mathbb{B}$. Therefore, Y_{tT} is a Gaussian random variable under $\mathbb{B}$, with zero mean and variance

$$\nu_{tT}^2 := \text{Var}^{\mathbb{B}}[Y_{tT}] = \frac{(T-t)(U-T)}{(U-t)}. \quad (4.1.16)$$

Next, we introduce a standard Gaussian random variable Y . Then we can write $Y_{tT} = \nu_{tT} Y$, and we obtain

$$B_{tT} = \frac{\mathbb{E} \left[\mathbb{E}^{\mathbb{B}} \left[f \left(T, \nu_{tT} Y + \frac{U-T}{U-t} I_{tU} \right) H \left(X_T, \nu_{tT} Y + \frac{U-T}{U-t} I_{tU} \right) \mid I_{tU}, I_{tT}, X_T \right] \mid I_{tU}, I_{tT} \right]}{f(t, I_{tU})}. \quad (4.1.17)$$

The random variable Y is independent of I_{tT} and X_T . Furthermore, we recall that Y is independent of I_{tU} under $\mathbb{B}$. Therefore, since Y is independent of the conditioning random variables, and the random variable I_{tU} , appearing in the arguments of $f(Y, I_{tU})$ and of $H(X_T, Y, I_{tU})$ is measurable, the inner conditional expectation reduces to a Gaussian integral over the range of Y :

$$B_{tT} = \frac{1}{f(t, I_{tU})} \int_{-\infty}^{\infty} f \left(T, \nu_{tT} y + \frac{U-T}{U-t} I_{tU} \right) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} y^2 \right) \times \mathbb{E} \left[H \left(X_T, \nu_{tT} y + \frac{U-T}{U-t} I_{tU} \right) \mid I_{tU}, I_{tT} \right] dy. \quad (4.1.18)$$

Owing to the independence of X_T and I_{tU} , we obtain

$$B_{tT} = \frac{1}{f(t, I_{tU})} \sum_{i=0}^n \pi_{it} \int_{-\infty}^{\infty} f \left(T, \nu_{tT} y + \frac{U-T}{U-t} I_{tU} \right) H \left(x_i, \nu_{tT} y + \frac{U-T}{U-t} I_{tU} \right) \times \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} y^2 \right) dy, \quad (4.1.19)$$

where π_{it} is given by (4.1.10). We refer to Chapter 2 for the derivation of the conditional probability $\{\pi_{it}\}$. $\square$

In the case where the payoff of the credit-risky bond is $H_T = H(X_T)$ where $H : \mathbb{R} \rightarrow [0, 1]$, we observe that the expression (4.1.9) factorizes into the product of the price of the sovereign discount bond with the same maturity, and a credit-risky component driven by the noisy information $\{I_{tT}\}$. In other words, we can express the price of the credit-risky bond by

$$B_{tT} = P_{tT} \sum_{i=0}^n \pi_{it} H(x_i), \quad (4.1.20)$$

where π_{it} is given by (4.1.10), and P_{tT} is modelled by formula (3.1.28). Here, the factorization is the result of the independence between the underlying discount bond system and the credit-risky component of the bond.

Example 4.1.1. Consider a credit-risky bond which pays a principal of $H(x_1)$ units of currency if there is no default, and $H(x_0)$ units of currency in the event of default, where

$$0 \leq H(x_0) < H(x_1) \leq 1. \quad (4.1.21)$$

We call such an instrument a “binary” bond (see Macrina [68]). The price of a binary bond is given by

$$B_{tT} = P_{tT} [\pi_{0t} H(x_0) + \pi_{1t} H(x_1)], \quad (4.1.22)$$

where P_{tT} is of the form (3.1.28). In particular, if $H(x_0) = 0$ and $H(x_1) = 1$, we call such a bond a “digital” bond (see [68]). The price of a digital bond is given by

$$B_{tT} = P_{tT} \pi_{1t}. \quad (4.1.23)$$

4.2 Bond yield spread

A measure for the excess return provided by a credit-risky bond over the return on a sovereign bond with the same maturity, is the bond yield spread, which we denote by $\{s_{tT}\}$. This measure is given by the difference between the yields-to-maturity on the defaultable bond and the sovereign bond, see for example Bielecki & Rutkowski [9]. That is:

$$s_{tT} = y_{tT}^D - y_{tT} \quad (4.2.1)$$

for $t < T$, where y_{tT} and y_{tT}^D are the yields associated with the sovereign bond and the credit-risky bond, respectively. Thus, we can write

$$s_{tT} = \frac{1}{T-t} (\ln P_{tT} - \ln B_{tT}). \quad (4.2.2)$$

For bonds with the payoff $H_T = H(X_T)$, we observe that the spread is not influenced by (i) noisy news that is circulating in the market about the factor X_U ,

and (ii) the choice of the function f , that is to say, the model for the pricing kernel. Thus for $0 \leq t < T$, the spread at time t depends solely on information concerning potential default. The bond yield spread between a digital bond and the sovereign bond is given by

$$s_{tT} = -\frac{1}{T-t} \ln \pi_{1t}. \quad (4.2.3)$$

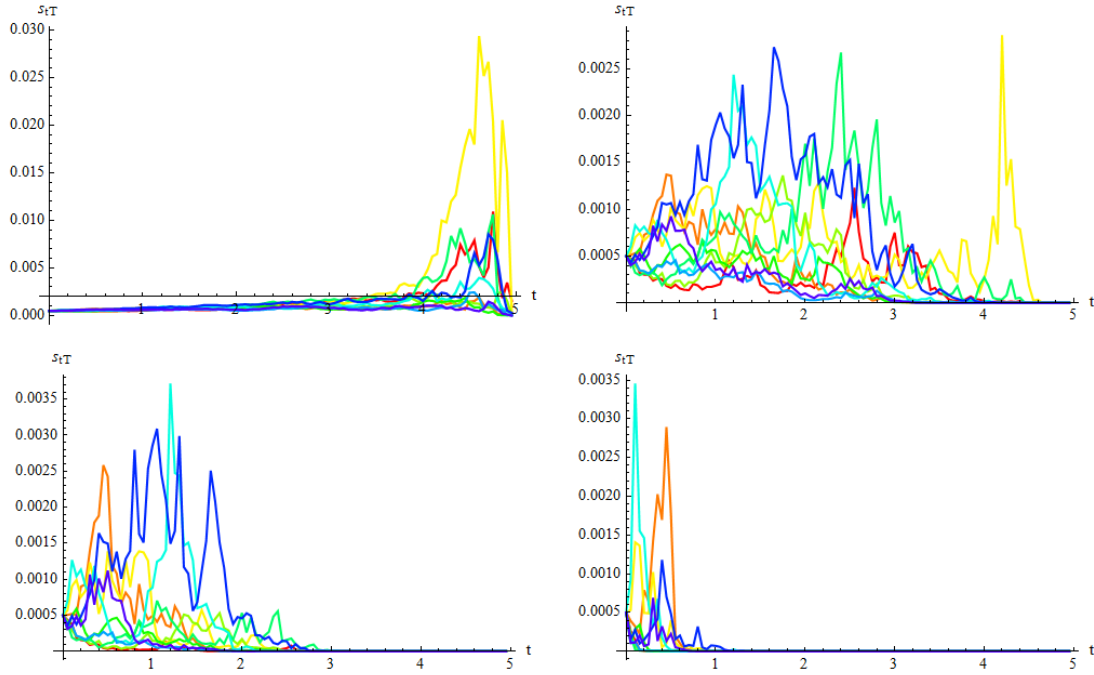


Figure 4.1: Bond yield spread between a digital bond, with all trajectories conditional on the outcome that $X_T = 1$, and a sovereign bond. The bonds have maturity $T = 5$ years. The *a priori* probability of default is taken to be $p_0 = 0.25\%$. We use (i) $\sigma_2 = 0.15$, (ii) $\sigma_2 = 0.75$, (iii) $\sigma_2 = 1.5$, and (iv) $\sigma_2 = 4$. (From left to right; top to bottom)

Figure 4.1 shows simulations of the bond yield spreads between a digital bond that is destined not to default, i.e. $X_T = 1$, and a sovereign bond. The maturities of the bonds are taken to be $T = 5$ years and the *a priori* probability of default is assumed to be $p_0 = 0.25\%$. The effect of different values of the information flow parameter is shown by considering $\sigma_2 = 0.15$, $\sigma_2 = 0.75$ and $\sigma_2 = 1.5$, $\sigma_2 = 4$. Since the paths of the digital bond are conditional on the outcome that default does not occur, the bond yield spreads should eventually drop to zero. The parameter σ_2 controls the magnitude of genuine information about potential default that is available to bondholders. For $\sigma_2 = 0.15$, the bondholder is, so to speak, “in the dark” about the outcome; the size of the bond yield spread close to maturity reflects this. For higher values of σ_2 , the bondholder is

better informed about the likely value that X_T will take. As σ_2 increases, the noisiness in the bond yield spreads becomes less pronounced before maturity. As Macrina [68] notes, this occurs because the signal term in the information process dominates the noise produced by the Brownian bridge. Moreover, we observe that if bondholders in the market are well-informed, they require a smaller premium for buying the credit-risky bond since its behaviour will be similar to that of the sovereign bond. It is worth noting that in the information-based asset pricing approach, an increased level of genuine information available to investors about their exposure, is manifestly equivalent to a sort of “securitisation” of the risky investments.

The case where the paths of the digital bond are conditional on default can also be simulated. Here, the effect of increasing the information flow rate parameter σ_2 is similar. However, as $t \rightarrow T$, the bondholder now requires an infinitely high reward for buying a bond that will be worthless at maturity. Thus the bond-yield spread grows very rapidly near the bond maturity.

4.3 Default-sensitive pricing kernel

We can generalize the pricing kernel model (4.1.5) by considering a pricing kernel $\{\pi_t\}$ of the form

$$\pi_t = M_t f(t, I_{tT}, I_{tU}). \quad (4.3.1)$$

By following the technique in the proof of Proposition 4.1.1, and by using the fact that at time T we have $I_{TT} = \sigma_2 X_T T$, we can show that the price of a credit-risky bond is

$$\begin{aligned} B_{tT} = \frac{1}{f(t, I_{tT}, I_{tU})} \sum_{i=0}^n \pi_{it} \int_{-\infty}^{\infty} f \left(T, \sigma_2 x_i T, \nu_{tT} y + \frac{U-T}{U-t} I_{tU} \right) \\ \times H \left(x_i, \nu_{tT} y + \frac{U-T}{U-t} I_{tU} \right) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} y^2 \right) dy. \end{aligned} \quad (4.3.2)$$

Here, we model the situation in which the pricing kernel in the economy is not only a function of information at that time about the macroeconomic (or other) fundamental random variable X_U , but is also dependent on noisy information about potential default of the firm leaked in the market through $\{I_{tT}\}$. We imagine that this type of model may be relevant when news circulating in financial markets about a potential large corporate default can affect interest rates, the market price of risk and asset prices. If investors have a large amount of genuine information about the default, interest rates and the market price of risk may reflect this early on. However, if investors are ignorant, default may come as a complete surprise,

bringing about more sudden moves. The bankruptcy of Lehman Brothers in September 2008 is a well-known example of a default that had a significant effect on interest rates and risk premia. In the aftermath of this event, the risk premium on short term inter-bank borrowing and risk premia on corporate bonds increased dramatically (see McKibbin & Stoeckel [72]). This is illustrated by Chart 3.3 in [72] where the risk premium on inter-bank borrowing³ and the risk premium on BAA rated corporate bonds⁴ are plotted over the period 2001-2009, and upward spikes are observed immediately after the Lehman collapse.

4.4 Recovery models

Let us consider the case in which the credit-risky bond pays $H_T = X_T$ where X_T is a discrete random variable that takes the values

$$X_T = x_i \quad (4.4.1)$$

($i = 0, 1, \dots, n$) with *a priori* probabilities p_i where

$$0 \leq x_0 < x_1 < \dots < x_{n-1} < x_n \leq 1. \quad (4.4.2)$$

Such a payoff spectrum is a model for random recovery where at bond maturity one out of a discrete number of recovery levels may be realised.

We can also consider credit-risky bonds with continuous random recovery in the event of default. In doing so, we introduce the notion of information-driven recovery. For instance, let X_U be a macroeconomic factor and, suppose that the payoff of the credit-risky bond is given by

$$H_T = X_T + (1 - X_T) R(I_{TU}), \quad (4.4.3)$$

where X_T takes the values $\{0, 1\}$ with *a priori* probabilities $\{p_0, p_1\}$. Here, $R : \mathbb{R} \rightarrow [0, 1)$ is a function of the information at time T about the factor X_U , and is to be viewed as the recovery level. In this case, if the credit-risky bond defaults at maturity T , the recovery level of the bond depends on the state of the economy at time U that is perceived in the market at time T . In other words, if the sentiment in the market at time T is that the economy will have good times ahead, then a firm in a state of default at T may have better chances to raise more capital from liquidation (or restructuring), thus increasing the level of recovery of the

³This is approximated in [72] by the difference between the rate on 1 month Eurodollar deposits and the Fed Funds rate.

⁴In [72], this is measured by the difference between the yield on BAA rated corporate bonds and the 10 year treasury bond yield.

issued bond. We can price the cash flow (4.4.3) by applying equation (4.1.9), with $n = 1$, $x_0 = 0$ and $x_1 = 1$. The result is:

$$B_{tT} = P_{tT}\pi_{1t} + \pi_{0t} \frac{1}{f(t, I_{tU})} \int_{-\infty}^{\infty} f\left(T, \nu_{tT}y + \frac{U-T}{U-t}I_{tU}\right) \times R\left(\nu_{tT}y + \frac{U-T}{U-t}I_{tU}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy, \quad (4.4.4)$$

where P_{tT} is given by equation (3.1.28). This model may be appropriate when the extent of recovery is determined by how difficult it is for the firm to raise capital by liquidating its assets, i.e. the exposure of the firm to the general economic environment.

The above model does not say much about how firm-specific characteristics may influence recovery in the event of default. This observation brings us to another model of recovery. Default of a firm may be triggered, for instance, by poor internal practices and (or) tough economic conditions. We now structure recovery by specifying the payoff of the credit-risky bond by

$$H_T = X_T^C [X_T^E + (1 - X_T^E)R_E] + (1 - X_T^C) [X_T^E R_C + (1 - X_T^E)R_{CE}], \quad (4.4.5)$$

where X_T^C and X_T^E are random variables taking values in $\{0, 1\}$ with *a priori* probabilities $\{p_0^C, p_1^C\}$ and $\{p_0^E, p_1^E\}$, respectively. Let X_U be a macroeconomic (or political) factor which is revealed at time U , and define X_T^C and X_T^E to be binary indicators at time T of good management of the company and a strong economy, respectively. We set R_C to be a continuous random variable assuming values in the interval $[0, 1)$. We take R_E to be a function of I_{TU} , and R_{CE} to be a function of I_{TU} and R_C , where both R_E and R_{CE} take values in the interval $[0, 1)$.

The payoff in equation (4.4.5) covers the following situations: First, we suppose that despite good overall management of the firm, default is triggered as a result of a depressed economy. Here, $X_T^C = 1$ and $X_T^E = 0$ which implies that $H_T = R_E$. Therefore, the recovery is dependent on noisy economic (or political) information about the factor X_U circulating in the market at time T ; this news may influence how easy it is for the firm to raise funds. It is also possible that a firm can default in otherwise favourable economic conditions, perhaps due to the management's negligence. In this case we have $X_T^E = 1$ and $X_T^C = 0$. Thus $H_T = R_C$. Here, the amount recovered varies with the condition of the company and perhaps, the type of contract or instrument issued; that is to say, it is firm-specific. Finally, we have the case in which a firm is poorly managed, i.e. $X_T^C = 0$, and difficult economic conditions prevail, i.e. $X_T^E = 0$. Recovery is given by the amount $H_T = R_{CE}$, which is dependent on both, the extent of mismanagement of the firm and information about X_U which may influence how much capital the firm can raise.

The payoff structure (4.4.5) is used in Macrina [68] to model the dependence structure between two credit-risky discount bonds that share market factors. Further investigation may include the situation where one models such dependence structures for bonds subject to stochastic interest rates and featuring recovery functions of the form (4.4.5). Acharya et al. [1] demonstrate that a model for recovery risk should “[stem] from firm-specific factors as well as systematic, industry-specific factors.” Even though equation (4.4.5) is a simple model, it incorporates both idiosyncratic and systematic variables.

It is worth mentioning that empirical evidence on corporate bonds may help us to construct models for recovery. For instance, the findings in [1] suggest that creditors of firms that have defaulted recover less if there is industry-wide distress, since prospective buyers cannot afford to pay high prices for the firm’s assets in difficult times. In addition, in [1] the authors provide evidence of a fire-sales⁵ effect on creditor recoveries whereby recovery levels fall if the industry is in distress and if

- fellow firms in the industry have liquidity problems; or
- the industry is characterized by particular assets that have limited use outside the industry; or
- the industry is concentrated, i.e. there are fewer fellow firms that have not defaulted, and thus, fewer potential bidders in asset auctions.

These observations suggest that we may construct further recovery models by considering additional X -factors relating to the financial condition of fellow firms, level of liquidity in the distressed asset market, or the concentration of the industry, for instance, in order to better capture observed features.

4.5 Call option on credit-risky bond

Let $\{C_{st}\}_{0 \leq s \leq t < T}$ be the price process of a European-style call option with maturity t and strike K , written on a credit-risky bond with price process $\{B_{tT}\}$. The price of such an option at time s is given by

$$C_{st} = \frac{1}{\pi_s} \mathbb{E} [\pi_t (B_{tT} - K)^+ | \mathcal{F}_s]. \quad (4.5.1)$$

We assume that the filtration $\{\mathcal{F}_t\}$ is generated by the independent information processes $\{I_{tT}\}$ and $\{I_{tU}\}$, where $T < U$, and that the pricing kernel $\{\pi_t\}$ is of

⁵A fire sale is essentially a forced sale of an asset at a “dislocated” price. Assets sold in fire sales can trade at greatly discounted prices, resulting in large losses to the seller (Shleifer & Vishny [94]).

the form

$$\pi_t = M_t f(t, I_{tU}), \quad (4.5.2)$$

with $\{M_t\}$ satisfying equation (4.1.6). By the tower rule, we can write

$$C_{st} = \frac{1}{M_s f(s, I_{sU})} \mathbb{E} \left[\mathbb{E} [M_t f(t, I_{tU}) (B_{tT} - K)^+ \mid \sigma(\mathcal{F}_s, I_{tU})] \mid \mathcal{F}_s \right]. \quad (4.5.3)$$

We recall that the information processes are Markov processes and use the martingale $\{M_t\}$ to change the measure as follows:

$$\begin{aligned} C_{st} &= \frac{1}{M_s f(s, I_{sU})} \mathbb{E} [M_t f(t, I_{tU}) \mathbb{E} [(B_{tT} - K)^+ \mid I_{sT}, I_{sU}, I_{tU}] \mid I_{sT}, I_{sU}] \\ &= \frac{1}{f(s, I_{sU})} \mathbb{E}^{\mathbb{B}_U} [f(t, I_{tU}) \mathbb{E} [(B_{tT} - K)^+ \mid I_{sT}, I_{sU}, I_{tU}] \mid I_{sT}, I_{sU}]. \end{aligned} \quad (4.5.4)$$

For clarity, we use the notation $\mathbb{B}_U$ to denote the bridge measure under which the process $\{I_{tU}\}$ has the law of a Brownian bridge over $[0, U]$.

Next, for simplicity, we consider the special case in which the payoff of the credit-risky bond is $H_T = X_T$. Here, the price of the bond at time t is

$$B_{tT} = P_{tT} \sum_{i=0}^n \pi_{it} x_i, \quad (4.5.5)$$

where P_{tT} is given by equation (3.1.28) and the conditional density π_{it} is defined in (4.1.10). Thus, we can write

$$C_{st} = \frac{1}{f(s, I_{sU})} \mathbb{E}^{\mathbb{B}_U} \left[f(t, I_{tU}) \mathbb{E} \left[\left(P_{tT} \sum_{i=0}^n \pi_{it} x_i - K \right)^+ \mid I_{sT}, I_{sU}, I_{tU} \right] \mid I_{sT}, I_{sU} \right]. \quad (4.5.6)$$

We first simplify the inner conditional expectation by following an analogous calculation to that in Brody et al. [19], Section 9. Let us introduce the process $\{\Phi_t\}_{0 \leq t < T}$ by

$$\Phi_t = \sum_{i=0}^n p_{it}, \quad (4.5.7)$$

where

$$p_{it} = p_i \exp \left[\frac{T}{T-t} \left(\sigma_2 x_i I_{tT} - \frac{1}{2} \sigma_2^2 x_i^2 t \right) \right]. \quad (4.5.8)$$

We can write the inner expectation as

$$\begin{aligned} \mathbb{E} \left[\left(P_{tT} \sum_{i=0}^n \pi_{it} x_i - K \right)^+ \mid I_{sT}, I_{sU}, I_{tU} \right] \\ = \mathbb{E} \left[\frac{1}{\Phi_t} \left(\sum_{i=0}^n (P_{tT} x_i - K) p_{it} \right)^+ \mid I_{sT}, I_{sU}, I_{tU} \right]. \end{aligned} \quad (4.5.9)$$

Macrina [68] proves that $\{\Phi_t^{-1}\}_{0 \leq t < T}$ is a positive martingale with dynamics

$$d\Phi_t^{-1} = -\frac{\sigma_2 T}{T-t} \mathbb{E}[X_T | I_{tT}] \Phi_t^{-1} d\widehat{W}_t, \quad (4.5.10)$$

where the innovations process $\{\widehat{W}_t\}_{0 \leq t < T}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion. Since $\Phi_0 = 1$, it follows that $\mathbb{E}[\Phi_t^{-1}] = 1$. Therefore, the process $\{\Phi_t^{-1}\}$ induces a change of measure from $\mathbb{P}$ to a bridge measure $\mathbb{B}_T$, under which the second information process $\{I_{tT}\}$ has the distribution of a Brownian bridge over $[0, T)$. This allows us to use Bayes' formula to express the expectation as follows:

$$\begin{aligned} \mathbb{E} \left[\frac{1}{\Phi_t} \left(\sum_{i=0}^n (P_{tT} x_i - K) p_{it} \right)^+ \middle| I_{sT}, I_{sU}, I_{tU} \right] \\ = \frac{1}{\Phi_s} \mathbb{E}^{\mathbb{B}_T} \left[\left(\sum_{i=0}^n (P_{tT} x_i - K) p_{it} \right)^+ \middle| I_{sT}, I_{sU}, I_{tU} \right]. \end{aligned} \quad (4.5.11)$$

In order to compute the expectation we introduce the Gaussian random variable Z_{st} , defined by

$$Z_{st} = \frac{I_{tT}}{T-t} - \frac{I_{sT}}{T-s}, \quad (4.5.12)$$

which is independent of $\{I_{uT}\}_{0 \leq u \leq s}$.

It is possible to find the critical value, for which the argument of the expectation vanishes, in closed form if it is assumed that the credit-risky bond is a binary bond. For $n = 1$, the critical value z^* is given by

$$z^* = \frac{\ln \left[\frac{\pi_{0s}(K - x_0 P_{tT})}{\pi_{1s}(x_1 P_{tT} - K)} \right] + \frac{1}{2} \sigma_2^2 (x_1^2 - x_0^2) \alpha_{st}^2 T^2}{\sigma_2 (x_1 - x_0) \alpha_{st} T}, \quad (4.5.13)$$

where $\alpha_{st}^2 = \text{Var}^{\mathbb{B}_T}[Z_{st}]$. The computation of the expectation amounts to two Gaussian integrals reducing to cumulative normal distribution functions, which we denote by $N[x]$. We obtain the following:

$$\begin{aligned} \mathbb{E} \left[\left(P_{tT} \sum_{i=0}^1 \pi_{it} x_i - K \right)^+ \middle| I_{sT}, I_{sU}, I_{tU} \right] &= \pi_{1s} (P_{tT} x_1 - K) N[d_s^+] \\ &\quad - \pi_{0s} (K - P_{tT} x_0) N[d_s^-], \end{aligned} \quad (4.5.14)$$

where

$$d_s^\pm = \frac{\ln \left[\frac{\pi_{1s}(x_1 P_{tT} - K)}{\pi_{0s}(K - x_0 P_{tT})} \right] \pm \frac{1}{2} \sigma_2^2 (x_1 - x_0)^2 \alpha_{st}^2 T^2}{\sigma_2 (x_1 - x_0) \alpha_{st} T}. \quad (4.5.15)$$

We can now insert this intermediate result into equation (4.5.6) with $n = 1$; we have

$$C_{st} = \frac{1}{f(s, I_{sU})} \mathbb{E}^{\mathbb{B}^U} [f(t, I_{tU}) [\pi_{1s}(P_{tT}x_1 - K)N[d_s^+] - \pi_{0s}(K - P_{tT}x_0)N[d_s^-]] | I_{sT}, I_{sU}]. \quad (4.5.16)$$

We emphasize that P_{tT} is given by a function $P(t, T, I_{tU})$ and thus is affected by the conditioning with respect to I_{sU} . To compute the expectation in equation (4.5.16), we introduce the Gaussian random variable Y_{st} , defined by

$$Y_{st} = I_{tU} - \frac{U-t}{U-s} I_{sU}, \quad (4.5.17)$$

with mean zero and variance $\nu_{st}^2 = \text{Var}^{\mathbb{B}^U}[Y_{st}]$. Thus, as shown in the previous sections, the conditional expectation reduces to a Gaussian integral:

$$\begin{aligned} C_{st} = \frac{1}{f(s, I_{sU})} \int_{-\infty}^{\infty} f\left(t, \nu_{st}y + \frac{U-t}{U-s} I_{sU}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \\ \times \left[\pi_{1s} \left(P\left(t, T, \nu_{st}y + \frac{U-t}{U-s} I_{sU}\right) x_1 - K \right) N[d_s^+(y)] \right. \\ \left. - \pi_{0s} \left(K - P\left(t, T, \nu_{st}y + \frac{U-t}{U-s} I_{sU}\right) x_0 \right) N[d_s^-(y)] \right] dy. \end{aligned} \quad (4.5.18)$$

Therefore we obtain a semi-analytical pricing formula for a call option on a credit-risky bond in a stochastic interest rate setting. Depending on the specification of the function $f(t, x)$, the resulting models may have different degrees of tractability.

4.6 Multi-dimensional pricing kernel models

So far, we have focused on the pricing of credit-risky bonds with stochastic discounting. The formalism presented in the previous sections can also be applied to the pricing of other types of securities. In particular, as an example of a hybrid security, we consider the valuation of an inflation-linked credit-risky discount bond. In addition, we discuss the pricing of a credit-risky discount bond denominated in a foreign currency.

These applications give us the opportunity to extend the thus far presented pricing models to the case where n independent information processes are employed. We shall call such models, “multi-dimensional pricing models”. In what follows, we consider three independent information processes, $\{I_{tT}\}$, $\{I_{tU_1}\}$ and $\{I_{tU_2}\}$, defined by

$$I_{tT} = \sigma t X_T + \beta_{tT}, \quad I_{tU_1} = \sigma_1 t X_{U_1} + \beta_{tU_1}, \quad I_{tU_2} = \sigma_2 t X_{U_2} + \beta_{tU_2}, \quad (4.6.1)$$

where $0 \leq t \leq T < U_1 \leq U_2$. The positive random variable X_T is used to model default and is taken to be a discrete random variable, while X_{U_1} and X_{U_2} are assumed to be continuous macroeconomic random variables. The market filtration $\{\mathcal{F}_t\}$ is generated jointly by the three information processes.

In what follows, we make use of an important relationship between pricing kernels associated with different countries, and currency prices, that must hold in an arbitrage-free setup. This relation states that the price of a foreign currency (GBP, say) in units of the domestic currency (USD, say) is given by the ratio of the pricing kernel denominated in the foreign currency to the pricing kernel denominated in the domestic currency (see Backus et al. [6] for details). Panigirtzoglou [81] observes that as long as the respective pricing kernel processes are specified, the exchange rate process is governed by this relation; that is to say, only two of the three components can be independently specified. In inflation modelling, an analogous formula dictates the relationship between the nominal and real pricing kernels and the price level process (see, for example, Brigo & Mercurio [13], Brody et al. [14], Hinnerich [50], Hughston [54], Mercurio [73]). This relation is known as the “foreign exchange analogy”. Here, the nominal pricing kernel and the real pricing kernel are viewed as being associated with the domestic currency (USD, say) and foreign currency (baskets of goods and services, say) respectively, with the price level process acting as the exchange rate.

4.6.1 Hybrid securities

We now derive an expression for the price of an inflation-linked credit risky (ILCR) bond. While such a security has inherent credit risk, it offers bondholders protection against inflation. We let $\{C_t\}$ represent a price level process, e.g., the process of the consumer price index. Then the price Q_{tT} , at time t , of an inflation-linked discount bond that pays C_T units of a currency at maturity T , is given by

$$Q_{tT} = \frac{\mathbb{E}[\pi_T C_T | \mathcal{F}_t]}{\pi_t}. \quad (4.6.2)$$

By the foreign exchange analogy, the process $\{C_t\}$ is expressed by the following ratio:

$$C_t = \frac{\pi_t^R}{\pi_t} \quad (4.6.3)$$

where $\{\pi_t\}$ is the nominal pricing kernel and $\{\pi_t^R\}$ is the real pricing kernel. For further details about the modelling of the real and the nominal pricing kernels, and the pricing of inflation-linked assets, the reader should refer to Hughston & Macrina [55]. In what follows, we make use of the method proposed in [55] to price an example of an ILCR discount bond that, at maturity T , pays a cash flow

$$H_T = C_T H(X_T, I_{TU_1}, I_{TU_2}). \quad (4.6.4)$$

The price H_{tT} at time $t \leq T$ of such a bond is

$$H_{tT} = \frac{1}{\pi_t} \mathbb{E} \left[\pi_T^R H(X_T, I_{TU_1}, I_{TU_2}) \mid \mathcal{F}_t \right], \quad (4.6.5)$$

where we have used relation (4.6.3). We choose to model the real and the nominal pricing kernels by

$$\pi_t = M_t^{(1)} M_t^{(2)} f(t, I_{tU_1}, I_{tU_2}) \quad \text{and} \quad \pi_t^R = M_t^{(1)} M_t^{(2)} g(t, I_{tU_1}, I_{tU_2}), \quad (4.6.6)$$

where $f(t, x_1, x_2)$ and $g(t, x_1, x_2)$ are positive functions. It follows that the expression for the price level process is given by

$$C_t = \frac{g(t, I_{tU_1}, I_{tU_2})}{f(t, I_{tU_1}, I_{tU_2})}. \quad (4.6.7)$$

Each of the processes $\{M_t^{(i)}\}_{0 \leq t \leq T < U_i}$ ($i = 1, 2$) is a martingale that induces a change of measure to a bridge measure $\mathbb{B}_i$. We recall that each information process $\{I_{tU_i}\}$ has the law of a Brownian bridge under the measure $\mathbb{B}_i$. In order to work out the expectation in (4.6.5) with the pricing kernel models introduced in (4.6.6), we define a process $\{M_t\}$ by

$$M_t = M_t^{(1)} M_t^{(2)}, \quad (4.6.8)$$

where $0 \leq t \leq T < U_1 \leq U_2$. Since the information processes $\{I_{tU_1}\}$ and $\{I_{tU_2}\}$ are independent, it follows that $M_t^{(1)}$ and $M_t^{(2)}$ are independent for all $t < U_1 \leq U_2$, and $\{M_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale with $\mathbb{E}[M_t] = 1$. Thus $\{M_t\}$ can be used to effect a change of measure from $\mathbb{P}$ to a “master” bridge measure $\mathbb{B}$, under which the random variables I_{tU_1} and I_{tU_2} have the distribution of a Brownian bridge for $0 \leq t \leq T < U_1$; see Section 3.1 for details. By use of $\{M_t\}$ and the Bayes’ formula, and the fact that $\{I_{tT}\}$, $\{I_{tU_1}\}$ and $\{I_{tU_2}\}$ are $\{\mathcal{F}_t\}$ -Markov processes, equation (4.6.5) reduces to

$$H_{tT} = \frac{1}{f(t, I_{tU_1}, I_{tU_2})} \times \mathbb{E} \left[\mathbb{E}^{\mathbb{B}} \left[g(T, I_{TU_1}, I_{TU_2}) H(X_T, I_{TU_1}, I_{TU_2}) \mid X_T, I_{tU_1}, I_{tU_2}, I_{tT} \right] \mid I_{tU_1}, I_{tU_2}, I_{tT} \right]. \quad (4.6.9)$$

Next we repeat an analogous calculation to the one leading from equation (4.1.14) to expression (4.1.18). For the ILCR discount bond under consideration, we obtain

$$H_{tT} = \frac{1}{f(t, I_{tU_1}, I_{tU_2})} \sum_{i=0}^n \pi_{it} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(T, z(y_1), z(y_2)) H(x_i, z(y_1), z(y_2)) \times \frac{1}{2\pi} \exp \left[-\frac{1}{2} (y_1^2 + y_2^2) \right] dy_1 dy_2. \quad (4.6.10)$$

Here the conditional density π_{it} is given by an expression analogous to equation (4.1.10) and, $z(y_k)$ is defined for $k = 1, 2$ by

$$z(y_k) = \nu_{tT}^{(k)} y_k + \frac{U_k - T}{U_k - t} I_{tU_k}, \quad \text{where} \quad \nu_{tT}^{(k)} = \sqrt{\frac{(T - t)(U_k - T)}{U_k - t}}. \quad (4.6.11)$$

In the special case when $H_T = H(X_T)$, where $H : \mathbb{R} \rightarrow [0, 1]$, the expression for the price at time t of the ILCR discount bond simplifies to

$$H_{tT} = Q_{tT} \sum_{i=0}^n \pi_{it} H(x_i). \quad (4.6.12)$$

We recall that Q_{tT} is the price of an inflation-linked discount bond that depends on the information processes $\{I_{tU_1}\}$ and $\{I_{tU_2}\}$. Here, Q_{tT} factorizes out, as in equation (4.1.20) in Section 4.1, since the inflation-linked bond price is independent of the credit-risky component. In particular, a formula similar to (4.5.18) can be derived for the price of a European-style call option written on an ILCR bond with price process given by (4.6.12) with $n = 1$.

4.6.2 Bonds denominated in a foreign currency

We define the exchange rate process $\{S_t\}$ as the price of one unit of foreign currency in units of the domestic currency at time t . By the foreign exchange relation, the exchange rate process $\{S_t\}$ is expressed by

$$S_t = \frac{\pi_t^F}{\pi_t}, \quad (4.6.13)$$

where $\{\pi_t\}$ and $\{\pi_t^F\}$ are the pricing kernels associated with the domestic and foreign currencies respectively. We model the respective pricing kernels by

$$\pi_t = M_t f(t, I_{tU_1}, \dots, I_{tU_n}) \quad \text{and} \quad \pi_t^F = M_t g(t, I_{tU_1}, \dots, I_{tU_n}), \quad (4.6.14)$$

where $\{M_t\}$ is defined as in Section 3.1, and $f(t, x_1, \dots, x_n)$ and $g(t, x_1, \dots, x_n)$ are positive functions. We observe that both of the countries are influenced by the same noisy news about the fundamental X -factors; however, the impact of this information varies across countries due to the country-specific specifications of the functions $f(t, x_1, \dots, x_n)$ and $g(t, x_1, \dots, x_n)$. A similar situation arises in the potential approach to foreign exchange by Rogers [88]. The exchange rate process simplifies to

$$S_t = \frac{g(t, I_{tU_1}, \dots, I_{tU_n})}{f(t, I_{tU_1}, \dots, I_{tU_n})}. \quad (4.6.15)$$

We note that similar pricing formulae to those in Section 4.6.1 can be derived for credit-risky discount bonds denominated in a foreign currency and for options

on such bonds. Furthermore, Hughston & Macrina [55] remark that in a multi-currency setting with a domestic currency, N foreign currencies, and n information processes, we require that $n \geq 2N + 1$ in order to obtain a realistic model.

4.7 Credit-risky coupon bonds

Let $\{T_k\}_{k=1,\dots,n}$ be a collection of fixed dates where $0 \leq t \leq T_1 \leq \dots \leq T_n$. We now consider the valuation of a credit-risky bond with maturity T_n , and coupon payments H_{T_k} at times T_k ($k = 1, \dots, n$). The bond is considered to be in a state of default as soon as a coupon payment does not occur. We denote the price process of the coupon bond by $\{B_{tT_n}\}_{0 \leq t \leq T_n}$, and introduce n independent random variables $X_{T_1}, \dots, X_{T_n}$ to construct the random cash flows H_{T_k} . We model the cashflows by

$$H_{T_k} = \mathbf{c} \prod_{j=1}^k X_{T_j}, \quad (4.7.1)$$

for $k = 1, \dots, n-1$, with

$$H_{T_n} = (\mathbf{c} + \mathbf{p}) \prod_{j=1}^n X_{T_j}. \quad (4.7.2)$$

Here $\mathbf{c}$ and $\mathbf{p}$ denote the coupon and principal payments, respectively. The random variables $\{X_{T_k}\}_{k=1,\dots,n}$ are assumed to take values in $\{0, 1\}$ with *a priori* probabilities $\{p_0^{(k)}, p_1^{(k)}\}$, that is

$$p_0^{(k)} = \mathbb{P}[X_{T_k} = 0] \quad (4.7.3)$$

and $p_1^{(k)} = 1 - p_0^{(k)}$. With each factor X_{T_k} we associate an independent information process $\{I_{tT_k}\}$ defined by

$$I_{tT_k} = \sigma_k t X_{T_k} + \beta_{tT_k}. \quad (4.7.4)$$

Furthermore we introduce another independent information process $\{I_{tU}\}$ given by

$$I_{tU} = \sigma t X_U + \beta_{tU} \quad (4.7.5)$$

for $t \leq T_n < U$, which we reserve for the modelling of the pricing kernel. The market filtration $\{\mathcal{F}_t\}$ is generated jointly by the $n+1$ information processes, that is $\{I_{tT_k}\}_{k=1,\dots,n}$ and $\{I_{tU}\}$. As in Section 4.1, we model the pricing kernel $\{\pi_t\}$ by

$$\pi_t = M_t f(t, I_{tU}), \quad (4.7.6)$$

where $\{M_t\}$ satisfies

$$dM_t = -\frac{\sigma U}{U-t} \mathbb{E}[X_U | I_{tU}] M_t dW_t, \quad (4.7.7)$$

and $\{W_t\}$ is given by

$$W_t = I_{tU} + \int_0^t \frac{I_{sU}}{U-s} ds - \int_0^t \frac{\sigma U}{U-s} \mathbb{E}[X_U | I_{sU}] ds. \quad (4.7.8)$$

We recall that $\{M_t\}$ is the density martingale which induces a change of measure to the bridge measure, under which $\{I_{tU}\}$ has the law of a Brownian bridge. We are now in a position to write down the formula for the price B_{tT_n} at time $t \leq T_n$ of the credit-risky coupon bond:

$$\begin{aligned} B_{tT_n} &= \frac{1}{\pi_t} \sum_{k=1}^n \mathbb{E} [\pi_{T_k} H_{T_k} | I_{tT_1}, \dots, I_{tT_k}, I_{tU}] \\ &= \frac{1}{M_t f(t, I_{tU})} \sum_{k=1}^n \mathbb{E} \left[M_{T_k} f(T_k, I_{T_k U}) \mathbf{c} \prod_{j=1}^k X_{T_j} \mid I_{tT_1}, \dots, I_{tT_k}, I_{tU} \right] \\ &\quad + \frac{1}{M_t f(t, I_{tU})} \mathbb{E} \left[M_{T_n} f(T_n, I_{T_n U}) \mathbf{p} \prod_{j=1}^n X_{T_j} \mid I_{tT_1}, \dots, I_{tT_n}, I_{tU} \right]. \end{aligned} \quad (4.7.9)$$

To compute the expectation, we use the approach presented in Section 4.1. Since the pricing kernel and the cash flow random variables H_{T_k} , $k = 1, \dots, n$, are independent, we conclude that the expression for the bond price B_{tT_n} simplifies to

$$\begin{aligned} B_{tT_n} &= \mathbf{c} \sum_{k=1}^n P_{tT_k} \mathbb{E} \left[\prod_{j=1}^k X_{T_j} \mid I_{tT_1}, \dots, I_{tT_k} \right] \\ &\quad + \mathbf{p} P_{tT_n} \mathbb{E} \left[\prod_{j=1}^n X_{T_j} \mid I_{tT_1}, \dots, I_{tT_n} \right]. \end{aligned} \quad (4.7.10)$$

Here the discount bond system $\{P_{tT_k}\}$ is given by

$$P_{tT_k} = \frac{1}{f(t, I_{tU})} \int_{-\infty}^{\infty} f \left(T_k, \nu_{tT_k} y_k + \frac{U - T_k}{U - t} I_{tU} \right) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} y_k^2 \right) dy_k, \quad (4.7.11)$$

where

$$\nu_{tT_k}^2 = (T_k - t)(U - T_k)/(U - t). \quad (4.7.12)$$

We note that formula (4.7.10) can be simplified further since the expectations therein can be worked out explicitly due to the independence of the information processes $\{I_{tT_1}\}, \dots, \{I_{tT_n}\}$. We can write

$$\mathbb{E} \left[\prod_{j=1}^k X_{T_j} \mid I_{tT_1}, \dots, I_{tT_k} \right] = \prod_{j=1}^k \pi_{1t}^{(j)}, \quad (4.7.13)$$

where the conditional density $\pi_{1t}^{(j)}$ at time t , that the random variable X_{T_j} takes the value one, is given by

$$\pi_{1t}^{(j)} = \frac{p_1^{(j)} \exp \left[\frac{T_j}{T_j - t} \left(\sigma_j I_{tT_j} - \frac{1}{2} \sigma_j^2 t \right) \right]}{p_0^{(j)} + p_1^{(j)} \exp \left[\frac{T_j}{T_j - t} \left(\sigma_j I_{tT_j} - \frac{1}{2} \sigma_j^2 t \right) \right]}. \quad (4.7.14)$$

Thus, the price B_{tT_n} at time t of the credit-risky coupon bond is

$$B_{tT_n} = \mathbf{c} \sum_{k=1}^n P_{tT_k} \prod_{j=1}^k \pi_{1t}^{(j)} + \mathbf{p} P_{tT_n} \prod_{j=1}^n \pi_{1t}^{(j)}. \quad (4.7.15)$$

At this stage, we remark that the price of a credit-risky coupon bond has been derived for the case in which the cash flow functions H_{T_k} , $k = 1, \dots, n$, do not depend on the information available at time T_k about the macroeconomic factor X_U , thereby leading to independence between the discount bond system and the credit-risky component of the coupon bond. This is generalized in a straightforward manner by considering cash flow functions of the form

$$H_{T_k} = H(X_{T_1}, \dots, X_{T_k}, I_{T_k U}), \quad (4.7.16)$$

for $k = 1, \dots, n$. The valuation at time t of a coupon bond with such cash flows can be treated similarly to the bond in Proposition 4.1.1.

