

https://dspace.org/

Developing computerised agent-based models of the stock market

Seedat, Shaheen

2019
https://hdl.handle.net/10539/29540

Downloaded from a DSpace repository. https://dspace.org/

Developing Computerised Agent-Based Models of the Stock Market

Shaheen Seedat (0700213F)



UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

A thesis submitted in fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

School of Computer Science and Applied Mathematics, Faculty of Science,
University of the Witwatersrand, Johannesburg

Johannesburg, May 2019

Declaration

I, Shaheen Seedat, declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

_____

Signature:

_____ day of _____ 20 _____ at _____

Abstract

This thesis, based on four research papers, develops three novel agent-based models with the aim of approximating stock markets with greater realism. Two methods are presented for examining parameter effects on an agent based model. The first is parameter variation Monte Carlo which we derive by applying the law of large numbers pointwise. The second method is to use (multivariate) regression using Maximum Likelihood estimation. These two methods are applied to the Santa-Fe model of LeBaron *et al.* (1999) which is shown to be a reasonable model. Subsequently, three new agent-based models of the stock market are developed which incorporate genetic learning. In the first model, a multiple stock model of the stock market is developed, which significantly generalises the Santa-Fe model. By Monte Carlo simulations and estimation of a Dynamic Conditional Correlation (DCC) model, stock bubbles are shown to exist and the relationship between agent beliefs and stock mispricing, stock volatility and cross-correlations is studied. The model is shown to be reasonable regarding the impact of parameters on simulated prices. Subsequently, significant evidence is presented that various time unconditional and conditional kernel distributions of stock market returns are a mixture of distributions, including during important financial economic periods such as the Great Depression, the Dot-Com bubble and the Great Recession. To explain these phenomena, mixture modelling is incorporated into the development of two agent-based mixture models. In the first of these two new research papers, smart (algorithmic) traders and portfolio managers trade and invest in an equally weighted average of stocks that underlie an equity index such as the S&P 500, have consistent beliefs about stock market dynamics and learn over time. It is shown that the model can produce kernel distributions of stock market returns that are observed in the real-world. In the second of these two new research papers, smart (algorithmic) traders and portfolio managers trade and invest in the *underlying equities that constitute an equity market index* and *optimise* their holdings in their portfolio optimisation, have consistent beliefs about stock market dynamics and learn over time. It is shown, by Monte Carlo simulations, that the model can produce kernel distributions of *aggregated* stock market returns observed in the real-world. To compute optimal stock holdings of traders and investors, the law of iterated expectations and the expectation of a log *mixed* random variable is applied. The latter result is developed and is novel to the best of our knowledge. In total, eleven new theorems are developed and proved and 172800 Monte Carlo simulations are conducted.

Acknowledgements

I would like to thank the following people:

- My Ph.D. thesis advisor, Professor Shirley Abelman, for her continued support and advice of this study including providing overall mentorship and guidance and for the reading of my manuscripts and thesis.
- My family for their continued support and love without which completing this thesis would have been impossbile.
- Everybody at the University of the Witwatersrand who has provided advice and support and the enabling environment to complete my studies.

Contents

1 Introduction **20**
1.1 Statement of Problem Investigated and Justificiation 20
1.2 Aims and Objectives 21
1.3 Literature Review 22
1.4 Thesis Outline 25
1.5 Statement of Sources of Data 27

