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Doctor Of Philosophy

Symmetry reductions and group-invariant solutions for models arising in water and contaminant transport

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May 23, 2018

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Abstract

In this work we consider convection-diffusion equation (CDE) arising in the theory of contamination of water by oil spill. Furthermore, these equations arise in solute transport and groundwater. Group classification of the one dimensional CDE which depends on time t and space x is performed. Lie point symmetries of the one-dimensional CDE are obtained. Group invariant solutions are constructed using admitted Lie point symmetries and these solutions are used to reduce the CDE to the ordinary differential equations (ODEs), which in most cases are solvable. In cases where a number of symmetries are obtained, we will construct the one-dimensional optimal systems of sub-algebras.

The two-dimensional and three dimensional CDE in solute transport with constant dispersion coefficient is considered. In some of these cases, double reduction methods will be used. Exact solutions are obtained using the Lie symmetry method in conjunction with the (G'/G)-expansion method and the substitution $w(z) = (z')^{-1}$. To further our studies, we apply the method of potential symmetries to determine group invariant solutions that cannot be obtained using point symmetries.

Finally, the non-classical symmetries are obtained and comparison study is done between the results obtained through nonlocal and nonclassical symmetry methods.

Declaration

I declare that the contents of this thesis are my own original work, except where due references have been made. It is being submitted as fulfilment of the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other institution.

_____

B.P. NTSIME

May 23, 2018

Dedications

I dedicate this work to my three kids, Pleasure, Kagiso and Keaoutumisa.

To my late father and mother, you left too soon!

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To my siblings, I don't have words to express how fortunate I am to be part of you. Together we will conquer the world. To my late parents, Ntate Mabule and Mme Nneneki, I would have never chosen any parent if I was asked to do so. Thank you for raising me to be this little old girl graduating at fifty five. I will forever cherish your words of wisdom "O ikoketse ngwanaka."

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have conquered the odds. You have been a pillar to me and your siblings. May the good Lord bless you abundantly.

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Figure 1: *I shall do all things through Christ who strengthens me, Philippians 4v13.* Image from *www.nap.ed*

Chapter 1

Introduction

In this chapter, a theoretical and physical account of contamination of water by different pollutants is given. Focus is on the oil spill as one form of water pollutant. The sources, effects, cleaning, removal and management of the oil spill are briefly outline here. The literature studies on the contamination of water is provided.

1.1 Physical account

Environmental contamination is one of the major challenges world wide. Due to this activity, essentials of life such as water (groundwater, sea, ocean and river) [1], air, environment and soil are negatively exposed to amongst others smoke, dust, baggage, refuse, salt and oil [2]. This exposure is the result of processes such as industrial activities, municipal dumping areas, precipitation, increase in concentration of nitrates and other chemical species, increase in harmful pathogens, agricultural activities such as the use of fertilizers, pesticides, herbicides and also radioactive wastes and oil spill [3]. The oil spill occurs either intentionally or accidentally through a number of occurrences. These occurrences include leaking oil tanks, barges, pipelines, refineries, drilling rigs and storage facilities; accidents from marine oil tankers or freighters; vessels sinking; operations from quays or ports; insufficient knowledge, oversight and ignorance of some managers and operators; technological; faulty facilities and equip-

ment; explosion and combustion and civil war [4]. The Torrey Canyon oil spill that happened on March, 18, 1967 in Cornwall, UK is an example of oversight and ignorance. It was mainly caused by overloading when a ship designed for 60 000 tons capacity was loaded with a 120 000 tons causing the ship to hit a reef. The oil move through spreading, advection and Langmuir circulation polluting over 180 miles of coastal land causing serious damages [5, 6].

Many countries have experienced disastrous oil spills in the past. To name a few, Exxon Valdez grounded and spilled oil in Alaska, March, 24, 1989 [7]; the drilling platform Well Ixtoc I Well exploded after catching fire in Campeche, Mexico, June, 3, 1979 [8]; the Hebei Spirit oil spill occurred in South Korea on December, 7, 2007 [9] and the British Petroleum (BP) oil spill in Mexico, April, 20, 2010 [10]. One of the most disastrous oil spill ever occurred was the Gulf of Mexico oil spill, 20th April 2010 where the Gulf collapsed into the sea killing eleven people [11, 12]. About 4.9 million of barrel of crude oil was lost. The cleaning took over five months. By the time cleaning was completed marine and wildlife habitats were polluted. South Africa is not immune from such disasters. It experienced two disastrous oil spill in 1994 and 2000 [11, 13]. In 1994, South Africa experienced the largest oil spill when MV Apollo sea sank near Cape Town. The oil leaked from the vessel and caused major environment disaster. Thousands of sea birds such as penguins were oiled. Some were rehabilitated and some could not survive. Substantial number of the rehabilitated penguins later died due to malaria or failure to adapt to the polluted environment [13]. In 2000, another oil spill occurred when MV Treasure spread towards Robin Island and Dessen Island killing over 2000 penguins [13].

Oil spills have serious negative effects, be it short and long term on the environment, the economy and social activities [14]. The environment and vegetation are adversely affected by the toxic chemicals from oil (benzene, toluene, poly-aromatic hydrocarbons and oxygenated polycyclic aromatic) [15]. Some of these chemicals do not only pollute the water but also the air in the form of vapour through evaporation. Human beings and other living animals also inhale these chemicals from the air. Once they reach the body, they cause lung and liver infections that can lead

to death. The water scent also changes and thus affect some life processes such as reproduction and nurturing young ones. Some marine mammals use the scent to find their young ones. Some young ones may be abandoned and rejected by their mothers leaving them stranded without care and ultimately die because they cannot trace each other [4]. The root areas of plants fail to absorb water from the soil. Sea birds' feathers and furs get coated with oil preventing birds to fly. Other affected lives include aquatic plants, marine mammals, marine ecosystem and the finally the food chain is interrupted [14, 15]. Economically, an enormous amount of oil get lost through spilling. The recovery, removal and cleaning processes are quite expensive. The company's reputation and credibility is also damaged. Companies pay lot of penalty fees and lawsuits, for example in case of the Gulf oil spill, BP was fined 18.7 billion US dollars for cleaning and other expenses [12]. The cleaners' health is negatively affected leading to huge medical bills due by government. The hospitality and tourism industry fail to attract tourists because it become impossible for them to do most of social activities such as swimming, sailing, rafting and fishing. This obviously lead to a huge loss of revenue [12].

Due to the pollution effect of oil in water and environment as stated above, oil spill must be treated as an emergency activity and must be removed as soon as possible. The cleaning and removal processes depend on the composition of compounds in the oil. For example it can be easy to remove petroleum oil than crude oil, because crude oil contains hydrocarbons (alkanes, cycloalkanes, various aromatic hydrocarbons), organic nitrogen (oxygen, sulphur) and some metals such as copper, iron, nickel and vanadium [16]. Cleaning and removal processes can be natural, mechanical or physical processes. The natural forces are the weathering processes such as evaporation, biodegradation, sedimentation, dissolution, emulsification, oxidation and spreading. These processes can either promote pollution or to a certain extend retard the spill, for an example evaporation reduces the amount of oil through vapour, on the other hand spreading makes it difficult to remove the oil until a barrier such as booms are erected [17].

Biodegradation happens at a later stage of the spill generally after evaporation and

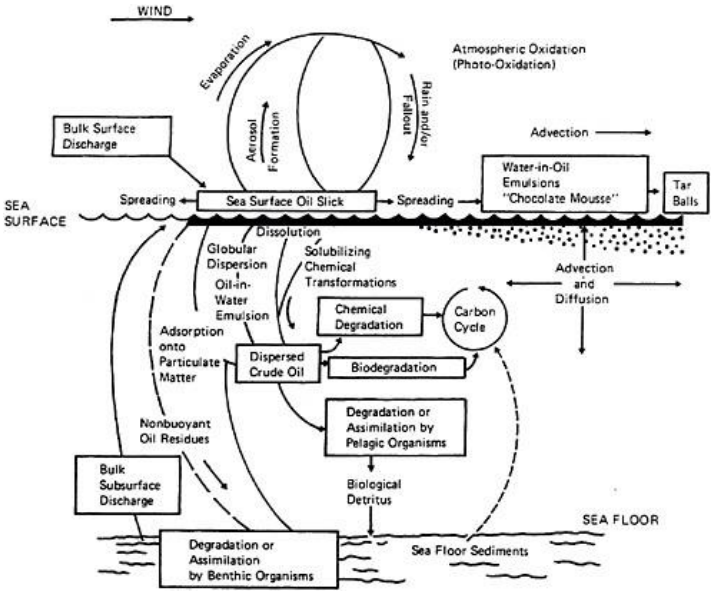


Figure 1.1: Weathering processes after oil spill. Image from https://ars.els-cdn.com/content/image/1-s2.

spreading. During this process, some micro-organisms or bacteria start to feed on oil hydrocarbons thereby reducing the oil content in water. Furthermore during sedimentation oil particles tend to settle out of the oil and get deposited against any barrier constructed. Gravity, acceleration, and electromagnetism play a vital role in this process [16]. On the other hand, dissolution happens and some oil chemicals are broken down into soluble small particles [16, 17]. Another weathering process is emulsification where the oil mixes with water. Once emulsification happens, the oil is collected in one area where it can be removed easily. During oxidation, oxygen combines with oil hydrocarbons to produce water soluble compounds [16, 17].

The physical and mechanical methods involve the installation of booms and skimmers, spraying sorbents and dispersants on the water surface. Temporary floating barriers called booms can be installed to stop the oil from spreading. The booms block the oil to pollute the shorelines [16, 18]. On the other hand, devices such as skimmers can be used to collect oil spilled in water. Once the oil is collected it

can be removed from the water using absorbent pads containing chemicals that can absorb or adsorp the oil from the source. These chemicals will change oil from liquid to semi-solid making it easy to collect or remove. The chemicals can either be inorganic mineral (include perlites, graphites, diatomites etc.) products, organic synthetic products(include polypropylene and polyurethane) or organic vegetable products [19]. Dispersants can also be used to break down the chemicals into droplets that readily mix with water. These droplets will be pushed down the water. Dispersants again expose the oil to some oil eating bacteria which then break it down to soluble droplets [18].

Rapid response from disaster management, government and communities is crucial for removal and prevention. Environmental scientists must do prevention planning that involves learning from the past events, even though the occurrences differ according to geographical, ecological, societal, and temporal context in which the spill occurred [16]. Larger spills in seas can be prevented by covering ships with Miko plaster that quickly repairs the leaks and cracks while floating [20] and implementing programmes such as the Oil Spill Prevention Containment and Countermeasures programme by the United States Environmental Protection Agency [11]. The removal can also involve methods of containment, dispersant and cleaning processes and prevention can be through containment method, corralling the oil and acting as a barrier [21, 22]. Different models of oil spill has been investigated and different computer programs such as COZOIL model [21], SINTEF OSCAR [23], GULFSPILL [24], ADIOS [25] have been developed as part of contingency plan to forecast the spreading.

1.1.1 Sources of contaminants

In their studies the authors [26–30] categorise the contaminants as follows: Organic contaminants from industrial wastes and spills, and aromatic hydrocarbons. Examples are chlorinated hydrocarbons such as chloroform and benzene respectively. Inorganic contaminants includes chemicals such as chloride, sulphate, nitrate and

sodium ions originating from radionuclide in uranium tailings and nuclear waste in storage sites. Pathogenic organisms are from poorly constructed septic tanks or improper disposal of water from hospitals, see Figure 1.2.

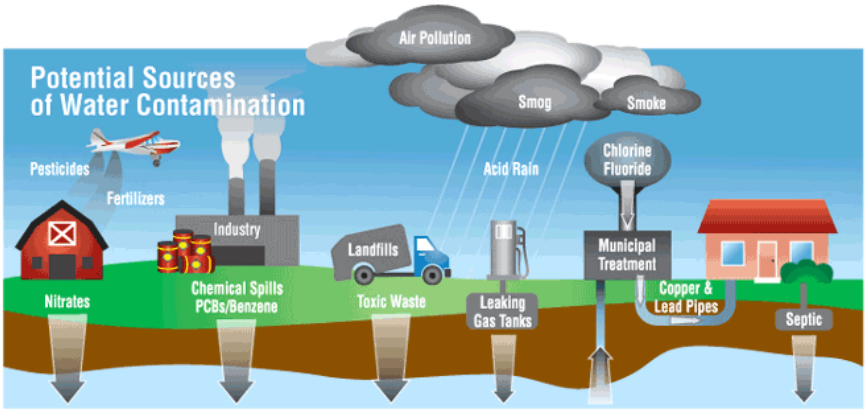


Figure 1.2: Some sources of contaminants. Image from ITOPF.com

1.1.2 Classification of sources of contaminants

According to [26, 27] contaminants are categorically classified as firstly sources designed to release substances from engineered structures for subsurface disposal of waste by percolation, example is septic tank. Secondly there are sources designed to store, treat and/or dispose of substances as well as discharge resulting from unplanned release. Examples are landfills of municipal and industrial waste, legal and illegal waste dumps, graveyards and animal burial sites [28]. Thirdly we have sources that transport and transfer hazardous and non-hazardous materials. There are other sources that discharge substances to the environment as part of various planned activities [29]. Examples are irrigation practices, pesticides, fertilizers, animal feeding operations, application of de-icing salts, urban run-off, precipitation, surface and subsurface mining and mining drainage operations. The other category includes wells and construction excavation, geothermal and heat recovery wells, water supply wells, monitoring wells and exploration wells [29]. The last category involves the naturally

occurring sources whose discharge is created and made worse by human activities. The activities include transporting oil and atmospheric pollutants. Other sources include the point sources and distributing source. Point sources include septic tanks, storage tanks, sanitary landfills, uncontrolled hazardous waste dumps, surface impoundments of liquids, abandoned wells, disposal of drilling mud, transportation accidents, storage of hazardous solid chemicals, tailing of coal, copper, uranium and other mines. Distributed source include agriculture and acid precipitation [28–30].

1.1.3 Long term effects on health to living species

The analysis of solute transport equations is of great interest to the environmental engineers and scientists; since groundwater contamination by hazardous pollutants and soil salination is a worldwide problem [31, 32].

Groundwater provides base flow to streams and supports wetlands and other groundwater dependent ecosystem. It is predominantly needed for agriculture, urban supplies, domestic and stock supply [33, 34]. Common health effects are diarrhea and nausea and vomiting. Chronic effects lead to cancer, liver and kidney damage, hair loss and reproductive difficulties. More information on water contamination can be found from [35–37].

1.1.4 Removal of other contaminants from the groundwater

Environmental scientists have to monitor and control any unhealthy activity by tracking down how these chemical and biological contaminants are transported through subsurface aquifer systems and come up with solutions either by preventing or reducing the contamination . To add to the methods stated above, contamination can be eradicated by identifying the pollutant source accurately and efficiently, removal of the contaminant from the aquifer contaminant, if possible abandon the use of the source and find the alternative source [38–41].

1.2 Literature review

The analysis of the reaction-diffusion and advection-dispersion equations arising in oil spill and solute transport continues to be of a scientific interest [42,43]. When oil is spilled into the water, the environment immediately experiences two main processes. The first one is caused by coastal currents due to advection, spreading and diffusion. The second one is due to oil-water and oil-air interfaces [44]. Nagheed and Kolahdoozan [44] investigated the transport of oil slick by using the binomial interpolation method. The authors developed a two-dimensional, two phased numerical method model to predict the transport of the slick. In their study, they considered the effects of spreading, evaporation, sedimentation, emulsification, and simulation processes. The wind and water currents effects were also considered during these processes. The numerical solutions illustrated that emusification and evaporation changes the viscocity and density of the oil. Jacque *et al* [45] studied the one-dimensional vertical flow in a variable saturated medium using Richard's equation and solute transport CDE. Gj*ö*stein [46] studied the oil spill spreading model in cold waters using finite difference method in a two-dimensional space. Pérez Guerrro *et al* [47] considered one, two and three-dimensional transient and steady state of the CDE. The authors used the generalized transform and integral transform techniques to change the variables leading to exact solutions of the CDE.

The oil model attracted other researchers from different mathematical approaches including the Lie symmetry methods of solutions. This method was introduced by Ovsiannikov [82] and it was welcomed by many researchers in solving convection and diffusion equations in different scientific fields, provided the Lie symmetries obtained are admitted by the equation. Dorodnitsyn [48, 49] focused on separation of variables of a generalized porous medium diffusion (heat) equation with a linear source. In these papers, classification was done on the nonlinear equation with a source. Different functional forms of the diffusivity were obtained. Arrigo *et al* [50] solved the convection-advection equation with dispersion coefficient using nonclassical symmetry method. Many researchers also classified and solved the CDE for

various functions of dispersion coefficient and velocity, (see for e.g., [51,52]). Jaiswal *et al* [53–55] obtained analytical solutions for a one-dimensional advection-diffusion equation with variable coefficients in a longitudinal finite initially solute free domain. In the paper [54] a temporally dependent solute dispersion along the uniform flow in a homogeneous domain is studied. In the paper [53] the velocity is considered as spatially dependent. The analytical solutions were obtained using Laplace transformation.

Recent study by Potsane and Moitsheki [56] performed Lie point symmetry analysis for a two-dimensional contamination transport under steady radial flow in saturated soils. In this case the point source, the dispersion coefficient and the velocity are given either as logarithmic and exponential functions. Further investigation when the dispersion is constant and velocity dependent. Group invariants were classified by the elements of the optimal systems leading to exact solutions. On the other paper, Moitsheki and Potsane [57] obtained exact solutions for convection-dispersion equation model describing contaminant transport in saturated soils under radial water flow using nonlocal symmetry analysis techniques.

Catania *et al* [58] also investigated the two-dimensional advection-dispersion equation in radial space. The influence of location, time of sampling and the number of samples on the estimates of dispersion and kinetic parameters are analyzed by means of Analysis of Variance and Fisher test. Ivanova and Sophocleous [59] determined the nonclassical symmetries of a generalized Huxley equations of the form $u_t = u_{xx} + k(x)u^2(1-u)$ where $k(x) \neq 0$.

Broadbridge *et al* [60] considered the diffusion-convection equation describing a steady water flow using admitted classical symmetries in a nontrivial volumetric water content background led to an invariant solution which could be obtained by a method of separation of variables. The infinitesimal of the time variable was assumed to be nonzero constant with arbitrary volumetric water content. Some nonclassical reductions were obtained which could not be obtained via classical techniques.

Sidauruk *et al* [61] used the inverse method to remedy the contamination. The

authors identified the parameters such as the amount of pollutants, its source location, time origin, dispersion coefficients *etc.*, to remedy the contamination. It was shown by means of two cases, two-dimensional instantaneous point source and two-dimensional continuous point source, that the inverse procedure can delineate the plume and allow tracing backwards in time and space. It should be noted that this algorithm uses explicit formulas hence there is no need to establishing any initial guess and the subsequent iterations to determine the unknown parameters. This on its own avoids the possibility of divergence or false convergence to local extrema. A similar approach is used by [62, 63].

Extensive research work has been conducted on heat, groundwater and deformation models. Solute transport models have become the most popular to be researched since they affect the quality of water. To prevent the water to be adversely affected, it has become necessary to develop a methodology to simulate the transport of contaminants through the groundwater systems [64]. A number of authors have dealt with this research, interested reader can visit the works of Sim and Chrysikopoulos [65, 66]. Moutsopoulos *et al* [67] investigated the turbulent and transient flows in confined porous media, two equations for both short and long time conditions were introduced. In addition, the inertia effects and non-Darcy flow conditions were also considered in some analytical solutions [68].

The application of CDE in radial water flow is also considered by Li *et al* [69]. This study is based on confined-unconfined aquifer flow model in the open-pit aquifer. In this model, the water movement in the confined aquifer is free, it can go up and down depending on the recharge or discharge whereas that of unconfined aquifer is restricted to a certain extend. Permeability, recharge or discharge of water, and pumping from wells then play an important role [70, 71]. In both situations usable water can be pumped.

Li *et al* [69] studied groundwater mainly determining the amount of water that can be withdrawn safely from the aquifer on a $65km^2$ area. The study involves the transmissible, storage coefficient, the extend of the aquifer and its hydraulic conductivity; leakage and the recharge and discharge conditions and the time t.

Interested readers can visit the works of Dawson [72]. Hydraulic conductivity is the rate at which the water of a certain kinetic viscosity is transmitted through a unit width of aquifer under a unit hydraulic gradient. The hydraulic head, which is the height to which the water will rise during pumping can decline over time. The product of the rate of decline of the hydraulic head and the storage coefficient over the area of influence results in the discharge. Since the water must be taken from the reduction of storage in the aquifer, the hydraulic head will continue to decline as long as the aquifer is infinite. This lead to unsteady or transient flow. The rate of decline decreases as the area expands [69, 70]. Some other interesting theoretical studies on groundwater can be found in [73–75].

Different methods such as numerical methods [76,77], analytical methods [65,66,78], Lie point symmetry analysis [79–83], the P*á*inleve approach [84–86], the nonclassical symmetry method [87–90], the nonlocal symmetry method [91, 92], the method of generalised conditional symmetries and group classification method [93,94], Noether symmetry [95–98] and classical symmetry method [99].

In this work, we consider three methods outlined below to solve problems arising from solute transport and population dynamics. These methods have been used and proven to yield exact solutions without any doubt because they can be validated and can also provide insight into solute transport processes, see for e.g., [100].

1.3 Research objectives

The aim of research carried out throughout the thesis is to determine the group classification of one-dimensional convection-diffusion equation. The one-dimensional optimal system is carried out. The partial differential equation (PDE) is reduced to a simpler ordinary differential equation (ODE) by group invariants obtained from the Lie sub-algebra. We construct the one-dimensional optimal systems of sub-algebras and perform reduction processes using admitted symmetries. Group invariant solutions will be constructed using admitted Lie point symmetries. We will employ the

double reduction methods which involve determining conservation laws and associated Lie point symmetries.

1.4 Outline of the thesis

The outline of the thesis is as follows. In chapter 2, a brief description of the generalised advection-dispersion equation is given. In chapter 3, the preliminaries based on Lie symmetry methods and their application. The use of potential symmetry method is also outlined. In chapter 4, the one dimensional convection-dispersion is classified and different forms of the diffusion coefficient and velocity are obtained. Eight cases for which Lie symmetries are admitted are considered. The commutator table and the adjoint table are calculated. Information from the two tables is used to determine the optimal system. Both the transient and steady state of CDE are investigated. Chapter 5 deals with the 2D model whereas chapter 6 deals with the 3D model of the oil spill. The method of change of variables is used extensively to further reduce the differential equations (DEs). In chapter 7, the one dimensional CDE in solute transport model is solved using potential symmetries. We investigate all six cases for possible existence of hidden symmetries. In chapter 8, focus is on the potential symmetries in radial water flow equation. The nonclassical symmetries are used to solve the CDE in radial form. We write the CDE in conserved form and also introduce a new variable. The results from nonlocal and nonclassical symmetry reductions are analysed and comparison study is done. The last chapter 9 gives the overall summary and conclusion on the work done through out the thesis.

Chapter 2

Formulation of adversion-dispersion equation in a contaminated groundwater

2.1 Introduction

In this chapter, we give a brief description of the generalised advection-dispersion equation. A brief outline of some work already done on this equation is also given. The importance and application of this equation is also indicated.

2.2 Descriptions of the mathematical model

Mathematical modelling has been used since 1800. It is derived from a differential equation that are based on the physical principles governing the process to be modelled. In this chapter, we consider groundwater flow model and the physical principles comply with Darcy's law and mass balance. We consider the transport of a $\gamma -$ contaminant within a fluid space or vector field $u = (u_x, u_y)$ at a volumetric fraction of the space θ given by the general equation

$$\frac{\partial \theta c^{\gamma}}{\partial t} = D_{ij}^{\gamma}(\theta)\frac{\partial c^{\gamma}}{\partial x_j} \tag{2.2.1}$$

where D is the coefficient of molecular diffusion within a space, c^{γ} is the concentration of a $\gamma-$contaminant and $\dfrac{\partial c^{\gamma}}{\partial t}$ is the advective flux of the contaminant. The flow can be one-dimensional, two-dimensional or three-dimensional. In all three the flow could be isotrophic, anistrophic and transient $\dfrac{\partial c^{\gamma}}{\partial t} \neq 0$ or steady $\dfrac{\partial c^{\gamma}}{\partial t} = 0$, [101, 102]. It should be noted that the Einstein summation convection is adopted in (2.2.1) [103].

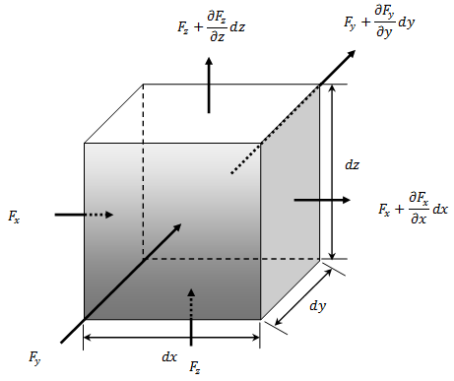


Figure 2.1: Convection-diffusion in a three-dimensional space. Image from *www.nap.ed*

For a 3D-model, where conductivity is heterogeneous (2.2.1) is given by

$$\frac{\partial \theta c^{\gamma}}{\partial t} = q_x\frac{\partial^2 c^{\gamma}}{\partial x^2} + q_y\frac{\partial^2 c^{\gamma}}{\partial y^2} + q_z\frac{\partial^2 c^{\gamma}}{\partial z^2}. \tag{2.2.2}$$

If the conductivity is isotrophic where $q_x = q_y = q_z = Q$ then (2.2.2) becomes

$$\frac{\theta}{Q}\frac{\partial c^{\gamma}}{\partial t} = \frac{\partial^2 c^{\gamma}}{\partial x^2} + \frac{\partial^2 c^{\gamma}}{\partial y^2} + \frac{\partial^2 c^{\gamma}}{\partial z^2}, \tag{2.2.3}$$

or equivalently

$$\nabla^2 c^{\gamma} = \frac{\theta}{Q}\frac{\partial c^{\gamma}}{\partial t},$$

where

$$\nabla^2() = \frac{\partial^2()}{\partial x^2} + \frac{\partial^2()}{\partial y^2} + \frac{\partial^2()}{\partial z^2}.$$

In a steady state (2.2.1) will be given by

$$\nabla^2 c^{\gamma} = 0.$$

If $q_x \neq 0$, $q_y \neq 0$ and $q_z = 0$ then (2.2.3) reduces to a 2D-model given by

$$\frac{\partial \theta c^\gamma}{\partial t} = q_x \frac{\partial^2 c^\gamma}{\partial x^2} + q_y \frac{\partial^2 c^\gamma}{\partial y^2}. \tag{2.2.4}$$

Similarly in an isotrophic flow and a steady state (2.2.4) becomes

$$\nabla^2 c^\gamma = \frac{\theta}{Q} \frac{\partial c^\gamma}{\partial t},$$

where

$$\nabla^2() = \frac{\partial^2()}{\partial x^2} + \frac{\partial^2()}{\partial y^2}.$$

This simplification of a 3D-model to a 2D-model is called the Dupuit-Forchheimer approximation [101, 103]. If $q_x \neq 0$, $q_y = 0$ and $q_z = 0$ then (2.2.3) reduces to a 1D-model given by

$$\frac{\partial \theta c^\gamma}{\partial t} = q_x \frac{\partial^2 c^\gamma}{\partial x^2}. \tag{2.2.5}$$

In this work, we shall only focus on two models, namely the 1D- and 2D-models [101, 103]. Future work will involve the 3D-models seen in the works of Tamsir [104] (see also [105–107]). We shall study the equations arising into the theory of oil spill model which was first developed by Fay (1969) [108] where spreading of the slick in a three-dimensional movement was considered in calm water. The spreading is characterised into three phases, the gravity-inertial spreading, gravity-viscous spreading and the viscous-surface tension [109]. As the oil flow, its concentration changes due to some physical and chemical processes known as weathering processes. Involving all these phases and processes will make the equation complex to solve. For this reason models such as spreading, evaporation *etc.*, are designed and investigated separately.

In this thesis, we investigate the two processes involved in the spreading of oil in water surface, the advection (convection) and dispersion (diffusion). Convection is the physical process by which the oil is transported by the ordered motion of the flow whereas dispersion is the physical process by which the oil is transported by the random motion of the molecules in the oil to a wider area of the water surface [110]. The two processes can be best described by CDE which is obtained from the work of Al-Rabeh and Mazaheri *et al* [24, 107] and it is given by

$$R\frac{\partial C}{\partial t} = \nabla\Big[D\nabla C - VC\Big] + \rho. \tag{2.2.6}$$

where $C(t, x, y, z)$ is the concentration of the oil in water at point (x, y, z) in the Cartesian coordinates at time t, V is the current field, D is the diffusion coefficient which involves the turbulent effects, R is the retardation term or rescaling of time and depends on the water chemistry and geochemical make up of the medium. and ρ is the source term or sink [24], the term $D\nabla C$ defines the diffusion term and the VC the convection [111]. Depending on the model, the CDE (2.2.6) can be one dimensional [53], two dimensional [78] or three dimensional [112].

Many researchers have already worked on this model using different numerical methods such as finite difference and finite element methods [112, 113], Gauss-Seidel numerical method [109] and the Newton-Rhapson method [114]. Other methods of solutions use are analytical [53, 54], Laplace transforms [115], Lagrangian discreet parcel algorithm [116], Hankel method [117], perturbation [118] and Green's method [119]. The numerical and analytical methods provide approximate solutions with a certain percentage of error. Alternative methods are classical Lie symmetry methods [60], nonlocal symmetries [57] and nonclassical [50, 100]. The CDE plays a very important role in various physical and chemical phenomena and behaves as an elliptic parabolic or hyperbolic PDE. It is because of this nature that most researchers get interested in solving the CDE. Results obtained have been applied in many areas of science and engineering including amongst others economics, fluid dynamics, forecasting, astrophysics, oceanography, meteorology, and population dynamics [120–122].

2.2.1 Radial water flow

This study is based on the theoretical background outlined by Li *et al* [69]. The following contaminant-governing equation is given by

$$\frac{\partial n\psi}{\partial t} = nD_r\frac{\partial^2\psi}{\partial r^2} - u_r\frac{\partial n\psi}{\partial r} - I. \tag{2.2.7}$$

where ψ is the Jilin Gaussian potential function which is also the hydraulic height,

$\varphi_0 = KMH_0 - \frac{1}{2}KM^2$; $r_0 = \sqrt{F/\pi} = 0.564\sqrt{F}$, where F is the area of mine (m^2); H_0 being the initial hydraulic head(m); K is the permeability coefficient (m/s); M is the confined aquifer thickness (m); Q is the flow rate of the whole pumping wells (m^3/s); r is the radius of groundwater(m); r_0 is initial radius in m, T_m is the different period corresponding to average hydraulic conductivity (m^2/s); t is the time in s; μ_m is the different period corresponding to average specific yield [69, 102]. Li *et al* [69] found analytical solution of (2.2.7) and intersting solutions can also be found in the works of Sen [123] and Wen *et al* [124, 125].

