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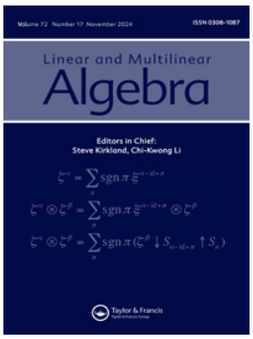

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Block determinants, partial determinants and the exponential map

A. Fošner[a], Y. Hardy [b,c] and B. Zinsou[b,c]

[a]Faculty of Management, University of Primorska, Koper, Slovenia; [b]School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa; [c]National Institute for Theoretical and Computational Sciences (NITheCS), Johannesburg, South Africa

ABSTRACT

The block trace and partial trace are closely related. However, the block determinant and partial determinant are quite different. We introduce a new determinant-like operation, the determinant-root, which has a partial operation that is more natural in the algebraic sense. The block determinant retains some of the algebraic properties of determinants, while the partial determinant-root retains the same algebraic properties for matrices which are Kronecker products. We investigate the equations $\det(D(A)) = \det(A), D(AB) = D(A)D(B)$ and $D(e^A) = e^{T(A)}$, where D is the block/partial/partial-root determinant and T is the corresponding block/partial/partial-normalized trace. These results yield a characterization of non-linear preservers of determinants of Kronecker products.

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1. Introduction

Let $\{E_{ij}^{[n]} : i,j \in \{1,\dots,n\}\}$ denote the standard basis in $M_n(\mathbb{F})$, the algebra of $n \times n$ matrices over a field $\mathbb{F}$. In other words, $E_{ij}^{[n]}$ is an $n \times n$ matrix with a 1 in the i-th row and the j-th column and all other entries are zero. The standard basis is given entry-wise by $(E_{ij}^{[n]})_{kl} = \delta_{ik}\delta_{jl}$, where δ_{ik} is the Kronecker delta

$$\delta_{ik} = \begin{cases} 1; & i = k, \\ 0; & i \neq k. \end{cases}$$

The transpose of the matrix $A \in M_n(\mathbb{F})$ will be denoted by A^T. Let $\otimes$ denote the Kronecker product of matrices, i.e. for $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ the matrix $A \otimes B$ is given block-wise by

$$A \otimes B := \begin{pmatrix} (A)_{11}B & (A)_{12}B & \cdots & (A)_{1m}B \\ (A)_{21}B & (A)_{22}B & \cdots & (A)_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ (A)_{m1}B & (A)_{m2}B & \cdots & (A)_{mm}B \end{pmatrix},$$

CONTACT A. Fošner ajda.fosner@fm-kp.si Faculty of Management, University of Primorska, Izolska vrata 2, SI-6000 Koper, Slovenia

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where $(A)_{ij}$ is the entry of A in the i-th row and j-th column. The Kronecker sum of $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ is the matrix $A \oplus B = A \otimes I_n + I_m \otimes B$, where I_k is the identity matrix in $M_k(\mathbb{F})$, $k = m, n$.

We have $M_{mn}(\mathbb{F}) = M_m(M_n(\mathbb{F})) = M_m(\mathbb{F}) \otimes M_n(\mathbb{F})$. Let $A \in M_{mn}(\mathbb{F})$, then we may write

$$A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}^{[n]}, \qquad A = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes A'_{ij},$$

where $A'_{ij} \in M_n(\mathbb{F})$ are the blocks of A, and $A_{ij} \in M_m(\mathbb{F})$ are the 'block like' parts of A induced by the isomorphism $M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) = M_n(\mathbb{F}) \otimes M_m(\mathbb{F})$.

Choi gives the definition of the partial determinants $\det_1(A)$ and $\det_2(B)$ as follows.

Definition 1.1 ([1]): Let $A \in M_{mn}(\mathbb{F})$ with

$$A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}^{[n]}, \quad A = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes A'_{ij}.$$

The partial determinants $\det_1(A)$ and $\det_2(A)$ are given by

$$\det_1(A) := \sum_{i,j=1}^{n} \det(A_{ij}) E_{ij}^{[n]}, \quad \det_2(A) := \sum_{i,j=1}^{m} \det(A'_{ij}) E_{ij}^{[m]}.$$

This definition is analogous to that of the partial trace. The partial trace features prominently in quantum information theory (see for example [2]).

Definition 1.2: Let $A \in M_{mn}(\mathbb{F})$ with

$$A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}^{[n]}, \quad A = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes A'_{ij}.$$

The partial traces $\mathrm{tr}_1(A)$ and $\mathrm{tr}_2(A)$ are given by

$$\mathrm{tr}_1(A) := \sum_{i,j=1}^{n} \mathrm{tr}(A_{ij}) E_{ij}^{[n]}, \quad \mathrm{tr}_2(A) := \sum_{i,j=1}^{m} \mathrm{tr}(A'_{ij}) E_{ij}^{[m]}.$$

Proposition 1.1: *Let $r \in \mathbb{N}$, $A_1, \ldots, A_r \in M_m(\mathbb{F})$ and $B_1, \ldots, B_r \in M_n(\mathbb{F})$. Then*

$$\mathrm{tr}_1\left(\sum_{j=1}^{r} A_j \otimes B_j\right) = \sum_{j=1}^{r} \mathrm{tr}(A_j) B_j,$$

and

$$\mathrm{tr}_2\left(\sum_{j=1}^{r} A_j \otimes B_j\right) = \sum_{j=1}^{r} \mathrm{tr}(B_j) A_j.$$

Proof: We have

$$\begin{aligned}\mathrm{tr}_1\left(\sum_{j=1}^{r} A_j \otimes B_j\right) &= \mathrm{tr}_1\left(\sum_{j=1}^{r}\sum_{k,l=1}^{n} A_j \otimes ((B_j)_{kl}E_{kl}^{[n]})\right)\\ &= \mathrm{tr}_1\left(\sum_{j=1}^{r}\sum_{k,l=1}^{n} ((B_j)_{kl}A_j) \otimes E_{kl}^{[n]}\right)\\ &= \sum_{j=1}^{r}\sum_{k,l=1}^{n} \mathrm{tr}((B_j)_{kl}A_j)E_{kl}^{[n]}\\ &= \sum_{j=1}^{r} \mathrm{tr}(A_j)\sum_{k,l=1}^{n} ((B_j)_{kl}E_{kl}^{[n]})\\ &= \sum_{j=1}^{r} \mathrm{tr}(A_j)B_j.\end{aligned}$$

■

The following proposition shows that tr_1 is the block trace of [3], i.e. the trace of A taken over the matrix ring $M_n(\mathbb{F})$.

Proposition 1.2: *Let $A \in M_{mn}(\mathbb{F})$ with*

$$A = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes A'_{ij}.$$

Then

$$\mathrm{tr}_1(A) = A'_{11} + A'_{22} + \cdots + A'_{mm}.$$

Proof: We have, by Proposition 1.1,

$$\mathrm{tr}_1(A) = \sum_{i,j=1}^{m} \mathrm{tr}(E_{ij}^{[m]})A'_{ij} = \sum_{i=1}^{m} A'_{ii}.$$

■

Consequently, we note the following corollary for any $A \in M_{mn}(\mathbb{F})$ and $B \in M_n(\mathbb{F})$.

Corollary 1.1: *Let $A \in M_{mn}(\mathbb{F})$, $B \in M_n(\mathbb{F})$ and $C \in M_m(\mathbb{F})$. Then*

$$\mathrm{tr}_1(A(I_m \otimes B)) = \mathrm{tr}_1(A)B \quad \text{and} \quad \mathrm{tr}_2(A(C \otimes I_n)) = \mathrm{tr}_2(A)C.$$

Proof: We have

$$\mathrm{tr}_1(A(I_m \otimes B)) = \mathrm{tr}_1\left(\left(\sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes A'_{ij}\right)(I_m \otimes B)\right)$$

$$
\begin{aligned}
&= \mathrm{tr}_1\left(\sum_{i,j=1}^{m}(E_{ij}^{[m]}I_m)\otimes(A'_{ij}B)\right)\\
&= \mathrm{tr}_1\left(\sum_{i,j=1}^{m}E_{ij}^{[m]}\otimes(A'_{ij}B)\right)\\
&= \sum_{i,j=1}^{m}\mathrm{tr}(E_{ij}^{[m]})(A'_{ij}B)\\
&= \sum_{j=1}^{m}A'_{jj}B = \mathrm{tr}_1(A)B,
\end{aligned}
$$

and similarly,

$$\mathrm{tr}_2(A(C\otimes I_n)) = \mathrm{tr}_2(A)C. \qquad \blacksquare$$

The partial trace is linear while the partial determinant is not linear. Additionally, the partial trace is 'partial' in the sense that the trace can be completed, i.e. using Proposition 1.1,

$$\mathrm{tr}(\mathrm{tr}_1(A\otimes B)) = \mathrm{tr}(\mathrm{tr}(A)B) = \mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(A\otimes B),$$

so that, by linearity of the trace and partial trace,

$$\mathrm{tr}(\mathrm{tr}_1(A)) = \sum_{i,j=1}^{n}\mathrm{tr}(\mathrm{tr}_1(A_{ij}\otimes E_{ij}^{[n]})) = \sum_{i,j=1}^{n}\mathrm{tr}(A_{ij}\otimes E_{ij}^{[n]}) = \mathrm{tr}(A).$$

Similarly,

$$\mathrm{tr}(\mathrm{tr}_1(A)) = \mathrm{tr}(\mathrm{tr}_2(A)) = \mathrm{tr}(A).$$

Thompson showed that if $A\in M_{mn}(\mathbb{C})$ is positive definite, then $\det(\det_2(A))\geq\det(A)$ and that

$$\det(\det_2(A)) = \det(A) \quad\Longleftrightarrow\quad A'_{ij} = \delta_{ij}A'_{ij},$$

where δ_{ij} denotes the Kronecker delta [4].

Let $R := M_n(\mathbb{F})$ be the associative ring of $n\times n$ matrices over $\mathbb{F}$. Then the trace tr_R in $M_m(R)$ is the block trace, i.e. $\mathrm{tr}_R(A) = \mathrm{tr}_1(A)$. The same relation does not hold for the block determinant $\det_R$, if we use the definition of $\det_R$ from [5, Definition 2.1].

Definition 1.3: Let $B\in M_m(R)$, where $R = M_n(\mathbb{F})$, with

$$B = \sum_{i,j=1}^{m}E_{ij}^{[m]}\otimes B_{ij}.$$

The block determinant $\det_R$ over R is given by

$$\det_R(B) = \sum_{\sigma\in\pi_m}\mathrm{sgn}(\sigma)B_{1\sigma(1)}\cdots B_{m\sigma(m)},$$

where the sum is taken over all permutations σ of $\{1, 2, \ldots, m\}$, i.e. the symmetric group π_m.

Here, $\det(\det_R(B)) = \det(B)$ when the blocks of B commute [6]. In fact, not all of the blocks need commute as shown in [5, Theorem 1.2].

Another important property to consider is whether the identity $\det(AB) = \det(A)\det(B)$ carries over to the partial determinant. In other words, what are the conditions on A and B such that $\det_1(AB) = \det_1(A)\det_1(B)$? The same question can be asked for the block determinant. In the case of the block determinant, we have $\det_R(AB) = \det_R(A)\det_R(B)$ (Theorem 2.1) under certain commutativity conditions on the entries of A and B in R.

Since $B \otimes C$ is a permutation similar to $C \otimes B$ via the vec-permutation matrix P (perfect shuffle matrix) [7, 8] so that $\det_2(A) = \det_1(PAP^T)$, we confine our attention to $\det_1(A)$. The characterization for arbitrary matrices (i.e. neither Kronecker products nor positive definite matrices) remains open. We also consider a (less restricted) monoid $D_m(\mathbb{F})$ and a determinant-root operation Det (see Section 4.2 of this text), such that the partial determinant-root obeys $\mathrm{Det}(A \otimes B) = \mathrm{Det}(\mathrm{Det}_1(A \otimes B))$ and $\mathrm{Det}_1((A \otimes B)(C \otimes D)) = \mathrm{Det}_1(A \otimes B)\,\mathrm{Det}_1(C \otimes D)$.

Finally, we note the connection between the determinant and trace. If $\mathbb{F}$ is the field of complex numbers or the field of real numbers and $A \in M_n(\mathbb{F})$, then

$$\det(e^A) = e^{\mathrm{tr}(A)}. \tag{1}$$

The exponential map also provides a connection between Kronecker sums and Kronecker products [9, p. 440], i.e.

$$e^{A \oplus B} = e^A \otimes e^B.$$

We investigate whether the exponential-determinant-trace relation may hold for $\det_1$ and tr_1. In the case of Kronecker sums, we have a straightforward answer. We also show that $\det_R(e^A) = e^{\mathrm{tr}_R(A)}$ in a certain factor group, and that for a Kronecker sum A, we have $\mathrm{Det}_1(e^A) = e^{\mathrm{Tr}_1(A)}$ over a certain factor group. Here, $\mathrm{Tr}(A) = 1/n\,\mathrm{tr}(A)$ in $M_n(\mathbb{F})$ and $\mathrm{Tr}_1(A) = 1/m\,\mathrm{tr}_1(A)$ in $M_{mn}(\mathbb{F})$. For partial determinants and partial determinant-roots of arbitrary matrices (see Section 4.2 of this text), we have not found an equivalent result.

