Differentiating this with respect to t gives

$$Df(x(t))\frac{dx}{dt} = \frac{f(x(t))a(t)}{\|f(x(t))\|} = -f(x(t))\lambda(t),$$

where $a(t)$ and $\lambda(t)$ are some scalar constants. Since they only indicate the speed at which we move along the path, their magnitude is unimportant. The sign of $\lambda(t)$ is important however, and an orientation argument shows that the sign is that of $\det(Df(x))$. Thus, beginning at almost any point on the boundary of D^n and following this differential equation, leads to an equilibrium.

Now suppose that $f : D^l \to \mathbb{R}^l$ is only continuous but still satisfies (SB). Define a new continuous map $f_0 : D^l_2 \to \mathbb{R}^l$ (where D^l_2 is the disk in $\mathbb{R}^l$ of radius 2) by

$$f_0(x) = \begin{cases} f(x) & \text{for } \|x\| \le 1 \\ -x & \text{for } \|x\| \ge 1. \end{cases}$$

Take a sequence of $\epsilon_i \to 0$. For each i construct a C^∞ approximation f_i of f_0, such that $\|f_i(x) - f_0(x)\| < \epsilon_i$ for all $x \in D^l_2$.

What we have shown then implies that the exists an $x_i \in D^l_2$ with $f_i(x_i) = 0$. Clearly $x_i \in D^l$ and also $x_i \to \{x \in D^l | f(x) = 0\}$ as $i \to \infty$. This proves the theorem in the case of the strong boundary condition (SB).

Finally, suppose $f : D^l \to \mathbb{R}^l$ is continuous and satisfies the weaker boundary condition (B_D). Define a continuous map $\hat{f} : D^l_2 \to \mathbb{R}^l$ such that $\hat{f}(x) = -x$ for all $x \in \partial D^l_2$ as

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } \|x\| \le 1, \\ (2 - \|x\|)f\left(\frac{x}{\|x\|}\right) + (\|x\| - 1)(-x) & \text{for } \|x\| \ge 1. \end{cases}$$

We have shown that there is an $x^* \in D^l_2$ with $\hat{f}(x^*) = 0$. But $\|x^*\| \le 1$ for otherwise the boundary condition would be violated. Thus $f(x^*) = 0$ and we are done. □

For the main results on the existence of equilibria we modify the theorem 6.1 from disks to simplicies. Define

$$\Delta_1 = \left\{p \in P \middle| \sum p^i = 1\right\}$$
$$\partial\Delta = \{p \in \Delta_1 | \text{ some } p^i = 0\}$$
$$\Delta_0 = \left\{z \in \mathbb{R}^l \middle| \sum z^i = 1\right\}$$
$$p_c = (1/l, \ldots, 1/l) \in \Delta_1$$

p_c being the centre of Δ_1.

Theorem 6.2. let $\phi : \Delta_1 \to \Delta_0$ be a continuous map satisfying the boundary condition

B: $\phi(p)$ is not of the form $\mu(p - p_c), \mu > 0$ for $p \in \partial\Delta_1$.

[c] If $Df(x)$ is non-singular, the Euler's method of discrete approximation yields

$$x_n = x_{n-1} \pm Df(x_{n-1})^{-1} f(x_{n-1}),$$

which with fixed sign, is Newton's method of solving $f(x) = 0$. Thus the Global Newton is indeed a global version of Newton's method in some reasonable sense.

Then there is a $p^* \in \Delta_1$ with $\phi(p^*) = 0$.

Proof. A "ray" preserving homeomorphism into the situation of theorem 6.1 is constructed.

Define $h : \Delta_1 \to \Delta_0$ by $h(p) = p - p_c$. Let $\beta : \Delta_0 \backslash 0 \to \mathbb{R}^+$ be the map

$$\beta(p) = -\left(\frac{1}{l}\right)\left(\frac{1}{\min_i p_i}\right).$$

Then let $D = D^l \cap \Delta_0; \psi : D \to f(\Delta_1)$ defined by $\psi(p) = \mu(p/\|p\|)p$. ψ is a ray preserving homeomorphism.

Consider the composition $\alpha : D \to \Delta_0$,

$$D \xrightarrow{\psi} h(\Delta_1) \xrightarrow{h^{-1}} \Delta_1 \xrightarrow{\phi} \Delta_0.$$

To show that α satisfies the boundary condition (B_D) of theorem 6.1. consider $q \in \partial D$ and let $r = \psi(q) + p_c = h^{-1}\psi(q)$. The boundary condition insures that there is no $\mu > 0$ with $\phi(p) = \mu(p - p_c)$ or with $\mu(p - p_c) = \alpha(p)$. Equivalently there is no $\mu > 0$ with $\alpha(q) = \mu\psi(q)$, and since ψ is ray preserving that means that $\alpha(q) \neq \mu q, \mu > 0$.

We now conclude from theorem 6.1 that there is a $q^* \in D$ with $\alpha(q^*) = 0$; or if $p^* = \psi(q^*) + p_c$ then $\phi(p^*) = 0$. □

Proof of theorem E. Define from the excess demand function of the theorem, $f : \bar{P}\backslash\{0\} \to \mathbb{R}^l$, a new map $\phi : \Delta_1 \to \Delta_0$ by $\phi(p) = f(p) - \left(\sum f^i(p)\right)p$. Since $\sum \phi^i(p) = \sum f^i(p) - \sum f^i(p) \sum p^i = 0$, clearly ϕ is well defined. ϕ is also continuous. If $p \in \partial\Delta_1, p^i = 0$ for some i and so $\phi^i(p) = f^i(p) \geq 0$. Thus (B) of theorem 6.2 is satisfied and there is an $p^* \in \Delta_1$ with $\phi(p^*) = 0$ or $f(p^*) = \sum f^i(p^*)p^*$. Taking the dot product of each side with $f(p^*)$ and applying Walras' law shows that $\|f(p^*)\|^2 = 0$ or $f(p^*) = 0$. □

The methods presented here have been successfully generalised by Hirsch and Smale [13], in particular to finding zeros of polynomial systems. Smale [32] has also considered the speed of convergence of some of these methods.