Example 4.7.1. As an illustration we consider the situation in which the bond pays a coupon $\mathbf{c}$ at T_k , $k = 1, \dots, n$, and the principal amount $\mathbf{p}$ at T_n . We define the functions $R_k : \mathbb{R} \rightarrow [0, 1)$. Upon default, market-dependent recovery given by $R_k(I_{T_k U})$ (as a percentage of coupon plus principal) is paid at T_k .

For simplicity, we consider $n = 2$. In this case, the random cash flows of the bond are given by

$$H_{T_1} = \mathbf{c} X_{T_1} + (\mathbf{c} + \mathbf{p}) R_1(I_{T_1 U})(1 - X_{T_1}), \quad (4.7.17)$$

$$H_{T_2} = (\mathbf{c} + \mathbf{p}) X_{T_1} [X_{T_2} + R_2(I_{T_2 U})(1 - X_{T_2})]. \quad (4.7.18)$$

By making use of the technique presented in Section 4.4, we can express the price of the credit-risky coupon bond by

$$\begin{aligned} B_{tT_2} = & \mathbf{c} P_{tT_1} \pi_{1t}^{(1)} + (\mathbf{c} + \mathbf{p}) P_{tT_2} \pi_{1t}^{(1)} \pi_{1t}^{(2)} \\ & + (\mathbf{c} + \mathbf{p}) \left[\pi_{0t}^{(1)} \frac{1}{f(t, I_{tU})} \int_{-\infty}^{\infty} f(T_1, m(y_1)) R_1(m(y_1)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y_1^2\right) dy_1 \right. \\ & \left. + \pi_{1t}^{(1)} \pi_{0t}^{(2)} \frac{1}{f(t, I_{tU})} \int_{-\infty}^{\infty} f(T_2, m(y_2)) R_2(m(y_2)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y_2^2\right) dy_2 \right], \end{aligned} \quad (4.7.19)$$

where, P_{tT_k} is defined by (4.7.11) for $k = 1, 2$, and

$$m(y_k) = \nu_{tT_k} y_k + \frac{U - T_k}{U - t} I_{tU}, \quad \nu_{tT_k} = \sqrt{\frac{(T_k - t)(U - T_k)}{U - t}}. \quad (4.7.20)$$

4.8 Debt-sensitive pricing kernels

We fix the dates U_1 and U_2 , where $U_1 \leq U_2$, to which we associate the economic factors X_{U_1} and X_{U_2} respectively. The first factor is identified with a debt payment at time U_1 . For example X_{U_1} could be a coupon payment that a country is obliged to make at time U_1 . The second factor, X_{U_2} , could be identified with the measured growth (possibly negative) in the employment level in the same country at time U_2 since the last published figure. In such an economy, with two random factors only, it is plausible that the prices of the treasuries fluctuate according to the noisy information market participants will have about the outcome of X_{U_1} and X_{U_2} . Thus the price of a sovereign bond with maturity T , where $0 \leq t \leq T < U_1 \leq U_2$, is given by:

$$P_{tT} = \frac{1}{f(t, I_{tU_1}, I_{tU_2})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \left(T, \nu_{tT}^{(1)} y_1 + \frac{U_1 - T}{U_1 - t} I_{tU_1}, \nu_{tT}^{(2)} y_2 + \frac{U_2 - T}{U_2 - t} I_{tU_2} \right) \times \frac{1}{2\pi} \exp \left[-\frac{1}{2} (y_1^2 + y_2^2) \right] dy_2 dy_1. \quad (4.8.1)$$

In particular, the resulting interest rate process in this model is subject to the information processes $\{I_{tU_1}\}$ and $\{I_{tU_2}\}$ making it fluctuate according to information (both genuine and misleading) about the economy's factors X_{U_1} and X_{U_2} .

We now ask the following question: *What type of model should one consider if the goal is to model a pricing kernel that is sensitive to an accumulation of losses? Or in other words, how should one model the nominal short rate of interest and the market price of risk processes if both react to the amount of debt accumulated by a country over a finite period of time?*

To treat this question we need to introduce a model for an accumulation process. We shall adopt the method developed by Brody et al. [20], where the idea of a gamma bridge accumulation process is introduced. It turns out that the use of such a cumulative process is suitable to provide an answer to the above question. Let $\{\gamma_t\}_{t \geq 0}$ be a (standard) gamma process with growth rate m , i.e.

$$\mathbb{P}[\gamma_t \in dx] = \frac{x^{mt-1} \exp(-x)}{\Gamma[mt]} dx, \quad (4.8.2)$$

where $\Gamma[a]$ is the gamma function for $a > 0$ defined by

$$\Gamma[a] = \int_0^{\infty} x^{a-1} e^{-x} dx. \quad (4.8.3)$$

Here, we have $\mathbb{E}[\gamma_t] = mt$ and $\text{Var}[\gamma_t] = mt$. We define the process $\{\gamma_{tU_1}\}_{0 \leq t \leq U_1}$ by

$$\gamma_{tU_1} = \frac{\gamma_t}{\gamma_{U_1}}. \quad (4.8.4)$$

We observe that $\gamma_{0U_1} = 0$ and $\gamma_{U_1U_1} = 1$. The process $\{\gamma_{tU_1}\}$ is a gamma bridge over $[0, U_1]$. It is shown in [20] that the density of γ_{tU_1} is given by

$$f(y) = \mathbb{1}_{\{0 < y < 1\}} \frac{y^{mt-1} (1-y)^{m(U_1-t)-1}}{B[mt, m(U_1-t)]}, \quad (4.8.5)$$

where $B[a, b]$ is the beta function defined by

$$B[a, b] = \int_0^1 y^{a-1} (1-y)^{b-1} dy. \quad (4.8.6)$$

The properties of the gamma bridge $\{\gamma_{tU_1}\}$ are described in great detail in [20]. If in the example above, the factor X_{U_1} is identified with the total accumulated debt at time U_1 , then the gamma bridge accumulation process $\{I_{tU_1}^\gamma\}$, defined by

$$I_{tU_1}^\gamma = X_{U_1} \gamma_{tU_1} \quad (4.8.7)$$

where $\{\gamma_{tU_1}\}_{0 \leq t \leq U_1}$ is a gamma bridge process that is independent of X_{U_1} , measures the level of the accumulated debt as of time t , $0 \leq t \leq U_1$. If the market filtration is generated, among other information processes, also by the debt accumulation process, then asset prices that are calculated by use of this filtration, will fluctuate according to the updated information about the level of the accumulated debt of a country. We now work out the price of a sovereign bond for which the price process reacts both to Brownian and gamma information.

We consider the time line $0 \leq t \leq T < U_1 \leq U_2 < \infty$. Time T is the maturity date of a sovereign bond with unit payoff and price process $\{P_{tT}\}_{0 \leq t \leq T}$. With the date U_1 we associate the factor X_{U_1} and with the date U_2 the factor X_{U_2} . The positive random variable X_{U_1} is independent of X_{U_2} , and both may be discrete or continuous random variables. Then we introduce the following information processes:

$$I_{tU_1}^\gamma = X_{U_1} \gamma_{tU_1}, \quad I_{tU_2} = \sigma t X_{U_2} + \beta_{tU_2}. \quad (4.8.8)$$

The process $\{I_{tU_1}^\gamma\}$ is a gamma bridge accumulation process, and it is taken to be independent of $\{I_{tU_2}\}$. We assume that the market filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is generated jointly by $\{I_{tU_1}^\gamma\}$ and $\{I_{tU_2}\}$.

In this setting, the pricing kernel reacts to the updated information about the level of accumulated debt and, for the sake of example, also to noisy information about the likely level of employment growth at U_2 . Thus we propose the following model for the pricing kernel:

$$\pi_t = M_t f(t, I_{tU_1}^\gamma, I_{tU_2}) \quad (4.8.9)$$

where the process $\{M_t\}$ is the change-of-measure martingale from the probability measure $\mathbb{P}$ to the Brownian bridge measure $\mathbb{B}$, satisfying

$$dM_t = -\frac{\sigma U_2}{U_2 - t} \mathbb{E}[X_{U_2} | I_{tU_2}] M_t dW_t. \quad (4.8.10)$$

Here $\{W_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion. The formula for the price of the sovereign bond is given by

$$P_{tT} = \frac{\mathbb{E}[M_T f(T, I_{TU_1}^\gamma, I_{TU_2}^\gamma) | \mathcal{F}_t]}{M_t f(t, I_{tU_1}^\gamma, I_{tU_2}^\gamma)}. \quad (4.8.11)$$

We recall that the information processes are independent and have the Markov property. By the tower rule and a change of measure, we can express the bond price by

$$P_{tT} = \frac{\mathbb{E}[\mathbb{E}^\mathbb{B}[f(T, I_{TU_1}^\gamma, I_{TU_2}^\gamma) | I_{TU_1}^\gamma, I_{tU_1}^\gamma, I_{tU_2}^\gamma] | I_{tU_1}^\gamma, I_{tU_2}^\gamma]}{f(t, I_{tU_1}^\gamma, I_{tU_2}^\gamma)}. \quad (4.8.12)$$

We now use the technique adopted in the preceding sections, where we introduce the Gaussian random variable Y_{tT} with mean zero and variance

$$\nu_{tT}^2 = \frac{(T-t)(U_2-T)}{U_2-t}, \quad (4.8.13)$$

and the standard Gaussian random variable Y . By following the approach taken in Section 4.1, we can compute the inner expectation explicitly since the conditional expectation reduces to a Gaussian integral over the range of the random variable Y . Thus we obtain:

$$P_{tT} = \int_{-\infty}^{\infty} \frac{\mathbb{E}_\gamma[f(T, I_{TU_1}^\gamma, \nu_{tT}Y + \frac{U_2-T}{U_2-t}I_{tU_2}^\gamma) | I_{tU_1}^\gamma, I_{tU_2}^\gamma]}{f(t, I_{tU_1}^\gamma, I_{tU_2}^\gamma)} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2) dy. \quad (4.8.14)$$

The feature of this model which sets it apart from those considered in preceding sections, is the fact that we have to calculate a gamma expectation $\mathbb{E}_\gamma$. To this end we refer to the work in [20], where the price process of an Arrow-Debreu security (for the case that it is driven by a gamma bridge accumulation process) is derived. We use this result and obtain the following expression for the Arrow-Debreu density process $\{A_{tT}\}$:

$$A_{tT}(y_\gamma) = \mathbb{E}[\delta(I_{TU_1}^\gamma - y_\gamma) | I_{tU_1}^\gamma] \quad (4.8.15)$$

$$= \frac{\mathbb{1}\{y_\gamma > I_{tU_1}^\gamma\} (y_\gamma - I_{tU_1}^\gamma)^{m(T-t)-1} \int_{y_\gamma}^{\infty} p(x) x^{1-mU_1} (x - y_\gamma)^{m(U_1-T)-1} dx}{B[m(T-t), m(U_1-T)] \int_{I_{tU_1}^\gamma}^{\infty} p(z) z^{1-mU_1} (z - I_{tU_1}^\gamma)^{m(U_1-t)-1} dz}, \quad (4.8.16)$$

where $\delta(y)$ is the Dirac distribution, $p(x)$ is the *a priori* probability density of X_{U_1} and $B[a, b]$ is the beta function. Following Macrina [68], Section 3.4, we consider a function $h(I_{TU_1}^\gamma)$ of the random variable $I_{TU_1}^\gamma$ and note that for a suitable function h we may write:

$$\mathbb{E}_\gamma [h(I_{TU_1}^\gamma) \mid I_{tU_1}^\gamma] = \int_0^\infty \mathbb{E}_\gamma [\delta(I_{TU_1}^\gamma - y_\gamma) \mid I_{tU_1}^\gamma] h(y_\gamma) dy_\gamma. \quad (4.8.17)$$

Next, we see that the conditional expectation in the integral is the Arrow-Debreu density (4.8.15) for which there is the closed-form expression (4.8.16). We can use (4.8.17) to calculate the gamma expectation in (4.8.14) since $I_{TU_1}^\gamma$ is independent of I_{tU_2} . We write:

$$\begin{aligned} \mathbb{E}_\gamma \left[f \left(T, I_{TU_1}^\gamma, \nu_{tT} y + \frac{U_2 - T}{U_2 - t} I_{tU_2} \right) \mid I_{tU_1}^\gamma, I_{tU_2} \right] \\ = \int_0^\infty A_{tT}(y_\gamma) f \left(T, y_\gamma, \nu_{tT} y + \frac{U_2 - T}{U_2 - t} I_{tU_2} \right) dy_\gamma. \end{aligned} \quad (4.8.18)$$

We are now in the position to write down the bond price (4.8.14) in explicit form by using equation (4.8.18). We thus obtain:

$$P_{tT} = \int_{-\infty}^\infty \int_0^\infty \frac{A_{tT}(y_\gamma) f \left(T, y_\gamma, \nu_{tT} y + \frac{U_2 - T}{U_2 - t} I_{tU_2} \right)}{f(t, I_{tU_1}^\gamma, I_{tU_2})} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy_\gamma dy. \quad (4.8.19)$$

The bond price can be written in a more compact form by defining

$$\begin{aligned} \tilde{f}(T, t, I_{tU_1}^\gamma, I_{tU_2}) &= \int_{-\infty}^\infty \int_0^\infty A_{tT}(y_\gamma) f \left(T, y_\gamma, \nu_{tT} y + \frac{U_2 - T}{U_2 - t} I_{tU_2} \right) \\ &\quad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy_\gamma dy. \end{aligned} \quad (4.8.20)$$

We thus have:

$$P_{tT} = \frac{\tilde{f}(T, t, I_{tU_1}^\gamma, I_{tU_2})}{f(t, I_{tU_1}^\gamma, I_{tU_2})}. \quad (4.8.21)$$

Future investigation in this line of research incorporates the construction of processes $\{f(t, I_{tU_1}^\gamma, I_{tU_2})\}$ such that the resulting pricing kernel (4.8.9) is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -supermartingale. The appropriate choice of $f(t, x_1, x_2)$ depends also on a suitable description of the economic interplay of the information flows modelled by $\{I_{tU_1}^\gamma\}$ and $\{I_{tU_2}\}$. One might begin with looking at the situation in which the price of the bond depreciates due to a rising debt level and a higher level of unemployment. We conclude by observing that the gamma bridge accumulation process may also be considered for the modelling of credit-risky bonds, where default is triggered by the firm's accumulated debt exceeding a specified threshold at bond maturity. Random recovery models may be constructed using the technique in Section 4.4.

Chapter 5

Explicit finite-time pricing kernel models

In this chapter, we present some explicit constructions of information-based pricing kernel models with a finite horizon. For convenience, we restrict our attention to one-dimensional models; however, multi-dimensional models can be constructed along similar lines. The underlying information process is assumed to be a Brownian bridge information process and the functions used to model the pricing kernel satisfy the PDI (3.1.29). We first build a novel example of a model which exhibits features of the rational lognormal model in a finite-time setting. Akahori & Macrina [3] generate examples of finite-time models by using the weighted heat kernel approach with time-inhomogeneous Markov processes for the underlying drivers: We discuss the quadratic model and the exponential-quadratic model and show some interesting features of these models. Recently, Macrina [69] has generalized the approach in [3]. We investigate some of the properties of the generalized heat kernel models in greater detail. Having considered models which ensure that interest rates are non-negative, we construct an information-based model which violates (3.1.29). This model resembles a Gaussian model and may be useful for modelling real interest rates which can become negative.

5.1 Information-based rational lognormal type model

Rational models for pricing kernels belong to the Flesaker-Hughston class of interest rate models (see [36]). The properties of these models have been studied by O'Brien [78], Nakamura & Yu [77] and Rutkowski [91] amongst others. In these models, the pricing kernel is of the form

$$\pi_t = \alpha_1(t)N_t + \alpha_2(t), \tag{5.1.1}$$

where $\{N_t\}_{t \geq 0}$ is a strictly positive martingale with respect to the real probability measure $\mathbb{P}$, with $N_0 = 1$, and $\alpha_1(t)$ and $\alpha_2(t)$ are strictly positive, non-increasing deterministic functions. The resulting expressions for interest rates and discount bonds are of a rational form; that is,

$$P_{tT} = \frac{\alpha_1(T)N_t + \alpha_2(T)}{\alpha_1(t)N_t + \alpha_2(t)}, \quad (5.1.2)$$

and since $r_t = -\partial_T \ln(P_{tT})|_{T=t}$, the short rate is given by

$$r_t = -\frac{(\partial_t \alpha_1)N_t + \partial_t \alpha_2}{\alpha_1(t)N_t + \alpha_2(t)}. \quad (5.1.3)$$

It is shown by O'Brien [78] that the model prices of discount bonds and the short rate lie between deterministic bounds given in terms of the functions $\alpha_1(t)$ and $\alpha_2(t)$. These bounds may be tighter than the usual bounds of $0 < P_{tT} \leq 1$ and $r_t \geq 0$ for non-negativity of nominal interest rates.

The rational lognormal model is a special type of rational model for which the underlying martingale is given by

$$N_t = \exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right) \quad (5.1.4)$$

where $\lambda \in \mathbb{R}$ and $\{B_t\}_{t \geq 0}$ is a $\mathbb{P}$ -Brownian motion. Here, $\{N_t\}$ is a lognormal martingale and therefore, the prices of bond options and swaptions are given by Black-Scholes type analytical formulae.

We now construct an information-sensitive finite-time model which has some common features with the rational lognormal model. We consider a Brownian bridge information process

$$I_{tU} = \sigma t X_U + \beta_{tU}, \quad (5.1.5)$$

and define the positive process $\{\Psi_t\}_{0 \leq t < U}$ by

$$\Psi_t = \exp\left[\frac{\lambda\sqrt{U}}{U-t} I_{tU} - \frac{1}{2} \frac{\lambda^2 t}{U-t}\right] \quad (5.1.6)$$

where $\lambda \in \mathbb{R}$. It follows that for each $t < U$, Ψ_t is a $\mathbb{B}$ -lognormal random variable for which

$$\mathbb{E}^{\mathbb{B}}[\ln(\Psi_t)] = -\frac{1}{2} \frac{\lambda^2 t}{U-t}, \quad \text{Var}^{\mathbb{B}}[\ln(\Psi_t)] = \frac{\lambda^2 t}{U-t}. \quad (5.1.7)$$

Here, as in Chapters 3 and 4, we model the pricing kernel by $\pi_t = M_t f(t, I_{tU})$ where $\{M_t\}_{0 \leq t < U}$ is the change-of-measure martingale from $\mathbb{P}$ to a measure $\mathbb{B}$, under which the information process $\{I_{tU}\}$ has the law of a Brownian bridge.

Proposition 5.1.1. *The process $\{\Psi_t\}_{0 \leq t < U}$ is an $(\{\mathcal{F}_t\}, \mathbb{B})$ -martingale, where $\{\mathcal{F}_t\}$ is the filtration generated by $\{I_{tU}\}$, and $\mathbb{B}$ is the Brownian bridge measure.*

Proof. For $s \leq t$, we define a random variable

$$Y_{st} = I_{tU} - \frac{U-t}{U-s} I_{sU}. \quad (5.1.8)$$

We recall from Chapters 3 and 4 that Y_{st} is a Gaussian random variable under $\mathbb{B}$, with zero mean and variance

$$\nu_{st}^2 = \frac{(U-t)(t-s)}{U-s}. \quad (5.1.9)$$

Furthermore, Y_{st} is $\mathbb{B}$ -independent of I_{sU} since $\text{Cov}^{\mathbb{B}}[Y_{st}, I_{sU}] = 0$. Therefore, we can write

$$\begin{aligned} \mathbb{E}^{\mathbb{B}}[\Psi_t | I_{sU}] &= e^{-\frac{1}{2} \frac{\lambda^2 t}{U-t}} \mathbb{E}^{\mathbb{B}} \left[\exp \left(\frac{\lambda \sqrt{U}}{U-t} \left[Y_{st} + \frac{U-t}{U-s} I_{sU} \right] \right) \middle| I_{sU} \right] \\ &= \exp \left(\frac{\lambda \sqrt{U}}{U-s} I_{sU} - \frac{1}{2} \frac{\lambda^2 t}{U-t} \right) \mathbb{E}^{\mathbb{B}} \left[\exp \left(\frac{\lambda \sqrt{U}}{U-t} Y_{st} \right) \right] \\ &= \exp \left(\frac{\lambda \sqrt{U}}{U-s} I_{sU} - \frac{1}{2} \frac{\lambda^2 t}{U-t} + \frac{1}{2} \frac{\lambda^2 U(t-s)}{(U-t)(U-s)} \right) \\ &= \Psi_s, \end{aligned} \quad (5.1.10)$$

by the properties of the lognormal distribution. Moreover, it can be shown that $\mathbb{E}^{\mathbb{B}}[|\Psi_t|] < \infty$ for $t < U$. $\square$

We define

$$f(t, I_{tU}) = \alpha_1(t) \Psi_t + \alpha_2(t) \quad (5.1.11)$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are positive, non-increasing deterministic functions. We can show that the function $f(t, x)$ satisfies the PDI (3.1.29) for non-negativity of interest rates since

$$\frac{I_{tU}}{U-t} \partial_x f - \frac{1}{2} \partial_{xx} f - \partial_t f = -(\partial_t \alpha_1) \Psi_t - \partial_t \alpha_2 \geq 0, \quad (5.1.12)$$

and hence, that $\{f(t, I_{tU})\}_{0 \leq t < U}$ is an $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale. It can be shown that the price of a discount bond is given by

$$P_{tT} = \frac{\alpha_1(T) \Psi_t + \alpha_2(T)}{\alpha_1(t) \Psi_t + \alpha_2(t)}, \quad (5.1.13)$$

where $P_{0T} = (\alpha_1(T) + \alpha_2(T)) / (\alpha_1(0) + \alpha_2(0))$. Thus, in order for the model to be calibrated to the initial discount curve, we have the freedom to specify only one of the functions $\alpha_1(t)$ or $\alpha_2(t)$. The expression for the short rate is

$$r_t = -\frac{(\partial_t \alpha_1) \Psi_t + \partial_t \alpha_2}{\alpha_1(t) \Psi_t + \alpha_2(t)}. \quad (5.1.14)$$

5.1.1 Derivation of bounds

By following the arguments of O'Brien [78] for rational models, we show that discount bond prices and the short rate are constrained by tighter bounds than $0 < P_{tT} \leq 1$ and $r_t \geq 0$.

Proposition 5.1.2. *For $0 \leq t \leq T < U$, the price of a discount bond maturing at time T is bounded by*

$$\min \left\{ \frac{\alpha_1(T)}{\alpha_1(t)}, \frac{\alpha_2(T)}{\alpha_2(t)} \right\} \leq P_{tT} \leq \max \left\{ \frac{\alpha_1(T)}{\alpha_1(t)}, \frac{\alpha_2(T)}{\alpha_2(t)} \right\}. \quad (5.1.15)$$

Proof. Let c_i ($i = 1, \dots, 5$) be constants and define

$$p(z) = \frac{c_1 \exp(c_2 z) + c_3}{c_4 \exp(c_2 z) + c_5}. \quad (5.1.16)$$

Then, it is evident that $\lim_{z \rightarrow -\infty} p(z) = c_3/c_5$ and $\lim_{z \rightarrow \infty} p(z) = c_1/c_4$. Moreover, since

$$\frac{dp(z)}{dz} = \frac{c_2 \exp(c_2 z)(c_1 c_5 - c_3 c_4)}{(c_4 \exp(c_2 z) + c_5)^2}, \quad (5.1.17)$$

the sign of $dp(z)/dz$ depends on that of $c_2(c_1 c_5 - c_3 c_4)$.

- (i) If $c_2(c_1 c_5 - c_3 c_4) > 0$ then $p(z)$ is increasing in z and $\frac{c_3}{c_5} < p(z) < \frac{c_1}{c_4}$.
- (ii) If $c_2(c_1 c_5 - c_3 c_4) < 0$ then $p(z)$ is decreasing in z and $\frac{c_1}{c_4} < p(z) < \frac{c_3}{c_5}$.
- (iii) If $c_2(c_1 c_5 - c_3 c_4) = 0$ then $p(z)$ is constant in z and $\frac{c_3}{c_5} = p(z) = \frac{c_1}{c_4}$.

Thus, we have

$$\min \left\{ \frac{c_1}{c_4}, \frac{c_3}{c_5} \right\} \leq p(z) \leq \max \left\{ \frac{c_1}{c_4}, \frac{c_3}{c_5} \right\}. \quad (5.1.18)$$

For fixed T, U and $t \leq T$, let

$$\begin{aligned} c_1 &= \alpha_1(T) \exp\left(-\frac{1}{2} \frac{\lambda^2 t}{U-t}\right), & c_2 &= \frac{\lambda \sqrt{U}}{U-t}, & c_3 &= \alpha_2(T), \\ c_4 &= \alpha_1(t) \exp\left(-\frac{1}{2} \frac{\lambda^2 t}{U-t}\right), & c_5 &= \alpha_2(t). \end{aligned}$$

Then for each $0 \leq t \leq T < U$, we have (5.1.15). $\square$

Proposition 5.1.3. *For $t < U$, the short rate is bounded by*

$$\min \left\{ -\frac{\partial_t \alpha_1}{\alpha_1(t)}, -\frac{\partial_t \alpha_2}{\alpha_2(t)} \right\} \leq r_t \leq \max \left\{ -\frac{\partial_t \alpha_1}{\alpha_1(t)}, -\frac{\partial_t \alpha_2}{\alpha_2(t)} \right\}. \quad (5.1.19)$$

Proof. Let

$$r(z) = -\frac{c_1 \exp(c_2 z) + c_3}{c_4 \exp(c_2 z) + c_5}. \quad (5.1.20)$$

Then it is evident that $\lim_{z \rightarrow -\infty} r(z) = -\frac{c_3}{c_5}$ and $\lim_{z \rightarrow \infty} r(z) = -\frac{c_1}{c_4}$. Furthermore,

$$\frac{dr(z)}{dz} = -\frac{c_2 \exp(c_2 z)(c_1 c_5 - c_3 c_4)}{(c_4 \exp(c_2 z) + c_5)^2} \quad (5.1.21)$$

and the sign of $dr(z)/dz$ depends on that of $c_2(c_1 c_5 - c_3 c_4)$.

- (i) If $c_2(c_1 c_5 - c_3 c_4) < 0$ then $r(z)$ is increasing in z and $-\frac{c_3}{c_5} < r(z) < -\frac{c_1}{c_4}$.
- (ii) If $c_2(c_1 c_5 - c_3 c_4) > 0$ then $r(z)$ is decreasing in z and $-\frac{c_1}{c_4} < r(z) < -\frac{c_3}{c_5}$.
- (iii) If $c_2(c_1 c_5 - c_3 c_4) = 0$ then $r(z)$ is constant in z and $-\frac{c_3}{c_5} = r(z) = -\frac{c_1}{c_4}$.

Thus, we have

$$\min \left\{ -\frac{c_1}{c_4}, -\frac{c_3}{c_5} \right\} \leq r(z) \leq \max \left\{ -\frac{c_1}{c_4}, -\frac{c_3}{c_5} \right\}. \quad (5.1.22)$$

For fixed U and $t < U$, let

$$\begin{aligned} c_1 &= (\partial_t \alpha_1) \exp \left(-\frac{1}{2} \frac{\lambda^2 t}{U-t} \right), & c_2 &= \frac{\lambda \sqrt{U}}{U-t}, & c_3 &= \partial_t \alpha_2, \\ c_4 &= \alpha_1(t) \exp \left(-\frac{1}{2} \frac{\lambda^2 t}{U-t} \right), & c_5 &= \alpha_2(t). \end{aligned}$$

Therefore, for each $t < U$, we must have that (5.1.19) holds. $\square$

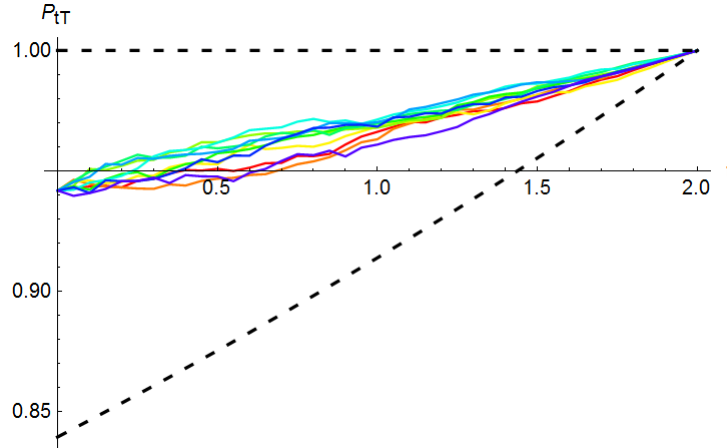


Figure 5.1: Sample paths for discount bond with $T = 2$ for rational lognormal type model. We let $U = 5$, $\sigma = 0.25$, $\lambda = 0.5$ and we assume that X_U has an *a priori* Bernoulli(p) distribution with $p = 0.5$. We assume that $\alpha_1(t) = \exp(-0.03t) - K$ and that $\alpha_2(t) = K$, where $K = \exp[-0.03(U + 10)]$. The bounds indicated by dashed lines are given by equation (5.1.15).

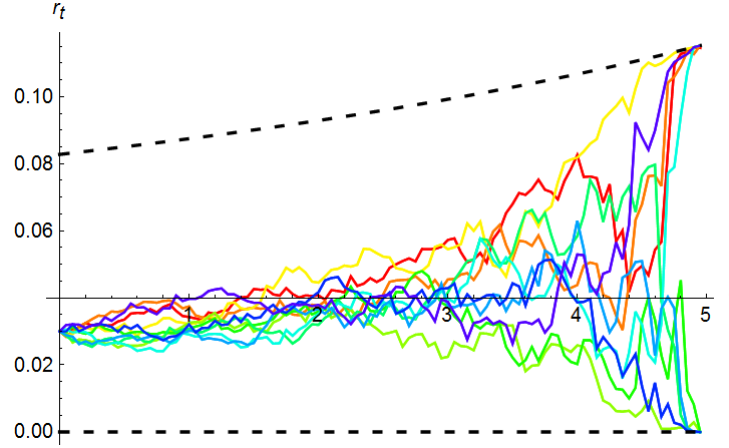


Figure 5.2: Short rate trajectories for rational lognormal type model. We let $U = 5$, $\sigma = 0.25$, $\lambda = 0.5$ and we assume that X_U has an *a priori* Bernoulli (p) distribution with $p = 0.5$. We assume that $\alpha_1(t) = \exp(-0.03t) - K$ and that $\alpha_2(t) = K$, where $K = \exp[-0.03(U + 10)]$. The bounds indicated by dashed lines are given by equation (5.1.19).

In the above plots, we simulate the case where a fundamental factor will become known in five years time. We assume that the factor can take the values $X_U = 0$ and $X_U = 1$, each with a 50% *a priori* probability. The parameter σ is a measure of the amount of genuine information released about X_U . The price of the discount bond maturing at $T = 2$ will vary with the dissemination of information. We see that, with time, the short rate trajectories will approach one of the bounds depending on the likely value taken by X_U . Thus, we can think of the outcome $X_U = 1$ as being associated with a higher interest rate environment and the outcome $X_U = 0$ as being associated with a decrease in interest rates.

5.1.2 Bond option and swaption pricing

We now consider the pricing of vanilla interest rate derivatives. The following calculations are similar to those in the case of the rational lognormal model by Rutkowski [91]. Let $\{C_{st}\}_{0 \leq s \leq t < T < U}$ be the price process of a European style call option with maturity t and strike K on a bond with price process $\{P_{tT}\}_{0 \leq t \leq T < U}$. We let

$$a_0 = \alpha_2(T) - K\alpha_2(t) \quad (5.1.23)$$

$$b_0 = K\alpha_1(t) - \alpha_1(T). \quad (5.1.24)$$

Then, it follows that the bond option price can be written as

$$C_{st} = \frac{\mathbb{E}^{\mathbb{B}}[(a_0 - b_0\Psi_t)^+ | I_{sU}]}{f(s, I_{sU})}. \quad (5.1.25)$$

First, we suppose that $b_0 > 0$ and $a_0 < 0$. Then the bond option is never exercised and is thus worthless, i.e. $C_{st} = 0$. Next, we suppose that $b_0 < 0$ and $a_0 > 0$. In this case, the bond option is always exercised and

$$C_{st} = \frac{\alpha_2(T) - K\alpha_2(t) + (\alpha_1(T) - K\alpha_1(t)) \Psi_s}{\alpha_1(s)\Psi_s + \alpha_2(s)}. \quad (5.1.26)$$

In both of these cases, the option price model is trivial since the outcome is certain. Therefore, for non-trivial bond option prices, we must have either

$$a_0 > 0 \text{ and } b_0 > 0 \quad \text{or} \quad a_0 < 0 \text{ and } b_0 < 0.$$

Proposition 5.1.4. *For the call bond option price to be non-trivial, the option strike price $K \in (K^{\text{LB}}, K^{\text{UB}})$, where*

$$K^{\text{LB}} = \min \left\{ \frac{\alpha_1(T)}{\alpha_1(t)}, \frac{\alpha_2(T)}{\alpha_2(t)} \right\}, \quad (5.1.27)$$

$$K^{\text{UB}} = \max \left\{ \frac{\alpha_1(T)}{\alpha_1(t)}, \frac{\alpha_2(T)}{\alpha_2(t)} \right\}. \quad (5.1.28)$$

Proof. First, we suppose that $\alpha_1(T)/\alpha_1(t) < K < \alpha_2(T)/\alpha_2(t)$. From (5.1.23) and (5.1.24), we see that $a_0 > 0$ and $b_0 > 0$. If $\alpha_2(T)/\alpha_2(t) < K < \alpha_1(T)/\alpha_1(t)$, then $a_0 < 0$ and $b_0 < 0$. This is also evident from the implicit bounds on the discount bond price given by (5.1.15). $\square$

Proposition 5.1.5. *The price of a call bond option with strike K is given by*

$$C_{st} = \begin{cases} \frac{1}{f(s, I_{sU})} [a_0 N(d_1) - b_0 \Psi_s N(d_2)] & \text{for } \frac{\alpha_1(T)}{\alpha_1(t)} < K < \frac{\alpha_2(T)}{\alpha_2(t)} \\ \frac{1}{f(s, I_{sU})} [a_0 N(-d_1) - b_0 \Psi_s N(-d_2)] & \text{for } \frac{\alpha_2(T)}{\alpha_2(t)} < K < \frac{\alpha_1(T)}{\alpha_1(t)} \end{cases}$$

where

$$d_1 = \frac{\ln \left(\frac{a_0}{b_0 \Psi_s} \right) + \frac{1}{2} \lambda^2 U \left(\frac{\nu_{st}}{U-t} \right)^2}{\lambda \sqrt{U} \frac{\nu_{st}}{U-t}} \quad \text{and} \quad d_2 = \frac{\ln \left(\frac{a_0}{b_0 \Psi_s} \right) - \frac{1}{2} \lambda^2 U \left(\frac{\nu_{st}}{U-t} \right)^2}{\lambda \sqrt{U} \frac{\nu_{st}}{U-t}}, \quad (5.1.29)$$

with $\nu_{st}^2 = (U-t)(t-s)/(U-s)$.

Proof. We can write

$$C_{st} = \frac{1}{f(s, I_{sU})} \mathbb{E}^{\mathbb{B}} \left[\left(a_0 - b_0 \exp \left[\frac{\lambda \sqrt{U}}{U-t} Y_{st} - \frac{1}{2} \lambda^2 U \left(\frac{\nu_{st}}{U-t} \right)^2 \right] \Psi_s \right)^+ \middle| I_{sU} \right], \quad (5.1.30)$$

where the random variable Y_{st} is given by (5.1.8), and has mean 0 and variance ν_{st}^2 under $\mathbb{B}$. First, we suppose that $\alpha_1(T)/\alpha_1(t) < K < \alpha_2(T)/\alpha_2(t)$. Then, if we let

$$y^* = \frac{\ln\left(\frac{a_0}{b_0\Psi_s}\right) + \frac{1}{2}\lambda^2 U \left(\frac{\nu_{st}}{U-t}\right)^2}{\frac{\lambda\sqrt{U}}{U-t}}, \quad (5.1.31)$$

the expression for the price of the bond option can be written as

$$C_{st} = \frac{1}{f(s, I_{sU})} \int_{-\infty}^{y^*} \left(a_0 - b_0 \exp \left[\frac{\lambda\sqrt{U}}{U-t} y - \frac{1}{2}\lambda^2 U \left(\frac{\nu_{st}}{U-t} \right)^2 \right] \Psi_s \right) \times \frac{1}{\sqrt{2\pi} \nu_{st}} e^{-\frac{1}{2}\left(\frac{y}{\nu_{st}}\right)^2} dy. \quad (5.1.32)$$

By expanding the integral and completing the square in the second term, we obtain

$$C_{st} = \frac{a_0}{f(s, I_{sU})} \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi} \nu_{st}} e^{-\frac{1}{2}\left(\frac{y}{\nu_{st}}\right)^2} dy - \frac{b_0\Psi_s}{f(s, I_{sU})} \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi} \nu_{st}} e^{-\frac{1}{2}\left(\frac{y}{\nu_{st}} - \lambda\sqrt{U} \frac{\nu_{st}}{U-t}\right)^2} dy. \quad (5.1.33)$$

This expression reduces to

$$C_{st} = \frac{1}{f(s, I_{sU})} [a_0 N(d_1) - b_0 \Psi_s N(d_2)]. \quad (5.1.34)$$

If $\alpha_2(T)/\alpha_2(t) < K < \alpha_1(T)/\alpha_1(t)$, then by a similar calculation we can show that

$$C_{st} = \frac{1}{f(s, I_{sU})} [a_0 N(-d_1) - b_0 \Psi_s N(-d_2)]. \quad (5.1.35)$$

□

Next, we consider the pricing of a payer swaption. We recall that the buyer of a payer swap enters into a contract in which he/she pays a fixed rate and receives a floating rate. We assume that the swap contract starts at time T_n and payments are made on the set of dates T_i , $i = n+1, \dots, N$, with $T_N < U$. For convenience, we assume that the swap has a unit notional and that fixed payments are made at the rate k . Then the value of the payer swap at time $t \leq T_n$ is given by

$$Swap(t) = (P_{tT_n} - P_{tT_N}) - k \sum_{i=n+1}^N \tau_{i-1,i} P_{tT_i}, \quad (5.1.36)$$

where $\tau_{i-1,i}$ represents the daycount fraction for the swap payment at time T_i . The forward par swap rate $y_{n,N}(t)$ is defined as the fixed rate for which the value of the (forward starting) swap equals zero, i.e.

$$y_{n,N}(t) = \frac{P_{tT_n} - P_{tT_N}}{\sum_{i=n+1}^N \tau_{i-1,i} P_{tT_i}}. \quad (5.1.37)$$

A payer swaption gives the holder the right to enter into a payer swap and has a payoff at maturity T_n of

$$[Swap(T_n)]^+ = \left((1 - P_{T_n T_N}) - k \sum_{i=n+1}^N \tau_{i-1,i} P_{T_n T_i} \right)^+. \quad (5.1.38)$$

A payer swaption can be viewed as a call option on a swap rate since the above payoff can also be expressed by

$$[Swap(T_n)]^+ = \sum_{i=n+1}^N \tau_{i-1,i} P_{T_n T_i} (y_{n,N}(T_n) - k)^+. \quad (5.1.39)$$

We can rewrite the payoff (5.1.38) in terms of the pricing kernel as

$$[Swap(T_n)]^+ = \left(1 - \frac{\mathbb{E}^\mathbb{P}[\pi_{T_N} | \mathcal{F}_{T_n}]}{\pi_{T_n}} - k \sum_{i=n+1}^N \tau_{i-1,i} \frac{\mathbb{E}^\mathbb{P}[\pi_{T_i} | \mathcal{F}_{T_n}]}{\pi_{T_n}} \right)^+. \quad (5.1.40)$$

Thus, the price of the payer swaption at $t \leq T_n$ is given by

$$PS_t = \frac{1}{\pi_t} \mathbb{E}^\mathbb{P} \left[\left(\pi_{T_n} - \mathbb{E}^\mathbb{P}[\pi_{T_N} | \mathcal{F}_{T_n}] - k \sum_{i=n+1}^N \tau_{i-1,i} \mathbb{E}^\mathbb{P}[\pi_{T_i} | \mathcal{F}_{T_n}] \right)^+ \middle| \mathcal{F}_t \right]. \quad (5.1.41)$$

Next, we assume that the pricing kernel is given by $\pi_t = M_t f(t, I_{tU})$ where $f(t, I_{tU})$ is modelled by (5.1.11), and we let

$$p_0 = k \sum_{i=n+1}^N \tau_{i-1,i} \alpha_1(T_i) + \alpha_1(T_N) - \alpha_1(T_n) \quad (5.1.42)$$

$$q_0 = \alpha_2(T_n) - \alpha_2(T_N) - k \sum_{i=n+1}^N \tau_{i-1,i} \alpha_2(T_i). \quad (5.1.43)$$

Then we can write

$$PS_t = \frac{1}{f(t, I_{tU})} \mathbb{E}^\mathbb{B}[(q_0 - p_0 \Psi_{T_n})^+ | I_{tU}]. \quad (5.1.44)$$

The same analysis can be carried out as for the bond option. We state the following results without proof.