Finally, we note that $\det_1(I_{mn}) = I_n$, $\det_R(I_{mn}) = I_n$ and $\mathrm{Det}_1(I_{mn}) = I_n$.

When $\mathbb{F}$ is the field of real numbers or the field of complex numbers, define $H_n \subset M_n(\mathbb{F})$ to be the Hermitian matrices in $M_n(\mathbb{F})$. Linear maps preserving properties of Kronecker products of matrices have received considerable attention in recent years. Such maps are closely connected to quantum information science (see, e.g. [10]). More recently, Ding et al. considered linear preservers of determinants of Kronecker products of Hermitian matrices [11], i.e. linear maps $\phi : H_{mn} \to H_{mn}$ satisfying

$$\det(\phi(A \otimes B)) = \det(A \otimes B),$$

where A and B are Hermitian. A few of the results in [11] are restricted to the case when A and B are positive or negative semidefinite matrices. In order to study this problem more

generally, we make use of the identity (1). In particular,

$$\det\left(e^A \otimes e^B\right) = e^{\mathrm{tr}(A\oplus B)},$$

where $\oplus$ is the Kronecker sum, i.e.

$$A \oplus B := A \otimes I_n + I_m \otimes B,$$

where I_k, $k = m, n$ denotes the $k \times k$ identity matrix. Under exponentiation of Hermitian matrices, the Kronecker sum arises naturally as the unique map $\oplus : H_m \times H_n \to H_m \otimes H_n$ satisfying

$$e^{A\oplus B} = e^A \otimes e^B, \quad A \in H_m, B \in H_n.$$

Moreover, $A \in M_n(\mathbb{C})$ is non-singular if and only if $A = e^B$ for some $B \in M_n(\mathbb{C})$ [9, Example 6.2.15]. Thus, in studying linear maps $\psi : M_{mn}(\mathbb{F}) \to M_{mn}(\mathbb{F})$ preserving determinants of (non-singular) Kronecker products

$$\det(\psi(A \otimes B)) = \det(A \otimes B)$$

of non-singular matrices A and B, it suffices to study maps $\phi : M_{mn} \to M_{mn}$ preserving the trace of Kronecker sums

$$\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B). \tag{2}$$

In this paper we will confine our attention to linear maps ϕ satisfying (2). We denote by $GL_n(\mathbb{F})$ and $SL_n(\mathbb{F})$ the general and special linear groups in $M_n(\mathbb{F})$ respectively. Thus, we are also interested in non-linear preservers of the determinant of Kronecker product in $GL_{mn}(\mathbb{F})$ in terms of linear preservers of the Kronecker sum. Non-linear preservers of the determinant have also been studied in [12], where preservers were observed under the additive structure of $M_n(\mathbb{F})$. We also investigate preservers in terms of additive structure, but due to the indirect (via exponentiation) approach, we are able to reframe the study in terms of linear preservers.

2. Block determinants

Note that $\det(\det_R(B)) = \det(B)$ when the blocks of B commute [6] or, more generally, under a sufficient commutativity condition [5]. Now we turn our attention to products and to the exponential map.

2.1. Products

Bourbaki [13, §6 Theorem 1] proved Theorem 2.1 in terms of exterior products. For completeness, we provide a proof using Definition 1.3.

Theorem 2.1: *Let $A, B \in M_{mn}(\mathbb{F})$ and let R be a commutative subring of $M_n(\mathbb{F})$. Then* $\det_R(AB) = \det_R(A)\det_R(B)$.

Proof: Since all of the blocks of A and B in R commute, the proof proceeds exactly as in the case of matrices over a field. We have

$$\begin{aligned}
&\det_R(A)\det_R(B)\\
&= \sum_{\sigma,\psi\in\pi_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\psi) A_{1,\sigma(1)}\cdots A_{m,\sigma(m)} B_{1,\psi(1)}\cdots B_{m,\psi(m)}\\
&= \sum_{\sigma,\psi\in\pi_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\psi) A_{1,\sigma(1)}\cdots A_{m,\sigma(m)} B_{\sigma(1),\psi\circ\sigma(1)}\cdots B_{\sigma(m),\psi\circ\sigma(m)}\\
&= \sum_{\sigma,\psi'\in\pi_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\psi'\circ\sigma^{-1}) A_{1,\sigma(1)}\cdots A_{m,\sigma(m)} B_{\sigma(1),\psi'(1)}\cdots B_{\sigma(m),\psi'(m)}\\
&= \sum_{\sigma,\psi'\in\pi_m} \operatorname{sgn}(\sigma)\operatorname{sgn}(\psi'\circ\sigma^{-1}) A_{1,\sigma(1)} B_{\sigma(1),\psi'(1)}\cdots A_{m,\sigma(m)} B_{\sigma(m),\psi'(m)}\\
&= \sum_{\sigma,\psi'\in\pi_m} \operatorname{sgn}(\psi') A_{1,\sigma(1)} B_{\sigma(1),\psi'(1)}\cdots A_{m,\sigma(m)} B_{\sigma(m),\psi'(m)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\det_R(AB) &= \sum_{\psi'\in\pi_m} \operatorname{sgn}(\psi')\,(AB)_{1,\psi'(1)}\cdots(AB)_{1,\psi'(m)}\\
&= \sum_{\psi'\in\pi_m}\sum_{k_1,\ldots,k_m=1}^{m} \operatorname{sgn}(\psi') A_{1,k_1}B_{k_1,\psi'(1)}\cdots A_{m,k_m}B_{k_m,\psi'(m)}
\end{aligned}$$

and the terms in this sum are zero whenever $k_i = k_j$, $i \neq j$. Let us consider the terms where $k_1 = k_2$. Then, since every permutation can be expressed as an even permutation composed with either the identity or the permutation which exchanges 1 and 2, we have

$$\begin{aligned}
\det_R(AB) &= \sum_{\psi'\in\pi_m}\sum_{k_1,\ldots,k_m=1}^{m} \operatorname{sgn}(\psi') A_{1,k_1}B_{k_1,\psi'(1)}\cdots A_{m,k_m}B_{k_m,\psi'(m)}\\
&= \sum_{\substack{\psi'\in\pi_m\\ \operatorname{sgn}(\psi')=1}}\sum_{k_1,\ldots,k_m=1}^{m} \operatorname{sgn}(\psi')\big[A_{1,k_1}B_{k_1,\psi'(1)}A_{2,k_2}B_{k_2,\psi'(2)}\cdots A_{m,k_m}B_{k_m,\psi'(m)}\\
&\quad - A_{1,k_1}B_{k_1,\psi'(2)}A_{2,k_2}B_{k_2,\psi'(1)}\cdots A_{m,k_m}B_{k_m,\psi'(m)}\big]\\
&= \sum_{\substack{\psi'\in\pi_m\\ \operatorname{sgn}(\psi')=1}} \operatorname{sgn}(\psi') \sum_{\substack{k_1,\ldots,k_m=1\\ k_1\neq k_2}}^{m} A_{1,k_1}B_{k_1,\psi'(1)}A_{2,k_2}B_{k_2,\psi'(2)}\cdots A_{m,k_m}B_{k_m,\psi'(m)}\\
&= \sum_{\psi'\in\pi_m} \operatorname{sgn}(\psi') \sum_{\substack{k_1,\ldots,k_m=1\\ k_1\neq k_2}}^{m} A_{1,k_1}B_{k_1,\psi'(1)}\cdots A_{m,k_m}B_{k_m,\psi'(m)}.
\end{aligned}$$

Proceeding in the same way for all $i \neq j$, the sum is taken over all terms with distinct k_1, $k_2, \ldots, k_n$, i.e. the sum is over all permutations σ, where we identify $\sigma(j) = k_j$,

$$\det_R(AB) = \sum_{\psi', \sigma \in \pi_m} \operatorname{sgn}(\psi') A_{1,\sigma(1)} B_{\sigma(1),\psi'(1)} \cdots A_{m,\sigma(m)} B_{\sigma(m),\psi'(m)}.$$

■

2.2. The exponential map

Note that $\exp_{\mathbb{F}}(A) = \exp_R(A)$, where $R = M_n(\mathbb{F})$, i.e. the power series for e^A yields the same matrix with multiplication in the field $\mathbb{F}$ and multiplication in the ring R. Our first proposition shows that, in the special case of Kronecker sums A, the block trace and block determinant obey $\det_R(e^A) = e^{\operatorname{tr}_R(A)}$.

Proposition 2.1: *Let $\mathbb{F}$ be the field of real numbers or the field of complex numbers. Let $A \in M_m(\mathbb{F})$, $B \in M_n(\mathbb{F})$ and let R be the matrix ring $R = M_n(\mathbb{F})$. Then $\det_R(e^{A \oplus B}) = e^{\operatorname{tr}_R(A \oplus B)}$.*

Proof: Since all of the blocks $(e^A)_{ij} e^B$ of $e^A \otimes e^B$ commute,

$$\begin{aligned} \det_R(e^{A \oplus B}) &= \det_R(e^A \otimes e^B) \\ &= (e^B)^m \det_{\mathbb{F}}(e^A) \\ &= e^{mB} e^{\operatorname{tr}(A)} \end{aligned}$$

and $e^{\operatorname{tr}_R(A \oplus B)} = e^{\operatorname{tr}(A) I_n + mB} = e^{mB} e^{\operatorname{tr}(A)}$. ■

In the following, we let R be a commutative ring, and consider the matrix ring $M_m(R)$. In particular, we will consider the case $R \subset M_n(\mathbb{F})$, i.e. R is a commutative subring of the $n \times n$ matrices over $\mathbb{F}$, and hence we will be considering block structured matrices in $M_m(R) \subset M_m(M_n(\mathbb{F}))$. We have a weaker result of Proposition 2.1 for matrices which are not Kronecker sums. Let R be a commutative ring and let $[M_m(R), M_m(R)]$ denote the R-submodule of $M_m(R)$ generated by all commutators in $M_m(R)$. If $\operatorname{tr}_R(A) = 0_n$, then A is in an R-submodule of $[M_m(R), M_m(R)]$ [14, Theorem 15]. The next lemma shows that matrices with zero trace, have unit determinant under the exponential map. By Theorem 2.1 we have

$$\det_R(e^A) \det_R(e^B) = \det_R(e^A e^B)$$

Now $e^A e^B$ is non-singular, so there exists $C \in M_m(R)$ such that $e^A e^B = e^C$ [9, Example 6.2.15]. Thus, if R is a commutative subring of $M_n(\mathbb{F})$, then the set

$$G = \{ \det_R(e^A) : A \in M_m(R) \}$$

is an abelian multiplicative group in R with $I_n = \det_R(e^{0_{mn}})$ and $(\det_R(e^A))^{-1} = \det_R(e^{-A})$.

Lemma 2.1: *Let $\mathbb{F}$ be the field of real numbers or the field of complex numbers. Let $A \in \ker(\mathrm{tr}_R)$, where R is a commutative subring (with identity) of $M_n(\mathbb{F})$. Then*

$$K_A \det_R(e^A) = K_A, \tag{3}$$

where K_A is the normal subgroup of G which is multiplicatively generated by A, i.e.

$$K_A = \{ \det_R(e^{jA}) \, : \, j \in \mathbb{Z} \}.$$

Furthermore, K_A is the smallest subgroup of G which satisfies Equation (3).

Proof: Clearly $I_n \in K_A$ and $(\det_R(e^A))^{-1} = \det_R(e^{-A}) \in K_A$. By construction, K_A is closed. Since R is commutative, K_A is a multiplicative normal subgroup of G. If $K' \subseteq G$ satisfies Equation (3), then $K_A \subseteq K'$ since $I_n \in K'$. ■

Theorem 2.2: *Let $\mathbb{F}$ be the field of real numbers or the field of complex numbers. Let $A \in M_m(R)$, where R is a commutative subring (with identity) of $M_n(\mathbb{F})$ such that $\exp(\frac{1}{m}\,\mathrm{tr}_R(A)) \in R$. Then*

$$K_{\mathrm{TZ}(A)} \det_R(e^A) = K_{\mathrm{TZ}(A)} e^{\mathrm{tr}_R(A)},$$

where

$$\mathrm{TZ}(A) := A - \frac{1}{m} I_m \otimes \mathrm{tr}_R(A), \quad \textit{and} \quad K_{\mathrm{TZ}(A)} := \{ \det_R(e^{j\mathrm{TZ}(A)}) \, : \, j \in \mathbb{Z} \}.$$

Proof: Since R is commutative, $\mathrm{TZ}(A)$ and $\frac{1}{m} I_m \otimes \mathrm{tr}_R(A)$ commute in $M_m(R)$ and

$$e^{\mathrm{TZ}(A)} e^{\frac{1}{m} I_m \otimes \mathrm{tr}_R(A)} = e^{\mathrm{TZ}(A) + \frac{1}{m} I_m \otimes \mathrm{tr}_R(A)}.$$

All of the blocks, in R, of $\mathrm{TZ}(A)$, $\frac{1}{m} I_m \otimes \mathrm{tr}_R(A)$ and the blocks of their exponentials commute. Using Theorem 2.1, Lemma 2.1 and the fact that $\mathrm{tr}_R(\mathrm{TZ}(A)) = 0$, we find

$$\begin{aligned} K_{\mathrm{TZ}(A)} \det_R(e^A) &= K_{\mathrm{TZ}(A)} \det_R \left(e^{\mathrm{TZ}(A) + \frac{1}{m} I_m \otimes \mathrm{tr}_R(A)} \right) \\ &= K_{\mathrm{TZ}(A)} \det_R \left(e^{\mathrm{TZ}(A)} e^{\frac{1}{m} I_m \otimes \mathrm{tr}_R(A)} \right) \\ &= K_{\mathrm{TZ}(A)} \det_R \left(e^{\mathrm{TZ}(A)} \right) \det_R \left(e^{\frac{1}{m} I_m \otimes \mathrm{tr}_R(A)} \right) \\ &= K_{\mathrm{TZ}(A)} \det_R \left(e^{\frac{1}{m} I_m \otimes \mathrm{tr}_R(A)} \right) \end{aligned}$$

and by bilinearity of the Kronecker product and [9, Exercise 6.2.13],

$$\begin{aligned} K_{\mathrm{TZ}(A)} \det_R(e^A) &= K_{\mathrm{TZ}(A)} \det_R \left(I_m \otimes e^{\frac{1}{m} \mathrm{tr}_R(A)} \right) \\ &= K_{\mathrm{TZ}(A)} e^{\mathrm{tr}_R(A)}. \end{aligned}$$

■

Corollary 2.1: *Let $\mathbb{F}$ be the field of real numbers or the field of complex numbers. Let R be a commutative subring (with identity) of $M_n(\mathbb{F})$ which is closed under scalar multiplication. Then for all $A \in M_m(R)$,*

$$K \det_R(e^A) = Ke^{\mathrm{tr}_R(A)}, \tag{4}$$

where $K = \{ \det_R(e^A) : A \in \ker(\mathrm{tr}_R) \}$. Moreover, K is the smallest subgroup of G satisfying (4).