7. Locally Stable Price Mechanisms

7.1. **Introduction.** An *effective price mechanism* is one, like the GNM whose solution curves always tend to an equilibrium of an economy. In §5 we showed that

$$\dot{p} = f(x) \tag{7.1}$$

was not effective, while in §6 we showed that effective price mechanisms do exist. The difference between the two mechanisms is their informational requirement. For 7.1 the market needs to know the price of each commodity; if one were to use the Global Newton method or other "quasi-Newton" methods, the information necessary for the dynamic increases to include the Jacobian matrix for the economy, that is, how a change in the j-th commodity effects the rate of change of demand for the k-th commodity for all j and k. For all practical purposes, this amount of information is to vast to be known. Consequently, the natural question is whether there exist effective price mechanisms with more modest demands on information content. This section begins to answer such questions.

The results presented here show that the informational requirements of effective price mechanisms cannot be significantly less than that of the Global Newton Method. We follow the paper of Jordan [14].

7.2. **Definitions and preliminary results.** Certain definitions from previous sections will be tailored so as to deal more effectively with the local perspective of this section. We will deal with an l good economy. Notationally it will often be convenient to let $n = l - 1$ and to denote the tangent space to S at a point p simply by $\mathbb{R}^n$.

Definition 7.1. Let Q be an open subset of S. The *space of market excess demand functions*, denoted $\mathcal{M}$, is the space of C^1 sectionfrom Q to TQ with the topology of C^1 uniform convergence on compact subsets of Q, that is, the C^1 weak topology $C^1_w(Q, TQ)$. For each $p \in Q$ let $\mathcal{M}_p = \{f \in \mathcal{M} | f(p) = 0 \text{ and } \det(Df(p)) \neq 0\}$.

The open set Q should be interpreted as small. In particular, if K is a compact cube in P and Q is a neighbourhood of $K \cap S$, then for any $f \in \mathcal{M}_p$, f can easily be extended to a function $\bar{f} : S \to \mathbb{R}^n$ satisfying

1. $\bar{f}$ is bounded from below;
2. for any sequence $\{p_k\}_{k=1}^{\infty} \in S$ with no cluster points in S, $\|\bar{f}(p_k)\| \to \infty$; and
3. $\bar{f}|_K = f|_K$.

This section covers only local stability and since the first two conditions are global in nature they are ignored in the definition of $\mathcal{M}$. Initially we shall not distinguish between the functions f and $\bar{f}$ even though working with $\bar{f}$ has the potentially negative side effect of introducing additional equilibria. This problem shall however be dealt with in §7.4.

Definition 7.2. A *price mechanism* is a function $M : Q \times \mathcal{M} \to TQ$ which maps each $(p, f) \in Q \times \mathcal{M}$ to a vector in $T_pQ (\approx \mathbb{R}^n)$ such that

1. $M(p, f) = 0$ if and only if $f(p) = 0$; and

2. There is an open set $\mathcal{U} \subset Q \times \mathcal{M}$ with $\{p\} \times \mathcal{M}_p \subset \mathcal{U}$ such that
   (a) M is continuous on $\mathcal{U}$; and
   (b) $M(\cdot, f)$ is C^1 on the open set $\{p | (p, f) \in \mathcal{U}\}$.

Given $f \in \mathcal{M}$ and $p' \in P$, a price mechanism determines the ordinary differential equation[17]

$$\dot{p} = M(p, f), \quad p(0) = p'. \tag{7.2}$$

Price mechanisms will be characterised by their response to perturbations of the excess demand functions. For fixed p' consider the two functions from $\mathcal{M} \backslash \mathcal{M}_{p'}$ to $\mathbb{R} \backslash \{0\}$ defined by

$$f \mapsto M(p', f)$$
$$f \mapsto \pi(\text{zero}(f) - p')$$

where zero(f) is an equilibrium point of f near p' and π is the projection of the vector into the space $T_{p'}Q$. The first map describes th initial movement under the price mechanism while the second map describes the dir tion of actual equilibrium. The first major step (theorem F) will be to show that the "price paths" generated by a "stable mechanism" trace out a homotopy between these two maps. This will then be used to establish a convenient characterisation of stable price mechanisms.

Definition 7.3. Given a price mechanism M, let $D = \{(p, f) | (7.2)$ has a unique solution on the entire real domain $[0, \infty)\}$. Define the function $q : D \times [0, \infty) \to P$ by setting $q(p, f; t)$ equal to the solution of (7.2) at time t. A price mechanism is *locally stable* if for each $p' \in Q$ and each neighbourhood U of p', there is an open neighbourhood $\mathcal{U}$ of $\{p'\} \times \mathcal{M}_{p'}$ such that for each $(p, f) \in \mathcal{U}$,

I. $q(p, f; t) \in U$ for all t; and
II. $\lim_{t \to \infty} q(p, f; t)$ exists and is in $\{p^* | f \in \mathcal{M}_{p^*}\}$.

We shall suppose, with out loss of generality, that $\mathcal{U}$ is small enough such that the second condition of definition 7.2 is satisfied.

Condition II is known traditionally as *asymptotic stability*. Condition I is a variation of *Lyapunov stability*. If f was fixed, the first condition would be the traditional definition of Lyapunov stability. Classically, Lyapunov and asymptotic stability together are known as local stability.

The motivation for our departure from the traditional definitions is that deviation from economic equilibria is more likely to come from a change in the demand function than from an exogenous change in the price system. Thus the first condition of definition 7.3 is also one of Lyapunov stability for small perturbations in f.

The two stability conditions together imply that q converges continuously to equilibrium. This fact will be used in order to construct a homeomorphism in theorem F.

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[17] For the "ordinary" Newton method mentioned previously $M(p, f) = -(Df(p))^{-1} f(p)$ which is well defined if $Df(p)$ is non-singular. The Newton method can be extended to $Q \times \mathcal{M}$ by specifying $M(p, f) = f(p)$ if $Df(p)$ is singular. This satisfies the two conditions of the definition and indicates why the regularity conditions (part 2 of the definition) are not imposed on all of $Q \times \mathcal{M}$. We shall show in proposition 7.19 that this awkwardness cannot be avoided.

Definition 7.4. Let $[0,\infty]$ be the one point compactification of $[0,\infty)$ with sets of the form $(a,\infty)\cup\{\infty\}$ forming a neighbourhood base at ∞.

Note that the function $g:[0,1]\to[0,\infty]$ defined by

$$g(x)=\begin{cases}\frac{x}{1-x} & \text{if } x<1,\\ \infty & \text{if } x=1\end{cases}$$

is a homeomorphism.

Lemma 7.5. Suppose that M is locally stable. Let $p'\in Q$, $f'\in\mathcal{M}_{p'}$ and let U be a neighbourhood of p' in Q. Let $\mathcal{U}$ be an open neighbourhood of $\{p'\}\times\mathcal{M}_{p'}$ as in definition 7.3 and for each $(p,f)\in\mathcal{U}$, define $q(p,f;\infty)=\lim_{t\to\infty}q(p,f;t)$. Then q is continuous on $\mathcal{U}\times[0,\infty]$.