Proposition 5.1.6. *For the payer swaption price to be non-trivial, the swaption strike price $k \in (k^{\text{LB}}, k^{\text{UB}})$, where*

$$k^{\text{LB}} = \min \left\{ \frac{\alpha_1(T_n) - \alpha_1(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_1(T_i)}, \frac{\alpha_2(T_n) - \alpha_2(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_2(T_i)} \right\} \quad (5.1.45)$$

$$k^{\text{UB}} = \max \left\{ \frac{\alpha_1(T_n) - \alpha_1(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_1(T_i)}, \frac{\alpha_2(T_n) - \alpha_2(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_2(T_i)} \right\}. \quad (5.1.46)$$

Proposition 5.1.7. *The price of a payer swaption with strike k is given by*

$$PS_t = \begin{cases} \frac{q_0 N(D_1) - p_0 \Psi_t N(D_2)}{f(t, I_{tU})} & \text{for } \frac{\alpha_1(T_n) - \alpha_1(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_1(T_i)} < k < \frac{\alpha_2(T_n) - \alpha_2(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_2(T_i)} \\ \frac{q_0 N(-D_1) - p_0 \Psi_t N(-D_2)}{f(t, I_{tU})} & \text{for } \frac{\alpha_2(T_n) - \alpha_2(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_2(T_i)} < k < \frac{\alpha_1(T_n) - \alpha_1(T_N)}{\sum_{i=n+1}^N \tau_{i-1,i} \alpha_1(T_i)}. \end{cases}$$

where

$$D_1 = \frac{\ln\left(\frac{q_0}{p_0 \Psi_t}\right) + \frac{1}{2} \lambda^2 U \left(\frac{\nu_{tT_n}}{U - T_n}\right)^2}{\lambda \sqrt{U} \left(\frac{\nu_{tT_n}}{U - T_n}\right)} \quad \text{and} \quad D_2 = \frac{\ln\left(\frac{q_0}{p_0 \Psi_t}\right) - \frac{1}{2} \lambda^2 U \left(\frac{\nu_{tT_n}}{U - T_n}\right)^2}{\lambda \sqrt{U} \left(\frac{\nu_{tT_n}}{U - T_n}\right)}, \quad (5.1.47)$$

and $\nu_{tT_n}^2 = (U - T_n)(T_n - t)/(U - t)$.

It is worth noting that the market usually prices both interest rate caplets (equivalently, bond options) and swaptions by using the Black 1976 formula. However, the use of Black's formula for the pricing of both of these types of derivatives is inconsistent. The reason for this is that the underlying lognormal assumption for both forward rates and swap rates simultaneously is inconsistent; see Rebonato [86] and the references therein. Both the rational lognormal model and the constructed information-sensitive model, produce consistent Black-Scholes type expressions for the prices for both sets of instruments by construction.

5.1.3 Dynamics

Proposition 5.1.8. *Let $\{W_t\}_{0 \leq t < U}$ be a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by*

$$W_t = I_{tU} + \int_0^t \frac{I_{sU}}{U - s} ds - \int_0^t \frac{\sigma U}{U - s} \mathbb{E}^\mathbb{P}[X_U | I_{sU}] ds. \quad (5.1.48)$$

Then the dynamics of the discount bond price are given by

$$\frac{dP_{tT}}{P_{tT}} = \left[r_t - \varsigma_{tt}(\varsigma_{tT} - \varsigma_{tt}) + \frac{\sigma U}{U - t} \mathbb{E}^\mathbb{P}[X_U | I_{tU}](\varsigma_{tT} - \varsigma_{tt}) \right] dt + (\varsigma_{tT} - \varsigma_{tt}) dW_t, \quad (5.1.49)$$

where the short rate $\{r_t\}_{0 \leq t < U}$ is given by (5.1.14) and

$$\varsigma_{tT} = \frac{\alpha_1(T) \frac{\lambda \sqrt{U}}{U - t} \Psi_t}{\alpha_1(T) \Psi_t + \alpha_2(T)}, \quad (5.1.50)$$

with $\varsigma_{tt} = \varsigma_{tT}|_{T=t}$.

Proof. It can be shown that the dynamics of $\{\Psi_t\}_{0 \leq t < U}$ are given by

$$\frac{d\Psi_t}{\Psi_t} = \frac{\lambda\sigma U^{\frac{3}{2}}}{(U-t)^2} \mathbb{E}^{\mathbb{P}}[X_U | I_{tU}] dt + \frac{\lambda\sqrt{U}}{U-t} dW_t, \quad (5.1.51)$$

where $\{W_t\}_{0 \leq t < U}$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by (5.1.48). We obtain the result by applying the Itô quotient rule to (5.1.13). $\square$

The discount bond dynamics can also be expressed by

$$\frac{dP_{tT}}{P_{tT}} = r_t dt + (\varsigma_{tT} - \varsigma_{tt}) d\widetilde{W}_t \quad (5.1.52)$$

where

$$d\widetilde{W}_t = dW_t + \left(\frac{\sigma U}{U-t} \mathbb{E}^{\mathbb{P}}[X_U | I_{tU}] - \varsigma_{tt} \right) dt \quad (5.1.53)$$

is a $\mathbb{Q}$ -Brownian motion. We can identify the measure $\mathbb{Q}$ as a risk-neutral measure, since the drift rate of discount bond prices under $\mathbb{Q}$ is given by the short rate.

We recall that the instantaneous forward rate is defined by $r_{tT} = -\partial_T \ln(P_{tT})$. Therefore, the forward rate is given by

$$r_{tT} = -\frac{\partial_T \alpha_1 \Psi_t + \partial_T \alpha_2}{\alpha_1(T) \Psi_t + \alpha_2(T)}. \quad (5.1.54)$$

Proposition 5.1.9. *Let $\{W_t\}_{0 \leq t < U}$ be defined by equation (5.1.48). The dynamics of the forward rate are*

$$dr_{tT} = \left[\varsigma_{tT} \partial_T \varsigma_{tT} - \partial_T \varsigma_{tT} \frac{\sigma U}{U-t} \mathbb{E}^{\mathbb{P}}[X_U | I_{tU}] \right] dt - \partial_T \varsigma_{tT} dW_t, \quad (5.1.55)$$

where ς_{tT} is given by equation (5.1.50), and $\varsigma_{tt} = \varsigma_{tT}|_{T=t}$.

Proof. We recall that the $\mathbb{P}$ -dynamics of $\{\Psi_t\}_{0 \leq t < U}$ are given by (5.1.51). We obtain the result by applying the Itô quotient rule to (5.1.54). We note that

$$\partial_T \varsigma_{tT} = -\frac{[\alpha_1(T) \partial_T \alpha_2 - \alpha_2(T) \partial_T \alpha_1] \frac{\lambda\sqrt{U}}{U-t} \Psi_t}{(\alpha_1(T) \Psi_t + \alpha_2(T))^2}. \quad (5.1.56)$$

and that

$$\varsigma_{tT} \partial_T \varsigma_{tT} = -\frac{\alpha_1(T) [\alpha_1(T) \partial_T \alpha_2 - \alpha_2(T) \partial_T \alpha_1] \frac{\lambda^2 U}{(U-t)^2} \Psi_t^2}{(\alpha_1(T) \Psi_t + \alpha_2(T))^3}. \quad (5.1.57)$$

$\square$

Furthermore, we can express the dynamical equation by

$$dr_{tT} = \Sigma_{tT} \partial_T \Sigma_{tT} dt - \partial_T \Sigma_{tT} d\widetilde{W}_t, \quad (5.1.58)$$

where

$$\Sigma_{tT} = \varsigma_{tT} - \varsigma_{tt} \quad (5.1.59)$$

and $\{\widetilde{W}_t\}_{0 \leq t < U}$ is a $\mathbb{Q}$ -Brownian motion defined by (5.1.53), where $\mathbb{Q}$ is a risk-neutral measure. Equation (5.1.58) is the usual HJM representation for the forward rate dynamics; see Heath et al. [49].

The constructed model shares some properties with the rational lognormal model by Flesaker & Hughston [36], such as the presence of inherent bounds and closed form analytical formulae for interest rate derivatives. We observe, however, that there are also a number of key differences between these models.

- The constructed model is a finite-time model with a fixed horizon $U > 0$.
- The stochastic process driving this model is a time-inhomogeneous Markov process.
- There is an underlying filtering problem as prices are dependent on partial information that is observed about a fundamental factor X_U , the value of which is revealed at the horizon U .
- In the rational lognormal model in [36], if the pricing kernel is modelled by (5.1.1), then discount bond prices are given by (5.1.2) where $\{N_t\}$ is a lognormal $\mathbb{P}$ -martingale. Even though the bond price (5.1.13) appears to be of the same form, it is worth noting that $\{\Psi_t\}$ is a lognormal $\mathbb{B}$ -martingale. Clearly, $\{\Psi_t\}$ is not a $\mathbb{P}$ -martingale since there is a drift term in equation (5.1.51).

In our closing remarks on this model, we briefly motivate our choice for the underlying function $f(t, x)$ in (5.1.11). The constructed model is developed from the following observations: First, we recall from Example 3.4.2 that the function

$$\ell(u, z) = \gamma_1(u) \exp\left(\lambda z - \frac{1}{2}\lambda^2 u\right) + \gamma_2(u) \quad (5.1.60)$$

is a space-time superharmonic function for Brownian motion if $\gamma_1(u)$ and $\gamma_2(u)$ are positive non-increasing functions. This is precisely the type of function that is used to construct the pricing kernel in the Flesaker-Hughston rational lognormal model. Next, if we make use of the variable transformation (3.4.20), we obtain the

following space-time superharmonic function for a Brownian bridge with length U and terminal value 0:

$$L(t, x) = \gamma_1 \left(\frac{t}{U-t} \right) \exp \left[\frac{\lambda \sqrt{U}}{U-t} x - \frac{1}{2} \frac{\lambda^2 t}{U-t} \right] + \gamma_2 \left(\frac{t}{U-t} \right). \quad (5.1.61)$$

Furthermore, we note that since $\frac{t}{U-t}$ is strictly increasing in t for $t < U$, we could just as well model the function $f(t, x)$ by

$$f(t, x) = \alpha_1(t) \exp \left[\frac{\lambda \sqrt{U}}{U-t} x - \frac{1}{2} \frac{\lambda^2 t}{U-t} \right] + \alpha_2(t) \quad (5.1.62)$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are positive non-increasing functions. Since the information process $\{I_{tU}\}$ has the distribution of a Brownian bridge under $\mathbb{B}$, the process $\{f(t, I_{tU})\}$ defined by (5.1.62) is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale. Thus, $\{f(t, I_{tU})\}$ produces an information-sensitive version of the rational lognormal model in a finite-time setting.

5.2 Quadratic model

We now turn our attention to one of the examples of pricing kernel models generated by Akahori & Macrina [3] using the weighted heat kernel approach. As shown in Chapter 3, the weighted heat kernel models driven by Brownian bridge information processes satisfy the PDI for non-negativity of the short rate, like the rational lognormal type model constructed in this chapter. In [3], it is shown that if we apply the weighted heat kernel approach with

$$F(t, x) = x^2, \quad (5.2.1)$$

$$w(t, u) = U - t - u, \quad (5.2.2)$$

this leads to the following expression:

$$f(t, I_{tU}) = \frac{1}{12}(U-t)^3 + \frac{1}{4}(U-t)^2 I_{tU}^2. \quad (5.2.3)$$

The process $\{f(t, I_{tU})\}$ can be used in the framework of Hughston & Macrina [55] to generate a pricing kernel model. This specification of $f(t, x)$ satisfies (3.1.29) since

$$\frac{x}{U-t} \partial_x f - \frac{1}{2} \partial_{xx} f - \partial_t f = (U-t)x^2 \geq 0. \quad (5.2.4)$$

In this model, the price of a discount bond is given by

$$P_{tT} = \frac{\frac{1}{12}(U-T)^3 + \frac{1}{4} \frac{(T-t)(U-T)^3}{(U-t)} + \frac{1}{4} \frac{(U-T)^4}{(U-t)^2} I_{tU}^2}{\frac{1}{12}(U-t)^3 + \frac{1}{4}(U-t)^2 I_{tU}^2}. \quad (5.2.5)$$

Here, we have made use of the so-called “propagation property” discussed in [3] in order to obtain (5.2.5). However, by substituting (5.2.3) into (3.1.28), and by using the integral representations for the mean and variance of a standard normal random variable, we recover exactly the above expression. This confirms the compatibility between the approaches of [3] and [55].

Moreover, the short rate in this model is given by

$$r_t = \frac{I_{tU}^2}{\frac{1}{12}(U-t)^2 + \frac{1}{4}(U-t)I_{tU}^2}. \quad (5.2.6)$$

First we derive the discount bond and short rate dynamics by making use of the Itô quotient rule.

Proposition 5.2.1. *The dynamics of the discount bond are given by*

$$\frac{dP_{tT}}{P_{tT}} = (r_t + \lambda_t \Omega_{tT}) dt + \Omega_{tT} dW_t, \quad (5.2.7)$$

where $\{W_t\}_{0 \leq t < U}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion defined by

$$W_t = I_{tU} + \int_0^t \frac{I_{sU}}{U-s} ds - \int_0^t \frac{\sigma U}{U-s} \mathbb{E}^{\mathbb{P}}[X_U | I_{sU}] ds, \quad (5.2.8)$$

and the market price of risk and bond volatility are, respectively,

$$\lambda_t = \frac{\sigma U}{U-t} \mathbb{E}[X_U | I_{tU}] - \frac{(U-t)^2 I_{tU}}{2f(t, I_{tU})} \quad (5.2.9)$$

$$\Omega_{tT} = \frac{(U-t)^2 I_{tU}}{2f(t, I_{tU})} \left[\frac{1}{P_{tT}} \left(\frac{U-T}{U-t} \right)^4 - 1 \right]. \quad (5.2.10)$$

Proposition 5.2.2. *The short rate is governed by the following dynamics:*

$$\frac{dr_t}{r_t} = \mu_t^r dt + v_t^r dW_t \quad (5.2.11)$$

where

$$\theta_t = \frac{\sigma U}{U-t} \mathbb{E}[X_U | I_{tU}], \quad (5.2.12)$$

$$\mu_t^r = \frac{1 + 2I_{tU}\theta_t}{I_{tU}^2} + \frac{1}{U-t} + \frac{36I_{tU}^2}{(3I_{tU}^2 + U-t)^2} + \frac{2(3I_{tU}\theta_t - 8)}{3I_{tU}^2 + U-t}, \quad (5.2.13)$$

$$v_t^r = \frac{2}{I_{tU}} - \frac{6I_{tU}}{3I_{tU}^2 + U-t}, \quad (5.2.14)$$

and $\{W_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by (5.2.8).

It can be verified that the expressions for the short rate and market price of risk are exactly of the form of those obtained in Section 3 of [55]. As $t \rightarrow U$, the absolute volatility of the short rate $V_t^r = v_t^r r_t \rightarrow 8/(3I_{UU}^3)$. We observe that the model is difficult to calibrate to the initial term structure owing to its rigidity. Models with no functional or parametric freedom (i.e. deterministic degrees of freedom) do not offer flexibility in calibration since the model determines what the initial term structure is. Additional degrees of freedom may be desirable to control the limiting dynamics of the interest rate process as t approaches the horizon U .

Next, we demonstrate that fundamental quantities such as discount bond prices and interest rates are also bounded in this model.

Proposition 5.2.3. *The price of a discount bond at t , with maturity $T \geq t$, is bounded by*

$$\left(\frac{U-T}{U-t}\right)^4 \leq P_{tT} \leq \frac{(U+3T-4t)(U-T)^3}{(U-t)^4}. \quad (5.2.15)$$

Proof. Let c_i ($i = 1, \dots, 4$) be constants and let z be a non-negative variable. We define

$$p(z) = \frac{c_1 + c_2 z}{c_3 + c_4 z}. \quad (5.2.16)$$

Then $\lim_{z \rightarrow 0^+} p(z) = c_1/c_3$ and $\lim_{z \rightarrow \infty} p(z) = c_2/c_4$. Furthermore,

$$\frac{dp(z)}{dz} = \frac{c_2 c_3 - c_1 c_4}{(c_3 + c_4 z)^2}. \quad (5.2.17)$$

For fixed T, U and $t \leq T$, let

$$\begin{aligned} c_1 &= \frac{1}{12}(U-T)^3 + \frac{1}{4} \frac{(T-t)(U-T)^3}{(U-t)}, & c_2 &= \frac{1}{4} \frac{(U-T)^4}{(U-t)^2}, \\ c_3 &= \frac{1}{12}(U-t)^3, & c_4 &= \frac{1}{4}(U-t)^2. \end{aligned} \quad (5.2.18)$$

Since

$$c_2 c_3 - c_1 c_4 = -\frac{1}{12}(T-t)(U-t)(U-T)^3 \leq 0, \quad (5.2.19)$$

it follows that $dp(z)/dz \leq 0$ and

$$\frac{c_2}{c_4} \leq p(z) \leq \frac{c_1}{c_3}. \quad (5.2.20)$$

The following inequality holds for all $t \leq T < U$:

$$\frac{\frac{1}{4} \frac{(U-T)^4}{(U-t)^2}}{\frac{1}{4}(U-t)^2} \leq P_{tT} \leq \frac{\frac{1}{12}(U-T)^3 + \frac{1}{4} \frac{(T-t)(U-T)^3}{(U-t)}}{\frac{1}{12}(U-t)^3}. \quad (5.2.21)$$

□

Proposition 5.2.4. *The short rate is bounded by*

$$0 \leq r_t < \frac{4}{U-t}. \quad (5.2.22)$$

Proof. Let c_i ($i = 1, 2$) be constants and let z be a non-negative variable. We define

$$r(z) = \frac{z}{c_1 + c_2 z}. \quad (5.2.23)$$

We note that $\lim_{z \rightarrow 0^+} r(z) = 0$ and $\lim_{z \rightarrow \infty} r(z) = 1/c_2$. Furthermore,

$$\frac{dr(z)}{dz} = \frac{c_1}{(c_1 + c_2 z)^2}. \quad (5.2.24)$$

For fixed U and $t < U$, let

$$c_1 = \frac{1}{12}(U-t)^2, \quad c_2 = \frac{1}{4}(U-t). \quad (5.2.25)$$

Since $c_1 > 0$, $dr(z)/dz > 0$. Furthermore, in (5.2.6) we see that at time 0 the short rate takes the value 0. Thus, $0 \leq r(z) < 1/c_2$, and it follows for all $t < U$ that

$$0 \leq r_t < \frac{4}{U-t}. \quad (5.2.26)$$

□

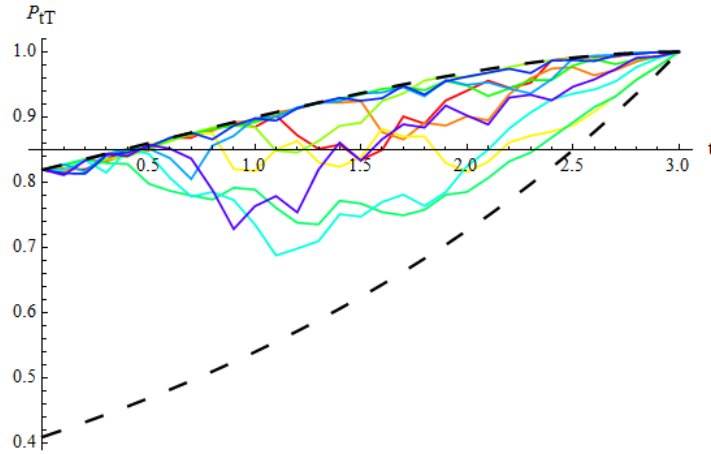


Figure 5.3: Simulation of discount bond trajectories for quadratic model. We let $U = 15$ and $T = 3$, and assume that $\sigma = 0.1$, and that X_U is distributed according to a $N(0, 0.025)$ distribution, *a priori*. The upper and lower bounds are given by Proposition 5.2.3.

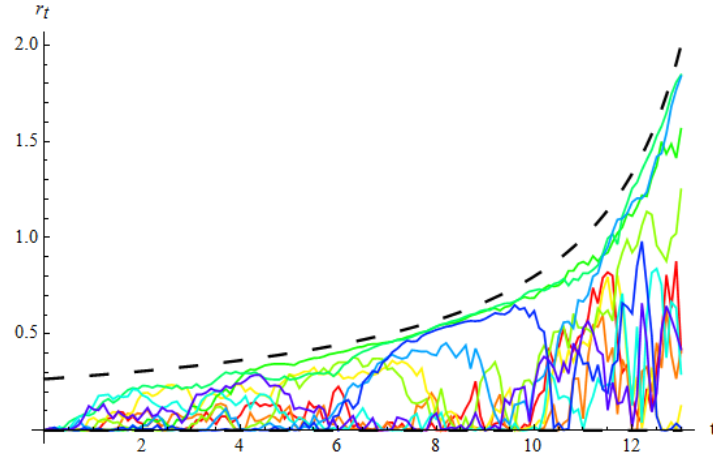


Figure 5.4: Simulation of corresponding short rate trajectories for quadratic model with $U = 15$. We assume that $\sigma = 0.1$, and that X_U is distributed according to a $N(0, 0.025)$ distribution, *a priori*. The upper and lower bounds are given by Proposition 5.2.4.

The quadratic model produces bounds that depend on time, the horizon U and, in the case of bonds, on the bond maturity T . While it may not be very restrictive for the short rate to lie in a time-dependent band over a fixed interval of time, since there is an absence of functional or parametric flexibility in the bounds, this model, as it stands, is quite rigid. Depending on the marginal distribution of X_U and the values taken by the random variable; that is to say the domain of X_U , the dynamics of the short rate may vary. Akahori & Macrina [3] show that the quadratic model leads to an analytical expression for the price of a bond option. A similar calculation shows that swaptions can be priced analogously. Much like in the model in Section 5.1, we expect that the respective strike prices must lie between bounds in order for the model prices of these vanilla interest rate derivatives to be non-trivial.

5.2.1 Digital bond pricing

As a simple application, we now consider the pricing of a digital bond where the evolution of stochastic interest rates is modelled using a quadratic model. From Section 4.1, we recall that the price of a digital bond is given by (4.1.23). In this case, we have independence between the underlying interest rates and the credit-risky component of the bond.

We simulate trajectories for a digital bond with maturity $T = 3$. Figure 5.5 shows the sample paths for a bond that is destined to default, i.e. $X_T = 0$. In Figure 5.6, the bond price paths are conditional on the outcome that $X_T = 1$,

that is, the bond is destined not to default. The *a priori* probability of default is assumed to be $p_0 = 10\%$. We recall that the parameter σ_2 controls the amount of genuine information about the random variable X_T that is available to market participants. We consider the following values: $\sigma_2 = 0.05$, $\sigma_2 = 0.75$, $\sigma_2 = 2$, $\sigma_2 = 5$. The effect of varying the information flow rate parameter σ_2 has already been discussed in detail in Section 4.2 in the context of the bond yield spread: The value of σ_2 is an inverse measure of surprise about the outcome at time T . It is worth remarking that in the case of digital bonds, the bond yield spread (4.2.3) is not influenced by the choice of the interest rate model. Therefore, to demonstrate the effect of the stochastic discounting we investigate the digital bond price process paths.

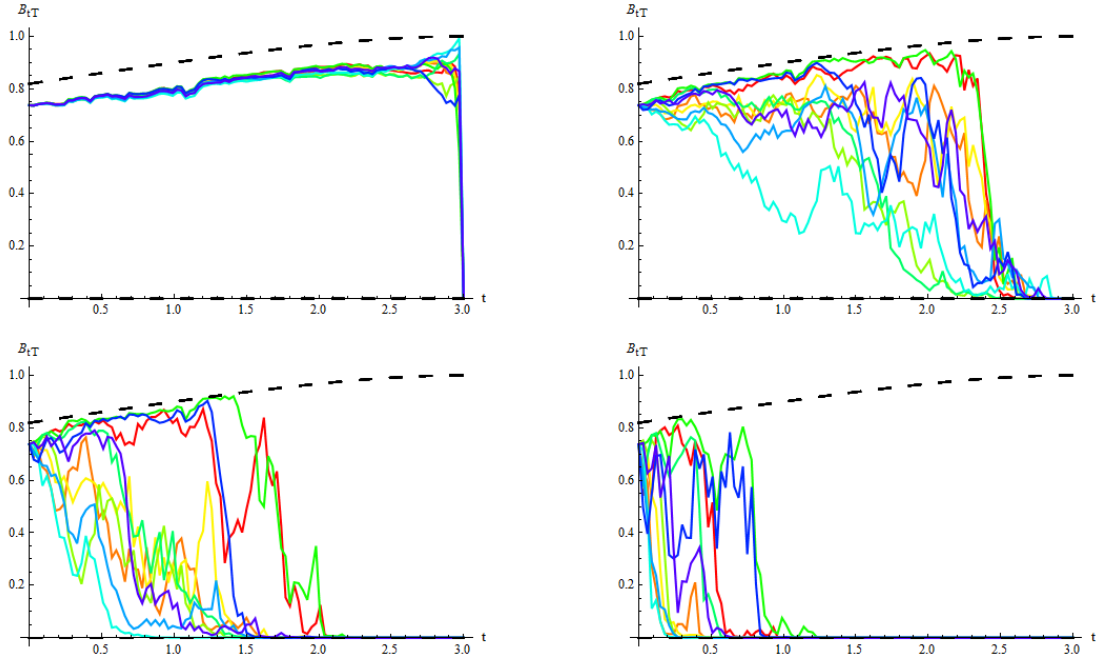


Figure 5.5: Digital bond price process with all trajectories conditional on $X_T = 0$. The bond has maturity $T = 3$. We take $p_0 = 10\%$, and let (i) $\sigma_2 = 0.05$, (ii) $\sigma_2 = 0.75$, (iii) $\sigma_2 = 2$, (iv) $\sigma_2 = 5$ (From left to right; top to bottom). For the underlying interest rates, we have used the quadratic model with $U = 15$, $\sigma_1 = 0.2$, where X_U has an *a priori* Normal $(0, 1)$ distribution. The dashed lines indicate implicit bounds given by (5.2.27).

We model interest rates by using the quadratic model with $U = 15$, and we assume that the fundamental factor X_U has an *a priori* Normal $(0, 1)$ distribution and that the corresponding information flow rate is $\sigma_1 = 0.2$. In Figures 5.5 and 5.6, we have simulated one sample path for the information process $\{I_{tU}\}$,

reflecting one possible stochastic interest rate scenario, and ten paths for $\{I_{tT}\}$, the information process concerning default. These simulations are a generalization of those in Chapter 2 of Macrina [68] for digital bonds in a deterministic interest rate setting. We recall that the quadratic model imposes deterministic bounds on discount bond prices. The price of a digital bond is lower than that of a sovereign discount bond with the same maturity (and its bond yield, higher) because of the inherent risk of default. In particular, since $0 \leq \pi_{1t} \leq 1$ and (5.2.15) holds, and since the expression for the digital bond is given by (4.1.23), it follows that the price of a digital bond is bounded by

$$0 \leq B_{tT} \leq \frac{(U + 3T - 4t)(U - T)^3}{(U - t)^4}. \quad (5.2.27)$$

These bounds are indicated by dashed lines in Figures 5.5 and 5.6.

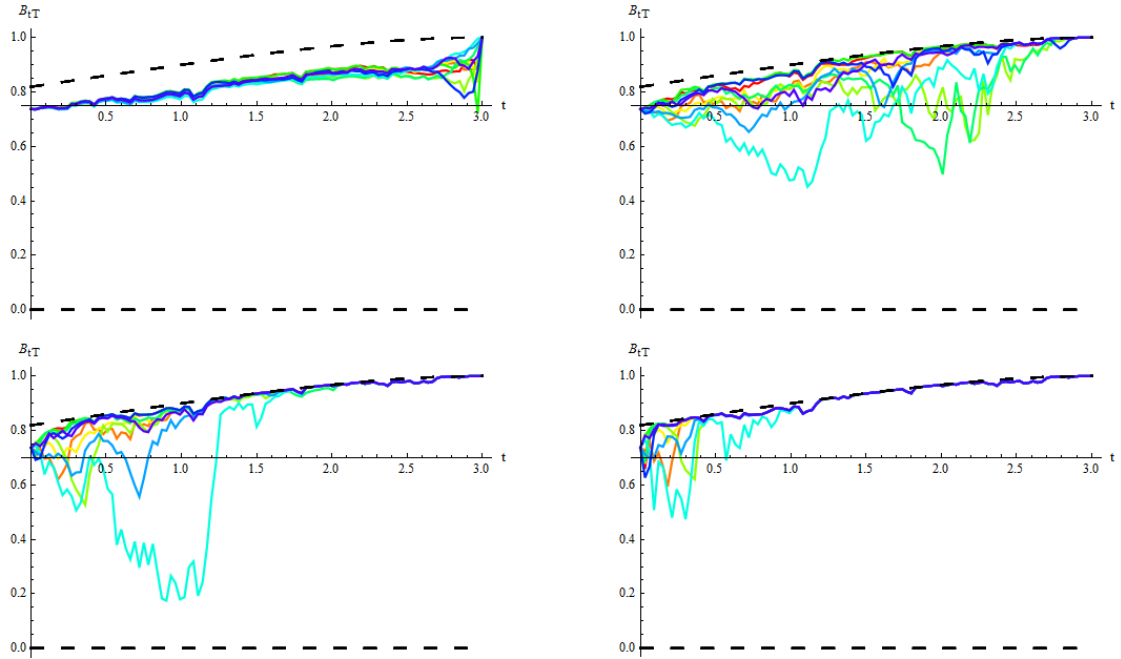


Figure 5.6: Digital bond price process with all trajectories conditional on $X_T = 1$. The bond has maturity $T = 3$. We take $p_0 = 10\%$, and let (i) $\sigma_2 = 0.05$, (ii) $\sigma_2 = 0.75$, (iii) $\sigma_2 = 2$, (iv) $\sigma_2 = 5$ (From left to right; top to bottom). For the underlying interest rates, we have used the quadratic model with $U = 15$, $\sigma_1 = 0.2$, where X_U has an *a priori* Normal $(0, 1)$ distribution. The dashed lines indicate implicit bounds given by (5.2.27).

5.3 Exponential-quadratic model

Akahori & Macrina [3] also construct a pricing kernel model with

$$f(t, I_{tU}) = g_0(t) + g_1(t)(U - t)^\eta \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right), \quad (5.3.1)$$

where $\{I_{tU}\}$ is a Brownian bridge information process, $g_0(t)$ and $g_1(t)$ are non-increasing deterministic functions and $\eta > 1/2$. It follows that the function $f(t, x)$ satisfies inequality (3.1.29) since

$$\begin{aligned} \frac{x}{U - t} \partial_x f - \frac{1}{2} \partial_{xx} f - \partial_t f \\ = \left(\eta - \frac{1}{2}\right) g_1(t)(U - t)^{\eta-1} \exp\left(\frac{1}{2} \frac{x^2}{U - t}\right) - \partial_t g_0 \\ - (\partial_t g_1)(U - t)^\eta \exp\left(\frac{1}{2} \frac{x^2}{U - t}\right) \geq 0. \end{aligned} \quad (5.3.2)$$

The price of a discount bond is given by

$$P_{tT} = \frac{g_0(T) + g_1(T)(U - T)^{\eta-\frac{1}{2}}(U - t)^{\frac{1}{2}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}{g_0(t) + g_1(t)(U - t)^\eta \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}, \quad (5.3.3)$$

and the short rate is given by

$$r_t = \frac{-\partial_t g_0 + \left[\left(\eta - \frac{1}{2}\right)g_1(t)(U - t)^{\eta-1} - (\partial_t g_1)(U - t)^\eta\right] \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}{g_0(t) + g_1(t)(U - t)^\eta \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}. \quad (5.3.4)$$

In the special case when $g_1(t) = (U - t)^{-(\eta-1/2)}$, the expressions for the discount bond price and short rate simplify to

$$P_{tT} = \frac{g_0(T) + (U - t)^{\frac{1}{2}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}{g_0(t) + (U - t)^{\frac{1}{2}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}, \quad (5.3.5)$$

$$r_t = -\frac{\partial_t g_0}{g_0(t) + (U - t)^{\frac{1}{2}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U - t}\right)}. \quad (5.3.6)$$

Proposition 5.3.1. *The dynamics of the discount bond are*

$$\frac{dP_{tT}}{P_{tT}} = (r_t + \lambda_t \Omega_{tT}) dt + \Omega_{tT} dW_t \quad (5.3.7)$$

where $\{W_t\}_{0 \leq t < U}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion defined by

$$W_t = I_{tU} + \int_0^t \frac{I_{sU}}{U - s} ds - \int_0^t \frac{\sigma U}{U - s} \mathbb{E}^\mathbb{P}[X_U | I_{sU}] ds. \quad (5.3.8)$$

Let

$$\Lambda_{tT} = \frac{g_1(T)(U-T)^{\eta-\frac{1}{2}}(U-t)^{-\frac{1}{2}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U-t}\right) I_{tU}}{g_0(T) + g_1(T)(U-T)^{\eta-\frac{1}{2}}(U-t)^{\frac{1}{2}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U-t}\right)} \quad (5.3.9)$$

with $\Lambda_{tt} = \Lambda_{tT}|_{T=t}$. Here, the short rate is given by (5.3.4), and the market price of risk and bond volatility, respectively, are

$$\lambda_t = \frac{\sigma U}{U-t} \mathbb{E}[X_U | I_{tU}] - \Lambda_{tt} \quad (5.3.10)$$

$$\Omega_{tT} = \Lambda_{tT} - \Lambda_{tt}. \quad (5.3.11)$$

Proposition 5.3.2. *We define*

$$\mathcal{E}_t = \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U-t}\right), \quad (5.3.12)$$

$$\theta_t = \frac{\sigma U}{U-t} \mathbb{E}[X_U | I_{tU}], \quad (5.3.13)$$

$$X = -\partial_t g_0 + \left[(\eta - \frac{1}{2})g_1(t)(U-t)^{\eta-1} - (\partial_t g_1)(U-t)^\eta\right] \mathcal{E}_t, \quad (5.3.14)$$

$$Y = g_0(t) + g_1(t)(U-t)^\eta \mathcal{E}_t, \quad (5.3.15)$$

$$\Sigma_t^X = I_{tU} \mathcal{E}_t (U-t)^{\eta-1} \left[\frac{(\eta - \frac{1}{2})g_1(t)}{U-t} - \partial_t g_1 \right], \quad (5.3.16)$$

$$\Sigma_t^Y = I_{tU} \mathcal{E}_t g_1(t) (U-t)^{\eta-1}, \quad (5.3.17)$$

$$\begin{aligned} \mu_t^X = & -\frac{1}{4(U-t)^2} \{ 4(U-t)^2 \partial_{tt} g_0 + \mathcal{E}_t (U-t)^\eta [4(t-U)((t-U)\partial_{tt} g_1 \\ & + (\partial_t g_1)(-I_{tU}\theta_t + 2\eta - 1)) + (2\eta - 1)g_1(t)(-2I_{tU}\theta_t + 2\eta - 3)] \} \end{aligned} \quad (5.3.18)$$

$$\mu_t^Y = \partial_t g_0 + \mathcal{E}_t (\partial_t g_1)(U-t)^\eta - \frac{1}{2} \mathcal{E}_t g_1(t) (-2\theta_t I_{tU} + 2\eta - 1)(U-t)^{\eta-1}. \quad (5.3.19)$$

Then, the short rate in (5.3.4) is governed by the following dynamics:

$$\frac{dr_t}{r_t} = \mu_t^r dt + v_t^r dW_t, \quad (5.3.20)$$

where

$$\mu_t^r = \frac{\mu_t^X}{X} - \frac{\mu_t^Y}{Y} - \frac{\Sigma_t^X \Sigma_t^Y}{XY} + \left(\frac{\Sigma_t^Y}{Y} \right)^2 \quad (5.3.21)$$

$$v_t^r = \frac{\Sigma_t^X}{X} - \frac{\Sigma_t^Y}{Y} \quad (5.3.22)$$

and $\{W_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion given by (5.3.8).

The exponential-quadratic model exhibits the following bounds:

Proposition 5.3.3. *The price of a discount bond at t , with maturity $T \geq t$, is bounded by*

$$P^{\text{LB}} \leq P_{tT} \leq P^{\text{UB}} \quad (5.3.23)$$

where

$$\begin{aligned} P^{\text{LB}} &= \min \left\{ \frac{g_0(T) + g_1(T)(U - T)^{\eta - \frac{1}{2}}(U - t)^{\frac{1}{2}}}{g_0(t) + g_1(t)(U - t)^\eta}, \frac{g_1(T)}{g_1(t)} \left(\frac{U - T}{U - t} \right)^{\eta - \frac{1}{2}} \right\} \\ P^{\text{UB}} &= \max \left\{ \frac{g_0(T) + g_1(T)(U - T)^{\eta - \frac{1}{2}}(U - t)^{\frac{1}{2}}}{g_0(t) + g_1(t)(U - t)^\eta}, \frac{g_1(T)}{g_1(t)} \left(\frac{U - T}{U - t} \right)^{\eta - \frac{1}{2}} \right\}. \end{aligned}$$

Proof. Let c_i ($i = 1, \dots, 5$) be constants and let z be a non-negative variable. We define

$$p(z) = \frac{c_1 + c_2 \exp(c_5 z)}{c_3 + c_4 \exp(c_5 z)}. \quad (5.3.24)$$

Then $\lim_{z \rightarrow 0^+} p(z) = (c_1 + c_2)/(c_3 + c_4)$ and $\lim_{z \rightarrow \infty} p(z) = c_2/c_4$. Furthermore,

$$\frac{dp(z)}{dz} = \frac{(c_2 c_3 - c_1 c_4) c_5 \exp(c_5 z)}{(c_3 + c_4 \exp(c_5 z))^2}. \quad (5.3.25)$$

For fixed T, U and $t \leq T$, let

$$\begin{aligned} c_1 &= g_0(T), & c_2 &= g_1(T)(U - T)^{\eta - \frac{1}{2}}(U - t)^{\frac{1}{2}}, & c_3 &= g_0(t), \\ c_4 &= g_1(t)(U - t)^\eta, & c_5 &= \frac{1}{2(U - t)}. \end{aligned} \quad (5.3.26)$$

The sign of $dp(z)/dz$ depends on that of $c_5(c_2 c_3 - c_1 c_4)$.

