C.2.1 Determination of the Modal Damping Coefficient

In order to select the material damping constant μ to achieve a prescribed relative proportional modal damping coefficient in the first mode ζ_1, the free response of the equation for $q(t)$ may be examined:

$$\ddot{q} + \frac{\omega_n^2}{c^2}\mu\dot{q} + \omega_n^2 q = 0$$

Thus:

$$2\zeta_n\omega_n = \frac{\omega_n^2\mu}{c^2}$$

If the material damping coefficient is selected on the basis of the modal damping ratio, then:

$$\mu = \frac{2c^2\zeta_1}{\omega_1}$$

and it follows that:

$$\zeta_n = \zeta_1\frac{\omega_n}{\omega_1}$$

This demonstrates that ζ_n is proportional to the ratio of the n^{th} natural frequency of the fundamental ω_1. Hence if μ is selected on the basis of the fundamental, the higher modes will become successively more damped and less significant in the response.

C.3 General Proportional Damping

Greenway[1989] provided a rudimentary damping estimate regarding the damping coefficient extracted from free response data captured during deceleration tests at Deelkraal Mine. This estimate was obtained from the logarithmic decrement of first longitudinal mode; Greenway[1989] showed that if a relative proportional damping model was applied, a material damping coefficient μ could be defined. A viscous relative proportional damping model results in the higher modes becoming successively more damped. Since the damping factor is critical to the steady state stability analysis, further site tests were conducted to determine if the damping estimate related to the higher modes followed such a relationship. In that test it was found that the higher modes were more lightly damped than the fundamental, and that the damping estimate obtained for the fundamental was dependent on the mean rope tension. These results are presented and discussed in detail in Appendix G. Greenway[1993] demonstrated that if the dependence of the first mode on mean tension was neglected, a general proportional viscous damping model could be constructed, which accounts for the lower damping estimates measured in the higher modes.

The analysis presented regarding the relative proportional damped longitudinal response, can be readily modified to include the case of general viscous proportional damping. For general viscous proportional damping, two material damping coefficients are introduced, μ_a, μ_b, where the first relates to a damping force which is proportional to the mass properties of the system, whilst the second relates to a damping force which is proportional to the stiffness properties of the system. This is analogous to a Rayleigh damping form in a discrete system. The linear longitudinal equation of motion is governed by:

$$u_{tt} + \mu_a u_t = \mu_b u_{t,ss} + c^2 u_{ss}$$

For a forced response at frequency ω:

$$u(s,t) = \phi(s)e^{i\omega t}$$

Thus:

$$\phi'' + (\frac{\omega^2 - i\mu_a\omega}{c^2 + i\mu_b\omega})\phi = 0$$

Thus:

$$\phi(s) = Acos\gamma s + Bsin\gamma s$$

where:

$$\gamma = (\frac{\omega^2 - i\mu_a\omega}{c^2 + i\mu_b\omega})^{\frac{1}{2}}$$

The modal damping factor can be obtained by applying a normal mode expansion, and considering the principal modes q_i.

$$\ddot{q}_i + (\mu_a + \mu_b(\frac{\omega_i}{c})^2)\dot{q}_i + \omega_i^2 q_i = 0$$

Thus it follows that:

$$\zeta_i = \frac{1}{2}(\frac{\mu_a}{\omega_i} + \frac{\mu_b}{c^2}\omega_i)$$

With reference to Appendix G, the material damping constants a, b would be related to μ_a, μ_b as:

$$\mu_a = a$$

$$\mu_b = c^2 b$$

The forced response for general proportional viscous damping can be obtained from the equations developed in section C.2 by appropriate substitution for γ.

C.4 Longitudinal Response - Kloof Mine

The steady state forced longitudinal response for the Kloof Mine is examined. This exercise demonstrates the difference between a relative proportional, and a general proportional damping model. Appendix G presents longitudinal damping factors extracted by Greenway[1989] from free decay results at Deelkraal Mine and by Constancon[1992] from free decay results obtained from Elandsrand Mine. Since no other data is available, this data will be utilised as being representative of the Kloof Mine rope damping properties.

C.4.1 Relative Proportional Damping Model

The relative proportional material damping co-efficient μ, is selected on the basis of the Deelkraal[1] data as presented in table G.1 of Appendix G. This data indicated that the material damping constant b increased roughly in proportion to the rope length. Thus the relative proportional damping coefficient μ is defined as:

$$\mu(s) = c^2 bs/l$$

where s represents the length of the rope.

The data tabulated in Appendix G table G.1 provides $b = 0.03sec$, at a rope length of $\frac{3}{4}l$ of the depth of the shaft, where $l = 2062m$. Thus

$$c = \sqrt{EA/m} = 3658m/s$$

$$\mu(s_2) = c^2 \frac{4 \times 0.03}{3 \times 2062} s_2 = 259 s_2$$

By selecting the damping constant μ, the modal damping factor of the n^{th} mode is set to $\zeta_n = \zeta_1 \frac{\omega_n}{\omega_1}$. The modal damping factors for the first four modes are presented in figure C.2. As is evident, in comparison to the fundamental, the second and higher modes are significantly damped.

[1]The data was recorded by Thomas et al.[1987], and the logarithmic decrement was extracted by Greenway[1989].

C.4.2 General Proportional Damping Model

A general proportional damping model is presented in Appendix G. This model was proposed by Greenway[1993], based on measurements captured by Constancon[1992] and Page[1992] at Elandsrand Gold Mine. Measurements were captured at 73 level, which represents an active rope length between the drum and skip of 2090m. Due to time limitations, further measurements were not obtained at a different level, and thus the variation of the damping constants with rope length could not be determined. However at 73 level, material damping coefficients of $\mu_a = 0.159$ and $\mu_b = 0.00164c^2$ were determined from the test data. As stated in Appendix G, this model provides a convenient fit to the measured data, rather than physical evidence of a general viscous damping mechanism. Further experimental results would be required to define the physical basis for such a model. Since the measurements at Deelkraal mine indicate that the relative damping coefficient b increases in proportion to rope length, a similar effect is assumed with regard to the proportional damping coefficient μ_b. Thus the damping model proposed is:

$$\mu_a = 0.159$$

$$\mu_b(s_2) = 0.001644c^2 \frac{s_2}{2090} = 10.49s_2$$

The modal damping coefficients based on this model are presented in figure C.3. It is evident in this figure that the fundamental mode is more damped than the higher modes. This emphasises the difference between the damping models, and consequently the need to accurately determine the damping mechanism.

C.4.3 Forced Longitudinal Response

The Kloof Mine system parameters are presented in table C.1. The natural frequencies of the first four modes, for a full ascending skip, are presented in figure C.4(a). In figure C.4(a), the horizontal lines represent the Lebus groove excitation frequencies at the first and second harmonic of the coil cross-over frequency.

The modulus of the longitudinal forced response at the sheave wheel due to the first and second harmonic excitation at the winder drum, is presented in figure C.4(b),(c) respectively. The response amplitude is dramatically affected by the damping model assumed. It is clear that the higher modes would be active if the general proportional viscous damping model applied. The damping models proposed have been formulated on the basis of extremely limited data, and as such are approximate. However, at this stage the general proportional damping model is the most representative model available, and will be utilised until further experimental studies provide a better description.