Proof. The continuity of q on $\mathcal{U}\times[0,\infty)$ follows from the second condition of definition 7.2 and standard results regarding the continuity of solutions to ordinary differential equations. To prove continuity at $t=\infty$, let $\{p_n,f_n;t_n\}_{n=1}^{\infty}\subset\mathcal{U}\times[0,\infty)$ converging to $(p,f;\infty)$ for some $(p,f)\in\mathcal{U}$ and let $\bar{p}=q(p,f;\infty)$ which is well defined by the asymptotic stability condition in definition 7.3. Let $\epsilon>0$ and using I, let $\mathcal{U}^*$ be a neighbourhood of $(\bar{p},f)$ such that for each $(p^*,f^*)\in\mathcal{U}^*$, $\rho(q(p^*,f^*;t)-\bar{p})<\epsilon$ for all t. Here (and throughout this section) ρ is the metric on the $l-1$ sphere. Again, by the asymptotic stability condition there is some $t^*>0$ such that $(q(p,f;t^*),f)\in\mathcal{U}^*$. Since $(p_n,f_n)\to(\bar{p},f)$, we have $(q(p_n,f_n;t^*),f_n)\in\mathcal{U}'$ for sufficiently large n. Hence for large n,

$$\rho(q(p_n,f_n;t_n);\bar{p})=\rho(q(q(p_n,f_n;t^*),f_n;t_n-t^*);\bar{p})<\epsilon.$$

□

Definition 7.6. If X is a topological space and $A\subset X$, let

$$X\backslash A=\{x\in X|x\notin A\}$$

with the topology it inherits as a subspace of X. If f and g are continuous functions from $X\backslash A$ to $Y\backslash B$, f and g are *homotopic*, written $f\simeq g$, if there is a continuous function $H:(X\backslash A)\times[0,1]\to Y\backslash B$ with $H(\cdot,0)=f$ and $H(\cdot,1)=g$.

Let $p'\in Q$, U' be a neighbourhood of p' in Q and let $\mathcal{U}$ be an open neighbourhood of $\{p'\}\times\mathcal{M}_{p'}$ as before, to remain fixed from now on. Let $\mathcal{M}'=\{f\in\mathcal{M}|(p',f)\in\mathcal{U}\}$. Then $\mathcal{M}'$ is an open neighbourhood of $\mathcal{M}_{p'}$. Define $q':\mathcal{M}'\to\mathbb{R}^n$ by $q'(f)=\pi(q(p',f,\infty)-p')$ where, as usual, π is the projection into the tangent plane at p'. Then $q':\mathcal{M}_{p'}\to\{0\}$ and $q':\mathcal{M}'\backslash\mathcal{M}_{p'}\to\mathbb{R}^n\backslash\{0\}$. Also, $M(p',\cdot):\mathcal{M}_{p'}\to\{0\}$ and $M(p',\cdot):\mathcal{M}'\backslash\mathcal{M}_{p'}\to\mathbb{R}^n\backslash\{0\}$. The function $f\mapsto M(p',f)$ describes the initial direction of movement dictated by the adjustment mechanism and $f\mapsto q'(f)$ gives the direction of actual equilibrium from the initial point p'. To characterise stable mechanisms the former shall be related to the latter. This is done by establishing a homotopy equivalence between the adjustment mechanism and the equilibrium function. This equivalence is established in the following theorem which is the basis for all the remaining results in this section.

Theorem F. The functions $q':\mathcal{M}'\backslash\mathcal{M}_{p'}\to\mathbb{R}^n\backslash\{0\}$ and $M(p',\cdot):\mathcal{M}'\backslash\mathcal{M}_{p'}\to\mathbb{R}^n\backslash\{0\}$ are homotopic[18].

Proof. Define the function $H : (\mathcal{M}'\backslash\mathcal{M}_{p'}) \times [0,\infty] \to \mathbb{R}^n\backslash\{0\}$ by

$$H(f,\lambda) = \begin{cases} \pi(q(p',f;\lambda) - p') & \text{if } \lambda \geq 1 \\ \pi\left(q(p',f;\lambda) - p'\right)/\lambda) & \text{if } 0 < \lambda < 1 \\ M(p',f) & \text{if } \lambda = 0. \end{cases}$$

Then $H(\cdot,0) = M(p',\cdot)$ and $H(\ ,\infty) = q'$. The continuity of H follows from part 2 of definition 7.2 and the definition of q'. Since $[0,\infty]$ is homeomorphic to $[0,1]$, the proof is complete. □

7.3. **Price Mechanisms at a Fixed Initial Condition.** As mentioned in the introduction to this section and proved in previous sections, the price mechanism $\dot{p} = f(p)$ is not 'effective', while that of the Global Newton method is "effective" (here effective means that the solution curve of equation 7.2 tends to a zero of f as $t \to \infty$). Also, as noted in the introduction to this section, the difference between these two mechanisms is their informational requirements : one depending on $f(p)$ and the other on both $f(p)$ and $Df(p)$. We now decompose a general price mechanism into components from which we seek to analyse the contribution of the various types of information in an effective general price mechanism. (For example, in the case of the Global Newton method, we wish to examine exactly how many terms in the Jacobian, $Df(p)$, are necessary if the price mechanism is to remain stable).

To this end, fix an initial price p', and decompose the function $M(\cdot,p')$ by additively decomposing the demand function f into the vector of excess demands $f(p')$ and the function $f - f(p')$:

Suppose that $A : \mathbb{R}^n \times \mathcal{M}_{p'}$ is a continuous function that satisfies for each $f \in \mathcal{M}'$

$$M(p',f) = A\left(f(p'), f - f(p')\right).$$

Note that for each $f \in \mathcal{M}_{p'}$, $A(\cdot,f) : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\}$.

Let $Gl(n)$ denote the space of linear functions from $\mathbb{R}^n\backslash\{0\}$ to $\mathbb{R}^n\backslash\{0\}$ with the topology of uniform convergence on compact subsets of $\mathbb{R}^n\backslash\{0\}$. Each $N \in Gl(n)$ is represented by a non-singular $n \times n$ matrix. The topology of uniform convergence on compact subsets agrees with the topology on $Gl(n)$ viewed as a subspace of $\mathbb{R}^{n^2}$. Define a function $i : \mathcal{M}_{p'} \to Gl(n)$ by $i(f) = -(Df(p'))^{-1}$. Clearly i is continuous.