If we consider $D_r = n = 1$, $u_r = -\frac{1}{r}$ and $I = 0$ then (2.2.7) becomes

$$\frac{T_m}{\mu_m}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad (r_0 \leq r \leq \infty,\ t > 0). \tag{2.2.8}$$

We further consider the equation (2.2.8) with initial and boundary conditions

$$\left.\begin{aligned} \psi(r,0) &= \psi_0, \\ \psi(\infty,t) &= \psi_0, \\ lim_{r\to r_0} 2\pi r\tfrac{\partial \psi}{\partial r} &= Q(cons). \end{aligned}\right\} \tag{2.2.9}$$

In this study, convection, dispersion and chemical reaction will be considered. The problem will be derived as a stable well flow problem of a planar radial flow field. The following conditions are assumed. Firstly, the pollutant is mainly distributed in a homogeneous, isotropic and equal-thickness confined aquifer; secondly the aquifer has a complete pumping well, constant flow pumping and an established stable well flow is centered on the well; thirdly the diffusion coefficient of pollutants in the medium is assumed to be constant; and lastly pollutants transfer into groundwater from the porous medium by solid phase chemical reactions [69, 102].

2.3 Conclusion

In this chapter, the mathematical description of contaminant transport has been outlined in terms of deterministic models that are given by the CDE. The radial

water flow model from the paper by Li *et al* [69] is also given. The CDE will be dealt with in chapter 4, 5 and 6. In Chapter 7 we use the nonlocal symmetry method to solve the one-dimensional CDE. Chapter 8 involves reductions by using both the nonlocal and nonclassical symmetry methods in the radial water flow model. The summary of the thesis is given in Chapter 9. Lastly, the computational procedure used to determine the nonclassical symmetries is given in the Appendix.

Chapter 3

Techniques for symmetry reductions

3.1 Introduction

In this chapter we provide the historical account and how Lie symmetry methods have evolved. We start with relating the partial differential equations (PDEs) with Lie symmetries. A PDE is an equation involving an unknown function of several variables that include time t as one of the independent variables. A second order PDE has one dependent and two independent variables in the form

$$F(t, x, u, u_t, u_x, u_{tx}, u_{xx}) = 0, \;\; (t, x) \in S,$$

where x and t are independent variables lying in some given domain $S \in \mathbb{R}^2$ and u is the dependent variable.

3.2 Symmetry methods

Group analysis of differential equations originated from the works of the outstanding mathematician of the nineteenth century Sophus Lie (1842-1899). The original problem of group analysis, namely, problem solvability of differential equation by

quadratures was actually solved by Lie himself, but it did not enjoy widespread acceptance. Group analysis augments an intuitive perception of symmetry and equips us with constructive methods for its application. Furthermore it guides scholars to properly formulate problems and often discloses possible approaches to solve them. Lie used his theory of continuous transformation groups to show that the majority of known methods of integration of differential equations (DEs), which until then had seemed artificial, could be derived in a unified manner. Lie group analysis is the only universal and effective method for solving nonlinear differential equations analytically [126].

Symmetry reduction methods have been widely used by numerous researchers to construct explicit solutions of partial differential equations [126–128]. In their recent work, Ntsime and Moitsheki [95] considered a nonlinear convection-dispersion equation arising from contaminant transport. A number of cases for which this equation admits Lie point symmetries were considered. In each case a partial differential equation (PDE) was reduced to ordinary differential equations (ODE). Noether symmetries can also be obtained for a PDE provided the Langrangian exists. This approach also reduces the PDE to quadratures which might become simple to solve. Numerous researchers such as Ibragimov *et al* [129], and many others [93, 96] used Noether symmetry method.

Further in this study, we consider the use and application of potential symmetries of the governing equation (2.2.6). This method was first introduced by Bluman and Kumei [130] who found the class of extra symmetries called potential or non local symmetries of a given partial differential equation provided the PDE is written in a conserved form. Several researchers used this method to reduce the most complicated and complex PDE into a simple ordinary differential equation that can be easily solved algebraically (see for e.g [131–134]).

3.2.1 Lie (local) point symmetries

In the late nineteenth century, a prominent Norwegian mathematician Sophus Lie (1842-1899) developed a new mathematical method known as Lie group analysis for solving differential equations. This approach coupled with majority of adhoc methods of integration of differential equations is used to solve both simple (heat and Burger's equation) and complex partial differential equations (convection-diffusion). Recently, a lot of literature in this field is displayed in many books. We mention a few here, Bluman and Kumei [130], Lie [135], Olver [136], Stephani [137], Ibragimov [138–142], Hydon [143], Cantwell [144], Mahomed [145] and Ovsiannikov [82]. Symbolic computer packages are also used to perform most of the calculations involved in Lie's theory. See, for example, Baumann [146], Head [147, 148], Sherring and Head [149], Diaz [150].

Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a),\ \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{3.2.1}$$

where a continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 3.1 A set G of transformations (3.2.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure).

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity).

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse).

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{3.2.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 3.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\, \phi(s, b)}{\partial b}\right|_{b=0}.$$

Prolongation of point transformations and group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{3.2.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{3.2.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (3.2.3). Thus the u_s^α are called differential variables [138].

We now consider a pth-order PDE(s), namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{3.2.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{3.2.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{3.2.7}$$

obtained from (3.2.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x, u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{3.2.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (3.2.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}\,. \tag{3.2.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (3.2.5), we say that X is an *admitted operator* of (3.2.5) or X is an *infinitesimal symmetry* of equation (3.2.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{3.2.10}$$

where $\bar{D}_j$ are the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}^\alpha_i = \bar{D}_j(u^\alpha), \qquad \bar{u}^\alpha_{ij} = \bar{D}_j(\bar{u}^\alpha_i) = \bar{D}_i(\bar{u}^\alpha_j), \cdots .$$

Now let us apply (3.2.10) and (3.2.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}^\alpha_j. \end{aligned} \tag{3.2.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u^\beta_i \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}^\alpha_j = \frac{\partial \phi^\alpha}{\partial x^i} + u^\beta_i \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{3.2.12}$$

The quantities $\bar{u}^\alpha_j$ can be represented as functions of $x, u, u_{(i)}$, for small a, i.e., (3.2.12) is locally invertible:

$$\bar{u}^\alpha_i = \psi^\alpha_i(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u^\alpha_i. \tag{3.2.13}$$

The transformations in $x, u, u_{(1)}$ space given by (3.2.6) and (3.2.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}^\alpha_i \approx u^\alpha_i + a\zeta^\alpha_i \tag{3.2.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (3.2.7) and (3.2.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (3.2.11).

Prolonged generators

Using (3.2.11) together with (3.2.7) and (3.2.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\ (\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\ u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j\text{)}. \end{aligned} \tag{3.2.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k\text{)}. \tag{3.2.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{p-1}\,j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j\text{)}. \tag{3.2.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha\text{)}, \\ &\cdot \\ &\cdot \\ &\cdot \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Group admitted by a PDE

Definition 3.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{3.2.18}$$

is a *point symmetry* of the pth-order PDE (3.2.5), if

$$X^{[p]}(E_\alpha) = 0, \tag{3.2.19}$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]}\, E_\alpha\big|_{E_\alpha=0} = 0, \tag{3.2.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 3.3 Equation (3.2.19) is called the determining equation of (3.2.5) because it determines all the infinitesimal symmetries of (3.2.5).

Definition 3.4 (Symmetry group) A one-parameter group G of transformations (3.2.1) is called a symmetry group of equation (3.2.5) if (3.2.5) is has the same form in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{3.2.21}$$

where the function E_α is the same as in equation (3.2.5).

Group invariants

Definition 3.5 A function $F(x,u)$ is called an invariant of the group of transformation (3.2.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x,u,a),\, \phi^\alpha(x,u,a)) = F(x,u), \tag{3.2.22}$$

identically in x, u and a.

Theorem 3.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x,u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{3.2.23}$$

It follows from the above theorem that every one-parameter group of point transformations (3.2.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}.$$

Theorem 3.3 If the infinitesimal transformation (3.2.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u^\alpha}}{da} = \eta^\alpha(\bar{x},\bar{u}), \tag{3.2.24}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0} = x, \quad \bar{u^\alpha}\big|_{a=0} = u\,.$$

Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 3.6 The commutator of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 3.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\dfrac{\partial}{\partial x^i} + \eta^\alpha(x,u)\dfrac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u},$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1),$$

is also an element of L. It follows that the commutator is

(a) Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], [X, aY + bZ] = a[X, Y] + b[X, Z];$$

(b) Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

(c) and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Adjoint representation

Let G be a Lie group. For each $g \in G$, group conjugation $K_g(h) \equiv ghg^{-1}$, $h \in G$, determines a diffeomorphism on G. Moreover, $K_g \circ K_{g'} = K_{gg'}$, $K_e = 1_G$, so K_g determines a global group action of G on itself, with each conjugacy map K_g being a group homomorphism: $K_g(hh') = K_g(h)K_g(h')$, etc. The differential $dK_g : TG|_h \to TG|_{K_g(h)}$ is readily seen to preserve the right-invariance of the vector fields, and hence determines a linear map on the Lie algebra of G, called adjoint representation:

$$Adg(X) \equiv dK_g(X), \;\; X \in g.$$

The adjoint representation gives a global linear action of G on g:

$$Ad(g \cdot g') = Ad\ g \circ Ad\ g', \;\; Ad\ e = 1.$$

If $X \in g$ generates the one-parameter subgroup $H = \{\exp(\varepsilon X) : \varepsilon \in \mathcal{R}\}$, then $Ad\ g(X)$ is easily seen to generate the conjugate one-parameter subgroup $K_g(H) \equiv gHg^{-1}$, (see [136]).

Linear combination

Let $\Gamma \in L^r$ be any operator and a_λ be any constant. Then, $\Gamma = a_1\Gamma_1 + \cdots + a_\lambda\Gamma_\lambda$ is called a linear combination of the $\Gamma_\lambda,\ r = 1, \cdots, \lambda$. For example, if $\Gamma_1 = \partial_x$ and $\Gamma_2 = \partial_u$ and $\Gamma_3 = \partial_x + \partial_F$ then $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3$ is called a linear combination of these three operators.

Inherited symmetries

If two subalgebras are connected with each other by a transformation of symmetry group, then their corresponding invariant solutions are connected with each other by the same transformations. Suppose a given equation admits a non-Abelian two dimensional subalgebra such that $[X_i, X_j] = \alpha X_i$ and $[X_i, X_j] = 0$ where $i, j, \alpha \in \mathbb{R}$. Reduction procedure begins with the original X_i and it then becomes automatic that the new equation admits X_j in new variables. The new symmetry X_j^* is referred to as an inherited symmetry.

Optimal system

Let the finite Lie algebra L^r be spanned by the operators Γ provide a possibility to find invariant solutions of a DE based on the one-dimensional subalgebras of the algebra L^r, that is on any operator $\Gamma \in L^r$ [136]. However there are infinite number of one-dimensional subalgebra operator can be written from L^* as a linear combination of the operator and hence depends on r arbitrary constants. The desire to minimise the search for invariants solutions by finding non-equivalent branches of solutions leads to the concept of an optimal system of one dimensional subalgebras. Similarly when all invariant solutions are obtained in principle by constructing the invariant solution for each member of the optimal system of subalgebras, the set of invariant solutions obtained in this manner is an optimal system of invariant solutions, (see e.g. [136]).

Contact symmetries

It has been noted that the point symmetry dependent t and x as independent and also on the dependent variables $u(t,x)$. Should the symmetry depend on the derivative of $u(t,x)$, such a symmetry is called a contact symmetry [126]. In that case

$$\Gamma = \xi(x,u,u')\partial_x + \eta(x,u,u')\partial_u + \zeta(x,u,u')\partial_{u'}.$$

Since $\Gamma^{[1]}$ will not possess any u'' we then require that

$$\begin{aligned} \zeta(x,u,u') &= \eta'(x,u,u') - u'\xi'(x,u,u'), \\ &= \frac{\partial\eta}{\partial x} + u'\frac{\partial\eta}{\partial u} + u''\frac{\partial\eta}{\partial u'} - u'(\frac{\partial\xi}{\partial x} + u'\frac{\partial\xi}{\partial u}) + u''\frac{\partial\xi}{\partial u'}, \\ &= \frac{\partial\eta}{\partial x} - u'\frac{\partial\xi}{\partial x} + u'(\frac{\partial\eta}{\partial u} - u'\frac{\partial\xi}{\partial u}) + u''(\frac{\partial\eta}{\partial u'} - u'\frac{\partial\xi}{\partial u'}). \end{aligned} \tag{3.2.25}$$

The last equation equates to zero. This simply means that Γ is regarded as a contact symmetry only if $u''=0$. That is when

$$\frac{\partial\eta}{\partial u'} = u'\frac{\partial\xi}{\partial u'}.$$

3.2.2 Potential (nonlocal) symmetry method

Nonlocal symmetries are defined as those symmetries in which the coefficient functions depend on integrals containing the derivatives and the dependent variables of differential equations (DEs) [151]. A hidden symmetry is described as a Lie symmetry which is admitted by the given DE after a change of variable; these symmetries cannot be directly calculated from the original equation. By definition, a PDE might possess extra hidden symmetries provided the PDE is written in integrated form. An infinitesimal generator

$$X = \xi^1(t,x,C,\omega)\partial_t + \xi^2(t,x,C,\omega)\partial_x + \eta(t,x,C,\omega)\partial_C + \zeta(t,x,C,\omega)\partial_\omega, \tag{3.2.26}$$

is called a potential symmetry of the original equation if and only if (ξ^i,η) depends on $\omega(t,x)$, otherwise X is a point symmetry [130]. On the other hand if

$$\xi_\omega^2 + \tau_\omega^2 + \eta_\omega^2 \neq 0, \tag{3.2.27}$$

then (3.2.26) is a potential symmetry of the original equation, otherwise X translate into a point symmetry [100].

Consider a PDE written in a conserved form $\mathcal{R}(t, x, C)$

$$D_x F(x, t, u, u_x, u_t) = D_t G(x, t, u, u_x, u_t), \tag{3.2.28}$$

where D_x and D_t are total derivative operators. We introduce an auxiliary variable ω, and form a potential system $\mathcal{S}(t, x, C, \omega)$:

$$\left.\begin{aligned} \omega_t &= F(t, x, u, u_x, u_t), \\ \omega_x &= G(t, x, u, u_x, u_t), \end{aligned}\right\} \tag{3.2.29}$$

which is equivalent to (3.2.28). The homogeneous linear system which characterises the generators is obtained from

$$X^1(\omega_t - F)_S = 0, \tag{3.2.30}$$

$$X^1(\omega_x - G)_S = 0. \tag{3.2.31}$$

To compute the potential symmetries we consider an infinitesimal generator (3.2.26) and its first prolongation

$$X^1 = X + \eta^t \partial_{C_t} + \eta^x \partial_{\omega_x} + \zeta^t \partial_{\omega_t} + \zeta^x \partial_{C_x}, \tag{3.2.32}$$

where

$$\eta^x = D_x\eta - C_x D_x\xi - C_t D_x\tau, \quad \eta^t = D_t\eta - C_x D_t\xi - C_t D_t\tau, \tag{3.2.33}$$

$$\zeta^x = D_x\zeta - \omega_x D_x\xi - \omega_t D_x\tau, \quad \zeta^t = D_t\zeta - \omega_x D_t\xi - \omega_t D_t\tau, \tag{3.2.34}$$

with the invariant surface conditions

$$\xi(t, x, C, \omega)C_x + \tau(t, x, C, \omega)C_t - \eta(t, x, C, \omega) = 0, \tag{3.2.35}$$

$$\xi(t, x, C, \omega)\omega_x + \tau(t, x, C, \omega)\omega_t - \zeta(t, x, C, \omega) = 0. \tag{3.2.36}$$

If $C(t, x)$ and $\omega(t, x)$ satisfy $\mathcal{S}(t, x, C, \omega)$ then $C(t, x)$ solves $\mathcal{R}(t, x, C)$ and $\omega(t, x)$ solves an integrated related equation denoted by $\mathcal{S}(t, x, C, \omega)$. We establish existence of potential symmetries based on the definition of potential symmetry stated above. Interesting research application of this method can be found from the works of [91, 100, 131].

3.3 Conclusion

Preliminaries relating to Lie symmetry methods has been discussed. The discussion covered concepts of Lie point symmetry, continuous one-parameter group, prolonged and extended groups. Definitions of symmetry group, group invariants, commutator and adjoint representations that lead to one-dimensional optimal systems are outlined. The procedure on how to use contact and potential symmetries are well expressed. These preliminaries are well used in chapters 4 to 8.

Chapter 4

Direct group classification and invariant solutions of the advection-dispersion equation

Some of the work in this chapter appeared in Ntsime B. P. and Moitsheki R. J., Symmetry reductions of a non-linear convection-dispersion model arising in contaminant transport theory, ICNAAM (2016), AIP, 1738, 330004, http://dx.doi.org/10.1063/1.4952116

4.1 Introduction

In this chapter, the one-dimensional convection-dispersion equation is analysed and different forms of the diffusion coefficient and velocity are obtained. Eight cases for which Lie symmetries are admitted are considered. The commutator table and the adjoint table are calculated. Information from the two tables is used to determine the optimal system. Both the transient and steady state of equation (2.2.6) are investigated.

4.2 Group classification

Investigation begins with the one-dimensional nonlinear convection-diffusion equation described by (2.2.6) with no source and with retardation factor $R = 1$. Some functional forms are provided by Yung *et al* in their paper [52] (see also [152]).
We consider the linear convection-diffusion PDE in one-dimensional space

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big[D(C)\Big(\frac{\partial C}{\partial x}\Big) - V(x)C\Big]. \tag{4.2.1}$$

where $D = D(C)$ is the dispersion coefficient, $V = V(x)$ is the transport velocity in the x-direction and $C = C(t, x)$ is the concentration of the contaminant.

Acting on the equation (4.2.1) with arbitrary D and V, by the appropriate prolongation of the symmetry generator (3.2.18) resulted in the admitted time translation symmetry. It turned out that V and D must satisfy the ODEs

$$DD'D''' - 2D(D'')^2 + (D')^2D'' = 0; \quad D' = 0; \quad V'' = 0;$$

$$V' = 0; \quad (a_1x + a_2)V' + a_3x + a_4 = 0,$$

for equation (4.2.1) to admit Lie point symmetries beyond the generic ones. A number of cases listed in the Table (4.1) for which equation (4.2.1) admit extra Lie point symmetries.

Table 4.1: Cases of D and V for which equation (2.2.6) admit extra symmetries

$D(C)$	$V(x)$
$(\alpha+\beta C)^m$	$n \in \mathcal{R}$ $x^n, \quad n=\pm 1$ $p+qx, \quad p,q \in \mathcal{R}$
$(\alpha+\beta C)^{-4/3}$	$n \in \mathcal{R}$
$e^{mC}; \quad m \in \mathcal{R}$	$n \in \mathcal{R}$ x^n $\alpha+\beta x, \quad \alpha,\beta \in \mathcal{R}.$

4.3 Case 1: $D(C)=(\alpha+\beta C)^m, \quad V=n.$

In this case, we consider the governing equation (4.2.1) given by

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left[\Big(\alpha+\beta C\Big)^m\frac{\partial C}{\partial x}-nC\right], \tag{4.3.1}$$

(see also for e.g., [153] for equation (4.3.1) with $\alpha=0$ and $\beta=1$). Some results for this equation are obtained in [154]. Here we review the results and extended the problem when the velocity is not a constant.

The principal Lie algebra of (4.3.1) is four-dimensional Lie algebra given by

$$\left.\begin{aligned} X_1&=\frac{\partial}{\partial t},\\ X_2&=2t\frac{\partial}{\partial t}+(x+nt)\frac{\partial}{\partial x},\\ X_3&=\frac{1}{m\beta^2}\Big\{-m\beta t\frac{\partial}{\partial t}-mn\beta t\frac{\partial}{\partial x}+(\alpha+\beta C)\frac{\partial}{\partial C}\Big\},\\ X_4&=\frac{\partial}{\partial x}. \end{aligned}\right\} \tag{4.3.2}$$

Remark: It is worth noting that symmetry X_3 can be written as

$$X_3 = m\beta t\frac{\partial}{\partial t} + mn\beta t\frac{\partial}{\partial x} - (\alpha + \beta C)\frac{\partial}{\partial C}$$

4.3.1 Construction of the commutator table.

In this section we construct a one-dimensional optimal system of sub-algebras which are spanned by the equation. We first construct a commutator table where the commutator of two symmetries X_i and X_j is given by

$$[X_i, X_j] = X_iX_j - X_jX_i \tag{4.3.3}$$

where the subscripts i and j takes the values from 1 to 4.
To demonstrate the calculation of the commutator we will use $[X_1, X_2]$.

$$\begin{aligned}
[X_1, X_2] &= X_1X_2 - X_2X_1 \\
&= \frac{\partial}{\partial t}\big[(x+nt)\frac{\partial}{\partial t} + 2t\frac{\partial}{\partial t}\big] - \big[(x+nt)\frac{\partial}{\partial t} + 2t\frac{\partial}{\partial t}\big](\frac{\partial}{\partial t}) \\
&= n\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} \\
&= nX_4 + 2X_1
\end{aligned}$$

It should be noted that $[X_i, X_j] = -[X_j, X_i]$ and if $i = j$ then $[X_i, X_j] = 0$.
The table 4.2 below is the summary of the commutators for the four-dimensional Lie algebra.

Table 4.2: Commutator table of the subalgebras for equation (4.3.1).

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$nX_4 + 2X_1$	$\frac{1}{\beta}(nX_4 - X_1)$	0
X_2	$nX_4 - 2X_1$	0	0	$-X_4$
X_3	$\frac{1}{\beta}[X_1 - nX_4]$	0	0	0
X_4	0	X_4	0	0

4.4 Adjoint representation of equation (4.3.2).

We now construct the adjoint representation to define equivalence relation on one-dimensional subalgebras which is generated through $X_i,\ i = 1, \cdots, 4$ by summarising the Lie series outlined (see [136]) as

$$\mathrm{Adj}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] \dots$$

where the subscripts j also takes the values from 1 to 4.
To demonstrate the calculation of the adjoint representation, we refer to the Lie brackets Table 4.2 above, where $[X_i, X_j] = X_iX_j - X_jX_i$ is the Lie bracket and $\epsilon \in \mathbb{R}$. We demonstrate the procedure below and deduce that

$$\mathrm{Adj}(j) = A(j, \epsilon)X_i, \tag{4.4.1}$$

where $A(j, \epsilon)$ is some n x n, (where $n = 4$) matrix corresponding to each X_j and the transposing yield the $ij - th$ entry summarised in Table 4.3.

$$\begin{aligned}\mathrm{Adj}(e^{\epsilon X_1})X_2 &= X_2 - \epsilon[X_1, X_2] + \frac{1}{2!}\epsilon^2[X_1, [X_1, X_2]] \\ &= X_2 - \epsilon[nX_4 + 2X_1] + \frac{\epsilon^2}{2!}[X_1, [4X_4 + 2X_1]] \\ &= X_2 - \epsilon nX_4 - 2\epsilon X_1.\end{aligned}$$

It is also worth showing adjoint representation acting on X_1 :

$$\begin{aligned}\mathrm{Adj}(e^{\epsilon X_2})X_1 &= X_1 - \epsilon[X_2, X_1] + \frac{\epsilon^2}{2!}[X_2, [X_2, X_1]] - \frac{\epsilon^3}{3!}[X_2, [X_2, [X_2, X_1]]] \dots \\ &= X_1 - \epsilon[-nX_4 - 2X_1] + \frac{\epsilon^2}{2!}[X_2, [-4X_4 - 2X_1]] \\ &\quad -\frac{1}{3!}\epsilon^3[X_2, [X_2, [-nX_4 - 2X_1]]] \dots \\ &= X_1 + \epsilon nX_4 + 2\epsilon X_1 + \frac{1}{2!}\epsilon^2[nX_4 - 2(-nX_4 - 2X_1)] \\ &\quad -\frac{1}{3!}\epsilon^3[X_2, nX_4 - 2(-nX_4 - 2X_1)] \dots \\ &= \Big(1 + 2\epsilon + \frac{4\epsilon^2}{2!} + \frac{8\epsilon^3}{3!}\Big)X_1 - \Big(n\epsilon + \frac{3n\epsilon^2}{2!} + \frac{7\epsilon^3}{3!}\Big)X_4 \dots \\ &= \mathrm{e}^{2\epsilon}X_1 - \varphi nX_4\end{aligned}$$

where $\varphi = n\sum_{k=1}^{\infty}\frac{2^k - 1}{k!}\epsilon^k$.

The Table 4.3 below provides the summary of adjoint representations calculated from Table 4.2.

Table 4.3: Adjoint table of sub-algebras for $D(C) = (\alpha + \beta C)^m$, $V = n$.

$Adj[X_i, X_j]X_j$	X_1	X_2	X_3	X_4
X_1	X_1	$X_2 + \epsilon[nX_4 - 2X_1]$	$X_3 + \frac{\epsilon}{\beta}(X_1 - nX_4)$	X_4
X_2	$\mathrm{e}^{-2\epsilon}X_1 + \varphi nX_4$	X_2	X_3	$\mathrm{e}^{\epsilon}X_4$
X_3	$\mathrm{e}^{\frac{\epsilon}{\beta}}X_1 + \frac{n}{\beta}(1 - \mathrm{e}^{-\epsilon})X_4$	X_2	X_3	X_4
X_4	X_1	$X_2 - \epsilon X_4$	X_3	X_4

4.5 One-dimensinal optimal system.

It is well known that reduction of independent variables of a PDE by one is feasible using any linear combination of symmetry $X_i,\ i = 1, \cdots, 4$. In this section, we show how to calculate the one-dimensional optimal system of the subgroup. We use the procedure outlined in (see e.g. [136]) and summarised in section 3.4 above. Many researchers applied this method to construct the optimal system of a sub-algebra to reduce the order of invariant solutions. Interesting calculations can be obtained from [155–157]. We note that the infinite-dimensional subalgebra X_∞ does not result in group invariants and it will not be considered here. We aim to simplify as much as possible so that we have a simpler X. To do this, we use the a non-zero vector

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \tag{4.5.1}$$

where a_i are arbitrary constants.

We begin by assuming that $a_3 = 0$ and $a_4 \neq 0$ and $a_4 = 1$.

Then we have referring to adjoint table, we act on X to eliminate X_3 Acting on X yields

$$\mathrm{Adj}\Big(\exp(a_3X_3)\Big)X = \tilde{a}_1X_1 + a_2X_2 + \tilde{a}_4X_4,$$

where $\tilde{a_4} = \frac{n}{\beta}a_1\big(1-\mathrm{e}^{-\alpha}\big) + a_4$ and $\tilde{a_1} = a_1\mathrm{e}^{\frac{\alpha}{\beta}}$.

Hence

$$X' = \tilde{a}_1X_1 + a_2X_2 + \tilde{a}_4X_4.$$

Next we eliminate X_2, acting on X' by $\mathrm{Adj}\Big(\exp(a_2X_2)\Big)X'$ yields

$$\mathrm{Adj}\Big(\exp(a_2X_2)\Big)X' = \tilde{\tilde{a}}_1X_1 + \tilde{\tilde{a}}_4X_4,$$

where $\tilde{\tilde{a}}_1 = \tilde{a}_1\mathrm{e}^{-2\alpha}$ and $\tilde{\tilde{a}}_4 = \tilde{a}_1n\varphi + \tilde{a}_4\mathrm{e}^{\epsilon}$.

Hence

$$X'' = \tilde{\tilde{a}}_1X_1 + \tilde{\tilde{a}}_4X_4.$$

Lastly we eliminate X_1 by acting on X'' by $\mathrm{Adj}\Big(\exp(\tilde{\tilde{a}}_1X_1)\Big)X'' = \tilde{\tilde{a}}_4X_4$.

We cannot eliminate any further therefore the corresponding element of the optimal system is $\pm X_4$.

Next we assume that $a_3 \neq 0$ and $a_3 = 1$. To eliminate X_4 we assume that $a_4 = 0$. Using the adjoint table, we act on X to eliminate X_4.

$$\mathrm{Adj}\Big(\exp(a_4X_4)\Big)X = a_1X_1 + a_2X_2 + X_3.$$

Further elimination is not possible. Hence the second optimal system is $a_1X_1 + a_2X_2 + X_3$.

Next we assume that $a_2 \neq 0$ and $a_2 = 1$. To eliminate a_3 and a_4 simultaneously, we let $a_3 = a_4 = 0$. The non-zero vector is then given by $X = a_1X_1 + a_2X_2$. We cannot proceed with the elimination. Hence $a_1X_1 + a_2X_2$ is the third optimal system. Next we assume that $a_1 \neq 0$ and $a_1 = 1$. We let $a_2 = a_3 = a_4 = 0$. The non-zero vector is then given by $X = a_1X_1$. We cannot proceed with the elimination. Hence a_1X_1 is the fourth optimal system.

An optimal system of one dimensional sub-algebras is provided by those generated by

$$\{X_1,\ \ X_2,\ \ \pm a_4X_4,\ \ a_1X_1 + a_2X_2,\ \ a_1X_1 + a_2X_2 + X_3\}. \tag{4.5.2}$$

4.5.1 Symmetry reductions and invariant solutions.

We now perform symmetry reductions using the elements of the optimal systems (4.5.2). We only show calculations for reduction under X_1 with characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{0}. \tag{4.5.3}$$

The invariants are given by

$$C(t, x) = F(x). \tag{4.5.4}$$

The derivatives of (4.5.4) reduce (4.3.1) to a second order ODE given by

$$(\alpha + \beta F)^m F'' + m\beta(\alpha + \beta F)^{m-1} F'^2 - nF' = 0. \tag{4.5.5}$$

The ODE (4.5.5) admits two symmetries

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{1}{\alpha}\Big(m\beta x\frac{\partial}{\partial x} + (\alpha + \beta F)\frac{\partial}{\partial F}\Big).$$

We consider reduction by Γ_2. The first prolongation of Γ_2 is given by

$$\Gamma_2^{[1]} = \frac{1}{\alpha}\Big(m\beta x\frac{\partial}{\partial x} + (\alpha + \beta F)\frac{\partial}{\partial F} + \beta(1 - m)F'\Big).$$

The characteristic equations are given by

$$\frac{dx}{m\beta x} = \frac{dF}{(\alpha + \beta F)} = \frac{dF'}{\beta(1 - m)F'} \tag{4.5.6}$$

and the corresponding characteristic equations yield

$$\left.\begin{aligned} I_1 &= x^{-1/m}F \\ I_2 &= x^{m-1/m}F' \end{aligned}\right\} \tag{4.5.7}$$

Let $z = x^{-1/m}F$ and $G = x^{m-1/m}F'$ where $G = G(z)$.

