Int. J. Appl. Comput. Math (2025) 11:48
https://doi.org/10.1007/s40819-025-01861-6

ORIGINAL PAPER

A Technique to Solve a Parabolic Equation by Point Symmetries that Incorporate Initial Data

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Accepted: 12 February 2025

Abstract

In this paper, we show how transformation techniques coupled with a convolution integral can be used to solve a generalised option-pricing model, including the Black–Scholes model. Such equations are parabolic and the special convolutions are extremely involved as they arise from an initial value problem. New symmetries are derived to obtain solutions through an application of the invariant surface condition. The main outcome is that the point symmetries are effective in producing exact solutions that satisfy a given initial condition, such as those represented by a call-option.

Keywords Option-pricing · Symmetries · Heat transfer · Initial conditions.

MSC2020: 35K05 · 35K15 · 37C79

Introduction

In this study, we consider a general pricing equation, given by

$$\frac{\partial u(x,t)}{\partial t} + \kappa x^p \frac{\partial^2}{\partial x\,\partial x} u(x,t) + \lambda(\beta - x)\frac{\partial}{\partial x} u(x,t) + \gamma x^q u(x,t) = 0, \quad x > 0,\ t \in [0,T] \tag{1.1}$$

where $p \geq 0, q \geq 0$, are arbitrary constants. The other constants, in various contexts, may be linked to the volatility of the stock (κ), the risk free interest rate (λ and β) and elasticity (γ). Also, x is the stock price at time t, T is the time of expiry and $u(x,t)$ is the value function, where $u : (0,\infty) \times [0,T] \to \mathbb{R}$.

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Published online: 27 February 2025

This approach shall work for many types of terminal conditions, but suppose we consider a European call-option, viz.

$$u(x, T) = \begin{cases} x - k & \text{if} \quad x > k \\ 0 & \text{if} \quad x \leq k, \end{cases} \tag{1.2}$$

where k is the strike price. This terminal equation is continuous and unsmooth at $x = k$.

Equation (1.1) is a generalisation of several popular models in financial mathematical applications.

Originally, many financial models such as the one above, appear as evolutionary partial differential equations with unknown coefficients. Black–Scholes [3] were the first to reduce a financial equation to the classical heat equation. In another study [4], Kolmogorov related equations which were transformable to the backward heat model were studied. However, along with Eq. (1.1), more complex models exist in the known literature, but many cannot be reduced to simple equations with known solutions.

Pricing models are a critical tool in studying financial instruments. By solving these equations, we can derive the price of options on stocks, bonds and commodities. Extended applications involve assessing risk and devising strategies for hedging, arbitrage and trading. Similar models are rising in popularity in currency markets and cryptocurrency derivatives, and there is a need by traders and investors to unlock the predictive power of such models. The methods presented here may be used to speculate the future prices of some derivatives.

In fact, in the nineteenth century, Lie [14] devised a way to classify certain classes of equations that reduce to the heat equation. However, this particular application of his work is certainly not trivial and not well-known. This is our approach in the present paper.

The equations that model financial instruments have long been an important research area. A parabolic model of this type was previously considered in [19, 24] to find solutions based on symmetry invariance, but did not consider the terminal condition. Symmetry analysis is commonly performed without any consideration of initial or boundary values. However, in this work we apply the symmetry generators in such a way that the above terminal condition is satisfied for the obtained solutions. We consider two cases, one of which is related to the popular equations mentioned above but the other case is introduced here, from certain assumptions of the free parameters. As a consequence of this study, we offer a mathematical way of determining how such free parameters of the model should be determined. This is a significant step in formulating equations for mathematical modelling.

We note that many studies of equations, in finance or otherwise, involve the Black–Scholes model or the famous heat transfer equation [2, 5, 8, 12, 13, 15, 16, 20, 22, 23], to mention a few. Recently, symmetries have been applied in various important contexts [1, 17, 18]. As we will uncover below, Lie symmetries are a versatile tool for manipulating initial or boundary conditions. The solution of initial data problems for parabolic models, such as (1.1) can be reduced to the construction of a particular solution with specific singularities known in the literature as fundamental solutions [6, 10]. According to [11], for certain families of equations, the fundamental solution is an invariant solution.

In Sect. 2, we define the unknown coefficients and specify the transformations to reduce the above equation to the one dimensional heat model. Section 3 describes the primary results of this investigation. We establish certain characteristics of the Cauchy problem, and thereafter provide analytical solutions of (1.1) given the terminal condition. We conclude our results in Sect. 4.