- (i) If $c_5(c_2 c_3 - c_1 c_4) > 0$ then $p(z)$ is increasing in z and $\frac{c_1 + c_2}{c_3 + c_4} < p(z) < \frac{c_2}{c_4}$.
- (ii) If $c_5(c_2 c_3 - c_1 c_4) < 0$ then $p(z)$ is decreasing in z and $\frac{c_2}{c_4} < p(z) < \frac{c_1 + c_2}{c_3 + c_4}$.
- (iii) If $c_5(c_2 c_3 - c_1 c_4) = 0$ then $p(z)$ is constant in z and $\frac{c_1 + c_2}{c_3 + c_4} = p(z) = \frac{c_2}{c_4}$.

Thus, we have

$$\min \left\{ \frac{c_1 + c_2}{c_3 + c_4}, \frac{c_2}{c_4} \right\} \leq p(z) \leq \max \left\{ \frac{c_1 + c_2}{c_3 + c_4}, \frac{c_2}{c_4} \right\}. \quad (5.3.27)$$

It follows that for all $t \leq T < U$, $P^{\text{LB}} \leq P_{tT} \leq P^{\text{UB}}$. $\square$

If we let $g_1(t) = (U - t)^{-(\eta - 1/2)}$, for example, then we have

$$\frac{g_0(T) + (U - T)^{\frac{1}{2}}}{g_0(t) + (U - t)^{\frac{1}{2}}} \leq P_{tT} \leq 1. \quad (5.3.28)$$

Proposition 5.3.4. *The short rate is bounded by*

$$r^{\text{LB}} \leq r_t \leq r^{\text{UB}}, \quad (5.3.29)$$

where

$$\begin{aligned} r^{\text{LB}} &= \min \left\{ \frac{-\partial_t g_0 + (\eta - \frac{1}{2})g_1(t)(U-t)^{\eta-1} - (\partial_t g_1)(U-t)^\eta}{g_0(t) + g_1(t)(U-t)^\eta}, \frac{\eta - \frac{1}{2}}{U-t} - \frac{\partial_t g_1}{g_1(t)} \right\} \\ r^{\text{UB}} &= \max \left\{ \frac{-\partial_t g_0 + (\eta - \frac{1}{2})g_1(t)(U-t)^{\eta-1} - (\partial_t g_1)(U-t)^\eta}{g_0(t) + g_1(t)(U-t)^\eta}, \frac{\eta - \frac{1}{2}}{U-t} - \frac{\partial_t g_1}{g_1(t)} \right\}. \end{aligned}$$

Proof. Let c_i ($i = 1, \dots, 5$) be constants and let z be a non-negative variable. We define

$$r(z) = \frac{c_1 + c_2 \exp(c_5 z)}{c_3 + c_4 \exp(c_5 z)}. \quad (5.3.30)$$

Then $\lim_{z \rightarrow 0^+} r(z) = (c_1 + c_2)/(c_3 + c_4)$ and $\lim_{z \rightarrow \infty} r(z) = c_2/c_4$. Furthermore,

$$\frac{dr(z)}{dz} = \frac{(c_2 c_3 - c_1 c_4) c_5 \exp(c_5 z)}{(c_3 + c_4 \exp(c_5 z))^2}. \quad (5.3.31)$$

For fixed U and $t < U$, let

$$\begin{aligned} c_1 &= -\partial_t g_0, & c_2 &= (\eta - \frac{1}{2})g_1(t)(U-t)^{\eta-1} - (\partial_t g_1)(U-t)^\eta, & c_3 &= g_0(t), \\ c_4 &= g_1(t)(U-t)^\eta, & c_5 &= \frac{1}{2(U-t)}. \end{aligned} \quad (5.3.32)$$

By a similar sign analysis to the case of the discount bond, it follows that

$$\min \left\{ \frac{c_1 + c_2}{c_3 + c_4}, \frac{c_2}{c_4} \right\} \leq r(z) \leq \max \left\{ \frac{c_1 + c_2}{c_3 + c_4}, \frac{c_2}{c_4} \right\}. \quad (5.3.33)$$

Therefore, for all $t < U$, $r^{\text{LB}} \leq r_t \leq r^{\text{UB}}$. $\square$

In the special case where $g_1(t) = (U-t)^{-(\eta-1/2)}$, the short rate is bounded by

$$0 < r_t < -\frac{\partial_t g_0}{g_0(t) + (U-t)^{\frac{1}{2}}}. \quad (5.3.34)$$

For a discount bond $\{P_{tT}\}$, the bond yield is the continuously compounded rate $\{y_{tT}\}$ for which

$$P_{tT} = \exp[-y_{tT}(T-t)]. \quad (5.3.35)$$

It follows that the bond yield on a discount bond with maturity $T < U$ is bounded by

$$-\frac{\ln P^{\text{UB}}}{T-t} \leq y_{tT} \leq -\frac{\ln P^{\text{LB}}}{T-t}. \quad (5.3.36)$$

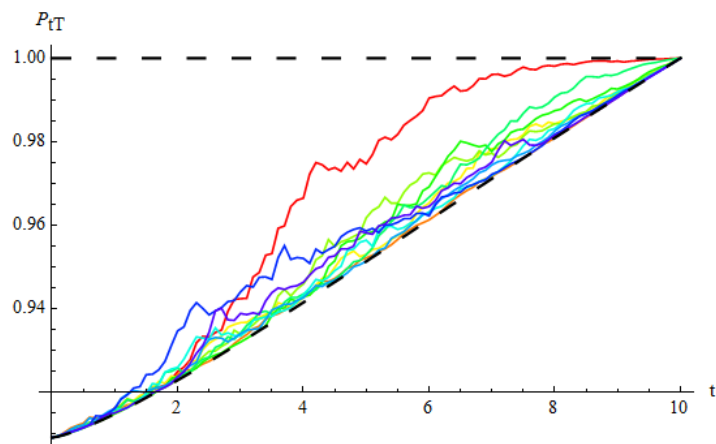


Figure 5.7: Discount bond price trajectories for exponential-quadratic model with $U = 30$ and $T = 10$. We let $\sigma = 0.5$ and X_U have a $\text{Uniform}(-1, 1)$ distribution a priori. We choose $g_0(t) = \exp(-rt^c)$ and $g_1(t) = (U - t)^{-(\eta - \frac{1}{2})}$ where $\eta = 2$, $r = 0.05$ and $c = 1.25$.

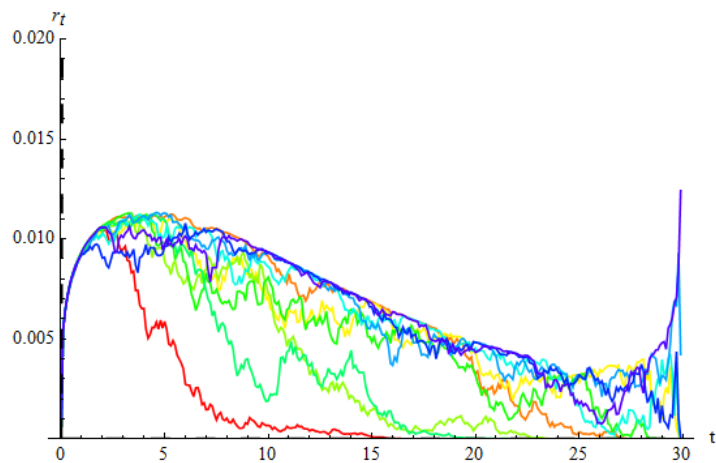


Figure 5.8: Short rate trajectories for exponential-quadratic model with $U = 30$. We let $\sigma = 0.5$ and X_U have a $\text{Uniform}(-1, 1)$ distribution a priori. We choose $g_0(t) = \exp(-rt^c)$ and $g_1(t) = (U - t)^{-(\eta - \frac{1}{2})}$ where $\eta = 2$, $r = 0.05$ and $c = 1.25$.

We observe that the exponential quadratic model is more flexible than the quadratic model as we have the freedom to specify the functions $g_0(t)$ and $g_1(t)$. Since these functions appear in the expressions for the discount bond, yield and short rate bounds, the bounds will vary with different specifications of $g_i(t)$ ($i = 0, 1$).

We summarize our key observations on the quadratic and exponential-quadratic models in the following table:

	Quadratic	Exponential-Quadratic
Interest rates	Non-negative by construction	Non-negative by construction
Vanilla interest rate derivatives	Analytical formulae for bond options and swaptions	Analytical formulae for bond options and swaptions
Boundedness	Rigid bounds on bonds, interest rates and yields	Bounds on bonds, interest rates and yields with functional or parametric freedom
Calibration	Model is rigid and “chooses” the initial term structure making calibration difficult	Model is more flexible and can fit observed initial term structure

Table 5.1: Comparison of Quadratic and Exponential-quadratic models

5.4 Generalized heat kernel models

It is worth mentioning that in equation (5.3.1), the supermartingale generated by the weighted heat kernel approach (with an exponential-quadratic function) is adjusted using positive non-increasing deterministic functions. This modification ensures that the stochasticity does not cancel out in the resulting expressions for the discount bond price and interest rates, but at the same time preserves the supermartingale property. In recent work, Macrina [69] generalizes the weighted heat kernel approach in [3] by using a similar idea, with the intent of introducing deterministic degrees of freedom to allow for greater flexibility for calibration of heat kernel models. This approach can be used to generate information-sensitive pricing kernels if the underlying time-inhomogeneous Markov process has the interpretation of an information process.

For convenience, we consider the case where the underlying time-inhomogeneous Markov process is a Brownian bridge information process $\{I_{tU}\}$, and we make explicit the connection between the approach in [69] and the framework of Hughston & Macrina [55]. Let

$$\pi_t = M_t f(t, I_{tU}) \quad (5.4.1)$$

where $\{M_t\}_{0 \leq t < U}$ is a martingale which induces a change of measure to the bridge measure $\mathbb{B}$, and let $\{\mathcal{F}_t\}$ be the filtration generated by $\{I_{tU}\}$. Suppose that we define

$$f(t, I_{tU}) = k_0(t) + k_1(t)h(t, I_{tU}), \quad (5.4.2)$$

where k_i ($i = 0, 1$) are positive non-increasing deterministic functions, and

$$h(t, I_{tU}) = \int_0^{U-t} w(t, u) \mathbb{E}^{\mathbb{B}}[F(t+u, I_{t+u,U}) | I_{tU}] du. \quad (5.4.3)$$

Here, $F(t, x)$ is a positive integrable function and $w(t, u)$ is a weight function. It follows from the discussion in Section 3.5 that $\{h(t, I_{tU})\}$ is a $\{\mathcal{F}_t, \mathbb{B}\}$ -supermartingale. Furthermore, if the function $h(t, x) \in C^{1,2}$, then $h(t, x)$ satisfies the PDI (3.1.29), and

$$\begin{aligned} & \frac{x}{U-t} \partial_x f - \frac{1}{2} \partial_{xx} f - \partial_t f \\ &= -\partial_t k_0 - (\partial_t k_1) h(t, x) + k_1(t) \left(\frac{x}{U-t} \partial_x h - \frac{1}{2} \partial_{xx} h - \partial_t h \right) \geq 0. \end{aligned} \quad (5.4.4)$$

Therefore, such functions $f(t, x)$ generate further examples of models which satisfy the PDI (3.1.29) for non-negative interest rates.

We can show that the price of a discount bond with maturity $T < U$, is given by

$$\begin{aligned} P_{tT} &= \frac{\mathbb{E}^{\mathbb{B}}[f(T, I_{TU}) | \mathcal{F}_t]}{f(t, I_{tU})} \\ &= \frac{k_0(T) + k_1(T) \mathbb{E}^{\mathbb{B}}[h(T, I_{TU}) | \mathcal{F}_t]}{k_0(t) + k_1(t) h(t, I_{tU})}. \end{aligned} \quad (5.4.5)$$

Akahori & Macrina [3] show that

$$Y_{tT} := \mathbb{E}^{\mathbb{B}}[h(T, I_{TU}) | \mathcal{F}_t] = \int_{T-t}^{U-t} w(T, u - T + t) \mathbb{E}^{\mathbb{B}}[F(t+u, I_{t+u,U}) | I_{tU}] du. \quad (5.4.6)$$

Therefore, we can write

$$P_{tT} = \frac{k_0(T) + k_1(T) Y_{tT}}{k_0(t) + k_1(t) Y_{tt}}. \quad (5.4.7)$$

Here, the degrees of freedom are (i) the choice of weight function, (ii) the function $F(t, x)$, and (iii) the function $k_1(t)$. In [69], it is noted that the function $k_0(t)$ can be expressed in terms of the initial term structure and the degrees of freedom in order for the model to be calibrated to the initial term structure, i.e.

$$k_0(T) = P_{0T} (k_0(0) + k_1(0)Y_{00}) - k_1(T)Y_{0T}. \quad (5.4.8)$$

Then we can define

$$y(t) = \frac{k_1(t)}{k_0(0) + k_1(0)Y_{00}}, \quad (5.4.9)$$

and the pricing kernel is given by

$$\pi_t = M_t \pi_0 [P_{0t} + y(t)(Y_{tt} - Y_{0t})]. \quad (5.4.10)$$

By substituting (5.4.8) into (5.4.7), we can show that the price of a discount bond is

$$P_{tT} = \frac{P_{0T} + y(T)(Y_{tT} - Y_{0T})}{P_{0t} + y(t)(Y_{tt} - Y_{0t})}. \quad (5.4.11)$$

In [69] it is shown that in certain special cases we are able to write

$$\pi_t = M_t \pi_0 [P_{0t} + b(t)A_t] \quad (5.4.12)$$

where $\{A_t\}$ denotes a $(\{\mathcal{F}_t\}, \mathbb{B})$ -martingale and $b(t)$ is a non-increasing deterministic function. Here, $\{A_t\}$ and $b(t)$ are chosen so that the pricing kernel remains strictly positive. In this case, the price of a discount bond can be expressed as

$$P_{tT} = \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t}, \quad (5.4.13)$$

and the short rate is given by

$$r_t = -\frac{\partial_t P_{0t} + (\partial_t b)A_t}{P_{0t} + b(t)A_t}. \quad (5.4.14)$$

We shall call these models the “ (bA) ” class of heat kernel models. It is worth mentioning that the expressions for discount bond prices and the short rate are similar in form to those in the rational models of Flesaker & Hughston [36]. However, there are some differences:

- In (5.4.13), $\{A_t\}$ is not a $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale, but instead, a $(\{\mathcal{F}_t\}, \mathbb{B})$ -martingale.
- The martingale $\{A_t\}$ does not have to be positive, as in the rational models.

It is interesting to note that the rational form of the (bA) models may result in boundedness for discount bond prices and interest rates in certain cases. We explore this in Section 5.4.1.

The quadratic model studied in Section 5.2 can be obtained using the approach in [69] by setting

$$\begin{aligned} k_0(t) &= 0 & k_1(t) &= 1 \\ w(t, u) &= U - t - u & F(t, x) &= x^2. \end{aligned} \quad (5.4.15)$$

This model is of type (bA) ; that is to say the resulting bond prices are given by (5.4.13), with

$$A_t = \frac{U}{(U-t)^2} I_{tU}^2 - \frac{t}{U-t} \quad b(t) = \frac{3(U-t)^4}{U^4}. \quad (5.4.16)$$

We recall that a limitation of the quadratic model is its rigidity. A generalization of the quadratic model which provides greater flexibility for calibration is obtained in [69]. This model is a (bA) model with

$$A_t = \frac{U}{(U-t)^2} I_{tU}^2 - \frac{t}{U-t} \quad b(t) = \frac{k_1(t)(U-t)^4}{4U(k_0(0) + \frac{1}{12}k_1(0)U^3)}. \quad (5.4.17)$$

The exponential-quadratic model described in Section 5.3 is already precisely a generalized heat kernel model with

$$\begin{aligned} k_0(t) &= g_0(t) & k_1(t) &= g_1(t) \\ w(t, u) &= (U-t-u)^{\eta-\frac{1}{2}} & F(t+u, x) &= \exp\left(\frac{1}{2} \frac{x^2}{U-t-u}\right). \end{aligned} \quad (5.4.18)$$

Here too, we can express the discount bond price by (5.4.13), with

$$A_t = \sqrt{\frac{U-t}{U}} \exp\left(\frac{1}{2} \frac{I_{tU}^2}{U-t}\right) - 1 \quad b(t) = \frac{(U-t)^{\eta-\frac{1}{2}} U^{\frac{1}{2}} g_1(t)}{g_0(0) + g_1(0)U^\eta}. \quad (5.4.19)$$

5.4.1 Observations on boundedness

Next, we show the presence of bounds for certain (bA) models. To this end, we first consider the behavior of the (bA) models for limiting values of $\{A_t\}$.

Proposition 5.4.1. *In the class of (bA) models, we observe that*

$$P_{tT} \rightarrow \frac{b(T)}{b(t)} \text{ as } A_t \rightarrow \infty \text{ and, as } A_t \rightarrow -\infty.$$

Moreover,

$$r_t \rightarrow -\frac{\partial_t b}{b(t)} \text{ as } A_t \rightarrow \infty \text{ and, as } A_t \rightarrow -\infty.$$

These are limiting bounds for discount bond prices and the short rate.

By following similar arguments to O'Brien [78], we provide sufficient conditions for a (bA) pricing kernel model to produce discount bond prices and interest rates that are bounded by deterministic functions.

Proposition 5.4.2. *A sufficient condition for the discount bond price process $\{P_{tT}\}$ to be bounded for $t \leq T$, is for $\{P_{tT}\}$ to be of the form*

$$P_{tT} = \frac{C_1(t, T) + C_2(t, T)Z_t}{C_1(t, t) + C_2(t, t)Z_t} \quad (5.4.20)$$

where $C_i(t, T)$ and $C_i(t, t)$ ($i = 1, 2$) are positive deterministic functions, and $\{Z_t\}$ is a non-negative process. In this case, we obtain

$$\min \left\{ \frac{C_1(t, T)}{C_1(t, t)}, \frac{C_2(t, T)}{C_2(t, t)} \right\} \leq P_{tT} \leq \max \left\{ \frac{C_1(t, T)}{C_1(t, t)}, \frac{C_2(t, T)}{C_2(t, t)} \right\}. \quad (5.4.21)$$

Proof. Let c_i ($i = 1, \dots, 4$) be constants and let z be a non-negative variable. We define

$$p(z) = \frac{c_1 + c_2 z}{c_3 + c_4 z}. \quad (5.4.22)$$

Then $\lim_{z \rightarrow 0^+} p(z) = c_1/c_3$ and $\lim_{z \rightarrow \infty} p(z) = c_2/c_4$. Moreover,

$$\frac{dp}{dz} = \frac{c_2 c_3 - c_1 c_4}{(c_3 + c_4 z)^2}. \quad (5.4.23)$$

For fixed T, U and $t \leq T < U$, let

$$\begin{aligned} c_1 &= C_1(t, T), & c_2 &= C_2(t, T), \\ c_3 &= C_1(t, t), & c_4 &= C_2(t, t). \end{aligned} \quad (5.4.24)$$

The sign of $dp(z)/dz$ depends on that of $c_2 c_3 - c_1 c_4$.

- (i) If $c_2 c_3 - c_1 c_4 > 0$ then $p(z)$ is increasing in z and $\frac{c_1}{c_3} < p(z) < \frac{c_2}{c_4}$.
- (ii) If $c_2 c_3 - c_1 c_4 < 0$ then $p(z)$ is decreasing in z and $\frac{c_2}{c_4} < p(z) < \frac{c_1}{c_3}$.
- (iii) If $c_2 c_3 - c_1 c_4 = 0$ then $p(z)$ is constant in z and $\frac{c_1}{c_3} = p(z) = \frac{c_2}{c_4}$.

Thus, we have

$$\min \left\{ \frac{c_1}{c_3}, \frac{c_2}{c_4} \right\} \leq p(z) \leq \max \left\{ \frac{c_1}{c_3}, \frac{c_2}{c_4} \right\}. \quad (5.4.25)$$

It follows that for all $t \leq T < U$, equation (5.4.21) holds. $\square$

Similarly, we obtain the following result for the short rate.

Proposition 5.4.3. *A sufficient condition for the short rate process $\{r_t\}$ to be bounded for $t < U$, is for $\{r_t\}$ to be of the form*

$$r_t = -\frac{\partial_u C_1(t, u)|_{u=t} + \partial_u C_2(t, u)|_{u=t} Z_t}{C_1(t, t) + C_2(t, t) Z_t} \quad (5.4.26)$$

where $C_i(t, t)$ ($i = 1, 2$) are positive deterministic functions, and $\{Z_t\}$ is a non-negative process. We obtain

$$r^{\text{LB}} \leq r_t \leq r^{\text{UB}}, \quad (5.4.27)$$

where

$$r^{\text{LB}} = \min \left\{ -\frac{\partial_u C_1(t, u)|_{u=t}}{C_1(t, t)}, -\frac{\partial_u C_2(t, u)|_{u=t}}{C_2(t, t)} \right\} \quad (5.4.28)$$

$$r^{\text{UB}} = \max \left\{ -\frac{\partial_u C_1(t, u)|_{u=t}}{C_1(t, t)}, -\frac{\partial_u C_2(t, u)|_{u=t}}{C_2(t, t)} \right\}. \quad (5.4.29)$$

For instance, in the case of the (bA) models for which

$$A_t = \ell_1(t) + \ell_2(t)F(t, I_{tU}), \quad (5.4.30)$$

where $\ell_i(t)$ ($i = 1, 2$) are deterministic functions and $F(t, x)$ is the positive integrable function in equation (5.4.3), we have that

$$b(u)A_t = q_1(t, u) + q_2(t, u)F(t, I_{tU}) \quad (5.4.31)$$

where $q_i(t, u)$ ($i = 1, 2$) are deterministic functions given by

$$q_1(t, u) = b(u)\ell_1(t) \quad q_2(t, u) = b(u)\ell_2(t). \quad (5.4.32)$$

Here, we can write

$$P_{tT} = \frac{P_{0T} + q_1(t, T) + q_2(t, T)F(t, I_{tU})}{P_{0t} + q_1(t, t) + q_2(t, t)F(t, I_{tU})}, \quad (5.4.33)$$

and we can show that the short rate is given by

$$r_t = -\frac{\partial_t P_{0t} + \partial_u q_1(t, u)|_{u=t} + \partial_u q_2(t, u)|_{u=t} F(t, I_{tU})}{P_{0t} + q_1(t, t) + q_2(t, t)F(t, I_{tU})}. \quad (5.4.34)$$

Proposition 5.4.4. *In the case where discount bonds are given by (5.4.33), we have*

$$P^{\min} \leq P_{tT} \leq P^{\max} \quad (5.4.35)$$

where

$$P^{\min} = \min \left\{ \frac{P_{0T} + q_1(t, T)}{P_{0t} + q_1(t, t)}, \frac{q_2(t, T)}{q_2(t, t)} \right\} \quad (5.4.36)$$

$$P^{\max} = \max \left\{ \frac{P_{0T} + q_1(t, T)}{P_{0t} + q_1(t, t)}, \frac{q_2(t, T)}{q_2(t, t)} \right\}. \quad (5.4.37)$$

The short rate given by (5.4.34) is bounded by

$$r^{\min} \leq r_t \leq r^{\max} \quad (5.4.38)$$

where

$$r^{\min} = \min \left\{ -\frac{\partial_t P_{0t} + \partial_u q_1(t, u)|_{u=t}}{P_{0t} + q_1(t, t)}, -\frac{\partial_u q_2(t, u)|_{u=t}}{q_2(t, t)} \right\} \quad (5.4.39)$$

$$r^{\max} = \max \left\{ -\frac{\partial_t P_{0t} + \partial_u q_1(t, u)|_{u=t}}{P_{0t} + q_1(t, t)}, -\frac{\partial_u q_2(t, u)|_{u=t}}{q_2(t, t)} \right\} \quad (5.4.40)$$

Proof. The proofs follow from Propositions 5.4.2 and 5.4.3. $\square$

We remark that the bound $q_2(t, T)/q_2(t, t)$ on the discount bond price is precisely the limiting bound $b(T)/b(t)$, and $-\partial_u q_2(t, u)|_{u=t}/q_2(t, t)$ is the limiting bound $-\partial_t b/b(t)$ for the short rate. These bounds vary with different choices of the degree of freedom $k_1(t)$. Furthermore, the other bounds in (5.4.35) and (5.4.38) also include the initial term structure P_{0t} . These bounds can be thought of as being market-sensitive as they depend on initial market data. Rebonato [87] has noted (in the case of rational models) that in certain situations, “the existence of bounds can be turned to modelling advantage”. These sentiments are echoed in [69], where it is also suggested that the bounds could be used to reflect the monetary policies of authorities.

Remark 5.4.1. The bounds in (5.4.35) and (5.4.38) may not be the tightest bounds on $\{P_{tT}\}$ and $\{r_t\}$, respectively. As we demonstrate later, in certain cases tighter bounds may be derived.

It turns out that the quadratic model, the generalized quadratic model and the exponential-quadratic model described previously are all models of type (bA) for which (5.4.31) holds,

Example 5.4.1. (Quadratic model in [3])

Here, we have

$$q_1(t, u) = -\frac{3t(U-u)^4}{U^4(U-t)} \quad (5.4.41)$$

$$q_2(t, u) = \frac{3(U-u)^4}{U^3(U-t)^2} \quad (5.4.42)$$

and $F(t, x) = x^2$. The bounds are given by

$$\frac{q_2(t, T)}{q_2(t, t)} \leq P_{tT} \leq \frac{P_{0T} + q_1(t, T)}{P_{0t} + q_1(t, t)}, \quad (5.4.43)$$

and

$$0 \leq r_t \leq -\frac{\partial_u q_2(t, u)|_{u=t}}{q_2(t, t)}, \quad (5.4.44)$$

and are in agreement with those in Section 5.2.

Example 5.4.2. (Generalized quadratic model in [69])

Here, we have

$$q_1(t, u) = -\frac{t k_1(u) (U - u)^4}{4U (U - t) (k_0(0) + \frac{1}{12}k_1(0)U^3)} \quad (5.4.45)$$

$$q_2(t, u) = \frac{k_1(u) (U - u)^4}{4(U - t)^2 (k_0(0) + \frac{1}{12}k_1(0)U^3)} \quad (5.4.46)$$

and $F(t, x) = x^2$. Thus, we obtain $P^{\min} \leq P_{tT} \leq P^{\max}$ and $r^{\min} \leq r_t \leq r^{\max}$, where the bounds are defined in Proposition 5.4.4.

Example 5.4.3. (Exponential-quadratic model)

Here, we have

$$\begin{aligned} q_1(t, u) &= -\frac{(U - u)^{\eta - \frac{1}{2}} U^{\frac{1}{2}} g_1(u)}{g_0(0) + g_1(0)U^\eta} \\ q_2(t, u) &= \frac{(U - u)^{\eta - \frac{1}{2}} U^{\frac{1}{2}} \sqrt{1 - \frac{t}{U}} g_1(u)}{g_0(0) + g_1(0)U^\eta} \end{aligned} \quad (5.4.47)$$

and $F(t, x) = \exp\left(\frac{1}{2} \frac{x^2}{U - t}\right)$. The bounds are given in Proposition 5.4.4. We note that these bounds are not, in fact, the tightest bounds on the discount bond and short rate processes. In Section 5.3, we obtain tighter bounds.

Let $\{C_{st}\}_{0 \leq s \leq t < T < U}$ be the price process of a European style call option with maturity t and strike K written on a bond with price process $\{P_{tT}\}_{0 \leq t \leq T < U}$. Then, in the considered subclass of (bA) models, we also obtain stricter bounds on the strike than $K \in (0, 1)$ for non-trivial bond option prices.

Proposition 5.4.5. *The strike price is bounded by*

$$P^{\min} \leq K \leq P^{\max}, \quad (5.4.48)$$

where $P^{\min}$ and $P^{\max}$ are given by (5.4.36) and (5.4.37), respectively, with tighter bounds possible in certain cases.

It is worth mentioning that in Macrina [69], (bA) -type equity models are also constructed. It can be shown that similar bounds exist if the asset price model with stochastic discounting is based on a single information process. However, it is more realistic to use a multi-dimensional model in this case. Multi-dimensional models may not produce explicit bounds for the prices of discount bonds and interest rates.

5.5 Conditional Gaussian interest rate model

Thus far, we have generated pricing kernel models in which the process $\{f(t, I_{tU})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -supermartingale. Here, the underlying positive function $f(t, x)$ satisfies (3.1.29) for non-negativity of the short rate. While the preceding models ensure non-negative nominal interest rates, they are not suitable for modelling real interest rates, which can become negative. Real interest rates have been negative in the US and in many European economies since the 2008 financial crisis.

In the real economy, the value of all goods and services are measured in terms of a representative basket of goods and services purchased by a typical consumer, rather than in units of currency. Let $\{P_{tT}^R\}_{0 \leq t \leq T}$ denote the price process of a real discount bond. Here, P_{tT}^R is the price at time t , in units of goods and services, for one basket of goods and services to be delivered at the bond maturity T (see Hughston [54]). We shall adopt a pricing kernel approach; that is, we write

$$P_{tT}^R = \frac{1}{\pi_t^R} \mathbb{E}[\pi_T^R | \mathcal{F}_t], \quad (5.5.1)$$

where $\{\pi_t^R\}$ is the real pricing kernel. We can think of the associated real short rate $\{r_t^R\}$ as being determined by the drift of $\{\pi_t^R\}$. Equivalently, $r_t^R = -\partial_T \ln(P_{tT}^R)|_{T=t}$. We now construct the following information-sensitive model for the real pricing kernel:

$$\pi_t^R = M_t g(t, I_{tU}), \quad (5.5.2)$$

where $\{M_t\}$ is the usual change-of-measure martingale and $\{I_{tU}\}$ is the Brownian bridge information process pertaining to the fundamental factor X_U . We may, for instance, take X_U to be a random variable related to (i) the accumulated debt of the economy (or possible default or restructuring of such debt), (ii) economic growth, (iii) prospective fiscal policy (spending cuts, tax increases, austerity plans etc.), (iv) monetary policy, or (v) demographic changes at time U . For simplicity, we consider a one-dimensional model; however, in reality, it is likely that information about multiple factors will affect the dynamics of real interest rates. We remark that the function $g(t, x)$ does not have to satisfy (3.1.29). We only require that $g(t, x)$ is a positive function for the real pricing kernel to be positive, thereby rendering negative prices impossible. This is equivalent to modelling the process $\{g(t, I_{tU})\}$ by a positive $(\{\mathcal{F}_t\}, \mathbb{B})$ -semimartingale. As an example, we let

$$g(t, I_{tU}) = \exp[\gamma_1(t)I_{tU} + \gamma_2(t)], \quad (5.5.3)$$

where $\gamma_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\gamma_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ are deterministic functions. It can be shown that the price of a real discount bond with maturity T is given by

$$P_{tT}^R = \exp \left[\gamma_2(T) - \gamma_2(t) + \frac{1}{2} \gamma_1^2(T) \nu_{tT}^2 + \left(\frac{U-T}{U-t} \gamma_1(T) - \gamma_1(t) \right) I_{tU} \right] \quad (5.5.4)$$

where $\nu_{tT}^2 = (U - T)(T - t)/(U - t)$, and the real short rate is of the form

$$r_t^R = -\partial_t \gamma_2 - \frac{1}{2} \gamma_1^2(t) + \left(\frac{\gamma_1(t)}{U - t} - \partial_t \gamma_1 \right) I_{tU}. \quad (5.5.5)$$

In this model, real discount bonds are lognormally distributed, conditional on the value of X_U . Similarly, the real short rate model is a conditional Gaussian model since the random variable I_{tU} is normally distributed, given the value of X_U . Therefore, the model assigns a positive probability to negative real interest rates. In the following simulations, we let

$$\gamma_1(t) = \frac{k(U - t)}{U^2} \quad (5.5.6)$$

$$\gamma_2(t) = \frac{k^2(U - t)^3}{3U^4}, \quad (5.5.7)$$

where $k \in \mathbb{R}$ is a constant parameter.

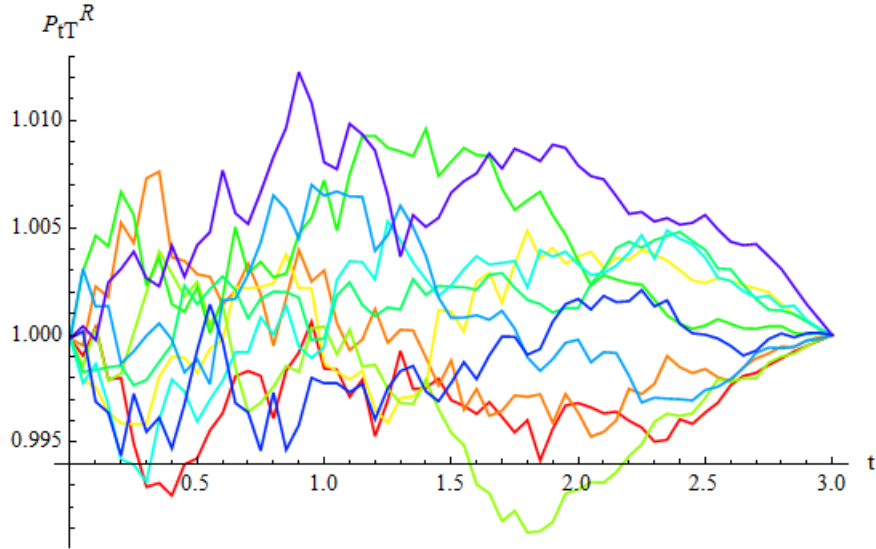


Figure 5.9: Real discount bond sample paths for conditional Gaussian interest rate model, in units of goods and services. We let $T = 3$, $U = 5$, $\sigma = 0.5$, $k = 0.05$ and we assume that X_U has an *a priori* Normal $(-0.5, 0.25)$ distribution.

Next, we assume that the nominal pricing kernel is modelled by an exponential-quadratic model with $f(t, I_{tU})$ given by (5.3.1) where

$$g_0(t) = c^t \quad (5.5.8)$$

$$g_1(t) = (U - t)^{-(\eta - \frac{1}{2})}, \quad (5.5.9)$$

with $\eta > \frac{1}{2}$ and $0 < c < 1$. In Figure 5.11 we simulate sample paths for the nominal short rate $\{r_t^N\}$.

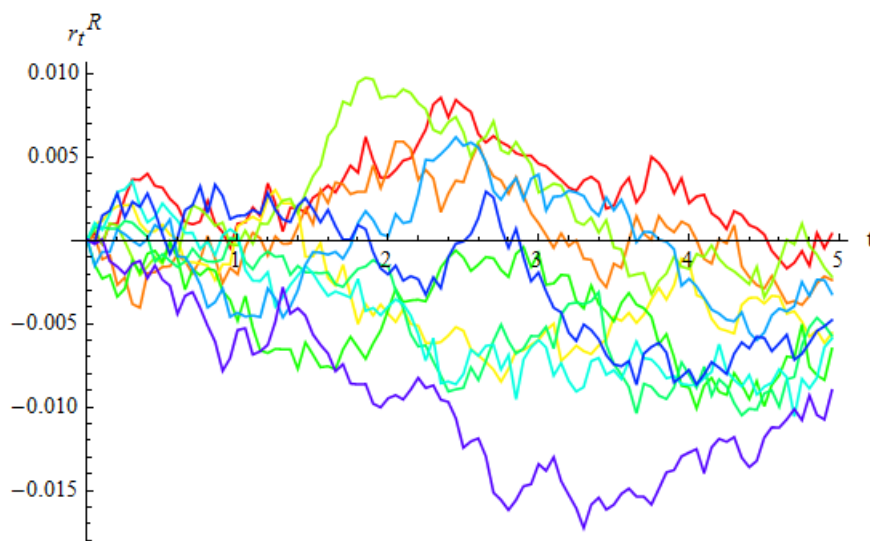


Figure 5.10: Real short rate sample paths for conditional Gaussian interest rate model. We let $U = 5$, $\sigma = 0.5$, $k = 0.05$ and we assume that X_U has an *a priori* Normal $(-0.5, 0.25)$ distribution.

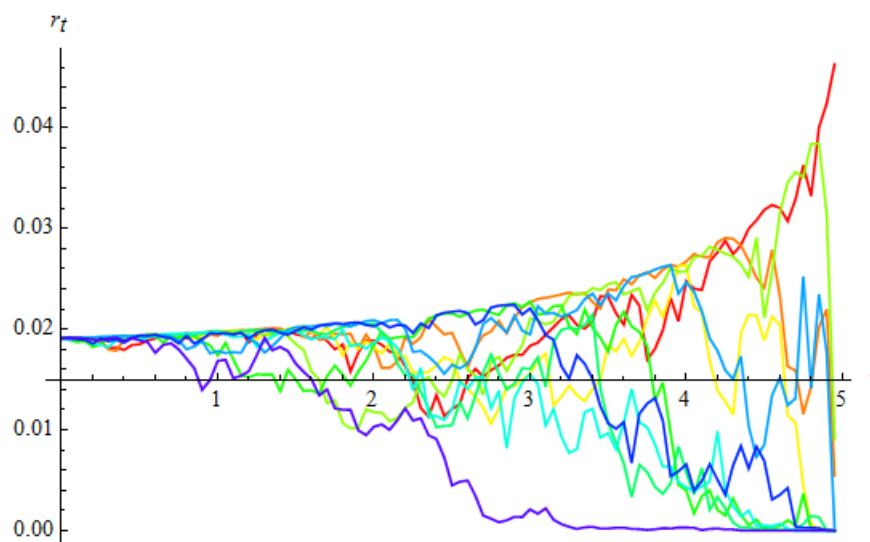


Figure 5.11: Sample paths for exponential-quadratic nominal short rate model. We let $U = 5$, $\sigma = 0.5$, $c = 0.92$, $\eta = 2$ and we assume that X_U has an *a priori* Normal $(-0.5, 0.25)$ distribution.

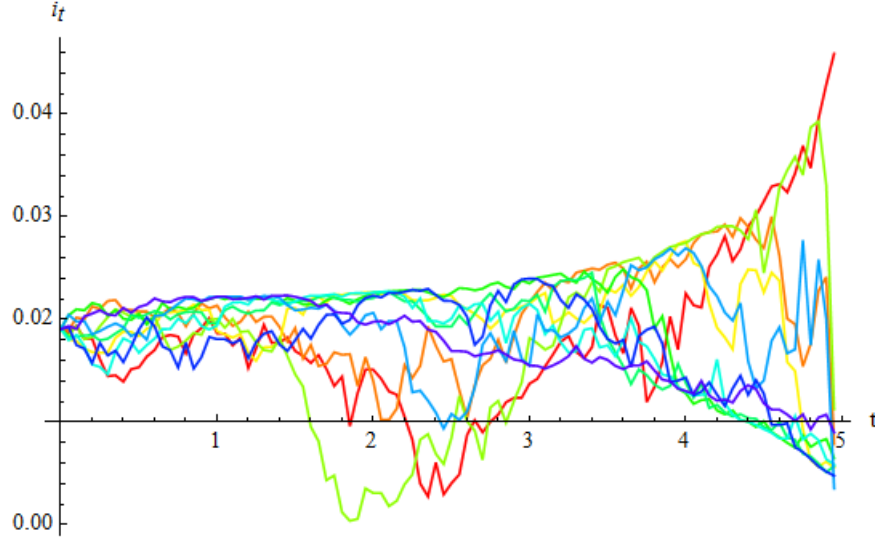


Figure 5.12: Sample paths for instantaneous inflation rate given by (5.5.14). We let $U = 5$ and we assume that X_U has an *a priori* Normal $(-0.5, 0.25)$ distribution where $\sigma = 0.5$. We use the above-mentioned exponential-quadratic model with $c = 0.92$ and $\eta = 2$, for the nominal short rate. We use the conditional Gaussian model with $k = 0.05$ for the real short rate.

We recall that we can express the dynamics for the nominal discount bond by

$$\frac{dP_{tT}^N}{P_{tT}^N} = (r_t^N + \lambda_t^N \Omega_{tT}^N) dt + \Omega_{tT}^N dW_t, \quad (5.5.10)$$

where λ_t^N is the nominal market price of risk process and Ω_{tT}^N is the nominal bond volatility process. Here, the $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion $\{W_t\}$ is given by (5.2.8). Hughston [54] shows that the real discount bond dynamics can be written as:

$$\frac{dP_{tT}^R}{P_{tT}^R} = (r_t^R + \lambda_t^R \Omega_{tT}^R) dt + \Omega_{tT}^R dW_t, \quad (5.5.11)$$

where λ_t^R is the real market price of risk process and Ω_{tT}^R is the real bond volatility process. By the “foreign exchange” analogy (see Section 4.6), we obtain the following stochastic differential equation for the consumer price index process $\{C_t\}$:

$$\frac{dC_t}{C_t} = [r_t^N - r_t^R + \lambda_t^N (\lambda_t^N - \lambda_t^R)] dt + (\lambda_t^N - \lambda_t^R) dW_t. \quad (5.5.12)$$

Since inflation measures the rate of increase in the price of a representative basket of goods and services, we can define the instantaneous inflation rate $\{i_t\}$ to be the drift of the CPI; that is

$$i_t = r_t^N - r_t^R + \lambda_t^N (\lambda_t^N - \lambda_t^R). \quad (5.5.13)$$

Under the nominal risk neutral measure, the nominal risk premium term satisfies $\lambda_t^N = 0$. Thus, we have

$$i_t = r_t^N - r_t^R. \quad (5.5.14)$$

This relation is known as the Fisher equation. Thus, we have presented a simple model in which movements in real and nominal interest rates and the instantaneous inflation rate occur according to the revelation of information about the fundamental risk factor X_U .