Proof: Since, for each $A \in M_n(R)$, $K_{\mathrm{TZ}(A)} \subseteq K$ is a subgroup of K, we have $K = KK_{\mathrm{TZ}(A)}$ and hence by Theorem 2.2,

$$K \det_R(e^A) = KK_{\mathrm{TZ}(A)} \det_R(e^A) = KK_{\mathrm{TZ}(A)} e^{\mathrm{tr}_R(A)} = Ke^{\mathrm{tr}_R(A)},$$

so that (4) holds. Now let K' be a subgroup of G which satisfies (4) for all $A \in M_n(R)$. Let $A \in \ker(\mathrm{tr}_R)$, then

$$K' \det_R(e^A) = K' e^{\mathrm{tr}_R(A)} = K'$$

and since $I_n \in K'$ it follows that $\det_R(e^A) \in K'$, i.e. $K \subseteq K'$. ∎

Of course, when R is the field of real numbers or the field of complex numbers, we have $K = \{1\}$ and $\det(e^A) = e^{\mathrm{tr}(A)}$.

3. Partial determinants

3.1. Kronecker products

Let I_n denote the $n \times n$ identity matrix, and 0_n the $n \times n$ zero matrix. First we note that $\det_1(I_{mn}) = \det_1(I_m \otimes I_n) = I_n$ analogous to $\det(I) = 1$.

Definition 3.1: Let $A, B \in M_n(R)$. The Hadamard product of A and B, $A \circ B$, is the entry-wise product,

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \circ \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}b_{n1} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

Let $m \in \mathbb{N}$. The m-th Hadamard power of A, denoted $A^{(m)}$ is the matrix defined inductively as the Hadamard product $A^{(m)} = A \circ A^{(m-1)}$, where $A^{(1)} := A$.

Lemma 3.1: *Let $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$. Then*

$$\det_1(A \otimes B) = \det(A) B^{(m)}.$$

Proof: The proof follows immediately from

$$A \otimes B = \sum_{i,j=1}^{n} ((B)_{ij} A) \otimes E_{ij}^{[n]},$$

where $B = ((B)_{ij})$. ∎

This is a direct consequence of the remark by Choi that $\det_1(A)$ may be computed as a determinant where the blocks are treated as scalars and instead of the usual scalar product we use the Hadamard product [1].

Theorem 3.1: *Let $A, C \in M_m(\mathbb{F})$ and $B, D \in M_n(\mathbb{F})$ with $B \circ D = 0_n$. Then*

$$\det_1(A \otimes B + C \otimes D) = \det_1(A \otimes B) + \det_1(C \otimes D).$$

***Proof*:** We have

$$A \otimes B + C \otimes D = \sum_{\substack{i,j=1 \\ (B)_{ij} \neq 0}}^{n} ((B)_{ij}A) \otimes E_{ij}^{[n]} + \sum_{\substack{i,j=1 \\ (D)_{ij} \neq 0}}^{n} ((D)_{ij}C) \otimes E_{ij}^{[n]}, \tag{5}$$

and since $B \circ D = 0_n$ if follows that $(B)_{ij} \neq 0 \Longrightarrow (D)_{ij} = 0$ and vice-versa. The two sums in (5) are over disjoint subsets of $\{1, \ldots, n\}^2$ and

$$\begin{aligned}\det_1(A \otimes B + C \otimes D) &= \sum_{\substack{i,j=1 \\ (B)_{ij} \neq 0}}^{n} \det(A)(B)_{ij}^{m} E_{ij}^{[n]} + \sum_{\substack{i,j=1 \\ (D)_{ij} \neq 0}}^{n} \det(C)(D)_{ij}^{m} E_{ij}^{[n]} \\ &= \det_1(A \otimes B) + \det_1(C \otimes D).\end{aligned}$$

■

Lemma 3.1 provides an immediate characterization of partial determinants that can be completed.

Theorem 3.2: *Let $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$. Then*

$$\det(\det_1(A \otimes B)) = \det(A \otimes B)$$

if and only if $\det(A) = 0$ *or* $\det(B^{(m)}) = \det(B)^m$.

Drnovšek considered a much stronger condition in [15] for matrices B (which also satisfy Theorem 3.2), namely that $B^{(r)} = B^r$ for all $r \in \mathbb{N}$. Under the determinant, we have a weaker condition which leads to somewhat trivial cases for triangular matrices and $(0, 1)$-matrices. For triangular matrices B, we have $\det(B^{(m)}) = \det(B^m) = \det(B)^m$.

Corollary 3.1: *Let $A \in M_m(\mathbb{F})$ be arbitrary and let $B \in M_n(\mathbb{F})$ be triangular. Then*

$$\det(\det_1(A \otimes B)) = \det(A \otimes B).$$

For $(0, 1)$-matrices B we have $B^{(m)} = B$, and hence the following corollary of Theorem 3.2.

Corollary 3.2: *Let $A \in M_m(\mathbb{F})$ be arbitrary and let $B \in M_n(\mathbb{F})$ be a $(0, 1)$-matrix. Then*

$$\det(\det_1(A \otimes B)) = \det(A \otimes B)$$

if and only if $\det(A)\det(B) = 0$ *or* $\det(B)$ *is an* $(m-1)$*-th root of unity.*

When $n = 2$, the characterization is straightforward since

$$\det\begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2$$

if and only if $b^2c^2 = abcd$, so that $bc = 0$ (triangular matrix) or $ad = bc \neq 0$ (rank-1 matrix) and both cases satisfy $\det(B^{(m)}) = \det(B)^m$. Here we used the fact that a rank-1 matrix $\mathbf{x}\mathbf{y}^T$ remains a rank-1 matrix $\mathbf{x}^{(m)}(\mathbf{y}^{(m)})^T$ under the Hadamard power, where $\mathbf{x}$ and $\mathbf{y}$ are column vectors. Now we consider transformations which preserve the partial determinant. The following theorem characterizes the multiplicative property of the partial determinant.

Theorem 3.3: *Let $A, B \in M_m(\mathbb{F})$ and $C, D \in M_n(\mathbb{F})$. Then*

$$\det_1((A \otimes C)(B \otimes D)) = \det_1(A \otimes C)\det_1(B \otimes D)$$

if and only if $\det(AB) = 0$ *or* $(CD)^{(m)} = C^{(m)}D^{(m)}$.

Proof: Applying Lemma 3.1 on both sides of the equation provides

$$\det(AB)(CD)^{(m)} = \det(AB)C^{(m)}D^{(m)}$$

and the result follows. ■

When $\mathbb{F} = GF(2)$, the Galois field of two elements, we have the following corollary.

Corollary 3.3: *Let $A, B \in M_m(GF(2))$ and $C, D \in M_n(GF(2))$. Then*

$$\det_1((A \otimes C)(B \otimes D)) = \det_1(A \otimes C)\det_1(B \otimes D).$$

Since a row or column permutation of a matrix commutes with the Hadamard power, as does multiplying each row by a constant, we have the following corollary to Theorem 3.3.

Corollary 3.4: *Let $A, B \in M_m(\mathbb{F})$ and $C, P \in M_n(\mathbb{F})$, where P is a permutation matrix or a diagonal matrix. Then*

$$\det_1((AB) \otimes (PC)) = \det_1(A \otimes P)\det_1(B \otimes C),$$
$$\det_1((AB) \otimes (CP)) = \det_1(A \otimes C)\det_1(B \otimes P).$$

Thus, if $B \in M_n(\mathbb{F})$ has no more than $n(n+1)/2$ non-zero entries, and there exist permutations $P, Q \in M_n(\mathbb{F})$ such that PBQ^T is triangular, then $\det(\det_1(A \otimes B)) = \det(A \otimes B)$. Of course, this is not true in general. For example,

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has $7 \leq 10$ non-zero entries, but no such P and Q exist. We note that multiplying by P preserves the number of non-zero entries in each column and multiplying by Q^T preserves the number of non-zero entries in each row.

3.2. The exponential map

Finally, when $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we have

$$\det(\exp(A)) = \exp(\mathrm{tr}(A))$$

for all square matrices A.

Theorem 3.4: *Let $B \in M_m(\mathbb{F})$ and $C \in M_n(\mathbb{C})$ and let A be the Kronecker sum $A = B \oplus C$. Then $\det_1(\exp(A)) = \exp(\mathrm{tr}_1(A))$ if and only if the m-th Hadamard and matrix powers of $\exp(C)$ coincide, i.e. $(\exp(C))^{(m)} = (\exp(C))^m$.*

The proof follows from $\exp(A) = \exp(B) \otimes \exp(C)$, Lemma 3.1 and the fact that $\det(\exp(B)) = \exp(\mathrm{tr}(B))$.

4. Determinant-roots

The monoids of matrices described in Theorem 3.3 also satisfy Theorem 3.2. Let $H_m \subseteq M_n(\mathbb{F})$ be a submonoid under matrix multiplication such that $A, B \in H_m$ satisfy $(AB)^{(m)} = A^{(m)}B^{(m)}$. Then H_m is a submonoid satisfying $\det(A^{(m)}) = \det(A)^m$ for all $A \in H_m$. This follows since

$$\begin{aligned}\det(I_n^{(m)}) &= \det(I_n)^m,\\ \det(AB)^m &= \det(A)^m \det(B)^m = \det(A^{(m)}) \det(B^{(m)}) = \det(A^{(m)}B^{(m)})\\ &= \det((AB)^{(m)}).\end{aligned}$$

This monoid structure motivates the definition of a determinant-like operation which satisfies theorems analogous to Theorems 3.2 and 3.3. If the underlying field is algebraically closed, then the analogous theorems hold in $M_n(\mathbb{F})$ (i.e. for all matrices). In this section we define monoids and a 'determinant-root' operation Det which is completable on Kronecker products of matrices in these monoids. First, let us define the m-th roots on the multiplicative group $\mathbb{F}_\times$ of $\mathbb{F}$.

4.1. Fields with m-th roots of unity

Let G be a multiplicative abelian group, and R_m be the subgroup

$$R_m = \{\, g \in G : g^m = 1 \,\} \tag{6}$$

of G. The cosets of R_m are the equivalence classes $a \cdot R_m$, where

$$a \cdot R_m = b \cdot R_m \text{ if and only if } a^m = b^m \tag{7}$$

for $a, b \in G$. In particular, if $G = \mathbb{F}_\times$ – the multiplicative group in $\mathbb{F}$ – we obtain the following definition.

Definition 4.1: Let $\mathbb{F}$ be a field and $a, b \in \mathbb{F}$, and define

$$R_m := \{\, g \in \mathbb{F} : g^m = 1, g \neq 0 \,\}.$$

We write $a \cdot R_m = \sqrt[m]{b}$ when $a^m = b$. If no such $a \in \mathbb{F}$ exists for a given $b \in \mathbb{F}$, then $\sqrt[m]{b}$ is undefined. Furthermore, we define $\sqrt[m]{0} := 0 \cdot R_m$.

We note that $\sqrt[m]{ab} = \sqrt[m]{a} \cdot \sqrt[m]{b}$ when the relevant m-th roots exist. The non-zero m-th roots of $\mathbb{F}_\times$ form a multiplicative group, and the matrices $M \in M_n(\mathbb{F})$ where $\sqrt[n]{\det(M)}$ exists form a monoid $D_n(\mathbb{F})$. In other words, since $M_n(\mathbb{F})$ forms a monoid under matrix multiplication, $D_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ is a submonoid since $I_n \in D_n(\mathbb{F})$ ($\sqrt[n]{\det(I_n)} = 1$ exists), and $\sqrt[n]{\det(AB)} = \sqrt[n]{\det(A)}\sqrt[n]{\det(B)}$ exists whenever $\sqrt[n]{\det(A)}$ and $\sqrt[n]{\det(B)}$ exist. Let $\sqrt[m]{a}$ be the m-th root of $a \in \mathbb{F}$. Since $\sqrt[m]{a}$ is in the (multiplicative) quotient $\mathbb{F}_\times/R_m$, we consider the n-th root $\sqrt[n]{\sqrt[m]{a}}$, when it exists. In this case, we may write $\sqrt[mn]{a} \equiv \sqrt[n]{\sqrt[m]{a}}$, since

$$G/R_{mn}(G) \cong (G/R_m(G)) \,/\, R_n(G/R_m(G)), \tag{8}$$

where $R_m(G)$ is the subgroup of the abelian group G as above. The isomorphism is simply

$$aR_{mn}(G) \mapsto (a \cdot R_m(G)) \cdot R_n(G/R_m(G)).$$

We will elaborate more on this isomorphism in Lemma 4.2. In the following we develop a hierarchy of fields from repeated quotients of the form $\mathbb{F}_\times/R_m$. This development holds for fields where every element has an m-th root (for example, when $\mathbb{F} = \mathbb{C}$ is the field of complex numbers), but also holds for commutative division rigs (commutative semirings with multiplicative inverse) where every element has an m-th root (for example, the non-negative real numbers).