Transformations and parameters

We first prove that it is possible to transform (1.1) to the classical heat equation, under specific restrictions on the free parameters of the model. The change of variables is important to set up the Cauchy problem and manipulate the solutions such that they satisfy the terminal condition (1.2). This idea provides a mathematical tool for choosing the parameters and formulate experimental models for further study. Therefore we will prove the following.

Theorem 1 *Eq. (1.1) is converted to*

$$\omega_{\hat{t}} - \omega_{yy} = 0, \tag{2.1}$$

which is the 1+1 heat equation, with the parameter choices:

(a) Case I: $p = 2,\ q = 0,\ \kappa = \frac{A^2}{2},\ \beta = 0,\ \gamma = -B$ *and* $\lambda = -B$, *(*A, B *are arbitrary constants). The transformations are*

$$\hat{t} = T - t, \tag{2.2}$$

$$y = \frac{\sqrt{2}\log(x)}{A}. \tag{2.3}$$

(b) Case II: $p = 0,\ q = 2,\ \kappa = \frac{1}{4},\ \beta = 0,\ \gamma = 4$ *and* $\lambda = 2$. *The transformations are*

$$\hat{t} = T - t, \tag{2.4}$$

$$y = 2x. \tag{2.5}$$

Proof Case I: an application of y and $\hat{t}$, reduces Eq. (1.1) to the form

$$a(y,\hat{t})u_y + c(y,\hat{t})u + u_{\hat{t}} - u_{yy} = 0, \tag{2.6}$$

where $u(y,\hat{t})$, and the functions are $a(y,\hat{t}) = \dfrac{A^2 - 2B}{\sqrt{2}A}$ and $c(y,\hat{t}) = B$.

Thereafter, the variable $u(y,\hat{t})$, is defined as

$$u(y,\hat{t}) = \omega(y,\hat{t}) \cdot e^{-\phi(y,\hat{t})}, \tag{2.7}$$

and

$$\phi(y,\hat{t}) = \frac{B^2\hat{t}}{2A^2} - \frac{y\left(A^2 - 2B\right)}{2\sqrt{2}A} + \frac{A^2\hat{t}}{8} + \frac{B\hat{t}}{2}. \tag{2.8}$$

This transformation procedure converts Eq. (2.6) to the heat equation (2.1)..

Case II is analogous, but with $a(y,\hat{t}) = -2y$ and $c(y,\hat{t}) = -y^2$ and similarly, we define (2.7) with

$$\phi(y,\hat{t}) = \hat{t} + \frac{y^2}{2}. \tag{2.9}$$

□

The values of the parameters in Theorem 1 result from a mathematical requirement to convert the model to the heat equation. However, these parameter values do not violate any real-world requirements of the model. The parameters may be interpreted in the following context. The p, q are power laws applied to stock prices and may cater for large price movements. A positive value of volatility κ indicates that there is uncertainty about the asset's future price changes. The interest rate λ generally increase the price of call-options, and the elasticity γ indicates that the price of the option is positively correlated with the price of the underlying asset.

Case I is equivalent to the Black–Scholes call-option model while Case II can be viewed as a modified pricing model. We note that the transformation of the Black–Scholes model to the 1+1 heat model is well known but we include it under our study to validate our approach below. Theorem 1 induces the following lemma involving the terminal condition.

Lemma 2 *The corresponding condition (1.2) is, via the transformations (2.2)-(2.9),*

$$\omega(y,\hat{t}) = max\left\{ \left(e^{\frac{Ay}{\sqrt{2}}} - k\right) \exp\left(\frac{B^2\hat{t}}{2A^2} - \frac{y\left(A^2-2B\right)}{2\sqrt{2}A} + \frac{A^2\hat{t}}{8} + \frac{B\hat{t}}{2}\right), 0\right\}, \tag{2.10}$$

for Case I, and

$$\omega(y,\hat{t}) = max\left\{ e^{\hat{t}+\frac{y^2}{2}}\left(\frac{y}{2} - k\right), 0\right\}, \tag{2.11}$$

for Case II.

The proof follows easily from the transformations given in Theorem 1.

The Cauchy problem

The 1+1 heat equation is one of the most unique models, whose fundamental solution is remarkable. In the work that follows, we illustrate how the fundamental solution, combined with symmetry properties, solves a Cauchy problem for Eq. (2.1), the heat equation. This process, once the transformations of Theorem 1 are reversed, leads to the solution of (1.1) subject to the terminal condition.

From Lemma 2, let $\omega(y,0) = F(y)$, where $F(y)$ is the *max* function.

