

**An Analysis of Symmetries and  
Conservation Laws of Nonlinear Partial  
Differential Equations Arising from  
Burgers' Hierarchy**

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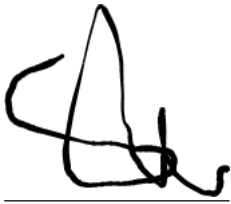
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## DECLARATION

I declare that the contents of this dissertation are original, except where due references have been made. It is submitted for the degree of Master of Science at the University of the Witwatersrand. It was not submitted before for any degree or examination to any other institution.

Usaamah Obaidullah



A handwritten signature in black ink, consisting of a large, stylized 'U' followed by a cursive 'O' and 'B'. The signature is written above a horizontal line.

Signed at Johannesburg on the 27<sup>th</sup> day of October 2020.

## Abstract

We investigate the nonlinear evolutionary partial differential equations (PDEs) derived from Burgers' hierarchy and give the exact solution of the complete hierarchy. The conservation laws of the hierarchy are studied and we proceed to establish the general  $n$ th conservation law. A transformation is used to render the hierarchy to a hierarchy of nonlinear ordinary differential equations (ODEs). These expressions are then linearised. Ultimately we give a novel exact solution of the entire Burgers' hierarchy, that is, for all values of  $n$ . Several members of the hierarchy are solved, and the graphical counterparts of their solutions are provided to illustrate the applicability of our formula. Next we extend our study to the hierarchy of ODEs linked to this hierarchy. One-parameter Lie group of transformations that leave the ODEs invariant are constructed, from which it is established that these symmetries arise from the  $(n + 1)$  complex roots of a certain polynomial. This gives us a formula to solve the ODE expressions, and finally we show how a more general exact solution of the complete hierarchy is obtained from this result.

*I almost wish I hadn't gone down that rabbit-hole  
—and yet—and yet—it's rather curious,  
you know, this sort of life!*

Lewis Carroll

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# Contents

<b>1</b>	<b>Introduction</b>	<b>7</b>
1.1	Outline of Chapters . . . . .	8
<b>2</b>	<b>Notation and Theory</b>	<b>10</b>
2.1	Symmetry Methods . . . . .	10
2.2	Conservation Laws . . . . .	11
2.3	The Total Differentiation Operator . . . . .	12
2.4	Multipliers . . . . .	12
2.5	Invariant Solutions . . . . .	12
2.6	Double Reduction of PDEs . . . . .	13
<b>3</b>	<b>A Computational Procedure for Exact Solutions of Burgers' Hierarchy of Nonlinear Partial Differential Equations</b>	<b>14</b>

3.1	The Burgers' Hierarchy of Nonlinear PDEs . . . . .	15
3.2	The Conserved Vector Transformation . . . . .	17
3.3	Applications . . . . .	22
3.4	Conclusion . . . . .	25
<b>4</b>	<b>On the Formulaic Solution of a <math>(n+1)^{th}</math> Order Differential Equation</b>	<b>28</b>
4.1	Introduction . . . . .	29
4.2	Main Results . . . . .	30
4.2.1	Point Generators . . . . .	31
4.2.2	Exact Solutions for $v(r)$ . . . . .	39
4.2.3	Applications to a PDE Hierarchy . . . . .	42
4.3	Conclusion . . . . .	44
	<b>Conclusion</b>	<b>47</b>
	<b>Bibliography</b>	<b>48</b>

# List of Figures

3.1	Graphical illustration of the analytical solutions are depicted, we select the parameter values $A_j = \bar{A}_j = 1$ : (a) solution of (52) where $c = 0.5$ ; (b) solution of (55) where $c = 2$ ; (c) solution of (60) where $c = 1$ ; and, (d) solution of (62) where $c = 0.5$ . . . . .	26
4.1	A plot of all complex roots of $P(z)$ for $c = k = 1$ and $n = 1, \dots, 23$ . . . . .	36
4.2	A plot of all complex roots of $P(z)$ for $k = 0$ and $n = 1, \dots, 23$ . . . . .	36
4.3	Progression of $v_1(r)$ in Eq. (77). Parameter values for Curve $j$ are $C_1 = C_2 = c = k = j$ where $j = 1, \dots, 4$ . . . . .	42
4.4	Progression of $v_2(r)$ in Eq. (79). Parameter values for the $j$ th Curve are $C_1 = C_4 = C_5 = p$ where $p = 1, \dots, 4$ . . . . .	43
4.5	The evolution of $u(x, t)$ based on $v_1(r)$ . We select the parameter values $C_1 = C_2 = c = k = 1$ . . . . .	44
4.6	The evolution of $u(x, t)$ based on the real version of $v_2(r)$ . We select the parameter values $C_1 = C_4 = C_5 = 1$ . . . . .	45

# List of Tables

4.1 Table of symmetries for  $k = c = 1$  . . . . . 37

4.2 Table of symmetries for  $k = 0$  . . . . . 38

# Chapter 1

## Introduction

Nonlinear PDEs are foundational to contemporary science, being used extensively in various fields to describe physical systems and natural phenomena. A few of these range from classical physics to quantum physics, biological modelling, financial derivatives and even the modelling of social systems. The nonlinear PDEs that will be investigated in this thesis arise from Burgers' hierarchy. This hierarchy provides myriads of PDEs, the first being the Burgers' equation. Burgers' equation has been extensively studied in the literature [1]-[5], and is an example of a nonlinear PDE that explains phenomena in fluid mechanics as it mimics the well-known Navier-Stokes equation. Although initially introduced in 1915 by H. Bateman, the physicist J.M. Burgers explained the mathematical modelling of turbulence through this equation [6]. Several other phenomena are modelled using the PDEs from this hierarchy. Although Burgers' hierarchy and its resulting PDEs have been extensively studied, the exact solutions that have been presented in literature are primarily for the first few members. In this study we provide an original formula to solve the full hierarchy using a novel approach involving conservation laws, hence the importance

of this study.

Interwoven with differential equations is the concept of Lie symmetry analysis. Pioneered by Norwegian mathematician Sophus Lie (1842-1899) who observed that the various methods to solve these equations are simply special cases of a more general procedure of integration, established on the invariance of the equation under a continuous group of symmetries [7]. Mathematically a symmetry occurs when there is invariance under transformation. Extended to PDEs, a symmetry is a transformation that maps any solution of the equation into another solution of the same equation. One can easily calculate the symmetries of a differential equation and then use them to obtain its analytic solutions in a methodical manner. New solutions can also be constructed from known ones using this technique. Thus, symmetry analysis has been one of the most widely used techniques for finding solutions to differential equations as well as interpreting the physical aspects of these solutions.

In physics, a conservation law states that a measurable quantity of a physical system remains constant as the system undergoes changes. This concept has significant relevance in several realms of physics, general relativity being one. In the case of differential equations conservation laws are significant tools, particularly in the linearisation and analysis of solutions.

## 1.1 Outline of Chapters

The goal of this thesis is to give exact solutions to the entire Burgers' hierarchy of nonlinear PDEs using Lie symmetries and conservation laws.

Chapter 2 lays down the fundamental theory and notations adopted in this thesis.

In Chapter 3, we investigate for the first time the conservation laws of Burgers' hierarchy to provide its general  $n$ th conservation law. The fundamental theorem of double reduction is used to obtain a ODE expressions via a coordinate transformation. To linearise the expressions, we employ yet another transformation. Consequently, we provide a novel formula that enables one to solve the entire Burgers' hierarchy for all values of  $n$ . We demonstrate the application of our formula with the first four members of the hierarchy and provide graphical illustrations of the solutions that are obtained.

The results of Chapter 3 have been published [8].

In Chapter 4, we provide a methodical procedure to obtain the solution of a  $(n+1)$ th order ODE that is linked to Burgers' hierarchy. Its symmetries are examined and we show that these are obtained from the  $(n+1)$  complex roots of a particular polynomial. We provide plots of these roots. A formula for solving the ODE using these roots is provided and its application demonstrated along with their graphical illustrations. Lastly, we establish a theorem, using this ODE, that gives the exact solutions of the entire Burgers' hierarchy, which differs from the solutions constructed in the previous chapter.

Chapter 4 has been submitted to a journal.

# Chapter 2

## Notation and Theory

### 2.1 Symmetry Methods

We outline the general procedure for determining point symmetries for an arbitrary system of equations. Consider an  $r$ th-order (in derivatives) system of partial differential equations of  $n$  independent variables  $x = (x^1, x^2, \dots, x^n)$  and  $m$  dependent variables  $u = (u^1, u^2, \dots, u^m)$ ,

$$G^\alpha(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \alpha = 1, \dots, m, \quad (1)$$

where  $u_{(1)}, u_{(2)}, \dots, u_{(r)}$  denote the collections of all first, second,  $\dots$ ,  $r$ th-order partial derivatives.

A one-parameter Lie group of transformations ( $\epsilon$  is the group parameter) that is invariant under (1) is given by

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \quad (2)$$

Expanding (2) around the identity  $\epsilon = 0$ , we can generate the following infinitesimal transformations:

$$\begin{aligned}\bar{x}^i &= x^i + \epsilon \xi^i(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u}^\alpha &= u^\alpha + \epsilon \eta^\alpha(x, u) + \mathcal{O}(\epsilon^2).\end{aligned}\tag{3}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}.\tag{4}$$

The action of  $X$  is extended to all derivatives appearing in the equation in question through the appropriate prolongation. The infinitesimal criterion for invariance is given by

$$X [G^\alpha(x, u^{(k)})] = 0, \quad \text{when} \quad G^\alpha(x, u^{(k)}) = 0.\tag{5}$$

Equation (5) yields an overdetermined system of linear homogeneous equation which can be solved algorithmically, more details can be found in [7] among other texts.