Using the method of differential invariants such that

$$D_G = G' D_z \tag{4.5.8}$$

we obtain (4.5.5) reduced in a first order ODE given by

$$(m\beta zG - z^{m+1})G' - (m - 1)z^m G + m^2\beta z^{m-1}G^2 - nG = 0. \tag{4.5.9}$$

Case (a) $m = 1, \quad \beta = 1.$

Equation (4.5.9) is given by

$$(G - z)zG' + G^2 - nG = 0. \tag{4.5.10}$$

The equation does not yield any solution.

Case (b) $m = 1, \quad \beta = 0.$

The equation (4.5.9) is given by

$$z^2G' + nG = 0,$$

and its solution is given by

$$G(z) = \mathrm{Aexp}(nz^{-1}),$$

where A is an arbitrary constant. The remaining reductions are summarised in Table 4.4 below.

Table 4.4: Reduction of equation (4.3.1) and some exact solution.

Symmetries	Reductions	$m,\ n,\ \alpha,\ \beta \in \mathbb{R}$	Solutions in original variables
X_1	$C(t,x) = F(x)$ $(\alpha+\beta F)^m F'' + m\beta(\alpha+\beta F)^{m-1}F'^2 - nF' = 0.$	$\beta = 0$ $\alpha^m F'' - nF' = 0$	$C = c_1 + c_2 \mathrm{e}^{\frac{nx}{\alpha^m}}$. where c_1 and c_2 are arbitrary constants.
		$\beta = 0,\ n = 0$ $F'' = 0$	$C(x,t) = c_1 x + c_2$ where c_1 and c_2 are arbitrary constants.
		$n = 0,\ m = 1,\ \alpha = 1,$ $F' + 2F^2 = 0$	$C(x) = \pm c_2\sqrt{2x + c_1}$ where c_1 and c_2 are arbitrary constants.
		$m = n = 1,\ \alpha = 0,\ \beta = -1,$ $FF'' - F'^2 - F' = 0$	$C(x,t) = c_2 \exp\big(c_1 x\big) + 1$ where c_1 and c_2 are arbitrary constants.
		$m = -\frac{1}{2},\ \beta = -1$, $2(\alpha+F)F'' - F'^2 - 2nF^{\frac{3}{2}}F' = 0$	The equation was not studied further.
		$m = -\frac{1}{2}, n = 0,\ \alpha = 0$, $2\beta FF'' - F'^2 = 0$	$C(t,x) = c_2 \mathrm{e}^{c_1 x}$ where c_1 and c_2 are arbitrary constants.

cont.

Symmetries	Invariant solutions and reduced ODE	$m,\ n,\ \alpha,\ \beta \in \mathbb{R}$	Solutions in original variables
X_2	$C(t,x) = F(\frac{x-nt}{\sqrt{t}})$ $2(\alpha+\beta F)^{m-1}F''+2m\beta(\alpha+\beta F)^m F'^2-\gamma F' = 0.$	$\beta = 0$ $\alpha^m F'' - nF' = 0$	$C = c_1 + c_2 \mathrm{e}^{\frac{nx}{\alpha^m}}$. where c_1 and c_2 are arbitrary constants.
		$m = 0,\ \alpha = 0$ $2F'' + \beta\gamma F F' = 0$	The equation was not studied further.
		$m = 1/2,\ \alpha = 0,$ $2F'' + \beta\gamma F^{1/2} F' = 0$	The equation was not studied further.
		$m = 1,\ \alpha = 0,$ $2F'' + \gamma F' = 0$	$C(x,t) = c_2 \mathrm{erf}\big(\frac{x-nt}{2\sqrt{t}}\big) + c_1.$ where c_1 and c_2 are arbitrary constants.

cont.

Symmetries	Invariant solutions and reduced ODE	$m,\ n,\ \alpha,\ \beta \in \mathbb{R}$	Solutions in original variables
$X_1 + a_2X_2, (a_2 = 1)$	$C(t,x) = F(\gamma) \quad \text{where} \quad \gamma = \frac{x-n(t+1)}{\sqrt{2t+1}}$ $(\alpha+\beta F)^m F'' + m\beta(\alpha+\beta F)^{m-1}F'^2 + \gamma F' = 0$	$m = 1, \ \ \beta = 0$ $\alpha F'' + \gamma F' = 0$	$C(t,x) = c_1 + \text{erf}\left(\frac{2i(x-n(t+1))}{\sqrt{4t+2}}\right)c_2$ where c_1 and c_2 are arbitrary constants.
$X_1 + X_2 + a_3X_3, (a_3 = 1).$	$\gamma = (\beta + (2\beta-1)t)^{-\frac{\beta}{(2\beta-1)}}(x - n(t+1)).$ $mF^{m+1}F'' - m^2F^mF'^2 - m\beta\gamma FF' + F^2 = 0$	$m = 1, \quad \beta = -1$ $FF'' - F'^2 - \gamma F' + F = 0$	The equation was not studied further.
		$m = -1, \quad \beta = 1$ $FF'' + F'^2 - \gamma F' + F = 0$	The equation was not studied further.

4.6 Case 2: $D(C) = (\alpha + \beta C)^{-4/3}$, $V = n$.

We note here that for $\alpha = 0$ and $\beta = 1$, this case has been considered by (see for e.g., [158]). The governing equation (4.2.1) becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big[(\alpha + \beta C)^{-4/3}\frac{\partial C}{\partial x} - nC\Big]. \tag{4.6.1}$$

The Lie algebra of equation (4.6.1) is five-dimensional and generated by operators

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= (x+nt)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ X_3 &= \frac{1}{12\beta^2}\Big(3(\alpha+\beta C)\frac{\partial}{\partial C} + 4n\beta t\frac{\partial}{\partial x} + 4\beta t\frac{\partial}{\partial t}\Big), \\ X_4 &= \frac{(x-nt)}{12\beta^2}\Big(3(\alpha+\beta C)\frac{\partial}{\partial C} - \beta(x-nt)\frac{\partial}{\partial x}\Big), \\ X_5 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{4.6.2}$$

An optimal system of one-dimensional sub-algebras is provided by

$$\{X_1 + a_2X_2 + a_3X_3,\ a_2X_2,\ X_2 + a_3X_3,\ X_4,\ X_3 \pm X_5,\ X_4 \pm a_5X_5,\ X_5\}.$$

4.6.1 Symmetry reductions and invariant solutions.

Reduction by $X_3 + X_5$.

The characteristic equations are given by

$$\frac{3\beta dt}{t} = \frac{3\beta dx}{nt + 3\beta} = \frac{12\beta^2 dC}{3(\alpha + \beta C)}. \tag{4.6.3}$$

The invariants are given by

$$C(t,x) = \frac{1}{\beta}\big(t^{\frac{3}{4}}F(\gamma) - \alpha\big), \quad \gamma = t^{-3\beta}\mathrm{e}^{x-nt}. \tag{4.6.4}$$

The derivatives of equation (4.6.4) reduce equation (4.6.1) to a second derivative given by

$$12\gamma^2 FF'' - 16\gamma^2 F'^2 + 12\gamma FF' + 36\beta\gamma F' - 9F^{10/3} = 0. \tag{4.6.5}$$

The remaining reductions are given in a summary form on Table 4.5.

4.7 Case 3 : $D(C) = (\alpha + \beta C)^m$, $V = p + qx$.

The governing equation (4.2.1) becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big[(\alpha + \beta C)^m \frac{\partial C}{\partial x} - (p + qx)C\Big]. \tag{4.7.1}$$

The power diffusivity of (4.7.1) has been considered by (see e.g. [51, 158]) with $\alpha = p = 0$ and $\beta = q = 1$.

The equation (4.7.1) admits two-dimensional Lie point symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = -e^{qt}\frac{\partial}{\partial x}.$$

4.7.1 Symmetry reductions and invariant solutions

A summary reduction is given in Table 4.6 below.

4.8 Case 4 : $D(C) = (\alpha + \beta C)^m$, $m = -3$, $V = x$.

The governing equation becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big[\alpha + \beta C)^{-3}\frac{\partial C}{\partial x} - xC\Big]. \tag{4.8.1}$$

The diffusivity of (4.8.1) has been considered previously (see for e.g., [52]).

The equation (4.8.1) admits three dimensional Lie point symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = -e^t\frac{\partial}{\partial x}, \quad X_3 = \frac{e^{-t}}{\beta}\Big(\beta\frac{\partial}{\partial t} + \beta x\frac{\partial}{\partial x} - (\alpha + \beta C)\frac{\partial}{\partial C}\Big).$$

Table 4.5: Symmetry reduction of equation (4.6.1)

Symmetries	Invariant solutions and reduced ODE	$\alpha,\ \beta \in \mathbb{R}$	Solutions in original variables
$X_1+a_2X_2+a_3X_3$	$C(t,x)=\frac{1}{\beta}\Big\{\Big((6\beta+1)t+3\beta\Big)^{\frac{3}{4(6\beta+1)}}F(\gamma)-\alpha\Big\}$		
	$\gamma=\Big[(6\beta+1)t+3\beta\Big]^{-\frac{3\beta}{6\beta+1}}\Big[x\Big((6\beta+1)t+3\beta\Big)^{\frac{6\beta}{6\beta+1}}-n(t+1)\Big]$	$\beta=1$	
		$12FF''-16F'^2+36\gamma F^{\frac{7}{3}}F'-9F^{\frac{10}{3}}=0$	The equation is not studied further.
	$12FF''-16F'^2+36\beta\gamma F^{\frac{7}{3}}F'-9\beta F^{\frac{10}{3}}=0$	$\beta=-1$	
		$12FF''-16F'^2-36\gamma F^{\frac{7}{3}}F'+9F^{\frac{10}{3}}=0$	The equation is not studied further.
X_2	$C(t,x)=F(\gamma),\ \ \gamma=\frac{x-nt}{\sqrt{t}}$	$\beta=0,\ \alpha=1$	
		$2F''+\gamma F'=0$	$C(t,x)=c_1+c_2\text{erf}\Big(\frac{x-nt}{\sqrt{t}}\Big)$ where c_1 and c_2 are arbitrary constants.
	$6(\alpha+\beta F)F''-8\beta F'^2+3\gamma F'(\alpha+\beta F)^{\frac{7}{3}}=0$	$\beta=\alpha=0$	
		$F''=0$	$C(t,x)=c_1(\frac{x-nt}{\sqrt{t}})+c_2.$ where c_1 and c_2 are arbitrary constants.
$X_2+a_3X_3.$	$C(t,x)=\frac{1}{\beta}\Big(t^{\frac{3\beta}{4(6\beta+1)}}F(\gamma)-\alpha\Big)$	$\beta=0$	
		$12F''F-16F'^2-36\gamma F'F^{\frac{7}{3}}-9F^{\frac{10}{3}}=0$	The equation is not studied further.
	$\gamma=(x-nt)t^{-\frac{3\beta}{6\beta+1}}$	$\beta=\frac{1}{6}$	
		$24F''F-32F'^2-36\gamma F'F^{\frac{7}{3}}-9F^{\frac{10}{3}}=0$	The equation is not studied further.
	$12F''F(6\beta+1)-16F'^2(6\beta+1)-36\gamma F'F^{\frac{7}{3}}-9F^{\frac{10}{3}}=0$	$\beta=-\frac{1}{6}$	
		$4\gamma F'+F=0$	$C(t,x)=-6\Big\{(x-nt)^{-\frac{1}{4}}-\alpha\Big\}$
$X_4\pm a_5X_5$	$C(t,x)=\frac{1}{\beta}\Big((x-nt)^2-12\beta^2\Big)^{-\frac{3}{2}}F(t)-\alpha\Big)$	$\beta=$ arbitrary	
		$F^{\frac{1}{3}}F'+36\beta^2=0$	$C(t,x)=\frac{1}{\beta}\Big((x-nt)^2-12\beta^2\Big)^{-\frac{3}{2}}F(t)-\alpha\Big)$ where $F(t)$ satisfies $F(t)^{\frac{4}{3}}+48\beta^t+c_1$ and c_1 and c_2 are arbitrary constants.
X_5	$C(t,x)=F(x)$	$\beta=0$	
		$F''-n\alpha^{\frac{4}{3}}\gamma F'$	$C(t,x)=c_1+c_2\text{exp}\Big(n\alpha^{\frac{4}{3}}x\Big)$ where c_1 and c_2 are arbitrary constants.
	$3(\alpha+\beta F)F''-4\beta F'^2-3n\gamma(\alpha+\beta F)^{\frac{7}{3}}F'=0$	$\alpha=\beta=$ arbitrary	
		$3(\alpha+\beta F)F''-4\beta F'^2-3n\gamma(\alpha+\beta F)^{\frac{7}{3}}F'=0$	The equation is not studied further.

Table 4.6: Symmetry reduction for (4.7.1).

Symmetries	Invariant solutions and reduced ODE	$m,\ p,\ q \in \mathbb{R}$	Solutions in original variables
X_1	$C(t,x) = F(x)$	$m = 1,\ \beta = 0$	
		$\alpha F'' - (p+qx)F' - qF = 0$	$C(x) = \exp\left(\frac{2px+qx^2}{2\alpha}\right)\left\{c_1 \text{erf}\left(\frac{2}{\sqrt{2\alpha}}\left(\sqrt{p}+\sqrt{q}x\right)\right) + c_2\right\}$ where c_1 and c_2 are arbitrary constants.
	$(\alpha+\beta F)^m F'' + m\beta(\alpha+\beta F)^{m-1}F'^2 - (p+qx)F' - qF = 0$	$m = 1,\ \beta = 0,\ \alpha = 0$	
		$(p+qx)F' + qF = 0$	$C(x) = \frac{c_1}{p+qx}$ where c_1 is arbitrary constant.
X_2	$C = F(t)$	$q = 0$	
		$F' = 0$	$C(x) = c_1$ where c_1 is arbitrary constant.
	$F' + qF' = 0$	$q = 1$	
		$F' + qF = 0$	$C(t,x) = A\text{e}^{-x}$ where A is arbitrary constant.
		$q = -1,$	
		$F' - F = 0$	$C(x) = c_1\text{e}^{x}$ where c_1 is arbitrary constant.
$X_1 + a_2X_2$	$C(t,x) = F(\gamma),\ \gamma = \frac{1}{q}\text{e}^{qt} + x$	$m = 2,\ \beta = 0$	
		$\alpha^2 F'' - (p+q\gamma)F' - qF = 0$	$C(t,x) = \exp\left(\frac{\text{e}^{qt}+qx}{2\alpha q}\left(\text{e}^{qt}+2p+qx\right)\right)\left\{c_1\text{erf}\left(\frac{2}{\sqrt{2\alpha q}}\left(\sqrt{pq}+\text{e}^{qt}+qx\right)\right) + c_2\right\}$ where c_1 and c_2 are arbitrary constants.
	$(\alpha+\beta F)^m F'' + m\beta(\alpha+\beta F)^{m-1}F'^2 - (p+q\gamma)F' - qF = 0$	$m = 1,\ \ q = \beta = 0$	
		$\alpha F F'' - pF' = 0$	$C(t,x) = c_1 + c_2\exp\left(\frac{p(\text{e}^{qt}+xq)}{\alpha q}\right)$ where c_1 and c_2 are arbitrary constants.
		$m = 1,\ \ \alpha = \beta = 0$	
		$(p+q\gamma)F' + qF = 0$	$C(t,x) = \frac{A}{p+\text{e}^{qt}+qx}$ where A is arbitrary constant.

An optimal system of one dimensional sub-algebras is provided by those generated by

$$\{X_1, \quad X_1 + a_2 X_2, \quad X_2 + X_3\}.$$

4.8.1 Symmetry reductions and invariant solutions

4.9 Case 5 : $D(C) = (\alpha + \beta C)^m$, $V = x^{-1}$.

Subsequent to substituting D and V, the governing equation (4.2.1) becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big[(\alpha + \beta C)^m \frac{\partial C}{\partial x} - x^{-1} C\Big]. \tag{4.9.1}$$

Some results for different values of m have been dealt with in [159] for $\alpha = p = 0$ and $\beta = q = 1$. The Lie algebra of equation (4.9.1) is two-dimensional and given by

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}.$$

Reductions of one-dimensional sub-algebras is carried by linear combination given by

$$\{X_1, \quad X_2, \quad X_1 + X_2\}.$$

4.9.1 Symmetry reductions and invariant solutions.

The summary reductions is given in Table 4.8 below.

4.10 Steady state problem.

The time translation symmetry lead to the steady state problem. The one-dimensional linear DE defining steady (see for e.g., [110]) state of equation (4.2.1) is

$$(D(C)C')' - (V(x)C)' = 0. \tag{4.10.1}$$

Notice that equation (4.10.1) maybe integrated once into a first order ODE which is hard to solve exactly. Here we analyze equation (4.10.1) using Lie point symmetry methods to attempt constructing some exact solutions.

Table 4.7: Symmetry reduction equation (4.8.1).

Symmetries	Invariant solutions and reduced ODE	$\alpha,\ \beta \in \mathbb{R}$	Solutions in original variables
$X_1 + a_2X_2$	$C(t,x) = F(\gamma), \quad \gamma = \mathrm{e}^t + x$	$\alpha = 0,\ \beta = 1$	
		$FF'' - 3F'^2 - \gamma F^4F' - F = 0$	
	$(\alpha+\beta F)F'' - 3\beta F'^2 - \gamma(\alpha+\beta F)^4F' - (\alpha+\beta F)^4F = 0$	$(3G+2z)G' - (14+3z^3)G - 3z^4 = 0.$	The equation is not studied further.
		$\beta = -1$,	
		$FF'' + 3\beta F'^2 + \gamma F^4F' + F = 0$	
		$(3G+2z)G' + (14+3z^3)G + 3z^4 = 0.$	The equation is not studied further.
		$\beta = 0,$	
		$\alpha F'' - \gamma F' - F = 0$	$C(t,x) = \exp\left(\frac{(x+\mathrm{e}^t)^2}{2}\right)\left\{c_1\mathrm{erf}\left(\frac{2(x+\mathrm{e}^t)}{\sqrt{2}}\right) + c_2\right\}$
X_1	$C(t,x) = F(x)$	$\alpha = 0,\ \beta = 0$	
		$FF'' - xF^4F' - F = 0$	$C(t,x) = \exp\left(\frac{x}{2}\right)\left\{c_1\mathrm{erf}\left(\frac{2x}{\sqrt{2}}\right) + c_2\right\}$ where c_1 and c_2 are arbitrary constants.
	$(\alpha+\beta F)F'' - 3\beta F'^2 - x(\alpha+\beta F)^4F' - (\alpha+\beta F)^4F = 0$	$\beta = -1$	
		$FF'' + 3F'^2 + xF^4F' + F^5 = 0$	The equation is not studied further.
$X_2 + X_3$	$C(t,x) = \frac{1}{\beta}((1+\mathrm{e}^{-t})F(\gamma) - \alpha),\ \gamma = \frac{1}{x(1+\mathrm{e}^{-t})}$	$\alpha = 0$	
	$\gamma^4FF'' + 2\gamma^3FF' + \gamma F'F^4 + 3\gamma^4F'^2 + \alpha F^4 - F^5 = 0$	$\gamma^4FF'' + 2\gamma^3FF' + \gamma F'F^4 + 3\gamma^4F'^2 - F^5 = 0$	The equation is not studied further.

Table 4.8: Reductions for equation (4.9.1).

Symmetries	Invariant solutions and reduced ODE	$\alpha,\ \beta \in \mathbb{R}$	Solutions in original variables
X_1	$C(t,x) = F(x)$	$\beta = 0$ $\beta x^2 F'' - xF' + F = 0$	$C(t,x) = x(c_1 + c_2 \ln x)$ where c_1 and c_2 are arbitrary constants.
	$x^2(\alpha+\beta F)^m F'' + m\beta x^2(\alpha+\beta F)^{m-1}F'^2 - xF' + F = 0$	$m = 1$ $x^2(\alpha+\beta F)F'' + m\beta x^2 F'^2 - xF' + F = 0$	The equation is not studied further.
X_2	$C(t,x) = F(\gamma),\ \gamma = \frac{x}{\sqrt{t}}$	$\beta = 0,\ m = 1$ $2\gamma^2 F'' - (\gamma^2+2\gamma)F' + 4F = 0$	$C(t,x) = \frac{x}{\sqrt{t}}\exp\left(\frac{x^2}{8t}\right)\left\{c_1 I_o\left(\frac{\lambda x^2}{4t}\right) + c_2 K_o\left(\frac{\lambda x^2}{4t}\right)\right\}$ where I_o and K_o are Bessel functions of t and x and also c_1 and c_2 are arbitrary constants.
		$\beta = 1,\ m = 2$, $2\gamma^2 F^2 F'' + 2\gamma^2 F F'^2 - (\gamma^3+2\gamma)F' + 2F = 0$	The equation is not studied further.
	$2\gamma^2(\alpha+\beta F)^m F'' + 2\gamma^2\beta m(\alpha+\beta F)^{m-1}F'^2 - (\gamma^3 + 2\gamma)F' + 4F = 0$	$\alpha = 0,\ \beta = 0,$ $(\gamma^3+2\gamma)F' - 4F = 0$	$C(t,x) = \frac{x^2}{x^2+2t}c_1$ where c_1 and c_2 are arbitrary constants.
$X_1 + a_2 X_2$.	$C(t,x) = F(\gamma),\ \gamma = \frac{x}{\sqrt{2t+1}}$	$\alpha = 1,\ \ \beta = 0$ $\gamma^2 F'' + \gamma(\gamma^2-1)F' + F = 0$	$C(t,x) = \frac{x}{\sqrt{2t+1}}\exp\left(-\frac{x^2}{\sqrt{2t+1}}\right)\left\{c_1 I_o\left(\frac{\lambda x^2}{2t+1}\right) + c_2 K_o\left(\frac{\lambda x^2}{2t+1}\right)\right\}$ where I_o and K_o are Bessel functions of t and x, and also c_1 and c_2 are arbitrary constants.
		$\alpha = 0,\ \ \beta = 0$ $\gamma(\gamma^2-1)F' + F = 0$	$C(t,x) = \frac{xc_1}{\sqrt{x^2-2t-1}}$ where c_1 is an arbitrary constant.

Case 6 $D(C) = e^{mC}$, $V(x) = n$, $m, n \in \mathbb{R}$

The CDE has the coefficient dispersion in exponential form and with constant velocity. The steady state equation is

$$\frac{d}{dx}\Big[\mathrm{e}^{mC}\frac{dC}{dx} + nC\Big] = 0. \tag{4.10.2}$$

which equivalently be written as

$$\frac{d^2C}{dx^2} + m\big(\frac{dC}{dx}\big)^2 + n\mathrm{e}^{-mC}\frac{dC}{dx} = 0. \tag{4.10.3}$$

In this case equation (4.10.1) admits a two dimensional Lie algebra spanned by the base vectors

$$X_1 = \frac{\partial}{\partial x} \quad \text{and} \quad X_2 = \frac{\partial}{\partial C} + mx\frac{\partial}{\partial x}. \tag{4.10.4}$$

It is possible to completely integrate a second order ODE using two admitted symmetries (see for e.g., [160]). The canonical variables obtained from these symmetries are given by

$$t = \mathrm{e}^{mC} \quad \text{and} \quad u = x + \mathrm{e}^{mC}.$$

Notice that X_1 and X_2 are given in terms of canonical variables by

$$X_1^* = \frac{\partial}{\partial u} \quad \text{and} \quad X_2^* = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

writing $u = u(t)$ we obtain the transformed ODE

$$u'' = \frac{n}{t}(u' - 1)^2,$$

which has an exact solution

$$u = \frac{e^{c_1}\mathbf{Ei_1}(-\ln t + c_1)}{x} + t + c_2,$$

where **Ei** is the exponential integral [31]. In terms of the original variables we obtain the implicit exact solution

$$x^2 - c_2 x = \mathrm{e}^{c_1}\mathbf{Ei_1}(-mC + c_1),$$

where c_1 and c_2 are arbitrary constants. We note here that equation (4.10.3) can be solved directly without using symmetries. These results appear in the paper by Ntsime and Moitsheki [95].

4.10.1 Case 7 : $D(C) = e^{mC}, \quad V = \alpha + \beta x.$

Substitution of D and V in the governing equation (4.2.1) leads to equation given by (see for e.g., [161])

$$\frac{d}{dx}\Big[\mathrm{e}^{mC}\frac{dC}{dx}\Big] - \frac{d}{dx}\Big[(\alpha+\beta x)C\Big] = 0. \tag{4.10.5}$$

If $m = 0$, then (4.10.5) becomes

$$C'' - (\alpha+\beta x)C' - \beta C = 0. \tag{4.10.6}$$

The solution to (4.10.6) is given by

$$C(x) = \exp\left(\frac{2\alpha x + \beta x^2}{2}\right)\Big[\mathrm{erf}\left(\frac{\sqrt{2\beta}x}{2} + \frac{\sqrt{2\alpha^2}}{2\sqrt{\beta}}\right)c_1 + c_2\Big],$$

where c_1 and c_2 are arbitrary constants.

4.10.2 Case 8 : $D(C) = e^{mC}$, $\quad V = x^n.$

Sub-Case 8.1 : $n = -1$

The governing equation (4.2.1) becomes (see for e.g [161])

$$\frac{\partial C}{\partial t} = \mathrm{e}^{mC}\frac{\partial^2 C}{\partial x^2} + m\mathrm{e}^{mC}\left(\frac{\partial C}{\partial x}\right)^2 - \left(x^{-1}\frac{\partial C}{\partial x}\right) + x^{-2}C, \tag{4.10.7}$$

which admits symmetries

$$X_1 = \frac{\partial}{\partial t}.$$

The Lie symmetry X_1 provides a steady environment that lead to equation

$$\mathrm{e}^{mC}x^2C'' + m\mathrm{e}^{mC}x^2C'^2 - xC' + C = 0, \tag{4.10.8}$$

which admits one symmetry

$$\Gamma = -x\frac{\partial}{\partial x}.$$

The first prolongation of Γ is given by

$$\Gamma^{[1]} = x\frac{\partial}{\partial x} + 0\frac{\partial}{\partial C} - C'\frac{\partial}{\partial C'}.$$

The invariants are given by

$$\gamma = C, \;\; F(\gamma) = xC'.$$

Using chain rule, the equation (4.10.8) is then reduced to

$$(F' - 1)F + mF^2 - \mathrm{e}^{-mF}F + \gamma \mathrm{e}^{-mF} = 0. \tag{4.10.9}$$

If $m = 0$, then equation (4.10.9) has a solution

$$F(\gamma) = \frac{\gamma(1 + \mathrm{LambertW}(c_1\gamma))}{\mathrm{LambertW}(c_1\gamma)}.$$

Substituting back and applying basic integration results in

$$C(x) = x(\ln|x| - 1 + \mathrm{A})\mathrm{B},$$

where A and B are the arbitrary constants of integration with $\mathrm{B} = \exp(c_2 - 1)/c_1$ and c_1 and c_2 are arbitrary constants.

4.11 Conclusion

In this chapter, a one-dimensional CDE is considered. Group classification lead to seven functional cases. Lie point symmetries for which different cases (4.3.1) admits were obtained. The commutator table and adjoint representations for each case are obtained. The example of the optimal system is well outlined. The cases $D = (\alpha + \beta C)^{-3}, \;\; V = x$ and $D = (\alpha + \beta C)^{m}, \;\; V = x^{-1}$ results in Error and Bessel functions of t and x. Steady state is considered in the last section.

Chapter 5

Two-dimensional solute transport with constant dispersion coefficient

5.1 Introduction

The model predicting the oil spill is generally described by the convection-diffusion equation

$$\frac{\partial C}{\partial t} = \nabla^2(D_o C) - \frac{\partial}{\partial x}\Big(V(x)C\Big), \tag{5.1.1}$$

where D_o, is a constant, say $D_o = 1$. The convection term includes the effect of wind, wave and ocean current. In this chapter, we consider the CDE in solute water formed by oil particles dissolving in either water or in air. These solute forms are caused by some of the weathering processes such as evaporation, dispersion and turbulent diffusion as a result of wind, surface current and waves [26]. During these processes the oil particles tend to move with the water surrounding causing break down of oil particles which dissolve in water. The particles tend to mix with air and water forming solutes. All these processes have impact on the concentration of the oil at the source or at the shoreline. We consider the effects of these weathering processes in a x and y space only [102]. The diffusive velocity involved moves in the horizontal direction.

A number of cases arise and they are investigated. We note that, equation (5.1.1)

has been considered by researchers [78, 162]. Here we focus on the cases where the velocity depends on spatial coordinate x.

5.2 Double reductions leading to exact solutions.

In this section, we apply the method of differential invariants. We note that, if for any non-Abelian subgroup, the commutator $[X_i, X_j] = \beta X_i$ and $[X_i, X_j] = 0$, then we perform double reductions of (5.1.1) first by X_i then later by X_j.

5.2.1 Case 1: $V(x) = x$.

In this case, equation (5.1.1) admits a seven-dimensional Lie algebra spanned by

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= C\frac{\partial}{\partial C}, \\ X_3 &= \frac{\partial}{\partial y}, \\ X_4 &= 2t\frac{\partial}{\partial y} - yC\frac{\partial}{\partial C}, \\ X_5 &= \mathrm{e}^{-t}\Big(\frac{\partial}{\partial x} + xC\frac{\partial}{\partial C}\Big), \\ X_6 &= \mathrm{e}^{t}\frac{\partial}{\partial x}, X_7 = F(t,x,y)\frac{\partial}{\partial F} \end{aligned}\right\} \tag{5.2.1}$$

where $C(t, x, y)$ satisfies the equation $F_t = F_{xx} + C_{yy} - xF - F$. (See for e.g., [152].) An optimal system of one-dimensional sub-algebras spanned by vectors in equation (5.2.1) is given by

$$\{X_2 + a_3X_3,\ \ X_1 + X_2 + X_3,\ \ X_1 + a_2X_2,\ X_4 + X_5,\ \ X_4,\ \ X_5,\ \ X_6\},\ \ a_i \in \mathcal{R}. \tag{5.2.2}$$

Symmetry reductions

In this section, we perform symmetry reductions using elements of the optimal system (5.2.2) repeated.