It is interesting to note that the use of an interest rate model in which rates are constrained to remain between deterministic bounds, may lead to a model for inflation, in which the inflation rate is also implicitly bounded. The interest rate bounds could represent a central bank's desired range for interest rates, and the related bounds for inflation could reflect a central bank's inflation target band.

Chapter 6

Lévy random bridge models

We have, for the most part, examined pricing kernel models which are based on Brownian bridge information. In Section 4.8 we briefly considered an example of a pricing kernel model which is sensitive to cumulative government debt over a finite time interval, modelled by a gamma bridge accumulation process. This model exhibits jumps. A pertinent question at this point is: *Can one extend the considered information-sensitive pricing kernels to incorporate jumps using a single overarching framework?* In this chapter we shall build on the ideas in Section 3.1, where the approach of Hughston & Macrina [55] is described, in order to construct finite-time information-driven pricing kernel models with jumps. To this end, we draw from the theory of Lévy random bridges (hereafter LRBs) which has been developed by Hoyle [52] and Hoyle et al. [53].

6.1 Lévy processes, Lévy bridges and Lévy random bridges

In this section, we introduce the tools we shall use to construct LRB-based pricing kernel models. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the measure $\mathbb{P}$ is an arbitrary probability measure with no specific interpretation at this point. Let us start by briefly examining the concept of a Lévy process. The following results can be found in Applebaum [5] and Cont & Tankov [29], for instance.

Definition 6.1.1. *An adapted stochastic process $\{L_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a **Lévy process** if:*

(i) $X_0 = 0$ $\mathbb{P}$ -a.s.

(ii) The paths of $\{L_t\}$ are càdlàg $\mathbb{P}$ -a.s.

- (iii) $\{L_t\}$ has independent increments, i.e. for every increasing sequence of times $t_0, t_1, \dots, t_n$ ($n \in \mathbb{N}$), the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent.
- (iv) $\{L_t\}$ has stationary increments, i.e. the distribution of $L_{s+t} - L_s$ does not depend on s , or $L_{s+t} - L_s \stackrel{d}{=} L_t$, where $\stackrel{d}{=}$ denotes equality in distribution.
- (v) $\{L_t\}$ is stochastically continuous, i.e. for all $\epsilon > 0$ and $t \geq 0$,

$$\lim_{t \rightarrow s} \mathbb{P}[|X_t - X_s| > \epsilon] = 0 \quad \mathbb{P} - a.s. \quad (6.1.1)$$

The characteristic function of a Lévy process $\{L_t\}$ is given by

$$\mathbb{E}[e^{iuL_t}] = \exp[\Psi(u)t] \quad (6.1.2)$$

where $\Psi(u)$ denotes the characteristic exponent. By the Lévy Khintchine representation, we can write

$$\Psi(u) = iua - \frac{1}{2}u^2\Sigma + \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz\mathbb{1}_{\{|z|<1\}}) \Pi(dz) \quad (6.1.3)$$

where $a \in \mathbb{R}$, $\Sigma \in \mathbb{R}_+$ and Π is a so-called Lévy measure satisfying

$$\Pi(\{0\}) = 0 \quad \int_{-\infty}^{\infty} (1 \wedge |z|^2) \Pi(dz) < \infty. \quad (6.1.4)$$

The triplet (a, Σ, Π) is called the *characteristic triplet* and fully characterizes the Lévy process $\{L_t\}$. In what follows, we shall assume that L_t possesses a density $\rho_t(x) : \mathbb{R} \rightarrow \mathbb{R}_+$, that is

$$\mathbb{P}[L_t \in dy] = \rho_t(y)dy. \quad (6.1.5)$$

Furthermore, for $0 \leq s \leq t$,

$$\mathbb{P}[L_t \in dy \mid L_s = x] = \rho_{t-s}(y - x)dy. \quad (6.1.6)$$

A Lévy process is a time-homogeneous Markov process (see [5]). Several examples of Lévy processes are used for the construction of pricing kernel models in Chapter 7.

Next, we consider the notion of a *Lévy bridge*. A bridge is a stochastic process that is fixed to a particular point at a specified future time ([52]). Let $\{L_{tU}^{(z)}\}_{0 \leq t \leq U}$ be a $\{L_t\}$ -bridge that is fixed to the value $z \in \mathbb{R}$ at time U . In [52], it is shown that the bridge $\{L_{tU}^{(z)}\}$ is a Markov process. For $0 < \rho_U(z) < \infty$, the marginal bridge density $\rho_{tU}(y; z)$ is given by

$$\rho_{tU}(y; z) = \frac{\rho_t(y)\rho_{U-t}(z - y)}{\rho_U(z)}. \quad (6.1.7)$$

We can write

$$\mathbb{P}[L_{tU}^{(z)} \in dy] = \rho_{tU}(y; z) dy \quad (6.1.8)$$

and

$$\mathbb{P}[L_{tU}^{(z)} \in dy \mid L_{sU}^{(z)} = x] = \rho_{t-s, U-s}(y - x; z - x) dy \quad (6.1.9)$$

for $0 \leq s < t < U$. While a Lévy process has stationary and independent increments, a Lévy bridge has stationary increments which are not independent, since it is conditioned to take a given value at a future time. The Brownian bridge and the gamma bridge, which have been used for the construction of models thus far, are examples of Lévy bridges. Other examples are provided in [52] and by Gulisashvili & van Casteren [48].

It turns out that both Lévy processes and Lévy bridges are LRBs. An LRB can be thought of as a Lévy process that is conditioned to have a particular marginal law at a specified future time. The following definition and results are taken from [52, 53].

Definition 6.1.2. *We say that the process $\{L_{tU}\}_{0 \leq t \leq U}$ is an **LRB** $([0, U], \rho_t, \nu)$ if the following are satisfied:*

- (i) L_{UU} has marginal law ν .
- (ii) There exists a Lévy process $\{L_t\}$ such that L_t has density $\rho_t(x)$ for all $t \in (0, U]$.
- (iii) ν concentrates mass where $\rho_U(z)$ is positive and finite, i.e. $0 < \rho_U(z) < \infty$ for ν -a.e. z .
- (iv) For every $n \in \mathbb{N}_+$, every $0 < t_1 < \dots < t_n < U$, every $(x_1, \dots, x_n) \in \mathbb{R}^n$, and ν -a.e. z , we have

$$\mathbb{P}[L_{t_1U} \leq x_1, \dots, L_{t_nU} \leq x_n \mid L_{UU} = z] = \mathbb{P}[L_{t_1} \leq x_1, \dots, L_{t_n} \leq x_n \mid L_U = z]. \quad (6.1.10)$$

The finite-dimensional distributions of $\{L_{tU}\}$ are given by

$$\mathbb{P}[L_{t_1U} \in dx_1, \dots, L_{t_nU} \in dx_n, L_{UU} \in dz] = \prod_{i=1}^n [\rho_{t_i-t_{i-1}}(x_i - x_{i-1}) dx_i] \psi_{t_n}(dz; x_n), \quad (6.1.11)$$

where, for $0 < t < U$, the un-normalised measure $\psi_t(dz; \xi)$ is given by

$$\psi_0(dz; \xi) = \nu(dz), \quad (6.1.12)$$

$$\psi_t(dz; \xi) = \frac{\rho_{U-t}(z - \xi)}{\rho_U(z)} \nu(dz). \quad (6.1.13)$$

We let

$$\psi_t(\mathbb{R}; \xi) = \int_{-\infty}^{\infty} \psi_t(dz; \xi). \quad (6.1.14)$$

Then, we have

$$\mathbb{P}[L_{tU} \in dy] = \rho_t(y) \psi_t(\mathbb{R}; y) dy. \quad (6.1.15)$$

For $0 \leq s < t < U$, the transition law of $\{L_{tU}\}$ is given by

$$\mathbb{P}[L_{tU} \in dy \mid L_{sU} = x] = \frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} \rho_{t-s}(y - x) dy, \quad (6.1.16)$$

$$\mathbb{P}[L_{UU} \in dy \mid L_{sU} = x] = \frac{\psi_s(dy; x)}{\psi_s(\mathbb{R}; x)}. \quad (6.1.17)$$

We shall consider LRBs with a continuous state-space. It can be shown that the LRBs are Markov processes. It is noted in [52] that all LRBs have stationary increments. However, in general, the increments of an LRB are not independent. This follows from the transition density (6.1.16). In [52] it is shown that the increments of an LRB are independent when the ratio

$$\frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} \quad (6.1.18)$$

is a function only of $t - s$ and $y - x$. In this case, the LRB is a Lévy process.

Hoyle [52] and Hoyle et al. [53] use LRBs to model the information processes in the BHM framework. Many of the results obtained in the case of Brownian bridge information processes and gamma bridge accumulation processes are generalized.

6.2 One-dimensional framework

In what follows, we shall use LRBs to construct information-sensitive pricing kernels. We note that related ideas have been considered by Macrina [69] and Akahori & Macrina [3]. The basic setup is very similar to that in Section 3.1. We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\mathbb{P}$ denotes the real probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the market filtration. For ease of exposition, we initially consider the case where a single random variable X_U which is revealed at time U . We assume that X_U has the *a priori* probability law ν . We recall that X_U plays the role of a fundamental factor. We model the associated information process $\{L_{tU}\}_{0 \leq t \leq U}$ by an LRB whose terminal value is X_U . Thus, L_{UU} has marginal law ν . We assume that the Lévy process $\{L_t\}_{0 \leq t \leq U}$, which generates the LRB, has density $\rho_t(x)$ for all $0 < t \leq U$. We construct the market filtration $\{\mathcal{F}_t\}$ as follows:

$$\mathcal{F}_t = \sigma(\{L_{sU}\}_{0 \leq s \leq t}). \quad (6.2.1)$$

In [52, 53], it is proved that there exists a measure $\mathbb{L}$ that is equivalent to $\mathbb{P}$ such that under $\mathbb{L}$, $\{L_{tU}\}_{0 \leq t < U}$ has the law of a Lévy process, and L_{tU} has density $\rho_t(x)$. Let $\{R_t\}_{0 \leq t < U}$ be a change-of-measure martingale defined by

$$R_t := \frac{d\mathbb{L}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \psi_t(\mathbb{R}; L_{tU})^{-1}. \quad (6.2.2)$$

Here,

$$\psi_t(\mathbb{R}; L_{tU}) = \int_{z=-\infty}^{\infty} \psi_t(dz; L_{tU}) = \int_{-\infty}^{\infty} \frac{\rho_{U-t}(z - L_{tU})}{\rho_U(z)} \nu(dz). \quad (6.2.3)$$

It is shown that $\{R_t\}$ is a suitable change-of-measure martingale in [52]. We include the following key calculation from [52]:

$$\begin{aligned} \mathbb{E}^{\mathbb{L}}[\psi_t(\mathbb{R}; L_{tU}) | \mathcal{F}_s] &= \mathbb{E}^{\mathbb{L}} \left[\int_{-\infty}^{\infty} \frac{\rho_{U-t}(z - L_{tU})}{\rho_U(z)} \nu(dz) \Big| L_{sU} \right] \\ &= \int_{y=-\infty}^{\infty} \int_{z=-\infty}^{\infty} \frac{\rho_{U-t}(z - L_{sU} - y)}{\rho_U(z)} \nu(dz) \rho_{t-s}(y) dy \\ &= \int_{z=-\infty}^{\infty} \frac{1}{\rho_U(z)} \int_{y=-\infty}^{\infty} \rho_{U-t}(z - L_{sU} - y) \rho_{t-s}(y) dy \nu(dz) \\ &= \int_{-\infty}^{\infty} \frac{\rho_{U-s}(z - L_{sU})}{\rho_U(z)} \nu(dz) \\ &= \psi_s(\mathbb{R}; L_{sU}). \end{aligned} \quad (6.2.4)$$

Therefore, $\{R_t^{-1}\}$ is a $(\{\mathcal{F}_t\}, \mathbb{L})$ -martingale. It follows that $\{R_t\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale, where $R_t > 0$ for $0 \leq t < U$ and $\mathbb{E}[R_t] = 1$.

Next, we assume the existence of a pricing kernel $\{\pi_t\}$, which we model by

$$\pi_t = R_t f(t, L_{tU}) \quad (6.2.5)$$

where $f(t, x)$ is a positive function and $\{R_t\}$ is given by (6.2.2). We consider the pricing of a discount bond with maturity $T < U$. By the bond pricing formula (2.1.4), and the Markov property of $\{L_{tU}\}$, we obtain

$$P_{tT} = \frac{\mathbb{E}[R_T f(T, L_{TU}) | L_{tU}]}{R_t f(t, L_{tU})}. \quad (6.2.6)$$

By changing measure, we can write

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{L}}[f(T, L_{TU}) | L_{tU}]}{f(t, L_{tU})}. \quad (6.2.7)$$

Since $\{L_{tU}\}$ is an $\mathbb{L}$ -Lévy process and $\{L_{tU}\}$ has density $\rho_t(x)$, the expression for the bond price reduces to the following integral:

$$P_{tT} = \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f(T, x) \rho_{T-t}(x - L_{tU}) dx. \quad (6.2.8)$$

The forward rate can be computed using the relation $r_{tT} = -\partial_T \ln P_{tT}$ and the short rate is given by $r_t = r_{tt}$. We note that it is sufficient to choose the function $f(t, x)$ such that the process $\{f(t, I_{tU})\}_{0 \leq t < U}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{L})$ -supermartingale, for the term structure of interest rates to remain non-negative. The weighted heat kernel approaches of [3] and [69] can be used for the construction of such $(\{\mathcal{F}_t\}, \mathbb{L})$ -supermartingales, and lead to explicit examples.

However, we may also be interested in constructing models of the type

$$\pi_t^R = R_t g(t, L_{tU}), \quad (6.2.9)$$

where $\{\pi_t^R\}$ is the real pricing kernel and $g(t, x)$ is a positive function. Since the real interest rate may be negative, we only require that $\{g(t, L_{tU})\}$ is a positive $(\{\mathcal{F}_t\}, \mathbb{L})$ -semimartingale. Here, we have much greater freedom in choosing the function $g(t, x)$. For this reason, at this stage, we impose very little structure on the function in the expression for the pricing kernel to keep the framework as general as possible.

6.3 A comparison in the Brownian case

To begin with, we recall the Hughston-Macrina model considered in Section 3.1. For convenience, in the discussion which follows, we restrict our attention to the one-dimensional setup. For the sake of comparison, we shall assume that the information process in this model is given by the Brownian random bridge (BRB)

$$L_{tU} = \frac{t}{U} X_U + \beta_{tU}, \quad (6.3.1)$$

where X_U is a random variable with *a priori* law ν , $\{\beta_{tU}\}_{0 \leq t \leq U}$ is an independent Brownian bridge, and $L_{UU} = X_U$. Here, the implicit information flow rate is $\sigma = 1/U$. Then the Hughston-Macrina pricing kernel is modelled by¹

$$\pi_t^{\text{HM}} = M_t f(t, L_{tU}), \quad (6.3.2)$$

where $f(t, x)$ is a positive function and $\{M_t\}_{0 \leq t < U}$ is given by

$$M_t = \left(\int_{-\infty}^{\infty} e^{\frac{U}{U-t}(\sigma z L_{tU} - \frac{1}{2} \sigma^2 z^2 t)} \nu(dz) \right)^{-1}, \quad (6.3.3)$$

where $\sigma = 1/U$. We recall that the price of a discount bond with maturity T is given by

$$P_{tT}^{\text{HM}} = \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f\left(T, \nu_{tT} y + \frac{U-T}{U-t} L_{tU}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2\right) dy. \quad (6.3.4)$$

¹For clarity, we use the superscript HM to indicate quantities related to the Hughston-Macrina approach.

We now construct a pricing kernel model based on the BRB (6.3.1) using the LRB approach in Section 6.2. The density of the Lévy process which generates the LRB $\{L_{tU}\}$ is given by

$$\rho_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right). \quad (6.3.5)$$

We model the pricing kernel by²

$$\pi_t^L = R_t f(t, L_{tU}) \quad (6.3.6)$$

where $\{R_t\}_{0 \leq t < U}$ is given by

$$R_t = \left(\sqrt{\frac{U}{U-t}} \int_{-\infty}^{\infty} e^{-\frac{(z-L_{tU})^2}{2(U-t)} + \frac{z^2}{2U}} \nu(dz) \right)^{-1} \quad (6.3.7)$$

from equations (6.2.2), (6.2.3) and (6.3.5). It follows from (6.2.8) that the discount bond price is

$$P_{tT}^L = \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f(T, y\sqrt{T-t} + L_{tU}) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy. \quad (6.3.8)$$

In both models, we can derive the dynamics of the pricing kernel and obtain expressions for the short rate and market price of risk from the drift and volatility of the pricing kernel.

It is worth noting that in the Brownian case, the pricing kernel model (6.3.6) is conceptually very similar to the Hughston-Macrina model (6.3.2). In both approaches, a convenient auxiliary measure is introduced in order to simplify the calculation of the conditional expectations. The key structural difference lies in the martingales $(\{M_t\}; \{R_t\})$ used and the measures $(\mathbb{B}; \mathbb{L})$ induced. The martingale $\{M_t\}$ induces a change of measure to $\mathbb{B}$ under which the BRB $\{L_{tU}\}$ is a Brownian bridge. On the other hand, the martingale $\{R_t\}$ induces a change of measure to $\mathbb{L}$ under which the BRB $\{L_{tU}\}$ is a Brownian motion. The precise relationship between these change-of-measure martingales is given in the following proposition.

Proposition 6.3.1. *The change-of-measure martingales $\{R_t\}$, given by (6.3.7), and $\{M_t\}$, given by (6.3.3) with $\sigma = 1/U$, are related as follows:*

$$R_t = \sqrt{\frac{U-t}{U}} e^{\frac{L_{tU}^2}{2(U-t)}} M_t. \quad (6.3.9)$$

²We use the superscript L to indicate quantities that are associated with the LRB approach described in Section 6.2.

Proof. We note that if $\sigma = 1/U$ then

$$\exp \left[-\frac{(z - L_{tU})^2}{2(U-t)} + \frac{z^2}{2U} \right] = \exp \left[\frac{U}{U-t} \left(\sigma L_{tU} z - \frac{1}{2} \sigma^2 z^2 t \right) - \frac{L_{tU}^2}{2(U-t)} \right]. \quad (6.3.10)$$

Therefore, it follows that

$$\begin{aligned} R_t &= \left(\sqrt{\frac{U}{U-t}} \int_{-\infty}^{\infty} e^{\frac{U}{U-t} (\sigma L_{tU} z - \frac{1}{2} \sigma^2 z^2 t)} e^{-\frac{L_{tU}^2}{2(U-t)}} \nu(dz) \right)^{-1} \\ &= \sqrt{\frac{U-t}{U}} e^{\frac{L_{tU}^2}{2(U-t)}} M_t. \end{aligned} \quad (6.3.11)$$

□

To verify this relationship, we consider the pricing of a discount bond. In the Hughston-Macrina approach, we start off by determining the following expectation:

$$P_{tT}^{\text{HM}} = \frac{\mathbb{E}[M_T f(T, L_{TU}) | L_{tU}]}{M_t f(t, L_{tU})}. \quad (6.3.12)$$

Instead of using $\{M_t\}$ to induce a change of measure to $\mathbb{B}$, we can use (6.3.9) to write

$$P_{tT}^{\text{HM}} = \frac{\mathbb{E} \left[R_T \sqrt{\frac{U}{U-T}} e^{-\frac{L_{TU}^2}{2(U-T)}} f(T, L_{TU}) | L_{tU} \right]}{R_t \sqrt{\frac{U}{U-t}} e^{-\frac{L_{tU}^2}{2(U-t)}} f(t, L_{tU})}. \quad (6.3.13)$$

We now use the methodology proposed in Section 6.2; that is we change measure to $\mathbb{L}$ to obtain

$$P_{tT}^{\text{HM}} = \frac{\mathbb{E}^{\mathbb{L}} \left[\sqrt{\frac{U}{U-T}} e^{-\frac{L_{TU}^2}{2(U-T)}} f(T, L_{TU}) | L_{tU} \right]}{\sqrt{\frac{U}{U-t}} e^{-\frac{L_{tU}^2}{2(U-t)}} f(t, L_{tU})}. \quad (6.3.14)$$

Under $\mathbb{L}$, the BRB $\{L_{tU}\}$ has the law of a Brownian motion. This allows us to express the discount bond price as the following integral:

$$\begin{aligned} P_{tT}^{\text{HM}} &= \frac{\sqrt{\frac{U-t}{U-T}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(U-T)}} f(T, x) \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-L_{tU})^2}{2(T-t)}} dx}{e^{-\frac{L_{tU}^2}{2(U-t)}} f(t, L_{tU})} \\ &= \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \nu_{tT} f(T, x) e^{-\frac{x^2}{2(U-T)} - \frac{(x-L_{tU})^2}{2(T-t)} + \frac{L_{tU}^2}{2(U-t)}} dx, \end{aligned} \quad (6.3.15)$$

where

$$\nu_{tT}^2 = \frac{(U-T)(T-t)}{U-t}. \quad (6.3.16)$$

By the variable change $x = \nu_{tT}y + (U-T)/(U-t)L_{tU}$, we obtain precisely equation (6.3.4). In doing so, we have shown how the approach proposed in Section 6.2 in the Brownian case relates to the results obtained previously using the Hughston-Macrina framework.

6.4 1/2-stable random bridge model

To demonstrate how the LRB approach in Section 6.2 can be used, we consider a simple example. We construct an interest rate model that is sensitive to an accumulation process which we model by a 1/2-stable random bridge (SRB). The reader is referred to [52] for further details on SRBs. Let $\{L_t\}$ be a 1/2-stable subordinator³ with activity parameter $c > 0$. Then the random variable L_t has a Lévy distribution with density

$$\rho_t(x) = \mathbb{1}_{\{x>0\}} \frac{ct}{\sqrt{2\pi}x^{\frac{3}{2}}} \exp\left(-\frac{1}{2} \frac{c^2 t^2}{x}\right). \quad (6.4.1)$$

Let $\{L_{tU}\}$ be an SRB $([0, U], \rho_t, \nu)$ where $L_{UU} = X_U$ has marginal law ν . We model the pricing kernel by equation (6.2.5). From equation (6.2.8), it follows that the price of a discount bond with maturity T is given by

$$P_{tT} = \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f(T, x) \mathbb{1}_{\{x-L_{tU}>0\}} \frac{c(T-t)}{\sqrt{2\pi}(x-L_{tU})^{\frac{3}{2}}} \exp\left(-\frac{1}{2} \frac{c^2(T-t)^2}{x-L_{tU}}\right) dx. \quad (6.4.2)$$

Suppose that we choose

$$f(t, x) = \frac{e^{-(\alpha+\beta)t}}{\alpha+\beta} + \frac{e^{-\alpha t - \gamma x}}{\alpha + c\sqrt{2\gamma}}, \quad (6.4.3)$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are constant parameters. Then, the discount bond price at $t \leq T$ simplifies to

$$P_{tT} = \frac{1}{f(t, L_{tU})} \left(\frac{e^{-(\alpha+\beta)T}}{\alpha+\beta} + \frac{e^{-(\alpha+c\sqrt{2\gamma})T+c\sqrt{2\gamma}t-\gamma L_{tU}}}{\alpha + c\sqrt{2\gamma}} \right). \quad (6.4.4)$$

We can determine the short rate by using the relation $r_t = -\partial_T \ln(P_{tT})|_{T=t}$:

$$r_t = \frac{(e^{\beta t} + e^{\gamma L_{tU}})(\alpha + \beta)(\alpha + c\sqrt{2\gamma})}{e^{\beta t}(\alpha + \beta) + e^{\gamma L_{tU}}(\alpha + c\sqrt{2\gamma})}. \quad (6.4.5)$$

³A subordinator is a Lévy process that is non-decreasing almost surely (see e.g. Applebaum [5]).

We note that the above construction is a non-negative interest rate model. We verify this by showing that for $t < U$, $\{f(t, L_{tU})\}$ is a $(\{\mathcal{F}_t\}, \mathbb{L})$ -supermartingale. In particular, for $s \leq t < U$,

$$\mathbb{E}^{\mathbb{L}}[f(t, L_{tU}) | L_{sU}] = \frac{e^{-(\alpha+\beta)t}}{\alpha + \beta} + \frac{e^{-\alpha t - \gamma L_{sU}}}{\alpha + c\sqrt{2\gamma}} \mathbb{E}^{\mathbb{L}}[e^{-\gamma(L_{tU} - L_{sU})} | L_{sU}] \quad (6.4.6)$$

We recall that $\{L_{tU}\}$ has the law of its generating Lévy process under the measure $\mathbb{L}$. That is, the SRB has the law of a 1/2-stable subordinator under $\mathbb{L}$. We note that if $\{S_t\}$ is a subordinator, then

$$\mathbb{E}[e^{-qS_t}] = e^{-t\Phi(q)}, \quad (6.4.7)$$

where $q \geq 0$ and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the Laplace exponent (or Bernstein function) of the subordinator. In the case of a 1/2-stable process with parameter c , we have that $\Phi(q) = c\sqrt{2q}$. Thus, for $s \leq t < U$, we can write

$$\begin{aligned} \mathbb{E}^{\mathbb{L}}[f(t, L_{tU}) | L_{sU}] &= \frac{e^{-(\alpha+\beta)t}}{\alpha + \beta} + \frac{e^{-\alpha t - \gamma L_{sU} - (t-s)c\sqrt{2\gamma}}}{\alpha + c\sqrt{2\gamma}} \\ &\leq \frac{e^{-(\alpha+\beta)s}}{\alpha + \beta} + \frac{e^{-\alpha s - \gamma L_{sU}}}{\alpha + c\sqrt{2\gamma}} \\ &= f(s, L_{sU}). \end{aligned} \quad (6.4.8)$$

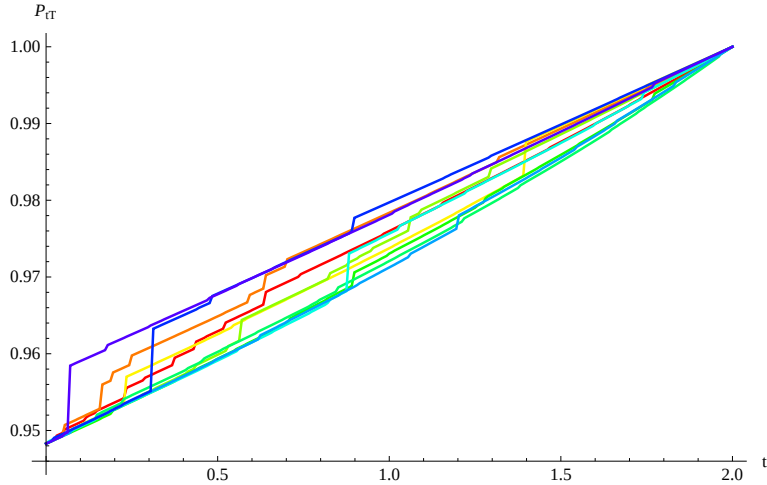


Figure 6.1: Discount bond price trajectories for 1/2-stable random bridge model. We set $T = 2$, $U = 4$, $c = 1$, $\alpha = 0.01$, $\beta = 0.01$, $\gamma = 1$. We assume that L_{UU} has a generalized Pareto distribution with scale parameter $\Sigma = 2$, location parameter $\mu = 3$ and shape parameter $\xi = 0.25$.

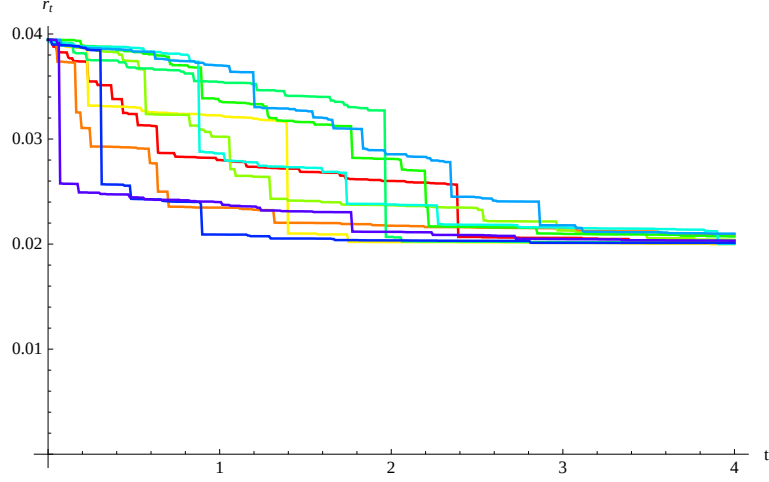


Figure 6.2: Short rate trajectories for 1/2-stable random bridge model. We set $U = 4$, $c = 1$, $\alpha = 0.01$, $\beta = 0.01$, $\gamma = 1$. We assume that L_{UU} has a generalized Pareto distribution with scale parameter $\Sigma = 2$, location parameter $\mu = 3$ and shape parameter $\xi = 0.25$.

In Figures 6.1 and 6.2, we have simulated sample paths for the discount bond price and the short rate given by equations (6.4.4) and (6.4.5).⁴

6.5 Bond option

Let $\{C_{st}\}_{0 \leq s \leq t < T < U}$ be the price process of a European style call option with maturity t and strike K on a bond with price process $\{P_{tT}\}_{0 \leq t \leq T < U}$. Then, we have

$$C_{st} = \frac{1}{\pi_s} \mathbb{E} [\pi_t (P_{tT} - K)^+ | \mathcal{F}_s]. \quad (6.5.1)$$

We recall that $\{L_{tU}\}$ is a Markov process. By substituting (6.2.5) for the pricing kernel and (6.2.8) for the discount bond price, we obtain

$$C_{st} = \frac{\mathbb{E} \left[R_t f(t, L_{tU}) \left(\frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f(T, x) \rho_{T-t}(x - L_{tU}) dx - K \right)^+ \middle| L_{sU} \right]}{R_s f(s, L_{sU})}. \quad (6.5.2)$$

Next, we use a change of measure to write

$$C_{st} = \frac{1}{f(s, L_{sU})} \mathbb{E}^{\mathbb{L}} \left[\left(\int_{-\infty}^{\infty} f(T, x) \rho_{T-t}(x - L_{tU}) dx - f(t, L_{tU}) K \right)^+ \middle| L_{sU} \right]. \quad (6.5.3)$$

⁴We have made use of *Mathematica* code written by A. E. V. Hoyle for the simulation of the 1/2-stable random bridge.

We recall that under $\mathbb{L}$, $\{L_{tU}\}$ is a Lévy process with density ρ_t . This allows us to write

$$C_{st} = \frac{1}{f(s, L_{sU})} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(T, x) \rho_{T-t}(x - y) dx - f(t, y)K \right)^+ \rho_{t-s}(y - L_{sU}) dy. \quad (6.5.4)$$

This is equivalent to

$$C_{st} = \frac{1}{f(s, L_{sU})} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [f(T, x) - f(t, y)K] \rho_{T-t}(x - y) dx \right)^+ \rho_{t-s}(y - L_{sU}) dy. \quad (6.5.5)$$

We define

$$A_t = \left\{ y \in \mathbb{R}; \int_{-\infty}^{\infty} [f(T, x) - f(t, y)K] \rho_{T-t}(x - y) dx > 0 \right\}. \quad (6.5.6)$$

Then, we obtain

$$C_{st} = \frac{1}{f(s, L_{sU})} \int_{A_t} \int_{-\infty}^{\infty} [f(T, x) - f(t, y)K] \rho_{T-t}(x - y) dx \rho_{t-s}(y - L_{sU}) dy. \quad (6.5.7)$$

6.6 Multi-dimensional framework

The ideas in Section 6.2 also lend themselves to a more general multi-dimensional setting as in Chapter 3. We now consider a set of fixed dates $U_1 \leq U_2 \leq \dots \leq U_n$. We assume that at each date U_k ($k = 1, \dots, n$), the value of a random variable X_{U_k} is revealed. The random variables are assumed to be mutually independent. We assume that X_{U_k} has the *a priori* probability law ν_k . Each random variable X_{U_k} is associated with an independent LRB $\{L_{tU_k}\}_{0 \leq t \leq U_k}$ with $L_{U_k U_k} = X_{U_k}$. In addition, we assume that each LRB $\{L_{tU_k}\}$ is generated by an independent Lévy process $\{L_t^{(k)}\}$ with density $\rho_t^{(k)}(x)$ for all $0 \leq t \leq U_k$. We construct the market filtration $\{\mathcal{F}_t\}$ as follows:

$$\mathcal{F}_t = \sigma(\{L_{sU_1}\}_{0 \leq s \leq t}, \dots, \{L_{sU_n}\}_{0 \leq s \leq t}). \quad (6.6.1)$$

We now model the pricing kernel by

$$\pi_t = R_t^{(1)} \dots R_t^{(n)} \cdot f(t, L_{tU_1}, \dots, L_{tU_n}) \quad (6.6.2)$$

where $f(t, x_1, \dots, x_n)$ is a positive function and, $\{R_t^{(k)}\}_{0 \leq t < U_k}$ ($k = 1, \dots, n$) is given by

$$R_t^{(k)} = \psi_t^k(\mathbb{R}; L_{tU_k})^{-1}. \quad (6.6.3)$$

Each $\{R_t^{(k)}\}$ is a martingale which induces a measure change to an auxiliary measure $\mathbb{L}^k$, under which $\{L_{tU_k}\}$ has the law of $\{L_t^{(k)}\}$. Here, we recall that

$$\psi_0^k(dz; \xi) = \nu_k(dz), \quad (6.6.4)$$

$$\psi_t^k(dz; \xi) = \frac{\rho_{U_k-t}^{(k)}(z - \xi)}{\rho_{U_k}^{(k)}(z)} \nu_k(dz), \quad (6.6.5)$$

and

$$\psi_t^k(\mathbb{R}; L_{tU_k}) = \int_{z=-\infty}^{\infty} \psi_t^k(dz; L_{tU_k}) = \int_{z=-\infty}^{\infty} \frac{\rho_{U_k-t}^{(k)}(z - L_{tU_k})}{\rho_{U_k}^{(k)}(z)} \nu_k(dz). \quad (6.6.6)$$

Proposition 6.6.1. *The process $\{R_t\}_{0 \leq t < U_1}$ defined by*

$$R_t := R_t^{(1)} \dots R_t^{(n)} \quad (6.6.7)$$

is a $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale.

Proof. The proof is similar to that of Proposition 3.1.1. $\square$

We observe that $R_t > 0$ for $0 \leq t < U_1$ and satisfies $\mathbb{E}[R_t] = 1$. The martingale $\{R_t\}_{0 \leq t < U_1}$ can be used to bring about a change of measure to a “master” measure $\mathbb{L}$ under which each of the independent LRBs has the law of its generating Lévy process. In this setup, the price of a discount bond is given by

$$P_{tT} = \frac{\mathbb{E}[R_T^{(1)} \dots R_T^{(n)} f(T, L_{TU_1}, \dots, L_{TU_n}) \mid L_{tU_1}, \dots, L_{tU_n}]}{R_t^{(1)} \dots R_t^{(n)} f(t, L_{tU_1}, \dots, L_{tU_n})}, \quad (6.6.8)$$

since each $\{L_{tU_k}\}$ ($k = 1, \dots, n$) is a Markov process. By applying Bayes’ formula, we obtain

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{L}}[f(T, L_{TU_1}, \dots, L_{TU_n}) \mid L_{tU_1}, \dots, L_{tU_n}]}{f(t, L_{tU_1}, \dots, L_{tU_n})}. \quad (6.6.9)$$

By the independence of the LRBs and since, under $\mathbb{L}$, each $\{L_{tU_k}\}$ has the density $\rho_t^{(k)}(x)$, we can express the bond price as

$$P_{tT} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{f(T, x_1, \dots, x_n)}{f(t, L_{tU_1}, \dots, L_{tU_n})} \prod_{k=1}^n \rho_{T-t}^{(k)}(x_k - L_{tU_k}) dx_1 \dots dx_n. \quad (6.6.10)$$

6.7 Credit-risky bond

We can also use the LRB pricing kernel framework for the pricing of credit-risky securities. For conciseness, we shall restrict our attention to the pricing of a credit-risky discount bond. However, since this approach generalizes the arguments in Chapter 4, it can also be used for the pricing of a variety of other instruments.

Once again, we shall assume that X_U is a fundamental factor with an arbitrary *a priori* probability distribution. The value of X_U is revealed at a fixed time U . Let $T < U$. We also introduce an independent idiosyncratic random variable X_T associated with a debt issuer. We associate the independent information processes $\{L_{tU}\}_{0 \leq t \leq U}$ and $\{L_{tT}\}_{0 \leq t \leq T}$ with X_U and X_T . We assume that $\{L_{tU}\}$ is an LRB($[0, U], \rho_t, \nu$), where $L_{UU} = X_U$, and $\{L_{tT}\}$ is an LRB($[0, T], \varrho_t, \nu$) where $L_{TT} = X_T$. We now construct the market filtration $\{\mathcal{F}_t\}$ as follows:

$$\mathcal{F}_t = \sigma(\{L_{sU}\}_{0 \leq s \leq t}, \{L_{sT}\}_{0 \leq s \leq t}). \quad (6.7.1)$$

We model the pricing kernel as in equation (6.2.5), where $\{R_t\}$ is defined by (6.2.2). We recall that $\{R_t\}$ induces a change of measure to $\mathbb{L}$, under which the LRB $\{L_{tU}\}$ has the distribution of a Lévy process with density ρ_t .

We now consider a credit-risky bond with maturity $T < U$. We assume that the payoff H_T of the credit-risky bond depends on the structural variable X_T and the market information about X_U available at time t ; that is

$$H_T = H(X_T, L_{tU}) \quad (6.7.2)$$

where $H : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$.

Proposition 6.7.1. *The price of the credit-risky discount bond is*

$$B_{tT} = \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f(T, x) \int_{-\infty}^{\infty} H(z, x) v_t(dz) \rho_{T-t}(x - L_{tU}) dx, \quad (6.7.3)$$

where

$$v_t(dz) := \mathbb{P}[L_{tT} \in dz \mid L_{tU}] = \frac{\bar{\psi}_t(dz; L_{tT})}{\bar{\psi}_t(\mathbb{R}; L_{tT})}, \quad (6.7.4)$$

and

$$\bar{\psi}_t(dz; L_{tT}) = \frac{\varrho_{T-t}(z - L_{tT})}{\varrho_T(z)} v(dz), \quad \bar{\psi}_t(\mathbb{R}; L_{tT}) = \int_{z=-\infty}^{\infty} \psi_t(dz; L_{tT}). \quad (6.7.5)$$

Proof. The expression for the price of the credit-risky discount bond is

$$B_{tT} = \frac{\mathbb{E}[R_T f(T, L_{tU}) H(X_T, L_{tU}) \mid \mathcal{F}_t]}{R_t f(t, L_{tU})}. \quad (6.7.6)$$

By the Tower property, we can write

$$B_{tT} = \frac{\mathbb{E}[\mathbb{E}[R_T f(T, L_{tU}) H(X_T, L_{tU}) \mid \sigma(\mathcal{F}_t, X_T)] \mid \mathcal{F}_t]}{R_t f(t, L_{tU})}. \quad (6.7.7)$$

We note that the LRBs $\{L_{tU}\}$ and $\{L_{tT}\}$ are Markov processes. We can now perform a change of measure in the inner conditional expectation. Then, we have

$$B_{tT} = \frac{\mathbb{E}[\mathbb{E}^{\mathbb{L}}[f(T, L_{tU}) H(X_T, L_{tU}) \mid L_{tU}, L_{tT}, X_T] \mid L_{tU}, L_{tT}]}{f(t, L_{tU})}. \quad (6.7.8)$$

Since the random variable L_{TU} is independent of L_{tT} and X_T , and since $\{L_{tU}\}$ has the distribution of its generating Lévy process under $\mathbb{L}$, the inner expectation can be expressed in integral form:

$$B_{tT} = \frac{\mathbb{E} \left[\int_{-\infty}^{\infty} f(T, x) H(X_T, x) \rho_{T-t}(x - L_{tU}) dx \mid L_{tU}, L_{tT} \right]}{f(t, L_{tU})}. \quad (6.7.9)$$

Furthermore, since v_t is the time t conditional law of L_{TT} , we can write

$$B_{tT} = \frac{1}{f(t, L_{tU})} \int_{-\infty}^{\infty} f(T, x) \int_{-\infty}^{\infty} H(z, x) v_t(dz) \rho_{T-t}(x - L_{tU}) dx. \quad (6.7.10)$$

where $v_t(dz)$ is given by (6.7.4). $\square$

6.7.1 Binary bond based on VGRB and BRB

We observe that if the payoff of the credit-risky bond is $H_T = H(X_T)$ where $H : \mathbb{R} \rightarrow [0, 1]$, the expression (6.7.3) factorizes into two independent components, i.e.

$$B_{tT} = P_{tT} \int_{-\infty}^{\infty} H(z) v_t(dz), \quad (6.7.11)$$

where the discount bond price is given by (6.2.8). As an example, we consider the pricing of a binary bond.