## 2.2 Conservation Laws

A current  $T = (T^1, \dots, T^n)$  is conserved if it satisfies

$$D_i T^i = 0,\tag{6}$$

along the solutions of (1). Eq. (6) is called a *local conservation law*.

## 2.3 The Total Differentiation Operator

**Definition 2.3.1.** *The total differentiation operator  $D_i$  with respect to  $x^i$  is given by*

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n. \quad (7)$$

## 2.4 Multipliers

Conservation laws may be constructed from multipliers (analogous to integrating factors). The multipliers  $\Lambda^\alpha$  of (1) satisfies the relation

$$D_i T^i = \Lambda^\alpha G^\alpha, \quad (8)$$

for all functions  $u^\alpha$ .

The overdetermined equations for  $\Lambda^\alpha$  are

$$\frac{\delta}{\delta u^\alpha} [\Lambda^\alpha G^\alpha] = 0. \quad (9)$$

## 2.5 Invariant Solutions

The operator in Equation (4) can be used to define the Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whose solution provides the zero-order invariants

$$W^{[0]}(x^i, u^\alpha). \quad (10)$$

These invariants can be used in order to reduce the order of the PDE.

## 2.6 Double Reduction of PDEs

This section describes a theory that enables a double reduction of PDEs or a system of PDEs with two independent variables. This is possible when the equations provide a symmetry that is associated with a conservation law [9].

**Definition 2.6.1.** *A symmetry generator  $X$  is associated with a conserved vector  $T$  of the system (1) if  $X$  and  $T$  satisfy the relation*

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^j) = 0. \quad (11)$$

**Theorem 2.6.1.** *A PDE  $F = 0$  of order  $q$  with two independent variables, which admits a symmetry  $X$  that is associated with a conserved vector  $T$ , is reduced to an ODE of order  $q - 1$ , namely  $T^r = k$ , where  $T^r$  is given by*

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}, \quad (12)$$

*for solutions invariant under  $X$ .*

## Chapter 3

# A Computational Procedure for Exact Solutions of Burgers' Hierarchy of Nonlinear Partial Differential Equations

Burgers' hierarchy is a fundamental example of nonlinear evolution equations and can be expressed in the form

$$u_t + D_x (D_x + u)^n u = 0, \quad n = 0, 1, 2, \dots \quad (14)$$

Solutions of certain members of equation (14) were discussed in many papers, see for example [10]-[15], and notably in [1], rational, periodic and solitary wave solutions were obtained for the hierarchy. Solutions of some related equations and their applications can be found in [16]-[21]. The significance of exact solutions in all areas of science can be seen in the recent collection of articles [22]-[31], while the prevalence

of conservation laws in analytical studies appears in the works [32] and [33].

This study considers the exact solution of Burgers' hierarchy of nonlinear evolution equations. We construct the general  $n$ th conservation law of the hierarchy and prove that these expressions may be transformed into ODEs. In particular, a coordinate transformation leads to the systematic reduction of the conservation law properties of the Burgers' hierarchy. Such an approach yields a nonlinear equation, where a second transformation is derived to linearise the expression. Consequently, this approach describes a procedure for finding the exact solutions of the hierarchy. A formula of the  $n$ th solution is provided. We note that the solutions obtained have not appeared elsewhere. In the literature, many solutions are obtained for only one member of the hierarchy, and for low values of  $n$ , as mentioned. On the other hand, the greatest advantage of our study is that it provides a solution for any  $n$ , that is, the full hierarchy. To demonstrate the application of our formula, we discuss the solution to several members of the nonlinear hierarchy.

### 3.1 The Burgers' Hierarchy of Nonlinear PDEs

This hierarchy is of order  $n + 1$  in derivatives where the value of  $n = 1$  gives the infamous Burgers' equation [6]

$$u_t + 2uu_x + u_{xx} = 0. \tag{15}$$

Applications of this equation include, but are not limited to, acoustics, traffic flow, gas dynamics etc. The Burgers' equation can be linearised by a Cole-Hopf transformation [34].

For the parameter  $n = 2$ , the hierarchy presents the Sharma - Tasso - Olver (STO) equation [35]

$$u_t + u_{xxx} + 3u_x^2 + 3uu_{xx} + 3u^2u_x = 0, \quad (16)$$

which is present in several areas of classical and quantum physics, including dispersive wave phenomena, quantum field theory and relativistic mechanics.

For the values  $n = 3$  and  $n = 4$ , the respective members of the hierarchy are

$$u_t + u_{xxxx} + 10u_xu_{xx} + 4uu_{xxx} + 12uu_x^2 + 6u^2u_{xx} + 4u^3u_x = 0, \quad (17)$$

and

$$\begin{aligned} u_t + u_{xxxxx} + 10u_{xx}^2 + 15u_xu_{xxx} + 5uu_{xxxx} + 15u_x^3 + 50uu_xu_{xx} \\ + 10u^2u_{xxx} + 30u^2u_x^2 + 10u^3u_{xx} + 5u^4u_x = 0, \end{aligned} \quad (18)$$

with applications involving wave phenomena.

The Lie point symmetries for Burgers' equation are

$$\begin{aligned} X_1 &= \partial_t, \quad X_2 = \partial_x, \\ X_3 &= 2t\partial_x + \partial_u, \\ X_4 &= 2t\partial_t - u\partial_u + x\partial_x, \end{aligned} \quad (19)$$

while for the STO equation the Lie point symmetries are

$$X_1, X_2, X_3 = 3t\partial_t - u\partial_u + x\partial_x. \quad (20)$$

If one repeats the Lie symmetry method for higher members, it is easy to see that Eq. (14) possesses the Lie point symmetries

$$X_1, X_2, X_n = (n + 1)t\partial_t - u\partial_u + x\partial_x. \quad (21)$$

## 3.2 The Conserved Vector Transformation

We now present the conserved vectors of the hierarchy, and via a transformation we obtain a nonlinear ODE. Subsequently, this equation is linearised and solved to provide a solution to the entire Burgers' hierarchy. To begin, we establish the  $n$ th conserved vector of Eq. (14) by the following theorem.

**Theorem 3.2.1.** *Burgers' hierarchy possesses the conserved vector  $T = (T^t, T^x) = (u, (D_x + u)^n u)$  along the solutions of Eq. (14).*

*Proof.* Suppose the conservation law is

$$D_t T^t + D_x T^x = 0, \quad (22)$$

where  $T^t$  is the conserved density and  $T^x$  is the conserved flux. Then, from Eq. (14) we have

$$\begin{aligned} u_t + D_x \left( (D_x + u)^n u \right) &= D_t(u) + D_x \left( (D_x + u)^n u \right) \\ &= 0, \end{aligned} \quad (23)$$

along the solutions of Eq. (14), and the result follows.  $\square$

Hence, the conserved density for every member of the hierarchy is  $T^t = u$  while the conserved flux is  $T^x = (D_x + u)^n u$ . For example, the fluxes for the first few members of the hierarchy, are

$$n = 1 \quad : \quad T^x = u^2 + u_x, \quad (24)$$

$$n = 2 \quad : \quad T^x = u^3 + 3uu_x + u_{xx}, \quad (25)$$

$$n = 3 \quad : \quad T^x = u_{xxx} + 3u_x^2 + 4uu_{xx} + 6u^2u_x + u^4, \quad (26)$$

$$\begin{aligned} n = 4 \quad : \quad T^x &= u_{xxxx} + 10u_xu_{xx} + u^5 + 5uu_{xxx} + 15uu_x^2 \\ &\quad + 10u^2u_{xx} + 10u^3u_x, \end{aligned} \quad (27)$$

etc.

To tie together the concept of conservation laws and point symmetries, we require an association condition between the two. Thus, a symmetry generator  $X = \xi^t(x, t, u) \partial_t + \xi^x(x, t, u) \partial_x + \eta(x, t, u) \partial_u$  is associated with a conserved vector  $T$  of Eq. (14) if  $X$  and  $T$  satisfies the condition (11) [36].

We apply this condition to the first few members of the hierarchy below.

- For  $n = 1$ , the Burgers' equation, we test each of the symmetries:

$X_1$ :

$$X_1 \begin{pmatrix} u \\ u^2 + u_x \end{pmatrix} = \underline{0}. \quad (28)$$

$X_2$ :

$$X_2 \begin{pmatrix} u \\ u^2 + u_x \end{pmatrix} = \underline{0}. \quad (29)$$

$X_3$ :

$$X_3^{[1]} \begin{pmatrix} u \\ u^2 + u_x \end{pmatrix} - \begin{pmatrix} 0 \\ 2u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \underline{0}. \quad (30)$$

$X_4$ :

$$X_4^{[1]} \begin{pmatrix} u \\ u^2 + u_x \end{pmatrix} + \begin{pmatrix} u \\ 2u^2 + 2u_x \end{pmatrix} = \begin{pmatrix} 0 \\ u_x \end{pmatrix} \neq \underline{0}. \quad (31)$$

- For  $n = 2$ , the STO equation:

$X_1$ :

$$X_1 \begin{pmatrix} u \\ u^3 + 3uu_x + u_{xx} \end{pmatrix} = \underline{0}. \quad (32)$$

$X_2$ :

$$X_2 \begin{pmatrix} u \\ u^3 + 3uu_x + u_{xx} \end{pmatrix} = \underline{0}. \quad (33)$$

$X_3$ :

$$X_3^{[1]} \begin{pmatrix} u \\ u^3 + 3uu_x + u_{xx} \end{pmatrix} + \begin{pmatrix} u \\ 3u^3 + 9uu_x + 3u_{xx} \end{pmatrix} = \begin{pmatrix} 0 \\ 3uu_x + 3u_{xx} \end{pmatrix} \neq \underline{0}. \quad (34)$$

When we repeat the above computations for higher members of the hierarchy it is easy to observe that the point symmetry  $X_1$ , when applied to condition (11) is

$$X_1 \begin{pmatrix} u \\ (D_x + u)^n u \end{pmatrix} = \underline{0}, \quad (35)$$

and for  $X_2$

$$X_2 \begin{pmatrix} u \\ (D_x + u)^n u \end{pmatrix} = \underline{0}. \quad (36)$$

Therefore, both  $X_1$  and  $X_2$  satisfy the association condition and we conclude that they are associated with every conservation law of Theorem 3.2.1, i.e. with any hierarchy member. Next, we recall the fundamental theorem on double reduction [9, 37], which states that there exist functions  $T^r$  such that

$$D_r T^r = 0. \quad (37)$$

The transformed conserved quantity may be expressed as (12), where  $r$  and  $s$  are similarity variables connected to an associated symmetry  $X$ .