1. *For invariance under* $X_2 + a_3X_3,\ a_3 = 1$, the characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{dC}{C}. \tag{5.2.3}$$

Solving equation (5.2.3) yields

$$C(t, x, y) = \mathrm{e}^y F(t, x). \tag{5.2.4}$$

The derivatives of equation (5.2.4) with respect to t, x and y reduces equation (5.1.1) into a simpler PDE

$$\frac{\partial^2 F}{\partial^2 x} - x\frac{\partial F}{\partial x} - \frac{\partial F}{\partial t} - F = 0. \tag{5.2.5}$$

The equation (5.2.5) admits three dimensional Lie algebra spanned by

$$\Gamma_1 = F\frac{\partial}{\partial F}, \quad \Gamma_2 = \mathrm{e}^{-t}\Big(\frac{\partial}{\partial x} + xF\frac{\partial}{\partial F}\Big), \quad \Gamma_3 = \mathrm{e}^t\frac{\partial}{\partial x} \tag{5.2.6}$$

and the one-dimensional optimal system is given by $\{\Gamma_2+\Gamma_3,\ \ \Gamma_1+\Gamma_3,\ \ \Gamma_1+\Gamma_3\}$.

For invariance under $\Gamma_2 + \Gamma_3$, the characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{\mathrm{e}^{-t} + \mathrm{e}^t} = \frac{dy}{x\mathrm{e}^{-t}F} = \frac{dC}{0}. \tag{5.2.7}$$

Solving equation (6.4.14) leaves equation (5.2.5) invariant under

$$F = \exp\Big(\frac{x^2}{2(1 + \mathrm{e}^{2t})}\Big)G(t).$$

The resulting equation being

$$(1 + \mathrm{e}^{2t})G' - G = 0.$$

The solution of equation (5.1.1) in its original variables is given by

$$C(t,x,y) = \frac{c_1}{\sqrt{1+\mathrm{e}^{2t}}}\exp\Big(\frac{x^2}{2(1+\mathrm{e}^{2t})} + t + y\Big), \tag{5.2.8}$$

where c_1 and c_2 are arbitrary constants. The solutions of equation (5.2.8) are represented graphically in Figures (5.1), (5.2), (5.3) and (5.4) below.

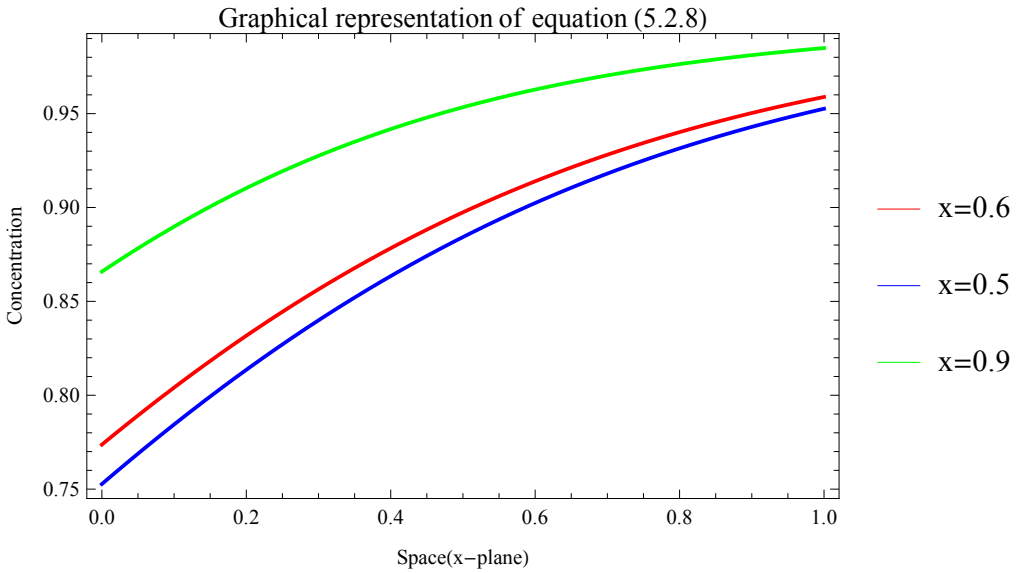


Figure 5.1: Concentration of oil over the x-plane.

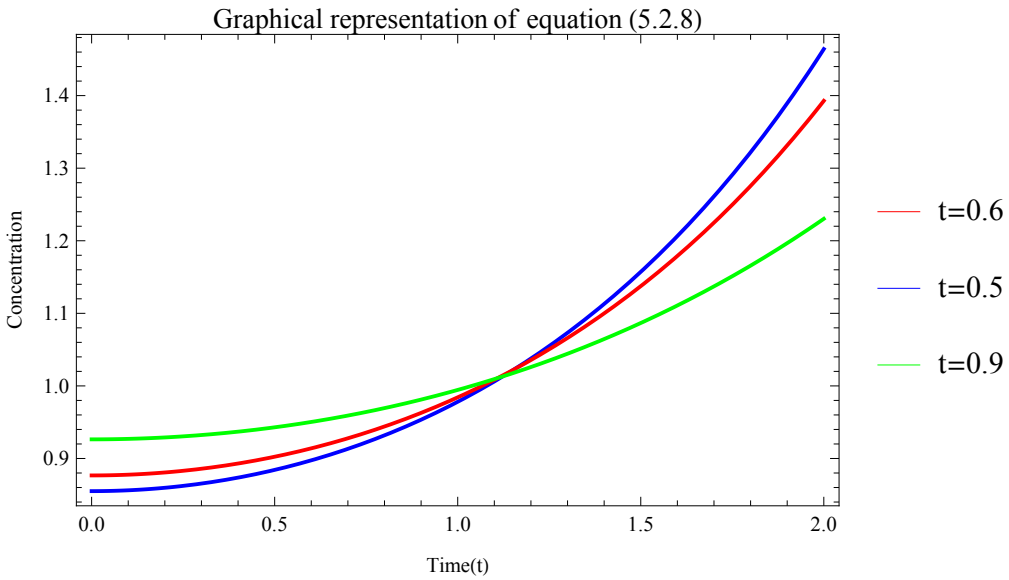


Figure 5.2: Concentration of oil over time.

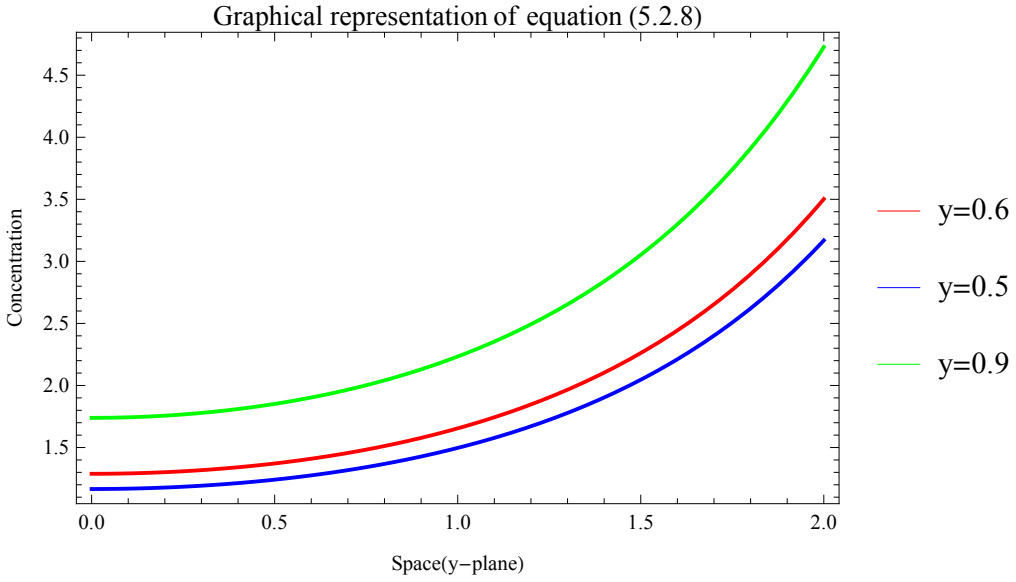


Figure 5.3: Concentration of oil over the y-space.

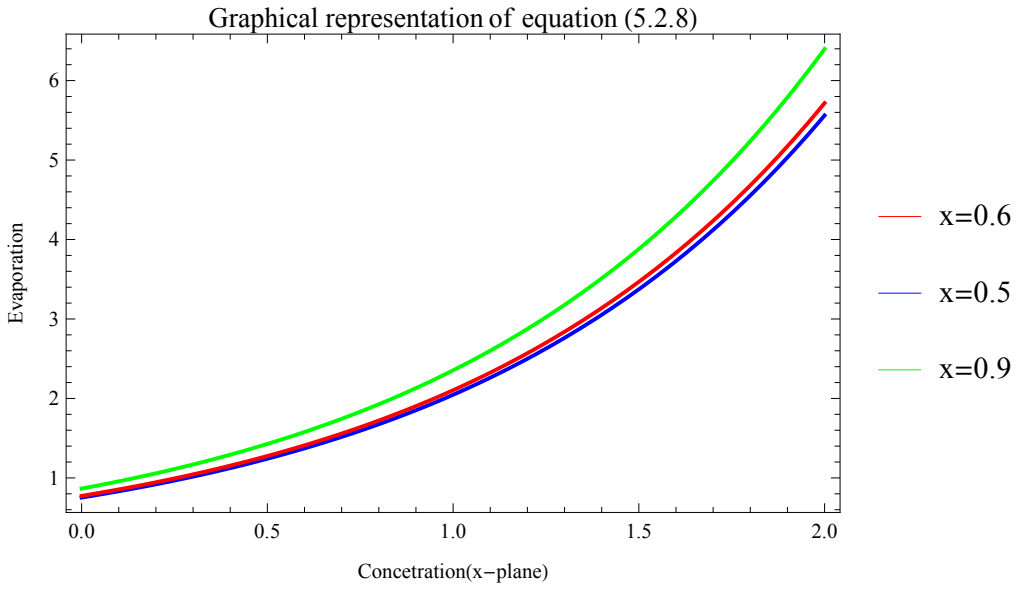


Figure 5.4: The effects of evaporation on the concentration of oil.

The remaining reductions are summarised in Table 5.1 below.

2. *For invariance under,* $X_1 + X_2 + X_3$ has the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{1} = \frac{dC}{C}. \tag{5.2.9}$$

The solution of equation (5.2.9) is

$$C(t, x, y) = F(x, \phi) \text{ where } \phi = y - t. \tag{5.2.10}$$

The derivatives of equation (5.2.10) with respect to x and ϕ, reduces equation (5.1.1) into a simpler PDE

$$\frac{\partial^2 F}{\partial^2 \phi} + \frac{\partial^2 F}{\partial^2 x} - x\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \phi} - F = 0. \tag{5.2.11}$$

Table 5.1: Summary of Symmetry reduction of equation (5.2.1).

Symmetry	Reduced Equations	Invariant Solutions in terms of original variables
$\Gamma_1 + \Gamma_3$	$G' - \mathrm{e}^{-2t}G = 0,$ $F = \exp\big(x\mathrm{e}^{-t}\big)G(t)$	$C(t,x,y) = c_1\exp\left(\dfrac{2x\mathrm{e}^{-t} - \mathrm{e}^{-2t} + 2y}{2}\right)$ where c_1 is an arbitrary constant.
$\Gamma_1 + \Gamma_2$	$G' - (1 + \mathrm{e}^{\frac{x^2+2x\mathrm{e}^t+4t}{2}})G = 0,$ $F = \exp\big(\dfrac{2x\mathrm{e}^t + x^2}{2}\big)G(t)$	$C(t,x,y) = c_1\exp\big(\dfrac{2x\mathrm{e}^t + x^2}{2}\big)\exp\big(\int(\exp(1 + \exp\,(\dfrac{x^2 + 2x\mathrm{e}^t + 4t}{2}))dt\big)$ where c_1 is an arbitrary constant.

The equation (5.2.11) further admits two symmetries

$$\Gamma_1 = -2\frac{\partial}{\partial \phi} + F\frac{\partial}{\partial F}, \qquad \Gamma_2 = \frac{\partial}{\partial \phi}.$$

Further reductions by $\{\Gamma_1, \quad \Gamma_2, \quad \Gamma_1 + \Gamma_2\}$ and possible exact solutions are provided in Table 5.2 below.

3. *For invariance under* $X_1 + a_2 X_2, \quad a_2 = 1$, the characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{dC}{C}. \tag{5.2.12}$$

Substituting the derivatives of equation (5.2.12) yield invariant solutions

$$C(t, x, y) = \mathrm{e}^t F(x, y). \tag{5.2.13}$$

The derivatives of equation (5.2.13) reduces equation (5.1.1) into a simpler PDE

$$\frac{\partial^2 F}{\partial^2 y} + \frac{\partial^2 F}{\partial^2 x} - \frac{\partial F}{\partial x} - 2F = 0. \tag{5.2.14}$$

The equation (5.2.14) further admits two symmetries X_2^* and X_3^*. Further reductions are outline in Table 5.3 below.

4. *For invariance under,* X_4 the characteristic equations given by

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{2t} = -\frac{dC}{yC}. \tag{5.2.15}$$

yield solutions that leaves non-linear partial differential equation (5.1.1) invariant under $C = \mathrm{e}^{-y^2/4t} F(t, x)$. The reduced equation is given by

$$2t\frac{\partial^2 F}{\partial^2 x} - 2tx\frac{\partial F}{\partial x} - 2t\frac{\partial F}{\partial t} - (2t+1)F = 0. \tag{5.2.16}$$

The equation (5.2.16) admits $\Gamma_4 = F\dfrac{\partial}{\partial F}$. Imposing the steady state on equation (5.2.16) lead to second order DE

$$2tF'' - 2t(x-1)F' - (2t+1)F = 0, \quad F = F(x). \tag{5.2.17}$$

Table 5.2: Summary reductions of (5.2.11) .

Symmetry	Reduced Equations	Invariant Solutions in terms of original variables
Γ_1	$4G'' - 4xG' - 5G = 0$ $F(x,\phi) = \exp\left(-\frac{\phi}{2}\right)G(x)$	$C(t,x,y) = x\exp\left(\frac{2y-2t-x^2}{2}\right)\left[c_1\mathrm{M}\left(\frac{9}{8},\frac{3}{2},\frac{x^2}{2}\right) + c_2\mathrm{U}\left(\frac{9}{8},\frac{3}{2},\frac{x^2}{2}\right)\right]$ where M_o and U_o are KummerM and KummerU of x respectively, and c_1 and c_2 are arbitrary constants.
Γ_2	$G'' - xG' - G = 0, \quad F(x,\phi) = G(x)$	$C(t,x,y) = \exp\left(\frac{x^2}{2}\right)\left[c_2 + c_1\mathrm{erf}\left(\frac{\sqrt{2}x}{2}\right)\right]$ where **erf** is Error function of x, and c_1 and c_2 are arbitrary constants.
$\Gamma_1 + \Gamma_2$	$G'' - xG' - G = 0, \quad F(x,\phi) = \mathrm{e}^{-\phi}G(x)$	$C(t,x,y) = \exp\left(\frac{x^2+2y-2y}{2}\right)\left[c_2 + c_1\mathrm{erf}\left(\frac{\sqrt{2}x}{2}\right)\right]$ where **erf** is Error function of x, and c_1 and c_2 are arbitrary constants.

Table 5.3: Reduction by equation (5.2.14).

Symmetries	Reduced Equations	Invariant Solutions in terms of original variables
Γ_1	$G'' - xG' - 2G = 0$ $F(x,y) = G(x)$	$C(t,x,y) = \mathrm{e}^t\left\{c_1 x\exp(\frac{x^2}{2}) + c_2(x\sqrt{2\pi}\exp(\frac{x^2}{2})\mathrm{erf}\left(\frac{\sqrt{2}x}{2}\right) + 2)\right\}$ where **erf** is Error function of x, and c_1 and c_2 are arbitrary constants.
$\Gamma_1 + \Gamma_2$	$G'' - xG' - G = 0$ $F(x,y) = \mathrm{e}^y G(x)$	$C(t,x,y) = \exp\left(\frac{x^2+2t+2y}{2}\right)\left\{c_2 + c_1\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right\}$ where **erf** is Error function of x, and c_1 and c_2 are arbitrary constants.

The solution of equation (5.1.1) in its original variables is given by

$$C(t,x,y) = x\mathrm{e}^{-\frac{y^2}{4t}}\left\{c_1 M_o\left(\frac{4t+1}{4t},\frac{3}{2},\frac{x^2}{2}\right) + c_2 U_o\left(\frac{4t+1}{4t},\frac{3}{2},\frac{x^2}{2}\right)\right\},$$

where M_o and U_o are KummerM and KummerU functions of x, and c_1 and c_2 are arbitrary constants.

5. *For invariance under* X_5, the characteristic equations given by

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{0} = \frac{dC}{xC}, \tag{5.2.18}$$

yield solutions that leaves non-linear partial differential equation (5.1.1) invariant under $C = \mathrm{e}^{x^2/2}F(t,y)$. The reduced equation is given by

$$\frac{\partial^2 F}{\partial^2 y} - \frac{\partial F}{\partial t} = 0, \tag{5.2.19}$$

which is just a well-known heat equation. The equation (5.2.19) admits X_1. This implies that equation (5.1.1) has invariant t. Further reduction by X_1 lead to

$$G'' = 0, \;\; G = G(y).$$

the solution given by $C = \mathrm{e}^{x^2/2}\big(c_1 y + c_2\big)$, and c_1 and c_2 are arbitrary constants. This defines evaporation in the upward direction.

6. *For invariance under* X_6, the characteristic equations given by

$$\frac{dt}{0} = \frac{dx}{\mathrm{e}^t} = \frac{dy}{0} = \frac{dC}{0}. \tag{5.2.20}$$

yield solutions that leaves non-linear partial differential equation (5.1.1) invariant under $C = F(t,y)$. The reduced equation is given by

$$\frac{\partial^2 F}{\partial^2 y} - \frac{\partial F}{\partial t} - F = 0. \tag{5.2.21}$$

The solutions of PDE (5.2.21) are well known. Notice that $[X_6, X_1] = 0$. In fact, this symmetry implies that equation (5.1.1) has invariant t. Further reduction by X_1 lead to

$$F'' - F = 0, \;\; F = F(y).$$

The solution of equation (5.1.1) in its original variables is given by

$$C(t,x,y) = \mathrm{e}^{y}\big\{c_1\mathrm{e}^{-2y} + c_2\big\},$$

and c_1 and c_2 are arbitrary constants. The solution defines an exponential movement of hydrocarbons in water. It is apparent that the evaporation, sedimentation and emulsification weather processes occurs simultaneously at a certain instant.

Steady state

The symmetry X_1 implies that equation (5.1.1) is independent of time variable. In this case, equation (5.2.1) is given by

$$C'' - xC' - C = 0.$$

The solution is thus given by $C(t,x,y) = \exp\big(x^2/2\big)\Big\{c_2 + c_1\mathrm{erf}\big(\sqrt{2}x/2\big)\Big\}$, and c_1 and c_2 are arbitrary constants.

5.2.2 Case 2: $V(x) = x^2$.

The two-dimensional solute transport in the spatial coordinate (5.1.1) is given by

$$\frac{\partial C}{\partial t} = \nabla(C) - \frac{\partial}{\partial x}\Big(x^2C\Big). \tag{5.2.22}$$

(see for e.g., [152]). Under these condition, the equation (5.1.1) admits four-dimensional Lie algebra spanned by

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= C\frac{\partial}{\partial C}, \\ X_3 &= \frac{\partial}{\partial y}, \\ X_4 &= 2t\frac{\partial}{\partial y} - yC\frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.2.23}$$

An optimal system of one-dimensional sub-algebras generated by these symmetries is given by

$$\{X_1 + a_2X_2, \quad X_3, \quad X_1 + X_2 + X_3, \quad X_4\}.$$

1. *For invariance under* X_4, The characteristic equations given by

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{2t} = -\frac{dC}{yC}. \tag{5.2.24}$$

yields solutions that leaves non-linear PDE (5.1.1) invariant under

$$C = \mathrm{e}^{-y^2/4t}F(t,x).$$

The reduced equation is given by

$$2t\frac{\partial^2 F}{\partial^2 x} - 2tx^2\frac{\partial F}{\partial x} - 2t\frac{\partial F}{\partial t} - (4xt+1)F = 0.$$

We note that further reduction by X_1 lead to

$$2tG'' - 2x^2tG' - (2tx+1)G = 0, \;\; G = G(x).$$

The solution of equation (5.1.1) in its original variables is given by

$$C(t,x,y) = \mathrm{A}\exp\left(\frac{x^3(72+\sqrt{(\sqrt[6]{613^5}-\sqrt[3]{63})^3})}{432}\right)$$
$$\left\{H_o\big(-\frac{3}{t(-\sqrt[3]{13}+\sqrt[6]{3^5})},0,0,-\frac{\sqrt{(\sqrt[6]{613^5}-\sqrt[3]{63})^3}x}{6}\big)\right\}$$
$$\left\{\int \frac{\exp\left(\frac{x^3\sqrt{(\sqrt[6]{613^5}-\sqrt[3]{63})^3}}{216}\right)}{H_o\big(-\frac{6}{\sqrt[6]{13^5}-\sqrt[3]{3}},0,0,-\frac{\sqrt{(\sqrt[6]{613^5}-\sqrt[3]{63})^3}x}{6}\big)^2}dx\right\}$$

where A $= c_1 + c_2$ is the arbitrary constant of integration, and H_o is the HeunT function of x. The summary of invariant solutions are outlined in Table 5.4 below.

Steady state

Given $V = x^2$, the steady state equation is given by

$$C'' - x^2 C' - 2xC = 0.$$

The solution is thus given by

$$C(t,x,y) = c_1 x^{-4} \exp\big(x^2/6\big)\left\{x^3 W_o\big(\frac{1}{6},\frac{2}{3},\frac{x^3}{3}\big) + 4W_o\big(\frac{7}{6},\frac{2}{3},\frac{x^3}{3}\big)\right\} + c_2 \exp\big(x^3/3\big).$$

where W_o is the WhittakerM functions of x, and c_1 and c_2 are arbitrary constants.

5.3 Conclusion

In this chapter, we considered the convection-dispersion equation with variable water velocity. Rich symmetry algebras are obtained for various cases of the water velocity. Reductions by the one-dimensional optimal system resulted in construction of exact solutions.

Table 5.4: Symmetry reductions of equation (5.2.22).

Symmetries	Reduced Equations	Invariant Solutions in terms of original variables
$a_1X_1 + X_2 + a_3X_3$	$4G'' - 4x^2G' - (8x+5)G = 0,$ $F(x,\phi) = \mathrm{e}^{-\frac{\phi}{2}}G(x)$ $\phi = y - t$	$C(t,x,y) = \exp\left(\frac{y-t}{2}\right)\exp\left(\frac{x^3(72+\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3})}{432}\right)$ $\left\{(c_1+c_2)H_o\big(-\frac{15}{-\sqrt[3]{23}+\sqrt[6]{213^5}},-3,0,-\frac{\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3}x}{6}\big)\right\}$ $\left\{\int \frac{\exp\left(\frac{x^3\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3}}{216}\right)}{H_o\big(-\frac{15}{-\sqrt[3]{23}+\sqrt[6]{213^5}},-3,0,-\frac{\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3}x}{6}\big)^2}dx\right\}$ where $\mathbf{H}_o$ are HeunT functions of x, and c_1 and c_2 are arbitrary constants.
$X_1 + a_2X_2$	$G'' - x^2G' - (2x+1)G = 0,$ $F(x,y) = \mathrm{e}^{-y}G(x)$	$C(t,x,y) = \mathrm{e}^{y}\exp\left(\frac{x^3(72+\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3})}{432}\right)$ $\left\{(c_1+c_2)H_o\big(-\frac{15}{-\sqrt[3]{23}+\sqrt[6]{213^5}},-3,0,-\frac{\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3}x}{6}\big)\right\}$ $\left\{\int \frac{\exp\left(\frac{x^3\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3}}{216}\right)}{H_o\big(-\frac{15}{-\sqrt[3]{23}+\sqrt[6]{213^5}},-3,0,-\frac{\sqrt{(-\sqrt[3]{63}+\sqrt[6]{613^5})^3}x}{6}\big)^2}dx\right\}$ where H_o are HeunT functions of x, and c_1 and c_2 are arbitrary constants.
X_3	$G'' - x^2G' - 2xG = 0,$ $F(t,x) = G(x),$	$C(t,x,y) = c_1x^{-\frac{5}{2}}\exp\left(\frac{x^2}{6}\right)\left\{x^3W_o\big(\frac{1}{6},\frac{2}{3},\frac{x^3}{3}\big) + 4W_o\big(\frac{7}{6},\frac{2}{3},\frac{x^3}{3}\big)\right\}$ $+c_2\exp\big(\frac{x^3}{3}\big)$ where W_o is the WhittakerM functions of x, and c_1 and c_2 are arbitrary constants.

Chapter 6

Two-dimensional solute transport with non-constant dispersion coefficient

6.1 Introduction

In this chapter, we look at the effects of some weathering processes that occur after oil spill in ocean/sea. The first process that occurs immediately is the evaporation forming oil-air interface caused by wind, dispersion which is caused by vertical movement and emulsification and dissolution as a result of breaking down of oil particles into small soluble particles forming water-in-oil solutes [102], (see figure 6.1). The concentration of the oil immediately after the spill and later after dispersion is considered. The velocity involved is in the horizontal direction. In this chapter we consider two-dimensional transient solute transport with non-constant dispersion coefficient

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(C)\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D(C)\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial x}\Big(V(x)C\Big). \tag{6.1.1}$$

The equation (6.1.1) attracted many researchers, see for example [112].

The method of change of variables is used extensively in this chapter to reduce

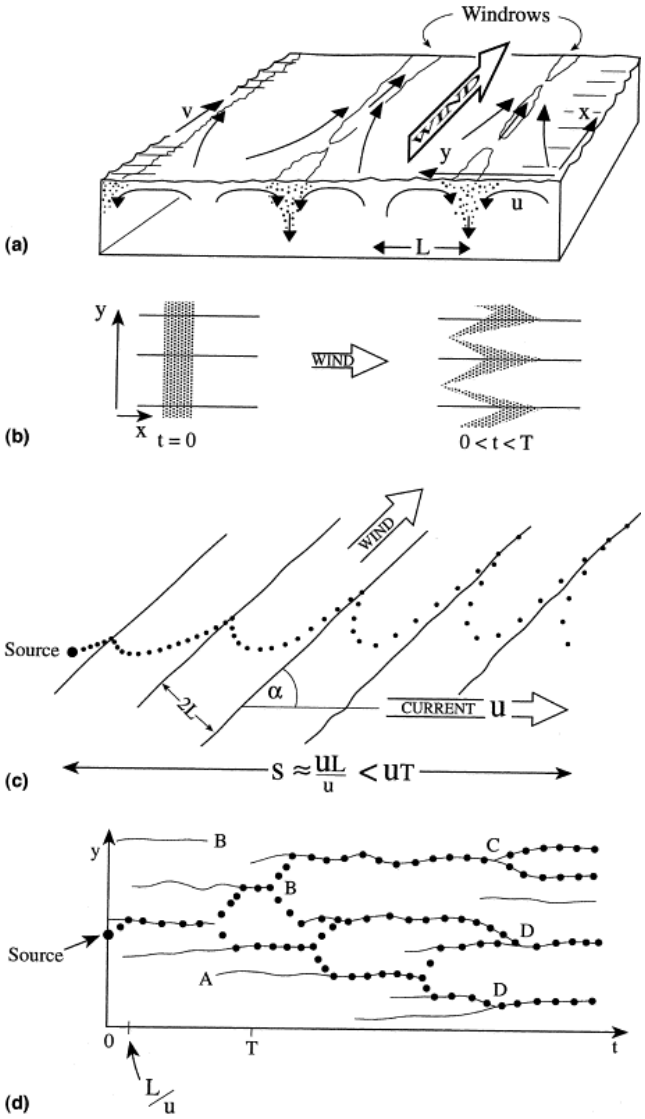


Figure 6.1: Two-dimensional convection-diffusion movement of oil in water. Image from *www.nap.ed*

(6.1.1) to simpler solvable DEs. In cases where the reduced ODE is solvable, we use transformation methods such as G'/G-method and substitution $w(z) = (z')^{-1}$.

We first reduce (6.1.1) using the symmetry X_i and further by X_j in all cases where

the Lie sub-algebra yields

$$[X_i, X_j] = \alpha X_i \quad \text{and} \quad [X_i, X_j] = 0.$$

Suppose the admitted Lie algebra is given by $[X_i, X_j] = \beta X_i$ or $[X_i, X_j] = 0$ where $\beta \in \mathbb{R}$ and i, j are from 1 to 6. First reduction by X_i is performed to reduce a PDE to a simple ODE, then the reduced equation automatically admit X_j. The summary of the remaining reductions are given in tabular form.

6.2 Case 1: $D = n, \quad V = 1.$

We note that this case with constant coefficient dispersion has extensively been dealt with by some works such as the one found in [162]. We provide this case to showcase the effectiveness of double reduction process from linear combinations. We shall therefore not provide all calculations from the linear combinations. We note here that equation (6.1.1) becomes

$$\frac{\partial C}{\partial t} = n\frac{\partial^2 C}{\partial x^2} + n\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial x}. \tag{6.2.1}$$

The equation (6.2.1) admits infinite symmetries

$$\left.\begin{aligned}
X_1 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4n}C(t^2 + x^2 - 2tx + 4nt + y^2)\frac{\partial}{\partial C} \\
X_2 &= \frac{1}{4n}\left(4nt\frac{\partial}{\partial t} + 2nx\frac{\partial}{\partial x} + 2ny\frac{\partial}{\partial y} + C(x-t)\frac{\partial}{\partial C}\right), \quad X_3 = \frac{\partial}{\partial t} \\
X_4 &= \frac{1}{2n}\Big(2nt\frac{\partial}{\partial y} - yC\frac{\partial}{\partial C}\Big), \quad X_5 = \frac{\partial}{\partial y}, \quad X_6 = C\frac{\partial}{\partial C} \\
X_7 &= \frac{1}{2n}\Big(2ny\frac{\partial}{\partial x} - 2nx\frac{\partial}{\partial y} + yC\frac{\partial}{\partial C}\Big), \quad X_8 = \frac{\partial}{\partial x} \\
X_9 &= \frac{1}{2n}\Big(2nt\frac{\partial}{\partial x} + (t-x)C\frac{\partial}{\partial C}\Big) \\
X_\infty &= H(t,x,y)\frac{\partial}{\partial C}
\end{aligned}\right\} \tag{6.2.2}$$

The commutator of (6.2.2) generators excluding X_∞ is given in table 6.1 below.

6.2.1 Double reduction of equation (6.2.1)

In this section we follow the procedure outlined in section 6.1 of this chapter. Specifically if $[X_7, X_3] = \alpha X_7$ then we can perform reduction first by X_7 and then X_3 is automatically admitted by the reduced PDE.