We model $\{L_{tT}\}$ by a variance-gamma random bridge (VGRB). We refer the reader to [52] for a detailed treatment of VGRBs. Here, the density of the generating Lévy process is given by

$$\varrho_t(x) = \sqrt{\frac{2}{\pi}} \frac{m^{mt}}{\Gamma[mt]} \left(\frac{x^2}{2m} \right)^{\frac{mt}{2} - \frac{1}{4}} K_{mt - \frac{1}{2}} \left[\sqrt{2mx^2} \right] \quad (6.7.12)$$

where $m > 0$ and is chosen such that $T > (2m)^{-1}$ for $0 < \varrho_T(x) < \infty$. Here, $\Gamma[\cdot]$ is the Gamma function and $K[\cdot]$ denotes the modified Bessel function of the third kind. We assume that the random variable $L_{TT} = X_T$ take the values $\{0, \sigma\}$ with *a priori* probabilities $\{p, 1 - p\}$ respectively. It is shown in [52] that if the payoff of the credit-risky bond is modelled by $H(X_T) = \frac{X_T}{\sigma}$, then the price of the bond is given by

$$B_{tT} = P_{tT} \left(1 + \frac{\varrho_T(\sigma) \varrho_{T-t}(-L_{tT}) p}{\varrho_T(0) \varrho_{T-t}(\sigma - L_{tT}) (1 - p)} \right)^{-1}. \quad (6.7.13)$$

We now assume that the evolution of stochastic interest rates is modelled using the quadratic model of Akahori & Macrina [3]. Let

$$L_{tU} = \sigma_U t X_U + \beta_{tU} \quad (6.7.14)$$

be a Brownian bridge information process. We model the discount factor P_{tT} in (6.7.13) by

$$P_{tT} = \frac{\frac{1}{12}(U-T)^3 + \frac{1}{4}\frac{(T-t)(U-T)^3}{(U-t)} + \frac{1}{4}\frac{(U-T)^4}{(U-t)^2}L_{tU}^2}{\frac{1}{12}(U-t)^3 + \frac{1}{4}(U-t)^2L_{tU}^2}. \quad (6.7.15)$$

In the following figures, we have simulated one sample path for the Brownian bridge information process $\{L_{tU}\}$ defined by (6.7.14) in order to model one possible stochastic interest rate scenario. Each of the five paths of the LRB $\{L_{tT}\}$, relate to different scenarios under which the binary bond defaults.⁵

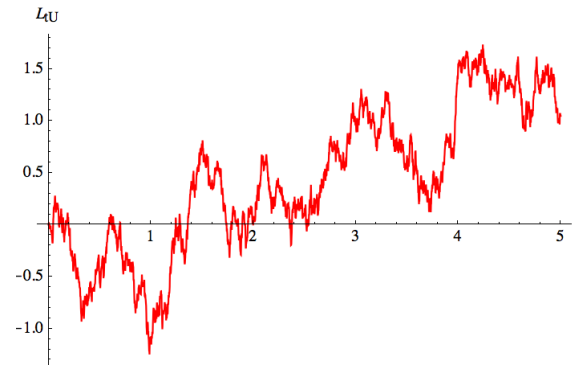


Figure 6.3: Sample path of Brownian bridge information process $\{L_{tU}\}$, where $U = 5$, X_U is *a priori* a standard normal random variable, and $\sigma_U = 0.5$.

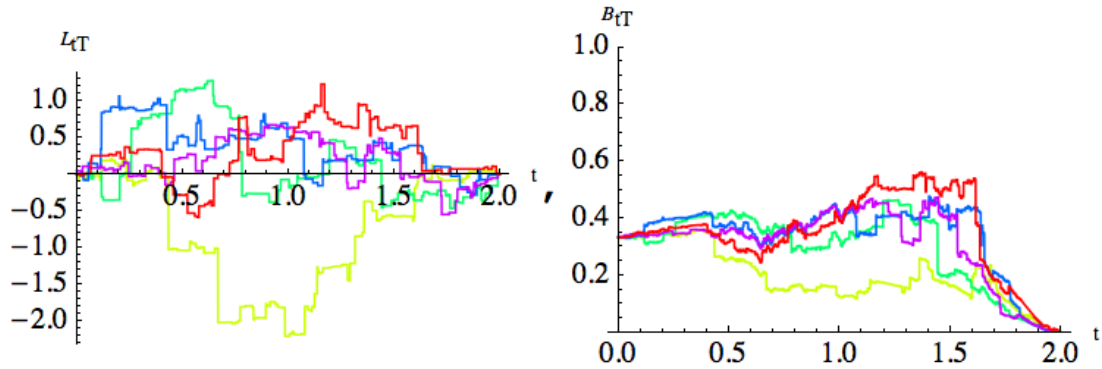


Figure 6.4: Sample paths of the VGRB $\{L_{tT}\}$ and a binary bond that is destined to default, where interest rates are modelled using the quadratic model. We let $T = 2$, $U = 5$, $\sigma_U = 0.5$, $p = 0.3$, $m = 10$ and $\sigma = 1$.

⁵In Figure 6.4, we have adapted *Mathematica* code written by A.E.V. Hoyle for the simulation of the VGRB and the binary bond.

In this chapter we have restricted our attention to LRBs with a continuous state space. However, analogous results hold for discrete state-space processes known as discrete random bridges (DRBs); see [52], where a Poisson random bridge, for example, is studied in detail. Therefore, in principle, we can also generate models for discount bonds and credit-risky bonds, which are sensitive to information modelled by a DRB. Further research may include the construction of explicit partial information pricing kernel models based on DRBs. It is interesting to note that such models may be natural candidates for credit risk applications since discrete state Markov processes, i.e. Markov chains, have been used extensively in the credit risk literature, for instance in the modelling of credit rating migrations. It may also be worthwhile investigating connections with Hidden Markov models for credit quality.

Chapter 7

Randomised mixture models

7.1 Summary

The material in this chapter appears in Macrina & Parbhoo [71] — however, additional insights are provided in Section 7.2. We develop interest rate models that offer consistent dynamics in the short, medium, and long term. Often interest rate models have valid dynamics in the short term, that is to say, over days or perhaps a few weeks. Such models may be appropriate for the pricing of securities with short time-to-maturity. For financial assets with long-term maturities, one requires interest rate models with plausible long-term dynamics, which retain their validity over years. Thus the question arises as to how one can create interest rate models which are sensitive to market changes over both short and long time intervals, so that they remain useful for the pricing of securities of various tenors. Ideally, one would have at one's disposal interest rate models that allow for consistent pricing of financial instruments expiring within a range of a few minutes up to years, and if necessary over decades. One can imagine an investor holding a portfolio of securities maturing over various periods of time, perhaps spanning several years. Another situation requiring interest rate models that are valid over short and long terms, is where illiquid long-term fixed-income assets need to be replicated with (rolled-over) liquid shorter-term derivatives. Here it is central that the underlying interest rate model possesses consistent dynamics over all periods of time in order to avoid substantial hedging inaccuracy. Insurance companies, or pension funds, holding liabilities over decades might have no other means but to invest in shorter-term derivatives, possibly with maturities of months or a few years, in order to secure enough collateral for their long-term liabilities reserves. Furthermore, such hedges might in turn need second-order liquid short-term protection, and so forth. Applying different interest rate models validated for the various investment periods, which frequently do not guarantee price and hedging consistency, seems undesirable. Instead, we propose a family of pricing kernel

models which may generate interest rate dynamics sufficiently flexible to allow for diverse behaviour over short, medium and long periods of time.

We imagine economies, and their associated financial markets, that are exposed to a variety of uncertainties, such as economic, social, political, environmental, or demographic ones. We model the degree of impact of these underlying factors on an economy (and financial markets) at each point in time by combinations of continuous-time stochastic processes of different probability laws. When designing interest rate models that are sensitive to the states an economy may take, subject to its response to the underlying uncertainty factors, one may wonder (i) how many stochastic factor processes ought to be considered, and (ii) what is the combination, or mixture, of factor processes determining the dynamics of an economy and its associated financial market. It is plausible to assume that the number of stochastic factors and their combined impact on a financial market continuously changes over time, and thus that any interest rate model designed in such a set-up is by nature time-inhomogeneous. The recipe used to construct interest rate models within the framework proposed in this chapter can be summarised as follows:

- (i) Assume that the response of a financial market to uncertainty is modelled by a family of stochastic processes, e.g. Markov processes.
- (ii) Consider a mixture of such stochastic processes as the basic driver of the resulting interest rate models.
- (iii) In order to explicitly design interest rate models, apply a method for the modelling of the pricing kernel associated with the economy, which underlies the considered financial market.
- (iv) Derive the interest rate dynamics directly from the pricing kernel models, or, if more convenient, deduce the interest rate model from the bond price process associated with the constructed pricing kernel.

The set of stochastic processes chosen to model an economy's response to uncertainty, the particular mixture of those, and the pricing kernel model jointly characterize the dynamics of the derived interest rate model. We welcome these degrees of freedom, for any one of them may abate the shortcoming (or may amplify the virtues) of another. For example, one might be constrained to choose Lévy processes to model the impact of uncertainty on markets. The fact that Lévy processes are time-homogeneous processes with independent increments, might be seen as a disadvantage for modelling interest rates for long time spans. However, a time-dependent pricing kernel function may later introduce time-inhomogeneity in the resulting interest rate model. The choice of a certain set of stochastic processes

implicitly determines a particular joint law of the modelled market response to the uncertainty sources. Although the resulting multivariate law may not coincide well with the law of the combined uncertainty impact, the fact that we can directly model a particular mixture of stochastic processes provides the desirable degree of freedom in order to control the dynamical law of the market's response to uncertainty. In this chapter, we consider “randomised mixing functions” for the construction of multivariate interest rate models with distinct response patterns to short-, medium-, and long-term uncertainties. Having a randomised mixing function enables us to introduce the concept of “partially-observable mixtures” of stochastic processes. We take the view that market agents cannot fully observe the actual combination of processes underlying the market. Instead they form best estimates of the randomised mixture given the information they possess; these estimates are continuously updated as time elapses. This feature introduces a feedback effect in the constructed pricing models.

Once again, the reason why we prefer to propose pricing kernel models in order to generate the dynamics of interest rates, as opposed to modelling the interest rates directly, is that the modelling of the pricing kernel offers an integrated approach to equilibrium asset pricing in general (see Cochrane [27], Duffie [32]), including risk management and thus the quantification of risk involved in an investment. The pricing kernel includes the *quantified* total response to the uncertainties affecting an economy or, in other words, the risk premium asked by an investor as an incentive for investing in risky assets. Our goal in this chapter is primarily to introduce a framework capable of addressing issues arising in interest rate modelling over short to long term time intervals. We first apply our ideas to the Flesaker-Hughston class of pricing kernels (see Flesaker & Hughston [36], Hunt & Kennedy [58], Cairns [25], Brigo & Mercurio [13]). We then conclude the chapter by introducing randomised weighted heat kernel models, along the lines of Akahori et al. [2] and Akahori & Macrina [3], which extend the class of pricing kernels developed in the first part of this chapter.

7.2 Randomised Esscher martingales

We begin by introducing the mathematical tools that we shall use to construct pricing kernel models based on randomised mixtures of Lévy processes. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ denotes the real probability measure.

Definition 7.2.1. *Let $\{L_t\}_{t \geq 0}$ be an n -dimensional Lévy process with independent components, and let $X : \Omega \rightarrow \mathbb{R}^m$ be an independent, m -dimensional vector of random variables. For $t, u \in \mathbb{R}_+$, the process $\{M_{tu}(X)\}$ is defined by*

$$M_{tu}(X) = \frac{\exp(h(u, X)L_t)}{\mathbb{E}[\exp(h(u, X)L_t) | X]}, \quad (7.2.1)$$

where $h : \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a measurable function such that $\mathbb{E}[|M_{tu}(X)|] < \infty$ for all $t \in \mathbb{R}_+$.

Proposition 7.2.1. *Let the filtration $\{\mathcal{H}_t\}_{t \geq 0}$ be given by $\mathcal{H}_t = \sigma(\{L_s\}_{0 \leq s \leq t}, X)$. Then the process $\{M_{tu}(X)\}$ is an $(\{\mathcal{H}_t\}, \mathbb{P})$ -martingale.*

We note that X is $\mathcal{H}_0$ -measurable and therefore, that $\{\mathcal{H}_t\}$ is an initial enlargement of the natural filtration of $\{L_t\}$ by the random variable X . Furthermore, $M_{0u}(X) = 1$ and $M_{tu}(X) > 0$ for all $t, u \in \mathbb{R}_+$.

Proof. The condition that $\mathbb{E}[|M_{tu}(X)|]$ be finite for all $0 \leq t < \infty$ is ensured by definition. It remains to be shown that

$$\mathbb{E}[M_{tu}(X) | \mathcal{H}_s] = M_{su}(X) \quad (7.2.2)$$

for all $0 \leq s \leq t < \infty$. We observe that the denominator in (7.2.1) is $\mathcal{H}_0$ -measurable so that we can write

$$\mathbb{E}[M_{tu}(X) | \mathcal{H}_s] = \frac{\mathbb{E}[\exp(h(u, X)L_t) | \mathcal{H}_s]}{\mathbb{E}[\exp(h(u, X)L_t) | X]}. \quad (7.2.3)$$

Next we expand the right-hand-side of the above equation to obtain

$$\frac{\mathbb{E}[\exp[h(u, X)(L_t - L_s)] \exp[h(u, X)L_s] | \mathcal{H}_s]}{\mathbb{E}[\exp[h(u, X)(L_t - L_s)] \exp[h(u, X)L_s] | X]}. \quad (7.2.4)$$

Given X , the expectation in the denominator factorizes since $L_t - L_s$ is independent of L_s . In addition, the factor $\exp[h(u, X)L_s]$ is $\mathcal{H}_s$ -measurable so that we may write

$$\mathbb{E}[M_{tu}(X) | \mathcal{H}_s] = \frac{\exp[h(u, X)L_s]}{\mathbb{E}[\exp[h(u, X)L_s] | X]} \frac{\mathbb{E}[\exp[h(u, X)(L_t - L_s)] | \mathcal{H}_s]}{\mathbb{E}[\exp[h(u, X)(L_t - L_s)] | X]}. \quad (7.2.5)$$

Since the increment $L_t - L_s$ and X are independent of L_s , the $\mathcal{H}_s$ -conditional expectation reduces to an expectation conditional on X . Thus, equation (7.2.5) simplifies to

$$\mathbb{E}[M_{tu}(X) | \mathcal{H}_s] = \frac{\exp[h(u, X)L_s]}{\mathbb{E}[\exp[h(u, X)L_s] | X]}, \quad (7.2.6)$$

which is $M_{su}(X)$. □

Example 7.2.1. Let $\{W_t\}_{t \geq 0}$ be a standard Brownian motion that is independent of X , and set $L_t = W_t$ in Definition 7.2.1. Then,

$$M_{tu}(X) = \exp[h(u, X)W_t - \frac{1}{2}h^2(u, X)t]. \quad (7.2.7)$$

Example 7.2.2. Let $\{\gamma_t\}_{t \geq 0}$ be a gamma process with rate parameter $m > 0$ and scale parameter $\kappa > 0$. Then $\mathbb{E}[\gamma_t] = \kappa m t$ and $\text{Var}[\gamma_t] = \kappa^2 m t$. We assume that $\{\gamma_t\}$ is independent of X . Set $L_t = \gamma_t$ in Definition 7.2.1. Then, if $h(u, X) < \kappa^{-1}$, we have

$$M_{tu}(X) = [1 - \kappa h(u, X)]^{mt} \exp[h(u, X) \gamma_t]. \quad (7.2.8)$$

At this point, a few remarks are necessary to put the formulated results into context. First, we recall that for a constant $h \in \mathbb{R}$ and a Lévy process $\{L_t\}_{t \geq 0}$ ¹ with characteristic triplet (a, Σ, Π) ,

$$\Phi_t(h) = \frac{\exp(hL_t)}{\mathbb{E}[\exp(hL_t)]} \quad (7.2.9)$$

is a positive unit-initialized martingale, called the *Esscher martingale* corresponding to h , provided that $\mathbb{E}[\exp(hL_t)] < \infty$. This martingale is generally used to define a change of measure by

$$\Phi_t(h) = \frac{d\mathbb{P}_h}{d\mathbb{P}} \Big|_{\mathcal{F}_t}, \quad (7.2.10)$$

known as the *Esscher transform*, where $\{\mathcal{F}_t\} = \sigma(\{L_t\})$. The Esscher transform has been used extensively in insurance mathematics, and is an important tool for derivative pricing when the underlying stock follows a geometric Lévy process (see Gerber & Shiu [43] and, for example, Yao [100]). It is shown, for instance in Pascucci [83], that $\{L_t\}$ is a Lévy process with a modified characteristic triplet (a_h, Σ, Π_h) under $\mathbb{P}_h$, where

$$a_h = a + h\Sigma + \int_{|z| < 1} (e^{hz} - 1) z \Pi(dz), \quad (7.2.11)$$

$$\Pi_h(dz) = e^{hz} \Pi(dz). \quad (7.2.12)$$

Thus, the effect of the Esscher transform is to exponentially tilt the Lévy measure and shift the drift component (see Kyprianou [67]).

The positive $\{\mathcal{H}_t\}$ -adapted martingale family $\{M_{tu}(X)\}$ constructed in (7.2.1) generalizes the idea of the Esscher martingale, where the constant h is replaced with a function $h(u, X)$, where $u \in \mathbb{R}_+$ and X is a random variable that is independent of $\{L_t\}$. We call the family of processes $\{M_{tu}(X)\}$ the “randomised Esscher martingales”, and we refer to $h(u, X)$ as the “random mixer”. In what follows, we shall use randomised Esscher martingales for the purpose of generating pricing kernel models. However, it is worth noting that the constructed martingales may also be useful for defining measure changes and, therefore, may have other interesting applications. This has been demonstrated recently in the work of

¹Here, for simplicity, we restrict our attention to the one-dimensional setting.

Brody et al. [23], where randomised Esscher martingales have also appeared. Here, $\{\xi_t\}$ is taken to be a $\mathbb{P}_0$ -Lévy process with characteristic triplet (a, Σ, Π) , X is an independent random variable and $\{\mathcal{H}_t\}$ is the filtration generated jointly by $\{\xi_t\}$ and X . A change of measure is defined using a randomised Esscher martingale (with $h(u, X) = X$) as follows:

$$M_t(X) = \frac{\exp(X\xi_t)}{\mathbb{E}^{\mathbb{P}_0}[\exp(X\xi_t) | X]} = \frac{d\mathbb{P}_X}{d\mathbb{P}_0} \Big|_{\mathcal{H}_t}. \quad (7.2.13)$$

It is shown that the change of measure generated by $M_t(X)$ alters the characteristic triplet of $\{\xi_t\}$ to (a_X, Σ, Π_X) , where

$$a_X = a + X\Sigma + \int_{|z|<1} (e^{Xz} - 1) z \Pi(dz), \quad (7.2.14)$$

$$\Pi_X(dz) = e^{Xz} \Pi(dz). \quad (7.2.15)$$

As [23] note, the change of measure gives rise to a random shift in the drift term and a random rescaling of the Lévy measure. Analogous results hold for the random mixer $h(u, X)$ in (7.2.1). In particular, it is shown in [23] that under $\mathbb{P}_X$, the Lévy process and the random variable X are fused together. That is to say, the process $\{\xi_t\}$ defines a Lévy information process carrying information about a signal random variable or “message” X that is perturbed by the original Lévy noise. This feature is used in [23] to develop a theory for signal processing with Lévy information. In what follows, we shall also view X as a signal random variable, and consider a filtering problem; however, along different lines.

7.3 Filtered Esscher martingales

In this section we construct a projection of the randomised Esscher martingales that can be interpreted as follows. Let us suppose that the exact combination of Lévy processes that forms the stochastic basis of the martingale family $\{M_{tu}(X)\}$ is unknown. That is, we may have little knowledge about how much each of the Lévy processes involved actually contributes to the stochastic evolution of $\{M_{tu}(X)\}$. The random vector $h(u, X)$ however, can naturally be interpreted as the quantity inside $\{M_{tu}(X)\}$ that determines at time u the random mixture of Lévy processes driving the martingale family. Given a certain set of information, the actual mixture might not be fully observable, though.

This leads us to the following construction that applies the theory of stochastic filtering. For simplicity, we focus on the case where X is a one-dimensional signal random variable. We introduce a standard Brownian motion $\{B_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and define the filtration $\{\mathcal{G}_t\}$ by

$$\mathcal{G}_t = \sigma(\{B_s\}_{0 \leq s \leq t}, \{L_s\}_{0 \leq s \leq t}, X), \quad (7.3.1)$$

where $\{B_t\}$ is taken to be independent of X and $\{L_t\}$. Let $\ell : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a well-defined function. We define the information (or observation) process $\{I_t\}_{t \geq 0}$ by

$$I_t = \int_0^t \ell(s, X) ds + B_t, \quad (7.3.2)$$

Next, we introduce the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ defined by

$$\mathcal{F}_t = \sigma(\{I_s\}_{0 \leq s \leq t}, \{L_s\}_{0 \leq s \leq t}), \quad (7.3.3)$$

where $\mathcal{F}_t \subset \mathcal{G}_t$. The filtration $\{\mathcal{F}_t\}$ provides full information about the Lévy process $\{L_t\}$, however it only gives partial information about the random variable X . Let us thus consider the filtering problem defined by

$$\widehat{M}_{tu} = \mathbb{E}[M_{tu}(X) | \mathcal{F}_t]. \quad (7.3.4)$$

We emphasize that X is not $\mathcal{F}_t$ -measurable and thus $\{M_{tu}(X)\}$ is not adapted to $\{\mathcal{F}_t\}$. The filtering problem (7.3.4) is solved in closed form by introducing

$$\mathcal{E}_t := \exp\left(-\int_0^t \ell(s, X) dB_s - \frac{1}{2} \int_0^t \ell^2(s, X) ds\right), \quad (7.3.5)$$

where for all $t > 0$

$$\mathbb{E}\left[\int_0^t \ell^2(s, X) ds\right] < \infty, \quad (7.3.6)$$

and

$$\mathbb{E}\left[\int_0^t \mathcal{E}_s \ell^2(s, X) ds\right] < \infty. \quad (7.3.7)$$

The process $\{\mathcal{E}_t\}$ is a $(\{\mathcal{G}_t\}, \mathbb{P})$ -martingale and it defines a change-of-measure density martingale from $\mathbb{P}$ to a new measure $\mathbb{M}$:

$$\mathcal{E}_t = \frac{d\mathbb{M}}{d\mathbb{P}} \Big|_{\mathcal{G}_t}. \quad (7.3.8)$$

The $\mathbb{M}$ -measure is characterised by the fact that $\{I_t\}$ is a $(\{\mathcal{G}_t\}, \mathbb{M})$ -Brownian motion. The Kallianpur-Striebel formula then states that

$$\mathbb{E}[M_{tu}(X) | \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{M}}[\mathcal{E}_t^{-1} M_{tu}(X) | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{M}}[\mathcal{E}_t^{-1} | \mathcal{F}_t]}. \quad (7.3.9)$$

This can be simplified to obtain:

$$\mathbb{E}[M_{tu}(X) | \mathcal{F}_t] = \int_{-\infty}^{\infty} M_{tu}(x) f_t(x) dx, \quad (7.3.10)$$

where the $\mathcal{F}_t$ -measurable conditional density $f_t(x)$ of the random variable X is given by

$$f_t(x) = \frac{f_0(x) \exp \left(\int_0^t \ell(s, x) dI_s - \frac{1}{2} \int_0^t \ell^2(s, x) ds \right)}{\int_{-\infty}^{\infty} f_0(y) \exp \left(\int_0^t \ell(s, y) dI_s - \frac{1}{2} \int_0^t \ell^2(s, y) ds \right) dy}. \quad (7.3.11)$$

A similar filtering system is considered in a different context in Filipović et al. [35]. Here, further conditions are imposed on the dynamics of the information process defined in (7.3.2), which may be regarded necessary from a modelling point of view.

Proposition 7.3.1. *Let $\{\mathcal{F}_t\}$ be given by (7.3.3), and define the projection*

$$\widehat{M}_{tu} = \mathbb{E} [M_{tu}(X) | \mathcal{F}_t], \quad (7.3.12)$$

where $\{M_{tu}(X)\}$ is given by (7.2.1). Then, for $t, u \in \mathbb{R}_+$, $\{\widehat{M}_{tu}\}$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$ -martingale family.

Proof. Recall that $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$. For $s \leq t$, we have

$$\begin{aligned} \mathbb{E} [\widehat{M}_{tu} | \mathcal{F}_s] &= \mathbb{E} [\mathbb{E} [M_{tu}(X) | \mathcal{F}_t] | \mathcal{F}_s], \\ &= \mathbb{E} [M_{tu}(X) | \mathcal{F}_s], \\ &= \mathbb{E} [\mathbb{E} [M_{tu}(X) | \mathcal{G}_s] | \mathcal{F}_s], \\ &= \mathbb{E} [M_{su}(X) | \mathcal{F}_s], \\ &= \widehat{M}_{su}, \end{aligned} \quad (7.3.13)$$

where we make use of the tower property of the conditional expectation, and the fact that $\{M_{tu}(X)\}$ is a $\{\mathcal{G}_t\}$ -martingale—since $\mathcal{H}_t \subset \mathcal{G}_t$ and $\{B_t\}$ is independent of X and $\{L_t\}$. $\square$

Filtered Brownian martingales. We consider Example 7.2.1, in which the total impact of uncertainties is modelled by a Brownian motion $\{W_t\}$. The corresponding filtered Esscher martingale is

$$\widehat{M}_{tu} = \int_{-\infty}^{\infty} f_t(x) \exp \left(h(u, x) W_t - \frac{1}{2} h^2(u, x) t \right) dx, \quad (7.3.14)$$

where the density process $\{f_t(x)\}$, given in (7.3.11), is driven by the information process defined by (7.3.2).

Proposition 7.3.2. *The filtered Brownian models have dynamics*

$$d\widehat{M}_{tu} = \int_{-\infty}^{\infty} M_{tu}(x) f_t(x) [h(u, x) dW_t + V_t(x) dZ_t] dx, \quad (7.3.15)$$

where

$$M_{tu}(x) = \exp \left[h(u, x)W_t - \frac{1}{2}h^2(u, x)t \right], \quad (7.3.16)$$

$$V_t(x) = \ell(t, x) - \mathbb{E}[\ell(t, X) | \mathcal{F}_t], \quad (7.3.17)$$

$$Z_t = I_t - \int_0^t \mathbb{E}[\ell(s, X) | \mathcal{F}_s] ds, \quad (7.3.18)$$

and $f_t(x)$ is defined in (7.3.11).

Proof. We first show that

$$dM_{tu}(x) = h(u, x)M_{tu}(x)dW_t. \quad (7.3.19)$$

In Filipović et al. [35] it is proven that

$$df_t(x) = f_t(x) (\ell(t, x) - \mathbb{E}[\ell(t, X) | \mathcal{F}_t]) dZ_t, \quad (7.3.20)$$

where $\{Z_t\}_{t \geq 0}$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$ -Brownian motion, defined by

$$Z_t = I_t - \int_0^t \mathbb{E}[\ell(s, X) | \mathcal{F}_s] ds. \quad (7.3.21)$$

Thus by the Itô product rule, we get

$$d[M_{tu}(x)f_t(x)] = f_t(x)dM_{tu}(x) + M_{tu}(x)df_t(x) \quad (7.3.22)$$

since $dW_t dZ_t = 0$. This simplifies to

$$d[M_{tu}(x)f_t(x)] = M_{tu}(x)f_t(x) \left[h(u, x)dW_t + \left(\ell(t, x) - \int_{-\infty}^{\infty} \ell(t, y)f_t(y)dy \right) dZ_t \right], \quad (7.3.23)$$

and we obtain

$$d\widehat{M}_{tu} = \int_{-\infty}^{\infty} M_{tu}(x)f_t(x) [h(u, x)dW_t + V_t(x)dZ_t] dx \quad (7.3.24)$$

where we define

$$V_t(x) = \ell(t, x) - \int_{-\infty}^{\infty} \ell(t, y)f_t(y) dy. \quad (7.3.25)$$

□

Remark 7.3.1. The dynamics of $\{\widehat{M}_{tu}\}$ can be written in the following form:

$$d\widehat{M}_{tu} = \mathbb{E}[M_{tu}(X)h(u, X) | \mathcal{F}_t] dW_t + \mathbb{E}[M_{tu}(X)V_t(X) | \mathcal{F}_t] dZ_t. \quad (7.3.26)$$

Filtered gamma martingales. Let us suppose that the total impact of uncertainties on an economy is modelled by a gamma process $\{\gamma_t\}$ with density

$$\mathbb{P}(\gamma_t \in dy) = \frac{y^{mt-1} \exp\left(-\frac{y}{\kappa}\right)}{\kappa^{mt} \Gamma[mt]} dy, \quad (7.3.27)$$

where m and κ are the rate and the scale parameter, respectively. The associated randomised Esscher martingale is given in Example 7.2.2, where $h(u, X) < \kappa^{-1}$. The corresponding filtered process takes the form

$$\widehat{M}_{tu} = \int_{-\infty}^{\infty} f_t(x) ([1 - \kappa h(u, x)]^{mt} \exp[h(u, x) \gamma_t]) dx \quad (7.3.28)$$

for $h(u, x) < \kappa^{-1}$, and where the density $f_t(x)$ is given by (7.3.11).

Filtered compound Poisson and gamma martingales. We now construct a model based on two independent Lévy processes: a gamma process (as defined previously) and a compound Poisson process. The idea here is to use the infinite activity gamma process to represent small frequently-occurring jumps, and to use the compound Poisson process to model jumps, which are potentially much larger in magnitude, and which may occur sporadically. Let $\{C_t\}_{t \geq 0}$ denote a compound Poisson process given by

$$C_t = \sum_{i=1}^{N_t} Y_i \quad (7.3.29)$$

where $\{N_t\}_{t \geq 0}$ is a Poisson process with rate λ . The independent and identically distributed random variables Y_i are independent of $\{N_t\}$. The moment generating function is given by

$$\mathbb{E}[\exp(\varrho C_t)] = \exp[\lambda t (M_Y(\varrho) - 1)] \quad (7.3.30)$$

where M_Y is the moment generating function of Y_i . For $h_1(u, X) < \kappa^{-1}$, we have

$$\begin{aligned} M_{tu}(X) &= \frac{\exp(h_1(u, X)\gamma_t + h_2(u, X)C_t)}{\mathbb{E}[\exp(h_1(u, X)\gamma_t + h_2(u, X)C_t) | X]} \\ &= \frac{\exp(h_1(u, X)\gamma_t)}{\mathbb{E}[\exp(h_1(u, X)\gamma_t) | X]} \cdot \frac{\exp(h_2(u, X)C_t)}{\mathbb{E}[\exp(h_2(u, X)C_t) | X]} \\ &= M_{tu}^{(\gamma)}(X) M_{tu}^{(C)}(X), \end{aligned} \quad (7.3.31)$$

where, conditional on X , $\exp(h_1(u, X)\gamma_t)$ and $\exp(h_2(u, X)C_t)$ are independent. Furthermore,

$$M_{tu}^{(\gamma)}(X) = (1 - \kappa h_1(u, X))^{mt} \exp(h_1(u, X)\gamma_t), \quad (7.3.32)$$

$$M_{tu}^{(C)}(X) = \exp[h_2(u, X)C_t - \lambda t (M_Y(h_2(u, X)) - 1)]. \quad (7.3.33)$$

Then, the filtered process takes the form

$$\begin{aligned} \widehat{M}_{tu} = \int_{-\infty}^{\infty} f_t(x) [1 - \kappa h_1(u, x)]^{mt} \\ \times \exp [h_1(u, x) \gamma_t + h_2(u, X) C_t - \lambda t (M_Y(h_2(u, X)) - 1)] dx, \end{aligned} \quad (7.3.34)$$

where $f_t(x)$ is given by (7.3.11).

7.4 Filtered Esscher martingales with Lévy information

Up to this point, we have considered a Brownian information process given by equation (7.3.2). However, the noise component in the information process may be modelled by a Lévy process with randomly sized jumps, that is independent of the Lévy process $\{L_t\}$ used to construct the randomised Esscher martingale. In what follows, we give an example of a signal random variable which is distorted by gamma-distributed pure noise.

Example 7.4.1. Let $\{\tilde{\gamma}_t\}_{t \geq 0}$ be a gamma process with rate and scale parameters $\tilde{m}$ and $\tilde{\kappa}$, respectively. We define the gamma information process by

$$I_t = X \tilde{\gamma}_t. \quad (7.4.1)$$

Brody & Friedman [16] consider such an observation process in a similar situation. We define the filtration $\{\mathcal{G}_t\}$ by

$$\mathcal{G}_t = \sigma(\{\tilde{\gamma}_s\}_{0 \leq s \leq t}, \{L_s\}_{0 \leq s \leq t}, X), \quad (7.4.2)$$

and $\{\mathcal{F}_t\}$ by

$$\mathcal{F}_t = \sigma(\{L_s\}_{0 \leq s \leq t}, \{I_s\}_{0 \leq s \leq t}) \quad (7.4.3)$$

where $\{I_t\}$ is given by (7.4.1). To derive the conditional density of X given $\mathcal{F}_t$, we first show that $\{I_t\}$ is a Markov process with respect to its own filtration. That is, for $a \in \mathbb{R}$,

$$\mathbb{P}[I_t < a \mid I_s, I_{s_1}, \dots, I_{s_n}] = \mathbb{P}[I_t < a \mid I_s] \quad (7.4.4)$$

for all $t \geq s \geq s_1 \geq \dots \geq s_n \geq 0$ and for all $n \geq 1$. It follows that

$$\begin{aligned} \mathbb{P}[I_t < a \mid I_s, I_{s_1}, \dots, I_{s_n}] &= \mathbb{P}\left[I_t < a \mid I_s, \frac{I_{s_1}}{I_s}, \dots, \frac{I_{s_n}}{I_{s_{n-1}}}\right] \\ &= \mathbb{P}\left[X \tilde{\gamma}_t < a \mid X \tilde{\gamma}_s, \frac{\tilde{\gamma}_{s_1}}{\tilde{\gamma}_s}, \dots, \frac{\tilde{\gamma}_{s_n}}{\tilde{\gamma}_{s_{n-1}}}\right]. \end{aligned} \quad (7.4.5)$$

It can be proven that $\tilde{\gamma}_{s_1}/\tilde{\gamma}_s, \dots, \tilde{\gamma}_{s_n}/\tilde{\gamma}_{s_{n-1}}$ are independent of $\tilde{\gamma}_s$ and $\tilde{\gamma}_t$ (see Brody et al. [20]). Furthermore, $\tilde{\gamma}_{s_1}/\tilde{\gamma}_s, \dots, \tilde{\gamma}_{s_n}/\tilde{\gamma}_{s_{n-1}}$ are independent of X . Thus we have

$$\mathbb{P}[I_t < a \mid I_s, I_{s_1}, \dots, I_{s_n}] = \mathbb{P}[I_t < a \mid I_s]. \quad (7.4.6)$$

We assume that the random variable X has a continuous *a priori* density $f_0(x)$. Then the conditional density of X ,

$$f_t(x) = \frac{d}{dx} \mathbb{P}[X \leq x \mid I_t], \quad (7.4.7)$$

is given by

$$\begin{aligned} f_t(x) &= \frac{f_0(x) p(I_t \mid X = x)}{\int_{-\infty}^{\infty} f_0(y) p(I_t \mid X = y) dy} \\ &= \frac{f_0(x) x^{-\tilde{m}t} \exp[-I_t/(\tilde{\kappa}x)]}{\int_{-\infty}^{\infty} f_0(y) y^{-\tilde{m}t} \exp[-I_t/(\tilde{\kappa}y)] dy}, \end{aligned} \quad (7.4.8)$$

where we have used the Bayes' formula. The filtered Esscher martingale is thus obtained by

$$\widehat{M}_{tu} = \mathbb{E}[M_{tu}(X) \mid \mathcal{F}_t]. \quad (7.4.9)$$

The result is:

$$\widehat{M}_{tu} = \int_{-\infty}^{\infty} M_{tu}(x) \frac{f_0(x) x^{-\tilde{m}t} \exp[-I_t/(\tilde{\kappa}x)]}{\int_{-\infty}^{\infty} f_0(y) y^{-\tilde{m}t} \exp[-I_t/(\tilde{\kappa}y)] dy} dx. \quad (7.4.10)$$

It is worth mentioning that in recent work, Brody et al. [23] consider a range of further examples of Lévy information processes.

7.5 Flesaker-Hughston pricing kernel models

The absence of arbitrage in a financial market is ensured by the existence of a pricing kernel $\{\pi_t\}_{t \geq 0}$ satisfying $\pi_t > 0$ almost surely for all $t \geq 0$. We consider, in general, an incomplete market. We recall that the price of a discount bond system with price process $\{P_{tT}\}_{0 \leq t \leq T < \infty}$ and payoff $P_{TT} = 1$ is given by the bond pricing formula

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}[\pi_T \mid \mathcal{F}_t]. \quad (7.5.1)$$

Flesaker & Hughston [36] provide a framework for constructing positive interest rate models, in which the pricing kernel is modelled by

$$\pi_t = \int_t^{\infty} \rho(u) m_{tu} du, \quad (7.5.2)$$

where $\{m_{tu}\}_{0 \leq t \leq u < \infty}$ is a family of positive unit-initialized martingales, and

$$\rho(t) = -\partial_t P_{0t}. \quad (7.5.3)$$

In what follows, we shall construct explicit Flesaker-Hughston models, which are driven by a randomised mixture of Lévy processes. We develop such a class of pricing kernels by setting

$$\pi_t = \int_t^\infty \rho(u) \widehat{M}_{tu} du \quad (7.5.4)$$

where the martingale family $\{\widehat{M}_{tu}\}_{0 \leq t \leq u < \infty}$ is defined by (7.3.4) with $\widehat{M}_{tu} > 0$ and $\widehat{M}_{0u} = 1$. Then, the discount bond system is given by

$$P_{tT} = \frac{\int_T^\infty \rho(u) \widehat{M}_{tu} du}{\int_t^\infty \rho(u) \widehat{M}_{tu} du}. \quad (7.5.5)$$

The associated instantaneous forward rate $\{r_{tT}\}_{0 \leq t \leq T}$ is defined by $r_{tT} = -\partial_T \ln P_{tT}$. We deduce that

$$r_{tT} = \frac{\rho(T) \widehat{M}_{tT}}{\int_T^\infty \rho(u) \widehat{M}_{tu} du}, \quad (7.5.6)$$

and that the short rate of interest $\{r_t\}_{t \geq 0}$ is given by the formula

$$r_t = \frac{\rho(t) \widehat{M}_{tt}}{\int_t^\infty \rho(u) \widehat{M}_{tu} du}, \quad (7.5.7)$$

where $r_t := r_{tt}$. The interest rate is positive by construction. We note here that the pricing kernel models proposed in Brody et al. [18] can be recovered by considering a special case of the random mixer, namely $h(u, X) = h(u)$.

7.6 Pricing kernel models driven by filtered Brownian martingales

In the case where the filtered martingales driving the pricing kernel are Gaussian processes, the dynamics of the discount bond system can be expressed by a diffusion equation of the form (7.6.2). Inserting the filtered Brownian martingale family (7.3.14) into (7.5.5), we obtain the price process of the discount bond in the Brownian set-up:

$$P_{tT} = \frac{\int_T^\infty \rho(u) \int_{-\infty}^\infty f_t(x) \exp \left[h(u, x) W_t - \frac{1}{2} h^2(u, x) t \right] dx du}{\int_t^\infty \rho(v) \int_{-\infty}^\infty f_t(y) \exp \left[h(v, y) W_t - \frac{1}{2} h^2(v, y) t \right] dy dv}. \quad (7.6.1)$$

A similar expression is obtained for the associated interest rate system by plugging (7.3.14) into (7.5.7).