Next, we construct the transformed conserved quantity for Burgers' hierarchy.

**Lemma 3.2.1.** *The conserved quantity of Eq. (14) can be reduced to*

$$\tilde{u} - c(cD_r + \tilde{u})^n \tilde{u}, \quad (38)$$

where  $\tilde{u}(r)$ .

*Proof.* Since  $X_1$  and  $X_2$  are associated with the conserved vector  $T$ , we consider the linear combination  $X = X_2 + cX_1$ , ( $c$  is a constant) to obtain the similarity transformation

$$r = cx - t, \quad s = x, \quad u(x, t) = \tilde{u}(r). \quad (39)$$

Hence, by (12) we have

$$T^r = \frac{u(-1) + (D_x + u)^n uc}{-1}. \quad (40)$$

In the new variables, Eq. (40) transforms to

$$T^r = \tilde{u} - c(cD_r + \tilde{u})^n \tilde{u}. \quad (41)$$

□

Based on Eq. (37), we have that  $T^r = k$ ,  $k$  constant. We set  $k = 0$  so that

$$\tilde{u} - c(cD_r + \tilde{u})^n \tilde{u} = 0. \quad (42)$$

Therefore, we seek the solution of the ODE (42), as it solves Burgers' hierarchy.

**Lemma 3.2.2.** *The general solution of Eq. (42) is  $\tilde{u}(r) = c \frac{v'(r)}{v(r)}$  where  $v(r)$  satisfies the ODE*

$$v^{(n+1)} c^{n+1} - v' = 0. \quad (43)$$

*Proof.* Firstly, for  $n = 1$ , the following expression holds

$$\begin{aligned} c \left( cD_r + c \frac{v'}{v} \right) c \frac{v'}{v} &= c^3 \left( \frac{v''}{v} + \frac{(v')^2}{v^2} - \frac{(v')^2}{v^2} \right) \\ &= c^3 \frac{v''}{v}. \end{aligned} \quad (44)$$

As an inductive proof, suppose that for  $n = m - 1$ , we have

$$c \left( cD_r + c \frac{v'}{v} \right)^{m-1} \frac{cv'}{v} = c^{m+1} \frac{v^{(m)}}{v}. \quad (45)$$

For  $n = m$ , the expression is

$$\begin{aligned} c \left( cD_r + c \frac{v'}{v} \right)^m c \frac{v'}{v} &= c \left( cD_r + \frac{cv'}{v} \right) \left( cD_r + c \frac{v'}{v} \right)^{m-1} c \frac{v'}{v} \\ &= \left( cD_r + c \frac{v'}{v} \right) c^{m+1} \frac{v^{(m)}}{v} \\ &= c^{m+2} \left( \frac{v^{(m+1)}}{v} + \frac{v^{(m)}v'}{v^2} - \frac{v^{(m)}v'}{v^2} \right) \\ &= c^{m+2} \frac{v^{(m+1)}}{v}. \end{aligned} \quad (46)$$

Hence, we linearise (42) via the transformation  $\tilde{u} = c \frac{v'}{v}$ , so that (42) becomes

$$c^{n+2} \frac{v^{(n+1)}}{v} - c \frac{v'}{v} = 0, \quad (47)$$

and the result follows.  $\square$

The above Lemma prompts the following theorem.

**Theorem 3.2.2.** *The solution of the  $n$ th member of Burgers' hierarchy Eq. (14) is*

$$u(x, t) = \frac{c^{1/n} \sum_{j=1}^n A_j \exp \left( -\frac{2j\pi(n+1)i}{n} \right) \exp \left( c^{-\frac{n+1}{n}} \exp \left( -\frac{2j\pi(n+1)i}{n} \right) (cx - t) \right)}{A_0 + \sum_{j=1}^n A_j \exp \left( c^{-\frac{n+1}{n}} \exp \left( -\frac{2j\pi(n+1)i}{n} \right) (cx - t) \right)}, \quad (48)$$

where  $A_0$  and  $A_j$  are arbitrary constants.

*Proof.* Eq. (43) has the root of zero and the set of roots  $c^{-\frac{n+1}{n}}$ . Therefore, the general solution of (43) is

$$v(r) = A_0 + \sum_{j=1}^n A_j \exp \left( c^{-(n+1)/n} \exp \left( -\frac{2j\pi(n+1)i}{n} \right) r \right), \quad (49)$$

which provides  $\tilde{u}(r)$  and from the transformations (39) we obtain the solution (48).  $\square$

We illustrate some examples in the next section.

### 3.3 Applications

Let us apply the above formulae to the first few members of the hierarchy.

- Burgers' equation  $n = 1$ . The conserved density  $T^t$  and flux (24), under the transformations (39) is

$$T^r = \tilde{u} - c\tilde{u}^2 - c^2\tilde{u}_r. \quad (50)$$

By the Lemma 3.2.2, we have the ODE

$$c^2v'' - v' = 0,$$

that has the solution

$$v(r) = A_0 + A_1 \exp \left( \frac{r}{c^2} \right).$$

From the transformation  $\tilde{u} = c\frac{v'}{v}$ , we find

$$\tilde{u}(r) = \frac{A_1 \exp \left( \frac{r}{c^2} \right)}{c \left( A_0 + A_1 \exp \left( \frac{r}{c^2} \right) \right)}, \quad (51)$$

and by Theorem 3.2.2, the solution of Burgers' equation is

$$u(x, t) = \frac{A_1 \exp\left(\frac{cx-t}{c^2}\right)}{c\left(A_0 + A_1 \exp\left(\frac{cx-t}{c^2}\right)\right)}. \quad (52)$$

- The STO equation  $n = 2$ . The conserved density  $T^t$  and flux (25), under the transformations (39) is

$$T^r = \tilde{u} - c\tilde{u}^3 - 3c^2\tilde{u}\tilde{u}_r - c^3\tilde{u}_{rr}. \quad (53)$$

Lemma 3.2.2 gives the following equation

$$c^3v''' - v' = 0,$$

which is solved to obtain

$$v(r) = A_0 + A_1 \exp\left(-\frac{r}{c^{3/2}}\right) + A_2 \exp\left(\frac{r}{c^{3/2}}\right).$$

From this solution, we may construct  $\tilde{u}(r)$ , viz.

$$\tilde{u}(r) = -\frac{-A_1 \exp\left(\frac{r}{c^{3/2}}\right) + A_2 \exp\left(\frac{r}{c^{3/2}}\right)}{\sqrt{c}\left(A_0 + A_1 \exp\left(-\frac{r}{c^{3/2}}\right) + A_2 \exp\left(\frac{r}{c^{3/2}}\right)\right)}, \quad (54)$$

and with Theorem 3.2.2 we get a solution of the STO equation

$$u(x, t) = \frac{-A_1 \exp\left(-\frac{cx-t}{c^{3/2}}\right) + A_2 \exp\left(\frac{cx-t}{c^{3/2}}\right)}{\sqrt{c}\left(A_0 + A_1 \exp\left(-\frac{cx-t}{c^{3/2}}\right) + A_2 \exp\left(\frac{cx-t}{c^{3/2}}\right)\right)}. \quad (55)$$

- The hierarchy member with  $n = 3$ , with transformation (39) for the conserved density  $T^t$  and flux (26), yields

$$T^r = \tilde{u} - c^4\tilde{u}_{rrr} - 3c^3\tilde{u}_r^2 - 4c^3\tilde{u}\tilde{u}_{rr} - 6c^2\tilde{u}^2\tilde{u}_r - c\tilde{u}^4. \quad (56)$$

Here, Lemma 3.2.2 gives us

$$c^4 v'''' - v' = 0,$$

that has the solution

$$v(r) = A_0 + A_1 \exp\left(-\frac{(i\sqrt{3}+1)r}{2c^{4/3}}\right) + A_2 \exp\left(\frac{(i\sqrt{3}-1)r}{2c^{4/3}}\right) + A_3 \exp\left(\frac{r}{c^{4/3}}\right). \quad (57)$$

To ensure a real solution, we let  $-i\bar{A}_2 = A_2 - A_1$  and  $\bar{A}_1 = A_1 + A_2$  and use Euler's formula to obtain

$$v(r) = A_0 + \exp\left(-\frac{r}{2c^{4/3}}\right) \left( \bar{A}_2 \cos\left(\frac{\sqrt{3}r}{2c^{4/3}}\right) + \bar{A}_1 \sin\left(\frac{\sqrt{3}r}{2c^{4/3}}\right) \right) + A_3 \exp\left(\frac{r}{c^{4/3}}\right). \quad (58)$$