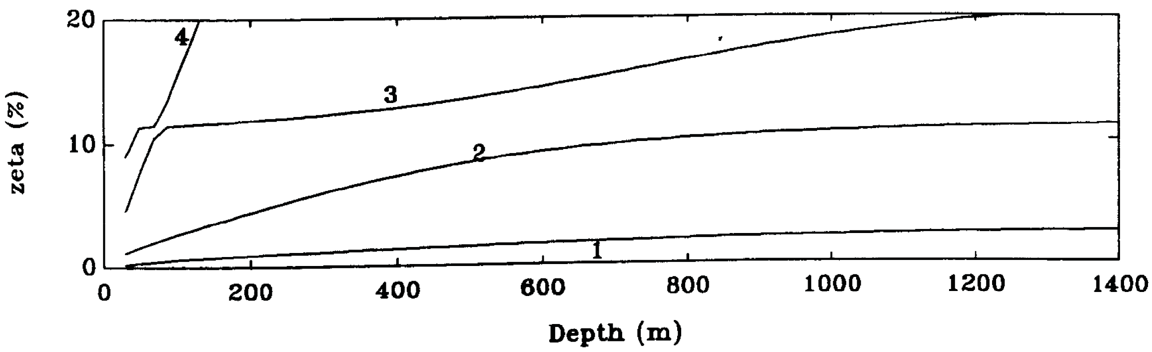

Figure C.2: Modal damping factors - Relative proportional damping

$$\zeta_n = \frac{1}{2}\frac{\mu}{c^2}\omega_n$$
$$\mu(s_2) = 259 s_2$$

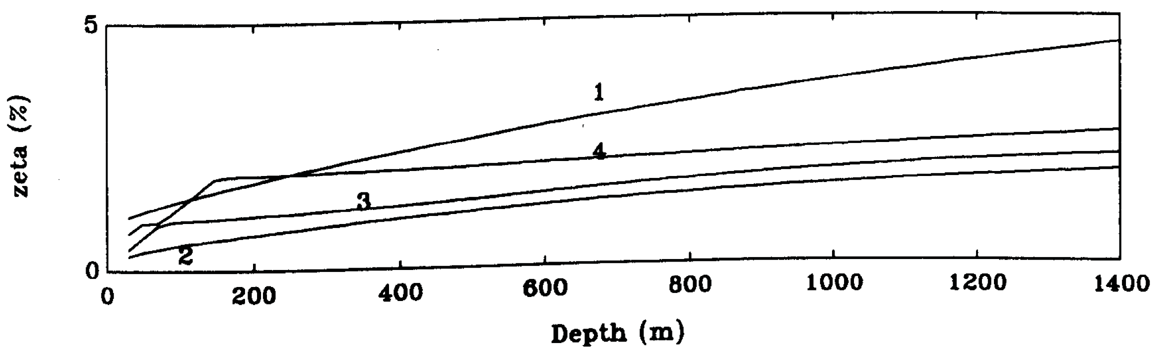

Figure C.3: Modal damping factors - General proportional damping

$$\zeta_n = \frac{1}{2}\left(\frac{\mu_a}{\omega_n} + \frac{\mu_b}{c^2}\omega_n\right)$$
$$\mu_a = 0.159 \qquad \mu_b = 10.49 s_2$$

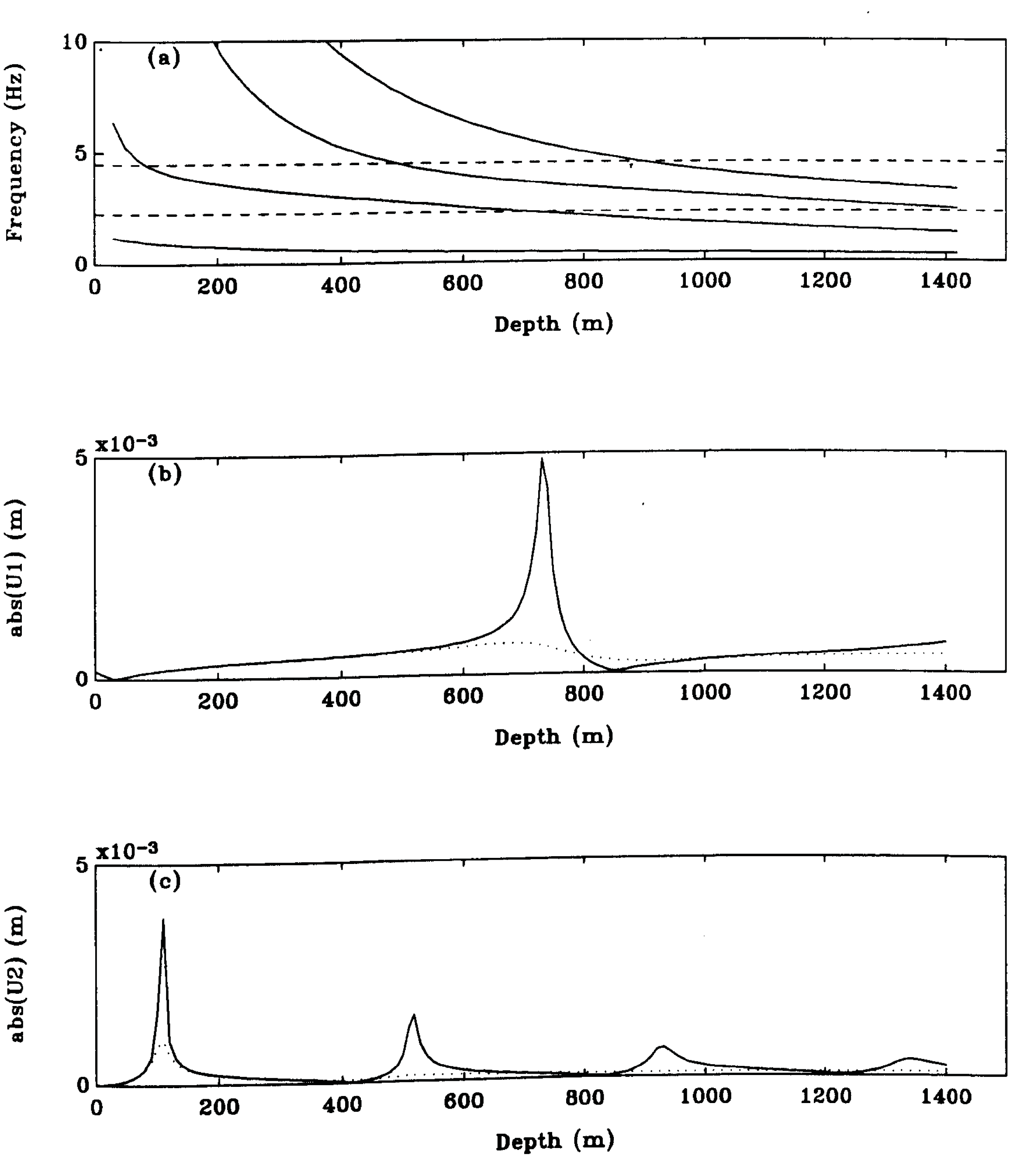

Figure C.4: Linear steady state forced response

(a) Longitudinal Natural Frequencies vs Shaft depth
(b) Forced Longitudinal Response at the Sheave Wheel - First Harmonic
(c) Forced Longitudinal Response at the Sheave Wheel - Second Harmonic
...... Relative proportional damping. —- General proportional damping.

$$u(0,t) = 2.89 \times 10^{-4} cos(\Omega_l t + \phi_1) + 1.05 \times 10^{-4} cos(2\Omega_l t + \phi_2)$$

$$\Omega_l = \frac{4V_e}{Dd} = 14.01 rad/s, \quad \phi_1 = 1.669 rad, \quad \phi_2 = -1.374 rad$$

Table C.1: Kloof Mine - System parameters

J	Sheave Inertia.	15200 kgm^2
M	Skip Mass.	7920 kg
M0	Skip Pay-load.	9664 kg
m	Linear Rope density.	8.4 kg/m
V	Nominal Winding Speed.	15 m/s
De	Depth of wind.	2100m
Lc	Catenary Length.	74.95 m
E	Effective Youngs Modulus of the rope.	$1.1 \times 10^{11} N/m^2$
Ax	Effective steel area of the rope.	$0.001028 m^2$
β	Cross over arc.	0.2 rad
Dd	Drum Diameter.	4.28 m
Ds	Sheave Diameter.	4.26 m
Dr	Rope Diameter.	0.048 m

Appendix D

Linear Lateral Catenary Response

This appendix presents the linear analysis of the catenary, in the absence of curvature, and longitudinal coupling. In this context, the equation of motion governing the in and out-of-plane motion reduces to the linear wave equation associated with a taut string. It is well established that the undamped wave equation is variable separable, ie $v(s,t) = \sum \phi_i(s) q_i(t)$. The natural frequency of the i^{th} principal mode of a string with pinned end conditions is given by $\omega_i = i\frac{c_l}{2l_1}$ where c_l represents the lateral wave speed which is related to the tension T and linear mass density of the rope m, by $c_l = \sqrt{T/m}$, and l_1 represents the chord length between the pinned ends.