2 Examining Parameter Impacts on the Computerised Agent-Based Santa Fe Artificial Stock Market Model **28**

3 A Computational Multivariate Agent-Based Model of the Stock Market **50**

4 Explaining Stock Return Distributions via an Agent-Based Model **85**

5 Explaining Aggregated Stock Return Distributions via a Multivariate Agent-Based Model **120**

6 Results, Conclusions and Future Work **179**
6.1 Results and Conclusions 179
6.2 Future Work 183

References **186**

List of Tables

1. Chapter 2. Table 1: Parameters of Agent-Based Model 1 (Santa Fe model). Page 34
2. Chapter 2. Table 2: Parameters of Agent-Based Model 2 (Santa Fe model). Page 35
3. Chapter 2. Table 3: Statistical model: regression using MLE (Santa Fe model). Page 46
4. Chapter 3. Table 1: Parameters: Base Case (Multivariate Model). Page 66
5. Chapter 3. Table 2: Statistical model: regression using MLE (Multivariate model) - stock 1. Page 75
6. Chapter 3. Table 3: Statistical model: regression using MLE (Multivariate model) - stock 2. Page 75
7. Chapter 3. Table 4: Statistical model: regression using MLE (Multivariate model) - stock market. Page 75
8. Chapter 4. Table 1: Parameters: Single-Stock Santa Fe Artificial Stock Market. Page 110
9. Chapter 4. Table 2: Parameters: Single-Stock Mixed Gaussian Artificial Stock Market. Page 111
10. Chapter 4. Table 3: Parameters: Base Case (Single-Stock Santa Fe Artificial Stock Market). Page 112
11. Chapter 4. Table 4: Statistical model: regression using MLE (Single Stock Mixed Gaussian artificial stock market). Page 117
12. Chapter 4. Table 5: Stock Mispricing Parameters: Base Case (Single Stock Mixed Gaussian artificial stock market). Page 117
13. Chapter 5. Table 1: Parameters: Multiple-Stock Santa Fe Artificial Stock Market for Density Estimation. Page 155
14. Chapter 5. Table 2: Parameters: Multiple-Stock Mixed Gaussian Artificial Stock Market. Page 156

15. Chapter 5. Table 3: Parameters: Base Case (Multiple-Stock Mixed Gaussian Artificial Stock Market). Page 157

16. Chapter 5. Table 4: Statistical model: regression using MLE (Multiple Stock Mixed Gaussian artificial stock market) - stock 1. Page 169

17. Chapter 5. Table 5: Statistical model: regression using MLE (Multiple Stock Mixed Gaussian artificial stock market) - stock 2. Page 169

18. Chapter 5. Table 6: Statistical model: regression using MLE (Multiple Stock Mixed Gaussian artificial stock market) - stock market. Page 170

19. Chapter 5. Table 7: Stock Mispricing Parameters: Base Case (Multiple-Stock Mixed Gaussian Artificial Stock Market). Page 171

List of Figures

1. Chapter 2. Figure 1: Impact of Parameters on the agent-based SFASM: Parameters impact the model in a plausible manner (Panel A). Page 40
2. Chapter 2. Figure 2: Impact of Parameters on the agent-based SFASM: Parameters impact the model in a plausible manner (Panel B). Page 41
3. Chapter 2. Figure 3: Agent-Based Model Stock Prices and Theoretical Stock Prices (Santa Fe model). Page 47
4. Chapter 3. Figure 1: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel A. Page 69
5. Chapter 3. Figure 2: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel B. Page 70
6. Chapter 3. Figure 3: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel A. Page 71
7. Chapter 3. Figure 4: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel B. Page 72
8. Chapter 3. Figure 5: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel A. Page 73
9. Chapter 3. Figure 6: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel B. Page 74
10. Chapter 3. Figure 7: Agent-Based Model Stock Dividend Process (Multivariate model). Page 77
11. Chapter 3. Figure 8: Agent-Based Model Stock Prices and Theoretical Stock Prices (Multivariate model). Page 78
12. Chapter 3. Figure 9: Multivariate Agent-Based Model Squared Stock Returns. Page 80

13. Chapter 3. Figure 10: Multivariate Agent-Based Model Volatility and Covariation. Page 81

14. Chapter 4. Figure 1: Density Estimate of S&P 500 Stock Market Returns (June 2010 - June 2015) - Single-Stock Mixed Gaussian Artificial Stock Market. Page 90

15. Chapter 4. Figure 2: Kernel Density Estimates of S&P 500 Stock Market Returns (Single-Stock Santa Fe Artificial Stock Market) - Panel A. Page 91