Lemma 4.1: *Let $\mathbb{F}$ denote a field with multiplication group $G = \mathbb{F}_\times$ such that every element has an m-th root and let R_m denotes the subgroup* (6)*. Then $\mathbb{F}_m := \mathbb{F}_\times/R_m \cup \{0 \cdot R_m\}$ is a field with operations*

$$(a \cdot R_m) \cdot (b \cdot R_m) = (ab) \cdot R_m, \quad (a \cdot R_m) + (b \cdot R_m) = \sqrt[m]{a^m + b^m} \cdot R_m.$$

Proof: In the following, $\sqrt[m]{x}$ denotes any m-th root of $x \in \mathbb{F}$ and the statements hold true for any m-th root. Then R_m is not empty since $1 \in R_m$. Moreover, since $a, b \in R_m$ we have $(ab^{-1})^m = a^m(b^m)^{-1} = 1$ since $\mathbb{F}_\times$ is commutative – i.e. $ab^{-1} \in R_m$. Clearly, R_m is a (commutative) normal subgroup of $\mathbb{F}_\times$. Hence, $\mathbb{F}_\times/R_m$ is a commutative group with operation $(a \cdot R_m) \cdot (b \cdot R_m) = (ab) \cdot R_m$. We will show that the multiplicative group may be extended to a field by defining an appropriate additive structure. We note that $a \cdot R_m = c \cdot R_m$ if and only if $ac^{-1} \in R_m$ if and only if $a^m c^{-m} = 1$ if and only if $a^m = c^m$. Addition is well defined since $a \cdot R_m$ is uniquely determined by a^m. For example, if $a \cdot R_m = c \cdot R_m$ and $b \cdot R_m = d \cdot R_m$, then

$$\begin{aligned} a \cdot R_m + b \cdot R_m &= \sqrt[m]{a^m + b^m} \cdot R_m \\ &= \sqrt[m]{c^m + d^m} \cdot R_m \\ &= c \cdot R_m + d \cdot R_m. \end{aligned}$$

Clearly $0 \cdot R_m$ is an identity element for addition. Now, additive inverses exist since $-1 \in \mathbb{F}_\times$ has an m-th root $\sqrt[m]{-1}$ and $\sqrt[m]{0} = 0$ is the unique m-th root of $0 \in \mathbb{F}$,

$$a \cdot R_m + (\sqrt[m]{-1}a) \cdot R_m = \sqrt[m]{a^m + (\sqrt[m]{-1}a)^m} \cdot R_m = \sqrt[m]{a^m + (-1)a^m} \cdot R_m = 0 \cdot R_m.$$

It remains to show that multiplicative distributivity holds. We have $a\sqrt[m]{x} \cdot R_m = \sqrt[m]{a^m x} \cdot$

R_m if and only if $(a \sqrt[m]{x})^m = (\sqrt[m]{a^m x})^m$ which holds true. Thus,

$$\begin{aligned}(a \cdot R_m)(b \cdot R_m + c \cdot R_m) &= (a \cdot R_m)(\sqrt[m]{b^m + c^m} \cdot R_m) \\ &= (a \sqrt[m]{b^m + c^m}) \cdot R_m \\ &= \sqrt[m]{(ab)^m + (ac)^m} \cdot R_m \\ &= (ab) \cdot R_m + (ac) \cdot R_m \\ &= (a \cdot R_m) \cdot (b \cdot R_m) + (a \cdot R_m) \cdot (c \cdot R_m).\end{aligned}$$

We also note that $\mathbb{F}_1 \equiv \mathbb{F}$. ■

Lemma 4.2: *Let $\mathbb{F}$ denote a field such that every element has an m-th root and an mn-th root. Then $\mathbb{F}_{mn} \cong (\mathbb{F}_m)_n$. Here $\mathbb{F}_m = \mathbb{F}_\times / R_m \cup \{0 \cdot R_m\}$ with operations as in Lemma* 4.1 *and similarly for $\mathbb{F}_{mn}$.*

Proof: First we must show that every element of $\mathbb{F}_m$ has an n-th root. Let $a \in \mathbb{F}$. Since a has mn-th roots there exists $b \in \mathbb{F}$ such that $b^{mn} = a$, and, hence, b^m is an n-th root of a and $b^m \cdot R_m$ is an n-th root of $a \cdot R_m$. It remains to demonstrate that there exists a field isomorphism $\phi : \mathbb{F}_{mn} \to (\mathbb{F}_m)_n$. Define

$$\phi(a \cdot R_{mn}) = (a \cdot R_m) \cdot R_n,$$

where $R_m = R_m(\mathbb{F})$ is the multiplicative subgroup (6) of $\mathbb{F}$ and $R_n = R_n(\mathbb{F}_m)$ is the multiplicative subgroup (6) of $\mathbb{F}_m$. The function ϕ is well-defined since for all $x \in a \cdot R_{mn}$, $(x^n)^m = (a^n)^m$ so that $x^n \cdot R_m = a^n \cdot R_m$ and, finally, $(x \cdot R_m) \cdot R_n = (a \cdot R_m) \cdot R_n$. Injectivity of ϕ follows similarly and surjectivity is immediate. Finally, ϕ is a ring homomorphism since

$$\begin{aligned}\phi((a \cdot R_{mn}) \cdot (b \cdot R_{mn})) &= [(a \cdot R_m) \cdot (b \cdot R_m)] \cdot R_n \\ &= [(a \cdot R_m) \cdot R_n] \cdot [(b \cdot R_m) \cdot R_n] \\ &= \phi(a \cdot R_{mn}) \cdot \phi(b \cdot R_{mn}), \\ \phi(a \cdot R_{mn} + b \cdot R_{mn}) &= \phi(\sqrt[mn]{a^{mn} + b^{mn}} \cdot R_{mn}) \\ &= (\sqrt[mn]{a^{mn} + b^{mn}} \cdot R_m) \cdot R_n \\ &= \sqrt[n]{a^n \cdot R_m + b^n \cdot R_m} \cdot R_n \\ &= (a \cdot R_m) \cdot R_n + (b \cdot R_m) \cdot R_n \\ &= \phi(a \cdot R_{mn}) + \phi(b \cdot R_{mn}).\end{aligned}$$

■

There is a natural embedding

$$h_{m,n} : a \cdot R_m \mapsto a^n \cdot R_{mn}$$

which is well defined since if $a \cdot R_m = b \cdot R_m$ then $a^m = b^m$ and hence,

$$(a^n)^{mn} = (b^n)^{mn},$$

which yields that $h_{m,n}(a \cdot R_m) = h_{m,n}(b \cdot R_m)$. We define the product $*$ as follows

$$(a \cdot R_m) * (b \cdot R_n) := h_{m,n}(a \cdot R_m) \cdot h_{n,m}(b \cdot R_n) = (a^n b^m) \cdot R_{mn}. \tag{9}$$

Consequently, $\sqrt[m]{a} * \sqrt[n]{b} = \sqrt[mn]{a^n b^m}$ when the given roots exist.

Lemma 4.3: *Let $\mathbb{F}$ denote a field such that every element has an m-th root and an mn-th root. Then $h_{m,n} : \mathbb{F}_m \to \mathbb{F}_{mn}$ is a (multiplicative) group homomorphism.*

Proof: The proof follows straightforwardly from the definitions

$$\begin{aligned} h_{m,n}((a \cdot R_m) \cdot (b \cdot R_m)) &= (ab)^n \cdot R_{mn} \\ &= a^n b^n \cdot R_{mn} \\ &= (a^n \cdot R_{mn}) \cdot (b^n \cdot R_{mn}) \\ &= h_{m,n}(a \cdot R_m) \cdot h_{m,n}(b \cdot R_m). \end{aligned}$$

■

The fields $\mathbb{F}_m$ are defined when all elements have m-th roots. Hence, it is necessary to consider a theory over the group (or binoid, if 0 is included) $\mathbb{F}_\times / R_m$ and additional results in $\mathbb{F}_m$ when it exists.

4.2. The determinant-root on $D_n(\mathbb{F})$

Recall that $D_n(\mathbb{F})$ denotes the matrices $A \in M_n(\mathbb{F})$ such that $\det(A)$ has an n-th root in $\mathbb{F}$.

Definition 4.2: We define the determinant-root Det by

$$\mathrm{Det}(M) := \sqrt[n]{\det(M)}$$

for all $M \in D_n(\mathbb{F})$.

Remark 4.1: When $n = 2$ and $\mathbb{F} = \mathbb{R}$, $\mathrm{Det}(M)$ is equivalent to the non-negative square root of $\det(M)$.

This definition provides the usual multiplicative property

$$\mathrm{Det}(AB) = \mathrm{Det}(A)\,\mathrm{Det}(B)$$

for $A, B \in D_n(\mathbb{F})$. With respect to scalar multiplication however, we have

$$\mathrm{Det}(\alpha A) = \sqrt[n]{\det(\alpha A)} = (\alpha \cdot R_n)\sqrt[n]{\det(A)} = (\alpha \cdot R_n)\,\mathrm{Det}(A).$$

By the embedding (9), we have

$$\mathrm{Det}(A \otimes C) = \mathrm{Det}(A) * \mathrm{Det}(C)$$

for $A \in D_m(\mathbb{F})$ and $C \in D_n(\mathbb{F})$ from $\det(A \otimes C) = (\det(A))^n(\det(C))^m$.

Now we are ready to consider partial determinant-roots and their properties. The partial determinant-root Det_2 is again the block-wise determinant-root.

Definition 4.3: Let $A, B \in M_{mn}(\mathbb{F}) = M_m(M_n(\mathbb{F})) = M_m(\mathbb{F}) \otimes M_n(\mathbb{F})$ with

$$A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}^{[n]}, \qquad B = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes B_{ij}$$

and assume that $A_{ij} \in D_m(\mathbb{F})$ and $B_{ij} \in D_n(\mathbb{F})$ for all i, j. The partial determinant-roots $\mathrm{Det}_1(A)$ and $\mathrm{Det}_2(B)$ are given by

$$\mathrm{Det}_1(A) := \sum_{i,j=1}^{n} \mathrm{Det}(A_{ij}) E_{ij}^{[n]}, \qquad \mathrm{Det}_2(B) := \sum_{i,j=1}^{m} \mathrm{Det}(B_{ij}) E_{ij}^{[m]}.$$

In general, we work in the group ring $\mathbb{Z}[\mathbb{F}_\times / R_n(\mathbb{F}_\times)]$, i.e. $\mathrm{Det}_2 : M_m(D_n(\mathbb{F})) \to M_m(\mathbb{Z}[\mathbb{F}_\times / R_m(\mathbb{F}_\times)])$ and similarly for Det_1. However, when we are concerned only with Kronecker products of matrices, we will express our results in $\mathbb{F}_\times / R_m(\mathbb{F}_\times)$ where possible. Additionally, when $\mathbb{F}_n$ exists, we assume $\mathrm{Det}_2 : M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) \to M_m(\mathbb{F}_n)$.

Lemma 4.4: *The partial determinant-root obeys*

$$\mathrm{Det}_1(A \otimes B) = \mathrm{Det}(A)\, B,$$

where $\mathrm{Det}(A)B$ *is the entry-wise product of the entries in* B *with* $\mathrm{Det}(A)$.

Proof: We have

$$A \otimes B = \sum_{i,j=1}^{n} ((B)_{ij} A) \otimes E_{ij}^{[n]},$$

where $B = (B)_{ij}$. It follows that

$$\begin{aligned} \mathrm{Det}_1(A \otimes B) &= \sum_{i,j=1}^{n} \mathrm{Det}(B_{ij} A) E_{ij}^{[n]} = \sum_{i,j=1}^{n} B_{ij} \,\mathrm{Det}(A) E_{ij}^{[n]} = \mathrm{Det}(A) \sum_{i,j=1}^{n} B_{ij} E_{ij}^{[n]} \\ &= \mathrm{Det}(A) B, \end{aligned}$$

since $\sqrt[m]{B_{ij}^m} = B_{ij} \cdot R_m(\mathbb{F}_\times)$. ∎

Theorem 4.1: *Let* $A \in D_m(\mathbb{F})$ *and* $C \in D_n(\mathbb{F})$. *Then*

$$\mathrm{Det}(A \otimes C) = \mathrm{Det}(\mathrm{Det}_1(A \otimes C)).$$

Proof: Since $\mathrm{Det}(A \otimes C) = \mathrm{Det}(A) * \mathrm{Det}(C)$ and (by the isomorphism (8))

$$\begin{aligned} \mathrm{Det}(\mathrm{Det}_1(A \otimes C)) &= \sqrt[n]{\det(\mathrm{Det}(A) C)} \\ &= \sqrt[n]{(\mathrm{Det}(A))^n \det(C)} \end{aligned}$$

$$\equiv \sqrt[mn]{(\det(A))^n(\det(C))^m}$$
$$= \mathrm{Det}(A) * \mathrm{Det}(C)$$

we obtain $\mathrm{Det}(A \otimes C) = \mathrm{Det}(\mathrm{Det}_1(A \otimes C))$. ■

Lemma 4.5: *Let $\mathbb{F}$ denote a field such that every element has an m-th root. Then*

$$\mathrm{Det}(A) = \mathrm{Det}(\mathrm{Det}_1(A)).$$

for all $A \in M_{mn}(\mathbb{F})$ if and only if

$$\mathrm{Det}(\mathrm{Det}_1(XY)) = \mathrm{Det}(\mathrm{Det}_1(X)\,\mathrm{Det}_1(Y))$$

for all $X, Y \in M_{mn}(\mathbb{F})$ and similarly for Det_2.