Remark 7.7. The function $M(p',\cdot)$ is continuous on the set $\mathcal{M}'$. The definition of A extends the domain of continuity to the set $\mathbb{R}^n \times \mathcal{M}_{p'}$. That is, $M(p,\cdot)$ is required to be continuous at all excess demand vectors $f(p')$ such that $Df(p')$ is non-singular. This definition does not place a substantive restriction on the class of price mechanisms and none of the subsequent results would be affected if we adopted the laborious process of keeping track of the proper domains of continuity.

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[18]This homotopy equivalence'is necessary by not sufficient for local stability. For example if $n = 2$ let M^1 denote the Newton method and for each $-1 \leq \lambda \leq 1$, let $R(\lambda)$ denote the rotation matrix

$$\begin{bmatrix} \lambda & -(1-\lambda^2)^{1/2} \\ (1-\lambda^2)^{1/2} & \lambda \end{bmatrix}$$

Then for each λ, the price mechanism defined by $M^\lambda(p,f) = R(\lambda)M^1(p,f)$ is homotopic to the Newton method but $M^{-1} = -M^1$ is explosive and M^0 is cyclic.

Theorem 7.8. Suppose that M is locally stable. Then for each $f \in \mathcal{M}_{p'}$, $A(\cdot, f)$ and $i(f)$ are homotopic as maps from $\mathbb{R}^n\backslash\{0\}$ to $\mathbb{R}^n\backslash\{0\}$. In particular, if $f, \bar{f} \in \mathcal{M}_{p'}$ with sign (det $(Df(p'))) \neq$ sign (det $(D\bar{f}(p')))$ then $A(\cdot, f)$ and $A(\cdot, \bar{f})$ do not coincide on any neighbourhood of 0 in $\mathbb{R}^n$.

Proof. Let $f \in \mathcal{M}_{p'}$, and let U be a closed ball about 0 in $\mathbb{R}^n$ such that $\{f + z | z \in U\} \subset \mathcal{M}'$, where $\mathcal{M}'$ is specified as before and $f + z$ denotes the function $x \mapsto f(x) + z$. By theorem F, there is a homotopy $H : (U\backslash\{0\}) \times [0, 1] \to \mathbb{R}^n\backslash\{0\}$ with $H(\,, 0) = A(\cdot, f)$ and $H(z, 1) = q'(f + z)$ for each $z \in U\backslash\{0\}$ and q' defined as before. (This can be seen by identifying the function $f_1 \in \mathcal{M}\backslash\mathcal{M}_{p'}$ from the homotopy in theorem F with the function $f = f_1 - z$ where $z = f_1(p') \neq 0$ in this theorem). Define the function $H' : (U\backslash\{0\}) \times [0, 1] \to \mathbb{R}^n$ by

$$H'(z, \lambda) = \begin{cases} q'(f + z) & \text{if } \lambda = 1, \\ \lambda^{-1} q'(f + \lambda z) & \text{if } 0 < \lambda < 1, \\ (-Df(p'))^{-1} z & \text{if } \lambda = 0, \end{cases}$$

and define the function $G : (U\backslash\{0\}) \times [0, 1] \to \mathbb{R}^n$ by

$$G(z, \lambda) = \begin{cases} H(z, 2\lambda) & \text{if } \lambda \leq \frac{1}{2}, \\ H'(z, 2 - 2\lambda) & \text{if } \lambda > \frac{1}{2}. \end{cases}$$

Then G is a homotopy between $A(\cdot, f)$ and $i(f)$ as maps on $U\backslash\{0\}$. Let ϵ denote the radius of U and define $A^*, q^* : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\}$ by

$$A^*(z) = \begin{cases} A(z, f) & \text{if } z \in U\backslash\{0\}, \\ A(\epsilon \|z\|^{-1} z, f) & \text{otherwise}, \end{cases}$$

and

$$q^*(z) = \begin{cases} -(Df(p'))^{-1} z & \text{if } z \in U\backslash\{0\}, \\ -(Df(p'))^{-1}(\epsilon \|z\|^{-1} z) & \text{otherwise}. \end{cases}$$

Modifying G in the same way gives a homotopy between A^* and q^*. The function $H'' : \mathbb{R}^n\backslash\{0\} \times [0, 1] \to \mathbb{R}^n\backslash\{0\}$ given by

$$H''(z, \lambda) = \begin{cases} A^*(z) & \text{if } z \in U\backslash\{0\}, \\ A\left(\lambda\epsilon \|z\|^{-1} z + (1 - \lambda) z, f\right) & \text{otherwise}, \end{cases}$$

shows that $A(\cdot, f) \simeq A^*$. Similarly, $q^* \simeq i(f)$ which proves the first assertion.

If sign (det $Df(p')) \neq$ sign (det $D\bar{f}(p')$), then $i(f)$ and $i(\bar{f})$, which are linear maps, also have determinants of differing sign by the definition of i. Therefore, for every neighbourhood V of 0, $i(f)$ and $i(\bar{f})$ are not homotopic as maps from $V\backslash\{0\}$ to $\mathbb{R}^n\backslash\{0\}$. Hence $A(\cdot, f)|_V \neq A(\cdot, \bar{f})|_V$. □

The implication that $A(\cdot, f)$ is sensitive to the sign of det $(Df(p'))$ is rather a coarse implication of theorem F. By using some more homotopy theory sharper implications can and shall be obtained later in this section. For now however, by strengthening the stability requirements in a conventional way, much stronger results will be obtained directly.

Given a price mechanism M and a demand function $f \in \mathcal{M}_{p'}$ we define a function $M' : Q \times \mathbb{R}^n \to \mathbb{R}^n$ by $M'(p, z) = M(p, f + z - f(p))$. Then (i) $M'(p', z) = A(z, f)$ for all z, (ii) $M'(p, f(p)) = M(p, f)$ for all p and (iii) $M'(p, 0) = 0$ for all p.

If we assume that M is sufficiently smooth so that M' is C^1 at $(p', 0)$ we obtain

$$
\begin{aligned}
D_p M(p', f) &= D_p M'(p', 0) + D_z M'(p', 0) Df(p') && \text{by (ii)} \\
&= D_z M'(p', 0) Df(p') && \text{by (iii)} \\
&= D_z A(0, f) Df(p') && \text{by (i)}.
\end{aligned}
$$

Hence the sufficient condition for local stability at p' (i.e. that $D_p M(p', f)$ has eigenvalues with negative real parts) becomes the requirement that all eigenvalues of $D_z A(0, f) Df(p')$ have negative real part. This motivates the following definition.