16. Chapter 4. Figure 3: Kernel Density Estimates of S&P 500 Stock Market Returns (Single-Stock Santa Fe Artificial Stock Market) - Panel B. Page 92

17. Chapter 4. Figure 4: Density Estimate of Artificial Stock Market returns (Single-Stock Mixed Gaussian Artificial Stock Market). Page 111

18. Chapter 4. Figure 5: Single Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables - Panel A. Page 114

19. Chapter 4. Figure 6: Single Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables - Panel B. Page 115

20. Chapter 4. Figure 7: Single Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables - Panel C. Page 116

21. Chapter 4. Figure 8: Single-Stock Mixed Gaussian Agent-Based Model Stock Prices and Theoretical Stock Prices. Page 118

22. Chapter 5. Figure 1: Density Estimate of Monthly S&P 500 Stock Market Returns (June 2010 - June 2015) - Multiple-Stock Mixed Gaussian Artificial Stock Market. Page 126

23. Chapter 5. Figure 2: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel A. Page 127

24. Chapter 5. Figure 3: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel B. Page 128

25. Chapter 5. Figure 4: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel C. Page 129

26. Chapter 5. Figure 5: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel D. Page 130

27. Chapter 5. Figure 6: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel E. Page 131

28. Chapter 5. Figure 7: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel F. Page 132

29. Chapter 5. Figure 8: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel G. Page 133

30. Chapter 5. Figure 9: Kernel Density Estimates of S&P 500 Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) - Panel H. Page 134

31. Chapter 5. Figure 10: Density Estimate of Artificial Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market) Page 156

32. Chapter 5. Figure 11: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel A. Page 160

33. Chapter 5. Figure 12: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel B. Page 161

34. Chapter 5. Figure 13: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel C. Page 162

35. Chapter 5. Figure 14: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel A. Page 163

36. Chapter 5. Figure 15: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2). - Panel B Page 164

37. Chapter 5. Figure 16: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel C. Page 165

38. Chapter 5. Figure 17: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel A. Page 166

39. Chapter 5. Figure 18: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel B. Page 167

40. Chapter 5. Figure 19: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel C. Page 168

41. Chapter 5. Figure 20: Agent-Based Model Stock Dividend Process (Multiple Stock Mixed Gaussian Artificial Stock Market). Page 171

42. Chapter 5. Figure 21: Multiple Stock Mixed Gaussian Artificial Stock Market Stock Prices and Theoretical Stock Prices. Page 173

43. Chapter 5. Figure 22: Multiple Stock Mixed Gaussian Artificial Stock Market Squared Stock Returns. Page 174

44. Chapter 5. Figure 23: Multiple Stock Mixed Gaussian Artificial Stock Market Volatility and Covariation. Page 175

List of Symbols

Chapter 2.

1. t	Time
2. p_t	Price of the stock at time t
3. d_t	Dividend paid at time t for the stock
4. $\bar{d}$	Constant mean of $D(t)$
5. ρ	First order autocorrelation coefficient of the process $D_t - \bar{D}$
6. u_t	Innovation term in the stock dividend process
7. $\sigma^2_{u_t}$	Stock dividend variation
8. λ	Constant absolute risk aversion coefficient
9. $X_{t,i}$	Stock holdings of agent i at time t
10. $X^*_{t,i}$	Optimal stock holdings of agent i at time t
11. r_f	Interest rate per period of the risk-free asset
12. $\mu_{t,i}$	Agent i's forecast of one-period ahead prices plus dividends
13. $\sigma^2_{p+d,i}$	Agent i's estimated variance of one-period ahead prices plus dividends
14. $\mathbb{E}$	Mathematical expectation operator
15. $a_{i,j}$	Factor loading in a linear forecast of future prices plus dividends for agent i and rule j
16. $b_{i,j}$	Constant parameter in a linear forecast of future prices plus dividends for agent i and rule j
17. $\sigma^2_{i,j}$	Agent i's estimated variance of its linear forecast of future prices plus dividends for rule j
18. τ	Parameter governing degree of smoothing in an EWMA