We now proceed to the application of Lie point symmetries This is an intricate process and often requires one to evaluate complicated integrals. For this purpose, the use of algebraic software is essential. The symmetry generators of (2.1), are given in the most general form as

$$X = \xi(y,\hat{t},\omega)\frac{\partial}{\partial y} + \eta(y,\hat{t},\omega)\frac{\partial}{\partial \hat{t}} + \bar{\phi}(y,\hat{t},\omega)\frac{\partial}{\partial \omega}, \tag{3.1}$$

or, equivalently

$$\begin{aligned} X = &\left(4c_1\hat{t}^2 + 2c_2\hat{t} + c_6\right)\frac{\partial}{\partial \hat{t}} + \left(4c_1 y\hat{t} + c_2 y + 2c_3\hat{t} + c_4\right)\frac{\partial}{\partial y} \\ &\left(\left(c_1(-2\hat{t} - y^2) - c_3 y + c_5\right)\omega + \alpha(y,\hat{t})\right)\frac{\partial}{\partial \omega}. \end{aligned} \tag{3.2}$$

In the above symmetry, $\alpha(y,\hat{t})$ is an arbitrary solution that satisfies the heat equation, i.e. $\alpha_{\hat{t}} = \alpha_{yy}$.

We apply the invariant surface condition (ISC) [21]

$$\xi\omega_y + \eta\omega_{\hat{t}} - \bar{\phi} = 0, \tag{3.3}$$

which holds at the boundary where it is imposed that $\hat{t} = 0$. This leads to the condition [9]

$$\frac{\bar{\phi}(y, 0, F(y)) - \xi(y, 0, F(y))F'(y)}{\eta(y, 0, F(y))} = \omega_{yy}. \tag{3.4}$$

If we evaluate condition (3.4), we have that

$$\alpha(y, 0) = \left(c_1 y^2 + c_3 y - c_5\right) F(y) + (c_2 y + c_4) F'(y) + c_6 F''(y). \tag{3.5}$$

Equation (3.5) is an ordinary differential equation which is connected to a general initial condition and it provides less restrictive criteria on the initial data so that they may be obtained by a specific symmetry. We now illustrate some solutions, where we do not provide an exhaustive list of solutions that can be found by the method described above, but rather we focus on several special conditions and solutions.

Recall that the function $\alpha(y, \hat{t})$ is also a solution to (2.1), so we construct an initial problem for (2.1), viz.

$$\begin{cases} \alpha_{\hat{t}} - \alpha_{yy} = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ \alpha(y, 0), & \text{on } \mathbb{R} \times \{\hat{t} = 0\}, \end{cases} \tag{3.6}$$

As mentioned previously, the above theory gives rise to various solutions to (2.1) and consequently, Eq. (1.1). We emphasise that all solutions are subject to the Cauchy problem, while solutions to (1.1) satisfy the terminal equation (1.2).

Noting that it is rare to find a symmetry study involving initial conditions, we start with Case I. Let $c_1 = c_2 = c_3 = c_4 = c_5 = 0$, $c_6 = 1$ to obtain the symmetry

$$X = \frac{\partial}{\partial \hat{t}} + \alpha(y, \hat{t}) \frac{\partial}{\partial \omega}$$

and function

$$\alpha(y, 0) = F''(y).$$

Thus, with (3.3), the solution to (2.1), the heat model is

$$\omega(y, \hat{t}) = \int \alpha(y, \hat{t}) d\hat{t} + H(y).$$

Without loss of generality, the arbitrary function may vanish, i.e. $H(y) = 0$ but $\alpha(y, \hat{t})$ is still unknown. However, we recall that the fundamental solution of (2.1) is

$$A(y, \hat{t}) := \begin{cases} \frac{1}{\sqrt{4\pi \hat{t}}} e^{-\frac{y^2}{4\hat{t}}}, & (y \in \mathbb{R}, \hat{t} > 0), \\ 0, & (y \in \mathbb{R}, \hat{t} < 0) \end{cases} \tag{3.7}$$

which is singular at (0,0). The function $A(y - \zeta, \hat{t})$ is also a solution of (2.1) for each fixed $y \in \mathbb{R}$, then consequently the convolution [7]

$$\alpha(y, \hat{t}) = \int_{\mathbb{R}} A(y - \zeta, \hat{t}) \alpha(\zeta, 0) d\zeta \quad (y \in \mathbb{R}, \hat{t} > 0), \tag{3.8}$$

is another solution to the heat model (2.1). Note that $(y, \hat{t}) \mapsto A(y, \hat{t})$ solves the heat equation away from the singularity at (0,0), and so too does $(y, \hat{t}) \mapsto A(y - \zeta, \hat{t})$ for each fixed $y \in \mathbb{R}$.