Proof: ($\Longrightarrow$) It follows by direct calculation

$$\mathrm{Det}(\mathrm{Det}_1(XY)) = \mathrm{Det}(XY) = \mathrm{Det}(X)\,\mathrm{Det}(Y) = \mathrm{Det}(\mathrm{Det}_1(X))\,\mathrm{Det}(\mathrm{Det}_1(Y))$$
$$= \mathrm{Det}(\mathrm{Det}_1(X)\,\mathrm{Det}_1(Y)).$$

($\Longleftarrow$) The proof is straightforward when $X = A$ and $Y = I_{mn}$, ■

Lemma 4.6: *Let $\mathbb{F}$ denote a field such that every element has an mn-th root. Then*

$$\mathrm{Det}(A) = \mathrm{Det}(\mathrm{Det}_2(A))$$

for all $A \in M_{mn}(\mathbb{F})$ if and only if

$$\mathrm{Det}(X + Y) = \mathrm{Det}(X) + \mathrm{Det}(Y) \text{ in } (\mathbb{F}_n)_m$$

for all $X, Y \in M_n(\mathbb{F})$.

Proof: ($\Longrightarrow$) Assume that $\mathrm{Det}(A) = \mathrm{Det}(\mathrm{Det}_2(A))$ for all $A \in M_{mn}(\mathbb{F})$. We will show that

$$\sqrt[m]{\mathrm{Det}(X + Y)} = \sqrt[m]{\mathrm{Det}(X) + \mathrm{Det}(Y)} \text{ in } (\mathbb{F}_n)_m$$

for all $X, Y \in M_n(\mathbb{F})$. By uniqueness of m-th roots in $(\mathbb{F}_n)_m$ we will conclude that

$$\mathrm{Det}(X + Y) = \mathrm{Det}(X) + \mathrm{Det}(Y) \text{ in } (\mathbb{F}_n)_m.$$

Using Definition 4.1 and the isomorphism in Lemma 4.2, we have

$$\sqrt[m]{\mathrm{Det}(X + Y)} = \mathrm{Det}\begin{pmatrix} X+Y & I_n & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I_n \end{pmatrix}$$

$$= \operatorname{Det}\begin{pmatrix} X & I_n & 0 & \cdots & 0 \\ -Y & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I_n \end{pmatrix} \operatorname{Det}\begin{pmatrix} I_n & 0 & 0 & \cdots & 0 \\ Y & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I_n \end{pmatrix}$$

and, by assumption $\operatorname{Det}(A) = \operatorname{Det}(\operatorname{Det}_2(A))$,

$$\sqrt[m]{\operatorname{Det}(X+Y)}$$

$$= \operatorname{Det}\left(\operatorname{Det}_2\begin{pmatrix} X & I_n & 0 & \cdots & 0 \\ -Y & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I_n \end{pmatrix}\right) \operatorname{Det}\left(\operatorname{Det}_2\begin{pmatrix} I_n & 0 & 0 & \cdots & 0 \\ Y & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I_n \end{pmatrix}\right)$$

$$= \operatorname{Det}\begin{pmatrix} \operatorname{Det}(X) & 1 & 0 & \cdots & 0 \\ -\operatorname{Det}(Y) & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \operatorname{Det}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \operatorname{Det}(Y) & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \operatorname{Det}\begin{pmatrix} \operatorname{Det}(X)+\operatorname{Det}(Y) & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \sqrt[m]{\operatorname{Det}(X)+\operatorname{Det}(Y)}.$$

($\Longleftarrow$) If $\operatorname{Det}(X+Y) = \operatorname{Det}(X) + \operatorname{Det}(Y)$ in $(\mathbb{F}_n)_m$, then Det is linear in the sense that $\operatorname{Det}(\alpha A) = (\alpha \cdot R_m)\operatorname{Det}(A)$ for all $\alpha \in \mathbb{F}_n$, Using the isomorphism in Lemma 4.2,

$$\operatorname{Det}(A) = \operatorname{Det}\left(\sum_{i,j=1}^{m} E_{ij} \otimes A'_{ij}\right) = \sum_{i,j=1}^{m} \operatorname{Det}\left(E_{ij} \otimes A'_{ij}\right) = \sum_{i,j=1}^{m} \operatorname{Det}(E_{ij}) * \operatorname{Det}(A'_{ij})$$

$$= \sum_{i,j=1}^{m} \sqrt[mn]{\det(E_{ij})^n \det(A'_{ij})^m} \cdot R_{mn} = \sum_{i,j=1}^{m} \sqrt[mn]{\left(\det(E_{ij})\sqrt[n]{\det(A'_{ij})}^m\right)^n} \cdot R_{mn}$$

$$= \sum_{i,j=1}^{m} \sqrt[mn]{\det\left(\sqrt[n]{\det(A'_{ij})}E_{ij}\right)^n} \cdot R_{mn}$$

$$\simeq \sum_{i,j=1}^{m} \sqrt[m]{\det(\operatorname{Det}(A'_{ij})E_{ij})} \cdot R_m = \sum_{i,j=1}^{m} \operatorname{Det}(\operatorname{Det}(A'_{ij})E_{ij})$$

$$= \operatorname{Det}\left(\sum_{i,j=1}^{m} \operatorname{Det}(A'_{ij})E_{ij}\right) = \operatorname{Det}(\operatorname{Det}_2(A)).$$

■

Corollary 4.1: *If*

$$\operatorname{Det}(A) = \operatorname{Det}(\operatorname{Det}_2(A))$$

for all $A \in M_{mn}(\mathbb{F})$, *then* $\operatorname{Det}(A) = 0$ *for all* $A \in M_{mn}(\mathbb{F})$.

Proof: We have, by Lemma 4.6,

$$\operatorname{Det}(A) = \operatorname{Det}\left(\sum_{i,j=1}^{m} E_{ij} \otimes A'_{ij}\right) = \sum_{i,j=1}^{m} \operatorname{Det}\left(E_{ij} \otimes A'_{ij}\right)$$
$$= \sum_{i,j=1}^{m} \operatorname{Det}(E_{ij}) * \operatorname{Det}(A'_{ij}) = 0.$$

■

Corollary 4.2: *Theorem* 4.1 *can only be extended to all matrices in a trivial sense, i.e.* $\operatorname{Det}(A) \equiv 0$.

4.3. Products

Next we show that partial determinant-roots are multiplicative on products of Kronecker products.

Theorem 4.2: *Let* $A, B \in D_m(\mathbb{F})$ *and* $C, D \in M_n(\mathbb{F})$. *Then*

$$\operatorname{Det}_1((A \otimes C)(B \otimes D)) = \operatorname{Det}_1(A \otimes C)\operatorname{Det}_1(B \otimes D).$$

Proof: Lemma 4.4 provides

$$\operatorname{Det}_1((A \otimes C)(B \otimes D)) = \operatorname{Det}(AB)\,(CD).$$

Similarly,

$$\begin{aligned}\operatorname{Det}_1(A \otimes C)\operatorname{Det}_1(B \otimes D) &= (\operatorname{Det}(A)\,C)(\operatorname{Det}(B)\,D)\\ &= (\operatorname{Det}(A)\operatorname{Det}(B))\,(CD)\\ &= \operatorname{Det}(AB)\,(CD).\end{aligned}$$

■

Lemma 4.7: *Let $\mathbb{F}$ denote a field such that every element has an n-th root. Then*

$$\mathrm{Det}_2(AB) = \mathrm{Det}_2(A)\,\mathrm{Det}_2(B).$$

for all $A, B \in M_{mn}(\mathbb{F})$ if and only if

$$\mathrm{Det}(X+Y) = \mathrm{Det}(X) + \mathrm{Det}(Y)$$

for all $X, Y \in M_n(\mathbb{F})$ and similarly for Det_1.

Proof: ($\Longrightarrow$) Assume that $\mathrm{Det}_2(AB) = \mathrm{Det}_2(A)\,\mathrm{Det}_2(B)$ for all $A, B \in M_{mn}(\mathbb{F})$. In particular, for all $X, Y \in M_n(\mathbb{F})$,

$$\begin{pmatrix} \mathrm{Det}(X)+\mathrm{Det}(Y) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Det}(X) & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mathrm{Det}(Y) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \mathrm{Det}_2 \begin{pmatrix} X & I_n & \cdots & I_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mathrm{Det}_2 \begin{pmatrix} I_n & 0 & \cdots & 0 \\ Y & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \mathrm{Det}_2 \begin{pmatrix} X+Y & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Det}(X+Y) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and, consequently, $\mathrm{Det}(X+Y) = \mathrm{Det}(X) + \mathrm{Det}(Y)$.
($\Longrightarrow$) Suppose that $\mathrm{Det}(X+Y) = \mathrm{Det}(X) + \mathrm{Det}(Y)$ for all $X, Y \in M_n(\mathbb{F})$. By direct calculation,

$$\mathrm{Det}_2(AB) = \mathrm{Det}_2\left(\sum_{i,j=1}^{m} E_{ij} \otimes \sum_{k=1}^{m} A'_{ik}B'_{kj}\right) = \sum_{i,j=1}^{m} \mathrm{Det}\left(\sum_{k=1}^{m} A'_{ik}B'_{kj}\right) E_{ij}$$

$$= \sum_{i,j,k=1}^{m} \mathrm{Det}\left(A'_{ik}B'_{kj}\right) E_{ij} = \sum_{i,j,k=1}^{m} \mathrm{Det}(A'_{ik})\,\mathrm{Det}(B'_{kj}) E_{ij}$$

$$= \left(\sum_{i,k=1}^{m} \mathrm{Det}(A'_{ik}) E_{ik}\right)\left(\sum_{j,k=1}^{m} \mathrm{Det}(B'_{kj}) E_{kj}\right) = \mathrm{Det}_2(A)\,\mathrm{Det}_2(B).$$

■

Corollary 4.3: *Theorem* 4.2 *can only be extended to all matrices in a trivial sense, i.e.* $\mathrm{Det}(A) \equiv 0$ *since,*

$$\mathrm{Det}(A) = \mathrm{Det}\left(\sum_{i,j=1}^{m} a_{ij}E_{ij}\right) = \sum_{i,j=1}^{m} \mathrm{Det}(a_{ij}E_{ij}) = 0.$$

4.4. The exponential map

Assume that $n \neq 0$ in $\mathbb{F}$. Define $\mathrm{Tr} : M_n(\mathbb{F}) \to \mathbb{F}$ by $\mathrm{Tr} : A \mapsto \frac{\mathrm{tr}(A)}{n}$ and define the partial operation Tr_1 in the natural way. For determinant-roots we obtain

$$\mathrm{Det}(\exp(A)) = \exp(\mathrm{Tr}(A)) \cdot R_n(\mathbb{F}_\times). \tag{10}$$

Partial determinant-roots obey the exponential-determinant-trace relation as follows.

Theorem 4.3: *Let* $\mathbb{F}$ *be the field* $\mathbb{F} = \mathbb{R}$ *of real numbers or the field* $\mathbb{F} = \mathbb{C}$ *of complex numbers and let* A *be the Kronecker sum* $A = B \oplus C$. *Then*

$$\mathrm{Det}_1(\exp(A)) = \exp(\mathrm{Tr}_1(A)) \cdot R_m(\mathbb{F}_\times).$$

Proof: The result follows from

$$\exp(\mathrm{Tr}_1(A)) = \exp(1/m\,\mathrm{tr}(B)I_n + C) = e^{1/m\,\mathrm{tr}(B)}e^C$$

and, since $e^B \in D_M(\mathbb{F})$ (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and using Lemma 4.4,

$$\begin{aligned}
\mathrm{Det}_1(\exp(A)) &= \mathrm{Det}_1(e^B \otimes e^C) \\
&= \mathrm{Det}(e^B)e^C \\
&= \sqrt[m]{\det(e^B)}\, e^C \\
&= \sqrt[m]{e^{\mathrm{tr}(B)}}\, e^C \\
&= e^{1/m\,\mathrm{tr}(B)}e^C \cdot R_m(\mathbb{F}_\times).
\end{aligned}$$

■

5. Non-linear preservers of determinants

In this section we discuss a class of preservers of determinants, i.e. maps $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ which satisfy $\det(\psi(A)) = \det(A)$. In particular, we study maps which may be written in the form $\psi(e^A) = e^{\phi(A)}$, where $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a linear map as described in (11) below. Naturally, this implies a restriction on the matrices A and we will find it convenient to reframe the problem in terms of the determinant-root Det from Definition 4.2.