Therefore,

$$\tilde{u}(r) = \frac{1}{\sqrt[3]{c}} \frac{A_3 e^{\frac{r}{c^{4/3}}} + \frac{1}{2} e^{-\frac{r}{2c^{4/3}}} \left( (\bar{A}_1 \sqrt{3} - \bar{A}_2) \cos\left(\frac{\sqrt{3}r}{2c^{4/3}}\right) - \sin\left(\frac{\sqrt{3}r}{2c^{4/3}}\right) (\bar{A}_2 \sqrt{3} + \bar{A}_1) \right)}{A_0 + A_3 e^{\frac{r}{c^{4/3}}} + e^{-\frac{r}{2c^{4/3}}} \left( \bar{A}_2 \cos\left(\frac{\sqrt{3}r}{2c^{4/3}}\right) + \bar{A}_1 \sin\left(\frac{\sqrt{3}r}{2c^{4/3}}\right) \right)}, \quad (59)$$

and the solution for the  $n = 3$  hierarchy is

$$u(x, t) = \frac{1}{\sqrt[3]{c}} \left( \frac{A_3 e^{\frac{cx-t}{c^{4/3}}} + \frac{1}{2} e^{-\frac{cx-t}{2c^{4/3}}} \left( (\bar{A}_1 \sqrt{3} - \bar{A}_2) \cos\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) - \sin\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) (\bar{A}_2 \sqrt{3} + \bar{A}_1) \right)}{A_0 + A_3 e^{\frac{cx-t}{c^{4/3}}} + e^{-\frac{cx-t}{2c^{4/3}}} \left( \bar{A}_2 \cos\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) + \bar{A}_1 \sin\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) \right)} \right) \frac{\sin\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) (\bar{A}_2 \sqrt{3} + \bar{A}_1)}{A_0 + A_3 e^{\frac{cx-t}{c^{4/3}}} + e^{-\frac{cx-t}{2c^{4/3}}} \left( \bar{A}_2 \cos\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) + \bar{A}_1 \sin\left(\frac{\sqrt{3}(cx-t)}{2c^{4/3}}\right) \right)}. \quad (60)$$

- The case of  $n = 4$  provides the transformed density and flux (27) as

$$T^r = \tilde{u} - c^5 \tilde{u}_{rrrr} - 10c^4 \tilde{u}_r \tilde{u}_{rr} - c\tilde{u}^5 - 5c^4 \tilde{u} \tilde{u}_{rrr} - 15c^3 \tilde{u} \tilde{u}_r^2 - 10c^3 \tilde{u}^2 \tilde{u}_{rr} - 10c^2 \tilde{u}^3 \tilde{u}_r. \quad (61)$$

In this case, Lemma 3.2.2 provides the expression

$$c^5 v'''' - v' = 0.$$

We solve to obtain

$$v(r) = A_0 + A_1 e^{\frac{-ir}{c^{5/4}}} + A_2 e^{-\frac{r}{c^{5/4}}} + A_3 e^{\frac{ir}{c^{5/4}}} + A_4 e^{\frac{r}{c^{5/4}}}.$$

Similar to the previous case, we ensure a real general solution by Euler's formula and the substitution  $\bar{A}_1 = A_1 + A_3$  and  $\bar{A}_3 = A_3 - A_1$ , namely

$$v(r) = A_0 + \bar{A}_1 \cos\left(\frac{r}{c^{5/4}}\right) + A_2 e^{-\frac{r}{c^{5/4}}} + \bar{A}_3 \sin\left(\frac{r}{c^{5/4}}\right) + A_4 e^{\frac{r}{c^{5/4}}}.$$

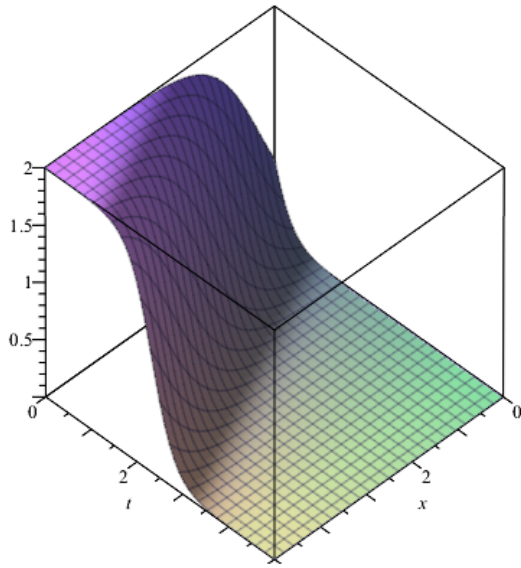
Finally, Theorem 3.2.2 states that the solution for the hierarchy member is

$$u(x, t) = \frac{1}{\sqrt[4]{C}} \frac{-\bar{A}_1 \sin\left(\frac{cx-t}{c^{5/4}}\right) - A_2 e^{-\frac{cx-t}{c^{5/4}}} + \bar{A}_3 \cos\left(\frac{cx-t}{c^{5/4}}\right) + A_4 e^{\frac{cx-t}{c^{5/4}}}}{A_0 + \bar{A}_1 \cos\left(\frac{cx-t}{c^{5/4}}\right) + A_2 e^{-\frac{cx-t}{c^{5/4}}} + \bar{A}_3 \sin\left(\frac{cx-t}{c^{5/4}}\right) + A_4 e^{\frac{cx-t}{c^{5/4}}}}. \quad (62)$$

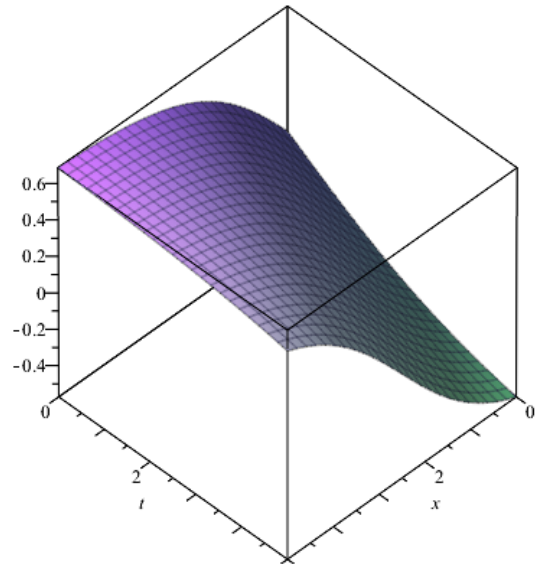
In Figure 3.1 we show the graphical solution of  $u(x, t)$  for the above cases.

### 3.4 Conclusion

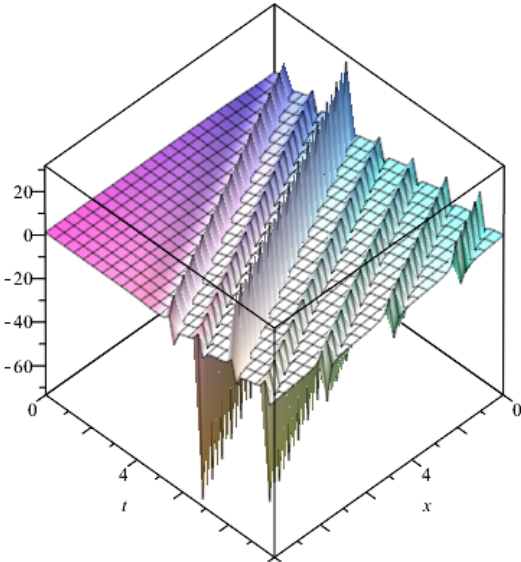
In this study, for the first time the conservation law of Burgers' hierarchy was investigated and its general form applied to solve the hierarchy. Firstly, a coordinate transformation was exploited to reduce the order of the conserved expression, followed by a second transformation that was selected to linearise the equation. Ultimately, we proved that this procedure leads to a solution of the entire hierarchy. Subsequently, we constructed a new formula for the solutions of any member of the hierarchy, consisting of exponential functions. The first four members of the hierarchy were solved to illustrate the method. It is possible to extend our approach



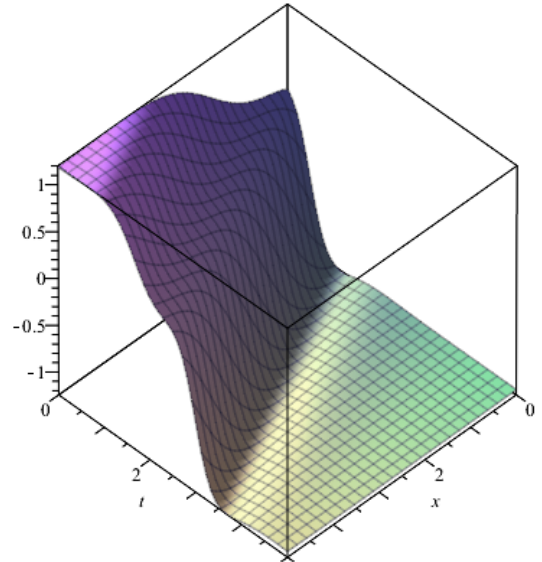
(a)



(b)



(c)



(d)

Figure 3.1: Graphical illustration of the analytical solutions are depicted, we select the parameter values  $A_j = \bar{A}_j = 1$  : (a) solution of (52) where  $c = 0.5$ ; (b) solution of (55) where  $c = 2$ ; (c) solution of (60) where  $c = 1$ ; and, (d) solution of (62) where  $c = 0.5$ .

to solve other ordinary or partial differential hierarchies. This would depend on the nature of the hierarchy and on condition that appropriate transformations are found where necessary. The present study forms an important framework for such an investigation.