19. A_{re} — Factor loading in a linear REE forecast of future prices plus dividends
20. B_{re} — Constant parameter in a linear REE forecast of future prices plus dividends
21. f — Factor loading in the linear price function as a function of dividends
22. e — Constant in a linear price function as a function of dividends
23. $\mathcal{M}(\theta)$ — A computerised agent-based model with parameters θ
24. θ — Parameters of the computerised agent-based model $\mathcal{M}(\theta)$
25. A range — Admissible parameter range for A_{re} in computational simulations.
26. B range — Admissible parameter range for B_{re} in computational simulations.
27. K — Time period after which agents re-learn
28. N — Number of agents (traders)
29. T — Number of time periods after which price estimates are recorded
30. N_1 — Number of simulations conducted for each parameter set
31. $\overset{a.s.}{\to}$ — Convergence almost surely

Chapter 3.

1. t — Time
2. n — Number of risky dividend-paying stocks
3. $P_{s,t}$ — Price of stock s at time t
4. $\mathbf{P}_t$ — Price vector of all n stocks at time t
5. $D_{s,t}$ — Dividend of stock s at time t
6. $\mathbf{D}_t$ — Dividend vector of all n stocks at time t
7. r_f — Interest rate per period of the risk-free asset
8. $\bar{\mathbf{D}}$ — Constant mean of $\mathbf{D}_t$
9. ρ — First order autocorrelation coefficient of the process $\mathbf{D}_t - \bar{\mathbf{D}}$
10. $\mathbf{u}_t$ — Innovation process of the process $\mathbf{D}_t - \bar{\mathbf{D}}$
11. $\mathbf{\Sigma}_{\mathbf{u}_t}$ — Variance-covariance matrix of the innovation process $\mathbf{u}_t$
12. $\mathbf{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{u_t}})$ — Multivariate Gaussian distribution with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{\Sigma}_{\mathbf{u}_t}$

13. $\mathbf{\Sigma_{P+D}}$ Agent i's estimated variance-covariance of one-period ahead prices plus dividends for all stocks
14. $\sigma^2_{s,p+d,i}$ Agent i's estimated variance of one-period ahead prices plus dividends for stock s
15. $\sigma^2_{u,s}$ Stock dividend variation of stock s
16. N Number of computationally intelligent agents (traders)
17. λ Constant absolute risk aversion coefficient
18. $W_{t,i}$ Wealth of agent i at time t
19. $X_{i,s,t}$ Stock holdings of stock s by agent i at time t
20. $\mathbf{X}_{i,t}$ Vector of stock holdings by agent i at time t
21. $X^*_{i,s,t}$ Optimal stock holdings of stock s by agent i at time t
22. $\mathbf{X}^*_{i,t}$ Vector of optimal stock holdings by agent i at time t
23. $\mathcal{I}_{t,i}$ Information set of agent i at time t
24. $\sim$ Distributed
25. $\mu_{t,i}$ Agent i's conditional forecast of one-period ahead prices plus dividends, i.e. of $\mathbf{P}_{t+1} + \mathbf{D}_{t+1}$
26. $\mathbf{\Sigma}_{t,i}$ Agent i's conditional variance-covariance estimate of one-period ahead prices plus dividends, i.e. of $\mathbf{P}_{t+1} + \mathbf{D}_{t+1}$
27. $U(W_{t,i})$ Utility of agent i at time t given wealth $W_{t,i}$
28. $\mathbb{E}$ Mathematical expectation operator
29. $\mathbb{E}_{t,i}$ Conditional expection of agent i at time t with respect to $\mathcal{I}_{t,i}$
30. A_{re} Factor loading in a linear REE forecast of future prices plus dividends
31. $\mathbf{B}_{re}$ Constant parameter in a linear REE forecast of future prices plus dividends
32. MA_v v-period prior moving average
33. $a_{i,j}$ Factor loading in a linear vector forecast of prices plus dividends for agent i and rule j
34. $\mathbf{b}_{i,j}$ Constant parameter in a linear vector forecast of prices plus dividends for agent i and rule j
35. $\mathbf{\Sigma}_{i,j}$ Agent i's variance-covariance estimate of their rule j linear vector forecast of prices plus dividends