The eigenfunction for this configuration is given by:

$$\phi_i(s) = sin(\frac{i\pi}{l_1}s)$$

Since curvature is neglected, the in-plane and out-of-plane eigenfunctions and natural frequencies are identical.

D.1 Forced Response

The lateral response of the catenary to an out-of-plane harmonic boundary excitation $We^{i\omega t}$ at the drum end, in the presence of relative proportional viscous damping is presented[1]. The equation of motion has the same form as that presented in Appendix C, governing the damped longitudinal behaviour of the rope. ie.

$$\frac{\partial^2 w}{\partial t^2} = \mu_l \frac{\partial^3 w}{\partial s_1^2 \partial t} + c_l^2 \frac{\partial^2 w}{\partial s_1^2} \tag{D.1}$$

where μ_l represents the coefficient of lateral dissipation, and co-ordinate s_1 represents the distance measured along the rope from the drum to the sheave. This equation is variable separable, and thus the solution may be expressed as:

$$w(s_1, t) = \phi_l(s_1)q(t)$$

$$\phi_l(s) = Acos\gamma_l s_1 + Bsin\gamma_l s_1$$

where:

$$\gamma_l = \frac{\frac{\omega}{c_l}}{(1 + (\frac{\mu_l \omega}{c_l^2})^2)^{1/4}} e^{-i\alpha_l/2}$$

$$\alpha_l = tan^{-1}\frac{\omega\mu_l}{c_l^2}$$

$$c_l^2 = \frac{T}{m}$$

The boundary conditions are:

$$w(0, t) = We^{i\omega t}$$

[1] W represents both phase and amplitude and is a complex entity

$$w(l_1, t) = 0$$

Under these boundary conditions, the solution can be shown to be:

$$w(s_1, t) = \frac{W}{sin\gamma_l l_1} sin(\gamma_l(l_1 - s_1))e^{i\omega t} = \phi_l(s_1)e^{i\omega t}$$

Since γ_l is complex, the response is phase shifted with respect to the excitation. Considering the first harmonic of the excitation as $w(0, t) = Re(We^{i\omega t})$, where ω is the excitation frequency, the response is:

$$w(s_1, t) = (Re(\phi_l)cos\omega t - Im(\phi_l)sin\omega t) = A_l(s_1)cos(\omega t + \Phi_l)$$

where:

$$A_l(s_1, t) = \sqrt{Re(\phi_l)^2 + Im(\phi_l)^2}$$

$$\Phi_l = tan^{-1}(\frac{Im(\phi_l)}{Re(\phi_l)})$$

The forced lateral catenary response due to a periodic lateral excitation at the winder drum can thus be constructed via superposition of the forced response due to each harmonic in the periodic excitation.

Appendix E

Linear Coupled System Response

This appendix presents the solution to the linearised stationary steady state system response as proposed in Chapter 4. The equations of motion developed in Chapter 4, considered the stationary steady state solution, for small lateral amplitudes in the absence of curvature, such that nonlinear terms were neglected in the lateral equations of motion, but retained in the longitudinal equation of motion. The in-plane lateral excitation is significantly less than the out-of-plane lateral and longitudinal excitation, and is consequently neglected. Thus planar response occurs in the $u - w$ plane, and consequently the equations of motion under consideration for the catenary section are:

$$\frac{\partial^2 u_1}{\partial t^2} = \mu \frac{\partial^3 u_1}{\partial s_1^2 \partial t} + c^2 \frac{\partial^2 u_1}{\partial s_1^2} + c^2 \frac{\partial w}{\partial s_1} \frac{\partial^2 w}{\partial s_1^2} \tag{E.1}$$

$$\frac{\partial^2 w}{\partial t^2} = \mu_l \frac{\partial^3 w}{\partial s_1^2 \partial t} + \bar{c}^2 \frac{\partial^2 w}{\partial s_1^2} \tag{E.2}$$

The vertical rope is laterally restrained and consequently the equation of motion describing longitudinal motion $u_2(s_2, t)$ is:

$$\frac{\partial^2 u_2}{\partial t^2} = \mu \frac{\partial^3 u_2}{\partial s_2^2 \partial t} + c^2 \frac{\partial^2 u_2}{\partial s_2^2} \tag{E.3}$$

where $c, \bar{c}$ represent the longitudinal and lateral wave speeds respectively, and s_1, s_2 represent the co-ordinates measured along the catenary from the drum end to the sheave, and along the vertical rope from the sheave to the conveyance respectively. $u_1(s_1, t), u_2(s_2, t)$ represent the longitudinal displacement in the catenary and vertical rope respectively.

Since the equation of motion for the lateral response is uncoupled from the longitudinal equation of motion, the lateral response due to forced boundary excitation $\sum W_n cos(n\omega t)$ can be derived in closed form[1], as presented in Appendix D. The lateral response couples independently to the longitudinal equation of motion, and consequently this coupling term may be viewed as a distributed forcing function over the catenary length.

The boundary conditions for the system, in the absence of longitudinal excitation at the drum end are:

$$u(0,t) = 0 \tag{E.4}$$

$$w(0,t) = \sum W_n cos(n\omega t) \tag{E.5}$$

$$\{AE\frac{\partial u_2}{\partial s_2} + \rho\mu A\frac{\partial^2 u_2}{\partial s_2 \partial t}\}|_{(l_2,t)} = -M\frac{\partial^2 u_2}{\partial t^2}|_{(l_2,t)} \tag{E.6}$$

In addition, since lateral motion exists, the inertial balance at the sheave is given as: :

$$\begin{aligned}
[\frac{I}{R^2}\frac{\partial^2 u_1}{\partial t^2}]|_{l_1,t} &= [EA\frac{\partial u_2}{\partial s_2} + \rho\mu A\frac{\partial^2 u_2}{\partial s_2 \partial t}]|_{(0,t)} \\
&- [EA\{\frac{\partial u_1}{\partial s_1} + \frac{1}{2}\frac{\partial^2 w}{\partial s_1^2}\} + \rho\mu A\frac{\partial^2 u_1}{\partial s_1 \partial t}]|_{(l_1,t)}
\end{aligned} \tag{E.7}$$

The quadratic nature of the coupling term $c^2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial s_1^2}$ in the longitudinal equation of motion (E.1), and $\frac{1}{2}(\frac{\partial w}{\partial s_1})^2$ in equation (E.7) describing the inertial balance across the sheave, results in the generation and transmission of longitudinal excitation induced by the forced lateral catenary motion, to the catenary and

[1] In subsequent sections, subscript 1,2 will refer to terms evaluated at the first and second harmonic of the excitation frequency respectively

vertical rope. This excitation consists of a time independent component, and a dynamic component at multiples and beats of the lateral response frequency. This can be demonstrated by considering the case where the forced lateral response is comprised of two frequencies, namely at the fundamental and the second harmonic of the Lebus excitation frequency. Thus the lateral forced response could be represented by:

$$w(s,t) = |w_1(s,t)|cos(\omega t + \phi_1) + |w_2(s,t)|cos(2\omega t + \phi_2)$$

where $|w_1(s_1,t)|, |w_2(s_1,t)|$ represent the maximum value of the response at any point along the catenary, due to the first and second harmonic of the excitation spectrum respectively.