## Chapter 4

# On the Formulaic Solution of a $(n + 1)^{th}$ Order Differential Equation

A systematic method to facilitate the solution of a  $(n + 1)^{th}$  order differential equation is introduced. This set of ODEs is unique in the sense that it defines a hierarchy whose solution can be manipulated to solve a partial differential hierarchy, called Burgers' hierarchy. The latter forms a highly nonlinear evolutionary class of equations. Our approach is twofold: a) we construct the one-parameter Lie group of transformations that leave the ODE invariant, and find that they are universally derived from the  $(n + 1)$  complex roots of a certain polynomial, and b) we then prove that the exact solution of the hierarchy for all values of  $n$ , is given by a simple integration formula. An important consequence of this study, is the extension of the formulaic method to exactly solve the entire Burgers' hierarchy. We exemplify the advantageous nature of our results through solving various members of the hierarchy.

## 4.1 Introduction

This study considers the ordinary differential hierarchy [8]

$$v(r)^{(n+1)}c^{n+1} - v'(r) - \frac{k}{c}v(r) = 0, \quad n = 1, 2, \dots \quad (63)$$

where  $k, c$  are constants, which is connected to Burgers' hierarchy of nonlinear equations (14). Specifically, this family of ODEs is a solution to Burgers' hierarchy, under the coordinate transformation  $r = cx - t$  and where

$$u(x, t) = c \frac{v'}{v}. \quad (64)$$

That is, we consider (38) in the same description as the previous chapter, but with  $k \neq 0$ .

It is our mission to analyse and solve the ODE (63) for all values of  $n$ . We undertake this study by exploring the one-parameter Lie group of transformations that leave (63) invariant. This is no easy task, as the complexity of (63) increases as  $n$  increases. Moreover, we find that the symbolic software [38]-[43], that is often used for the determination of the Lie point symmetries, fails to provide symmetry vectors for higher-order  $n$ . We shall tackle this problem below and establish a result that presents the symmetries of (63), for all values of  $n$ . It becomes apparent that all the cases have symmetries of exponential type. What is fascinating about our findings is that for  $n \geq 2$ , the ODE (63) admits  $n + 3$  symmetries, giving rise to a theorem. The  $n + 1$  symmetries are those in exponential nonlocal form and also contain the complex roots of a polynomial which will be discussed in this study.

One of the primary aims of this study is to extend our solutions for (63) to find a solution to the entire Burgers' hierarchy,  $u(x, t)$ , with the help of relation (64). In order to achieve this, we seek to formulate a method to find the analytical solutions to (63) i.e  $v(r)$ . Our proposed method uses a linear combination of the Lie point symmetries of (63) to construct characteristic equations, which are integrated to find the general solution for  $v(r)$  for any  $n$ . We iterate that the solutions for the Burgers' hierarchy, previously obtained in literature, are for few members of the hierarchy or for low values of  $n$ . Hence, the results of this paper equips us to find solutions for the entire Burgers' hierarchy by finding exact solutions of the ODE (63) for any  $n$ . We shall illustrate all theoretical results, with the use of examples and graphically.

Section 4.2 is devoted to the construction of our main results. We start-off by presenting the Lie point symmetries for (63) for  $n = 1$  and then introduce a theorem on the exponential symmetries that solve the Burgers' hierarchy. We illustrate this theorem using examples for different values of  $n$  and we present a graphical representation of the roots of the polynomials that are used. In subsection 4.2.2 we produce a theorem that gives the solutions for  $v(r)$  and highlight the applications of this theorem in finding the (real) solutions of different values of  $n$ . Several plots are given of  $v(r)$ . The solutions give rise to our final theorem which is introduced in the next subsection, and provides the exact solutions of Burgers' hierarchy. With the help of this theorem we plot some solutions of  $u(x, t)$ . Concluding remarks are given in section 4.3.

## 4.2 Main Results

The main results are split as follows. We first establish the point symmetries of Eq.(63)  $\forall n$ . For low values of the differential equation, i.e.  $n < 3$ , algebraic

software can easily provide the symmetries. However, we show that the Lie point symmetries are admitted by the exponents of roots of a  $(n+1)^{th}$  degree polynomial. These exponential symmetries provide Lie invariants which we exploit to construct solutions of the main equation, Eq.(63), once again for all values of  $n$ . Lastly, as mentioned previously, the solutions in terms of the function  $v(r)$  satisfy a relation that connects them to the solution of the PDE hierarchy, the Burgers' hierarchy.

## 4.2.1 Point Generators

A straightforward calculation for the Lie point symmetries of the case  $n = 1$ , yields that Eq. (63) is maximally symmetric and possesses the 8 dimensional Lie algebra of symmetries:

$$\begin{aligned}
X_r &= \partial_r, \\
X_v &= v\partial_v, \\
X_1^1 &= e^{1/2 \frac{(1+\sqrt{4ck+1})r}{c^2}} \partial_v, \\
X_2^1 &= e^{-1/2 \frac{(-1+\sqrt{4ck+1})r}{c^2}} \partial_v, \\
X_3^1 &= e^{1/2 \frac{(-1+\sqrt{4ck+1})r}{c^2}} v\partial_r + \frac{1}{2} \frac{v^2 (1 + \sqrt{4ck+1}) e^{1/2 \frac{(-1+\sqrt{4ck+1})r}{c^2}}}{c^2} \partial_v, \\
X_4^1 &= e^{-1/2 \frac{(1+\sqrt{4ck+1})r}{c^2}} v\partial_r - \frac{1}{2} \frac{v^2 (-1 + \sqrt{4ck+1}) e^{-1/2 \frac{(1+\sqrt{4ck+1})r}{c^2}}}{c^2} \partial_v, \\
X_5^1 &= \sin\left(\frac{\sqrt{-4ck-1}r}{c^2}\right) \partial_r + \\
&\quad \frac{\left(\left(-2ck - \frac{1}{2}\right) \cos\left(\frac{\sqrt{-4ck-1}r}{c^2}\right) + \frac{1}{2} \sqrt{-4ck-1} \sin\left(\frac{\sqrt{-4ck-1}r}{c^2}\right)\right) v}{\sqrt{-4ck-1}c^2} \partial_v,
\end{aligned}$$

$$X_6^1 = \cos\left(\frac{\sqrt{-4ck-1}r}{c^2}\right)\partial_r + \frac{\left((2ck + \frac{1}{2})\sin\left(\frac{\sqrt{-4ck-1}r}{c^2}\right) + \frac{1}{2}\sqrt{-4ck-1}\cos\left(\frac{\sqrt{-4ck-1}r}{c^2}\right)\right)v}{\sqrt{-4ck-1}c^2}\partial_v.$$

As  $n$  increases, an iteration of the above calculation produces voluminous, over-determined symmetry determining systems, that are difficult to solve. This problem has inspired the following result.

**Theorem 4.2.1.** *Eq. (63) admits  $n + 3$  symmetries for  $n \geq 2$ , viz.*

$$X_r = \partial_r,$$

$$X_v = v\partial_v,$$

and the  $n + 1$  symmetries

$$X_i^n = e^{z_i n r} \partial_v, \quad i = 1, \dots, n + 1,$$

where we denote  $z_{i,n}$  as the  $n + 1$  roots of the polynomial  $P(z) = z^{n+1}c^{n+2} - zc - k$ .

*Proof.* We recall the symmetry determining relation Eq. (5). The operation of  $X_r$  on Eq. (63) is zero and  $X_v$  is known as the linear symmetry as it appears for every linear equation. Moreover, for the exponential nonlocal symmetries, we have that

$$\begin{aligned} X_i^n \left( v(r)^{(n+1)}c^{n+1} - v'(r) - \frac{k}{c}v(r) \right) &= (e^{z_i n r} \partial_v + z_{i,n} e^{z_i n r} \partial_{v'} + \dots + z_{i,n}^{n+1} e^{z_i n r} \partial_{v^{(n+1)}}) \\ &\quad \times \left( v(r)^{(n+1)}c^{n+1} - v'(r) - \frac{k}{c}v(r) \right) \\ &= e^{z_i n r} (z^{n+1}c^{n+2} - zc - k) \\ &= 0, \end{aligned}$$

for the roots of  $P(z)$ . □

Furthermore, the  $n + 3$  symmetries above, possess the Lie bracket relations

$$\begin{aligned} [X_r, X_v] &= 0, \\ [X_r, X_i^n] &= e^{z_i, n} X_i^n, \\ [X_v, X_i^n] &= -X_i^n, \\ [X_i^n, X_j^n] &= 0, \quad j = 1, \dots, n + 1. \end{aligned}$$

Note that for  $n = 1$ , the symmetries  $X_1^1$  and  $X_2^1$ , also originate from the roots of the polynomial

$$P(z) = c^3 z^2 - cz - k, \quad (65)$$

that is, the  $z_{i,1}$  roots are

$$z_{1,1} = \frac{c - c\sqrt{4ck + 1}}{2c^3}, \quad z_{2,1} = \frac{c + c\sqrt{4ck + 1}}{2c^3}. \quad (66)$$