Proposition 7.6.1. *The dynamical equation of the discount bond process is given by*

$$\frac{dP_{tT}}{P_{tT}} = [r_t - \theta_{tt}(\theta_{tT} - \theta_{tt}) - \nu_{tt}(\nu_{tT} - \nu_{tt})] dt + (\theta_{tT} - \theta_{tt})dW_t + (\nu_{tT} - \nu_{tt})dZ_t \quad (7.6.2)$$

where

$$\theta_{tT} := \frac{\int_T^\infty \rho(u) \mathbb{E}[M_{tu}(X)h(u, X) | \mathcal{F}_t] du}{\int_T^\infty \rho(u) \widehat{M}_{tu} du}, \quad (7.6.3)$$

$$\nu_{tT} := \frac{\int_T^\infty \rho(u) \mathbb{E}[M_{tu}(X)V_t(X) | \mathcal{F}_t] du}{\int_T^\infty \rho(u) \widehat{M}_{tu} du}, \quad (7.6.4)$$

$\theta_{tt} = \theta_{tT}|_{T=t}$, and $\nu_{tt} = \nu_{tT}|_{T=t}$.

Proof. First we have

$$d \left[\int_T^\infty \rho(u) \widehat{M}_{tu} du \right] = \int_T^\infty \rho(u) d\widehat{M}_{tu} du \quad (7.6.5)$$

where $d\widehat{M}_{tu}$ is given by (7.3.26). Also,

$$d \left[\int_t^\infty \rho(u) \widehat{M}_{tu} du \right] = \int_t^\infty \rho(u) d\widehat{M}_{tu} du - \rho(t) \widehat{M}_{tt} dt. \quad (7.6.6)$$

We then apply the Itô quotient rule to obtain the dynamics of $\{P_{tT}\}$. We observe that the discount bond volatilities are given by

$$\Omega_{tT}^{(1)} = \theta_{tT} - \theta_{tt}, \quad (7.6.7)$$

$$\Omega_{tT}^{(2)} = \nu_{tT} - \nu_{tt}. \quad (7.6.8)$$

The market price of risk associated with $\{W_t\}$ is $\lambda_t^{(1)} := -\theta_{tt}$; the one associated with $\{Z_t\}$ is $\lambda_t^{(2)} := -\nu_{tt}$. The product between the bond volatility vector $\Omega_{tT} = (\Omega_{tT}^{(1)}, \Omega_{tT}^{(2)})$ and the market price of risk vector $\lambda_t = (\lambda_t^{(1)}, \lambda_t^{(2)})$ gives us the risk premium associated with an investment in the discount bond, that is,

$$\Omega_{tT} \cdot \lambda_t = -\theta_{tt}(\theta_{tT} - \theta_{tt}) - \nu_{tt}(\nu_{tT} - \nu_{tt}). \quad (7.6.9)$$

□

Proposition 7.6.2. *Let $\{M_{tu}(X)\}$ be of the class (7.2.7), and let $\{\widehat{M}_{tu}\}$ in (7.5.6) be given by the martingale family (7.3.14). Then the dynamical equation of the forward rate is given by*

$$dr_{tT} = [\theta_{tT} \partial_T \theta_{tT} + \nu_{tT} \partial_T \nu_{tT}] dt - \partial_T \theta_{tT} dW_t - \partial_T \nu_{tT} dZ_t \quad (7.6.10)$$

where

$$\theta_{tT} := \frac{\int_T^\infty \rho(u) \mathbb{E}[M_{tu}(X)h(u, X) | \mathcal{F}_t] du}{\int_T^\infty \rho(u) \widehat{M}_{tu} du}, \quad (7.6.11)$$

and

$$\nu_{tT} := \frac{\int_T^\infty \rho(u) \mathbb{E}[M_{tu}(X)V_t(X) | \mathcal{F}_t] du}{\int_T^\infty \rho(u) \widehat{M}_{tu} du} \quad (7.6.12)$$

where $V_t(X)$ is defined by (7.3.17).

Proof. We apply the Itô quotient rule to (7.5.6) to obtain the forward rate dynamics. We make the observations that

$$\partial_T \theta_{tT} = r_{tT} \left(\theta_{tT} - \frac{\mathbb{E}[M_{tT}(X)h(T, X) | \mathcal{F}_t]}{\widehat{M}_{tT}} \right), \quad (7.6.13)$$

and that

$$\partial_T \nu_{tT} = r_{tT} \left(\nu_{tT} - \frac{\mathbb{E}[M_{tT}(X)V_t(X) | \mathcal{F}_t]}{\widehat{M}_{tT}} \right). \quad (7.6.14)$$

□

In particular, if we set

$$\Sigma_{tT} = \theta_{tT} - \theta_{tt}, \quad (7.6.15)$$

$$\Lambda_{tT} = \nu_{tT} - \nu_{tt}, \quad (7.6.16)$$

then we can express the risk-neutral dynamics of the forward rate by

$$dr_{tT} = [\Sigma_{tT} \partial_T \Sigma_{tT} + \Lambda_{tT} \partial_T \Lambda_{tT}] dt - \partial_T \Sigma_{tT} d\widetilde{W}_t - \partial_T \Lambda_{tT} d\widetilde{Z}_t, \quad (7.6.17)$$

where $\{\widetilde{W}_t\}_{t \geq 0}$ and $\{\widetilde{Z}_t\}_{t \geq 0}$ are Brownian motions defined by the Girsanov relations

$$\begin{aligned} d\widetilde{W}_t &= dW_t + \lambda_t^{(1)} dt, \\ d\widetilde{Z}_t &= dZ_t + \lambda_t^{(2)} dt. \end{aligned} \quad (7.6.18)$$

The dynamical equation (7.6.17) has the form of the HJM dynamics for the forward rate under the risk-neutral measure, see Heath et al. [49].

Example 7.6.1. As a first illustration, let us consider the following information process:

$$I_t = \sigma X t + B_t, \quad (7.6.19)$$

where σ is a positive constant. It can be proven that this is a Markov process (see Brody et al. [22]). Equation (7.6.19) is a special case of the observation process

(7.3.2). Let $\{W_t\}$ be a standard Brownian motion that is independent of X . Then from Example 7.2.1, we have

$$M_{tu}(X) = \exp \left[h(u, X)W_t - \frac{1}{2} h^2(u, X)t \right]. \quad (7.6.20)$$

We suppose that the *a priori* distribution of X is uniform over the interval (a, b) , where $a \geq 0$ and $b > 0$. We choose to model the random mixer by

$$h(u, X) = c \exp(-uX) \quad (7.6.21)$$

where $c \in \mathbb{R}$. Here X can be interpreted as the random rate of the exponential decay in $h(u, X)$. We obtain the following expressions for the bond price

$$P_{tT} = \frac{\int_T^\infty \rho(u) \int_a^b \exp \left[\sigma x I_t + ce^{-ux} W_t - \frac{1}{2} (\sigma^2 x^2 + c^2 e^{-2ux}) t \right] dx du}{\int_t^\infty \rho(u) \int_a^b \exp \left[\sigma y I_t + ce^{-uy} W_t - \frac{1}{2} (\sigma^2 y^2 + c^2 e^{-2uy}) t \right] dy du}, \quad (7.6.22)$$

and the associated interest rate

$$r_t = \frac{\rho(t) \int_a^b \exp \left[\sigma x I_t + ce^{-tx} W_t - \frac{1}{2} (\sigma^2 x^2 + c^2 e^{-2tx}) t \right] dx}{\int_t^\infty \rho(u) \int_a^b \exp \left[\sigma y I_t + ce^{-uy} W_t - \frac{1}{2} (\sigma^2 y^2 + c^2 e^{-2uy}) t \right] dy}. \quad (7.6.23)$$

Since the model is constructed from a single Lévy process, it is not — strictly speaking — a mixture model as described previously. However, it can be viewed as a kind of two-factor Brownian model owing to the presence of the observation process $\{I_t\}$. The bond price and the associated interest rate are functions of time and the two state variables W_t and I_t . Thus, it is straightforward to generate simulated sample paths:

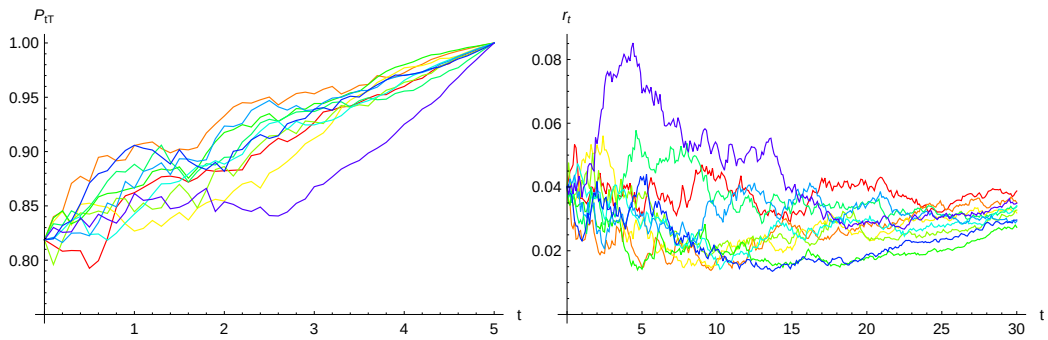


Figure 7.1: Sample paths of discount bond with $T = 5$ and short rate. We use the filtered Brownian model with $h(u, X) = c \exp(-uX)$ and $X \sim U(a, b)$. We set $a = 0$, $b = 0.1$, $\sigma = 0.1$, $c = 0.5$ and $P_{0t} = \exp(-0.04t)$.

The parameters a and b influence the rate at which $\exp(-uX)$ decays, and together with c determine the impact of the Brownian motion $\{W_t\}$ on the bond

and interest rate evolution. When c is close to zero, the impact of $\{W_t\}$ is very small. For sufficiently large values of $b - a$, σ or $|c|$, the numerical integration in the calculation of the pricing kernel may fail to converge. For large values of t , we observe that the sample paths of the short rate revert to r_0 . Thus, there is built-in reversion to the initial level of the short rate.

7.7 Bond prices driven by filtered gamma martingales

Let $\{\gamma_t\}$ denote a gamma process with $\mathbb{E}[\gamma_t] = \kappa mt$, and $\text{Var}[\gamma_t] = \kappa^2 mt$. We consider a bond price model based on a pricing kernel that is driven by a family of filtered gamma martingales given by (7.3.28). Then, equation (7.5.5) for the bond price gives the following expression:

$$P_{tT} = \frac{\int_T^\infty \rho(u) \int_{-\infty}^\infty f_t(x) [1 - \kappa h(u, x)]^{mt} \exp[h(u, x)\gamma_t] dx du}{\int_t^\infty \rho(v) \int_{-\infty}^\infty f_t(y) [1 - \kappa h(v, y)]^{mt} \exp[h(v, y)\gamma_t] dy dv}. \quad (7.7.1)$$

We now investigate this bond price model in more detail, and in particular show the effects of the various model components on the behaviour of the bond price.

Example 7.7.1. Let the information process $\{I_t\}$, driving the conditional density $\{f_t(x)\}$ be of the form

$$I_t = \sigma t X + B_t, \quad (7.7.2)$$

where X is a binary random variable taking the values $X = 1$ with *a priori* probability $f_0(1)$, and $X = 0$ with probability $f_0(0)$. We choose the random mixer

$$h(u, X) = c \exp[-bu(1 - X)], \quad (7.7.3)$$

where $c < \kappa^{-1}$ and $b > 0$. Then the expression for the filtered gamma martingale simplifies to

$$\widehat{M}_{tu} = f_t(0) \exp(c e^{-bu} \gamma_t) (1 - \kappa c e^{-bu})^{mt} + f_t(1) \exp(c \gamma_t) (1 - \kappa c)^{mt}, \quad (7.7.4)$$

where

$$f_t(0) = \frac{f_0(0)}{f_0(0) + f_0(1) \exp(\sigma I_t - \frac{1}{2} \sigma^2 t)} \quad f_t(1) = \frac{f_0(1) \exp(\sigma I_t - \frac{1}{2} \sigma^2 t)}{f_0(0) + f_0(1) \exp(\sigma I_t - \frac{1}{2} \sigma^2 t)}. \quad (7.7.5)$$

There are a number of degrees of freedom in this model which have a significant impact on the behaviour of the trajectories. In what follows, we analyse the degrees of freedom one by one.

A priori probability: When $f_0(1) = 0$, the diffusion $\{I_t\}$ plays no role. The sample paths of the discount bond and the short rate are driven solely by the pure jump process. The size of the jumps decays over time. As $f_0(1)$ increases, there is a greater amount of diffusion in the sample paths. Furthermore, there is a higher likelihood of obtaining sample paths for which the size of the jumps do not decay over time. If $f_0(1) = 1$, then $\{\widehat{M}_{tu}\}$ is no longer u dependent. This yields a stochastic pricing kernel, but flat short rate and deterministic discount bond prices, see Figure 7.2.

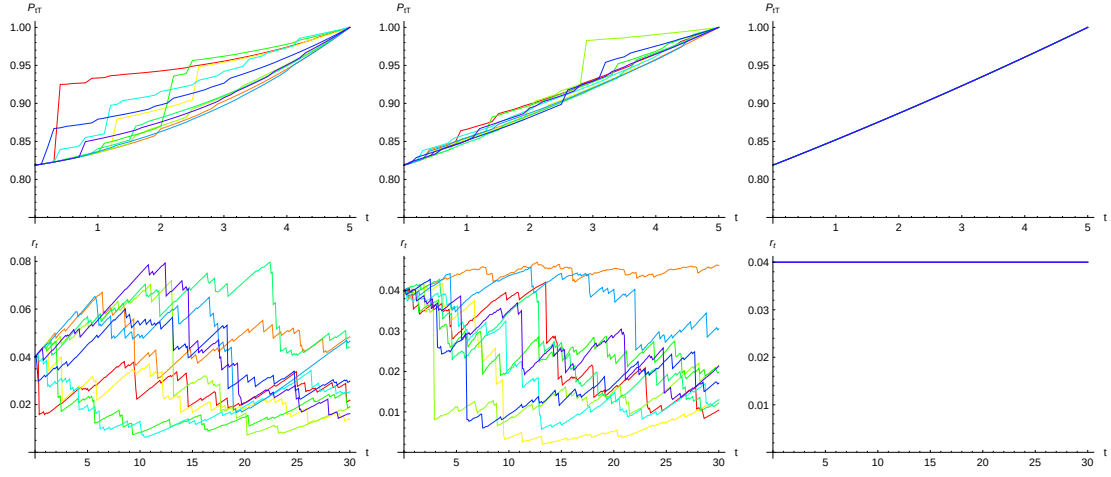


Figure 7.2: Sample paths for discount bond with $T = 5$, and associated short rate. We use the Brownian-gamma model with $h(u, X) = c \exp[-bu(1 - X)]$ where $X = \{0, 1\}$ with $m = 0.5$, $\kappa = 0.5$, $\sigma = 0.1$, $c = -2$, $b = 0.03$ and $P_{0t} = \exp(-0.04t)$. We let (i) $f_0(1) = 0$, (ii) $f_0(1) = 0.65$ and (iii) $f_0(1) = 1$.

Information flow rate σ : As the information flow rate increases, the investor becomes more knowledgeable at an earlier stage about whether the random variable may take the value $X = 0$ or $X = 1$, see Figure 7.3.

Parameters of the gamma process m and κ : The rate parameter m controls the rate of jump arrivals. The scale parameter κ controls the jump size.

Parameters of the random mixer b and c : The magnitude of c influences the impact of the jumps on the interest rate dynamics. When $c = 0$, the pricing kernel, and thus the short rate of interest, is deterministic. The sign of c affects the direction of the jumps. For $0 < c < \kappa^{-1}$, the short rate (discount bond) sample paths have upward (downward) jumps. The opposite is true for $c < 0$. It should be noted that $\exp(c \exp[-bu(1 - X)] \gamma_t)$, and $(1 - \kappa c \exp[-bu(1 - X)])^{mt}$ behave antagonistically in c . For large t , one term will eventually dominate the other. Thus, for both $c > 0$ and $c < 0$, the drift of the short rate trajectories

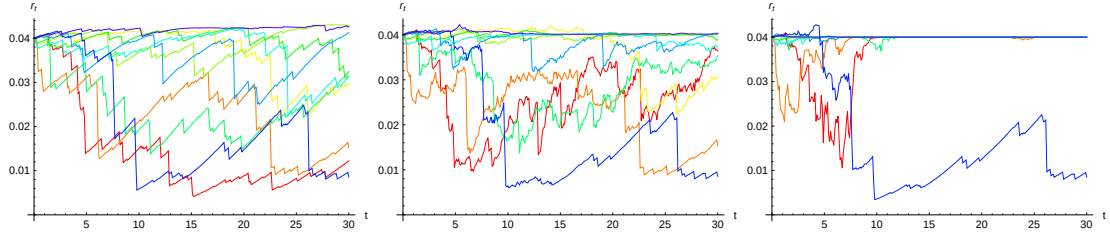


Figure 7.3: Short rate sample paths for the Brownian-gamma model with $h(u, X) = c \exp[-bu(1 - X)]$ and $X = \{0, 1\}$. We choose $m = 0.5$, $\kappa = 0.5$, $f_0(1) = 0.8$, $c = -2$, $b = 0.03$ and $P_{0t} = \exp(-0.04t)$. We set (i) $\sigma = 0.005$, (ii) $\sigma = 0.4$ and (iii) $\sigma = 1.2$.

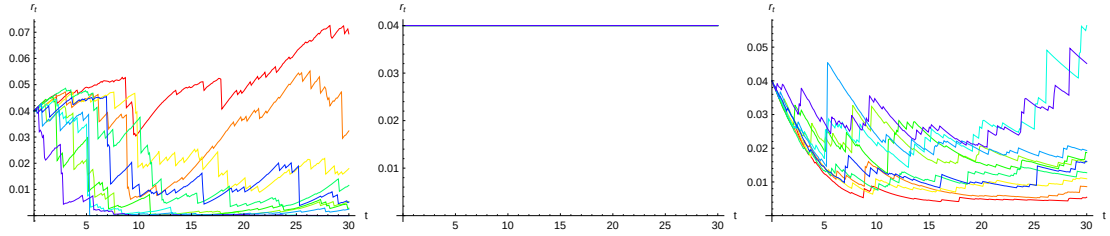


Figure 7.4: Short rate sample paths for the Brownian-gamma model with $h(u, X) = c \exp[-bu(1 - X)]$ and $X = \{0, 1\}$. We set $m = 0.5$, $\kappa = 0.5$, $f_0(1) = 0.5$, $\sigma = 0.1$, $b = 0.03$ and $P_{0t} = \exp(-0.04t)$. We choose (i) $c = -5$, (ii) $c = 0$ and (iii) $c = 1.5$.

is initially negative and then becomes positive for large t , see Figure 7.4. The parameter b determines how quickly the jumps are “killed off”. Alternatively, b can be viewed as the rate of reversion to the initial level of the interest rate. The interest rate process approaches the initial rate more rapidly for high values of b . When $b = 0$, $\widehat{M}_{tu}$ is no longer u dependent, and we obtain a stochastic pricing kernel, but flat short rate and deterministic discount bond prices, see Figure 7.5.

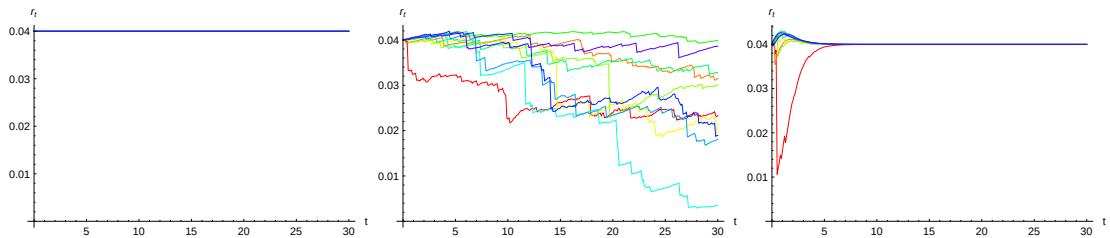


Figure 7.5: Short rate sample paths for the Brownian-gamma model with $h(u, X) = c \exp[-bu(1 - X)]$ and $X = \{0, 1\}$. We let $m = 0.5$, $\kappa = 0.5$, $f_0(1) = 0.5$, $\sigma = 0.1$, $c = -2$ and $P_{0t} = \exp(-0.04t)$. We choose (i) $b = 0$, (ii) $b = 0.005$ and (iii) $b = 1$.

Compared to Example 7.6.1, this model is more robust to variation in the values of the parameters. An analysis of the sample trajectories suggests that for large t , the short rate reverts to the initial level r_0 .

7.8 Bond prices driven by filtered variance-gamma martingales

We let $\{L_t\}$ denote a variance-gamma process. We define the variance-gamma process as a time-changed Brownian motion with drift (see Carr et al. [26]), that is

$$L_t = \theta\gamma_t + \Sigma B_{\gamma_t} \quad (7.8.1)$$

with parameters $\theta \in \mathbb{R}$, $\Sigma > 0$ and $\nu > 0$. Here $\{\gamma_t\}$ is a gamma process with rate and scale parameters $m = 1/\nu$ and $\kappa = \nu$ respectively, and $\{B_{\gamma_t}\}$ is a subordinated Brownian motion. The randomised Esscher martingale is expressed by

$$M_{tu}(X) = \exp[h(u, X)L_t] \left(1 - \theta\nu h(u, X) - \frac{1}{2}\Sigma^2\nu h^2(u, X)\right)^{t/\nu}, \quad (7.8.2)$$

and the associated filtered Esscher martingale is of the form

$$\widehat{M}_{tu} = \int_{-\infty}^{\infty} f_t(x) \exp[h(u, x)L_t] \left(1 - \theta\nu h(u, x) - \frac{1}{2}\Sigma^2\nu h^2(u, x)\right)^{t/\nu} dx, \quad (7.8.3)$$

where $f_t(x)$ may be given for example by (7.3.11) or a special case thereof, or by (7.4.8) depending on the type of information used to filter knowledge about X . This leads to the following expression for the discount bond price process:

$$P_{tT} = \frac{\int_T^{\infty} \rho(u) \int_{-\infty}^{\infty} f_t(x) \exp[h(u, x)L_t] \left(1 - \theta\nu h(u, x) - \frac{1}{2}\Sigma^2\nu h^2(u, x)\right)^{t/\nu} dx du}{\int_t^{\infty} \rho(v) \int_{-\infty}^{\infty} f_t(y) \exp[h(v, y)L_t] \left(1 - \theta\nu h(v, y) - \frac{1}{2}\Sigma^2\nu h^2(v, y)\right)^{t/\nu} dy dv}. \quad (7.8.4)$$

We can also obtain an expression for the short rate of interest by substituting (7.8.3) into (7.5.7). We now present another explicit bond pricing model.

Example 7.8.1. We assume that X is a random time, and hence a positive random variable taking discrete values $\{x_1, \dots, x_n\}$ with *a priori* probabilities $\{f_0(x_1), \dots, f_0(x_n)\}$. We suppose that the information process $\{I_t\}$ is independent of $\{L_t\}$, and that it is defined by

$$I_t = \sigma X t + B_t. \quad (7.8.5)$$

We take the random mixer to be

$$h(u, X) = c \exp[-b(u - X)^2] \quad (7.8.6)$$

where $b > 0$ and $c \in \mathbb{R}$. We see in Figure 7.6 that the random mixer, and thus the weight of the variance-gamma process, increases (in absolute value) until the random time X , and decreases (in absolute value) thereafter.

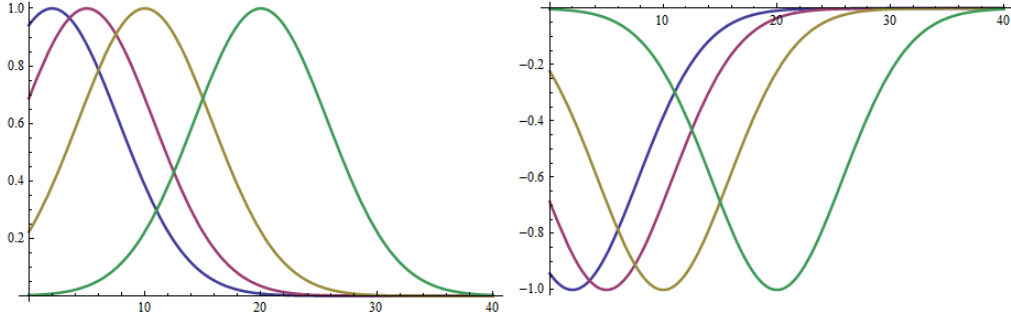


Figure 7.6: Plot of $h(u, x_i)$ for $x_1 = 2$, $x_2 = 5$, $x_3 = 10$ and $x_4 = 20$, where $b = 0.015$ and $c = 1$ (left) and $c = -1$ (right).

The associated bond price and interest rate processes have the following sample paths:

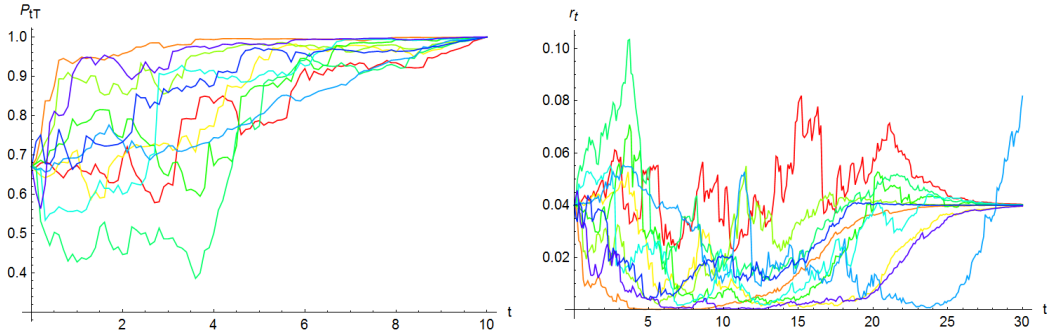


Figure 7.7: Sample paths for a discount bond with $T = 10$ and the short rate. We use the variance-gamma model with $h(u, X) = c \exp[-b(u - X)^2]$. We let $\theta = -1.5$, $\Sigma = 2$ and $\nu = 0.25$. We set $f_0(x_1) = 0.2$, $f_0(x_2) = 0.35$, $f_0(x_3) = 0.35$, $f_0(x_4) = 0.1$ and $x_1 = 2$, $x_2 = 5$, $x_3 = 10$, $x_4 = 20$. We choose $\sigma = 0.1$, $c = 0.5$, $b = 0.015$ and the initial term structure is $P_{0t} = \exp(-0.04t)$.

We observe that over time the sample paths of the interest rate process revert to the initial level r_0 . However, some paths may revert to r_0 at a later time than others, depending on the realized value of the random variable X .

7.9 Chameleon random mixers

The functional form of the random mixer $h(u, X)$ strongly influences the interest rate dynamics. The choice of $h(u, X)$ also affects the robustness of the model: there are choices in which the numerical integration in the calculation of the pricing kernel does not converge. So far, we have constructed examples based on an exponential-type random mixer. However, one may wish to introduce other functional forms for $h(u, X)$ for which we can observe different behaviour in the interest rate dynamics, while maintaining robustness. For instance we may consider a random piecewise function of the form

$$h(u, X) = g_1(u) \mathbb{1}_{\{u \leq X\}} + g_2(u) \mathbb{1}_{\{u > X\}} \quad (7.9.1)$$

where $g_j : \mathbb{R}_+ \rightarrow \mathbb{R}$ for $j = 1, 2$. The random mixer now has a “chameleon form”: initially appearing to be g_1 , and switching its form to g_2 at $X = u$. This results in the martingale $\{\widehat{M}_{tu}\}$, and the resulting interest rate sample paths, exhibiting different hues over time, depending on the choices of g_j ($j = 1, 2$). We can extend this idea further by considering (i) multiple g_j , or (ii) a multivariate random mixer of the form

$$h(u, X, Y_1, Y_2) = g_1(u, Y_1) \mathbb{1}_{\{u \leq X\}} + g_2(u, Y_2) \mathbb{1}_{\{u > X\}}, \quad (7.9.2)$$

where $X > 0$, Y_1 and Y_2 are independent random variables with associated information processes. In this case, the g_j are themselves random-valued functions. Here X can be regarded as the primary mixer which determines the timing of the regime switch. The variables Y_i ($i = 1, 2$) can then be interpreted as the secondary mixers determining the weights of the Lévy processes over two distinct time intervals.

Example 7.9.1. We now present what may be called the “Brownian-gamma chameleon model”. We consider the filtered gamma martingale family (7.3.28) in the situation where the random mixer $h(u, X)$ has the form

$$h(u, X) = c_1 \sin(\alpha_1 u) \mathbb{1}_{\{u \leq X\}} + c_2 \exp(-\alpha_2 u) \mathbb{1}_{\{u > X\}} \quad (7.9.3)$$

where $c_1, c_2 < \kappa^{-1}$ and $\alpha_2 > 0$. The information process $\{I_t\}$ associated with X is taken to be of the form

$$I_t = \sigma t X + B_t. \quad (7.9.4)$$

We assume that X is a positive discrete random variable taking values $\{x_1, x_2, \dots, x_n\}$ with *a priori* probabilities $f_0(x_i)$, $i = 1, 2, \dots, n$. That is, the function $h(u, X)$ will switch once from sine to exponential behaviour at one of the finitely many

random times. Inserting (7.3.28), with the specification (7.9.3), in the expression for the bond price (7.5.5), we obtain

$$P_{tT} = \frac{\int_T^\infty \rho(u) \sum_{i=1}^n f_t(x_i) [1 - \kappa h(u, x_i)]^{mt} \exp[h(u, x_i) \gamma_t] du}{\int_t^\infty \rho(v) \sum_{i=1}^n f_t(y_i) [1 - \kappa h(v, y_i)]^{mt} \exp[h(v, y_i) \gamma_t] dv}, \quad (7.9.5)$$

where $h(u, x_i)$ is given by (7.9.3) for $X = x_i$, and

$$f_t(x_i) = \frac{f_0(x_i) \exp[\sigma x_i I_t - \frac{1}{2} \sigma^2 x_i^2 t]}{\sum_{i=1}^n f_0(y_i) \exp[\sigma y_i I_t - \frac{1}{2} \sigma^2 y_i^2 t]}. \quad (7.9.6)$$

Since the sine function oscillates periodically within the interval $[-1, 1]$, the integrals in (7.9.5) may not necessarily converge to one value. However, at some finite random time $u = X$, the sine behaviour is replaced by an exponential decay; this ensures the integrals in the expression for the bond price converge. Such a behaviour may be viewed as a regime switch at a random time. In the simulation below, the analysis of the model parameters is analogous to the one in Example 7.7.1. It is worth emphasizing nevertheless that (i) the *a priori* probabilities $f_0(x_i)$, $i = 1, 2, \dots, n$ have a direct influence on the length of the time span during which the sine function in the chameleon mixer is activated, (ii) the magnitude of α_1 determines the frequency of the sine wave, while α_2 affects the rate at which reversion to the initial interest rate (in the simulation below $r_0 = 4\%$) occurs, and (iii) the size of c_1 determines the amplitude of the sine, and it significantly impacts the convergence of the numerical integration. We find that reasonable results are obtained for $-\kappa^{-1} < c_1 < \kappa^{-1}$.

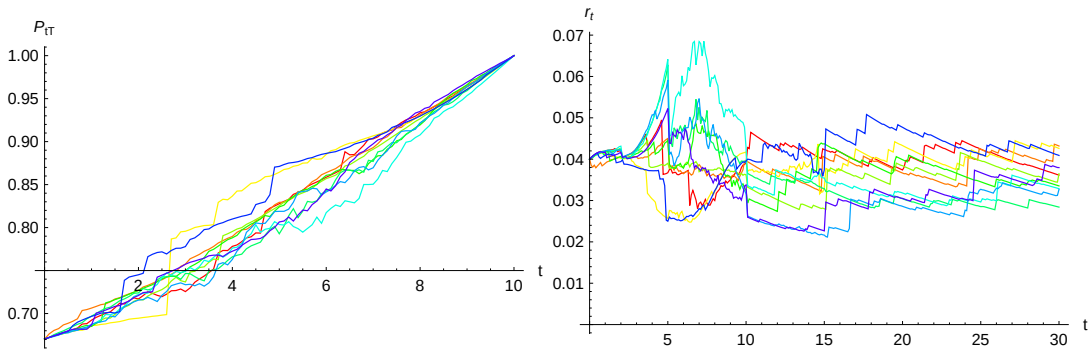


Figure 7.8: Sample paths of discount bond with $T = 10$ and short rate trajectories. We use the Brownian-gamma chameleon model with $h(u, X) = c_1 \sin(\alpha_1 u) \mathbb{1}_{\{u \leq X\}} + c_2 \exp(-\alpha_2 u) \mathbb{1}_{\{u > X\}}$. Let X take the values $\{x_1 = 2, x_2 = 5, x_3 = 10, x_4 = 15\}$ with *a priori* probabilities $\{f_0(x_1) = 0.2, f_0(x_2) = 0.35, f_0(x_3) = 0.35, f_0(x_4) = 0.1\}$. We set $m = 0.5$, $\kappa = 0.5$, $\sigma = 0.1$, $c_1 = 0.2625$, $c_2 = 0.75$, $\alpha_1 = 0.75$, $\alpha_2 = 0.02$ and $P_{0t} = \exp(-0.04t)$.

7.10 Model-generated yield curves

The yield curve at any time is defined as the range of yields that investors in sovereign debt can expect to receive on investments over various terms to maturity. For a calendar date t and a time to maturity τ , we let $Y_{t,t+\tau}$ be the continuously compounded zero-coupon spot rate for time to maturity τ , that is, the map $\tau \mapsto Y_{t,t+\tau}$. We write

$$P_{t,t+\tau} = \exp(-\tau Y_{t,t+\tau}). \quad (7.10.1)$$

There are four main shapes of yields curves that are observed in markets (see, for instance, Fabozzi [34]):

- Flat curves in which the yields for all maturities are similar;
- Normal or positively sloping curves in which yields are higher for longer maturities;
- Inverted or negatively sloping curves in which yields are lower for longer maturities;
- Humped curves which are positively sloping for a range of maturities and negatively sloping, thereafter (or inversely).

Typically, the following yield curve movements are observed:

- Parallel shifts of the yield curve where there is an equal increase in yields across all maturities.
- Steepening (resp. flattening) of the yield curve where the difference between the yields for longer-dated bonds and shorter-dated bonds widens (resp. narrows).
- Changes in the curvature and overall shape of the yield curve where the yield curve becomes more (less) humped.

The terms *shift*, *twist* and *butterfly*, respectively are also used to describe these yield curve movements. As shown in Figure 7.10, the two-factor Brownian-gamma model set-up in Example 7.7.1 is indeed too rigid to allow for significant changes in the shape of the yield curve. For $f_0(1) = 1$, the yield curve is flat at all times. For $0 \leq f_0(1) < 1$, this model can generate flat, upward sloping yield curves and in certain cases, slightly inverted yield curves. The variance-gamma model (see Figure 7.12) and the Brownian-gamma chameleon model (see Figure 7.14) show more flexibility, where changes of slope and different yield curve shapes are observed. These model may generate flat, positive sloping, inverted and humped

yield curves. We emphasize that these classes of models are able to capture all three types of yield curve movements.

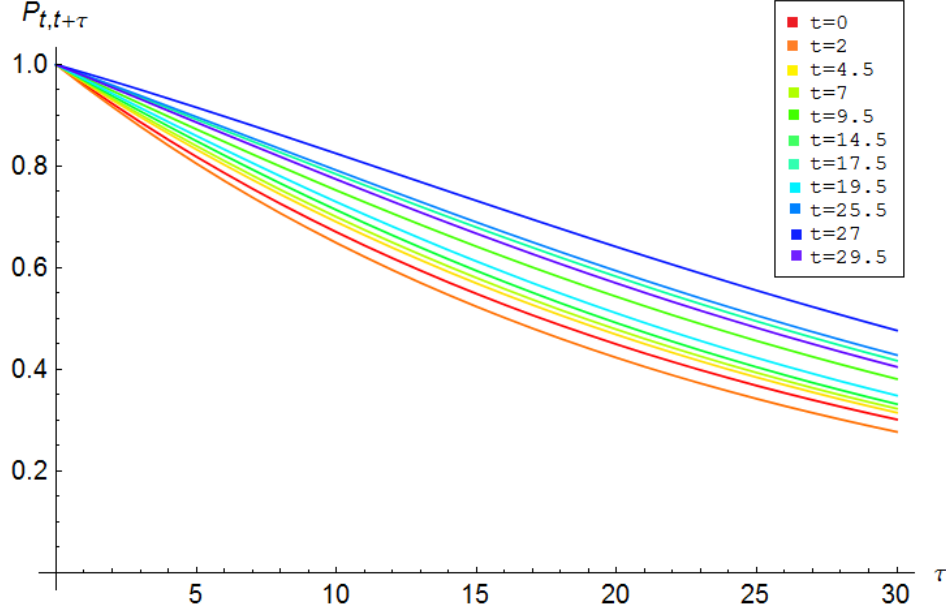


Figure 7.9: Discount bond curves for the Brownian-gamma model. We let $X = \{0, 1\}$ with $f_0(1) = 0.3$. We let $m = 2$, $\kappa = 0.2$, $\sigma = 0.1$, $c = -2$, $b = 0.03$, $P_{0t} = \exp(-0.04t)$.

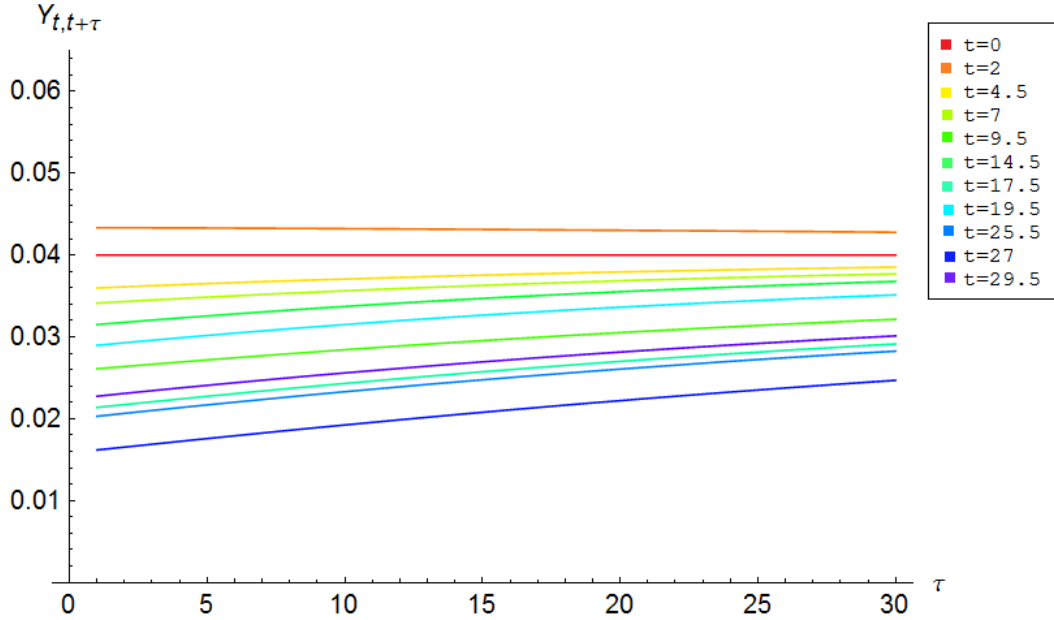


Figure 7.10: Yield curves for the Brownian-gamma model. We let $X = \{0, 1\}$ with $f_0(1) = 0.3$. We let $m = 2$, $\kappa = 0.2$, $\sigma = 0.1$, $c = -2$, $b = 0.03$, $P_{0t} = \exp(-0.04t)$.

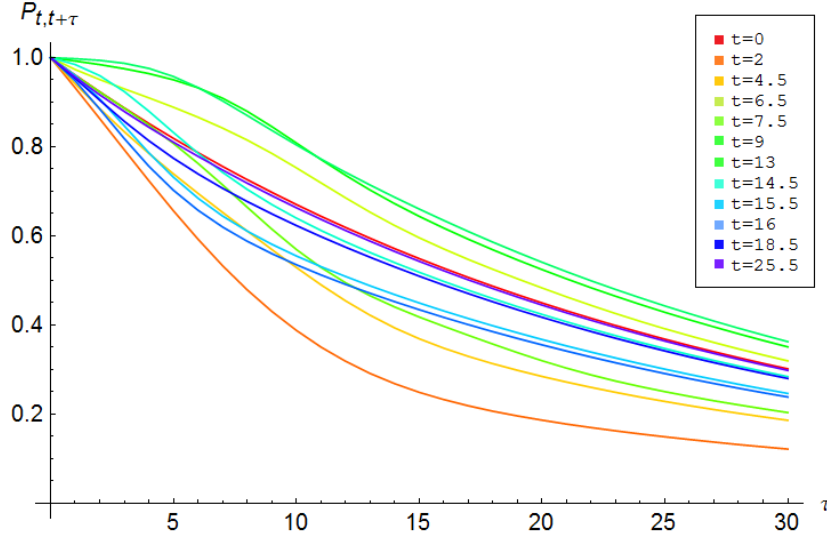


Figure 7.11: Discount bond curves for the variance-gamma model with $h(u, X) = c \exp[-b(u - X)^2]$. We let $\theta = -1.5$, $\Sigma = 2$ and $\nu = 0.25$. We set $f_0(x_1) = 0.2$, $f_0(x_2) = 0.35$, $f_0(x_3) = 0.35$, $f_0(x_4) = 0.1$ and $x_1 = 2$, $x_2 = 5$, $x_3 = 10$, $x_4 = 20$. We choose $\sigma = 0.1$, $c = 0.5$, $b = 0.015$ and the initial term structure is $P_{0t} = \exp(-0.04t)$.