(a) *Invariance under*

$$X_7 = \frac{1}{2n}\Big(2ny\frac{\partial}{\partial x} - 2nx\frac{\partial}{\partial y} + yC\frac{\partial}{\partial C}\Big).$$

The symmetry X_7 has characteristic equations

$$\frac{dt}{0} = \frac{dx}{2ny} = -\frac{dy}{2nx} = \frac{dC}{yC}. \tag{6.2.3}$$

Solving (6.2.3) yields invariant solutions given by

$$C(t,x,y) = \exp\Big(\frac{x}{2n}\Big)F(\gamma,\phi), \quad \gamma = t, \quad \phi = \frac{x^2+y^2}{2}. \tag{6.2.4}$$

The derivative of equation (6.2.4) with respect to γ and ϕ reduce (6.2.1) to

$$8n^2\phi\frac{\partial^2 F}{\partial\phi^2} + 8n^2\frac{\partial F}{\partial\phi} - 4n\frac{\partial F}{\partial t} - F = 0. \tag{6.2.5}$$

The equation (6.2.5) *inherits* $X_3^* = \partial/\partial t$ that leads to second reduction and equation (6.2.5) becomes

$$8n^2\phi F'' + 8n^2F' - F = 0.$$

The solution of (6.2.1) given in its original variables is

$$C(t,x,y) = \exp\Big(\frac{x}{2n}\Big)\left\{c_1 I_o\left(\frac{\lambda\sqrt{x^2+y^2}}{2n}\right) + c_2 K_o\left(\frac{\lambda\sqrt{x^2+y^2}}{2n}\right)\right\}.$$

where I_o and K_o are Bessel functions of x and y, and c_1 and c_2 are arbitrary constants. The other second reduction is carried out by the *inherited* X_2^* in

Table 6.1: Commutator table of sub-algebras of (6.2.2)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	$-4nX_1$	$-2X_2 + \frac{1}{2n}X_6$	0	$-X_4$	0	0	$-X_9$	0
X_2	$4nX_1$	0	$\frac{1}{4n}X_6 - X_3$	X_4	$-\frac{1}{2}X_5$	0	0	$-\frac{1}{2}X_8 - \frac{1}{4n}X_6$	$\frac{1}{4n}X_9$
X_3	$2X_2 - \frac{1}{2n}X_6$	$-\frac{1}{4n}X_6 + X_3$	0	X_5	0	0	0	0	$X_8 + \frac{1}{2n}X_6$
X_4	0	$-X_4$	$-X_5$	0	$\frac{1}{2n}X_6$	0	X_9	0	0
X_5	X_4	$\frac{1}{2}X_5$	0	$-\frac{1}{2n}X_6$	0	0	$X_8 + \frac{1}{2n}X_6$	0	0
X_6	0	0	0	0	0	0	0	0	0
X_7	0	0	0	$-X_9$	$-X_8 - \frac{1}{2n}X_6$	0	0	X_5	X_4
X_8	X_9	$\frac{1}{2}X_8 + \frac{1}{4n}X_6$	0	0	$-X_5$	0	0	0	$-\frac{1}{2n}X_6$
X_9	0	$-\frac{1}{4n}X_9$	$-X_8 - \frac{1}{2n}X_6$	0	0	0	$-2X_4$	$-\frac{1}{2n}X_6$	0

new variables on equation (6.2.5). We observe that through change of variables, X_2 becomes

$$X_2^* = \frac{1}{4n}\left(4nt\frac{\partial}{\partial t} + 4n\phi\frac{\partial}{\partial \phi} - tF\frac{\partial}{\partial F}\right).$$

The characteristic equations are given by

$$\frac{dt}{t} = \frac{d\phi}{\phi} = -\frac{4ndF}{tF}$$

which yield invariant solutions

$$F(t,\phi) = t^{-\frac{1}{4n}}G(z), \quad z = t^{-1}\phi. \tag{6.2.6}$$

The derivatives of (6.2.6) with respect to t and z reduce equation (6.2.5) to

$$2nzG'' + (2n+z)G' = 0. \tag{6.2.7}$$

It can be shown that

$$G(z) = c_1 + \mathbf{E_i}\Big(1, \frac{z}{2n}\Big)c_2$$

satisfies (6.2.7). The solution of (6.2.1) in its original variables is given by

$$C(t,x,y) = t^{-\frac{1}{4n}}\exp\Big(\frac{x}{2n}\Big)\Big\{c_1 + \mathbf{E_i}\Big(1, \frac{x^2+y^2}{4nt}\Big)c_2\Big\},$$

and c_1 and c_2 are arbitrary constants.

It is important to note that if $[X_i, X_j] = \alpha X_\rho + \beta X_r$ the reduction by $\alpha X_\rho + \beta X_r$ is feasible. We shall carry out reductions based on this concept.

(b) *Invariance under* $X_3 + \alpha X_6 = \dfrac{\partial}{\partial t} + C\dfrac{\partial}{\partial C}$, $\alpha = 1$,

with the characteristic equations given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{dC}{C}. \tag{6.2.8}$$

Solving (6.2.8) yields invariant solution

$$C(t,x,y) = \mathrm{e}^t F(x,y). \tag{6.2.9}$$

Substituting the derivatives of (6.2.9) into equation (6.2.1) results in

$$n\frac{\partial^2 F}{\partial x^2} + n\frac{\partial^2 F}{\partial y^2} - \frac{\partial F}{\partial x} - F = 0. \tag{6.2.10}$$

Note that equation (6.2.10) admits the inherited symmetry given by

$$X_2^* = \frac{1}{2n}\Big(- 2nx\frac{\partial}{\partial x} + 2ny\frac{\partial}{\partial y} - yF\frac{\partial}{\partial F}\Big)$$

The characteristic equations are given by

$$\frac{dx}{x} = -\frac{dy}{y} = -\frac{2ndF}{yF}. \tag{6.2.11}$$

Solving (6.2.11) we obtain invariant solutions

$$F(x,y) = \exp\Big(\frac{x}{2n}\Big)G(\gamma), \quad \gamma = \frac{x^2+y^2}{2}. \tag{6.2.12}$$

Differentiating equation (6.2.12) with respect to x and y and substituting the derivatives in (6.2.10) we obtain

$$4n^2G'' - G = 0.$$

The solution of (6.2.1) in its original variables is then given by

$$C(t,x,y) = \exp\Big(\frac{2nt-y}{2n}\Big)\Big\{c_1\exp\Big(- \frac{y}{2n}\Big) + c_2\exp\Big(\frac{y}{2n}\Big)\Big\},$$

and c_1 and c_2 are arbitrary constants.

(c) *For invariance under* $\alpha X_2 + \beta X_6$, the characteristic equations lead to invariants

$$C(t,x,y) = \sqrt{t}\exp\Big(\frac{2x-t}{4n}\Big)F(\gamma,\phi), \quad \gamma = \frac{x}{\sqrt{t}}, \quad \phi = \frac{y}{t}. \tag{6.2.13}$$

The derivatives of (6.2.13) reduces (6.2.1) to

$$4n\gamma F'' + 2(\gamma+2n)F' - F = 0 \tag{6.2.14}$$

The solution of (6.2.1), given in its original variables is

$$C(t,x,y) = \sqrt{t}\exp\Big(\tfrac{4xt-2t^2-x^2-y^2}{8nt}\Big)\Bigg\{\Big\{c_1\big(\tfrac{x^2+y^2+4nt}{2t}\big)I_o\big(0,\tfrac{x^2+y^2}{8nt}\big)+\big(\tfrac{x^2+y^2}{2t}\big)I_o\big(1,\tfrac{x^2+y^2}{8nt}\big)\Big\}+$$
$$\Big\{c_2\big(- \tfrac{x^2+y^2+4nt}{2t}\big)K_o\big(0,\tfrac{x^2+y^2}{8nt}\big) + \big(\tfrac{x^2+y^2}{2t}\big)K_o\big(1,\tfrac{x^2+y^2}{8nt}\big)\Big\}\Bigg\}$$

where K_o and I_o are BesselK and BesselI functions of t, x and y respectively, and c_1 and c_2 are arbitrary constants.

6.3 Case 2: $D = C, \quad V = 1.$

The equation (6.1.1) becomes (see for e.g., [51])

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(C\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(C\frac{\partial C}{\partial y}\right) - \frac{\partial C}{\partial x}. \tag{6.3.1}$$

The equation (6.3.1) admits four dimensional Lie algebra spanned by

$$\left.\begin{aligned} X_1 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + C\frac{\partial}{\partial C}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \frac{\partial}{\partial y}, \\ X_4 &= y\frac{\partial}{\partial x} + (t-x)\frac{\partial}{\partial y}, \\ X_5 &= (t-x)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2C\frac{\partial}{\partial C}, \\ X_6 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{6.3.2}$$

The commutator of the generators in equation (6.3.2) is given in Table 6.2 below.

Table 6.2: Commutator Table of Lie algebra of equation (6.3.1)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$-X_2$	$-X_3$	0	0	$-X_6$
X_2	X_2	0	0	X_3	X_6	0
X_3	X_3	0	0	X_6	X_6	0
X_4	0	$-X_3$	$-X_6$	0	0	X_3
X_5	0	$-X_6$	$-X_6$	0	0	X_6
X_6	X_6	0	0	$-X_3$	$-X_6$	0

6.3.1 Double reduction of (6.3.1) using G'/G-transformation.

In this section we follow the procedure outlined in section 6.1 of this chapter. Most importantly we note that if $[X_4, X_5] = 0$ then we can perform reduction first by X_4 and then X_5 is automatically admitted by the reduced PDE. We shall show the double reductions of equation (6.3.1) based on this notion and any other linear combination.

(a) *For invariance under* $X_4 = y\dfrac{\partial}{\partial x} + (t-x)\dfrac{\partial}{\partial y}$,
the characteristic equations

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{t-x} = \frac{dC}{0}$$

which yields invariant solutions

$$C(t,x,y) = F(t,\phi), \;\; I_1 = t, \;\; \phi = \frac{y^2 + (t-x)^2}{2}. \tag{6.3.3}$$

the derivatives of (6.3.3) transform equation (6.3.1) to

$$2\phi F\frac{\partial^2 F}{\partial \phi^2} + 2\phi\left(\frac{\partial F}{\partial \phi}\right)^2 + 2F\left(\frac{\partial F}{\partial \phi}\right) - \frac{\partial F}{\partial t} = 0 \tag{6.3.4}$$

It can be shown, through change of variables, that equation (6.3.4) inherits three symmetries in new variables X_1, X_2 and X_5. The inherited symmetries are given by

$$\left.\begin{aligned} X_1^* &= t\frac{\partial}{\partial t} + 2\phi\frac{\partial}{\partial \phi} + F\frac{\partial}{\partial F}, \\ X_2^* &= \frac{\partial}{\partial t}, \\ X_5^* &= \phi\frac{\partial}{\partial \phi} + F\frac{\partial}{\partial F}. \end{aligned}\right\} \tag{6.3.5}$$

We perform reduction by *inherited* $X_2^* = \dfrac{\partial}{\partial t}$. The characteristic equations are given by

$$\frac{dt}{1} = \frac{d\phi}{0} = \frac{dF}{0}$$

with invariants

$$F(t,\phi) = G(\phi), \;\; \phi = \frac{y^2 + (t-x)^2}{2}$$

The derivatives reduces (6.3.4) to ODE

$$\phi G G'' + \phi G'^2 + G G' = 0 \tag{6.3.6}$$

We employ the G'/G-method (see for e.g., [163]) to further reduce (6.3.6). Let $h = G'/G$ where $h = h(\phi)$. It can be shown that (6.3.6) can be reduced to

$$\phi h' + 2\phi h^2 + h = 0. \tag{6.3.7}$$

which is satisfied by

$$h = \frac{1}{(2\ln\phi + c_1)\phi}. \tag{6.3.8}$$

The solution (6.3.1) in its original form is given by

$$C(t,x,y) = c_2\sqrt{2\ln\big(\frac{y^2 + (t-x)^2}{2}\big) + c_1}. \tag{6.3.9}$$

Performing reduction by *inherited* $X_1^* = t\dfrac{\partial}{\partial t} + 2\phi\dfrac{\partial}{\partial \phi} + F\dfrac{\partial}{\partial F}$, with characteristic equations given by

$$\frac{dt}{t} = \frac{d\phi}{\phi} = \frac{dF}{F}. \tag{6.3.10}$$

Solving equation (6.3.10) give invariant solutions

$$F = tG(z), \;\; \text{where} \;\; z = t^{-1}\phi. \tag{6.3.11}$$

The derivatives of equation (6.3.11) with respect t and z reduce (6.3.4) to

$$2zGG'' + 2zG'^2 + 2GG' - 2zG' - F = 0. \tag{6.3.12}$$

The equation (6.3.12) admits

$$\Gamma_1 = z\frac{\partial}{\partial z} + G\frac{\partial}{\partial G}.$$

which is prolonged to

$$\Gamma_1^{[1]} = z\frac{\partial}{\partial z} + G\frac{\partial}{\partial G} + 0\frac{\partial}{\partial G'}.$$

The characteristic equations are defined as

$$\frac{dz}{z} = \frac{dG}{G} = \frac{dG'}{0}.$$

The invariant solutions are given by

$$G'(z) = H(\psi), \;\; \psi = z^{-1}G.$$

Applying the method of differential equations (see equation (4.5.8)), we note that equation (6.3.12) is reduced to

$$2\psi(H-1)H' + 2H(H+\psi-1) - \psi = 0.$$

The general solution is given implicitly by

$$\frac{-\psi K_o\big(0, \sqrt{\psi(\psi+2H-2)}\big) + \sqrt{\psi(\psi+2H-2)}K_o\big(1, -\sqrt{\psi(\psi+2H-2)}\big)}{\psi I_o\big(0, \sqrt{\psi(\psi+2H-2)}\big) + \sqrt{\psi(\psi+2H-2)}I_o\big(1, \sqrt{\psi(\psi+2H-2)}\big)} + c_1 = 0$$

where K_o and I_o are BesselK and BesselI functions of ψ respectively, and c_1 and c_2 are arbitrary constants.

On the other hand, the *inherited symmetry* X_5^* gives the characteristic equations given by

$$\frac{dt}{0} = \frac{d\phi}{\phi} = \frac{dF}{F},$$

which leaves equation (6.3.4) invariant under

$$F = \phi G(t). \tag{6.3.13}$$

The derivatives of equation (6.3.13) transforms equation (6.3.4) to

$$G' - 4G^2 = 0. \tag{6.3.14}$$

The solution of (6.3.1) in its original variables is given by

$$C(t,x,y) = \frac{y^2 + (t-x)^2}{2(c_1 - 4t)},$$

and c_1 is an arbitrary constant.

(b) *For invariance under* X_1, the characteristic equations are given as

$$\frac{dt}{t} = \frac{dx}{x} = \frac{dy}{y} = \frac{dC}{C},$$

which yield invariant solutions

$$C(t, x, y) = tF(\gamma, \phi), \;\; \gamma = \frac{x}{t}, \;\; \phi = \frac{y}{t}. \tag{6.3.15}$$

Substituting the derivatives of (6.3.15) into equation (6.3.1) leads to

$$\frac{\partial}{\partial\phi}\Big(F\frac{\partial F}{\partial\phi}\Big) + \frac{\partial}{\partial\gamma}\Big(F\frac{\partial F}{\partial\gamma}\Big) + \phi\frac{\partial F}{\partial\phi} + \gamma\frac{\partial F}{\partial\gamma} - \frac{\partial F}{\partial\gamma} - F = 0. \tag{6.3.16}$$

Referring to commutator Table 6.2 we note that equation (6.3.16) *inherits two symmetries* X_4 and X_5. It can be shown through change of variables that X_4 and X_5 in new variables are given as

$$\left.\begin{aligned} X_4^* &= \phi\frac{\partial}{\partial\gamma} + (1-\gamma)\frac{\partial}{\partial\phi}, \\ X_5^* &= (\gamma-1)\frac{\partial}{\partial\gamma} + \phi\frac{\partial}{\partial\phi} + 2F\frac{\partial}{\partial F}. \end{aligned}\right\} \tag{6.3.17}$$

We perform reduction by *inherited symmetry* X_4^*. The characteristic equations are given by

$$\frac{d\gamma}{\phi} = \frac{d\phi}{(1-\gamma)} = \frac{dF}{0},$$

which yield invariant solutions

$$F(\gamma, \phi) = G(z), \;\; z = \frac{\phi^2 + (1-\gamma)^2}{2}. \tag{6.3.18}$$

The derivatives of equation (6.3.18) reduce (6.3.16) to an ODE

$$2zGG'' + 2zG'^2 + 2zG' + 2GG' - F' = 0. \tag{6.3.19}$$

This equation does not yield any symmetry.

Next we perform reduction by *inherited symmetry* X_5^* on equation (6.3.16). The characteristic equations are given by

$$\frac{d\gamma}{\gamma-1} = \frac{d\phi}{\phi} = \frac{dF}{2F}. \tag{6.3.20}$$

The equation (6.3.20) yield invariant solutions

$$F(\gamma,\phi)=(\gamma-1)^2G(\psi),\ \ \psi=(\gamma-1)^{-1}\phi. \tag{6.3.21}$$

The derivatives (6.3.21) reduces (6.3.16) to an ODE

$$(\psi^2+1)GG''+(\psi^2+1)G'^2-6GG'+6G^2+G=0.$$

This equation does not yield any symmetry. We also note that generally this case does not yield any solution. We therefore provide the summary of reductions only in Table 6.3. The interested reader can use other methods to solve these reduced equations.

Table 6.3: Double reduction of equation (6.3.1).

Combinations	Reduced Equations
$[X_2, X_1]$	$(1+\gamma^2)(GG''+G'^2)+(1-2F)\gamma G'+(G-1)G=0$ $C(t,x,y)=F(x,y)=xG(\gamma),\ \ \gamma=\dfrac{y}{x}$
$[X_3, X_1]$	$GG''+G'^2+(1+\gamma)G'-G=0$ $C(t,x,y)=F(t,x)=tG(\gamma),\quad \gamma=\dfrac{x}{t}$
$[X_4, X_1]$	$2z(GG''+G'^2+G')+(2G'-1)G=0$ $C(t,x,y)=F(t,\gamma)=tG(\gamma),\quad \gamma=\dfrac{y^2+(t-x)^2}{2t^2}$
$[X_5, X_1]$	$(1+\gamma^2)FF''+(1+\gamma^2)F'^2-6\gamma FF'+6F^2+F=0$ $C(t,x,y)=(x-t)^2F(t,\gamma),\quad \gamma=(x-t)^{-1}y$
$[X_6, X_1]$	$\gamma^2(FF''+F'^2+F')+4\gamma FF'+F^2=0$ $C(t,x,y)=C(t,y)=yF(\gamma),\quad \gamma=\dfrac{y}{t}$

6.4 Case 3: $D = C^2, \quad V = 1.$

In this case we consider the power-law diffusivity with fixed exponent and constant velocity. The equation (6.1.1) is then given by (see for e.g., [153])

$$\frac{\partial}{\partial x}\left(C^2\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(C^2\frac{\partial C}{\partial y}\right) - \frac{\partial C}{\partial x} - \frac{\partial C}{\partial t} = 0, \tag{6.4.1}$$

which admits four dimensional Lie algebra spanned by

$$\left.\begin{aligned} X_1 = 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + C\frac{\partial}{\partial C},\ \ X_2 = \frac{\partial}{\partial t},\ \ X_3 = \frac{\partial}{\partial y}, \\ X_4 = y\frac{\partial}{\partial x} + (t-x)\frac{\partial}{\partial y},\ \ X_5 = (t-x)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - C\frac{\partial}{\partial C}, \\ X_6 = \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{6.4.2}$$

The commutator table for Lie algebra (6.4.2) is given in Table 6.4 below.

Table 6.4: Commutator table of Lie algebra of equation (6.4.2)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$-2X_2$	$-2X_3$	$-2X_3$	0	$-2X_6$
X_2	$2X_2$	0	0	X_3	X_6	0
X_3	$2X_3$	0	0	X_6	$-X_3$	0
X_4	0	$2X_2$	$-X_6$	0	0	X_6
X_5	0	$-X_6$	X_3	0	0	X_6
X_6	$2X_6$	0	0	$-X_6$	$-X_6$	0

6.4.1 Solutions of equation (6.4.1) using (F'/F)-expansion method.

In this subsection we use the $(F'/F)-$expansion method (see for e.g., [163]) and obtain some exact solutions of the corresponding ODE. This will result in the exact

solutions of the equation (6.4.1). We now perform *reduction by* X_5. The characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{t-x} = -\frac{dy}{y} = -\frac{dC}{C}. \tag{6.4.3}$$

The invariant solutions of (6.4.3) are determined to be

$$C(t,x,y) = (t-x)F(t,\phi), \quad \phi = \frac{y}{t-x}. \tag{6.4.4}$$

Substituting the derivatives of (6.4.4) into (6.4.1) we obtain a PDE with two variables t and ϕ given by

$$\begin{aligned}&(\phi^2+1)F^2\Big(\frac{\partial^2 F}{\partial \phi^2}\Big) + 2(\phi^2+1)F\Big(\frac{\partial F}{\partial \phi}\Big)^2 - \Big(\frac{\partial F}{\partial t}\Big)^2 \\ &-4\phi F^2\Big(\frac{\partial F}{\partial \phi}\Big) + 2F^3 = 0.\end{aligned} \tag{6.4.5}$$

The equation (6.4.5) admits the *inherited symmetry* $X_2^* = \partial/\partial t$ which then reduces (6.4.5) to a ODE

$$(\phi^2+1)F^2F'' + 2(\phi^2+1)FF'^2 - 4\phi F^2F' + 2F^3 = 0, \;\; F(t,\phi) = F(\phi). \tag{6.4.6}$$

Let $h = F'/F$. Using the quotient rule of differentiation equation (6.4.6) is reduced to a first order ODE

$$(\phi^2+1)\Big(h' + 3h^2\Big) - 4\phi h + 2 = 0. \tag{6.4.7}$$

It can be shown that the solution of (6.4.7) is

$$h(\phi) = \frac{c_1\phi^2 - 2\phi - c_1}{c_1\phi^3 - 3\phi^2 - 3\phi c_1 + 1}. \tag{6.4.8}$$

For $c_1 = 1$ and c_2 arbitrary and through standard integration process, it can be verified that

$$C(t,x,y) = c_2\sqrt[3]{y^3 - 3y^2(t-x) - 3y(t-x)^2 + (t-x)^3}, \;\; c_1 = 1$$

satisfies equation (6.4.1).

We now perform *reduction by* X_4. The characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{t-x} = \frac{dC}{0}$$

The invariant solutions are

$$C(t,x,y) = F(t,\phi), \quad \phi = \frac{y^2 + (t-x)^2}{2}. \tag{6.4.9}$$

The derivatives of (6.4.9) with respect to t, x and y reduces (6.4.1) to

$$2\phi F^2\Big(\frac{\partial^2 F}{\partial \phi^2}\Big) + 4\phi F\Big(\frac{\partial F}{\partial \phi}\Big)^2 + 2F^2\frac{\partial F}{\partial \phi} - \frac{\partial F}{\partial t} = 0. \tag{6.4.10}$$

The equation (6.4.10) inherits the Lie symmetry X_2 given now as $X_2^* = 2\partial/\partial t$. The invariant solution $F(t,\phi) = F(\phi)$ leaves (6.4.10) unchanged and the resultant ODE is then given by

$$2\phi F^2 F'' + 4\phi F F'^2 + 2F^2 F' = 0. \tag{6.4.11}$$

It can be shown, through substitution $h = F'/F$ that the equation (6.4.11) can be reduced to

$$\phi h' + 3\phi h^2 + h = 0 \tag{6.4.12}$$

Back substitution indicate that the solution of (6.4.1) is

$$C(t,x,y) = \sqrt[3]{\ln\Big(\frac{y^2 + (t-x)^2}{2}\Big)^3 + c_1}, \tag{6.4.13}$$

and c_1 is arbitrary constant.

6.4.2 Reduction by using $w = (z')^{-1}$, where $z = F$.

We employ the method used by (see for e.g., [52]) to transform the PDE (6.4.15) to a simpler DEs. The linear combination

$$X_2 + X_3 + X_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x},$$

leads to the characteristic equations given by

$$\frac{dt}{1} = \frac{dx}{1} = \frac{dy}{1} = \frac{dC}{0}$$

which yield invariant solutions given by

$$C(t,x,y) = F(\gamma,\phi), \;\; \gamma = x - t, \quad \phi = y - t. \tag{6.4.14}$$

The partial derivatives of (6.4.14) reduces (6.4.1) to

$$F^2\frac{\partial^2 F}{\partial\gamma^2}+2F\Big(\frac{\partial F}{\partial\gamma}\Big)^2+F^2\Big(\frac{\partial^2 F}{\partial\phi^2}\Big)+2F\Big(\frac{\partial F}{\partial\phi}\Big)^2+\frac{\partial F}{\partial\phi}=0. \tag{6.4.15}$$

which admits three symmetries

$$\left.\begin{aligned}\Gamma_1&=2\gamma\frac{\partial}{\partial\gamma}+2\phi\frac{\partial}{\partial\phi}+F\frac{\partial}{\partial F},\\ \Gamma_2&=\frac{\partial}{\partial\gamma},\\ \Gamma_3&=\frac{\partial}{\partial\phi}.\end{aligned}\right\} \tag{6.4.16}$$

Reduction by Γ_1 results in transforming equation (6.4.15) to

$$4(z^2+1)G^2G''+8(z^2+1)GG'^2-4zG^2G'+4G'+G^3=0 \tag{6.4.17}$$

where $F(\gamma,\phi)=\gamma^{1/2}G(z),\ \ z=\phi/\gamma$ are invariant solutions. The equation (6.4.17) does not yield any solution.

However, *symmetries* Γ_2 and Γ_3 allows further reductions by using the substitution $w=(z')^{-1}$, where $z=F$.

We note that reduction by Γ_2 leads to a DE

$$F^2F''+2FF'^2+F'=0 \ \text{ where } \ F=F(\gamma). \tag{6.4.18}$$

Let

$$w=(z')^{-1} \ \text{ where } \ z=F. \tag{6.4.19}$$

It can be shown through standard rule of differentiation that

$$z''=-wz'^3. \tag{6.4.20}$$

Substituting (6.4.19) and (6.4.20) into (6.4.18), equation (6.4.18) transforms to

$$z^2w'-w^2-2zw=0 \ \text{ where } \ w=w(z). \tag{6.4.21}$$

It can be verified that the solution (6.4.21) is given by

$$w(z)=\frac{z^2}{c_1-z},$$

where c_1 is a arbitrary constant. Similarly, *reduction by* Γ_3 leads to

$$F^2F'' + 2F'^2 = 0, \tag{6.4.22}$$

which by transformations (6.4.19) and (6.4.20) is reduced to

$$z^2w' + 2w = 0. \tag{6.4.23}$$

The equation is satisfied by

$$w(z) = c_1 \mathrm{e}^{\frac{2}{z}},$$

where c_1 is a arbitrary constant. Back substitution yield an implicit solution of (6.4.1). We note here that the solution of (6.4.22) can be obtained directly using MAPLE and it is given in the form of LambertW functions of z.

6.4.3 Symmetry reduction of equation (6.4.1)

(a) *Reduction by* $X_1 = 2t\dfrac{\partial}{\partial t} + 2x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y} + C\dfrac{\partial}{\partial C}$.

The characteristic equations are defined by

$$\frac{dt}{2t} = \frac{dx}{2x} = \frac{dy}{2y} = \frac{dC}{C}. \tag{6.4.24}$$

Solving (6.4.24) we obtain invariants solutions given by

$$C(t, x, y) = t^{\frac{1}{2}}F(\gamma, \phi), \quad \gamma = \frac{x}{t}, \quad \phi = \frac{y}{t}. \tag{6.4.25}$$

Differentiating equation (6.4.25) with respect to t, x and y reduces (6.4.1) to

$$\begin{aligned} &\frac{\partial}{\partial\gamma}\Big(2F^2\frac{\partial F}{\partial\gamma}\Big) + \frac{\partial}{\partial\phi}\Big(2F^2\frac{\partial F}{\partial\phi}\Big) + 2\gamma\frac{\partial F}{\partial\gamma} \\ &+ 2\phi\frac{\partial F}{\partial\phi} - 2\frac{\partial F}{\partial\gamma} - F = 0. \end{aligned} \tag{6.4.26}$$

From the commutator Table 6.4 we note that (6.4.26) inherits four symmetries

$$\left.\begin{aligned} X_2^* &= 2\gamma\frac{\partial}{\partial\gamma} + 2\phi\frac{\partial}{\partial\phi} + F\frac{\partial}{\partial F}, \\ X_3^* &= \frac{\partial}{\partial\phi}, \\ X_5^* &= (1-\gamma)\frac{\partial}{\partial\gamma} - \phi\frac{\partial}{\partial\phi} - F\frac{\partial}{\partial F}, \\ X_6^* &= \frac{\partial}{\partial\gamma}. \end{aligned}\right\} \tag{6.4.27}$$

Reduction by inherited symmetry X_2^* leads to

$$4(z^2+1)G^2G'' + 8(z^2+1)GG'^2 + 4z(1-G^2)G' + G^3 - 2G = 0, \tag{6.4.28}$$

where $F(\gamma,\phi) = \gamma^{1/2}G(z)$, $z = \phi/\gamma$ are invariant solutions. The equation (6.4.28) does not admit any symmetry.

Reduction by inherited symmetry X_3^* leads to

$$2F^2F'' + 4FF'^2 - 2F'(1-\gamma) - F = 0. \tag{6.4.29}$$

The equation (6.4.29) admits $\Gamma_1 = (\gamma-1)\dfrac{\partial}{\partial\gamma} + F\dfrac{\partial}{\partial F}$.
Using the method of differential equations, it can be shown that the admitted symmetry will leave the ODE invariant under $F' = G(z)$, $z = (\gamma-1)^{-1}F$. These invariant solutions transform (6.4.29) to a first order DE given by

$$2z^2(F-z)F' + 4zF^2 + 2F - z = 0. \tag{6.4.30}$$

Unfortunately equation (6.4.30) does not yield any solution.
Using the same procedure, *reduction by inherited symmetry* X_5^* leads to

$$2(z^2+1)\big(GG'' + 2G'^2\big) - 8zGG' + 4G^2 + 1 = 0, \quad F = G(z) \tag{6.4.31}$$

where $F = (\gamma-1)G(z)$, $z = (\gamma-1)\phi$. The equation (6.4.31) did not yield any solution.

Reduction by inherited symmetry X_6^* leads to invariant solution given by

$$2F^2F'' + 4FF'^2 - 2\phi F' - F = 0, \quad F = F(\phi). \tag{6.4.32}$$

The equation (6.4.32) now admits $\Gamma_1 = \phi\dfrac{\partial}{\partial\phi} + F\dfrac{\partial}{\partial F}$.

Using the method of differential equations, it can be shown that the admitted symmetry will leave the ODE invariant under $F' = G(z),\ z = \phi^{-1}F$. The invariant solutions transform (6.4.32) to a first order DE given by

$$2z^2(G - z^2)G' + 4zG^2 - 2G - z = 0. \tag{6.4.33}$$

The equation (6.4.33) does not yield any solution.