In this setting, we may study new classes of maps satisfying

$$\mathrm{Det}(\psi(A \otimes B)) = \mathrm{Det}(A \otimes B),$$

i.e. we will continue the study of [11] in this new setting. Since the determinant, trace and Kronecker product are related by the equations

$$\det(e^A) = e^{\mathrm{tr}(A)}, \qquad e^A \otimes e^B = e^{A \oplus B},$$

over the field $\mathbb{R}$ or $\mathbb{C}$, we will first study the Kronecker sum $\oplus$ and its linear preservers.

5.1. RT-Symmetry

Noting that $M_n(\mathbb{F})$ is an n^2-dimensional space, for an arbitrary linear map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$, we have

$$\phi(A) = \sum_{j,k,u,v=1}^{n} \alpha_{jk;uv} E_{jk}^{[n]} A E_{uv}^{[n]} \tag{11}$$

for some $\alpha_{jk;uv} \in \mathbb{F}$ in the underlying field.

Definition 5.1: Let $\mathbb{F}$ be an arbitrary field and $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear map in the form (11). We define the linear transform $' : \phi \mapsto \phi'$, by

$$\phi'(A) := \sum_{j,k,u,v=1}^{n} \alpha_{uv;jk} E_{jk}^{[n]} A E_{uv}^{[n]} \equiv \sum_{j,k,u,v=1}^{n} \alpha_{jk;uv} E_{uv}^{[n]} A E_{jk}^{[n]}. \tag{12}$$

When $\mathbb{F} = \mathbb{C}$ is the field of complex numbers, we denote by $\phi^* := \overline{\phi'}$ the complex conjugate composed with ϕ', i.e. $\phi^*(A) = \overline{\phi'(A)}$.

The following two lemmas follow immediately.

Lemma 5.1: *Let $\phi, \psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be linear maps. Then*

(a) $(\phi')' = \phi$,
(b) $(\alpha\phi + \psi)' = \alpha(\phi') + \psi'$ *for all* $\alpha \in \mathbb{F}$.

Lemma 5.2: *Let $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be linear maps. Then*

(a) $(\phi^*)^* = \phi$,
(b) $(\phi + \psi)^* = \phi^* + \psi^*$.

In the following, we will associate to every map ϕ a matrix Φ, which is defined in terms of the vec operator. Given a matrix A, let $\mathrm{vec}(A)$ be the column vector obtained by stacking the columns of A on top of each other [7]. The following lemma shows that the vec operator maps rank-one matrices to Kronecker products of column vectors.

Lemma 5.3: *Let $a, b \in \mathbb{F}^n$ be column vectors with entries from the field $\mathbb{F}$. Then* $\mathrm{vec}(ab^T) = b \otimes a$.

Proof: Straightforward calculation yields that

$$\operatorname{vec}(ab^T) = \operatorname{vec}\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix} = \begin{pmatrix} a_1b_1 \\ a_2b_1 \\ \vdots \\ a_nb_1 \\ \vdots \\ \vdots \\ a_1b_n \\ a_2b_n \\ \vdots \\ a_nb_n \end{pmatrix} = b \otimes a. \qquad \blacksquare$$

Let Φ be the matrix representing the linear map ϕ in the standard basis, i.e. $\Phi \operatorname{vec}(A) = \operatorname{vec}(\phi(A))$. Similarly, let Φ' be the matrix representing ϕ'. Since

$$\phi(E_{pq}^{[n]}) = \sum_{j,v=1}^{n} \alpha_{jp;qv} E_{jv}^{[n]}, \quad \phi'(E_{pq}^{[n]}) = \sum_{j,v=1}^{n} \alpha_{qv;jp} E_{jv}^{[n]},$$

it follows by Lemma 5.3 that

$$\Phi = \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} E_{vq}^{[n]} \otimes E_{jp}^{[n]}, \quad \Phi' = \sum_{j,p,q,v=1}^{n} \alpha_{qv;jp} E_{vq}^{[n]} \otimes E_{jp}^{[n]}.$$

Definition 5.2 (Perfect shuffle matrix [7,8]): The perfect shuffle matrix P is the unique permutation matrix which satisfies $P^T(A \otimes B)P = B \otimes A$ for every $A, B \in M_n(R)$, i.e.

$$P = \sum_{j=1}^{n} \sum_{k=1}^{n} E_{jk}^{[n]} \otimes E_{kj}^{[n]}.$$

The perfect shuffle matrix is also known as the vec-permutation matrix on $\mathbb{F}^n \otimes \mathbb{F}^n$. It follows that

$$\Phi^T = P^T \Phi' P. \tag{13}$$

Definition 5.3 (Rearrangement operator [16]): The rearrangement operator R is linear and defined by

$$R(A \otimes B) = (\operatorname{vec} A)(\operatorname{vec} B)^T,$$

where vec is the vec operator [7].

The rearrangement operator provides the characterization of ϕ which obeys $\phi = \phi'$.

Lemma 5.4: *Let $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear map and Φ be the matrix satisfying*

$$\Phi \operatorname{vec}(A) = \operatorname{vec}(\phi(A)).$$

Then the following are equivalent:

(a) $\phi = \phi'$,
(b) $\Phi^T = P^T \Phi P$,
(c) $R(\Phi)^T = R(\Phi^T)$,

where P is the perfect shuffle matrix (Definition 5.2) and R is the rearrangement operator (Definition 5.3).

Proof: The proof (a) $\Longleftrightarrow$ (b) follows since Φ is uniquely determined by $P^T(A \otimes B)P = B \otimes A$, and by (13). By simple calculation, if $\phi = \phi'$, then using the forms of ϕ in (11) and of ϕ' in (12) we have that $\alpha_{jp;qv} = \alpha_{qv;jp}$,

$$\begin{aligned} R(\Phi^T) &= \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} R(E_{pj}^{[n]} \otimes E_{qv}^{[n]}) = \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} \operatorname{vec}(E_{pj}^{[n]}) \operatorname{vec}(E_{qv}^{[n]})^T \\ &= \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} E_{jv}^{[n]} \otimes E_{pq}^{[n]} = \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} E_{qp}^{[n]} \otimes E_{vj}^{[n]}, \\ R(\Phi) &= \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} R(E_{jp}^{[n]} \otimes E_{vq}^{[n]}) = \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} \operatorname{vec}(E_{jp}^{[n]}) \operatorname{vec}(E_{vq}^{[n]})^T \\ &= \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} E_{pq}^{[n]} \otimes E_{jv}^{[n]} = R(\Phi^T)^T. \end{aligned}$$

On the other hand, comparing coefficients of $E_{pq}^{[n]} \otimes E_{jv}^{[n]}$ in $R(\Phi)^T = R(\Phi^T)$ yields $\alpha_{jp;qv} = \alpha_{qv;jp}$ so that $\phi = \phi'$. We have proved (a) $\Longleftrightarrow$ (c). ■

Lemma 5.4(c) motivates the following definitions.

Definition 5.4: A linear map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfying $\phi = \phi'$ is said to be *RT-symmetric*. If $\phi = -\phi'$ then ϕ is said to be *skew RT-symmetric*.

Definition 5.5: A linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $\phi = \overline{\phi'}$ is said to be *RT-Hermitian*. If $\phi = -\overline{\phi'}$ then ϕ is said to be *skew RT-Hermitian*.

The following lemma follows immediately from $\phi = \frac{1}{2}(\phi + \phi') + \frac{1}{2}(\phi - \phi')$.

Lemma 5.5: *If the underlying field has characteristic not equal to 2, then every linear map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is the sum of an RT-symmetric map and a skew RT-symmetric map.*

5.2. Linear trace preservers of Kronecker sums

We may write a linear map $\phi : M_{mn} \to M_{mn}$ in the operator-sum form

$$\phi(M) = \sum_{i=1}^{r} P_i M Q_i$$

for some matrices P_i, Q_i, $i = 1, \ldots, r$, of the appropriate sizes. If ϕ preserves the trace of a Kronecker sum M, then the cyclic property of the trace yields

$$\operatorname{tr}(M) = \operatorname{tr}(\phi(M)) = \sum_{i=1}^{r} \operatorname{tr}(Q_i P_i M)$$

and so we need only consider preservers of the form

$$\phi(M) = \sum_{i=1}^{r} Q_i P_i M = PM,$$

where

$$P := \sum_{i=1}^{r} Q_i P_i \tag{14}$$

and the remaining preservers are all obtained by representations (14) of P. First we consider maps of the form $\phi(M) = PM$, $M \in M_{mn}$.

Theorem 5.1: *Let $\phi : M_{mn} \to M_{mn}$ be a map given by $\phi : M \mapsto PM$ for some $P \in M_{mn}$. Then $\operatorname{tr}(\phi(A \oplus B)) = \operatorname{tr}(A \oplus B)$ for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if*

$$\operatorname{tr}_1(P) = \operatorname{tr}_1(I_{mn}) \quad \text{and} \quad \operatorname{tr}_2(P) = \operatorname{tr}_2(I_{mn}).$$

Proof: First, let us write P in block matrix form, $P = (P'_{kl})$ where each $P'_{kl} \in M_n(\mathbb{F})$ for $k, l = 1, \ldots, m$. In other words,

$$P = \sum_{k,l=1}^{m} E_{kl}^{[m]} \otimes P'_{kl}.$$

Since ϕ is linear, the map ϕ preserves the trace of Kronecker sums if and only if ϕ preserves traces of Kronecker products of the form $E_{ij}^{[m]} \oplus 0 = E_{ij}^{[m]} \otimes I_n$ and of the form $0 \oplus E_{kl}^{[n]} = I_m \otimes E_{kl}^{[n]}$. Thus, we have

$$n\delta_{ij} = \operatorname{tr}(E_{ij}^{[m]} \otimes I_n) = \operatorname{tr}(\phi(E_{ij}^{[m]} \otimes I_n)) \tag{15}$$

and

$$\begin{aligned} \operatorname{tr}(\phi(E_{ij}^{[m]} \otimes I_n)) &= \sum_{k,l=1}^{m} \operatorname{tr}((E_{kl}^{[m]} E_{ij}^{[m]}) \otimes P'_{kl}) \\ &= \sum_{k,l=1}^{m} \delta_{il} \operatorname{tr}(E_{kj}^{[m]}) \operatorname{tr}(P'_{kl}) = \sum_{k,l=1}^{m} \delta_{jk}\delta_{il} \operatorname{tr}(P'_{kl}) = \operatorname{tr}(P'_{ji}). \end{aligned} \tag{16}$$

It follows, by comparing (15) and (16), that

$$\mathrm{tr}_2(P) = \sum_{k,l=1}^{m} \mathrm{tr}(P'_{kl}) E_{kl}^{[m]} = \sum_{k,l=1}^{m} n\delta_{kl} E_{kl}^{[m]} = nI_m.$$

For Kronecker products of the form $I_m \otimes E_{kl}^{[n]}$, we find

$$m\delta_{kl} = \mathrm{tr}(I_m \otimes E_{kl}^{[n]}) = \mathrm{tr}(\phi(I_m \otimes E_{kl}^{[n]})),$$

and since $\mathrm{tr}(P_{ij} E_{kl}^{[n]}) = (P'_{jj})_{lk}$,

$$m\delta_{kl} = \mathrm{tr}(\phi(I_m \otimes E_{kl}^{[n]})) = \sum_{i,j=1}^{m} \mathrm{tr}(E_{ij}^{[m]} \otimes (P'_{ij} E_{kl}^{[n]})) = \sum_{i,j=1}^{m} \delta_{ij} \mathrm{tr}(P'_{ij} E_{kl}^{[n]}) = \sum_{j=1}^{m} (P'_{jj})_{lk}.$$

Consequently,

$$\mathrm{tr}_1(P) = \sum_{j=1}^{m} P'_{jj} = mI_n.$$

Conversely, suppose that $\mathrm{tr}_1(P) = mI_n$ and $\mathrm{tr}_2(P) = nI_m$. Then, by Corollary 1.1,

$$\begin{aligned} \mathrm{tr}(\phi(A \oplus B)) &= \mathrm{tr}(\mathrm{tr}_2(P(A \otimes I_n))) + \mathrm{tr}(\mathrm{tr}_1(P(I_m \otimes B))) \\ &= \mathrm{tr}(\mathrm{tr}_2(P)A) + \mathrm{tr}(\mathrm{tr}_1(P)B) = n\,\mathrm{tr}(A) + m\,\mathrm{tr}(B) = \mathrm{tr}(A \oplus B). \end{aligned}$$ ■

Corollary 5.1: *Let $\phi : M_{mn} \to M_{mn}$ be a map given by $\phi : M \mapsto PM$ for some $P \in M_{mn}$, where*

$$P = I_{mn} + \sum_{j=1}^{r} A_j \otimes B_j.$$

Here, r is the tensor rank of $P - I_{mn}$ over $M_m(\mathbb{F}) \otimes M_n(\mathbb{F})$ and $A_j \in M_m(\mathbb{F})$, $B_j \in M_n(\mathbb{F})$ for $j = 1, \ldots, r$. Then $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if $\mathrm{tr}(A_j) = \mathrm{tr}(B_j) = 0$ for $j = 1, \ldots, r$.