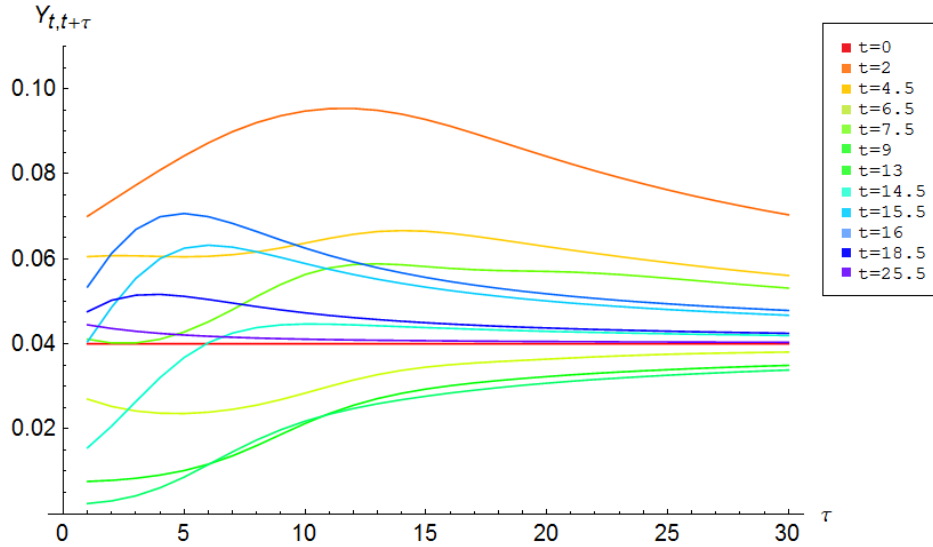


Figure 7.12: Yield curves for the variance-gamma model where $h(u, X) = c \exp[-b(u - X)^2]$. We let $\theta = -1.5$, $\Sigma = 2$ and $\nu = 0.25$. We set $f_0(x_1) = 0.2$, $f_0(x_2) = 0.35$, $f_0(x_3) = 0.35$, $f_0(x_4) = 0.1$ and $x_1 = 2$, $x_2 = 5$, $x_3 = 10$, $x_4 = 20$. We choose $\sigma = 0.1$, $c = 0.5$, $b = 0.015$ and the initial term structure is $P_{0t} = \exp(-0.04t)$.

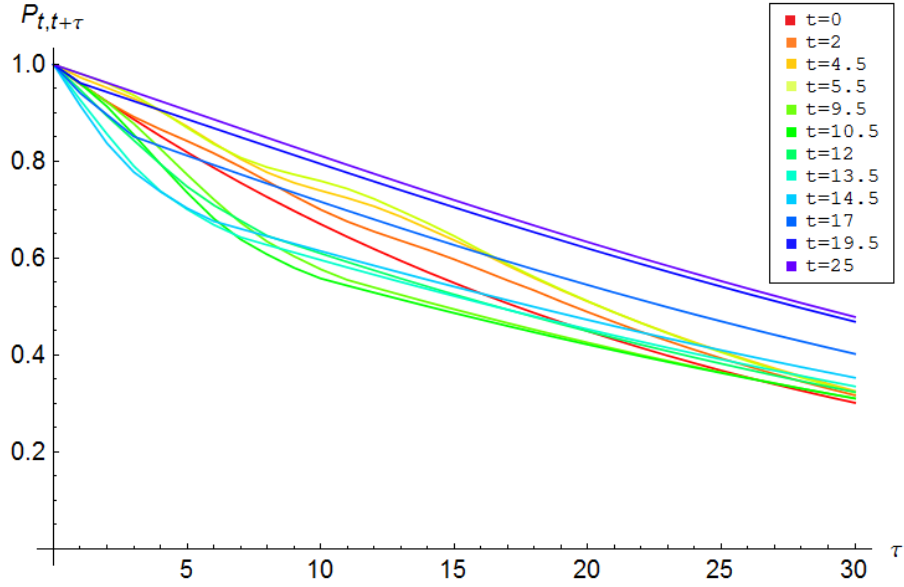


Figure 7.13: Discount bond curves for the Brownian-gamma chameleon model. We let $X = \{x_1 = 2, x_2 = 5, x_3 = 10, x_4 = 20\}$ with $f_0(x_1) = 0.15$, $f_0(x_2) = 0.35$, $f_0(x_3) = 0.35$, $f_0(x_4) = 0.15$. We let $m = 0.5$, $\kappa = 0.5$, $\sigma = 0.1$, $c_1 = -0.4375$, $c_2 = -1.25$, $\alpha_1 = 0.75$, $\alpha_2 = 0.02$, $P_{0t} = \exp(-0.04t)$.

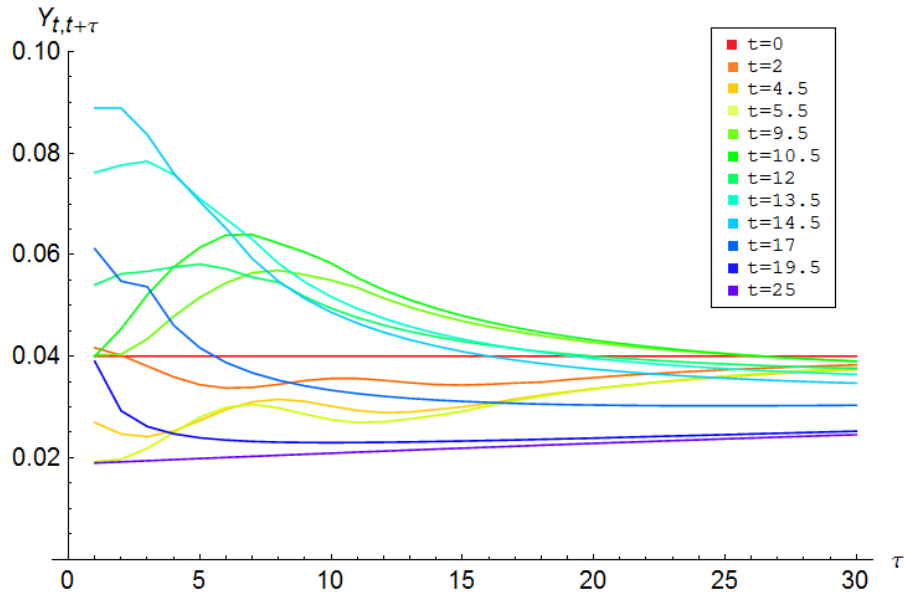


Figure 7.14: Yield curves for the Brownian-gamma chameleon model. We let $X = \{x_1 = 2, x_2 = 5, x_3 = 10, x_4 = 20\}$ with $f_0(x_1) = 0.15$, $f_0(x_2) = 0.35$, $f_0(x_3) = 0.35$, $f_0(x_4) = 0.15$. We let $m = 0.5$, $\kappa = 0.5$, $\sigma = 0.1$, $c_1 = -0.4375$, $c_2 = -1.25$, $\alpha_1 = 0.75$, $\alpha_2 = 0.02$, $P_{0t} = \exp(-0.04t)$.

7.11 Pricing of European-style bond options

Let $\{C_{st}\}_{0 \leq s \leq t < T}$ be the price process of a European call option with maturity t and strike $0 < K < 1$, written on a discount bond with price process $\{P_{tT}\}_{0 \leq t \leq T}$. The price of the option at time s is given by

$$C_{st} = \frac{1}{\pi_s} \mathbb{E} [\pi_t (P_{tT} - K)^+ | \mathcal{F}_s]. \quad (7.11.1)$$

By substituting (7.5.4) and (7.5.5) into (7.11.1), we obtain

$$C_{st} = \frac{1}{\pi_s} \mathbb{E} \left[\left(\int_T^\infty \rho(u) \widehat{M}_{tu} du - K \int_t^\infty \rho(u) \widehat{M}_{tu} du \right)^+ \middle| \mathcal{F}_s \right]. \quad (7.11.2)$$

In the single-factor models that we have considered with a Markovian information process $\{I_t\}$, we can define the region $\mathcal{V}$ by

$$\mathcal{V} := \left\{ y, z : \int_T^\infty \rho(u) \widehat{M}_{tu}(L_t = y, I_t = z) du - K \int_t^\infty \rho(u) \widehat{M}_{tu}(L_t = y, I_t = z) du > 0 \right\}. \quad (7.11.3)$$

It follows that the price of the call option is

$$C_{st} = \frac{1}{\pi_s} \int \int_{\mathcal{V}} \left(\int_T^\infty \rho(u) \widehat{M}_{tu}(y, z) du - K \int_t^\infty \rho(u) \widehat{M}_{tu}(y, z) du \right) q_s(y, z) dy dz \quad (7.11.4)$$

where

$$q_s(y, z) = \frac{\partial^2}{\partial y \partial z} \mathbb{P}[L_t \leq y, I_t \leq z | \mathcal{F}_s]. \quad (7.11.5)$$

We can use Fubini's theorem to write this more compactly in the form

$$C_{st} = \frac{1}{\pi_s} \left(\int_T^\infty \rho(u) \Phi_{tu} du - K \int_t^\infty \rho(u) \Phi_{tu} du \right), \quad (7.11.6)$$

where

$$\Phi_{tu} = \int \int_{\mathcal{V}} \widehat{M}_{tu}(y, z) q_s(y, z) dy dz. \quad (7.11.7)$$

We apply Monte Carlo techniques to simulate option price surfaces. A large number of iterations is required to obtain accurate estimates. To increase precision, variance reduction techniques or quasi-Monte Carlo methods can be considered (see Boyle et al. [12]). The choice of the random mixer affects the shape of the resulting option price surface. The simulations in Figure 7.15 are based on (i) the Brownian-gamma model constructed in Example 7.7.1, and (ii) the Brownian-gamma chameleon model in Example 7.9.1. The wave across the second option

price surface is produced by the sine function that defines part of the chameleon random mixer.

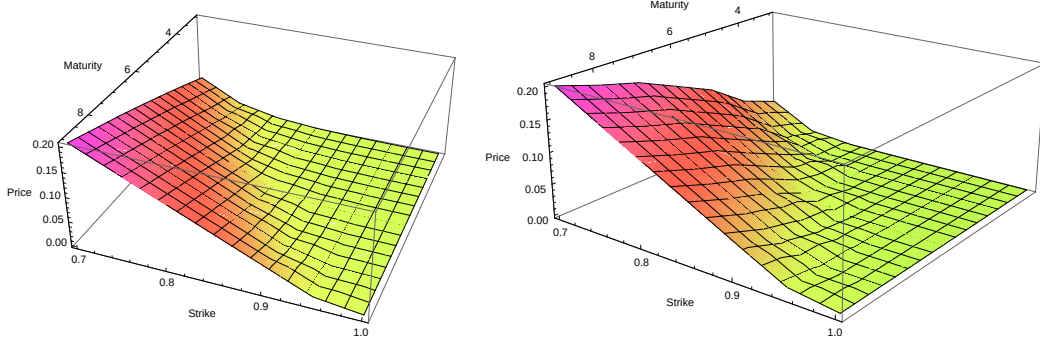


Figure 7.15: Option price surface at $s = 2$ of call options on a discount bond with $T = 10$. (i) Simulation based on the Brownian-gamma model. We set $X = \{0, 1\}$ with $f_0(1) = 0.5$, $m = 0.5$, $\kappa = 0.5$, $\sigma = 0.1$, $c = -2$, $b = 0.03$ and $P_{0t} = \exp(-0.04t)$. (ii) Simulation based on the Brownian-gamma chameleon model. We set $X = \{x_1 = 2, x_2 = 5, x_3 = 10, x_4 = 20\}$ with $f_0(x_1) = 0.15$, $f_0(x_2) = 0.35$, $f_0(x_3) = 0.35$, $f_0(x_4) = 0.15$, $m = 0.5$, $\kappa = 0.5$, $\sigma = 0.1$, $c_1 = 0.35$, $c_2 = 1$, $\alpha_1 = 3$, $\alpha_2 = 0.03$, and $P_{0t} = \exp(-0.04t)$.

7.12 Randomised heat kernel models

In Sections 7.2 and 7.3, we constructed martingales based on Lévy processes and an Esscher-type formulation. We recall that the pricing kernel is modelled by

$$\begin{aligned} \pi_t &= \int_t^\infty \rho(u) \mathbb{E}[M_{tu}(X, L_t) \mid \mathcal{F}_t] du \\ &= \int_{-\infty}^\infty \int_t^\infty \rho(u) M_{tu}(x, L_t) du f_t(x) dx. \end{aligned} \quad (7.12.1)$$

The process $\{M_{tu}(X, L_t)\}$ is a unit-initialized positive $\{\mathcal{G}_t\}$ -martingale, and the process

$$S_t(X, L_t) := \int_t^\infty \rho(u) M_{tu}(X, L_t) du \quad (7.12.2)$$

is a positive $\{\mathcal{G}_t\}$ -supermartingale. The projection of a positive $\{\mathcal{G}_t\}$ -supermartingale onto $\{\mathcal{F}_t\}$, that is

$$\pi_t := \mathbb{E}[S_t(X, L_t) \mid \mathcal{F}_t], \quad (7.12.3)$$

is an $\{\mathcal{F}_t\}$ -supermartingale (Föllmer & Protter [40], Theorem 3).

We now model the impact of uncertainty on a financial market by a process that has the Markov property with respect to its natural filtration, and which

we denote $\{Y_t\}_{t \geq 0}$. Of course, the case where $\{Y_t\}$ is a Lévy process, which is a Markov process of Feller type, is included (see Applebaum [5]). Let $\{n_t\}_{t \geq 0}$ be a pure noise process representing the observation noise, and let the filtration $\{\mathcal{G}_t\}$ be generated by

$$\mathcal{G}_t = \sigma(\{Y_s\}_{0 \leq s \leq t}, \{n_s\}_{0 \leq s \leq t}, X), \quad (7.12.4)$$

where $\{Y_t\}$, $\{n_t\}$, and the random variable X are all independent. We refrain from specifying the observation noise $\{n_t\}$ precisely since this level of detail is not required here. The noise $\{n_t\}$ could be Brownian noise or we may consider a setup with jumps such as, for instance, in Section 7.4.

Definition 7.12.1. *Let $\{Y_t\}$ be a Markov process with respect to its natural filtration. A measurable function $p : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a propagator if it satisfies*

$$\mathbb{E}[p(t, v, Y_t) \mid Y_s] = p(s, v + t - s, Y_s) \quad (7.12.5)$$

for $(v, t) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $0 \leq s \leq t$.

Let $G(\cdot)$ be a positive bounded function², and let $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$. Then we set

$$p(t, v, Y_t, X) := \mathbb{E}[G(h(t + v, X), Y_{t+v}) \mid \mathcal{G}_t]. \quad (7.12.6)$$

This is a $\{\mathcal{G}_t\}$ -propagator since X is $\mathcal{G}_0$ -measurable. It follows that

$$S_t(X, Y_t) := \int_0^\infty w(t, v) \mathbb{E}[G(h(t + v, X), Y_{t+v}) \mid \mathcal{G}_t] dv \quad (7.12.7)$$

is a $\{\mathcal{G}_t\}$ -supermartingale, see Akahori et al. [2]. Here $w(t, v)$ is a positive function that satisfies

$$w(t, v - s) \leq w(t - s, v) \quad (7.12.8)$$

for arbitrary $t, v \in \mathbb{R}_+$ and $s \leq t \wedge v$. Now we define the market filtration $\{\mathcal{F}_t\}$ by

$$\mathcal{F}_t = \sigma(\{Y_s\}_{0 \leq s \leq t}, \{I_s\}_{0 \leq s \leq t}), \quad (7.12.9)$$

where $\{I_t\}$ carries information about X , which is distorted by the pure noise $\{n_t\}$. We have that $\mathcal{F}_t \subset \mathcal{G}_t$. Then, by Föllmer & Protter [40] Theorem 3, the projection

$$\pi_t := \mathbb{E}[S_t(X, Y_t) \mid \mathcal{F}_t] \quad (7.12.10)$$

²As we stated in Section 2.1.2, once a Markov process $\{Y_t\}$ has been chosen, it may be sufficient to relax the boundedness condition, and choose $G(\cdot)$ to be integrable.

is an $\{\mathcal{F}_t\}$ -supermartingale. It follows that

$$\begin{aligned}\pi_t &= \mathbb{E} \left[\int_0^\infty w(t, v) \mathbb{E} [G(h(t+v, X), Y_{t+v}) | \mathcal{G}_t] dv \middle| \mathcal{F}_t \right], \\ &= \int_0^\infty w(t, v) \mathbb{E} [\mathbb{E} [G(h(t+v, X), Y_{t+v}) | \mathcal{G}_t] | \mathcal{F}_t] dv, \\ &= \int_0^\infty w(t, v) \mathbb{E} [G(h(t+v, X), Y_{t+v}) | \mathcal{F}_t] dv.\end{aligned}\quad (7.12.11)$$

We emphasize that in equation (7.12.11), $\mathbb{E} [G(h(t+v, X), Y_{t+v}) | \mathcal{F}_t]$ is not an $\{\mathcal{F}_t\}$ -propagator when $\{I_t\}$ is not a Markov process. Nevertheless, $\{\pi_t\}$ is a valid model for the pricing kernel, subject to regularity conditions.

7.13 Quadratic model based on the Ornstein-Uhlenbeck process

In this section, we generate term structure models by using Markov processes with dependent increments. We emphasize that such models cannot be constructed based on the filtered Esscher martingales. Let us suppose that $\{Y_t\}$ is an Ornstein-Uhlenbeck (OU) process with dynamics

$$dY_t = \delta(\beta - Y_t) dt + \Upsilon dW_t, \quad (7.13.1)$$

where δ is the speed of reversion, β is the long-run equilibrium value of the process and Υ is the volatility. Then, for $s \leq t$, the conditional mean and conditional variance are given by

$$\mathbb{E} [Y_t | Y_s] = Y_s \exp [-\delta(t-s)] + \beta (1 - \exp [-\delta(t-s)]). \quad (7.13.2)$$

$$\text{Var} [Y_t | Y_s] = \frac{\Upsilon^2}{2\delta} (1 - \exp [-2\delta(t-s)]). \quad (7.13.3)$$

Let us suppose, for a well-defined positive function $h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, that

$$G(h(v, X), Y_v) = h(v, X) Y_v^2. \quad (7.13.4)$$

Since X is $\mathcal{G}_0$ -measurable, and by applying (7.13.2) and (7.13.3), it follows that

$$\begin{aligned}p(u, t, Y_t, X) &= \mathbb{E} [h(t+u, X) Y_{t+u}^2 | \mathcal{G}_t], \\ &= h(t+u, X) \mathbb{E} [(Y_{t+u} - \mathbb{E} [Y_{t+u} | Y_t] + \mathbb{E} [Y_{t+u} | Y_t])^2 | Y_t], \\ &= h(t+u, X) [\text{Var} [Y_{t+u} | Y_t] + \mathbb{E} [Y_{t+u} | Y_t]^2], \\ &= h(t+u, X) \left[\frac{\Upsilon^2}{2\delta} (1 - e^{-2\delta u}) + [Y_t e^{-\delta u} + \beta (1 - e^{-\delta u})]^2 \right].\end{aligned}\quad (7.13.5)$$

The pricing kernel is then given by (7.12.11), and we obtain

$$\pi_t = \int_0^\infty w(t, u) \left[\frac{\Upsilon^2}{2\delta} (1 - e^{-2\delta u}) + [Y_t e^{-\delta u} + \beta (1 - e^{-\delta u})]^2 \right] \times \int_{-\infty}^\infty h(t + u, x) f_t(x) dx du. \quad (7.13.6)$$

It follows that the price of a discount bond is expressed by

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E} \left[\int_0^\infty w(T, v) \mathbb{E} [G(h(T + v, X), Y_{T+v}) | \mathcal{F}_T] dv \middle| \mathcal{F}_t \right], \quad (7.13.7)$$

where $\{\pi_t\}$ is given in (7.13.6), and the conditional expectation can be computed to obtain

$$\int_0^\infty w(T, v) \left[\frac{\Upsilon^2}{2\delta} (1 - e^{-2\delta(T+v-t)}) + [Y_t e^{-\delta(T+v-t)} + \beta (1 - e^{-\delta(T+v-t)})]^2 \right] \times \int_{-\infty}^\infty h(T + v, x) f_t(x) dx dv. \quad (7.13.8)$$

Example 7.13.1. We assume that X is a positive random variable that takes discrete values $\{x_1, \dots, x_n\}$ with *a priori* probabilities $\{f_0(x_1), \dots, f_0(x_n)\}$. We suppose that the information flow $\{I_t\}$ is governed by

$$I_t = \sigma X t + B_t. \quad (7.13.9)$$

We choose the random mixer to be

$$h(t + u, X) = c_1 \exp[-c_2(t + u - X)](t + u), \quad (7.13.10)$$

where $c_1 > 0$ and $c_2 > 0$, and we assume that the weight function is

$$w(t, u) = \exp[-j(u + t)] \quad (7.13.11)$$

for $j > 0$. Later, in Proposition 7.14.1, we show that this model belongs to the Flesaker-Hughston class. Therefore, the short rate of interest takes the form

$$r_t = \frac{e^{-jt} \mathbb{E} [G(h(t, X), Y_t) | \mathcal{F}_t]}{\int_0^\infty e^{-j(t+v)} \mathbb{E} [G(h(t + v, X), Y_{t+v}) | \mathcal{F}_t] dv}. \quad (7.13.12)$$

Next we simulate the trajectories of the discount bond and the short rate process. We refer to Iacus [59] for the simulation of the OU process using an Euler scheme. We observe oscillations in the sample paths owing to the mean-reversion in the Markov process.

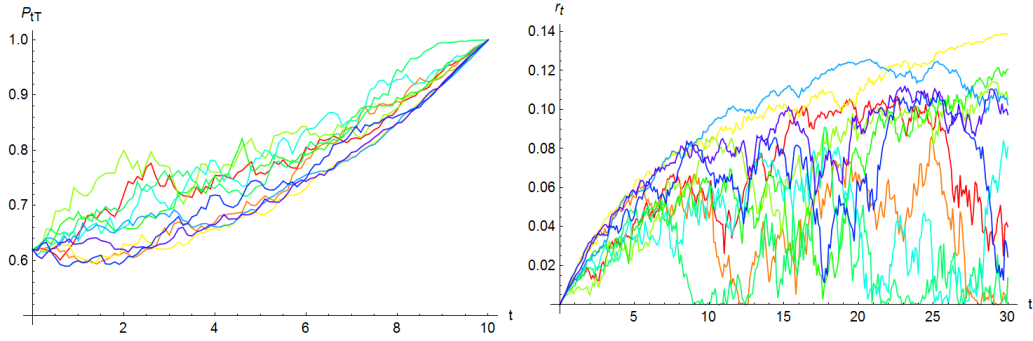


Figure 7.16: Sample paths for a discount bond with $T = 10$ and the short rate for the quadratic OU-Brownian model with $h(t+u, X) = c_1 \exp(-c_2(t+u-X))(t+u)$ with $c_1 = 0.02$ and $c_2 = 0.1$. We let $\delta = 0.02$, $\beta = 0.5$, $\Upsilon = 0.2$ and $Y_0 = 1$. We let $x_1 = 1$ and $x_2 = 2$ where $f_0(x_1) = 0.3$ and $f_0(x_2) = 0.7$ and $\sigma = 0.1$. The weight function is given by $w(t, u) = \exp[-0.04(t+u)]$.

The model-generated yield curves follow. In this example, we mostly observe changes of slope and shifts. However, it should be possible to produce changes of shape in the yield curve by varying the choices of $G(\cdot)$ and $h(\cdot)$.

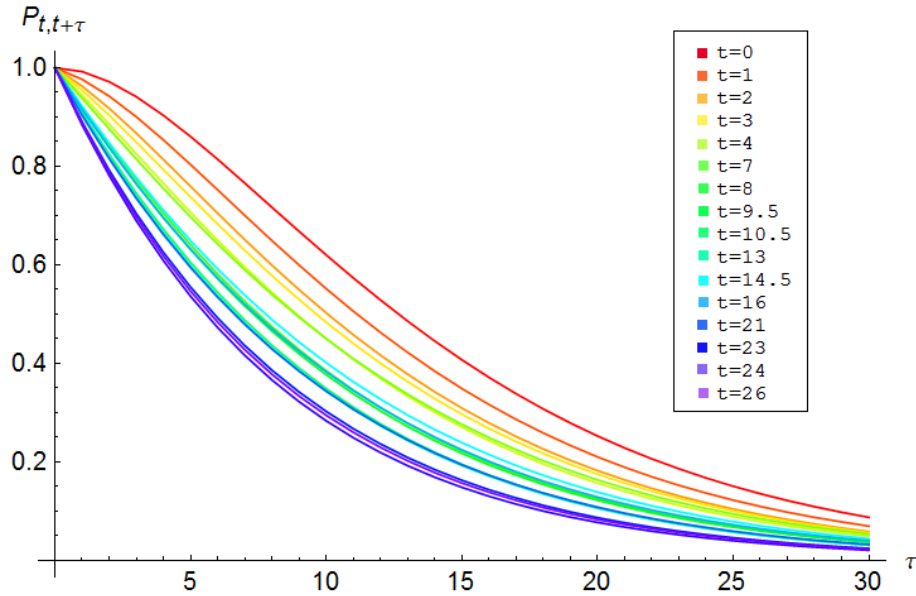


Figure 7.17: Discount bond curves for the quadratic OU-Brownian model with $h(t+u, X) = c_1 \exp(-c_2(t+u-X))(t+u)$ with $c_1 = 0.01$ and $c_2 = 0.1$. We let $\delta = 0.02$, $\beta = 0.5$, $\Upsilon = 0.2$ and $Y_0 = 1$. We let $x_1 = 1$ and $x_2 = 2$ where $f_0(x_1) = 0.5$ and $f_0(x_2) = 0.5$ and $\sigma = 0.1$. The weight function is given by $w(t, u) = \exp[-0.04(t+u)]$.

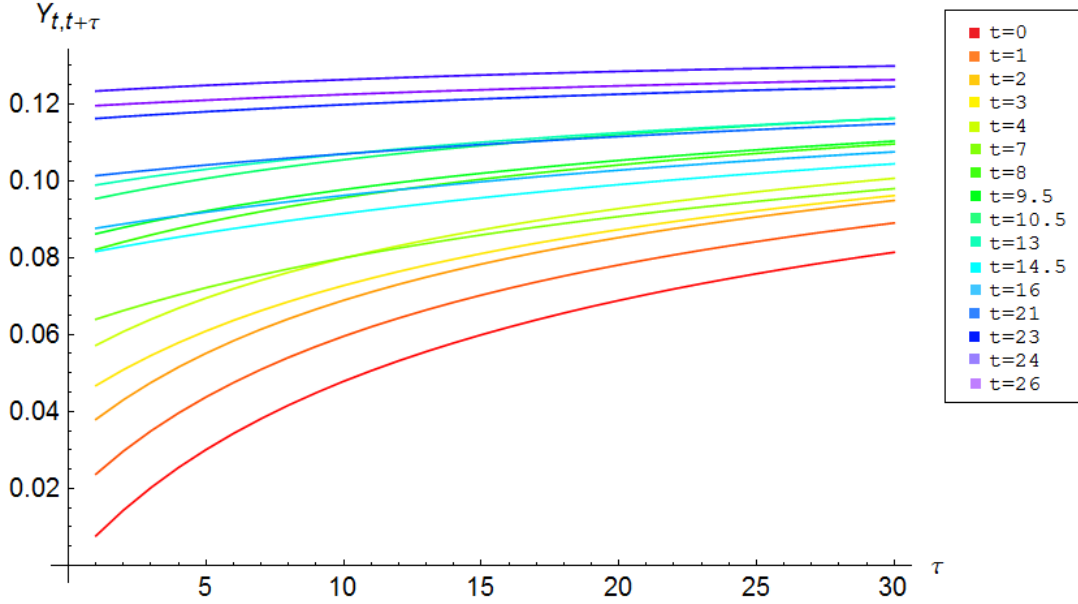


Figure 7.18: Yield curves for the quadratic OU-Brownian model with $h(t+u, X) = c_1 \exp(-c_2(t+u-X))(t+u)$ with $c_1 = 0.01$ and $c_2 = 0.1$. We let $\delta = 0.02$, $\beta = 0.5$, $\Upsilon = 0.2$ and $Y_0 = 1$. We let $x_1 = 1$ and $x_2 = 2$ where $f_0(x_1) = 0.5$ and $f_0(x_2) = 0.5$ and $\sigma = 0.1$. The weight function is given by $w(t, u) = \exp[-0.04(t+u)]$.

7.14 Classification of interest rate models

In what follows, we show that, under certain conditions, the constructed pricing kernels based on weighted heat kernel models belong to the Flesaker-Hughston class.

Proposition 7.14.1. *Let $\{Y_t\}$ be a Markov process, and let the weight function be given by*

$$w(t, v) = \bar{w}(t + v), \quad (7.14.1)$$

where $\bar{w} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded, non-increasing function. We assume that

$$\int_0^\infty \bar{w}(t + v) \mathbb{E}[G(h(t + v, X), Y_{t+v})] dv < \infty. \quad (7.14.2)$$

Then, the pricing kernel is given by

$$\pi_t = \int_0^\infty \bar{w}(t + v) \mathbb{E}[G(h(t + v, X), Y_{t+v}) | \mathcal{F}_t] dv. \quad (7.14.3)$$

It is sufficient for $\bar{w}(s) \rightarrow 0$ as $s \rightarrow \infty$ for (7.14.3) to be a potential. Moreover, (7.14.3) is a potential generated by

$$A_t = \int_0^t \bar{w}(u) \mathbb{E}[G(h(u, X), Y_u) | \mathcal{F}_u] du, \quad (7.14.4)$$

that is, a potential of class (D). Thus, the pricing kernel is of the Flesaker-Hughston type. We can write (7.14.3) in the Flesaker-Hughston form:

$$\pi_0 \int_t^\infty \rho(u) m_{tu} du, \quad (7.14.5)$$

where

$$\rho(u) = \frac{\bar{w}(u) \mathbb{E}[G(h(u, X), Y_u)]}{\pi_0} \quad \text{and} \quad m_{tu} = \frac{\mathbb{E}[G(h(u, X), Y_u) | \mathcal{F}_t]}{\mathbb{E}[G(h(u, X), Y_u)]}. \quad (7.14.6)$$

Here, $\{m_{tu}\}$ is a positive unit-initialized $\{\mathcal{F}_t\}$ -martingale for each fixed $u \geq t$. The constant π_0 is a scaling factor.

Let us now suppose that $\{Y_t\}$ is a Lévy process. We note that the class of Esscher randomised mixture models presented in this chapter, for which

$$M_{tu}(X, L_t) := \frac{\exp[h(u, X)L_t]}{\mathbb{E}[\exp[h(u, X)L_t] | X]}, \quad (7.14.7)$$

cannot be exactly constructed using the weighted heat kernel approach. We see this by setting

$$G(h(v, X), L_{t+v}) = \frac{\exp[h(v, X)L_{t+v}]}{\mathbb{E}[\exp[h(v, X)L_{t+v}] | X]}, \quad (7.14.8)$$

and by observing that $\mathbb{E}[G(h(v, X), L_{t+v}) | \mathcal{G}_t]$ is not a $\{\mathcal{G}_t\}$ -propagator. However, the weighted heat kernel approach can give rise to other kinds of interesting randomised mixture models, which cannot be produced starting from (7.14.7). As we mentioned earlier, the class of models introduced by Brody et al. [18] is included in the class of Esscher randomised mixture models. The following is a diagrammatic representation of the considered classes of non-negative interest rate models based on randomised mixtures:

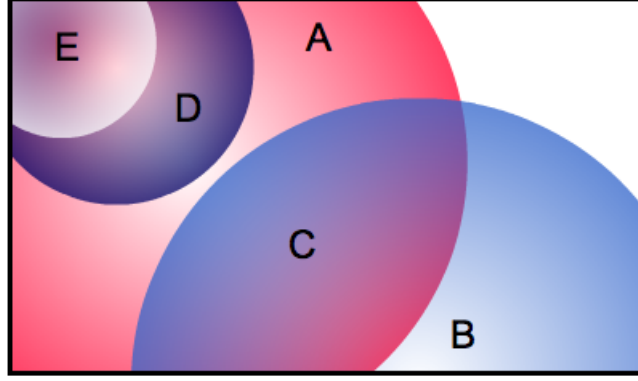


Figure 7.19: Classes of randomised mixture models with non-negative interest rates. Region A represents the Flesaker-Hughston class of models driven by randomised mixtures of stochastic processes. The pricing kernel is of the form of (7.5.4) where $\{\widehat{M}_{tu}\}_{t \leq u}$ is given by (7.3.4), and $\{M_{tu}(X)\}$ is a positive unit-initialized family of martingales with respect to an appropriate filtration which is larger than the market filtration $\{\mathcal{F}_t\}$. Region B represents the weighted heat kernel models driven by randomised mixtures of Markov processes. Here, the pricing kernel is modelled by (7.12.11). Region C is one overlapping area between the randomised mixture Flesaker-Hughston and weighted heat kernel models. From Proposition 7.14.1, such models can be generated by modelling the pricing kernel by equation (7.14.3), where $w(t, u) = \bar{w}(t + u)$. Here $\bar{w} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded, non-increasing function satisfying $\lim_{s \rightarrow \infty} \bar{w}(s) = 0$. Region D represents the class of Esscher randomised mixture models. These models are generated by the martingale family (7.2.1), where the pricing kernel is modelled by (7.5.4) and $\{\widehat{M}_{tu}\}$ is given by (7.3.4). Region E represents the class of models considered in [18] which can be recovered by setting $h(u, X) = h(u)$ in (7.2.1), and by modelling the pricing kernel as in Region D.

We conclude with the following observations. The pricing kernel models proposed in this chapter are versatile by construction, and potentially allow for many more investigations. For instance, we can think of applications to the modelling of foreign exchange rates where two pricing kernel models are selected—perhaps of different types to reflect idiosyncrasies of the considered domestic and foreign economies. In this context, it might be of particular interest to investigate dependence structures among several pricing kernel models for all the foreign economies involved in a polyhedron of foreign exchange rates. We expect the mixing function $h(u, X)$ to play a central role in the construction of dependence models. Furthermore, a recent application by Crisafi [30] of the randomised mixtures models to the pricing of inflation-linked securities may be developed further.

Appendix A

Auxiliary results

A.1 Estimate with the minimum quadratic error

Suppose that we are interested in estimating the value of a random variable ξ based on the information represented by a sigma-field $\mathcal{F}$. The following result from Xiong [99] shows that, of all $\mathcal{F}$ -measurable random variables, the conditional expectation $\mathbb{E}[\xi | \mathcal{F}]$ is the estimate with the minimum quadratic error.

Lemma A.1.1. *Let $L^2(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of all $\mathcal{F}$ -measurable square-integrable random variables. Let ξ be a square-integrable random variable in the probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Let $\mathcal{F}$ be a sub-sigma-field of $\mathcal{G}$. Then*

$$\mathbb{E} [(\xi - \mathbb{E}[\xi | \mathcal{F}])^2] = \min \{ \mathbb{E} [(\xi - \eta)^2] : \eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \}. \quad (\text{A.1.1})$$

Proof. Let $Z := \mathbb{E}[\xi | \mathcal{F}]$. Then, we obtain

$$\begin{aligned} \mathbb{E} [(\xi - \eta)^2] - \mathbb{E} [(\xi - Z)^2] &= \mathbb{E} [(Z - \eta) (2\xi - \eta - Z)] \\ &= \mathbb{E} [\mathbb{E} [(Z - \eta) (2\xi - \eta - Z) | \mathcal{F}]] \\ &= \mathbb{E} [(Z - \eta) \mathbb{E} [(2\xi - \eta - Z) | \mathcal{F}]] \\ &= \mathbb{E} [(Z - \eta)^2] \geq 0. \end{aligned} \quad (\text{A.1.2})$$

Here, we have made use of the tower property of conditional expectation and the fact that $Z - \eta$ is $\mathcal{F}$ -measurable. $\square$

A.2 Result used to derive the Zakai equation

The following result appears in van Handel [96]. We provide a detailed proof.

Lemma A.2.1. *Let $\{b_t\}$ and $\{w_t\}$ be independent $(\{\mathcal{H}_t\}, \mathbb{P})$ -Brownian motions. Let $\{F_t\}$ be a $\{\mathcal{H}_t\}$ -adapted process satisfying*

$$\int_0^t \mathbb{E}[F_s^2] ds < \infty. \quad (\text{A.2.1})$$

We define the sub-filtration $\mathcal{H}_t^w = \sigma(\{w_s\}_{s \leq t}) \subset \mathcal{H}_t$. Then

$$\mathbb{E} \left[\int_0^t F_s dw_s \middle| \mathcal{H}_t^w \right] = \int_0^t \mathbb{E}[F_s | \mathcal{H}_s^w] dw_s, \quad (\text{A.2.2})$$

$$\mathbb{E} \left[\int_0^t F_s db_s \middle| \mathcal{H}_t^w \right] = 0, \quad (\text{A.2.3})$$

$$\mathbb{E} \left[\int_0^t F_s ds \middle| \mathcal{H}_t^w \right] = \int_0^t \mathbb{E}[F_s | \mathcal{H}_s^w] ds. \quad (\text{A.2.4})$$

Proof. We choose any $A \in \mathcal{H}_t^w$. By the Itô representation theorem we can write

$$\mathbb{1}_A = \mathbb{P}(A) + \int_0^t H_s dw_s \quad (\text{A.2.5})$$

for some $\{\mathcal{H}_t^w\}$ -adapted process $\{H_t\}$ that satisfies $\int_0^t \mathbb{E}[H_s^2] ds < \infty$. It follows that

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_A \int_0^t F_s dw_s \right] &= \mathbb{E} \left[\int_0^t F_s H_s ds \right] \\ &= \mathbb{E} \left[\int_0^t \mathbb{E}[F_s | \mathcal{H}_s^w] H_s ds \right] \\ &= \mathbb{E} \left[\left(\int_0^t H_s dw_s \right) \left(\int_0^t \mathbb{E}[F_s | \mathcal{H}_s^w] dw_s \right) \right] \\ &= \mathbb{E} \left[\mathbb{1}_A \int_0^t \mathbb{E}[F_s | \mathcal{H}_s^w] dw_s \right], \end{aligned} \quad (\text{A.2.6})$$

by applying the Itô isometry. The second equality follows by Fubini's theorem and the tower property of conditional expectation. Furthermore, by the tower rule and since $A \in \mathcal{H}_t^w$, we can write

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_A \int_0^t F_s dw_s \right] &= \mathbb{E} \left[\mathbb{E} \left[\mathbb{1}_A \int_0^t F_s dw_s \middle| \mathcal{H}_t^w \right] \right] \\ &= \mathbb{E} \left[\mathbb{1}_A \mathbb{E} \left[\int_0^t F_s dw_s \middle| \mathcal{H}_t^w \right] \right]. \end{aligned} \quad (\text{A.2.7})$$

From (A.2.6) and (A.2.7), we see that (A.2.2) holds by the properties of the conditional expectation. By similar reasoning we obtain

$$\mathbb{E} \left[\mathbb{1}_A \int_0^t F_s \, db_s \right] = 0 = \mathbb{E} \left[\mathbb{1}_A \mathbb{E} \left[\int_0^t F_s \, db_s \mid \mathcal{H}_t^w \right] \right]. \quad (\text{A.2.8})$$

In so doing, we show that (A.2.3) holds. Finally, to prove (A.2.4), we note that

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_A \int_0^t F_s \, ds \right] &= \mathbb{E} \left[\mathbb{E} \left[\mathbb{1}_A \int_0^t F_s \, ds \mid \mathcal{H}_t^w \right] \right] \\ &= \mathbb{E} \left[\mathbb{1}_A \int_0^t \mathbb{E}[F_s \mid \mathcal{H}_t^w] \, ds \right] \\ &= \mathbb{E} \left[\mathbb{1}_A \int_0^t \mathbb{E}[F_s \mid \mathcal{H}_s^w] \, ds \right] \end{aligned} \quad (\text{A.2.9})$$

for $A \in \mathcal{H}_t^w$. Here, we have used the tower property of the conditional expectation and Fubini's theorem. The last equality follows since $\{F_s\}$ is $\{\mathcal{H}_s\}$ -measurable and is independent of $\mathcal{H}_{t,s}^w = \sigma(\{w_r - w_s\}_{s \leq r \leq t})$, where $\mathcal{H}_t^w = \sigma(\mathcal{H}_s^w, \mathcal{H}_{t,s}^w)$. $\square$

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