The following statements hold regarding the solution of the initial value problem, see [7] pg. 47: Let $F(y)$ be a continuous and bounded function, and the solution $\alpha(y, \hat{t})$ is given by the convolution (3.8), then

(i) $\alpha \in C^{\infty}(\mathbb{R} \times (0, \infty))$,
(ii) $\alpha_{\hat{t}}(y, \hat{t}) - \alpha_{yy}(y, \hat{t}) = 0, \quad y \in \mathbb{R}, \hat{t} > 0$, and
(iii) $\lim_{(y,\hat{t}) \to (y^0,0)} \alpha(y, \hat{t}) = F(y^0)$ for each point $y^0 \in \mathbb{R}$.

For uniqueness of the solution given by convolution, see Thikonov [25].

Hence form integrating the convolution, we find

$$\alpha(y,\hat{t}) = \frac{\exp\left(\frac{\left(A^2-2B\right)\left(A^2\hat{t}-2\sqrt{2}Ay-2B\hat{t}\right)}{8A^2}\right)\left(\left(A^2+2B\right)^2 e^{\frac{Ay}{\sqrt{2}}+B\hat{t}} - k\left(A^2-2B\right)^2\right)}{8A^2}, \tag{3.9}$$

and another integration to find $\omega(y, \hat{t})$ subject to (3.6), we calculate that

$$\omega(y,\hat{t}) = \exp\left(\frac{\left(A^2-2B\right)\left(A^2\hat{t}-2\sqrt{2}Ay-2B\hat{t}\right)}{8A^2}\right)\left(e^{\frac{Ay}{\sqrt{2}}+B\hat{t}} - k\right). \tag{3.10}$$

To construct the solution to the original model (1.1), we induce the transformations from Theorem 1 to find

$$u(x,t) = x - ke^{B(t-T)}, \tag{3.11}$$

subject to (1.2).

As a final illustration, consider Case II. For simplicity, let $c_4 = 1, c_1 = c_2 = c_3 = c_5 = c_6 = 0$. Hence

$$\alpha(y, 0) = F'(y),$$

and

$$X = \frac{\partial}{\partial y} + \alpha(y,\hat{t})\frac{\partial}{\partial \omega}.$$

Thus, via the ISC (3.3), we obtain that the solution to (2.1) is

$$\omega(y,\hat{t}) = \int \alpha(y,\hat{t})dy + H(\hat{t}).$$

We let $H(\hat{t}) = 0$, and determine $\alpha(y, \hat{t})$.

Thus, repeating the procedure, we obtain

$$\alpha(y,\hat{t}) = -\frac{e^{-\frac{y^2}{4\hat{t}+2}}\left(-2k(2\hat{t}y+y)-2\hat{t}+y^2-1\right)}{2\sqrt{\frac{1}{\hat{t}}+2}\sqrt{\hat{t}}(2\hat{t}+1)^2}, \tag{3.12}$$

the heat equation solution subject to (3.6) is

$$\omega(y,\hat{t}) = \frac{e^{-\frac{y^2}{4\hat{t}+2}}(y-2k(2\hat{t}+1))}{2\sqrt{\frac{1}{\hat{t}}+2}\sqrt{\hat{t}}(2\hat{t}+1)} \tag{3.13}$$

and the pricing equation has solution, subject to (1.2), as

$$u(x,t) = \frac{e^{-\frac{4x^2}{4(T-t)+2}-t+T+2x^2}(2x-2k(2(T-t)+1))}{2\sqrt{2(T-t)+1}(2(T-t)+1)}. \tag{3.14}$$