The longitudinal excitation induced via the lateral response $w(s,t)$ is:

$$\begin{aligned} c^2\frac{\partial w}{\partial s_1}\frac{\partial^2 w}{\partial s_1^2} = & \tfrac{1}{2}c^2\Big[(w_1^{'}w_1^{''} + w_2^{'}w_2^{''}) + w_1^{'}w_1^{''}cos(2\omega t + 2\phi_1) \\ & +(w_1^{'}w_2^{''} + w_2^{'}w_1^{''})\{cos(\omega t + \phi_2 - \phi_1) + cos(3\omega t + \phi_1 + \phi_2)\} + \\ & w_2^{'}w_2^{''}cos(4\omega t + 2\phi_2)\Big] \end{aligned} \qquad (E.8)$$

The first term on the right hand side of equation (E.8) is independent of time, and represents a static distributed force, which induces drift in the longitudinal response. Effectively this drift represents the mean displacement about which longitudinal motion occurs. The longitudinal excitation induced at frequencies $\omega, 2\omega, 3\omega, 4\omega$ demonstrates the autoparametric nature of the system, whereby the lateral motion induces longitudinal excitation.

The longitudinal system response due to the dynamic component of the excitation induced by the lateral motion follows, whilst the response due to the drift term is considered in section E.1.

Consider the longitudinal excitation induced by the linear lateral response of the catenary to the fundamental lateral excitation $W_1cos(\omega t)$. The longitudinal excitation occurs due to the coupling term $\frac{\partial w_1}{\partial s_1}\frac{\partial^2 w_1}{\partial s_1^2}$ in the longitudinal equation of motion. Since:

$$w_1(s_1,t) = Re\left[\frac{W_1}{sin\gamma_{l_1}}sin(\gamma_{l_1}(l_1 - s_1))e^{i\omega t}\right]$$

Thus it can be shown that[2].

$$Re(\frac{\partial w_1}{\partial s_1})Re(\frac{\partial^2 w_1}{\partial s_1^2}) = Re\left[\chi_1 sin2\gamma_{l_1}(l - s_1)e^{j(2\omega t)}\right] + D_1(s_1)$$

where $D_1(s_1)$ represents the static or drift excitation term. This term will be considered further in section E.1.

$$\chi_1 = \frac{W_1^2\gamma_{l_1}^3}{(2sin\gamma_{l_1}l_1)^2}$$

The longitudinal system response, considering the relevant boundary conditions described in equations (E.4),(E.6),(E.7) , is:

$$u_1(s,t) = Re\left[e^{2j\omega t}(Asin2\gamma_{l_1}(l_1 - s_1) + A_1 cos\gamma_2 s_1 + B_1 sin\gamma_2 s_1)\right]$$

$$u_2(s,t) = Re\left[e^{2j\omega t}(A_2 cos\gamma_2 s_1 + B_2 sin\gamma_2 s_1)\right]$$

where[3] $A = \frac{c^2\chi_1}{4(\gamma_{l_1}^2 c^2 - \omega^2) + 8i\mu\omega\gamma_{l_1}^2}$ and $u_1(s_1,t)$ and $u_2(s_2,t)$ represent the forced longitudinal motion of the catenary and vertical rope at frequency 2ω respectively.

The constants A_i, B_i are determined by the solution to the equations.

$$[\mathbf{A}]\{x\} = \{b\}$$

where:

[2] $\frac{\partial Re(w_1)}{\partial s_1}\frac{\partial^2 Re(w_1)}{\partial s_1^2} = Re(\frac{\partial w_1}{\partial s_1})Re(\frac{\partial^2 w_1}{\partial s_1^2})$

[3] In the case of general viscous damping, as introduced in Appendix G, $A = \frac{c^2\chi_1}{4(\gamma_{l_1}^2 c^2 - \omega^2) + j\omega(8\mu_s\gamma_{l_1}^2 + \mu_a)}$.

$[\mathbf{A}] =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ cos\gamma_2 l_1 & sin\gamma_2 l_1 & -1 & 0 \\ \frac{\Gamma^2}{\lambda}cos\gamma_2 l_1 + sin\gamma_2 l_1 & \frac{\Gamma^2}{\lambda}sin\gamma_2 l_1 - cos\gamma_2 l_1 & 0 & 1 \\ 0 & 0 & \Lambda^2 cos\gamma_2 l_2 + \lambda sin\gamma_2 l_2 & \Lambda^2 sin\gamma_2 l_2 - \lambda cos\gamma_2 l_2 \end{bmatrix}$$

$$\{x\} = \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{Bmatrix}$$

$$\{b\} = \begin{Bmatrix} -Asin2\gamma_{l_1} l_1 \\ 0 \\ (\frac{W_1 \gamma_{l_1}}{2\lambda_2 sin\gamma_{l_1} l_1})^2 - \frac{2\gamma_{l_1} A}{\gamma_2} \\ 0 \end{Bmatrix}$$

where:

$$\Lambda^2 = \frac{4M\omega^2}{AE} \qquad \Gamma^2 = \frac{4I\omega^2}{AER^2} \qquad \gamma_2 = \frac{\frac{2\omega}{c}}{(1+(\frac{2\mu\omega}{c^2})^2)^{1/4}}e^{-i\alpha/2}$$

$$\alpha = tan^{-1}\frac{2\omega\mu}{c^2} \qquad \zeta_2 = \frac{2\omega\theta\gamma_2}{AE} \qquad \theta = \rho A\mu$$

$$\lambda_2 = \gamma_2 + i\zeta_2 \qquad \gamma_{l_1} = \frac{\frac{\omega}{c_l}}{(1+(\frac{\mu_l\omega}{c_l^2})^2)^{1/4}}e^{-i\alpha_l/2} \qquad \alpha_{l1} = tan^{-1}\frac{\omega\mu_l}{c_l^2}$$

$$c^2 = \frac{E}{\rho} \qquad c_l^2 = \frac{T}{m}$$

The longitudinal response induced by the forced lateral response due to the second harmonic of the lateral excitation $W_2 cos2\omega t$, is obtained by appropriate substitution into the above equations.

The longitudinal response due to the interaction of the first and second lateral harmonic motions is developed along the same lines below. In this case time independent terms do not arise, but rather dynamic terms at ω and 3ω arise.

$$Re(\frac{\partial w_1}{\partial s_1})Re(\frac{\partial^2 w_2}{\partial s_1^2}) + Re(\frac{\partial w_2}{\partial s_1})Re(\frac{\partial^2 w_1}{\partial s_1^2}) =$$

$$\frac{1}{4}Re\left[\left(\frac{\gamma_{l_1}^* \gamma_{l_2} W_1^* W_2}{sin\gamma_{l_1}^* l_1 sin\gamma_{l_2} l_1}\right)\{(\gamma_{l_2} + \gamma_{l_1}^*)sin[(\gamma_{l_2} + \gamma_{l_1}^*)(l_1 - s)] + (\gamma_{l_2} - \gamma_{l_1}^*)sin[(\gamma_{l_2} - \gamma_{l_1}^*)(l_1 - s)]\}\, e^{j\omega t}\right] +$$

$$\frac{1}{4}Re\left[\left(\frac{\gamma_{l_1} \gamma_{l_2} W_1 W_2}{sin\gamma_{l_1} l_1 sin\gamma_{l_2} l_1}\right)\{(\gamma_{l_2} + \gamma_{l_1})sin[(\gamma_{l_2} + \gamma_{l_1})(l_1 - s)] + (\gamma_{l_2} - \gamma_{l_1})sin[(\gamma_{l_2} - \gamma_{l_1})(l_1 - s)]\}\, e^{3j\omega t}\right]$$

where $\gamma_{l_1}^*, W_1^*$ refers to the complex conjugate of γ_{l_1}, W_1.