6.5 Case 4: $D = C^m, \quad V(x) = x.$

The DE involves the power function of D with the spatial variable (see for e.g., [162]) and the governing equation thus becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(C^m\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(C^m\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial x}\Big(xC\Big). \tag{6.5.1}$$

The equation (6.4.1) admits Lie algebra spanned by

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= \frac{1}{m}\Big(mx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y} + 2C\frac{\partial}{\partial C}\Big), \\ X_4 &= -\mathrm{e}^t\frac{\partial}{\partial x}. \end{aligned}\right\} \tag{6.5.2}$$

An optimal system of one dimensional sub-algebras is provided by

$$\{X_1 + X_4,\ \ X_3,\ \ X_1 + X_2 + X_4\}.$$

6.5.1 Symmetry Reductions

(a) *Reduction by* $X_1 + X_4$.

The characteristic equations are given by

$$\frac{dt}{1} = -\frac{dx}{\mathrm{e}^t} = \frac{dy}{0} = \frac{dC}{0}.$$

The invariant solutions obtained are

$$C(t, x, y) = F(y, \phi), \quad \phi = x + \exp(t)$$

which after differentiating with respect to t, x and y reduces equation (6.5.1) to

$$\frac{\partial}{\partial \phi}\left(F^m \frac{\partial F}{\partial \phi}\right) + \frac{\partial}{\partial y}\left(F^m \frac{\partial F}{\partial y}\right) - \frac{\partial}{\partial \phi}\Big(\phi F\Big) = 0. \tag{6.5.3}$$

The equation (6.5.3) admits one $\Gamma_1 = \partial/\partial y$. This implies that F is independent of y, and hence

$$F^m F'' + mF^{m-1}F'^2 - \phi F' - F = 0, \quad F = F(\phi). \tag{6.5.4}$$

For $m = 0$, the equation (6.5.4) becomes

$$F'' - \phi F' - F = 0,$$

which yield the solution given by

$$C(t, x, y) = c_1 \mathrm{erf}\Big(\frac{\sqrt{2}(x + \mathrm{e}^t)}{2}\Big)\exp\Big(\frac{(x + \mathrm{e}^t)^2}{2}\Big) + c_2 \exp\Big(\frac{(x + \mathrm{e}^t)^2}{2}\Big),$$

where c_1 and c_2 are arbitrary constants.

We note here that no exact solutions are obtained for $m \neq 0$. Hence we provide only the reduced forms of equation (6.5.4) given in Table 6.5.

Table 6.5: Symmetry reduction of equation (6.5.4).

$m-$ value	Reduced ODE
$m=1$	$FF''+F'^2-\phi F'-F=0$
$m=2$	$F^2F''+2FF'^2-\phi F'-F=0$
$m=-2$	$FF''-2F'^2-\phi F^3F'-F^4=0$
$m=-3/2$	$2FF''-3F'^2-2\phi F^{5/2}F'-2F^{7/2}=0$

(b) *Reduction by* $X_1+X_2+X_4$.

The corresponding symmetry is given by

$$X_1+X_2+X_3=\frac{\partial}{\partial t}+\frac{\partial}{\partial y}-\mathrm{e}^t\frac{\partial}{\partial x},$$

with invariants given by

$$C(t,x,y)=F(\gamma,\phi),\quad \gamma=x+\mathrm{e}^t,\quad \phi=y-t,$$

which reduce (6.5.1) to

$$\frac{\partial F}{\partial \phi}\left(F^m\frac{\partial F}{\partial \phi}\right)+\frac{\partial F}{\partial \gamma}\left(F^m\frac{\partial F}{\partial \gamma}\right)-\frac{\partial F}{\partial \gamma}\left(\gamma F\right)=0. \tag{6.5.5}$$

Equation (6.5.5) admits $\Gamma_1=\partial/\partial\phi$. This implies that F is independent of ϕ, and hence

$$F^mF''+mF^{m-1}F'^2-\gamma F'-F=0,\quad F=F(\gamma). \tag{6.5.6}$$

A summary of reduced forms of ODE is given in Table 6.6 below.

(c) *Reduction by* X_3.

The characteristic equations are given by

$$\frac{dt}{0}=\frac{dx}{mx}=\frac{dy}{my}=\frac{dC}{2C}.$$

The invariant solutions obtained are

$$C(t,x,y)=F(t,\phi),\ \ \phi=\frac{y}{x} \tag{6.5.7}$$

Table 6.6: Symmetry reduction of equation (6.5.6).

$m-$ value	Reduced ODE
$m=1$	$FF''+F'^2-\phi F'-F=0$
$m=2$	$F^2F''+2FF'^2-\phi F'-F=0$
$m=-2$	$FF''-2F'^2-\phi F^3F'-F^4=0$
$m=-3/2$	$2FF''-3F'^2-2\phi F^{5/2}F'-2F^{7/2}=0$

Differentiating (6.5.7) with respect to t, x and y reduces (6.5.1) to

$$\begin{aligned} &m^2(\phi^2+1)\frac{\partial^2}{\partial^2\phi}F^{m+1}+m^3(\phi^2+1)F^m\left(\frac{\partial F}{\partial\phi}\right)^2\\ &-m\phi(4F^m+2mF^{m+1}-mF)\frac{\partial F}{\partial\phi}+m^2F\frac{\partial F}{\partial t}\\ &+2(m+2)F^{m+2}+m(m+2)F^2=0. \end{aligned} \tag{6.5.8}$$

Equation (6.5.8) admits one symmetry $\Gamma_1=\partial/\partial t$. This implies that F is independent of t. Hence

$$\begin{aligned} &m^2(\phi^2+1)F^{m+1}F''+m^3(\phi^2+1)F^mF'^2-m\phi(4F^m+2mF^{m+1}-mF)F'\\ &+2(m+2)F^{m+2}+m(m+2)F^2=0,\quad F=F(\phi). \end{aligned} \tag{6.5.9}$$

For $m=-2$, equation (6.5.9) becomes

$$4(\phi^2+1)FF''-8(\phi^2+1)F'^2+4\phi(F^3+2F+1)F'=0,\quad F=F(\phi). \tag{6.5.10}$$

Unfortunately equation (6.5.10) does not yield any symmetry and no further reduction could be done. We mention here that, equation (6.5.10) can be solved using different mathematical methods. Interested reader is referred to use the numerical, analytical and other mathematical methods to solve equation (6.5.10) (see for e.g., [53–55]).

Case 5: $D = C, \quad V(x) = x.$

In this case, we have a constant coefficient and spatial velocity (see for e.g., [162]) and equation (6.1.1) becomes

$$\frac{\partial}{\partial x}\Big(C\frac{\partial C}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(C\frac{\partial C}{\partial y}\Big) - \frac{\partial}{\partial x}\Big(xC\Big) - \frac{\partial C}{\partial t} = 0. \tag{6.5.11}$$

which admits four-dimensional Lie algebra spanned by

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2C\frac{\partial}{\partial C}, \\ X_4 &= -\mathrm{e}^t\frac{\partial}{\partial x}. \end{aligned}\right\} \tag{6.5.12}$$

6.5.2 Symmetry reductions.

To perform double reduction, we start with reduction by X_2 followed by X_3. Considering the Lie group X_2 we note that (6.5.11) becomes a DE depending on t and x only, that is $C = C(t, x)$. The equation (6.5.11) is then reduced to

$$\frac{\partial}{\partial x}\Big(C\frac{\partial C}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(xC\Big) - \frac{\partial C}{\partial t} = 0. \tag{6.5.13}$$

Since $[X_2, X_3] = \alpha X_2$, the equation (6.5.13) automatically admits X_3 whose characteristic equations yield invariant solution

$$C(t, x) = x^2 F(t). \tag{6.5.14}$$

Substituting the derivatives (6.5.14) into (6.5.13) leads to

$$F' - 6F^2 + 3F = 0.$$

It can be verified that $C(t, x, y) = \dfrac{x^2}{A\mathrm{e}^{3t} + 2}$ is the solution of equation (6.5.11) and where A ia an arbitrary constant.

We continue with reduction on equation (6.5.13) using X_1. We observe that equation (6.5.13) becomes

$$FF'' - F'^2 - xF' - F. \tag{6.5.15}$$

This case is the same as Case 4 above for $m = 1$.

Since $[X_2, X_1] = 0$, the equation (6.5.13) automatically admits X_1 which makes C independent of t. Hence the equation (6.5.13) becomes

$$CC'' + C'^2 - xC' - C = 0. \tag{6.5.16}$$

The equation does not yield any solution.

Considering *reduction by* $X_1 + a_2X_2 + X_4,\ a_2 = 1$. The characteristic equations are given by

$$\frac{dt}{1} = -\frac{dx}{\mathrm{e}^t} = \frac{dy}{1} = \frac{dC}{0}.$$

Hence

$$C = F(\gamma, \phi), \quad \gamma = x + \mathrm{e}^t,\ \ \phi = y - t.$$

Substituting the derivatives of C into (6.5.11) we obtain a PDE of two variables given by

$$\frac{\partial F}{\partial \phi}\Big(F\frac{\partial F}{\partial \phi}\Big) + \frac{\partial F}{\partial \gamma}\Big(F\frac{\partial F}{\partial \gamma}\Big) - \frac{\partial F}{\partial \gamma}\Big(\gamma F\Big) + \frac{\partial F}{\partial \phi} = 0 \tag{6.5.17}$$

which admits one symmetry $\Gamma = \partial/\partial\phi$. This implies that F is independent of ϕ, and hence

$$GG'' + G'^2 - \gamma G' - G = 0, \quad G = G(\gamma). \tag{6.5.18}$$

We note here that this case does not yield any solution for any combination. We recommend that the interested reader can use other methods such as numerical methods to solve the above ODEs.

Case 6: $D = C^m$, $\quad V(x) = x^2$.

In this case, we have a power diffusivity coefficient with spatial velocity (see for e.g., [51, 162]) and equation (6.1.1) becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial y}\Big(C^m \frac{\partial C}{\partial y}\Big) + \frac{\partial}{\partial x}\Big(C^m \frac{\partial C}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(x^2 C\Big) \tag{6.5.19}$$

which admits two symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}. \end{aligned}\right\} \tag{6.5.20}$$

(a) *Reduction by* $X_1 + X_2 = \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial y}$.

The characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{1} = \frac{dC}{0}.$$

Equation (6.5.19) is invariant under

$$C(t, x, y) = F(x, \phi), \quad \phi = y - t.$$

Equation (6.5.21) is reduced to

$$\frac{\partial F}{\partial \phi}\left(F^m \frac{\partial F}{\partial \phi}\right) + \frac{\partial F}{\partial x}\left(F^m \frac{\partial F}{\partial x}\right) - \frac{\partial F}{\partial x}\left(x^2 F\right) + \frac{\partial F}{\partial \phi} = 0 \tag{6.5.21}$$

which admits one symmetry $\Gamma_1 = \partial/\partial\phi$. This implies that F is independent of ϕ, and hence

$$G^m G'' + m G^{m-1} G'^2 - x^2 G' - 2xG = 0, \;\; G = G(x). \tag{6.5.22}$$

If $m = 0$, equation (6.5.22) has the solution

$$C(x) = x^{-\frac{5}{2}}\Big\{\exp\big(c_1 \frac{x^3}{6}\big)\Big[x^3 W_o\big(\frac{1}{6}, \frac{2}{3}, \frac{x^3}{3}\big) + 4W_o\big(\frac{7}{6}, \frac{2}{3}, \frac{x^3}{3}\big)\Big] + c_2 \exp\big(\frac{x^3}{3}\big)\Big\}.$$

where W_o is the Whittaker functions of x, and c_1 and c_2 are arbitrary constants. The remaining reduced forms do not yield any solution(s), however they are provided in Table 6.7 below. Interested reader can use other methods to obtain their solutions.

Table 6.7: Symmetry reduction of equation (6.5.22).

$m-$ value	Reduced ODE
$m=1$	$GG''-G'^2-x^2G^2G'-2xG=0$
$m=-2$	$GG''-2G'^2-x^2G^3G'-2xG^4=0$
$m=-4/3$	$3GG''-4G'^2-3x^2G^{7/3}G'-6xG^{10/3}=0$

(b) *Reduction by* $X_2=\dfrac{\partial}{\partial y}$.

The characteristic equations are given by

$$\frac{dt}{0}=\frac{dx}{0}=\frac{dy}{1}=\frac{dC}{C}. \tag{6.5.23}$$

The solution of equation (6.5.23) leaves (6.5.19) invariant, hence $C(t,x,y)=C(t,x)$. We thus have

$$\frac{\partial}{\partial x}\Big(C^m\frac{\partial C}{\partial x}\Big)-\frac{\partial}{\partial x}\Big(x^2C\Big)-\frac{\partial C}{\partial t}=0. \tag{6.5.24}$$

The equation (6.5.24) inherits symmetry X_1. Under this symmetry, equation (6.5.24) is further reduced to

$$C^mC''+mC^{m-1}C'^2-x^2C'-2xC=0. \tag{6.5.25}$$

Further on, equation (6.5.24) admits

$$\Gamma_2=-mt\frac{\partial}{\partial t}+mx\frac{\partial}{\partial x}+3C\frac{\partial}{\partial C}.$$

The characteristic equations are given by

$$C(t,x,y)=t^{-3/m}F(\gamma),\ \ \gamma=xt. \tag{6.5.26}$$

Differentiating (6.5.26) with respect to t and z and substituting them in (6.5.24) we obtain

$$mF^{m+1}F''-m^2F^mF'^2+\gamma^2F^2+m\gamma F'F=0. \tag{6.5.27}$$

Case 7: $D = e^{mC}$, $\quad V(x) = x.$

The governing equation is

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(\mathrm{e}^{mC}\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathrm{e}^{mC}\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial x}\Big(xC\Big). \tag{6.5.28}$$

which admits three symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= -\mathrm{e}^t\frac{\partial}{\partial x}. \end{aligned}\right\} \tag{6.5.29}$$

6.5.3 Symmetry reductions.

(a) *Reduction by* $X_2 + X_3$.

The characteristic equations are given by

$$\frac{dt}{0} = -\frac{dx}{\mathrm{e}^t} = \frac{dy}{1} = \frac{dC}{0}. \tag{6.5.30}$$

The invariant solution is

$$\left.\begin{aligned} C &= F(t,\phi), \\ I_1 &= t, \\ I_2 = \phi &= y + x\mathrm{e}^{-t}. \end{aligned}\right\} \tag{6.5.31}$$

Hence (6.5.28) reduces to

$$\mathrm{e}^{mF}\big(\mathrm{e}^{-2t}+1\big)\frac{\partial^2 F}{\partial\phi^2} + m\mathrm{e}^{mF}\big(\mathrm{e}^{-2t}+1\big)\left(\frac{\partial F}{\partial\phi}\right)^2 - \frac{\partial F}{\partial t} - F = 0. \tag{6.5.32}$$

Equation (6.5.32) admits two symmetries

$$\Gamma_1 = \frac{\partial F}{\partial t}, \quad \Gamma_2 = \frac{\partial F}{\partial \phi}.$$

Hence (6.5.32) is further reduced by Γ_1 to

$$\mathrm{e}^{mF}\big(\mathrm{e}^{-2t}+1\big)F''+m\mathrm{e}^{mF}\big(\mathrm{e}^{-2t}+1\big)F'^2-F=0,\quad F=F(\gamma). \tag{6.5.33}$$

If $m=0$ then equation (6.5.33) becomes

$$\big(\mathrm{e}^{-2t}+1\big)F''-F=0. \tag{6.5.34}$$

The solution of equation (6.5.34) in its original variables is given by

$$C(t,x,y)=c_1\exp\bigg(\exp\bigg(\frac{(2t-1)(y+x\mathrm{e}^{-t})}{2}\bigg)\bigg)$$
$$+c_2\exp\bigg(-\exp\bigg(\frac{(2t-1)(y+x\mathrm{e}^{-t})}{2}\bigg)\bigg),$$

where c_1 and c_2 are arbitrary constants. We note that the symmetry Γ_2 reduces equation (6.5.33) to

$$F'+F=0,\quad F(t). \tag{6.5.35}$$

The solution of equation (6.5.35) in its original variables is given by

$$C(t)=c_1\mathrm{e}^{-t},$$

where c_1 is an arbitrary constant. The summary reductions is given in Table 6.8 below.

Table 6.8: Symmetry reduction of equation (6.5.28).

Symmetries	Reduced equations	Solutions in original variable
X_1+X_2	$\mathrm{e}^{mF}F''+m\mathrm{e}^{mF}F'^2-xF'-F=0$ $C=F(x)$	$C(x)=\exp\big(\frac{x}{\sqrt{2}}\big)\Big\{c_1\mathrm{erf}\big(\frac{x}{2}\big)+c_2\Big\}$ where c_1 and c_2 are arbitrary constants.
$X_1+X_2+X_3$	$F''-\gamma F'-F=0.$ $F=F(\gamma),\ \ \gamma=x+\mathrm{e}^t$	$C(t,x)=\exp\big(\frac{x+\mathrm{e}^t}{\sqrt{2}}\big)\Big\{c_1\mathrm{erf}\Big(\frac{x+\mathrm{e}^t}{2}\Big)+c_2\Big\}$ where and c_1 and c_2 are arbitrary constants.

6.6 Conclusion

The concept of inherited symmetry is applied here. Double reduction is performed using firstly, linear combination of symmetries obtained from commutators and later inherited symmetries. In some cases where equations admitted two-dimensional Lie algebra, following reduction by one, the other symmetry is used to reduce the two-dimensional PDE to ODE. Transformations $h = G'/G$ and $w(z) = (z')^{-1}$ are used successfully to reduce the second DE to the first order DE. The case where $D = C^2$, $V = 1$, $D = C^m$ do not yield any solutions using symmetry methods. However, we refer the interested reader to use other mathematical methods such as numerical and analytical to solve the reduced ODE.

Chapter 7

Nonlocal symmetries and a family of general solutions of a convection-dispersion equation

7.1 Introduction

In this chapter, we further consider the one-dimensional CDE in solute transport model defined by (2.2.6). We investigate existence of possible hidden symmetries called potential symmetries.

The theoretical background of our focus is a time dependent evaporation, spreading and emulsification of oil concentration $C(t, x)$ along the sea. Our study is based on a one-dimensional transient model studied by Shen [3]. Assumption is that the movement is very slow, making the dynamic pressure quite negligible. It is also assumed that the water behaves as Newtonian fluid allowing the viscosity in the oil relatively small. For this study we focus on spreading only. Other weathering processes are neglected making the density and viscosity constant [46]. The retardation $R = 1$ and the source term ρ are ignored. The diffusion coefficient depends on the concentration of oil only, that is $D = D(C)$. Under these conditions, equation (4.2.1) becomes a

convection-diffusion equation given by

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial C}{\partial x} - VC\right], \tag{7.1.1}$$

This equation has been studied in [53–55] with appropriate initial and boundary conditions. Direct group classification of D and V leads six cases summarised in the table below, for which potential symmetries can exist.

Table 7.1: Cases of D and V for which equation (7.1.1) admit extra symmetries

$D(C)$	$V(x)$
$(\alpha+\beta C)^m,\ \ m \in \mathcal{R}$	$n \in \mathcal{R}$
$(\alpha+\beta C)^m,\ \ m \in \mathcal{R}$	$c_1x + c_2,\ \ \text{where}\ \ c_1, c_2 \in \mathcal{R}$
1	$n \in \mathcal{R}$
1	$c_1x + c_2,\ \ \text{where}\ \ c_1, c_2 \in \mathcal{R}$
$e^{mC},\ \ m \in \mathcal{R}$	$n \in \mathcal{R}$
$e^{mC},\ \ m \in \mathcal{R}$	$c_1x + c_2,\ \ c_1, c_2 \in \mathcal{R}$

We note that the case $(\alpha+\beta C)^m$, $V = n \in \mathcal{R}$ leads to admitted Lie point symmetries which has been considered in Case 1 of Section 4.3. This include $m = 0,\ n = 1$ cases. These symmetries were obtained in [154]. Two cases where $D = 1$ and $V = c_1x + c_2$ or $V = n$, for $n \in \mathcal{R}$ possess only one potential symmetry. Once the potential symmetries are obtained, they are used to reduce a PDE to a simpler ordinary differential equation which is easier to solve. This approach is outlined in Subsection 7.3 below, has been used by numerous researchers such as [91, 131].

7.2 Calculation of potential symmetries

7.2.1 Case 1: $D = (\alpha + \beta C)^m$, $V = n$.

In this case (7.1.1) becomes (see for e.g., [48, 49] with $\alpha = 0$ and $\beta = 1$.)

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big((\alpha + \beta C)^m \frac{\partial C}{\partial x} - nC)\Big), \tag{7.2.1}$$

Introducing a new variable ω lead to $\mathcal{S}(t, x, C, \omega)$:

$$\left.\begin{aligned} \omega_x &= C, \\ \omega_t &= (\alpha + \beta C)^m C_x - nC, \end{aligned}\right\} \tag{7.2.2}$$

which admits five symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + (x + nt)\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial \omega}, \\ X_3 &= \frac{1}{\alpha}\Big(- \beta(mt + 2t)\frac{\partial}{\partial t} - \beta(x + mnt + nt)\frac{\partial}{\partial x} \\ &\quad + (\alpha + \beta C)\frac{\partial}{\partial C} + \alpha(x - nt)\frac{\partial}{\partial \omega}\Big), \\ X_4 &= \frac{\partial}{\partial \omega}, \\ X_5 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{7.2.3}$$

The infinitesimal generators (7.2.3) are not potential symmetries of (7.2.1) because the condition (3.2.27) is not satisfied.

7.2.2 Case 2: $D = (\alpha + \beta C)^m$, $V = c_1 x + c_2$.

In this case (7.1.1) becomes (see for e.g., [51] with $\alpha = 0$ and $\beta = 1$.)

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big((\alpha + \beta C)^m \frac{\partial C}{\partial x} - (c_1 x + c_2)C)\Big), \tag{7.2.4}$$

Introducing a new variable ω lead to $\mathcal{S}(t,x,C,\omega)$:

$$\left.\begin{aligned}\omega_x &= C,\\ \omega_t &= (\alpha+\beta C)^m C_x - (c_1 x + c_2)C,\end{aligned}\right\} \tag{7.2.5}$$

which admits three symmetries.

$$\left.\begin{aligned}X_1 &= \frac{\partial}{\partial t},\\ X_2 &= \frac{\partial}{\partial \omega},\\ X_3 &= -\mathrm{e}^t \frac{\partial}{\partial x}.\end{aligned}\right\} \tag{7.2.6}$$

The infinitesimal generators (7.2.6) are not potential symmetries of (7.2.8) because the condition (3.2.27) is not satisfied.

7.2.3 Case 3: $D = 1, \quad V = n.$

We consider the solute model where diffusion coefficient and the velocity are both constants. Equation (7.1.1) in conserved form is $\mathcal{R}(t,x,C,\omega)$ is given by (see for e.g., [162]):

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big[\frac{\partial C}{\partial x} - nC\Big]. \tag{7.2.7}$$

Introducing a new variable ω lead to $\mathcal{S}(t,x,C,\omega)$:

$$\left.\begin{aligned}\omega_x &= C,\\ \omega_t &= C_x - nC,\end{aligned}\right\} \tag{7.2.8}$$

which admits infinite symmetries

$$\left.\begin{aligned} X_1 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + (2\omega(nt-x) + C(2ntx - n^2t^2 - 6t - x^2))\frac{\partial}{\partial C} \\ +\omega(2ntx - n^2t^2 - 2t - x^2)\frac{\partial}{\partial \omega}, \\ X_2 = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + n(\omega + xC - ntC)\frac{\partial}{\partial C} + \omega(nx - n^2t + 2)\frac{\partial}{\partial \omega}, \\ X_3 = \frac{\partial}{\partial t},\ \ X_4 = C\frac{\partial}{\partial C} + \omega\frac{\partial}{\partial \omega},\ \ X_5 = \frac{\partial}{\partial x}, \\ X_6 = 2t\frac{\partial}{\partial x} + tC(1-n)\frac{\partial}{\partial C} + \omega(nt - x - 1)\frac{\partial}{\partial \omega}, \\ X_\infty = T(t,x,\omega)\frac{\partial}{\partial C} + H(t,x,\omega)\frac{\partial}{\partial \omega}. \end{aligned}\right\} \quad (7.2.9)$$

By definition of potential symmetry, and the fact that condition (3.2.27) is met, it is clear that only symmetries and X_1 and X_2 are the only potential symmetry.

Sub-Case(a) Reduction by X_1.

The corresponding characteristic system related to the invariant conditions :

$$\begin{aligned} 4t^2C_t + 4txC_x - 2\omega(nt-x) - (2ntx - n^2t^2 - 6t - x^2)C = 0 \\ 4t^2\omega_t + 4tx\omega_x - (2ntx - n^2t^2 - 2t - x^2)\omega = 0, \end{aligned} \quad (7.2.10)$$

and the characteristic equations are given by

$$\frac{dt}{4t^2} = \frac{dx}{4tx} = \frac{dC}{(2\omega(nt-x) + C(2ntx - n^2t^2 - 6t - x^2)} = \frac{d\omega}{\omega(2ntx - n^2t^2 - 2t - x^2)}. \quad (7.2.11)$$

Solving the characteristic system (7.2.11) gives the following system of integrals $\mathcal{S}(t,x,C,v)$:

$$\begin{aligned} I_1 &= \frac{x}{t}, \\ I_2 &= \omega x^{\frac{1}{2}}\exp\left(-\frac{2ntx - n^2tx - x^2}{4t}\right), \\ I_3 &= C(t,x)x^{\frac{3}{2}}\exp\left(-\frac{2ntx - n^2t^2 - x^2}{4t}\right) + \frac{x(x-nt)}{2t}I_2. \end{aligned} \quad (7.2.12)$$

Let $I_1 = z$ be the similarity variable with $I_2 = h_1(z)$ and $I_3 = h_2(z)$ then

$$z = \frac{x}{t}, \tag{7.2.13}$$
$$\omega = h_1(z)x^{-\frac{1}{2}}\exp\left(\frac{2ntx - n^2t^2 - x^2}{4t}\right),$$
$$C(t,x) = x^{-\frac{3}{2}}\exp\left(\frac{2ntx - n^2t^2 - x^2}{4t}\right)\left[h_2(z) - \frac{x(x-nt)}{2t}h_1(z)\right].$$

Elimination of ω from the invariant surface conditions (7.2.10) leads to a partial differential equation

$$\begin{aligned}\Omega^* : &16t^4C_{tt} + 16t^2x^2C_{xx} + 32t^3xC_{tx} + (8n^2t^4 - 16nt^3x + 8t^2x^2\\ &+48t^3)C_t + (8n^2t^3x - 16nt^2x^2 + 48t^2x + 8tx^3)C_x + (n^4t^4 - 4n^3t^3x\\ &+12n^2t^3 + 6n^2t^2x^2 - 24nt^2x - 4ntx^3 + 12t^2 + 12tx^2 + x^4)C = 0. \end{aligned}\tag{7.2.14}$$

We introduce $(7.2.13)_3$ into (7.2.7) and arrive at a second order ordinary differential equation $T(t,x,C,\omega)$:

$$\begin{aligned}&4(z^3 - z^4)h_1'' - 4(nz^2 + z^3)h_1' + (z^2 + 3nz)h_1 + 8z^2h_2\\ &+t(8z^2h_2'' + 30h_2' - 24zh_2) = 0,\end{aligned}\tag{7.2.15}$$

which must hold for any value of t. In this way we obtain a system

$$\begin{aligned}4z^2h_2'' + 15h_2' - 12zh_2 &= 0,\\ 4(z^3 - z^4)h_1'' - 4(nz^2 + z^3)h_1' + (z^2 + 3nz)h_1 + 8z^2h_2 &= 0.\end{aligned}\tag{7.2.16}$$

Solving (7.2.16) yields

$$\begin{aligned}h_2 &= t^{-5/2}x^{3/2}\big(b_1x + b_2t\big),\\ h_1 &= \frac{3(a_2z^{\frac{1}{2}} + a_1z^{\frac{3}{2}}) - z^{\frac{5}{2}}(3b_2 + b_1z)}{3(n-z)},\end{aligned}\tag{7.2.17}$$

where $z = \dfrac{x}{t}$.

The family of solutions S_F^* are therefore given by

$$\begin{aligned}C(t,x) = x^{-\frac{3}{2}}&\exp\left(\frac{2ntx - n^2t^2 - x^2}{4t}\right)\left[t^{-5/2}x^{3/2}\big(b_1x + b_2t\big) - \right.\\ &\left.\frac{x(x-nt)}{2t}\left(\frac{3(a_2z^{\frac{1}{2}} + a_1z^{\frac{3}{2}}) - z^{\frac{5}{2}}(3b_2 + b_1z)}{3(n-z)}\right)\right].\end{aligned}\tag{7.2.18}$$

Substituting (7.2.13) into the auxiliary system and obtain the solutions $\mathcal{S}_F$:

$$2zh_1' - h_1 - 2h_2 = 0, \tag{7.2.19}$$
$$2zh_2' - 3h_2 = 0.$$

It can be verified that

$$h_2 = b_1 z^{3/2}, \tag{7.2.20}$$
$$h_1 = (b_1 z + a_1) z^{\frac{1}{2}},$$

are solutions of (7.2.19). The solution S_F is therefore given by

$$C(t,x) = \left\{ \frac{2b_1 t - (x-nt)(b_1 x + a_1 t)}{2t^{\frac{5}{2}}} \right\} \exp\left(- \frac{(x-nt)^2}{4t} \right), \tag{7.2.21}$$

where a_1 and b_2 are arbitrary constants and equation (7.2.21) satisfies

$$\Omega : 4t^2(2ntx - n^2t^2 - 2t - x^2)C_t + (8t^2x^2n - 4t^3xn^2 - 4tx^3 - 8nt^3)C_x$$
$$+(4n^3t^3x + 4ntx^3 - x^4t^4 - 12t^2 - x^4 - 6n^2t^2x^2)C = 0. \tag{7.2.22}$$

Sub-Case(b) Reduction by X_2.