Definition 7.9. A price mechanism is *hyperbolically locally stable* (HLS) if

1. for each $f \in \mathcal{M}_{p'}$, the function $z \mapsto A(z, f)$ is C^1,
2. the function $f \mapsto D_z A(0, f)$ is continuous on $\mathcal{M}_{p'}$, and
3. for each $f \in \mathcal{M}_{p'}$, all the eigenvalues of the matrix $D_z A(0, f) Df(p')$ have negative real parts.

Note that the property of hyperbolic local stability is a property of A, that is $M(p', \cdot)$. The behaviour of M at prices other that p' is not formally involved.

A mathematically convenient HLS price mechanism called the orthogonal Newton is now defined. This mechanism will be shown to contain the "minimum information" required for a price mechanism to be HLS. It is defined as a modification of the (obviously HLS) Newton method.

If $Df(p)$ is non-singular, Let $D^* f(p)$ denote the orthonormal matrix obtained by applying the Gram-Schmidt process to the columns of $Df(p)$. This process is defined inductively as follows : The first column of $D^* f(p)$ is the first column of $Df(p)$ normalised to Euclidean norm 1. The $(k+1)st$ column of $D^* f(p)$ is the normalised projection of the $(k+1)st$ column of $Df(p)$ projected onto the orthogonal complement of the hyperspace spanned by the first k columns of $D^* f(p)$.

Define $M^*(p, f) = (-D^* f(p'))^T f(p)$ where the superscript denotes the transpose. At p' we have $A^* : \mathbb{R}^n \times \mathcal{M}_{p'} \to \mathbb{R}^n$ defined by $A^*(z, f) = (-D^* f(p')^T z)$. Also, $D_z A^*(z, f) Df(p') = (-D^* f(p'))^T Df(p')$ and the right hand side is an upper triangular matrix whose j-th diagonal element is $-\|c^j\|$ where c^j is the j-th column of $D^* f(p')$ before normalisation. This implies that M^* is HLS. M^* is called the *Orthogonal Newton Mechanism*.

Theorem 7.10. If A is HLS then $D_z A(0, \cdot)$ and i are homotopic as maps from $\mathcal{M}_{p'}$ to $Gl(n)$.

Proof. Given $f \in \mathcal{M}_{p'}$, consider the autonomous linear differential equation on the space of $n \times n$ matrices,

$$\dot{B} = D_z A(0, f) Df(p') B + D_z A(0, f), \quad B(0) = 0 \text{ (the zero matrix)}.$$

Let $B(t, f)$ denote the solution at time t. Since all the eigenvalues of $D_z A(0, f) Df(p')$ have negative real parts, $B(t, f)$ converges to $(-Df(p'))^{-1} = i(f)$ and the convergence is continuous in f [19]. So, the function $B : [0, \infty] \times \mathcal{M}_{p'} \to \mathbb{R}^{n^2}$ obtained by setting $B(\infty, f) = i(f)$ is continuous. Define $H : [0, \infty] \times \mathcal{M}_{p'} \to \mathbb{R}^{n^2}$ by

$$H(t,f) = \begin{cases} B(t,f) & \text{if } t \geq 1, \\ t^{-1}B(t,f) & \text{if } 0 < t < 1, \\ D_z A(0,f) & \text{if } t = 0. \end{cases}$$

Then H is continuous, $H(0, f) = D_z A(0, f)$ and $H(\infty, f) = i(f)$, and it only remains to show that $H(t, f) \in Gl(n)$, that is $H(t, f)$ is non-singular for all $0 < t < \infty$. It suffices to show that $B(t, f)$ is non-singular for all $t > 0$, so let c be a non-zero vector in $\mathbb{R}^n$ and consider the differential equation on $\mathbb{R}^n$ given by

$$\dot{y} = D_z A(0, f) Df(p')y + D_z A(0, f)c, \; y(0) = 0,$$

with solution $y(t)$. Again this equation is stable and since it is autonomous, if $y(t) = 0$ for some $t > 0$, then $y(\cdot) \equiv 0$. However, the solution is given by $y(t) = B(t, f)c$, so $y(\infty) \neq 0$. Hence $B(t, f)c \neq 0$ for all $t > 0$ which proves the theorem. □

This theorem is a much stronger result on the dependence of M on $f - f(p')$ than that of theorem 7.8. Here we see that up to homotopy, $D_z A(0, \cdot)$ is a one to one correspondence on $\mathcal{M}_{p'}$. This implication will be established in the corollary which follows.

Let O denote the subset of $Gl(n)$ consisting of orthonormal matrices. In example 2.28 and theorem 2.37 we have seen that O is a smooth compact $(n(n-1))/2$-dimensional manifold consisting of two diffeomorphic components : $\{N \in O | \det N = 1\}$ and $\{N \in O | \det N = -1\}$.

Definition 7.11. Topological spaces Y and Z are *homotopy equivalent*, written $Y \simeq Z$, if there exists continuous functions $a : Y \to Z$ and $b : Z \to Y$ with $b \circ a \simeq 1_Y$ and $a \circ b \simeq 1_Z$ where 1_Y (resp. 1_Z) represents the identity map on Y (resp. Z). $\simeq$ is an equivalence relation and the functions a and b are called *homotopy equivalences*.

It is immediate that any map homotopic to a homotopy equivalence is itself a homotopy equivalence.

Homotopy equivalence is not the most transparent of characterisations. We shall see, however, that it is still useful in the analysis of locally stable price mechanisms since it is not always possible to strengthen this characterisation.

The Newton method maps $\mathcal{M}_{p'}$ onto the space $Gl(n)$ of non-singular matrices, which is an open set of dimension n^2 while the Orthogonal Newton method maps $\mathcal{M}_{p'}$ onto the space O of orthonormal matrices which is an $n(n-1)/2$ dimensional manifold. These two manifolds are however homotopically equivalent (the Gram-Schmidt process is a "strong deformation retraction").

Proposition 7.12. Define $j : Gl(n) \to \mathcal{M}_{p'}$ by $j(N)(p) = -N^{-1}(p - p')$. Then $i \circ j = 1_{Gl(n)}$ and $j \circ i \simeq 1_{\mathcal{M}_{p'}}$. Let $g : Gl(n) \to O$ denote the normalised Gram-Schmidt projection. Then g is a homotopy equivalence, so we have $\mathcal{M}_{p'} \simeq Gl(n) \simeq O$.