We provide several illustrations of the Theorem 4.2.1 up to  $n = 3$ . Due to the nature of the roots, the results consume a lot of space. In Figures 4.1 and 4.2, we demonstrate the complex roots for higher values of  $n$ . In the case of  $n = 2$ , the polynomial is

$$P(z) = c^4 z^3 - cz - k, \quad (67)$$

which has the roots

$$z_{1,2} = \frac{\sqrt[3]{\frac{2}{3}c}}{\sqrt[3]{9c^8k + \sqrt{3}\sqrt{27c^{16}k^2 - 4c^{15}}}} + \frac{\sqrt[3]{9c^8k + \sqrt{3}\sqrt{27c^{16}k^2 - 4c^{15}}}}{\sqrt[3]{23^{2/3}c^4}}, \quad (68)$$

$$z_{2,2} = -\frac{(1 + i\sqrt{3})c}{2^{2/3}\sqrt[3]{3}\sqrt[3]{9c^8k + \sqrt{3}\sqrt{27c^{16}k^2 - 4c^{15}}}} - \frac{(1 - i\sqrt{3})\sqrt[3]{9c^8k + \sqrt{3}\sqrt{27c^{16}k^2 - 4c^{15}}}}{2\sqrt[3]{23^{2/3}c^4}}, \quad (69)$$

$$z_{3,2} = -\frac{(1 - i\sqrt{3})c}{2^{2/3}\sqrt[3]{3}\sqrt[3]{9c^8k + \sqrt{3}\sqrt{27c^{16}k^2 - 4c^{15}}}} - \frac{(1 + i\sqrt{3})\sqrt[3]{9c^8k + \sqrt{3}\sqrt{27c^{16}k^2 - 4c^{15}}}}{2\sqrt[3]{23^{2/3}c^4}}. \quad (70)$$

Therefore, the Lie point symmetries of Eq. (63) are  $X_r$ ,  $X_v$  and

$$X_1^2 = e^{z_1, 2r} \partial_v, \quad X_2^2 = e^{z_2, 2r} \partial_v, \quad X_3^2 = e^{z_3, 2r} \partial_v.$$

Similarly, for  $n = 3$ , we have

$$P(z) = c^5 z^4 - cz - k, \quad (71)$$

that has the roots

$$\begin{aligned} z_{1,3} = & -\frac{1}{2} \left( \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{2}3^{2/3}c^5} - \frac{4\sqrt[3]{\frac{2}{3}}k}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} \right)^{\frac{1}{2}} \\ & - \frac{1}{2} \left( \frac{4\sqrt[3]{\frac{2}{3}}k}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} - \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{2}3^{2/3}c^5} \right. \\ & \left. - \frac{2}{c^4 \sqrt{\frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{2}3^{2/3}c^5} - \frac{4\sqrt[3]{\frac{2}{3}}k}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}} \right)^{\frac{1}{2}}, \quad (72) \end{aligned}$$

$$\begin{aligned} z_{2,3} = & \frac{1}{2} \left( \frac{4\sqrt[3]{\frac{2}{3}}k}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} - \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{2}3^{2/3}c^5} - \right. \\ & \left. \frac{2}{c^4 \sqrt{\frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{2}3^{2/3}c^5} - \frac{4\sqrt[3]{\frac{2}{3}}k}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}} \right)^{\frac{1}{2}} \\ & - \frac{1}{2} \left( \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{2}3^{2/3}c^5} - \frac{4\sqrt[3]{\frac{2}{3}}k}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} \right)^{\frac{1}{2}}, \quad (73) \end{aligned}$$

$$\begin{aligned}
z_{3,3} = & \frac{1}{2} \left( \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{23^{2/3}c^5}} - \frac{4\sqrt[3]{\frac{2}{3}k}}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} \right)^{\frac{1}{2}} \\
& - \frac{1}{2} \left( \frac{4\sqrt[3]{\frac{2}{3}k}}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} - \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{23^{2/3}c^5}} \right. \\
& \left. + \frac{2}{c^4 \sqrt{\frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{23^{2/3}c^5}} - \frac{4\sqrt[3]{\frac{2}{3}k}}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}}} \right)^{\frac{1}{2}}, \quad (74)
\end{aligned}$$

$$\begin{aligned}
z_{4,3} = & \frac{1}{2} \left( \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{23^{2/3}c^5}} - \frac{4\sqrt[3]{\frac{2}{3}k}}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} \right)^{\frac{1}{2}} \\
& + \frac{1}{2} \left( \frac{4\sqrt[3]{\frac{2}{3}k}}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}} - \frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{23^{2/3}c^5}} \right. \\
& \left. + \frac{2}{c^4 \sqrt{\frac{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}{\sqrt[3]{23^{2/3}c^5}} - \frac{4\sqrt[3]{\frac{2}{3}k}}{\sqrt[3]{9c^7 + \sqrt{3}\sqrt{256c^{15}k^3 + 27c^{14}}}}}} \right)^{\frac{1}{2}}. \quad (75)
\end{aligned}$$

Consequently, the  $n + 3$  Lie point symmetries of Eq. (63) are  $X_r$ ,  $X_v$  and

$$X_1^3 = e^{z_1,3r} \partial_v, \quad X_2^3 = e^{z_2,3r} \partial_v, \quad X_3^3 = e^{z_3,3r} \partial_v, \quad X_4^3 = e^{z_4,3r} \partial_v.$$

In application, the usefulness of the Theorem 4.2.1 is that it provides symmetries for all  $n$ , which can then be used to solve Eq. (63) again for any  $n$ . We discuss this idea in the next section.

Figure 4.1: A plot of all complex roots of  $P(z)$  for  $c = k = 1$  and  $n = 1, \dots, 23$

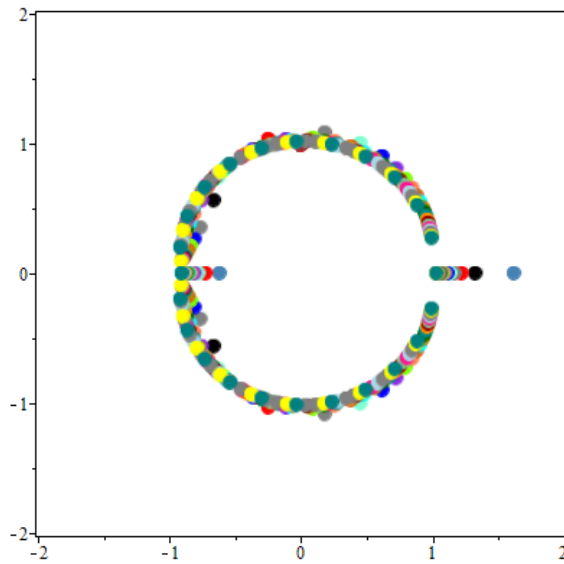
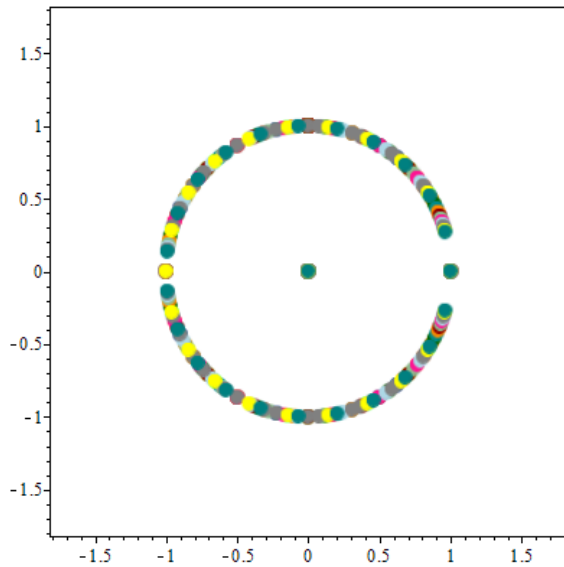


Figure 4.2: A plot of all complex roots of  $P(z)$  for  $k = 0$  and  $n = 1, \dots, 23$



In the Tables 4.1 and 4.2, we list the exponential symmetries in terms of their approximate numerical root values.

Table 4.1: Table of symmetries for  $k = c = 1$

<b>n</b>	<b>Symmetries of Eq. (63)</b>	<b>n</b>	<b>Symmetries of Eq. (63)</b>
<b>n=2</b>	$X_r,$ $X_v,$ $X_1 = e^{1.324717957 r} \partial_v,$ $X_2 = e^{(-0.6623589786+0.5622795121 i)r} \partial_v,$ $X_3 = e^{(-0.6623589786-0.5622795121 i)r} \partial_v.$	<b>n=4</b>	$X_r,$ $X_v,$ $X_1 = e^{1.167303978 r} \partial_v,$ $X_2 = e^{(0.1812324445+1.083954101 i)r} \partial_v,$ $X_3 = e^{(-0.7648844336+0.3524715460 i)r} \partial_v,$ $X_4 = e^{(-0.7648844336-0.3524715460 i)r} \partial_v,$ $X_5 = e^{(0.1812324445-1.083954101 i)r} \partial_v.$
<b>n=3</b>	$X_r,$ $X_v,$ $X_1 = e^{1.220744085 r} \partial_v,$ $X_2 = e^{(-0.2481260628+1.033982061 i)r} \partial_v,$ $X_3 = e^{-0.7244919590 r} \partial_v,$ $X_4 = e^{(-0.2481260628-1.033982061 i)r} \partial_v.$	<b>n=5</b>	$X_r,$ $X_v,$ $X_1 = e^{1.134724138 r} \partial_v,$ $X_2 = e^{(0.4510551586+1.002364572 i)r} \partial_v,$ $X_3 = e^{(-0.6293724285+0.7357559530 i)r} \partial_v,$ $X_4 = e^{-0.7780895987 r} \partial_v,$ $X_5 = e^{(-0.6293724285-0.7357559530 i)r} \partial_v,$ $X_6 = e^{(0.4510551586-1.002364572 i)r} \partial_v.$