The corresponding characteristic system related to the invariant conditions :

$$4tC_t + 2xC_x - n\omega - (nx - n^2t)C = 0, \tag{7.2.23}$$
$$4t\omega_t + 2x\omega_x - (nx - n^2t + 2)\omega = 0,$$

is given by

$$\frac{dt}{4t} = \frac{dx}{2x} = \frac{dC}{n\omega + (x-nt)nC} = \frac{d\omega}{\omega(nx - n^2t + 2)}. \tag{7.2.24}$$

Solving the characteristic system (7.2.24) gives the following three integrals

$$I_1 = \frac{x}{\sqrt{t}}, \tag{7.2.25}$$

$$I_2 = \omega x^{-1} \exp\left(\frac{n^2t - 2nx}{4} \right), \tag{7.2.26}$$

$$I_3 = C(t,x)\exp\left(\frac{n^2t - 2nx}{4} \right) - \frac{nx}{2} I_2. \tag{7.2.27}$$

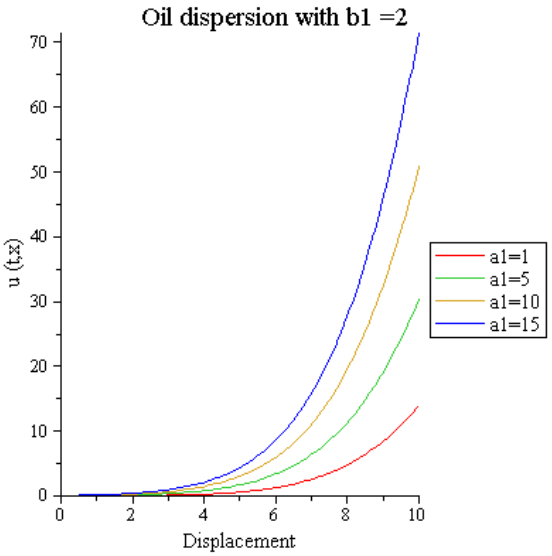


Figure 7.1: Oil dispersion for equation (7.2.21) after $t = 10$

Let $I_1 = z$ be the similarity variable with $I_2 = h_1(z)$ and $I_3 = h_2(z)$ then the solution of the invariant surface condition is

$$C(t,x) = \exp\left(\frac{2nx - n^2t}{4}\right)\left(h_2(z) + \frac{nx}{2}h_1(z)\right), \tag{7.2.28}$$

$$\omega = h_1(z)x\exp\left(\frac{2nx - n^2t}{4}\right), \tag{7.2.29}$$

$$z = \frac{x}{\sqrt{t}}. \tag{7.2.30}$$

Elimination of ω from the invariant surface conditions (7.2.23) leads to a partial differential equation

$$\Omega^* : 16t^2C_{tt} + 4x^2C_{xx} + 16txC_{tx} + 8t(n^2t - nx + 1)C_t + 4nx(nt - x)C_x + n^2(x^2 + 2t - 2ntx + n^2t^2)C = 0. \tag{7.2.31}$$

The family of solutions $\mathcal{S}_F^*$ for (7.2.31) is given by $(7.2.28)_1$. To find this family of solutions, we introduce $(7.2.28)_1$ into (7.2.7) to reduce it to a second order ODE

given by

$$4h_2'' + 2\sqrt{t}nzh_1'' + 2zh_2' + (4+z^2)n\sqrt{t}h' = 0, \tag{7.2.32}$$

which must hold for all positive values of $t \in \mathcal{R}$. The system obtained is then given by

$$\begin{aligned} 2h_2'' + zh_2' &= 0, \\ 2zh_1'' + (4+z^2)h_1' &= 0. \end{aligned} \tag{7.2.33}$$

Solving (7.2.33) yields

$$h_2 = \text{erf}\big(\frac{x}{2\sqrt{t}}\big)b_2 + b_1, \tag{7.2.34}$$

$$h_1 = \left(\frac{2\sqrt{t}}{x}\exp\big(-\frac{x^2}{4t}\big) + \sqrt{\pi}\text{erf}\big(\frac{x}{2\sqrt{t}}\big)\right)a_2 + a_1,$$

where a_i and b_i are arbitrary constants. The family of solutions S_F^* are therefore given by

$$C(t,x) = \exp\left(\frac{2nx - n^2t}{4}\right)\left\{\text{erf}\big(\frac{x}{2\sqrt{t}}\big)b_2 + b_1 \right. \tag{7.2.35}$$

$$\left. + \frac{a_2 nx}{2}\Big[\frac{2\sqrt{t}}{x}\exp\big(-\frac{x^2}{4t}\big) + \sqrt{\pi}\text{erf}\big(\frac{x}{2\sqrt{t}}\big)\Big] + \frac{a_1 nx}{2}\right\},$$

where erf is an error function of x and t and a_i and b_i are arbitrary constants. The solution of equation (7.2.35) is obtained by substituting equation (7.2.28) into the auxiliary system (7.2.8) and obtain a system

$$zh_1' + h_1 - h_2 = 0, \tag{7.2.36}$$

$$z^2h_1' + 2h_2' = 0, \tag{7.2.37}$$

with solutions given by

$$h_1 = \left(\frac{2\sqrt{t}}{x}\exp\left(-\frac{x^2}{4t}\right) + \sqrt{\pi}\text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)a_1 \tag{7.2.38}$$

$$h_2 = \left(\sqrt{\pi}\text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)a_1.$$

The solution S_F is therefore given by

$$C(t,x) = \exp\Big(\frac{2nx - n^2t}{4}\Big)\Big\{\sqrt{\pi}\mathrm{erf}\big(\frac{x}{2\sqrt{t}}\big)a_1 \tag{7.2.39}$$

$$+\frac{a_2 nx}{2}\Big(\frac{2\sqrt{t}}{x}\exp\big(-\frac{x^2}{4t}\big)\Big)\Big\},$$

where a_i and b_i are arbitrary constants. It can be shown that equation (7.2.39) is a solution of

$$\Omega : 4t(nx - n^2t + 2)C_t + (2nx^2 - 2n^2tx + 4x - 4nt)C_x \tag{7.2.40}$$

$$+(6n^2t - 4nx - n^2x^2 + 2n^3tx - n^4t^2)C = 0.$$

Case 4: $D = 1, \quad V = c_1x + c_2.$

We write (7.1.1) in conversed form (see for e.g., [164]):

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial C}{\partial x} - (c_1x + c_2)C\Big), \tag{7.2.41}$$

and with introduction of a new variable ω we obtain

$$\left.\begin{aligned} \omega_x &= C, \\ \omega_t &= C_x - nC, \end{aligned}\right\} \tag{7.2.42}$$

which admits four symmetries. We consider the case where $c_1 = 1$ and $c_2 = 0$.

$$\left.\begin{aligned} X_1 &= \omega\frac{\partial}{\partial\omega} + C\frac{\partial}{\partial C} \\ X_2 &= \mathrm{e}^t\Big\{\frac{\partial}{\partial x} + (\omega + xC)\frac{\partial}{\partial C} + x\omega\frac{\partial}{\partial\omega}\Big\} \\ X_3 &= \mathrm{e}^{-t}\frac{\partial}{\partial x} \\ X_\infty &= T(\tau,x,\omega)\frac{\partial}{\partial\omega} + H(\tau,x,\omega)\frac{\partial}{\partial\omega} \end{aligned}\right\} \tag{7.2.43}$$

By definition of potential symmetries, we conclude that X_2 is the only hidden symmetry of equation (7.2.42).

Sub-Case(a) Reduction by X_2.

The corresponding characteristic system related to the invariant conditions :

$$\left.\begin{aligned} C_x - \omega - xC = 0, \\ \omega_x - x\omega = 0. \end{aligned}\right\} \tag{7.2.44}$$

is given by

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dC}{\omega + xC} = \frac{d\omega}{x\omega}. \tag{7.2.45}$$

Solving the characteristic system (7.2.45) gives the following system of integrals

$$I_1 = t, \tag{7.2.46}$$

$$I_2 = \omega \exp\left(-\frac{x^2}{2}\right).$$

$$I_3 = C(t,x)\exp\left(-\frac{x^2}{2}\right) - xI_2.$$

Let $I_1 = z$ be the similarity variable with $I_2 = h_1(z)$ and $I_3 = h_2(z)$ then

$$\begin{aligned} z &= t, \\ \omega &= h_1(z)\exp\left(\frac{x^2}{2}\right), \\ C(t,x) &= \exp\left(\frac{x^2}{2}\right)\left(h_2(z) + xh_1(z)\right). \end{aligned} \tag{7.2.47}$$

Elimination of ω in the invariant surface conditions leads to a partial differential equation

$$\Omega^* : C_{xx} - 2xC_x + (x-1)C = 0. \tag{7.2.48}$$

We introduce $(7.2.47)_3$ into (7.2.7) and arrive at first order ODE

$$h_2' - h_2 + xh_1' - 2xh_1 = 0, \tag{7.2.49}$$

which must hold for any value of t. In this way we obtain a system

$$\begin{aligned} h_2' - h_2 = 0, \\ h_1' - 2h_1 = 0. \end{aligned} \tag{7.2.50}$$

Solving (7.2.50) yields

$$h_2 = b_1 \mathrm{e}^t,$$
$$h_1 = a_1 \mathrm{e}^{2t}. \tag{7.2.51}$$

The family of solution S_F^* is therefore given by

$$C(t,x) = \exp\left(\frac{x^2}{2}\right)\left(b_1 \mathrm{e}^t + x a_1 \mathrm{e}^{2t}\right), \tag{7.2.52}$$

where a_1 and b_1 are arbitrary constants. The solution of equation (7.2.3) is obtained by substituting (7.2.47) into the axillary system and obtain

$$h_2 = 0, \tag{7.2.53}$$
$$h_1' - h_1 = 0.$$

It can be verified that

$$h_2 = 0, \tag{7.2.54}$$
$$h_1 = a_1 \mathrm{e}^t.$$

The solution S_F is therefore given by

$$C(t,x) = \exp\left(\frac{x^2}{2}\right)\left\{a_1 \mathrm{e}^t + x a_2 \mathrm{e}^{2t}\right\}, \tag{7.2.55}$$

where a_i and b_i are arbitrary constants and equation (7.2.55) satisfies

$$\Omega : xC_x - (x^2+1)C = 0.$$

7.2.4 Case 5: $D = e^{mC}$, $V = n$.

In this case (7.1.1) becomes (see for e.g., [161])

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial C}{\partial x} - nC\right), \tag{7.2.56}$$

which admits infinite symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + (nt+x)\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial \omega}, \\ X_3 &= -mt\frac{\partial}{\partial t} - mnt\frac{\partial}{\partial x} + \frac{\partial}{\partial C} + (x-nt)\frac{\partial}{\partial \omega}, \\ X_4 &= \frac{\partial}{\partial \omega}, \\ X_5 &= \frac{\partial}{\partial x}, \\ X_\infty &= T(t,x,\omega)\frac{\partial}{\partial C} + H(t,x,\omega)\frac{\partial}{\partial \omega}. \end{aligned}\right\} \tag{7.2.57}$$

The infinitesimal generators (7.2.57) are not potential symmetries of (7.2.8) because the condition (3.2.27) is not satisfied.

7.2.5 Case 6: $D = e^{mC}$, $V = c_1 x + c_2$.

In this case (7.1.1) becomes (see for e.g., [161])

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial C}{\partial x} - (c_1 x + c_2)C\Big), \tag{7.2.58}$$

which admits infinite symmetries given by

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t} \\ X_2 &= \omega\frac{\partial}{\partial \omega} + C\frac{\partial}{\partial C} \\ X_\infty &= T(\tau,x,\omega)\frac{\partial}{\partial \omega} + H(\tau,x,\omega)\frac{\partial}{\partial \omega} \end{aligned}\right\} \tag{7.2.59}$$

The infinitesimal generators (7.2.59) are not potential symmetries of (7.2.8) because the condition (3.2.27) is not satisfied.

7.3 Steady state solution.

In this section, we consider the integrable steady equation with $D = 1$ and $V = a_1x + a_2$ given by

$$\frac{\partial}{\partial x}\Big[\frac{\partial C}{\partial x} - VC\Big] = 0. \tag{7.3.1}$$

Krukier *et al* [113] considered the numerical solution of the steady convection-diffusion equation with dominant convection. Other researchers [55, 112] also considered the steady state of (7.3.1). The solution of (7.3.1) is then given by

$$C' - (a_1x + a_2)C = c_1, \tag{7.3.2}$$

where c_1 is an arbitrary constant. A number of sub-cases occur. We consider three sub-cases where $a_1 = 0$, $a_2 = 0$ and $a_1 = a_2 \neq 0$.

Case(a) $a_1 = 0$.

$$C' - a_2C = c_1,\ a_2 = 1, \tag{7.3.3}$$

The solution of (7.3.3) is given by

$$C = c_2\mathrm{e}^x + c_1, \tag{7.3.4}$$

where c_1 is an arbitrary constant and the solution of equation (7.3.4) is represented graphically as shown in Figure 7.2 below.

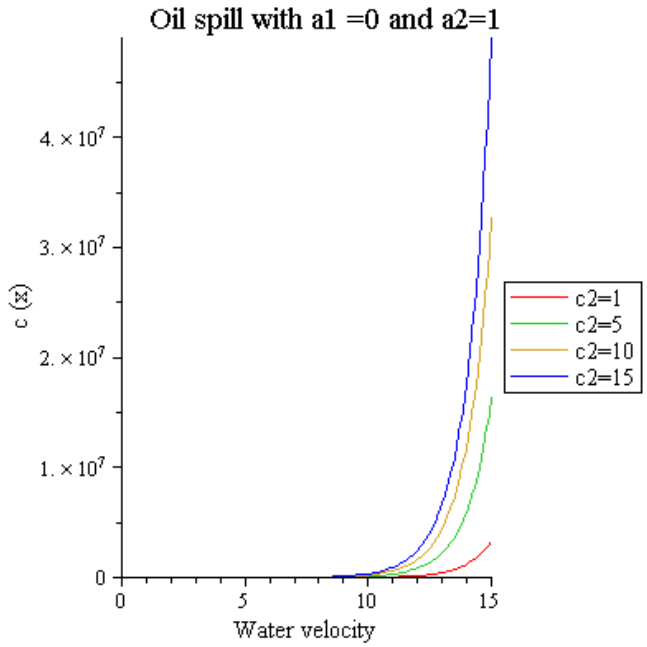


Figure 7.2: Steady state oil dispersion for equation (7.3.2).

where c_1 and c_2 are arbitrary constants.

Case(b) $a_2 = 0$.

The equation (7.3.2) becomes

$$C' - a_1 x C = c_1.$$

The solution is given in terms of Error functions

$$C(x) = \mathrm{e}^{x^2/2}\Big\{\mathrm{erf}\big(\frac{2x}{\sqrt{2}}\big)c_1 + c_2\Big\}, \tag{7.3.5}$$

where c_1 and c_2 are arbitrary constants and erf is the Error function of x and t. The solutions of equation (7.3.5) is represented in Figures 7.3 and 7.4 below

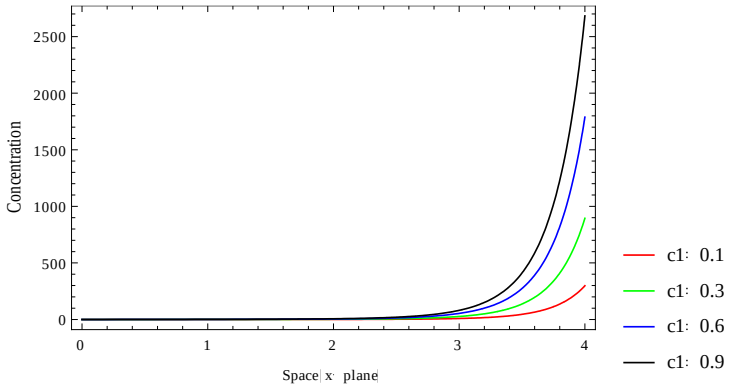


Figure 7.3: Graphical representation of equation (7.3.5), c2=0.

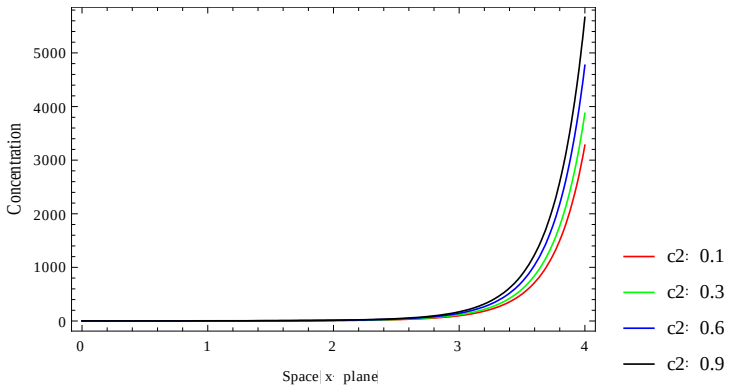


Figure 7.4: Graphical representation of equation (7.3.5), c1=1.

Case(c) $a_1 = a_2 \neq 0$.

In this case, the solution is

$$C(x) = \exp\Big(\frac{a_1x^2 + 2a_2x}{2}\Big)\Big\{\frac{\sqrt{2\pi}}{2\sqrt{a_1}}e^{{a_2}^2/a_1}\mathrm{erf}\Big(\frac{\sqrt{2}(a_1x + 2a_2)}{2\sqrt{a_1}}\Big)c_1 + c_2\Big\}, \qquad (7.3.6)$$

where a_1, a_2, c_1 and c_2 are arbitrary constants. The solutions of equation (7.3.6) is represented in Figures 7.5 and 7.6 below

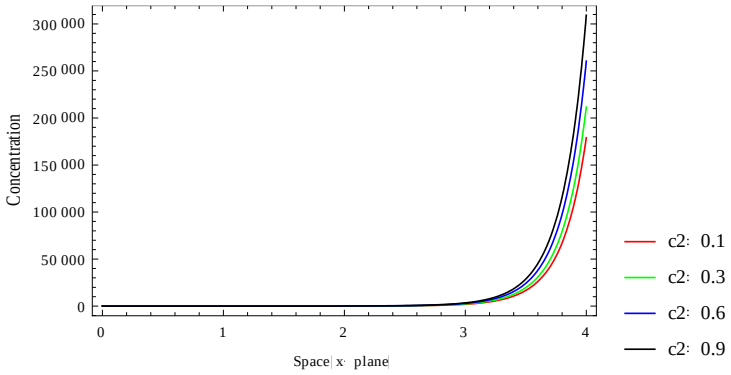


Figure 7.5: Graphical representation of equation (7.3.6), a1=a2=c1=1.

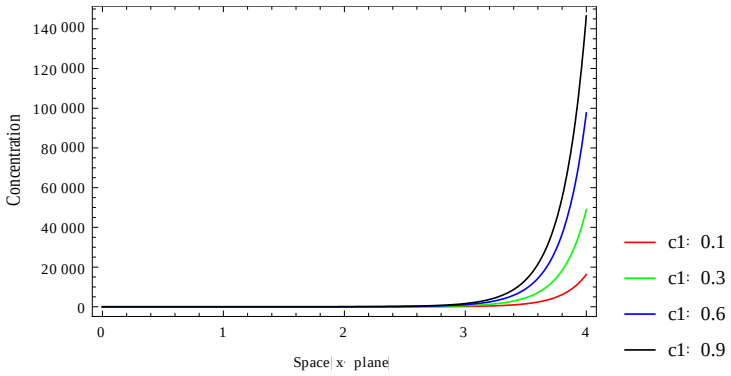


Figure 7.6: Graphical representation of equation (7.3.6), a1=a2=1,c2=0.

7.4 Conclusion

In this chapter, we considered all six cases in finding the potential symmetries. It came out that only two cases of solute transport possess potential symmetries, even if the equations were written in conserved form. One may recall that the prescribed boundary conditions are not invariant under Lie/nonlocal symmetries. However, it is mathematically interesting to obtain new general solutions. Reduced equations may be solved numerically.

Chapter 8

Comparison of nonlocal and nonclassical symmetry solutions for a linear water flow model

8.1 Introduction

In this chapter, we investigate the use the potential symmetries in radial water flow equation given by

$$\frac{\partial}{\partial t}\Big(\alpha_m r\psi\Big) = \frac{\partial}{\partial r}\Big(r\psi_r\Big). \tag{8.1.1}$$

The nonclassical symmetries are used to solve (8.1.1). Additional variable is introduced and (8.1.1) is written in conserved form. The results from nonlocal and nonclassical symmetry reductions are analysed and comparison study is done.

8.2 Radial water flow

To determine the potential symmetries of (2.2.8) we first write the equation in its conserved form, (see for e.g., [69]). We introduce v as an auxiliary variable and write equation (8.1.1) as

$$\left.\begin{aligned} v_r &= r\alpha_m\psi, \\ v_t &= r\psi_r, \end{aligned}\right\} \tag{8.2.1}$$

or equivalently

$$rv_{rr} - v_r - \alpha_m r v_t = 0, \tag{8.2.2}$$

which is obtained by substituting the first equation of (8.2.1) into the second equation.

8.3 Results and discussion.

Equation (8.1.1) admits infinite symmetries and only one symmetry X_1 is the genuine potential symmetry

$$\left.\begin{aligned} X_1 &= 4t^2\frac{\partial}{\partial t} + 4tr\frac{\partial}{\partial r} - v\Big(\alpha_m r^2\Big)\frac{\partial}{\partial v} - \Big(2v + \alpha_m r^2\psi + 8t\psi\Big)\frac{\partial}{\partial \psi}, \\ X_2 &= 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2v\frac{\partial}{\partial v}, \\ X_3 &= \frac{\partial}{\partial t} \\ X_4 &= -\frac{1}{\alpha_m}\Big(\psi\frac{\partial}{\partial \psi} + v\frac{\partial}{\partial v}\Big), \\ X_\infty &= T(t,r,v)\frac{\partial}{\partial v} + H(t,r,v)\frac{\partial}{\partial \psi}. \end{aligned}\right\} \tag{8.3.1}$$

where $\alpha_m = \dfrac{\mu_m}{T_m}$.

8.3.1 Reduction by X_1.

The corresponding characteristic system related to the invariant surface conditions

$$\left.\begin{aligned} 4t^2\psi_t + 4tr\psi_r + (\alpha_m r^2 + 8t)\psi + 2v = 0, \\ \\ 4t^2 v_t + 4trv_r + \alpha_m r^2 v = 0, \end{aligned}\right\} \tag{8.3.2}$$

associated with symmetry X_1 are

$$\frac{dt}{4t^2} = \frac{dr}{4tr} = \frac{dv}{-\alpha_m r^2 v} = \frac{d\psi}{-(2v + \alpha_m r^2\psi + 8t\psi)}. \tag{8.3.3}$$

The integrals of equation (8.3.3) are given by

$$I_1 = \frac{r}{t}, \tag{8.3.4}$$

$$I_2 = v\exp\left(\frac{\alpha_m r^2}{4t}\right),$$

$$I_3 = r^2\exp\left(\frac{\alpha_m r^2}{4t}\right)\psi + \frac{r^2}{2t}h_1.$$

Let $I_1 = z$ be the similarity variable with $I_2 = h_1(z)$ and $I_3 = h_2(z)$ then

$$z = \frac{r}{t}, \tag{8.3.5}$$

$$h_1(z) = v\exp\left(\frac{\alpha_m r^2}{4t}\right),$$

$$h_2(z) = r^2\exp\left(\frac{\alpha_m r^2}{4t}\right)\psi + \frac{r^2}{2t}h_1.$$

The solution of equation (8.3.3) is then given by

$$\left.\begin{aligned} z = \frac{r}{t}, \quad v = h_1(z)\exp\left(-\frac{\alpha_m r^2}{4t}\right), \\ \\ \psi = r^{-2}\exp\left(-\frac{\alpha_m r^2}{4t}\right)\left(h_2(z) - \frac{r^2}{2t}h_1(z)\right). \end{aligned}\right\} \tag{8.3.6}$$

The partial derivatives of ψ with respect to r and t are given as

$$\psi_t = \frac{r^{-2}}{4t^3}\exp\left(-\frac{\alpha_m r^2}{4t}\right)\left[\alpha_m r^2\left(th_2 - 4r^2 h_1\right) + 4rt\left(h_1' + rh_1 - h_2'\right)\right], \tag{8.3.7}$$

and

$$\psi_r = \frac{1}{2t^3r^3}\exp\left(-\frac{\alpha_m r^2}{4t}\right)\left[\left(2rth_2'-2r^5h_1'-(4r^3t^2-4r^2t-\alpha_m r^4)h_1-(4t^2-\alpha_m r^2 t)h_2\right)\right], \tag{8.3.8}$$

also

$$\psi_{rr} = \frac{r^{-2}}{t^2}h_2'' - \frac{1}{t^3}h_1'' \; + \; h_2'\left(\frac{-4r^{-3}}{t} - \frac{\alpha_m r^{-1}}{t^2}\right) \tag{8.3.9}$$

$$+\; h_1'\left(\frac{4r^{-1}}{t^2} + \frac{\alpha_m r}{4t^3} - \frac{2r^{-1}}{t^2}\right) \; + \; h_2\left(6r^{-4} + \; \frac{3\alpha_m r^{-2}}{2t} + \frac{\alpha_m^2}{4t^2}\right)$$

$$+\; h_1\left(6r^{-4} + \frac{3\alpha_m r^{-2}}{2t} + \frac{\alpha_m^2}{t^2} + \frac{3r^{-2}}{t} + \frac{\alpha_m}{t^2}\right).$$

To find a family of solutions Υ^* we introduce the derivatives of ψ in equation (8.1.1) and obtain a second order ODE given by

$$rz(z^2h_1'' - zh_1' + 2\alpha_m h_2) + 2(z^2h_2'' - 3zh_2' + 4h_2) = 0. \tag{8.3.10}$$

The equation (8.3.10) holds for any value of r such that

$$\left.\begin{aligned} z^2h_2'' - 3zh_2' + 4h_2 = 0, \\ z^2h_1'' + zh_1' - 2\alpha_m h_2 = 0. \end{aligned}\right\} \tag{8.3.11}$$

Solving (8.3.11) yields

$$\left.\begin{aligned} h_2(z) = z^2(b_1 + b_2\ln z), \\ h_1(z) = \frac{\ln(z)}{2}\big(2a_1 + \alpha_m z^2 b_2\big) + \frac{\alpha_m z^2}{2}\big(b_1 - b_2\big) + a_2. \end{aligned}\right\} \tag{8.3.12}$$

The family of solutions Υ^* is then given by

$$\psi(t,r) = r^{-2}\exp\big(-\frac{\alpha_m r^2}{4t}\big)\Big\{z^2(b_1 + b_2\ln z) - \tag{8.3.13}$$

$$\frac{r^2}{2t}\big(\frac{\ln(z)}{2}\big(2a_1 + \alpha_m z^2 b_2\big) + \frac{\alpha_m z^2}{2}\big(b_1 - b_2\big) + a_2\big)\Big\},$$

where a_i and b_i are arbitrary constants. It can be verified that (8.3.13) is a family solutions of

$$\ddot{\eta} = 16t^4\psi_{tt} + 16t^2r^2\psi_{rr} + 32t^3r\psi_{tr} + (64t^3 + 8\alpha_m t^2 r^2)\psi_t \tag{8.3.14}$$

$$+(64t^2r + 8\alpha_m t r^3)\psi_r + (32t^2 + 16tr^2\alpha_m + \alpha_m^2 r^4)\psi = 0.$$

Substituting equation (8.3.6) into equation (8.2.1) and obtain the system S_F given by

$$z^3 t h_1' - \alpha_m z^2 t h_2 + 2z h_2' - 4h_2 = 0, \tag{8.3.15}$$

which holds for all values of t such that

$$z h_1' - \alpha_m h_2 = 0, \tag{8.3.16}$$

$$z h_2' - 2h_2 = 0. \tag{8.3.17}$$

Equation (8.3.16) yields the solutions Υ given by

$$\left.\begin{aligned} h_2(z) &= b_1 z^2, \\ h_1(z) &= \tfrac{1}{2}\alpha_m b_1 z^2 + a_1. \end{aligned}\right\} \tag{8.3.18}$$

The solution is therefore given by

$$\psi(t,r) = r^{-2}\exp\Big(-\frac{\alpha_m r^2}{4t}\Big)\Big\{\frac{b_1 r^2}{t^2} - \frac{r^2}{2t}\Big(\frac{\alpha_m b_1 r^2}{2t^2} + a_1\Big)\Big\}, \tag{8.3.19}$$

where a_1 and b_1 are arbitrary constant. It can be shown that equation (8.3.19) satisfy the equation

$$\eta = 4\alpha_m t^2 r^2 \psi_t + (4\alpha_m t r^3 - 8t^2 r)\psi_r + \alpha_m^2 r^4 \psi = 0. \tag{8.3.20}$$

It is worth noting that the equation (8.1.1) can be solved algorithmically and obtain the exact solution given by

$$\psi = \exp\Big(\frac{\alpha t}{\lambda}\Big)\Big[c_1 + c_2 \ln r\Big],$$

where c_1 and c_2 are arbitrary constants.

8.3.2 Physical example

Suppose that we impose the physical conditions on (8.3.19):

$$\left.\begin{aligned} t = 0, \;\; \psi &\to \infty, \\ r \to \infty, \;\; \psi &\to 0, \\ r \to 0, \;\; \psi &= \frac{a_1}{t} - \frac{b_1}{2}, \end{aligned}\right\} \tag{8.3.21}$$

then one may choose without loss of generality the constant a_1 and b_1 to be unity. The results are plotted in the Figures 8.1, 8.2, 8.3 and 8.4. Notice that the concentration at $r = 0$ is infinity. This may be interpreted as, the source of contamination, and it vanishes at large radius.

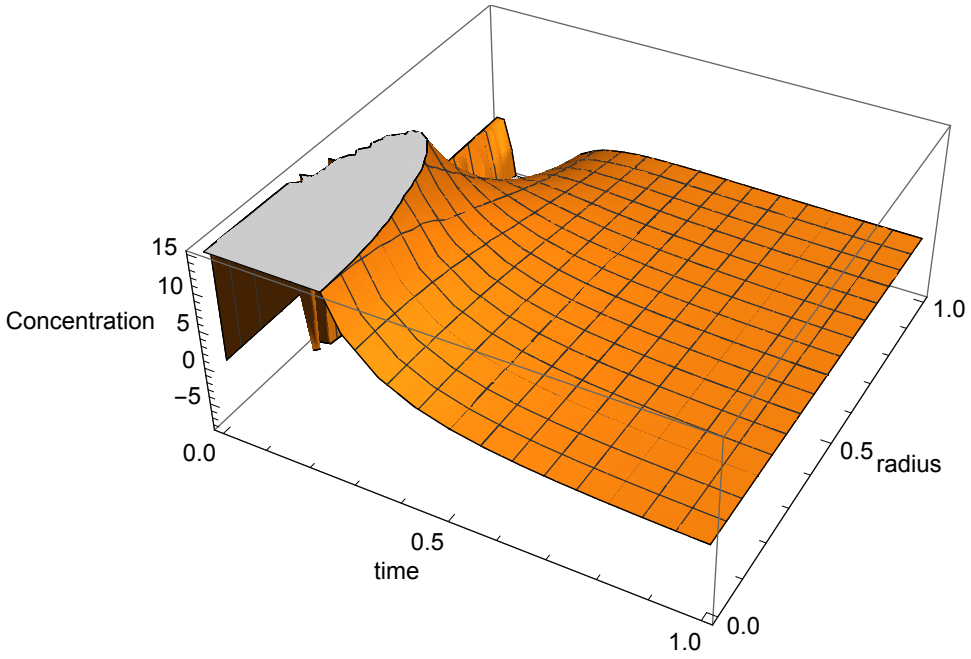


Figure 8.1: Concentration of oil over the surface from equation (8.3.19).