[19] If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an inevitable linear operator and $c \in \mathbb{R}^n$ is a nonzero constant vector, the change of co-ordinates of the form $x = T^{-1}(y - c)$ transforms the nonhomogenous equation $\dot{x} = Tx + c$ into the homogenous equation $\dot{y} = Ty$ which, if T has eigenvalues with negative real part, converges to the point 0. Since y tends to 0, x must tend to $-T^{-1}(c)$.

Proof. Since $i(f) = (-Df(p'))^{-1}$ it is immediate that $j \circ i \simeq 1_{Gl(n)}$. For each $f \in \mathcal{M}_{p'}$, $j \circ i(f)$ is the function $p \mapsto Df(p')(p - p')$ Define $H : \mathcal{M}_{p'} \times [0,1] \to \mathcal{M}_{p'}$ by

$$H(\lambda, f)(p) = \begin{cases} f(p) & \text{if } \lambda = 1 \\ \lambda^{-1} f(p' + \lambda(p - p')) & \text{if } 0 < \lambda < 1 \\ Df(p')(p - p') & \text{if } \lambda = 0. \end{cases}$$

Then H is the desired homotopy between $j \circ i$ and $1_{\mathcal{M}_{p'}}$. If $\iota : O \to Gl(n)$ is the inclusion map then, we have $g \circ \iota = 1_O$, and there is a geometrically obvious homotopy between $1_{Gl(n)}$ and $\iota \circ g$. Hence g is a homotopy equivalence which completes the proof. □

Corollary 7.13. *If A is HLS, then $D_z A(0, \cdot)$ is a homotopy equivalence. In particular, $j \circ D_z A(0, \cdot) \simeq 1_{\mathcal{M}_{p'}}$.*

Proof. By theorem 7.10, $D_z A(0, \cdot) \simeq i$ and by proposition 7.12 $j \circ i \simeq 1_{\mathcal{M}_{p'}}$. □

We would like to establish that among the class of HLS, the Orthogonal Newton method uses the least information about $f - f(p')$. The most transparent way to do this would be to show that for any HLS mechanism A, there is a function h such that $D_z A^*(0, f) = h(D_z A(0, f))$. This, however, turns out to be impossible.

The eigenvalue condition on $D_z A(0, f) Df(p')$ will remain satisfied for small changes in $Df(p')$ even if $D_z A(0, f)$ is held constant. Hence any HLS mechanism A may be modified so that $D_z A(0, \cdot)$ is held constant on a C^1 open neighbourhood of any given $f \in \mathcal{M}_{p'}$. For this reason, a characterisation of stable mechanisms must be stated in terms of their global behaviour on $\mathcal{M}_{p'}$ and the tools of homotopy are well suited to this purpose.

The informational minimality of the Orthogonal Newton method can be established in a coarser but still standard form. The question asked is how many parameters would be needed to parametrise the knowledge of f required by $D_z A(0, f)$. To answer this the derivative $D_z A(0, f)$ is represented as being obtained initially by projecting f to a "summary statistic[20]" $\pi_X(f) \in X$ - where X is a manifold - and then computing this derivative as a continuous function on X. The smallest dimension of the manifold X to support this representation is a measure of the information required by $D_z A(0, \cdot)$.

So, let X be a topological space, let $\pi_X : \mathcal{M}_{p'} \to X$ be continuous and let $B : \mathbb{R}^n \times X \to \mathbb{R}^n$ be a continuous function satisfying

1. for each $x \in X$, $B(\cdot, x)$ is C^1;
2. the function $x \mapsto D_z B(0, x)$ is continuous and
3. $B(z, \pi_X(f)) = A(z, f)$ for all $(z, f) \in \mathbb{R}^n \times \mathcal{M}_{p'}$.

The following theorem shows that the Orthogonal Newton method is informationally minimal.

Theorem G. *If A is HLS and X is a manifold, then $\dim X \geq n(n-1)/2 = \dim O$.*

[20] By summary statistic we mean some elements which capture certain desired information without having to contain all information from the system

Proof. Let $m = n(n-1)/2$. Let $H_m(Y, \mathbb{Z}_2)$ denote the m-th order homology group of a space Y with coefficients in the integer mod 2. If the space Y is connected has a triangulation and is of dimension m, then we have seen (see proposition 2.60) that this top order homology group $H_m(Y, \mathbb{Z}_2) = \mathbb{Z}_2$. O, being such a manifold has dimension m and two components. Therefore $H_m(O, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since homotopic spaces have isomorphic homology, proposition 7.12 implies $H_m(\mathcal{M}_{p'}, \mathbb{Z}_2) = H_m(O, \mathbb{Z}_2) = H_m(Gl(n), \mathbb{Z}_2)$ and i induces an isomorphism from $H_m(\mathcal{M}_{p'}, \mathbb{Z}_2)$ to $H_m(Gl(n), \mathbb{Z}_2)$. Let $X^\circ = \{x \in X | D_z B(0, x) \text{ is non-singular }\}$. Then $\pi_X(\mathcal{M}_{p'}) \subset X^\circ$ and X° is an open subset of X, and thus a manifold of the same dimension. By theorem 7.10, the induced isopmorphism factors through $H_m(X^\circ, \mathbb{Z}_2)$ so $H_m(X^\circ, \mathbb{Z}_2) \neq 0$ and hence $\dim X \geq m$. □

To show the Orthogonal Newton method achieves the minimum dimension, take $X = O, \pi_X(f) = -D^* f(p')^T$ and $B(z, N) = Nz$.

Corollary 7.13 implies that all of the homotopy groups of O factor through the corresponding groups of X, so that X is homotopically at least as rich as O. The next corollary shows that, if π_X is sufficiently like a projection that π_X^{-1} admits a continuous selection, then X and O are homotopy equivalent.

Corollary 7.14. Suppose that A is HLS and there is a continuous function $\gamma : X \to \mathcal{M}_{p'}$ such that $\pi_X \circ \gamma = 1_X$. Then $X \simeq O$.