The longitudinal system response at $e^{j\omega t}$ is thus obtained as:

$$u_1(s,t) = Re\left[e^{j\omega t}(\mathcal{A} sin(\gamma_{l_2} + \gamma^*_{l_1})(l_1 - s_1) + \mathcal{B} sin(\gamma_{l_2} - \gamma^*_{l_1})(l_1 - s_1) + A_1 cos\gamma s_1 + B_1 sin\gamma s_1)\right]$$

$$u_2(s,t) = Re\left[e^{j\omega t}(A_2 cos\gamma s_1 + B_2 sin\gamma s_1)\right]$$

where[4]:

$$\mathcal{A} = \frac{c^2\gamma^*_{l_1}\gamma_{l_2}W_1^*W_2(\gamma^*_{l_1} + \gamma_{l_2})}{4sin\gamma^*_{l_1}l_1 sin\gamma_{l_2}l_1[(\gamma_{l_2} + \gamma^*_{l_1})^2(c^2 + j\mu\omega) - \omega^2]}$$

$$\mathcal{B} = \frac{c^2\gamma^*_{l_1}\gamma_{l_2}W_1^*W_2(\gamma_{l_2} - \gamma^*_{l_1})}{4sin\gamma^*_{l_1}l_1 sin\gamma_{l_2}l_1[(\gamma_{l_2} - \gamma^*_{l_1})^2(c^2 + j\mu\omega) - \omega^2]}$$

and the loading vector $\{b\}$ is:

$$\{b\} = \left\{ \begin{array}{c} -\mathcal{A} sin(\gamma_{l_2} + \gamma^*_{l_1})l_1 - \mathcal{B} sin(\gamma_{l_2} - \gamma^*_{l_1})l_1 \\ 0 \\ \frac{W_1^*\gamma^*_{l_1}W_2\gamma_{l_2}}{2\lambda_1 sin\gamma^*_{l_1}l_1 sin\gamma_{l_2}l_1} - \frac{(\gamma_{l_2}+\gamma^*_{l_1})\mathcal{A}+(\gamma_{l_2}-\gamma^*_{l_1})\mathcal{B}}{\gamma_1} \\ 0 \end{array} \right\}$$

By appropriate substitution, the longitudinal response at the third harmonic of the Lebus excitation frequency can be obtained.

The total steady state longitudinal response may thus be obtained as a linear combination of that due to the longitudinal excitation at the winder drum,

[4]In the case of general viscous damping as described in Appendix G:

$$\mathcal{A} = \frac{c^2\gamma^*_{l_1}\gamma_{l_2}W_1^*W_2(\gamma^*_{l_1} + \gamma_{l_2})}{4sin\gamma^*_{l_1}l_1 sin\gamma_{l_2}l_1[(\gamma_{l_2} + \gamma^*_{l_1})^2(c^2 + j\mu_b\omega) + 2j\omega\mu_a - \omega^2]}$$

$$\mathcal{B} = \frac{c^2\gamma^*_{l_1}\gamma_{l_2}W_1^*W_2(\gamma_{l_2} - \gamma^*_{l_1})}{4sin\gamma^*_{l_1}l_1 sin\gamma_{l_2}l_1[(\gamma_{l_2} - \gamma^*_{l_1})^2(c^2 + j\mu\omega) + 2j\mu_a\omega - \omega^2]}$$

as presented in Appendix C and that induced by the lateral response of the catenary[5].

Clearly, the direct longitudinal excitation at the winder drum results in axial response at integer multiples of the coil cross over frequencies $\omega, 2\omega$, whilst that due to lateral catenary coupling accentuates these components, and introduces additional excitation at $3\omega, 4\omega$, as well as a drift term.

The drift in the longitudinal response as derived above causes a change in the average tension in the rope. As a consequence of this, the natural frequencies related to the lateral modes of the the variational equations of motion presented in Chapter 4, increase.

[5]It is noted that the results presented consider the case where the lateral excitation harmonics have a zero phase angle. Different phase angles can easily be accommodated in the analysis by including the phase information in the harmonic amplitudes W_n. Thus in general W_n would be a complex number.

E.1 Longitudinal Drift Term

The drift induced in the longitudinal response due to the static component of the excitation $D(s_1)$, is examined in this section. This occurs as a result of the quadratic term $Re[w_{,s}]Re[w_{,ss}]$ in the longitudinal equation of motion. Consider the steady state lateral response w_1 due to the fundamental of the lateral excitation at the drum, $Re[W_1 e^{j\Omega t}]$. The static component of the excitation applied to the longitudinal system is:

$$D(s_1) = \frac{1}{2}Re[w_{1,s_1} w_{1,s_1 s_1}] = \frac{1}{4}[w_{1,s_1} w^*_{1,s_1}]_{,s_1}$$

The longitudinal response to the drift term is obtained by satisfying the equations:

$$\begin{aligned} \frac{\partial^2 u_1}{\partial s_1^2} &= -D(s_1) \quad 0 < s_1 < l_1 \\ \frac{\partial^2 u_2}{\partial s_2^2} &= 0 \qquad l_1 < s_2 < l_2 \end{aligned} \tag{E.9}$$

and the boundary conditions:

$$\begin{aligned} u_1(0,t) &= 0 \\ u_2(0,t) &= u_1(l_1,t) \\ \frac{\partial u_1}{\partial s_1} + \frac{1}{2}\left(\frac{\partial w}{\partial s_1}\right)^2\Big|_{(l_1,t)} &= \frac{\partial u_2}{\partial s_2}\Big|_{(0,t)} \end{aligned}$$

The resulting drift in the longitudinal response is given by integrating equation (E.9) and satisfying the boundary conditions. This leads to:

$$u^D_{,ss} = -\frac{1}{4}[w_{,s}w^*_{,s}]_{,s} \tag{E.10}$$

$$u^D_{,s} = -\frac{1}{4}[w_{,s}w^*_{,s}] \tag{E.11}$$

$$u^D(s_1) = -\frac{1}{4}\int_0^{s_1}[w_{,s}w^*_{,s}]ds \tag{E.12}$$

Evaluating equation (E.12), and accounting for the first and second harmonic of the lateral response leads to:

$$u^D(s_1) = \chi_1 \left\{ \frac{1}{Re(\gamma_{l_1})^2} sin[2Re(\gamma_{l_1})(l - s_1)]cos\phi_1 + \frac{1}{Im(\gamma_{l_1})^2} sinh[2Im(\gamma_{l_1})(l - s_1)]sin\phi_1 \right\}$$
$$+\chi_2 \left\{ \frac{1}{Re(\gamma_{l_2})^2} sin[2Re(\gamma_{l_2})(l - s_1)]cos\phi_2 + \frac{1}{Im(\gamma_{l_2})^2} sinh[2Im(\gamma_{l_2})(l - s_1)]sin\phi_2 \right\}$$

$$\chi_1 = \frac{|\gamma_{l_1}|^3 |W_1|^2}{16|sin\gamma_{l_1} l_1|^2}$$

$$\chi_2 = \frac{|\gamma_{l_2}|^3 |W_2|^2}{16|sin\gamma_{l_2} l_1|^2}$$

The drift in the longitudinal response as derived above causes a change in the average tension in the rope. As a consequence of this, the natural frequencies related to the lateral modes of the the variational equations of motion presented in Chapter 4, increase. The drift induced by the lateral response due to the second harmonic of the lateral excitation at the winder drum can be evaluated by appropriate substitution. Thus the total longitudinal response of the system consists of a static drift to account for the lateral motion, and a dynamic component. The influence of the drift terms on the variational equations is considered further in Appendix F.3.

Appendix F

Parametric coupling Matrices

Chapter 4 presented a discussion which led to the proposition that the stability of the coupled linear steady state motion, as determined in appendices C,D,E, could serve as a criterion for determining design parameters so as to minimise the potential for nonlinear system behaviour. The homogeneous component of the linearised variational equations pertaining to the system are presented below, where the boundary conditions at the sheave and conveyance have been introduced with the use of the dirac delta function $\delta(s-l)$ and the Heaviside step function $H(s-l)$.

$$[1+\zeta\delta(s-l_1)+\eta(s-l_2)]\overline{u}_{tt} = \mu\overline{u}_{t,ss}+c^2\overline{u}_{,ss}+c^2\{w_{,s}\overline{w}_{,s}\}_{,s}[H(s)-H(s-l_1)] \quad \text{(F.1)}$$

$$\overline{v}_{,tt} = \mu_l\overline{v}_{t,ss} + \overline{c}^2\overline{v}_{,ss} + c^2\{(\overline{v}_{,s}u_{,s})_{,s} + \frac{1}{2}(w_{,s}^2\overline{v}_{,s})_{,s}\} \quad \text{(F.2)}$$

$$\overline{w}_{,tt} = \mu_l\overline{w}_{t,ss} + \overline{c}^2\overline{w}_{,ss} + c^2\{(\overline{w}_{,s}u_{,s})_{,s} + (w_{,s}\overline{u}_{,s})_{,s} + \frac{3}{2}(w_{,s}^2\overline{w}_{,s})_{,s}\} \quad \text{(F.3)}$$

$\zeta = I/\rho AR^2, \quad \eta = M/\rho A$

where c^2, $\overline{c}^2$ represent the longitudinal and lateral wave speeds respectively. $u(s,t)$ represents the total longitudinal steady state response due to the longitudinal excitation at the winder drum, and that induced by the lateral catenary motion, as derived in Appendices C, E respectively. $w(s,t)$ represents the steady state out-of-plane lateral catenary response as derived in Appendix D. $\overline{u}, \overline{v}, \overline{w}$ represents the variation of the motion about the steady state linear response in the longitudinal, in-plane lateral and out-of-plane lateral directions respectively. l_1 represents the catenary length, whilst l_2 represents the total cable length. The longitudinal equation (F.1) is defined over the entire length of the rope $0 \leq s \leq l_2$, whilst the lateral equations (F.2),(F.3) are defined only over the catenary length $0 \leq s \leq l_1$.