Proof: By Theorem 5.1, we only need to show that $\mathrm{tr}_1(P) = mI_n$ and $\mathrm{tr}_2(P) = nI_m$ if and only if $\mathrm{tr}(A_j) = \mathrm{tr}(B_j) = 0$ for $j = 1, \ldots, r$. The proof of ($\Leftarrow$) is immediate. For ($\Rightarrow$), suppose $\mathrm{tr}_1(P) = mI_n$ and $\mathrm{tr}_2(P) = nI_m$. It follows that

$$\mathrm{tr}_1(I_{mn}) = mI_n = \mathrm{tr}_1(P) = \mathrm{tr}_1(I_{mn}) + \sum_{j=1}^{r} \mathrm{tr}(A_j) B_j \quad \Longleftrightarrow \quad \sum_{j=1}^{r} \mathrm{tr}(A_j) B_j = 0_n,$$

$$\mathrm{tr}_2(I_{mn}) = nI_m = \mathrm{tr}_2(P) = \mathrm{tr}_2(I_{mn}) + \sum_{j=1}^{r} \mathrm{tr}(B_j) A_j \quad \Longleftrightarrow \quad \sum_{j=1}^{r} \mathrm{tr}(B_j) A_j = 0_m.$$

Since r is the tensor rank of $P - I_{mn}$, the set $\{B_1, \ldots, B_r\}$ is a linearly independent set and $\mathrm{tr}(A_j) = 0$ for $j = 1, \ldots, r$. Similarly, $\mathrm{tr}(B_j) = 0$ for $j = 1, \ldots, r$. ■

As a consequence of Theorem 5.1, we have that $\phi : M \mapsto PM$ satisfies $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ if and only if

$$\mathrm{tr}_1(\phi(I_{mn})) = \mathrm{tr}_1(P) = \mathrm{tr}_1(I_{mn}) \quad \text{and} \quad \mathrm{tr}_2(\phi(I_{mn})) = \mathrm{tr}_2(P) = \mathrm{tr}_2(I_{mn}).$$

In general, this statement is true modulo a traceless matrix. We note that any linear map $\phi : M_{mn} \to M_{mn}$ can be written in the form

$$\phi(M) = M + \sum_{j=1}^{s} (A_j \otimes C_j) M (B_j \otimes D_j),$$

where $A_j, B_j \in M_m(\mathbb{F})$ and $C_j, D_j \in M_n(\mathbb{F})$ for $j = 1, \ldots, s$. In the following we denote by $[A, B] = AB - BA$ the Lie product (the commutator of matrices A and B of the appropriate sizes).

Lemma 5.6: *Let $\phi : M_{mn} \to M_{mn}$ be a linear map given by*

$$\phi(M) = M + \sum_{j=1}^{s} (A_j \otimes C_j) M (B_j \otimes D_j).$$

Then $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if

$$\mathrm{tr}_1(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{s} \mathrm{tr}(A_j B_j)[C_j, D_j]$$

and

$$\mathrm{tr}_2(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{s} \mathrm{tr}(C_j D_j)[A_j, B_j].$$

***Proof*:** Since $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ if and only if $\mathrm{tr}(\phi(A \otimes I_n)) = \mathrm{tr}(A \otimes I_n)$ and $\mathrm{tr}(\phi(I_m \otimes B)) = \mathrm{tr}(I_m \otimes B)$ for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$, we consider these two cases separately. In the first case we have by the cyclic property of the trace,

$$\begin{aligned}
\mathrm{tr}(\phi(A \otimes I_n)) &= \mathrm{tr}(A \otimes I_n) + \sum_{j=1}^{s} \mathrm{tr}((A_j \otimes C_j)(A \otimes I_n)(B_j \otimes D_j)) \\
&= n\,\mathrm{tr}(A) + \sum_{j=1}^{s} \mathrm{tr}\left(((B_j A_j) \otimes (D_j C_j))(A \otimes I_n)\right) \\
&= n\,\mathrm{tr}(A) + \sum_{j=1}^{s} \mathrm{tr}\left(\mathrm{tr}_2(((B_j A_j) \otimes (D_j C_j))(A \otimes I_n))\right) \\
&= n\,\mathrm{tr}(A) + \sum_{j=1}^{s} \mathrm{tr}\left(\mathrm{tr}_2((B_j A_j) \otimes (D_j C_j))A\right) \qquad \text{(Corollary 1.1)}
\end{aligned}$$

$$= n\,\mathrm{tr}(A) + \mathrm{tr}\left(\left(\sum_{j=1}^{s} \mathrm{tr}(C_jD_j)B_jA_j\right)A\right) \qquad \text{(Proposition 1.1)}$$

for all $A \in M_m(\mathbb{F})$. Thus, $n\,\mathrm{tr}(A) = \mathrm{tr}(A \otimes I_n) = \mathrm{tr}(\phi(A \otimes I_n))$ holds for all A if and only if

$$0_m = \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)B_jA_j = \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)([B_j, A_j] + A_jB_j). \tag{17}$$

We note that

$$\mathrm{tr}_2(\phi(I_{mn})) - \mathrm{tr}_2(I_{mn}) = \sum_{j=1}^{s} \mathrm{tr}_2((A_jB_j) \otimes (C_jD_j)) = \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)A_jB_j.$$

Defining the traceless matrix Q (i.e. $\mathrm{tr}(Q) = 0$) by

$$Q := \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)[A_j, B_j]$$

provides the condition equivalent to (17)

$$0_m = -Q + \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)A_jB_j = -Q + \mathrm{tr}_2(\phi(I_{mn})) - \mathrm{tr}_2(I_{mn}).$$

Hence, $\mathrm{tr}(A \otimes I_n) = \mathrm{tr}(\phi(A \otimes I_n))$ for all A if and only if

$$\mathrm{tr}_2(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)[A_j, B_j].$$

Similarly, the second case yields that $\mathrm{tr}(\phi(I_m \otimes B)) = \mathrm{tr}(I_m \otimes B)$ for all B if and only if

$$\sum_{j=1}^{s} \mathrm{tr}(A_jB_j)[C_j, D_j] = \mathrm{tr}_1(\phi(I_{mn})) - \mathrm{tr}_1(I_{mn}).$$

■

The commutators in the above lemma highlight the traceless character [17].

Moreover, Lemma 5.6 shows that the partial traces of the identity matrix must be preserved modulo a traceless matrix. However, this traceless matrix is not arbitrary but precisely defined in terms of ϕ. The following theorem shows that ϕ', defined as in Equation (12), plays a fundamental role in the characterization of ϕ and provides a succinct characterization for RT-symmetric and skew RT-symmetric maps in the subsequent two corollaries.

Theorem 5.2: *Let $\phi : M_{mn} \to M_{mn}$ be a linear map. Then $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if*

$$\mathrm{tr}_1(\phi'(I_{mn})) = \mathrm{tr}_1(I_{mn}) \quad \text{and} \quad \mathrm{tr}_2(\phi'(I_{mn})) = \mathrm{tr}_2(I_{mn}).$$

Proof: Using the representation of ϕ from Lemma 5.6 provides

$$\mathrm{tr}_1(\phi(I_{mn})) = \mathrm{tr}_1(I_{mn}) + \sum_{j=1}^{s} \mathrm{tr}(A_jB_j)C_jD_j,$$

$$\mathrm{tr}_1(\phi'(I_{mn})) = \mathrm{tr}_1(I_{mn}) + \sum_{j=1}^{s} \mathrm{tr}(B_jA_j)D_jC_j$$

and subtracting these two equations yields

$$\mathrm{tr}_1((\phi - \phi')(I_{mn})) = \sum_{j=1}^{s} \mathrm{tr}(A_jB_j)[C_j, D_j].$$

Similarly,

$$\mathrm{tr}_2((\phi - \phi')(I_{mn})) = \sum_{j=1}^{s} \mathrm{tr}(C_jD_j)[A_j, B_j].$$

From Lemma 5.6, $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if

$$\mathrm{tr}_1(\phi(I_{mn}) - I_{mn}) = \mathrm{tr}_1((\phi - \phi')(I_{mn})) \quad \text{and}$$
$$\mathrm{tr}_2(\phi(I_{mn}) - I_{mn}) = \mathrm{tr}_2((\phi - \phi')(I_{mn}))$$

if and only if

$$\mathrm{tr}_1(\phi'(I_{mn})) = \mathrm{tr}_1(I_{mn}) \quad \text{and} \quad \mathrm{tr}_2(\phi'(I_{mn})) = \mathrm{tr}_2(I_{mn}).$$

■

Corollary 5.2: *Let $\phi : M_{mn} \to M_{mn}$ be an RT-symmetric map. Then* $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ *for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if*

$$\mathrm{tr}_1(\phi(I_{mn})) = \mathrm{tr}_1(I_{mn}) \quad \text{and} \quad \mathrm{tr}_2(\phi(I_{mn})) = \mathrm{tr}_2(I_{mn}).$$

Corollary 5.3: *Let $\phi : M_{mn} \to M_{mn}$ be a skew RT-symmetric map. Then* $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ *for all $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$ if and only if*

$$\mathrm{tr}_1(\phi(I_{mn})) = -\mathrm{tr}_1(I_{mn}) \quad \text{and} \quad \mathrm{tr}_2(\phi(I_{mn})) = -\mathrm{tr}_2(I_{mn}).$$

It is straightforward to extend Corollaries 5.2 and 5.3 to the RT-Hermitian and skew RT-Hermitian cases since $\mathrm{tr}_1(\phi'(I_{mn})) = \mathrm{tr}_1(I_{mn})$ if and only if $\mathrm{tr}_1(\overline{\phi'(I_{mn})}) = \mathrm{tr}_1(I_{mn})$.

Now we are ready to consider the connection with the work in [11]. The connection is provided by the exponential map, i.e.

$$\det(e^A \otimes e^B) = e^{\mathrm{tr}(A \oplus B)}.$$

5.3. Determinant preservers of Kronecker products

Denote by $GL_n(\mathbb{C})$ the general linear group of $n \times n$ matrices over $\mathbb{C}$ with non-zero determinant and by $SL_n(\mathbb{C})$ the group of matrices with determinant equal to 1. In the following we will follow the notational convention $R_n := R_n(\mathbb{C})$, i.e. R_n is the zero of the field $\mathbb{C}_n$ of Lemma 4.1. The condition given in Lemma 5.6 implies that we may characterize a class of determinant preservers of Kronecker products in terms of partial determinants. However, the relationship between the partial trace and the partial determinant is not straightforward. If we restrict our attention to matrices over the complex numbers, $M_{mn}(\mathbb{C})$, then we have from (10)

$$\mathrm{Det}(e^A \otimes e^B) = e^{\mathrm{Tr}(A \oplus B)} \cdot R_{mn},$$

where $\mathrm{Det}(A) := \sqrt[n]{\det(A)} R_n$ and $\mathrm{Tr}(A) := \mathrm{tr}(A)/n$ for $A \in M_n(\mathbb{C})$, and R_n is the multiplicative group of n-th roots of unity in $\mathbb{C}$. Furthermore, we showed in Theorem 4.3 that

$$\mathrm{Det}_1(e^A \otimes e^B) = e^{\mathrm{Tr}_1(A \oplus B)} \cdot R_m \quad \text{and} \quad \mathrm{Det}_2(e^A \otimes e^B) = e^{\mathrm{Tr}_2(A \oplus B)} \cdot R_n.$$

Definition 5.6: Let $\Omega_n \subset M_n(\mathbb{C})$ denote the set of matrices $A \in M_n(\mathbb{C})$ with the imaginary part of each eigenvalue $\lambda(A)$ satisfying $\mathrm{Im}(\lambda(A)) \in (-\pi, \pi]$.

Thus, we associate with every non-singular matrix A a unique matrix $M \in \Omega_n$ such that $A = e^M$ [9, p. 474]. Next, we wish to consider linear maps restricted to Ω_n. The following lemma shows that trace preservers on Ω_n are equivalent to trace preservers on $M_n(\mathbb{C})$.

Lemma 5.7: *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map obeying $\phi(\Omega_n) \subseteq \Omega_n$. Then ϕ satisfies*

$$\mathrm{tr}(\phi(A)) = \mathrm{tr}(A) \text{ for all } A \in M_n(\mathbb{C})$$

if and only if $\mathrm{tr}(\phi(B)) = \mathrm{tr}(B)$ *for all* $B \in \Omega_n$.

Proof: It suffices to show that each standard basis element $E_{ij}^{[n]} \in M_n(\mathbb{C})$ is also in Ω_n, which is easily verified. ■

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map obeying $\phi(\Omega_n) \subseteq \Omega_n$ and let $\psi : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ be the non-linear map

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_n.$$

The map is well defined since $M \in \Omega_n$ is uniquely determined for every matrix in $GL_n(\mathbb{C})$. We have

$$\det \psi(e^M) = \det e^{\phi(M)} = e^{\mathrm{tr}(\phi(M))}$$

so that $\det \psi(e^M) = \det(e^M)$ if and only if $e^{\mathrm{tr}(\phi(M))} = e^{\mathrm{tr}(M)}$. Moreover, if $\phi(M) \in \Omega_n$ and $M \in \Omega_n$, then $e^{\mathrm{Tr}(\phi(M))} = e^{\mathrm{Tr}(M)}$ if and only if $\mathrm{tr}(\phi(M)) = \mathrm{tr}(M)$. The following theorem motivates the remainder of the study.