Proof. Let B^* denote the function $D_z B(0, \cdot)$. Since $B^* \circ \pi_X = D_z A(0, \cdot)$, corollary 7.13 implies that $j \circ B^* \circ \pi_X \simeq 1_{\mathcal{M}_{p'}}$. Hence $j \circ B^* \circ \pi_X \circ \gamma \simeq \gamma$, and since $\pi_X \circ \gamma = 1_X, j \circ B^* \simeq \gamma$. Therefore $\pi_X \circ (j \circ B^*) \simeq \pi_X \circ \gamma = 1_X$, so π_X and $j \circ B^*$ establish that $X \simeq \mathcal{M}_{p'}$. Now by proposition 7.12, $\mathcal{M}_{p'} \simeq O$ which completes the proof. □

7.4. **Stability at Unique Equilibria.** Up to now price mechanisms have been required to be stable at p' whenever p' is a regular equilibrium. Since nothing is special about the price p', this amounts to requiring every regular equilibrium to be locally stable which may seem unreasonable. At the other extreme, probably the least that could be expected of a stable mechanism is that every regular demand function have at least one stable equilibrium. In particular, unique equilibria, if regular, should be locally stable.

Dierker [6] has shown that if $\det(-Df(p')) < 0$ for a regular f, then f must have at least two other equilibria. In this section, we will only require stability at p' if p' is the unique equilibria. This is the same as requiring stability when $\det(-Df(p')) > 0$ in the sense that if f satisfies this condition, then there is a demand function $\bar{f}$ which coincides with f near p' and has p' as its unique equilibrium (this will be established in 7.16). Since the results here depend only on the behaviour of demand functions near p', this criteria for uniqueness is sufficient. Thus all of the results of the previous section will carry over with $\mathcal{M}_{p'}$ replaced by $\mathcal{M}^+_{p'} = \{f \in \mathcal{M}_{p'} | \det(-Df(p')) > 0\}$ and O replaced by the rotational group $SO(n) = \{N \in O | \det N = 1\}$. Of course the result that $A(\cdot, f)$ must be sensitive to the sign of $\det D(p')$ is obviated but the results for HLS mechanisms retain their strength because O is simply the disjoint union of $SO(n)$ with its mirror image. In particular, the informational minimality of the Orthogonal Newton Mechanism is retained with no change in the minimum dimension.

Definition 7.15. A price mechanism M is called *locally stable at a unique equilibria* if definition 7.3 is satisfied with $\mathcal{M}_{p'}$ replaced by the set $\mathcal{M}^+_{p'}$. A mechanism M (more precisely A) is called *HLS at a unique equilibria* if definition 7.9 is satisfied with $\mathcal{M}_{p'}$ replaced by $\mathcal{M}^+_{p'}$

Let $Gl(n)^+ = \{N \in Gl(n) | \det N > 0\}$.

Proposition 7.16. Let $f \in \mathcal{M}^+_{p'}$. Then there is an open neighbourhood N of p' and a C^1 function $\bar{f} : P \to \mathbb{R}^n$ satisfying :

1. $\bar{f}|_N = f|_N$;
2. $\bar{f}$ is bounded from below;
3. for any sequence $\{p_k\}_{k=1}^{\infty} \in S$ with no cluster points in S, $\|\bar{f}(p_k)\| \to \infty$; and
4. $\bar{f}(p) \neq 0$ if $p \neq p'$

Proof. Since $Df(p')$ is non-singular, there is an ϵ-ball $B_\epsilon \subset Q$, (with a slight abuse of notation we will let B_ϵ be the intersection of a ball, centred p' with radius ϵ, in $\mathbb{R}^l$ and Q) with $\epsilon < \min\{p^j : 1 \leq j \leq n\}/3$ and $f(p) \neq 0$ for all $p \in B_{2\epsilon}$ with $p \neq p'$. Let $\sigma : B_{2\epsilon} \to [0,1]$ be a C^1 function satisfying :

a. $\sigma(p) = 1$ if $\rho(p' - p) \leq \epsilon$; and
b. $\sigma(p) = 0$ and $D\sigma(p) = 0$ if $\rho(p' - p) = 2\epsilon$.

Now define $f^* : B_{2\epsilon} \to \mathbb{R}^n$ by

$$f^*(p) = \begin{cases} f(p' + \sigma(p)(p - p'))/\sigma(p) & \text{if } \sigma(p) \neq 0, \\ Df(p')(p - p') & \text{if } \sigma(p) = 0. \end{cases}$$

Then f^* is a C^1 function on $B_{2\epsilon}$ which agrees with f on B_ϵ and $f^*(p) \neq 0$ if $p \neq p'$. Also, (b) implies that for each $p \in \partial B_{2\epsilon}$, $f^*(p) = Df(p')(p - p')$ and $Df^*(p) = Df(p')$. Since $\det(-Df(p')) > 0$, sign $(\det(Df(p')))$ = sign $(\det(-I))$ where I is the identity matrix. Therefore (see theorem 2.38) there is a continuous function $\tau : [0, \epsilon] \to Gl(n)$ satisfying

c. $\tau(0) = Df(p')$ and $\tau(\epsilon) = -I$.

Since $Gl(n)$, as an open subset of $\mathbb{R}^{n^2}$ is a smooth manifold, τ can be turned into a C^1 function satisfying, in addition to the previous constraint,

d. $D\tau(0) = D\tau(\epsilon) = 0$.

Now extend f^* to $B_{3\epsilon}$ by defining $f^*(p) = \tau(\rho(p-p') - 2\epsilon)(p-p')$ if $2\epsilon \leq \rho(p-p') \leq 3\epsilon$. Then on $B_{3\epsilon}$, f^* is a C^1 function satisfying $f^*(p) \neq 0$ and $Df^*(p) = -I$. Now extend f^* to P by setting $f^*(p) = p' - p$ if $p \neq p'$, and for all $p \in \partial B_{3\epsilon}$, $f^*(p) = p' - p$ if $p \notin B_{3\epsilon}$. For each $1 \leq j \leq n$, let $h_j : (-\infty, p'_j) \to \mathbb{R}$ be a C^1 strictly increasing function satisfying

e. h_j is bounded from below;
f. $\lim_{x \to p'_j} h_j(x) = \infty$; and

g. $h_j(x) = x$ for all $x \in [-p'_j/2, p'_j/2]$.

Finally, define $\bar{f} : P \to \mathbb{R}^n$ by $\bar{f}(p) = (h_j(f_j(p)))_{j=1}^n$. Assuming that ϵ was chosen small enough so that $|f_j(p)| \leq p'_j/2$ for all $p \in B_\epsilon$, $\bar{f} = f(p)$ for all $p \in B_\epsilon$. Hence the first condition of the theorem is satisfied with $N = \mathrm{int}B_\epsilon$. Also, (e) yields the second and (f) the fourth condition. The third and fifth conditions follow from the definition of f^*, (g) and the fact that each h_j is strictly increasing. Therefore $\bar{f}$ is the desired extension. □

7.5. **Market Mechanisms.** Actual markets are not likely to adjust prices according to the Newton method, orthogonal or otherwise, so we still need to determine whether there is a stable mechanism which can be interpreted as a market adjustment process. This is of course an ambiguous criteria, but it seems reasonable to require that if the excess demand for commodity j is zero then $\dot{p}^j = 0$. Unfortunately, theorem 7.18 shows that no such mechanism can be locally stable at unique equilibria.