By applying a normal mode technique, where:

$$\overline{u} = \sum \phi_i(s) p_i(t)$$

$$\overline{v} = \sum \Phi_i(s) q_i(t)$$

$$\overline{w} = \sum \Phi_i(s) r_i(t) \tag{F.4}$$

the equations of motion were reduced to a set of coupled linear parametrically excited ordinary differential equations of the form:

$$[I]\{\ddot{y}\} + [2\zeta_n \omega_n]\{\dot{y}\} + \left[[\omega_n^2] + [A_d] + \sum_{n=1}^{4} [P(n\Omega t)] \right] \{y\} = 0 \tag{F.5}$$

where:

$$\{y\}^T = (p_i, q_i, r_i)$$

$$\sum_{n=1}^{4} [P(n\Omega t)] = \begin{bmatrix} 0 & 0 & U_{uw}(\Omega t, 2\Omega t) \\ 0 & V_{vv}(\Omega t, 2\Omega t, 3\Omega t, 4\Omega t) & 0 \\ W_{uw}(\Omega t, 2\Omega t) & 0 & W_{ww}(\Omega t, 2\Omega t, 3\Omega t, 4\Omega t) \end{bmatrix}$$

Where $[A_d]$ represents an initial stress matrix. This matrix describes the change in the variational natural frequencies as a result in a change in the average tension in the catenary due to the forced excitation. $[A_d]$ is defined in section F.3.

The parametric coupling matrices are obtained on orthogonalising equations (F.1),(F.2),(F.3) with respect to the undamped longitudinal and lateral mode shapes $\phi_i(s), \Phi_i(s)$. It is evident from the above equation that parametric excitation occurs at multiples of the coil cross-over frequency Ω.

The submatrices $[U_{uv}(n\Omega t)], [V_{vv}(n\Omega t)], [W_{vu}(n\Omega t)], [W_{ww}(n\Omega t)]$ are derived in subsequent sections.

F.1 Longitudinal Parametric Coupling

In order to extract the submatrix $[U_{uv}(n\Omega t)]$ in the parametric coupling matrix $[P(n\Omega t)]$, a normal mode approximation is applied to the longitudinal equation of motion (F.1). This is accomplished by assuming:

$$\overline{u} = \sum \phi_i(s)p_i(t)$$

$$\overline{w} = \sum \Phi_i(s)r_i(t)$$

Where the undamped longitudinal and lateral mode shapes of the i^{th} longitudinal and lateral mode are $\phi_i(s), \Phi_i(s)$ respectively, where:

$$\begin{aligned}
&\phi_i(s) = sin\gamma_i s && 0 \leq s \leq l_1 \\
&\phi_i(s) = sin\gamma_i l_1 cos\gamma_i(s-l_1) + (cos\gamma_i l_1 - \tfrac{\Gamma^2}{\lambda} sin\gamma_i l_1) sin\gamma_i(s-l_1) && l_1 \leq s \leq l_2 \\
&\Phi_i(s) = sin\delta_i s && 0 \leq s \leq l_1
\end{aligned}$$

where $\delta_i = i\pi/l_1, \gamma_i^2 = w_i^2/c^2$, where ω_i is the i^{th} longitudinal natural frequency. Substituting the above equations into equation (F.1), and pre-multiplying by the i^{th} undamped longitudinal mode ϕ_i, and integrating over the domain of the rope leads to:

$$\begin{aligned}
\int_0^{l_2} [1+\zeta\delta(s-l_1)+\eta\delta(s-l_2)]\phi_i \sum \phi_j \ddot{p}_j ds &= c^2 \int_0^{l_2} \phi_i \sum \phi_j'' p_j ds \\
&+ c^2 \int_0^{l_1} \phi_i \left\{ \tfrac{\partial w}{\partial s} \sum \Phi_j'' r_j + \tfrac{\partial^2 w}{\partial s^2} \sum \Phi_j' r_j \right\} ds \\
&+ \lim_{l_{1-} \to l_{1+}} c^2 \int_{l_{1-}}^{l_{1+}} \phi_i \tfrac{\partial}{\partial s} \left\{ \tfrac{\partial w}{\partial s} \sum \Phi_j' r_j \right\} ds \qquad \text{(F.6)}
\end{aligned}$$

The integrals $\int_0^{l_2}[1+\zeta\delta(s-l_1)+\eta\delta(s-l_2)]\phi_i\phi_j ds$ and $c^2 \int_0^{l_2} \phi_i \phi_j'' ds$ are orthogonal for the normal modes of a system by definition. These expressions are termed the modal mass and modal stiffness m_{ii}, k_{ii} respectively, where the ratio k_{ii}/m_{ii} is equal to the square of the undamped natural frequency ω_i^2. Thus it is only necessary to calculate the modal mass and the remaining integrals on the right side of equation of equation (F.6). ie:

$$m_{ii} = \int [1+\zeta\delta(s-l_1)+\eta\delta(s-l_2)]\phi_i\phi_i ds$$

$$k_{ii} = c^2 \int \phi_i \phi_i'' ds = \omega_i^2 m_{ii}$$

The last integral on the right hand side of equation (F.6) accounts for the dirac delta function employed to represent the boundary condition at the sheave wheel, coupling the catenary to the vertical rope. This term may be integrated by parts as:

$$\lim_{l_{1-} \to l_{1+}} c^2 \int_{l_{1-}}^{l_{1+}} \phi_i \frac{\partial}{\partial s} \left\{ \frac{\partial w}{\partial s} \sum \Phi_j' r_j \right\} ds = -c^2 \phi_i \left\{ \frac{\partial w}{\partial s} \sum \Phi_j' \right\} |_{l_1} r_j$$

Thus the submatrix $[U_{uw}(n\Omega t)]$ is evaluated as:

$$[U_{uw}(n\Omega t)]_{i,j} = -\frac{c^2}{m_{ii}} \left[\int_0^{l_1} [\phi_i \Phi_j'' \frac{\partial w}{\partial s} + \phi_i \Phi_j' \frac{\partial^2 w}{\partial s^2}] ds - \frac{1}{2} \phi_i \left\{ \frac{\partial w}{\partial s} \Phi_j' \right\} |_{l_1} \right]$$

$$\begin{aligned} m_{ii} &= \frac{1}{2}(l_1 - \frac{1}{2\gamma_i} sin 2\gamma_i l_1) + \frac{1}{2}(A_i^2 + B_i^2) l_2 + \frac{1}{4\gamma_i}(A_i^2 - B_i^2) sin 2\gamma_i l_2 + \frac{A_i B_i}{\gamma_i} sin^2 \gamma_i l_2 \\ &+ \zeta \phi_i(l_1)^2 + \eta \phi_i(l_2)^2 \end{aligned}$$

$$A_i = sin\gamma_i l_1 \quad B_i = cos\gamma_i l_1 - \frac{\Gamma^2}{\lambda} sin\gamma_i l_1$$
$$\gamma_i = \omega_i / c \quad \eta = \frac{M}{\rho A} \quad \zeta = \frac{I}{\rho A R^2}$$

Where ω_i is the i^{th} undamped natural frequency of the longitudinal system.