Table 4.2: Table of symmetries for  $k = 0$

<b>n</b>	<b>Symmetries</b>	<b>n</b>	<b>Symmetries</b>
$n = 3$	$X_r,$ $X_v,$ $X_1 = \partial_v,$ $X_2 = e^{\frac{r}{c^{4/3}}} \partial_v,$ $X_3 = e^{-1/2 \frac{(i\sqrt{3}+1)r}{c^{4/3}}} \partial_v,$ $X_4 = e^{1/2 \frac{(i\sqrt{3}-1)r}{c^{4/3}}} \partial_v.$	$n = 5$	$X_r,$ $X_v,$ $X_1 = \partial_v,$ $X_2 = e^{\frac{r}{c^{6/5}}} \partial_v,$ $X_3 = e^{1/4 \frac{(-i\sqrt{2}\sqrt{5+\sqrt{5}}+\sqrt{5}-1)r}{c^{6/5}}} \partial_v,$ $X_4 = e^{-1/4 \frac{(i\sqrt{2}\sqrt{5-\sqrt{5}}+\sqrt{5}+1)r}{c^{6/5}}} \partial_v,$ $X_5 = e^{-1/4 \frac{(-i\sqrt{2}\sqrt{5-\sqrt{5}}+\sqrt{5}+1)r}{c^{6/5}}} \partial_v,$ $X_6 = e^{1/4 \frac{(i\sqrt{2}\sqrt{5+\sqrt{5}}+\sqrt{5}-1)r}{c^{6/5}}} \partial_v.$
$n = 4$	$X_r,$ $X_v,$ $X_1 = \partial_v,$ $X_2 = e^{\frac{r}{c^{5/4}}} \partial_v,,$ $X_3 = e^{-\frac{r}{c^{5/4}}} \partial_v,$ $X_4 = e^{\frac{-ir}{c^{5/4}}} \partial_v,$ $X_5 = e^{\frac{ir}{c^{5/4}}} \partial_v.$	$n = 6$	$X_r,$ $X_v,$ $X_1 = \partial_v,$ $X_2 = e^{\frac{r}{c^{7/6}}} \partial_v,$ $X_3 = e^{-\frac{r}{c^{7/6}}} \partial_v,$ $X_4 = e^{-1/2 \frac{(i\sqrt{3}-1)r}{c^{7/6}}} \partial_v,$ $X_5 = e^{-1/2 \frac{(i\sqrt{3}+1)r}{c^{7/6}}} \partial_v,$ $X_6 = e^{1/2 \frac{(i\sqrt{3}-1)r}{c^{7/6}}} \partial_v,$ $X_7 = e^{1/2 \frac{(i\sqrt{3}+1)r}{c^{7/6}}} \partial_v.$

## 4.2.2 Exact Solutions for $v(r)$

These Lie point symmetries are interesting because they are ‘solution’ symmetries. That is, they can be used to find invariant functions that solve the main ODE. Hence we establish the next theorem.

**Theorem 4.2.2.** *A solution of Eq. (63) is*

$$v(r) = \int e^{z_i n r} dr, \quad i = 1, \dots, n + 1. \quad (76)$$

*Proof.* Suppose we take the linear combination of symmetries  $X_r + X_i^n$ , for some  $i$ . By the method of invariants, the characteristic equation becomes

$$\frac{dr}{1} = \frac{dv}{e^{z_i n r}},$$

and by integration the result follows.  $\square$

A demonstration of the applicability of the above Theorem is given next, as we show how solutions of  $v(r)$  may be constructed using the roots of  $P(z)$  and the  $n + 1$  symmetries  $X_i^n$ . Suppose we take the first several solutions, presented as linear combinations with arbitrary constants  $C_j$ . For  $n = 1$ , based on the point symmetries  $X_1^1$  and  $X_2^1$  comprised of the roots  $z_{1,1}$  and  $z_{2,1}$ , Theorem 4.2.2 provides that the solution of Eq. (63) is

$$v_1(r) = \frac{\left(2c^2 e^{\frac{r(\sqrt{4ck+1}+1)}{2c^2}}\right)}{\sqrt{4ck+1}+1} C_1 - \frac{\left(2c^2 e^{-\frac{r(\sqrt{4ck+1}-1)}{2c^2}}\right)}{\sqrt{4ck+1}-1} C_2. \quad (77)$$

Similarly for  $n = 2$ , Theorem 4.2.2 in conjunction with the symmetries  $X_1^2$ ,  $X_2^2$  and  $X_3^2$  comprised of the roots  $z_{1,2}$ ,  $z_{2,2}$  and  $z_{3,2}$  respectively, we find the solution

$$\begin{aligned}
v_2(r) = & \frac{6^{2/3} c^4 \sqrt[3]{\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k} \exp\left(\frac{r \left(2 \sqrt[3]{3} c^5 + \sqrt[3]{2} (\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k)^{2/3}\right)}{6^{2/3} c^4 \sqrt[3]{\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k}}\right)}{2 \sqrt[3]{3} c^5 + \sqrt[3]{2} (\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k)^{2/3}} C_1 \\
& + \left( \left( \frac{\exp\left(-\frac{r \left(2 \sqrt[3]{2} \sqrt[6]{3} (\sqrt{3} + 3i) c^5 + \sqrt[3]{3} (1 - i\sqrt{3}) (2\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 18c^8 k)^{2/3}\right)}{12c^4 \sqrt[3]{\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k}}\right)}{i \sqrt[3]{2} \sqrt[6]{3} (\sqrt{3} + i) (\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k)^{2/3} - 2 (\sqrt{3} + 3i) c^5}\right) \right. \\
& \times \left. \left( \frac{2^1 2^{2/3} 3^{5/6} c^4 \sqrt[3]{\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k}}{i \sqrt[3]{2} \sqrt[6]{3} (\sqrt{3} + i) (\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k)^{2/3} - 2 (\sqrt{3} + 3i) c^5}\right) \right) C_2 \\
& + \left( \left( \frac{\exp\left(-\frac{r \left(2 \sqrt[3]{2} \sqrt[6]{3} (\sqrt{3} - 3i) c^5 + \sqrt[3]{3} (1 + i\sqrt{3}) (2\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 18c^8 k)^{2/3}\right)}{12c^4 \sqrt[3]{\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k}}\right)}{-2 (\sqrt{3} - 3i) c^5 - i \sqrt[3]{2} \sqrt[6]{3} (\sqrt{3} - i) (\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k)^{2/3}}\right) \right. \\
& \times \left. \left( \frac{2^1 2^{2/3} 3^{5/6} c^4 \sqrt[3]{\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k}}{-2 (\sqrt{3} - 3i) c^5 - i \sqrt[3]{2} \sqrt[6]{3} (\sqrt{3} - i) (\sqrt{3} \sqrt{c^{15} (27ck^2 - 4)} + 9c^8 k)^{2/3}}\right) \right) C_3.
\end{aligned} \tag{78}$$

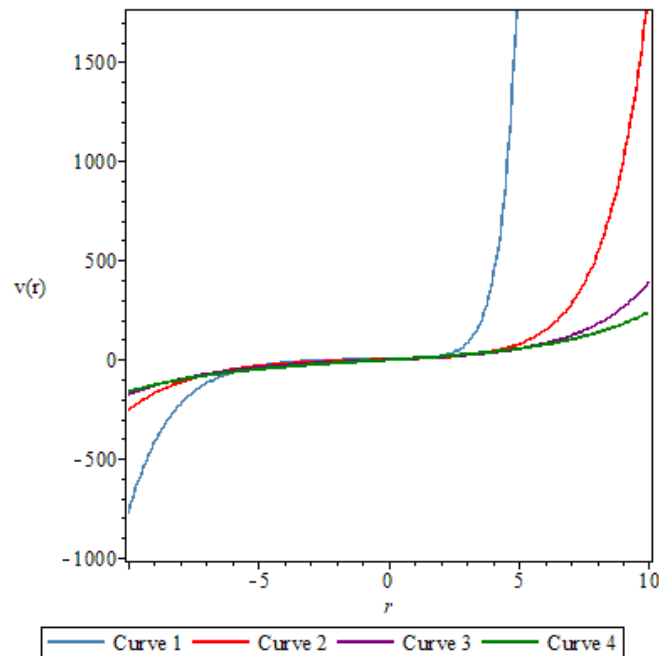
With the help of Euler's formula, one can convert the complex solutions to real

solutions. For instance, the solution Eq. (78) for  $c = k = 1$ , in real form is

$$\begin{aligned}
v_2(r) = & \frac{1}{2} \frac{C_1 12^{2/3} \sqrt[3]{\sqrt{3}\sqrt{23} + 9}}{\sqrt[3]{12} + (\sqrt{3}\sqrt{23} + 9)^{2/3}} e^{1/6 \frac{\sqrt[3]{12} \left( \sqrt[3]{12} + (\sqrt{3}\sqrt{23} + 9)^{2/3} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}}} \\
& - \frac{C_4}{4} \frac{12^{2/3} \sqrt[3]{\sqrt{3}\sqrt{23} + 9} e^{-1/12 \frac{\sqrt[3]{12} \left( (\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}}}}{-\sqrt[3]{12} (\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt{3}\sqrt{23} \sqrt[3]{\sqrt{3}\sqrt{23} + 9} + 12^{2/3} + 9 \sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \\
& \times \left( \sin \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) (\sqrt{3}\sqrt{23} + 9)^{2/3} \sqrt{3} \right. \\
& + \cos \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) (\sqrt{3}\sqrt{23} + 9)^{2/3} \\
& - \sin \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) \sqrt[3]{12}\sqrt{3} \\
& \left. + \cos \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) \sqrt[3]{12} \right) \\
& - \frac{C_5}{4} \frac{12^{2/3} \sqrt[3]{\sqrt{3}\sqrt{23} + 9} e^{-1/12 \frac{\sqrt[3]{12} \left( (\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}}}}{-\sqrt[3]{12} (\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt{3}\sqrt{23} \sqrt[3]{\sqrt{3}\sqrt{23} + 9} + 12^{2/3} + 9 \sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \\
& \times \left( -\sqrt{3} \cos \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) (\sqrt{3}\sqrt{23} + 9)^{2/3} \right. \\
& + \sqrt{3} \cos \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) \sqrt[3]{12} \\
& + \sin \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) (\sqrt{3}\sqrt{23} + 9)^{2/3} \\
& \left. + \sin \left( 1/12 \frac{\sqrt[3]{12}\sqrt{3} \left( -(\sqrt{3}\sqrt{23} + 9)^{2/3} + \sqrt[3]{12} \right) r}{\sqrt[3]{\sqrt{3}\sqrt{23} + 9}} \right) \sqrt[3]{12} \right), \quad (79)
\end{aligned}$$

where we have adjusted the constants  $C_2 + C_3 = C_4$  and  $C_2 - C_3 = -iC_5$ . A