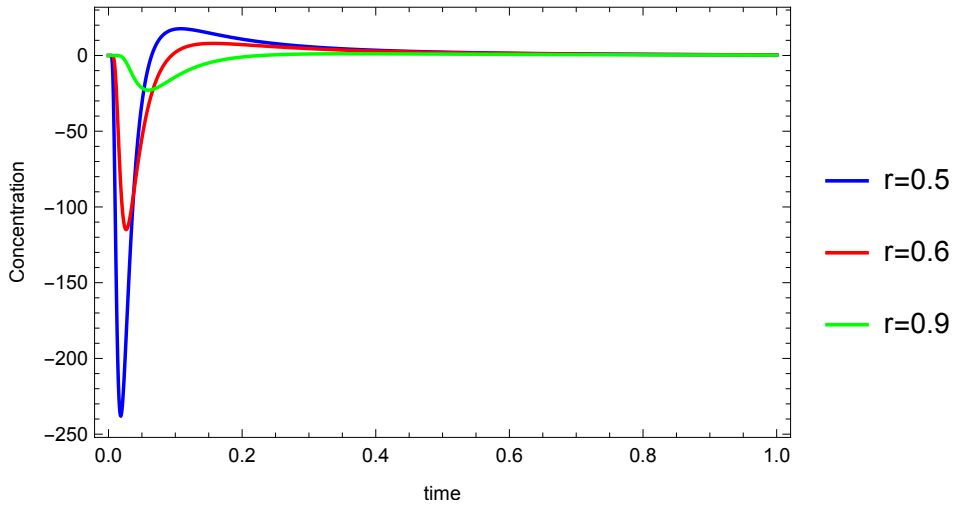


Figure 8.2: Concentration of oil over time from equation (8.3.19).

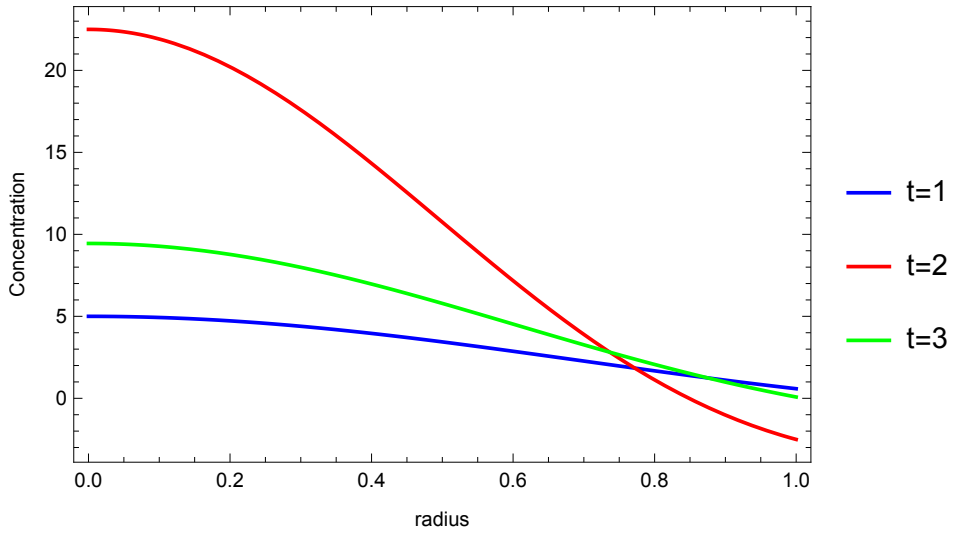


Figure 8.3: Concentration of oil along the sea from equation (8.3.19).

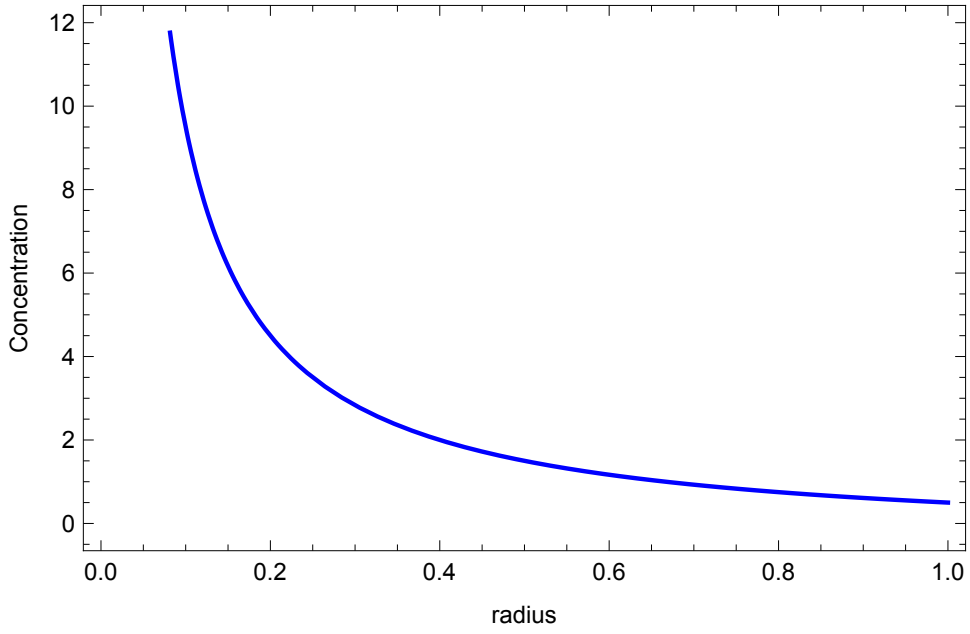


Figure 8.4: Concentration of oil when $r \to 0$ and $t \to 0$ from equation (8.3.19).

8.4 Nonclassical symmetries admitted by a groundwater flow model.

In this section, we employ the nonclassical symmetry techniques. We use the interactive program/code in Maple (see Appendix A). Solving the determine equations,

it turned out that the equation (8.1.1) admits two symmetries

$$X_1 = \frac{\partial}{\partial t} + \Big[c_1\psi + \exp\Big(-\frac{\lambda^2 t}{\alpha_m}\Big)\Big(c_2\mathrm{J}_0(\lambda r) + c_3\mathrm{Y}_0(\lambda r)\Big)\Big], \tag{8.4.1}$$

$$X_2 = \frac{\partial}{\partial t} + \Big[c_1\psi + \exp\Big(\frac{\lambda^2 t}{\alpha_m}\Big)\Big(c_2\mathrm{I}_0(\lambda r) + c_3\mathrm{K}_0(\lambda r)\Big)\Big],$$

where c_1 and c_2 are arbitrary constants, J_0, Y_0, I_0 and K_0 are Bessel functions of the first and second kinds.

8.4.1 Reduction by X_1.

Case(a) If $c_2 = c_3 = 0$.

The characteristic equation is given by

$$\frac{dt}{1} = \frac{dr}{0} = \frac{d\psi}{c_1\psi}.$$

The invariant solutions are

$$F(I_1) = \psi \mathrm{e}^{-c_1 t}, \qquad I_1 = r. \tag{8.4.2}$$

The derivatives of equation (8.4.2) reduce equation (8.1.1) to

$$rF''(r) + F'(r) - \alpha_m c_1 F(r) = 0. \tag{8.4.3}$$

The solution of equation (8.4.3) is given by

$$F(r) = A\mathrm{J}_0(ir\sqrt{\alpha_m c_1}) + B\mathrm{Y}_0(ir\sqrt{\alpha_m c_1}). \tag{8.4.4}$$

The solution ψ in its original variables is given by

$$\psi(r,t) = \Big[A\mathrm{J}_0(ir\sqrt{\alpha_m c_1}) + B\mathrm{Y}_0(ir\sqrt{\alpha_m c_1})\Big]\mathrm{e}^{c_1 t}.$$

where $i = \sqrt{-1}$, J_0 and Y_0 are Bessel functions of the first and second kind respectively, and A, B and c_1 are arbitrary constants.

Case(b) **If** $c_1 = 0$.

$$\frac{dt}{1} = \frac{dr}{0} = \frac{\exp\left(\frac{\lambda^2 t}{\alpha_m}\right) d\psi}{c_2 \mathrm{J}_0(\lambda r) + c_3 \mathrm{Y}_0(\lambda r)}.$$

The invariant solutions are

$$F(I_1) = \psi + \frac{\alpha_m}{\lambda^2} \exp\left(- \frac{\lambda^2 t}{\alpha_m} \right) \Big[c_2 \mathrm{J}_0(\lambda r) + c_3 \mathrm{Y}_0(\lambda r) \Big], \quad I_1 = r. \tag{8.4.5}$$

The derivatives of equation (8.4.5) reduce equation (8.1.1) to

$$rF''(r) + F'(r) = 0. \tag{8.4.6}$$

The solution of equation (8.4.6) is given by

$$F(r) = A + B \ln r, \tag{8.4.7}$$

The solution ψ in its original variables is given by

$$\psi(r,t) = -\frac{\alpha_m \exp\left(- \frac{\lambda^2 t}{\alpha_m} \right)}{\lambda^2} \Big[c_2 \mathrm{J}_0(\lambda r) + c_3 \mathrm{Y}_0(\lambda r) \Big] + B \ln r + A.$$

where J_0 and Y_0 are Bessel functions of the first and second kind respectively, and A, B, c_2 and c_3 are arbitrary constants.

Case(c) **If** $c_1 = c_3 = 0$.

$$\frac{dt}{1} = \frac{dr}{0} = \frac{\exp\left(\frac{\lambda^2 t}{\alpha_m}\right) d\psi}{c_2 \mathrm{J}_0(\lambda r)}.$$

The invariant solutions are

$$F(I_1) = \psi + \frac{\alpha_m}{\lambda^2} \exp\left(- \frac{\lambda^2 t}{\alpha_m} \right) \Big[c_2 \mathrm{J}_0(\lambda r) \Big], \quad I_1 = r.$$

the derivatives of which reduces equation (8.1.1) to

$$rF''(r) + F'(r) = 0. \tag{8.4.8}$$

The solution of equation (8.4.8) is given by

$$F(r) = B \ln r + A. \tag{8.4.9}$$

The solution ψ in its original variables is given by

$$\psi = \Big\{ -\frac{\alpha_m}{\lambda^2}\exp\Big(\frac{-t\big(\lambda^2 + c_1\alpha_m\big)}{\alpha_m}\Big)\Big[c_2\mathrm{J}_0(\lambda r)\Big] + B\ln r + A$$

where $i = \sqrt{-1}$, J_0 is the Bessel function of the first kind, and A, B, c_1 and c_2 are arbitrary constants.

Case(d) If $c_1 = c_2 = 0$.

$$\frac{dt}{1} = \frac{dr}{0} = \frac{\exp\big(\frac{\lambda^2 t}{\alpha_m}\big)d\psi}{c_3\mathrm{Y}_0(\lambda r)}.$$

The invariant solutions are

$$F(I_1) = \psi + \frac{\alpha_m}{\lambda^2}\exp\Big(-\frac{\lambda^2 t}{\alpha_m}\Big)\Big[c_3\mathrm{Y}_0(\lambda r)\Big], \quad I_1 = r.$$

the derivatives of which reduce (8.1.1) to

$$rF''(r) + F'(r) = 0. \tag{8.4.10}$$

The solution of equation (8.4.10) is given by

$$F(r) = B\ln r + A. \tag{8.4.11}$$

The solution ψ in its original variables is given by

$$\psi = -\frac{\alpha_m}{\lambda^2}\exp\Big(-\frac{\lambda^2 t}{\alpha_m}\Big)\Big[c_3\mathrm{Y}_0(\lambda r)\Big] + B\ln r + A$$

where $i = \sqrt{-1}$, and Y_0 is a Bessel function of the second kind, A, B and c_3 are arbitrary constants.

8.4.2 Reduction by X_2.

Case(a) If $c_2 = c_3 = 0$.

This case is the same as **Case1**(**a**), subsection 8.4.1 above.

Case(b) If $c_1 = 0$.

$$\frac{dt}{1} = \frac{dr}{0} = \frac{\exp\big(-\frac{\lambda^2 t}{\alpha_m}\big)d\psi}{c_2 \mathrm{I}_0(\lambda r) + c_3 \mathrm{K}_0(\lambda r)}.$$

The invariant solutions are

$$F(I_1) = \psi - \frac{\alpha_m}{\lambda^2}\exp\bigg(-\frac{\lambda^2 t}{\alpha_m}\bigg)\Big[c_2 \mathrm{I}_0(\lambda r) + c_3 \mathrm{K}_0(\lambda r)\Big], \quad I_1 = r.$$

the derivatives of which reduce (8.1.1) to

$$rF''(r) + F'(r) = 0. \tag{8.4.12}$$

The solution of (8.4.12) is given by

$$F(r) = B\ln r + A. \tag{8.4.13}$$

The solution ψ in its original variables is given by

$$\psi(r,t) = \frac{\alpha_m \exp\big(-\frac{\lambda^2 t}{\alpha_m}\big)}{\lambda^2}\Big[c_2 \mathrm{I}_0(\lambda r) + c_3 \mathrm{K}_0(\lambda r)\Big] + B\ln r + A.$$

where $i = \sqrt{-1}$, I_0 and K_0 are Bessel functions of the first and second kind respectively, and A, B $c_2 = 0$ and $c_3 = 0$ are arbitrary constants. The remaining two cases, c_2 and c_3 can be considered by the interested reader.

8.5 Conclusion

The radial water flow model of the oil spill is considered. The equation was written in conserved form. It came out that not all cases considered yielded potential symmetries even though they were written in conserved form. Again not all symmetries obtained are potential symmetries. Nonclassical symmetries were also obtained and used to reduce (8.1.1). Results obtained are presented. In a few examples it is possible that nonclassical symmetries yield the same invariant solution as the nonlocal symmetries. It turned out that this is not the case in this chapter.

Chapter 9

Conclusion

In conclusion, we reflect back on our precious work. Chapter 1, entailed the theoretical background on the oil spill model of CDE. Detailed information on the incidences where, the spill occurred and its effects on health, environment, economy etc., was given. It has been studied that many countries are affected by oil spill especially in coastal areas. Our country is not immune from this contamination. The sources are indicated. Prevention, cleaning, removal as management procedures are stated. The literature review on the previous work was highlighted.

The mathematical model of the CDE was described in Chapter 2. We note that, we only considered one and two-dimensional models of CDE.

In Chapter 3, relevant techniques for symmetry reduction procedures were satisfactorily outlined.

Chapter 4 dealt with group classification of a one-dimensional CDE. Different functional forms for the coefficient dispersion coefficient and diffusivity velocity were given. Some results/solutions were obtained through the optimal systems. In cases where solutions could not be obtained through symmetry methods, the interested reader is referred to other methods of solutions such as numerical and other analytical methods. The steady state was also analysed.

In Chapter 5, the solute transport with constant dispersion coefficient in a one-dimensional CDE was dealt with. Interesting results in the form of special mathematical functions such as Kummer, Error, HeunT and WhittakerM were obtained.

In Chapter 6, consideration of solute transportation of oil in water was made. The evaporation, dispersion, diffusion, and the dissolution were considered as movements of the oil in water.

The potential symmetry reduction method was used in Chapter 7. It came out clear that these symmetries are hidden and can only be obtained when the equation in written in the conserved form. Six cases were analysed and only two cases produced potential symmetries. These cases were dealt with and interesting results in the form of Error functions were obtained.

In Chapter 8, the potential and nonclassical symmetries for radial water flow were obtained and used to reduce equation (8.1.1).

This chapter gives the concluding remarks as stated above. In the outlook we aspire to construct approximate analytical solutions using differential transformation methods.

Chapter 10

Appendix A

restart;

de1: = ut - $\frac{uxx}{a} - \frac{ux}{x}$;

dx: = proc(f)

diff(f,x)+ diff(f,u)*ux + diff(f,ux)*uxx + diff(f,ut)*uxt + diff(f,uxx)*uxxx;

end;

dt: = proc(f)

diff(f,t) + diff(f,u)*ut + diff(f,ux)*uxt + diff(f,ut)*utt;

end;

pr2: = proc(f)

diff(f,x)*cx(x,t,u) + diff(f,t)*ct(x,t,u) +diff(f,u)*cu(x,t,u)+ diff(f,ux)*cux + diff(f,ut)*cut

+ diff(f,uxx)*cuxx + diff(f,uxt)*cuxt + diff(f,utt)*cutt;

end;

cux: = dx(cu(x,t,u)) - dx(cx(x,t,u))*ux - dx(ct(x,t,u))*ut:

cut: = dt(cu(x,t,u)) - dt(cx(x,t,u))*ux - dt(ct(x,t,u))*ut:

cuxx: = dx(cux) - dx(cx(x,t,u))*uxx - dx(ct(x,t,u))*uxt:

cuxt: = dt(cux) - dt(cx(x,t,u))*uxx - dt(ct(x,t,u))*uxt:

cutt: = dt(cut) - dt(cx(x,t,u))*uxt - dt(ct(x,t,u))*utt:

eqn1: = pr2(de1):

uxt: = dx(cu(x,t,u) - cx(x,t,u)*ux);

ut: = cu(x,t,u) - cx(x,t,u)*ux;

uxx: = a*ut -$\frac{ux}{x}$;
$ct(x,t,u)$: = 1;

$$del := ut - \frac{uxx}{a} - \frac{ux}{ax}$$

$dx := \mathbf{proc}(f)diff(f,x)+(diff(f,u))*ux+(diff(f,ut))*uxt+(diff(f,uxx))*uxxx$ **endproc**

$dt := \mathbf{proc}(f)diff(f,t)+(diff(f,u))*ut+(diff(f,ux))*uxt+(diff(f,ut))*utt$ **endproc**

$pr2 := \mathbf{proc}(f)(diff(f,x)) * cx(x,t,u) + (diff(f,t)) * ct(x,t,u) + (diff(f,u)) * cu(x,t,u) + (di$

endproc

$$uxt := \frac{\partial}{\partial x}cu(x,t,u) - \Big(\frac{\partial}{\partial x}cx(x,t,u)\Big)ux + \Big[\frac{\partial}{\partial u}cu(x,t,u) - \Big(\frac{\partial}{\partial u}(cx(x,t,u)\Big) * ux)\Big]ux - cx(x,t,u)uxx \tag{10.0.2}$$

$$ut := cu(x,t,u) - cx(x,t,u)ux$$

$$uxx := a(cu(x,t,u) - cx(x,t,u)ux) - \frac{ux}{x}$$

$$ct := (x,t,u) \to 1$$

$$eqn1 := \frac{numer(eqn1)}{x}; \tag{10.0.3}$$

$$\begin{aligned} eqn1 := \frac{1}{x}\Big[&cx(x,t,u)ux - x\Big(\frac{\partial}{\partial x}cu(x,t,u)\Big) - \Big(\frac{\partial^2}{\partial x^2}cu(x,t,u)\Big)x^2 \\ &-uxax^2\Big(\frac{\partial}{\partial t}cx(x,t,u)\Big) + 2\Big(\frac{\partial}{\partial x}cx(x,t,u)\Big)ax^2cu(x,t,u) \\ &+2uxax^2\Big(\frac{\partial}{\partial u}cx(x,t,u)\Big)cu(x,t,u) \\ &-2ux^2ax^2\Big(\frac{\partial}{\partial u}cx(x,t,u)\Big)cx(x,t,u) \\ &-2\Big(\frac{\partial}{\partial x}cx(x,t,u)\Big)ax^2cx(x,t,u)ux + \Big(\frac{\partial}{\partial t}cu(x,t,u)\Big)ax^2 \\ &-xux\Big(\frac{\partial}{\partial x}cx(x,t,u)\Big) \\ &-2x\Big(\frac{\partial}{\partial u}cx(x,t,u)\Big)ux^2 - 2\Big(\frac{\partial^2}{\partial x\partial u}cu(x,t,u)\Big)uxx^2 \\ &+x^2ux\Big(\frac{\partial^2}{\partial x^2}cx(x,t,u)\Big) + 2x^2ux^2\Big(\frac{\partial^2}{\partial x\partial u}cx(x,t,u)\Big) \\ &-x^2ux^2\Big(\frac{\partial^2}{\partial u^2}cu(x,t,u)\Big) + x^2ux^3\Big(\frac{\partial^2}{\partial u^2}cx(x,t,u)\Big)\Big] \end{aligned} \tag{10.0.4}$$

$e0 := coeff(eqn1, ux, 0);$

$$e0 := \frac{1}{x}\Big[- x(\frac{\partial}{\partial x}cu(x,t,u) - (\frac{\partial^2}{\partial x^2}cu(x,t,u)x^2 + 2(\frac{\partial}{\partial x}cx(x,t,u)ax^2cu(x,t,u) + (\frac{\partial}{\partial t}cu(x,t,u)ax^2)\Big] \tag{10.0.5}$$

$e1 := coeff(eqn1, ux, 1);$

$$e1 := \frac{1}{x}\Big[cx(x,t,u) - ax^2(\frac{\partial}{\partial t}cx(x,t,u) + 2ax^2(\frac{\partial}{\partial u}cx(x,t,u)cu(x,t,u) -2(\frac{\partial}{\partial x}cx(x,t,u)ax^2cx(x,t,u) - x(\frac{\partial}{\partial x}cx(x,t,u) - 2(\frac{\partial^2}{\partial x\partial u}cu(x,t,u)x^2 +x^2(\frac{\partial^2}{\partial x^2}cx(x,t,u))\Big] \tag{10.0.6}$$

$e2 := coeff(eqn1, ux, 2);$

$$e2 := \frac{1}{x}\Big[- 2ax^2\Big(\frac{\partial}{\partial u}cx(x,t,u)\Big)cx(x,t,u) - 2x\Big(\frac{\partial}{\partial u}cx(x,t,u)\Big) + 2x^2\Big(\frac{\partial^2}{\partial x\partial u}cx(x,t,u)\Big) -x^2\Big(\frac{\partial^2}{\partial u^2}cu(x,t,u)\Big)\Big] \tag{10.0.7}$$

$e3 := coeff(eqn1, ux, 3);$

$$e3 := x\Big(\frac{\partial^2}{\partial u^2}cx(x,t,u)\Big) \tag{10.0.8}$$

$cx(x,t,u) := f1(x,t)u + f2(x,t);$

$$cx := (x,t,u) \to f1(x,t)u + f2(x,t) \tag{10.0.9}$$

$simplify(e3);$

$$0 \tag{10.0.10}$$

$e2;$

$$\frac{1}{x}\Big[- 2ax^2f1(x,t)(f1(x,t)u + f2(x,t)) - 2xf1(x,t) + 2x^2(\frac{\partial}{\partial x}(f1(x,t)) -x^2(\frac{\partial^2}{\partial u^2}(cu(x,t,u)\Big] \tag{10.0.11}$$

$pdsolve(e2, cu);$

$$cu(x,t,u) = -(1/3)af1(x,t)^2u^3 - af1(x,t)f2(x,t)u^2 - f1(x,t)u^2/x +(\frac{\partial}{\partial x}f1(x,t)u^2 +_F 1(x,t)u +_F 2(x,t) \tag{10.0.12}$$

$$cu(x,t,u) := -\frac{1}{3}(f1(x,t)^2 a u^3 - f1(x,t) f2(x,t) a u^2 + (\frac{\partial}{\partial x} f1(x,t)) u^2$$
$$-\frac{f1(x,t)u^2)}{x} + f3(x,t)u + f4(x,t); \tag{10.0.13}$$

$$cu := (x,t,u) \to -\frac{1}{3} f1(x,t)^2 a u^3 - f1(x,t) f2(x,t) a u^2 + (\frac{\partial}{\partial x}(f1(x,t)) u^2$$
$$-\frac{f1(x,t)u^2}{x} + f3(x,t)u + f4(x,t) \tag{10.0.14}$$

$simplify(e2);$

$$0 \tag{10.0.15}$$

$ee := numer(e1);$

$$cee := x^2\Big[-9 f1(x,t)u + 3 f2(x,t) - 3ax^2\Big(\frac{\partial}{\partial t} f1(x,t)\Big)u - 3ax^2\Big(\frac{\partial}{\partial t} f2(x,t)\Big)$$
$$-2a^2x^2 f1(x,t)^3 u^3 - 6a^2x^2 f1(x,t)^2 f2(x,t)u^2 + 12a f1(x,t)u^2\Big(\frac{\partial}{\partial x}(f1(x,t)\Big)x^2$$
$$-6xa f1(x,t)^2 u^2 + 6ax^2 f1(x,t) f3(x,t)u + 6ax^2 f1(x,t) f4(x,t)$$
$$+6a\Big(\frac{\partial}{\partial x} f1(x,t) f2(x,t)\Big)ux^2 + 6a f1(x,t)\Big(\frac{\partial}{\partial x} f2(x,t)\Big)ux^2 - 6ax^2\Big(\frac{\partial}{\partial x} f2(x,t)\Big) f2(x,t)$$
$$+9x\Big(\frac{\partial}{\partial x} f1(x,t)\Big)u - 3x\Big(\frac{\partial}{\partial x} f2(x,t)\Big) - 9\Big(\frac{\partial^2}{\partial x^2} f1(x,t)\Big)ux^2 - 6\Big(\frac{\partial}{\partial x} f3(x,t))\Big)x^2$$
$$+3x^2\Big(\frac{\partial}{\partial^2 x^2} f2(x,t)\Big)\Big] \tag{10.0.16}$$

$eu3; coeff(ee,u,3);$

$$-2x^4 a^2 f1(x,t)^3 \tag{10.0.17}$$

$f1(x,t) := 0;$

$$f1 := f(x,t) \to 0 \tag{10.0.18}$$

$ee;$

$$x^2\Big[(3 f2(x,t) - 3ax^2\Big(\frac{\partial}{\partial t} f2(x,t)\Big) - 6ax^2\Big(\frac{\partial}{\partial x} f2(x,t) f2(x,t)\Big) - 3x\Big(\frac{\partial}{\partial x} f2(x,t)\Big)$$
$$-6\Big(\frac{\partial}{\partial x} f3(x,t)\Big)x^2 + 3x^2\Big(\frac{\partial^2}{\partial x^2} f2(x,t)\Big)\Big] \tag{10.0.19}$$

$eu1 := coeff(e0,u,1)$

$$eu1 := \frac{1}{x}\Big[-x\Big(\frac{\partial}{\partial x} f3(x,t)\Big) - \Big(\frac{\partial^2}{\partial x^2}(f3(x,t)\Big)x^2$$
$$+2\Big(\frac{\partial}{\partial x}(f2(x,t)\Big)ax^2 f3(x,t) + \Big(\frac{\partial}{\partial t}(f3(x,t)\Big)ax^2 \tag{10.0.20}$$

$eu0 := coeff(e0, u, 0);$

$$eu0 := \frac{1}{x}\Big[- x\Big(\frac{\partial}{\partial x} f4(x,t)\Big) - \Big(\frac{\partial^2}{\partial x^2}(f4(x,t)\Big)x^2 \\ +2\Big(\frac{\partial}{\partial x} f2(x,t)\Big)ax^2 f4(x,t) + \Big(\frac{\partial}{\partial t} f4(x,t)\Big)ax^2\Big] \tag{10.0.21}$$

$f2(x,t) := h(t);$

$$f2 := (x,t) \to h(t) \tag{10.0.22}$$

$ee;$

$$x^2\Big(3h(t) - 3ax^2\Big(\frac{d}{dt}h(t)\Big) - 6\Big(\frac{\partial}{\partial x}(f3(x,t)\Big)x^2 \tag{10.0.23}$$

$pdsolve(ee, f3(x,t));$

$$f3(x,t) = -\frac{1}{2}a\Big(\frac{d}{dt}h(t)\Big)x - \frac{1}{2a}h(t) +_F 1(t) \tag{10.0.24}$$

$f3(x,t) := -\frac{1}{2}\Big(\frac{d}{dt}h(t)\Big)ax - \frac{1}{2x}h(t) + p(t);\ simplify(ee);$

$$0 \tag{10.0.25}$$

$numer(eu1);$

$$-x\Big[- h(t) + a^2x^4\Big(\frac{d^2}{dt^2}h(t)\Big) - 2ax^3\Big(\frac{d}{dt}p(t)\Big)\Big] \tag{10.0.26}$$

$h(t) := 0; p(t) := c1;$

$$h := t \to 0$$
$$p := t \to c1 \tag{10.0.27}$$

$$0 \tag{10.0.28}$$

$eu0;$

$$\frac{1}{x}\Big[- x\Big(\frac{\partial}{\partial x} f4(x,t)\Big) - \Big(\frac{\partial^2}{\partial x^2} f4(x,t)\Big)x^2 + \Big(\frac{\partial}{\partial t} f4(x,t)\Big)ax^2\Big] \tag{10.0.29}$$

$pdsolve(eu0);$

$$\Big(f4(x,t) = F1(x)F2(t)\Big)where\Big[\Big\{\frac{d}{dt}F2(t) = \Big(\frac{F2(t)c[1]}{a}\Big)\Big(\frac{d^2}{dx^2}(F1(x)\Big) = \\ F1(x)c[1] - \Big(\frac{\frac{d}{dx}F1(x)}{x}\Big)\Big\}\Big] \tag{10.0.30}$$

$f4(x,t) := \exp\left(-\frac{la*^2 t}{a}\right)\left(c2\ \mathrm{BesselJ}(0, la*x) + c3\ \mathrm{BesselY}(0, la*x)\right);$

$$f4 := (x,t) \to \mathrm{e}^{\left(-\frac{\lambda^2 t}{a}\right)}\left(c2\ \mathrm{BesselJ}(0,\lambda x) + c3\ \mathrm{BesselY}(0,\lambda x)\right) \tag{10.0.31}$$

$simplify(eu0);$

$$0 \tag{10.0.32}$$

$f4(x,t) := \exp\left(\frac{la*^2 t}{a}\right)\left(c2\ \mathrm{BesselJ}(0, la*x) + c3\ \mathrm{BesselY}(0, la*x)\right);$

$$f4 := (x,t) \to \mathrm{e}^{\left(\frac{\lambda^2 t}{a}\right)}\left(c2\ \mathrm{BesselJ}(0,\lambda x) + c3\ \mathrm{BesselY}(0,\lambda x)\right) \tag{10.0.33}$$

$cx(x,t,u);$

$$0 \tag{10.0.34}$$

$ct(x,t,u);$

$$1 \tag{10.0.35}$$

$cu(x,t,u);$

$$c1*u + \mathrm{e}^{\frac{la^2 t}{a}}(c2\ \mathrm{BesselI}(0, la*x) + c3\ \mathrm{BesselK}(0, la*x)\Big) \tag{10.0.36}$$

Chapter 11

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