If one considers the response due to the n^{th} lateral harmonic of the out-of-plane excitation at the winder drum, $W_n e^{jn\Omega t}$, where the response $w_n(s,t)$ in complex form is given by[1]:

$$w_n(s,t) = \frac{W_n}{sin\gamma_{l_n} l_1} sin(\gamma_{l_n}(l_1 - s)) e^{in\Omega t} \qquad \text{(F.7)}$$

[1] W_n represents the amplitude and phase of the excitation and is generally a complex entity.

where

$$\gamma_{l_n} = \frac{\frac{n\Omega}{\bar{c}}}{(1 + (\frac{n\mu_l\Omega}{\bar{c}^2})^2)^{\frac{1}{4}}} e^{-\frac{\alpha_{l_n}}{2}}$$

$$\alpha_{l_n} = tan^{-1} \frac{\mu_l n\Omega}{\bar{c}^2}$$

Evaluation of the integral for $[U_{uw}(n\Omega t)]$ leads to:

$$[U_{uw}(n\Omega t)]_{i,j} = \frac{c^2}{m_{ii}} \left\{ \frac{\delta_j \gamma_{l_n} W_n}{sin(\gamma_{l_n} l_1)} \left[\frac{\gamma_i \gamma_{l_n} sin(\gamma_{l_n} l_1)\{\delta_j^2 + \gamma_i^2 - \gamma_{l_n}^2\} - (-1)^j sin(\gamma_i l_1)\{\gamma_i^2(\delta_j^2 + \gamma_{l_n}^2) - (\delta_j^2 - \gamma_{l_n}^2)^2\}}{[(\delta_j + \gamma_i)^2 - \gamma_{l_n}^2][(\delta_j - \gamma_i)^2 - \gamma_{l_n}^2]} \right] - \frac{1}{2}(-1)^j \gamma_{l_n} \delta_j W_n \frac{sin\gamma_i l_1}{sin\gamma_{l_n} l_1} \right\} e^{in\Omega t}$$

The parametric coupling matrix $[U_{uw}(n\Omega t)]$ is complex, hence the form of the matrix due to an excitation $Re\{W_n e^{in\Omega t}\}$ will be $Re([U_{uw}(n\Omega t)])$. If the first and second harmonic of the Lebus excitation frequency are considered, two sets of parametric coupling matrices arise, namely $Re\{[U_{uw}(\Omega t)]\}$, $Re\{[U_{uw}(2\Omega t)]\}$.

F.2 Lateral Parametric Coupling

This section presents the lateral parametric coupling matrices $[W_{uw}(n\Omega t)]$, $[W_{ww}(n\Omega t)]$, $[V_{vv}(n\Omega t)]$. The lateral out-of-plane equation of motion is:

$$\overline{w}_{,tt} = \mu_l \overline{w}_{t,ss} + \overline{c}^2 \overline{w}_{,ss} + c^2\{(\overline{w}_{,s}u_{,s})_{,s} + (w_{,s}\overline{u}_{,s})_{,s} + \frac{3}{2}(w_{,s}^2\overline{w}_{,s})_{,s}\} \tag{F.8}$$

Substituting the normal mode approximations (F.4) into equation (F.8), and pre-multiplying by the i^{th} undamped lateral mode Φ_i, and integrating the equation over the domain of the catenary[2] leads to:

$$\begin{aligned}
\int_0^{l_1} \Phi_i \sum \Phi_j \ddot{r}_j ds &= \overline{c}^2 \int_0^{l_1} \Phi_i \sum \Phi_j^{''} r_j ds \\
&+ c^2 \int_0^{l_1} \Phi_i \left\{ \frac{\partial w}{\partial s} \sum \phi_j^{''} p_j + \frac{\partial^2 w}{\partial s^2} \sum \phi_j^{'} p_j \right. \\
&+ \left. \frac{\partial u}{\partial s} \sum \Phi_j^{''} r_j + \frac{\partial^2 u}{\partial s^2} \sum \Phi_j^{'} r_j \right\} ds \\
&+ \frac{3}{2} \int_0^{l_1} \{ (\frac{\partial w}{\partial s})^2 \sum \Phi_j^{'} r_j \}_{,s} ds
\end{aligned} \tag{F.9}$$

Since the normal modes of the catenary are orthogonal by definition and hence the integrals $\int_0^{l_1} \Phi_i \Phi_i ds = m_{ii}$, $\overline{c}^2 \int_0^{l_1} \Phi_i \Phi_i^{''} ds = k_{ii}$, where the terms m_{ii}, k_{ii} refer to the modal mass and stiffness of the catenary, and the ratio k_{ii}/m_{ii} is represents the square of the i^{th} undamped natural frequency of lateral vibration of the catenary $\overline{\omega}_i^2$. Thus it is necessary to calculate the modal mass and the remaining integrals in the last two terms on the right hand side of equation(F.9).

$$m_{ii} = \int_0^{l_1} \Phi_i \Phi_i ds = l_1/2$$

$$k_{ii} = \overline{c}^2 \int_0^{l_1} \Phi_i \Phi_i^{''} ds = \overline{\omega}_i^2 m_{ii}$$

[2]In this case, the domain reduces to the catenary length, as the lateral mode is only defined for the catenary

The remaining integrals are defined by:

$$[W_{uw}(n\Omega t)]_{i,j} = -\frac{c^2}{m_{ii}}\int_0^{l_1}\Phi_i\left\{\frac{\partial w}{\partial s}\sum\phi_j'' + \frac{\partial^2 w}{\partial s^2}\sum\phi_j'\right\}ds$$

$$[W_{ww}(n\Omega t)]_{i,j} = -\frac{c^2}{m_{ii}}\int_0^{l_1}\Phi_i\left\{\frac{\partial u}{\partial s}\sum\Phi_j'' + \frac{\partial^2 u}{\partial s^2}\sum\Phi_j'\right\}ds - \frac{3c^2}{2m_{ii}}\int_0^{l_1}\{(\frac{\partial w}{\partial s})^2\sum\Phi_j'\}_{,s}ds$$

<u>Parametric coupling matrix $[W_{uw}(n\Omega t)]$</u>

Substituting the solution for the lateral out-of-plane response w_n from equation (F.7), due to the n^{th} harmonic of the lateral out-of-plane excitation at the Lebus drum, $W_n e^{in\Omega t}$, and carrying out the integration for the first integral above leads to:

$$[W_{uw}(n\Omega t)]_{j,i} = \frac{c^2\delta_i\gamma_j\gamma_{l_n}W_n}{m_{ii}sin(\gamma_{l_n}l_1)}\left[\frac{(-1)^i\gamma_j sin(\gamma_j l_1)(\delta_i^2 - \gamma_j^2 + \gamma_{l_n}^2) + \gamma_{l_n}sin(\gamma_{l_n}l_1)(\delta_i^2 + \gamma_j^2 - \gamma_{l_n}^2)}{[\delta_i^2 - (\gamma_j - \gamma_{l_n})^2][\delta_i^2 - (\gamma_j + \gamma_{l_n})^2]}\right]e^{in\Omega t}$$

<u>Parametric coupling matrix $[W_{ww}(n\Omega t)]$</u>

The second integral above for $[W_{ww}(n\Omega t)]$ is simplified into two separate components. The integral depends on the forced response u due to the longitudinal excitation at the drum and as a result of the lateral catenary motion, and on the forced lateral response w.