Figure 4.3: Progression of  $v_1(r)$  in Eq. (77). Parameter values for Curve  $j$  are  $C_1 = C_2 = c = k = j$  where  $j = 1, \dots, 4$ .



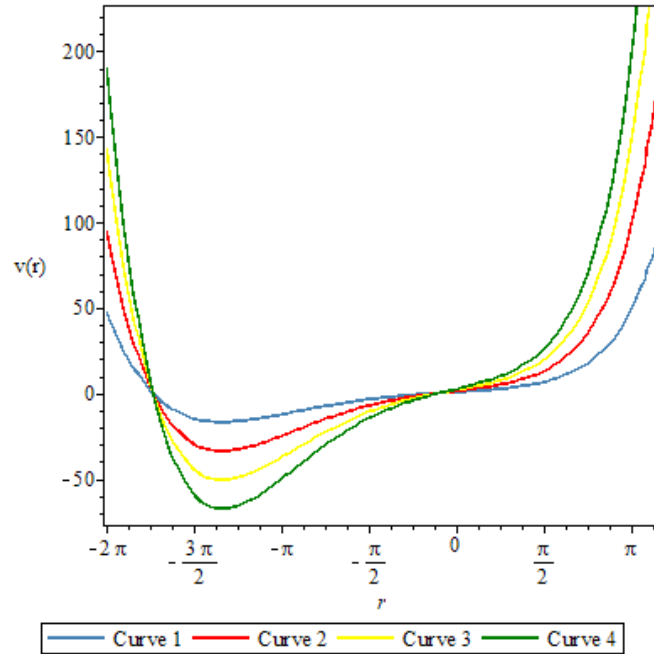
graphical representation of the above solutions of Eq. (63) can be found in Figures 4.3 and 4.4.

### 4.2.3 Applications to a PDE Hierarchy

At this stage, a question arises about how this connects to Burgers' hierarchy. We recall the salient members of this PDE hierarchy that has attracted much attention from Chapter 3.

One of our primary purposes of solving for  $v(r)$ , is to solve for Burgers' hierarchy. Given the theory stated in the preceding sections, we are able to state a general

Figure 4.4: Progression of  $v_2(r)$  in Eq. (79). Parameter values for the  $j$ th Curve are  $C_1 = C_4 = C_5 = p$  where  $p = 1, \dots, 4$ .



solution for Burgers' hierarchy Eq. (14). The Theorems 4.2.1 and 4.2.2 and the relation (64) facilitates the construction of solutions of the PDE hierarchy, for any value of  $n$ . In particular, our study gives rise to the following theorem for Burgers' hierarchy.

**Theorem 4.2.3.** *The exact solutions of Burgers' hierarchy of nonlinear partial differential equations, Eq. (14) are*

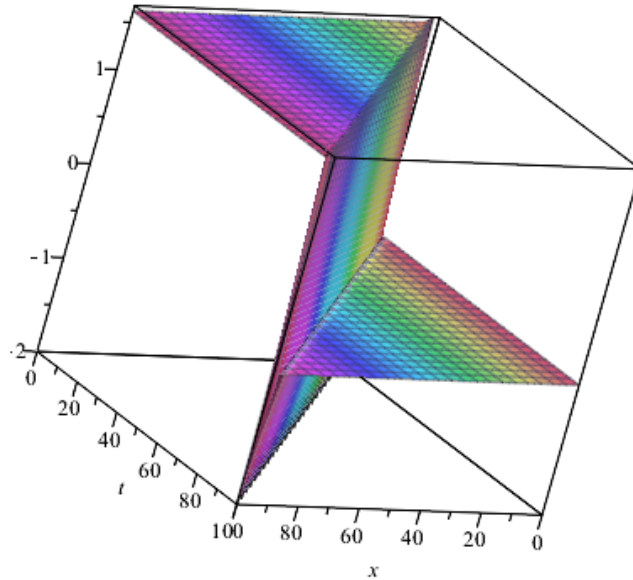
$$u(x, t) = c \frac{e^{z_{i,n}}}{\int e^{z_{i,n}r} dr}, \quad (80)$$

where  $r = cx - t$ .

We remark that this formula is easy in application on condition that the roots  $z_{i,n}$  of  $P(z)$  are derived first.

The explicit solutions of some of the  $u(x, t)$  occupy too much space to appear here, but one can easily view them from the solution  $v_1(r)$  and the real solution of  $v_2(r)$ , or view their behaviour in the Figures 4.5 and 4.6.

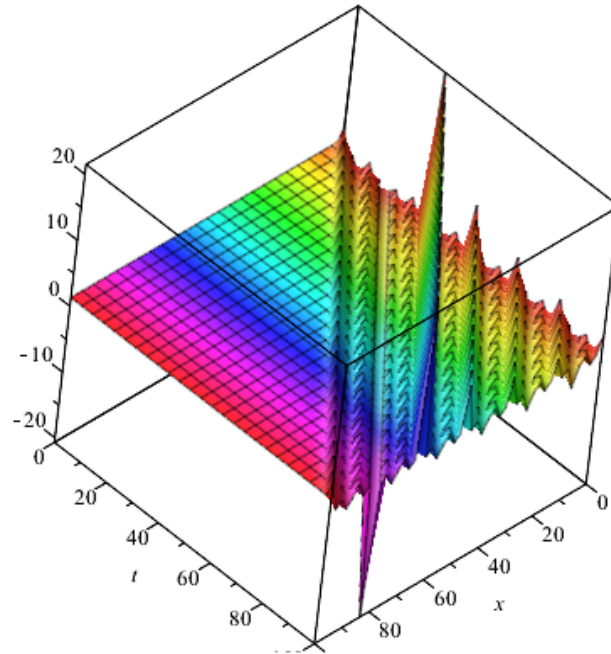
Figure 4.5: The evolution of  $u(x, t)$  based on  $v_1(r)$ . We select the parameter values  $C_1 = C_2 = c = k = 1$ .



### 4.3 Conclusion

Indeed, symmetry analysis is of paramount importance in the mathematical analysis of differential equations, many of which arise from mathematical physics as they explain most phenomena in the physical world. The principal motivation of this study was to provide a universal procedure to find the solutions of a  $(n + 1)th$ -order ODE (63) with the help of its exponential symmetry properties. This proved to be

Figure 4.6: The evolution of  $u(x, t)$  based on the real version of  $v_2(r)$ . We select the parameter values  $C_1 = C_4 = C_5 = 1$ .



an exceptional result as it led to the establishment of a novel theorem, that one can exploit or extend to construct solutions of the entire Burgers' PDE hierarchy. Several examples of the hierarchies were examined and graphical representations of the real solutions of  $v(r)$  and  $u(x, t)$  were given, to further illustrate our findings.

# Conclusion

Conservation laws are imperative and widely used in analysing differential equations notably in linearisation and for the analysis of solutions and their physical implications. Thus, several works of mathematics and physics are devoted to them. In this study we investigated the conservation laws of the Burgers' hierarchy. We provided the  $n$ th conserved vector that the hierarchy possess and used it to test the symmetries of the hierarchy for association. A coordinate transformation was used and we obtained a nonlinear ODE, accompanied by another transformation to linearise the equation. In the final analysis, this procedure gave us an original solution to the entire hierarchy i.e for all values of  $n$ . Several hierarchy members were solved with this formula and corresponding 3D plots shown. The novelty and advantage of our formula is that it bridges a gap in this particular area of study, since in the literature, there exist many works that specialised in solving only certain members of the hierarchy.

Symmetry analysis is exceptionally practical in formulating solutions to differential equations. In the final chapter we used symmetries to solve a  $(n + 1)$ th order ODE closely linked to Burgers' hierarchy. These symmetries were particularly interesting to us as they are universally derived from the  $(n + 1)$  complex roots of a certain polynomial. Plots of these complex roots were provided. This knowledge of the

roots proved to be useful in solving our ODE with a simple integration formula, for all values of  $n$ . To exemplify this formula, we applied it to a few cases of the ODE and provided figures showing the progression of these solutions. Finally, the association this formula has in giving new exact solutions to the entire Burgers' hierarchy was explored, and to demonstrate this we gave a few 3D plots of these hierarchy solutions.

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