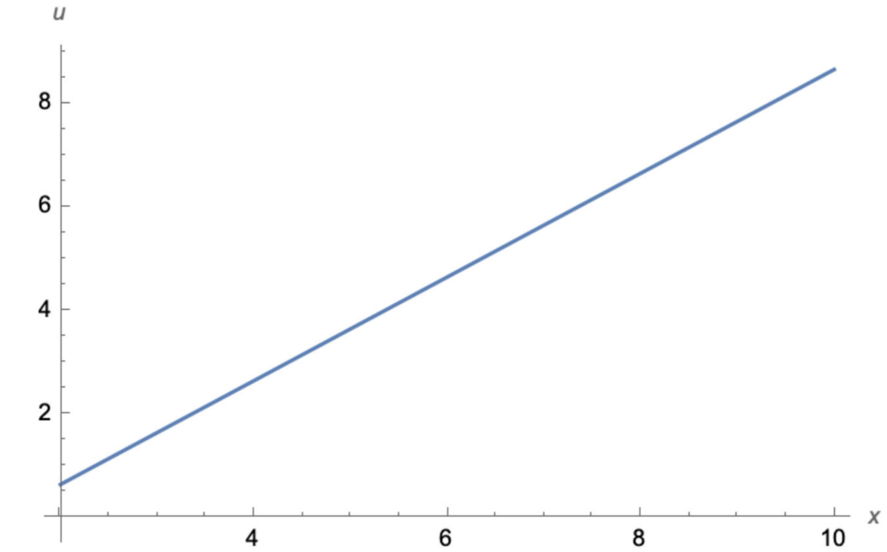

Fig. 1 Graphical 2D illustration of the analytical solution (3.11) with $k = 2$, $T = 2$, $t = 1.9$, $B = 4$

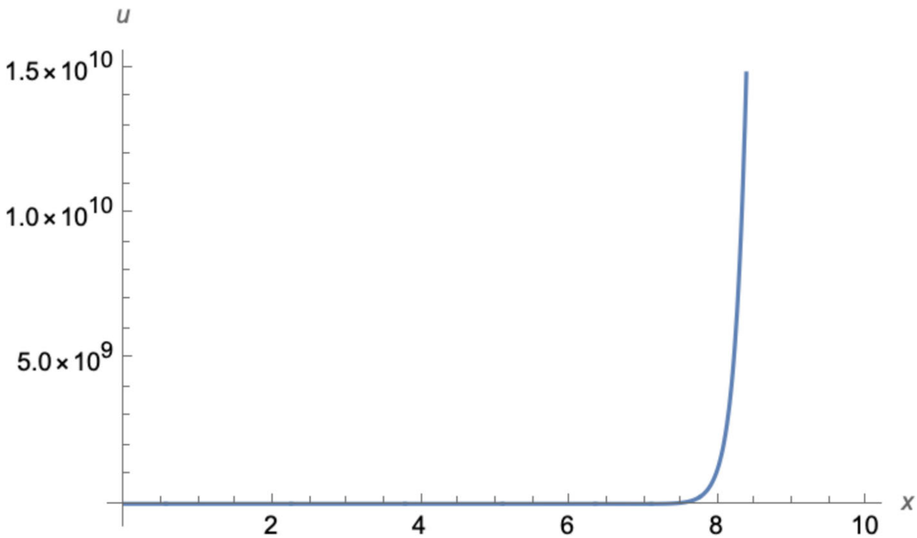

Fig. 2 Graphical 2D illustration of the analytical solution (3.14) with $k = 6$, $T = 2$, $t = 1.9$

More solution can be found in this manner. If we let $t = T$ the time of expiry, we see that (3.11) and (3.14) simplifies to

$$u(x, T) = x - k,$$

which shows that the solutions are subject to the terminal condition. We plot the solution of (3.11) which shows a steady growth in option value, but (3.14) predicts a steep option value as asset price rises to a particular value (Figs. 1, 2).

Conclusion

A general equation composed of European options has been solved for certain parameters. The unknown parameters were chosen to effect a transformation to the heat equation. Consequently the main pricing equation was analysed subject to its terminal condition. We have set up a Cauchy problem to the heat equation and used Lie symmetry generators to obtain new and non-trivial solutions.

Two solutions were illustrated. The solution for Case I gives the current price of the underlying asset minus a time-decaying term which accounts for the discounting of the strike price over time, as the option is closer to its expiration. Note that in general, the option can only be exercised at maturity, not before. The solution at expiry represents the intrinsic value of the option or the difference between the asset price and the strike price.

Case II is a noteworthy solution which shows a high option value. Such high values are sought after because they represent higher potential profits for investors. The value of an option is driven by various factors, but this solution suggests that the option value will rise significantly, and offers a target asset price for this phenomenon.

This work makes several key contributions:

- The innovative approach introduced provides a new perspective on solution methods by applying novel transformations that simplify a general finance model into the heat equation. These transformations are simple, making them available to a broad audience.
- Our conclusions are of practical significance, as they may have important implications for real-world financial theories.
- The examination of a specialized topic like call-options is expected to inspire further research and exploration in new directions.

Most of the equations that model financial derivatives are of the type considered here, and therefore this approach may be adopted for many of the models. Such models are often plagued with difficulties in finding solutions, however we have demonstrated how effective transformations are in simplifying these models in order to obtain solutions. In further applications, we plan to extend this study to other fields.

Acknowledgements Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

Author Contributions All authors have contributed equally.

Funding Open access funding provided by University of the Witwatersrand. None.

Data Availibility Not Applicable.

Code Availability Not Applicable.

Declarations

Conflict of interest None.

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