Theorem 5.3: *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map obeying $\phi(\Omega_n) \subseteq \Omega_n$. The map $\psi : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_n,$$

satisfies $\det(\psi(e^{M/n})) = \det(e^{M/n})$ *if and only if* $\mathrm{tr}(\phi(M)) = \mathrm{tr}(M)$.

Proof: Let $M \in \Omega_n$, then $\phi(M) \in \Omega_n$ so that $\mathrm{Im}(\mathrm{Tr}(\phi(M))), \mathrm{Im}(\mathrm{Tr}(M)) \in (-\pi, \pi]$. Thus, $\exp(\mathrm{Tr}(\phi(M))) = \exp(\mathrm{Tr}(M))$ if and only if $\mathrm{Tr}(\phi(M)) = \mathrm{Tr}(M)$. Hence,

$$\begin{aligned}
\mathrm{tr}(\phi(M)) = \mathrm{tr}(M) \quad &\iff \quad \exp(\mathrm{Tr}(\phi(M))) = \exp(\mathrm{Tr}(M)) \\
&\iff \quad \exp(\mathrm{tr}(\phi(M/n))) = \exp(\mathrm{tr}(M/n)) \\
&\iff \quad \det(\exp(\phi(M/n)) = \det(\exp(M/n)) \\
&\iff \quad \det(\psi(\exp(M/n))) = \det(\exp(M/n)).
\end{aligned}$$

■

The condition $\det(\psi(e^{M/n})) = \det(e^{M/n})$ implies $(\det(\psi(e^{M/n}))^n = (\det(e^{M/n}))^n$. Furthermore, from $\det(e^{M/n})^n = \det(e^M)$ we have, by the uniqueness of n-th roots in R_n, that $\mathrm{Det}(e^M) = \det(e^{M/n}) \cdot R_n$. Similarly, by linearity of ϕ, we have $\mathrm{Det}(\psi(e^M)) = \det(\psi(e^{M/n})) \cdot R_n$. Thus,

$$\mathrm{Det}(\psi(e^M)) = \det(\psi(e^{M/n})) \cdot R_n = \det(e^{M/n}) \cdot R_n = \mathrm{Det}(e^M),$$

for all $M \in \Omega_n$. Hence, the following corollary.

Corollary 5.4: *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map such that $\phi(\Omega_n) \subseteq \Omega_n$. If $\mathrm{tr}(\phi(M)) = \mathrm{tr}(M)$ for all $M \in \Omega_n$, then the map $\psi : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_n,$$

satisfies $\mathrm{Det}(\psi(e^M)) = \mathrm{Det}(e^M)$ *for all* $M \in \Omega_n$.

Lemma 5.8: *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map such that $\phi(\Omega_n) \subseteq \Omega_n$. Then the map $\psi : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \qquad M \in \Omega_n,$$

satisfies $\psi(e^M e^M) = \psi(e^N)\psi(e^N)$ for all $2M, N \in \Omega_n$ such that $e^{2\phi(M)} = e^{2\phi(N)}$.

Proof: If $M, N \in \Omega_n$ such that $e^{2\phi(M)} = e^{2\phi(N)}$,

$$\psi(e^{2M}) = e^{2\phi(M)} = e^{2\phi(N)} = e^{\phi(N)}e^{\phi(N)} = \psi(e^N)\psi(e^N).$$ ■

Theorem 5.4: *Let $m, n \geq 2$ and $\phi : M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a linear map such that $\phi(\Omega_{mn}) \subseteq \Omega_{mn}$. If $\mathrm{tr}(\phi(A \oplus B)) = \mathrm{tr}(A \oplus B)$ for all $A \oplus B \in \Omega_{mn}$, then the map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by*

$$\psi(e^M) := e^{\phi(M)}, \quad M \in \Omega_{mn},$$

satisfies $\mathrm{Det}(\psi(A \otimes B)) = \mathrm{Det}(A \otimes B)$ *for all* $A \in GL_m(\mathbb{C})$ *and* $B \in GL_n(\mathbb{C})$.

Proof: If $A \in GL_m(\mathbb{C})$, then there exists $M_A \in \Omega_m$ such that $A = e^{M_A}$, and, similarly, if $B \in GL_n(\mathbb{C})$, there exists $M_B \in \Omega_n$ such that $B = e^{M_B}$. We have $A \otimes B = \exp(M_A \oplus M_B)$. We note that $M_A \oplus 0_n = M_A \otimes I_n \in \Omega_{mn}$ and $0_m \oplus M_B = I_m \otimes M_B \in \Omega_{mn}$, so

$$\begin{aligned}\operatorname{tr}(\phi(M_A \oplus M_B)) &= \operatorname{tr}(\phi(M_A \oplus 0_n)) + \operatorname{tr}(\phi(0_m \oplus M_B)) = \operatorname{tr}(M_A \oplus 0_n) + \operatorname{tr}(0_m \oplus M_B)\\ &= \operatorname{tr}(M_A \oplus M_B).\end{aligned}$$

Let λ be an eigenvalue of M_A and μ be an eigenvalue of M_B. Then $(\lambda + \mu)/2$ is an eigenvalue of $\frac{1}{2}(M_A \oplus M_B)$, and all eigenvalues of $\frac{1}{2}(M_A \oplus M_B)$ have this form [9, Theorem 4.4.5]. Moreover, $\frac{1}{2}(M_A \oplus M_B) \in \Omega_{mn}$ since $\operatorname{Im}(\lambda + \mu)/2 \in (-\pi, \pi]$ whenever $\operatorname{Im}(\lambda), \operatorname{Im}(\mu) \in (-\pi, \pi]$. There exists a unique $C \in \Omega_{mn}$ such that $A \otimes B = e^C$. We have,

$$A \otimes B = e^C = e^{M_A \oplus M_B}, \qquad \det(A \otimes B) = e^{\operatorname{tr}(C)} = e^{\operatorname{tr}(M_A \oplus M_B)}.$$

Thus, $\det(e^{C/2})^2 = \det(e^{(M_A \oplus M_B)/2})^2$. By Lemma 5.8,

$$\psi(e^C) = \psi(e^{C/2}e^{C/2}) = \psi(e^{\frac{1}{2}(A \oplus B)})\psi(e^{\frac{1}{2}(A \oplus B)}),$$

and hence

$$\begin{aligned}\operatorname{Det}(\psi(A \otimes B)) &= \operatorname{Det}(\psi(e^C))\\ &= \operatorname{Det}(\psi(e^{\frac{1}{2}(M_A \oplus M_B)}))\operatorname{Det}(\psi(e^{\frac{1}{2}(M_A \oplus M_B)}))\\ &= e^{\frac{1}{2}\operatorname{Tr}(\phi(M_A \oplus M_B))}e^{\frac{1}{2}\operatorname{Tr}(\phi(M_A \oplus M_B))} \cdot R_{mn}\\ &= e^{\operatorname{Tr}(M_A \oplus M_B)} \cdot R_{mn}\\ &= \operatorname{Det}(e^{M_A \oplus M_B}) = \operatorname{Det}(A \otimes B).\end{aligned}$$

■

Theorem 5.2 and its corollaries provide the following corollaries.

Corollary 5.5: *Let $m, n \geq 2$ and $\phi : M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a linear map such that $\phi(\Omega_{mn}) \subseteq \Omega_{mn}$. If $\operatorname{tr}_1(\phi'(I_{mn})) = \operatorname{tr}_1(I_{mn})$ and $\operatorname{tr}_2(\phi'(I_{mn})) = \operatorname{tr}_2(I_{mn})$, then the map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_{mn},$$

satisfies $\operatorname{Det}(\psi(A \otimes B)) = \operatorname{Det}(A \otimes B)$ for all $A \in GL_m(\mathbb{C})$ and $B \in GL_n(\mathbb{C})$.

Lemma 5.9: *Suppose that $\phi'(I_{mn})$ is a Kronecker sum. Then the following are equivalent:*

(1) $\operatorname{tr}_1(\phi'(I_{mn})) = \operatorname{tr}_1(I_{mn})$ *and* $\operatorname{tr}_2(\phi'(I_{mn})) = \operatorname{tr}_2(I_{mn})$,
(2) $\phi'(I_{mn}) = I_{mn}$,
(3) $\operatorname{tr}(\phi(E_{jk}^{[mn]})) = \delta_{jk}$ *for all* $j, k \in \{1, \ldots, mn\}$.

Proof: ((1)$\Longleftrightarrow$(2)) Suppose that $\phi'(I_{mn}) = A \oplus B$ is a Kronecker sum. If $\mathrm{tr}_1(\phi'(I_{mn})) = \mathrm{tr}_1(I_{mn})$, then

$$\mathrm{tr}(A)I_n + mB = mI_n. \tag{18}$$

Similarly, if $\mathrm{tr}_2(\phi'(I_{mn})) = \mathrm{tr}_2(I_{mn})$ then

$$\mathrm{tr}(B)I_m + nA = nI_m. \tag{19}$$

Hence, A and B are both scalar matrices. Taking the trace of Equations (18) and (19) both yield that

$$n\,\mathrm{tr}(A) + m\,\mathrm{tr}(B) = mn,$$

so that

$$B = \left(1 - \frac{\mathrm{tr}(A)}{m}\right) I_n, \quad A = \frac{\mathrm{tr}(A)}{m} I_m.$$

It follows that

$$\phi'(I_{mn}) = \frac{\mathrm{tr}(A)}{m} I_m \otimes I_n + I_m \otimes \left(1 - \frac{\mathrm{tr}(A)}{m}\right) I_n = I_m \otimes I_n = I_{mn}.$$

The converse is immediate. ((2)$\Longleftrightarrow$(3)) Suppose that ϕ has the form

$$\phi(A) = \sum_{j,k,u,v=1}^{mn} \alpha_{jk;uv} E_{jk}^{[mn]} A E_{uv}^{[mn]} \tag{20}$$

for some $\alpha_{jk;uv} \in \mathbb{C}$. Then

$$\phi'(I_{mn}) = \sum_{j,k,u,v=1}^{mn} \alpha_{uv;jk} E_{jk}^{[mn]} E_{uv}^{[mn]} = \sum_{j,v=1}^{mn} \left(\sum_{k=1}^{mn} \alpha_{kv;jk} \right) E_{jv}^{[mn]} = I_{mn} \tag{21}$$

if and only if $\sum_{k=1}^{mn} \alpha_{kv;jk} = \delta_{jv}$, where δ_{jv} is the Kronecker delta. On the other hand,

$$\begin{aligned} \mathrm{tr}(\phi(E_{jk}^{[mn]})) &= \mathrm{tr}\left(\sum_{s,t,u,v=1}^{mn} \alpha_{st;uv} E_{st}^{[mn]} E_{jk}^{[mn]} E_{uv}^{[mn]} \right) \\ &= \mathrm{tr}\left(\sum_{s,v=1}^{mn} \alpha_{sj;kv} E_{sv}^{[mn]} \right) \\ &= \sum_{s=1}^{mn} \alpha_{sj;ks} = \delta_{jk} \end{aligned}$$

if and only if $\sum_{k=1}^{mn} \alpha_{kv;jk} = \delta_{jv}$. ∎

Corollary 5.6: *Let $m, n \geq 2$ and $\phi : M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a linear map such that $\phi(\Omega_{mn}) \subseteq \Omega_{mn}$. If $\operatorname{tr}(\phi(E_{jk}^{[mn]})) = \delta_{jk}$ for all $j, k \in \{1, \ldots, mn\}$, then the map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_{mn},$$

satisfies $\operatorname{Det}(\psi(A \otimes B)) = \operatorname{Det}(A \otimes B)$ for all $A \in GL_m(\mathbb{C})$ and $B \in GL_n(\mathbb{C})$.

Corollary 5.7: *Let $m, n \geq 2$ and $\phi : M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be an RT-symmetric map such that $\phi(\Omega_{mn}) \subseteq \Omega_{mn}$. If $\operatorname{tr}_1(\phi(I_{mn})) = \operatorname{tr}_1(I_{mn})$ and $\operatorname{tr}_2(\phi(I_{mn})) = \operatorname{tr}_2(I_{mn})$, then the map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_{mn},$$

satisfies $\operatorname{Det}(\psi(A \otimes B)) = \operatorname{Det}(A \otimes B)$ for all $A \in GL_m(\mathbb{C})$ and $B \in GL_n(\mathbb{C})$.

Corollary 5.8: *Let $m, n \geq 2$ and $\phi : M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a skew RT-symmetric map such that $\phi(\Omega_{mn}) \subseteq \Omega_{mn}$. If $\operatorname{tr}_1(\phi(I_{mn})) = -\operatorname{tr}_1(I_{mn})$ and $\operatorname{tr}_2(\phi(I_{mn})) = -\operatorname{tr}_2(I_{mn})$, then the map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by*

$$\psi(e^M) = e^{\phi(M)}, \quad M \in \Omega_{mn},$$

satisfies $\operatorname{Det}(\psi(A \otimes B)) = \operatorname{Det}(A \otimes B)$ for all $A \in GL_m(\mathbb{C})$ and $B \in GL_n(\mathbb{C})$.

6. Conclusion

We have considered three different notions of the determinant operation in $M_{mn}(\mathbb{F})$, namely the block determinant, partial determinant and determinant-root. The block determinant and partial determinant-root retain the multiplicative property and exponential-determinant-trace relation, although only for Kronecker products in the case of the determinant-root. The results offer a method for obtaining non-linear preservers of determinant-roots of Kronecker products by utilizing the linear preservers of Kronecker sums.

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ORCID

Y. Hardy http://orcid.org/0000-0003-0264-8874

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