Considering the first part of the integral which is due to the total longitudinal response u.ie:

$$[W_{ww}(n\Omega t)]_{i,j} = -\frac{c^2}{m_{ii}}\int_0^{l_1}\Phi_i\left\{\frac{\partial u}{\partial s}\sum\Phi_j'' + \frac{\partial^2 u}{\partial s^2}\sum\Phi_j'\right\}ds$$

The total dynamic longitudinal response including the fundamental and the second harmonic longitudinal and lateral excitation at $\Omega, 2\Omega$ is:

$$\begin{aligned} u_l(s,t) = \; & Re\Big[R_1e^{i\Omega t} + R_2e^{2i\Omega t} + R_3e^{3i\Omega t} + R_4e^{4i\Omega t} + \\ & \{\mathcal{A}_1 sin[(\gamma_{l2} + \gamma_{l_1}^*)(l_1 - s) + \mathcal{B}_1 sin[(\gamma_{l2} - \gamma_{l_1}^*)(l_1 - s)\}e^{i\Omega t} \\ & + \mathcal{A}_2 sin2\gamma_{l_1}(l_1 - s)e^{2i\Omega t} + \\ & \{\mathcal{A}_3 sin[(\gamma_{l2} + \gamma_{l_1})(l_1 - s) + \mathcal{B}_3 sin[(\gamma_{l2} - \gamma_{l_1})(l_1 - s)\}e^{3i\Omega t} + \\ & \mathcal{A}_4 sin2\gamma_{l2}(l_1 - s)e^{4i\Omega t}\Big] \end{aligned}$$

where A_n, B_n are determined from Appendix E as:

$$A_1 = \frac{\gamma_{l_1}^2 \gamma_{l_2} W_1^2 W_2 (\gamma_{l_1}^2 + \gamma_{l_2})}{4 sin(\gamma_{l_1} l_1) sin(\gamma_{l_2} l_1)[(\gamma_{l_1} + \gamma_{l_2})^2 (c^2 + i\mu\Omega) - \Omega^2]} \qquad B_1 = \frac{\gamma_{l_1}^2 \gamma_{l_2} W_1^2 W_2 (\gamma_{l_1}^2 - \gamma_{l_2})}{4 sin(\gamma_{l_1} l_1) sin(\gamma_{l_2} l_1)[(\gamma_{l_1} - \gamma_{l_2})^2 (c^2 + i\mu\Omega) - \Omega^2]}$$

$$A_2 = \frac{\gamma_{l_1}^4 W_1^2 sin(2\gamma_{l_1}(l_1 - s))}{(2 sin(\gamma_{l_1}))^2 [(2\gamma_{l_1})^2 (c^2 + 2i\mu\Omega) - 4\Omega^2]}$$

$$A_3 = \frac{\gamma_{l_1} \gamma_{l_2} W_1 W_2 (\gamma_{l_1} + \gamma_{l_2})}{4 sin(\gamma_{l_1} l_1) sin(\gamma_{l_2} l_1)[(\gamma_{l_1} + \gamma_{l_2})^2 (c^2 + 3i\mu\Omega) - 9\Omega^2]} \qquad B_3 = \frac{\gamma_{l_1} \gamma_{l_2} W_1 W_2 (\gamma_{l_1} - \gamma_{l_2})}{4 sin(\gamma_{l_1} l_1) sin(\gamma_{l_2} l_1)[(\gamma_{l_1} - \gamma_{l_2})^2 (c^2 + 3i\mu\Omega) - 9\Omega^2]}$$

$$A_4 = \frac{\gamma_{l_2}^4 W_2^2 sin(2\gamma_{l_2}(l_1 - s))}{(2 sin(\gamma_{l_2}))^2 [(2\gamma_{l_2})^2 (c^2 + 4i\mu\Omega) - 16\Omega^2]}$$

$$R_n = A_n cos\gamma_n s + B_n sin\gamma_n s$$

and A_n, B_n, γ_n are obtained from the solutions to the forced longitudinal system response as defined in Appendix E.

The parametric coupling matrix relating to the longitudinal response will be evaluated as two component matrices. ie

$$[W_{uw}(n\Omega t)]_{n=1,2,3,4} = [W_{uw}(n\Omega t)]^1_{n=1,2,3,4} + [W_{uw}(n\Omega t)]^2_{n=1,2,3,4}$$

Where $[W_{uw}(n\Omega t)]^1_{n=1,2,3,4}$ results from the terms $R_n e^{in\Omega t}$ in the longitudinal response, whilst $[W_{uw}(n\Omega t)]^2_{n=1,2,3,4}$ results from the remaining terms.

Considering the coupling matrix $[W_{uw}(n\Omega t)]^1$ due to the longitudinal response term $R_n e^{in\Omega t}$, the integral $[W_{uw}(n\Omega t)]^1$ is given by:

$$[W_{uw}(n\Omega t)]^1_{ij} = \delta_i \delta_j \gamma_n^2 \beta_n \frac{c^2}{m_{ii}} \left[\frac{\delta_i^2 + \delta_j^2 - \gamma_n^2}{[(\delta_i + \delta_j)^2 - \gamma_n^2][(\delta_i - \delta_j)^2 - \gamma_n^2]} \right] e^{in\Omega t}$$

$$\beta_n = A_n - (-1)^{i+j} \{A_n cos(\gamma_n l_1) + B_n sin(\gamma_n l_1)\}$$

The parametric coupling matrix $[W_{uw}(n\Omega t)]^2_{ij}$ is obtained by considering the term $A_2 sin 2\gamma_l(l_1 - s) e^{2i\omega t}$ separately. Carrying out the appropriate integration leads to:

$$[W_{uw}(2\Omega t)]^2_{ij} = 4 A_2 \delta_i \delta_j \gamma_{l_1}^2 sin(2\gamma_{l_1} l_1) \frac{c^2}{m_{ii}} \left[\frac{\delta_i^2 - \delta_j^2 - 4\gamma_{l_1}^2}{[(\delta_i + \delta_j)^2 - 4\gamma_{l_1}^2][(\delta_i - \delta_j)^2 - 4\gamma_{l_1}^2]} \right] e^{2i\Omega t}$$

The matrices $[W_{ww}(n\Omega t)]^2$ for $n = 1, 3, 4$ can be evaluated by appropriate substitution in the above equation.

The coupling matrix due to the forced lateral response w is defined by:

$$[W_{ww}(n\Omega t)]^3 = -\frac{3c^2}{2m_{ii}} \int_0^{l_1} \Phi_i \{ (\frac{\partial w}{\partial s})^2 \sum \Phi_j^{'} \}_{,s} ds \tag{F.10}$$

If one considers the forced lateral response due to the first and second harmonic of the Lebus frequency then:

$$w = Re(\sum_{n=1}^{2} \frac{W_n}{sin\gamma_{l_n} l_1} sin\gamma_{l_n}(l_1 - s) e^{in\Omega t}) = w_1 + w_2$$

Consequently dynamic[3] components of $(w_{,s})^2$ occur at $n\Omega$, $n = 1 \cdots 4$, given respectively as:

$$(w_{,s})^2 = \frac{1}{2}[2Re(\frac{\partial w_1}{\partial s}\frac{\partial w_2^*}{\partial s}) + Re(\frac{\partial w_1}{\partial s})^2 + 2Re(\frac{\partial w_1}{\partial s}\frac{\partial w_2}{\partial s}) + Re(\frac{\partial w_2}{\partial s})^2]$$

Consider the second component $Re(\frac{\partial w_1}{\partial s})^2$:

$$Re(\frac{\partial w_1}{\partial s})^2 = \frac{1}{2} Re((\frac{\gamma_{l_1} W_1}{sin\gamma_{l_1} l_1})^2 cos^2\gamma_{l_1}(l_1 - s) e^{2i\Omega t})$$

Substituting this expression into equation (F.10) and performing the integration:

$$[W_{ww}(2\Omega)]^3 = \frac{3c^2}{2m_{ii}} \{\frac{\gamma_{l_1} W_1}{sin\gamma_{l_1} l_1}\}^2 \left[\frac{\delta_i \delta_j \gamma_{l_1} sin2\gamma_{l_1} l_1}{[(\delta_i - \delta_j)^2 - (2\gamma_{l_1})^2][(\delta_i + \delta_j)^2 - (2\gamma_{l_1})^2]} - \Lambda \right] e^{i\Omega t}$$

where $\Lambda = \frac{l_1}{8}$ if $i = j$, and $\Lambda = 0$ if $i \neq j$.

[3]The static component of this term affects the lateral variational frequencies, and is considered further in section F.3.