Definition 7.17. A price mechanism is a *market mechanism* if there is a continuous function[21] $A : \mathbb{R}^n \times \mathcal{M}_{p'} \to \mathbb{R}^n$ such that for each $f \in \mathcal{M}'$,

1. $M(p', f) = A(f(p'), f - f(p'))$; and
2. for each $1 \leq j \leq n$, $A^j(f(p'), f - f(p')) = 0$ if $f^j(p') = 0$, where A^j (resp. f^j) denotes the j-th co-ordinate of A (resp. f).

Theorem 7.18. If $n \geq 2$ there does not exist a market mechanism which is locally stable at unique equilibria.

Proof. Define the function $d : SO(n) \to \mathcal{M}_{p'}$ by setting $d(N)$ equal to the demand function defined by $d(N)(p) = -N^{-1}(p - p')$. Then d is continuous and since $SO(n)$ is compact, $d(SO(n))$ is compact. It follows that there is some $\epsilon > 0$ such that $\{f + (\epsilon, 0, \ldots, 0) | f \in d(SO(n))\} \subset \mathcal{M}'$. Let $\epsilon^\circ = (\epsilon, 0, \ldots, 0)$. Then for each $N \in SO(n)$, $q'(d(N) + \epsilon^\circ) = N\epsilon^\circ = \epsilon N^1$, where N^1 is the first column of N. Let M be a market mechanism and suppose by way of contradiction that M is locally stable at unique equilibria. Then by theorem F, $A(\epsilon^\circ, d(\cdot))$ and $q'(d(\cdot) + \epsilon^\circ)$ are homotopic as maps from $SO(n)$ to $\mathbb{R}^n \backslash \{0\}$. By the second condition definition 7.17, $A(\epsilon^\circ, d(SO(n))$ is a compact subset of $\{y \in \mathbb{R}^n \backslash \{0\} | y_j = 0 \text{ for all } j > 1\}$ which is contractible in $\mathbb{R}^n \backslash \{0\}$ since $n \geq 2$. Therefore, $A(\epsilon^\circ, d(\cdot))$ is homotopic to a constant function. However, since the map $N \mapsto N^1$ is the projection in the standard fibre bundle representation of $SO(n)$, and is not homeomorphic to a constant (see remark 2.35). Hence $p'(d(\cdot) + \epsilon^\circ)$ and $A(\epsilon^\circ, d(\cdot))$ are not homotopic and this contradiction completes the proof. □

This result does not require the characteristic root condition. In other words, local stability unique equilibria requires the adjustment $\dot{p}^j$ to depend directly on the excess demand on other markets. In fact, it must depend directly on the excess demand on all other markets. This suggests that unless restricted to special classes of excess demand functions, competitive markets can operate satisfactorily only if price adjustments are made by a central planning agency. On the other hand theorem 7.18 may rely too heavily on the dubious identification of commodities and

[21] See remark 7.7.

markets. Perhaps different excess demands give rise to different market structures so that the class of natural adjustment mechanisms differs across demand functions. Of course we cannot pursue the evolution of markets here except to note a possible qualification to theorem 7.18 which is presented now.

7.6. **Generalised Newton Methods.** The Newton and Orthogonal Newton methods depend on $f - f(p')$ only through the derivative $Df(p')$. Thus the derivative contains enough information to ensure stability, so it is natural to investigate mechanisms of the form

$$M(p,f) = A(f(p), Df(p)), \quad \text{for all } (p,f) \in Q \times \mathcal{M},$$

where $Df(p)$ is non-singular and $A : \mathbb{R}^n \times Gl(n) \to \mathbb{R}^n$ is a C^1 function. We call these *generalised Newton methods*. It follows that a generalised Newton method is HLS (resp. HLS at unique equilibria) if for every $N \in Gl(n)$ (resp. $N \in Gl(n)^+$), $D_zA(0,n)$ is non-singular and all characteristic roots of $D_zA(0,n)N$ have negative real parts.

Saari and Simon [21] have studied such mechanisms for the purpose of determining whether any entries of the matrix $Df(p)$ can be ignored. An ignorable entry is a partial derivative which need not be evaluated in computing $\dot{p}$, so the maximum number of ignorable entries may have practical significance. Before turning to this question, we show in proposition 7.19 that such mechanisms, if locally stable at unique equilibria, cannot be extended from the space of non-singular matrices $Df(p)$ to the space of all matrices. Thus the somewhat awkward domain restriction in definition 7.2 is essential. Proposition 7.20 notes that if a generalised Newton method is HLS then it is locally stable, strengthening the motivation for studying hyperbolic local stability.

Proposition 7.19. Suppose that there is a continuous function $\bar{A} : (\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^{n^2} \to \mathbb{R}^n\backslash\{0\}$ which agrees with A on $(\mathbb{R}^n\backslash\{0\}) \times Gl(n)$. Then M is not locally stable at unique equilibria.

Proof. Let $\epsilon > 0$ and $\epsilon^\circ = (\epsilon, 0, \ldots, 0)$. Since R^{n^2} is contractible to 0, $\bar{A}(\epsilon^\circ, \cdot)$ is homotopic to the constant function $n \mapsto \bar{A}(\epsilon^\circ, 0)$. However, as in the proof of theorem 7.18, theorem F can be applied to show that this is inconsistent with stability at unique equilibria. □

Proposition 7.20. If a generalised Newton method is HLS (resp. HLS at a unique equilibria) then it is locally stable (resp. locally stable at unique equilibria).

Proof. This is a straight forward application of a standard result in the theory of ordinary differential equations. □

Definition 7.21. An entry ij is *ignorable* if there is some neighbourhood U of 0 in $\mathbb{R}^n$ such that for every $z \in U$, every $N \in Gl(n)$ and every $N_{i,j} \in \mathbb{R}$,

$$A(z, N) = A(z, N/N'_{i,j})$$

where $N/N'_{i,j} \in Gl(n)$ is the matrix obtained by substituting $N'_{i,j}$ for $N_{i,j}$.

Two entries ij and km are *rc-distinct* if $i \neq k$ and $j \neq m$.