36. $\mathbf{a}_i$ Factor loading in a linear vector forecast of prices plus dividends for agent i (not rule specific)

37. $\mathbf{b}_i$ Constant parameter in a linear vector forecast of prices plus dividends for agent i and rule j (not rule specific)

38. $a_{s,i,j}$ Factor loading in a linear forecast of prices plus dividends for agent i, rule j and stock s

39. $b_{s,i,j}$ Constant parameter in a linear forecast of prices plus dividends for agent i, rule j and stock s

40. ${\sigma^2}_{s,i,j}$ Agent i's variance estimate of their rule j linear vector forecast of prices plus dividends for stock s

41. a_i^* Agent i's optimal vector factor loading in a linear vector forecast of prices plus dividends for agent i

42. b_i^* Agent i's optimal vector constant parameter loading in a linear vector forecast of prices plus dividends for agent i

43. τ Parameter governing degree of smoothing in an EWMA

44. $\mathcal{M}(\theta)$ A computerised agent-based model with parameters θ

45. θ Parameters of the computerised agent-based model $\mathcal{M}(\theta)$

46. $\overset{a.s.}{\rightarrow}$ Convergence almost surely

47. N_1 Number of simulations conducted for each parameter set

48. A range Admissible parameter range for A_{re}.

49. B range Admissible parameter range for B_{re}.

50. N Number of agents (i.e. traders/investors)

51. T Number of time periods after which price estimates are recorded

52. K Time period after which agents re-learn

Chapter 4.

1. t Time

2. s_t Market state at time t

3. P_{t,s_t} Price of the stock at time t and in state s_t

4. D_{t,s_t} Dividend of the stock at time t and in state s_t

5. r_f Interest rate per period of the risk-free asset
6. $Q(j)$ Probability of state j occuring
7. $\bar{D}_{s_t}$ Constant mean of D_{t,s_t}
8. ρ First order autocorrelation coefficient of the process $D_{t,s_t} - \bar{D}_{s_t}$
9. u_t Innovation process in the stock dividend process $D_{t,s_t} - \bar{D}_{s_t}$
10. ${\sigma^2}_{u_t}$ Variance of the innovation process u_t
11. $\mathcal{I}_{t,s_t}$ Conditional information set of publicly available information at time t and in state s_t
12. N Number of smart traders/investors
13. λ Constant absolute risk aversion coefficient
14. $W_{t,s_t,i}$ Wealth of agent i at time t and in state s_t
15. $X_{t,s_t,i}$ Stock holdings of agent i at time t and in state s_t
16. $X^*_{t,s_t,i}$ Optimal stock holdings by agent i at time t and in state s_t
17. $\mathcal{I}_{t,s_t,i}$ Conditional information set of agent i at time t and in state s_t
18. $\mathbb{E}$ Mathematical expectation operator
19. $\mathbb{V}$ Mathematical variance operator
20. Cov Mathematical covariance operator
21. $\mathbb{E}_{t,i}$ Agent i's conditional expectation at time t over all random variables, conditional on $\mathcal{I}_{t,s_t,i}$
22. $\mathbb{V}_{t,i}$ Agent i's conditional variance estimate at time t over all random variables, conditional on $\mathcal{I}_{t,s_t,i}$
23. $\tilde{\mathbb{E}}_{t,i}$ Agent i's conditional expectation at time t over all random variables except s_t, conditional on $\mathcal{I}_{t,s_t,i}$
24. $\tilde{\mathbb{V}}_{t,i}$ Agent i's conditional variance estimate at time t over all random variables except s_t, i.e. conditional on $\mathcal{I}_{t,s_t,i}$
25. MG Mixed Gaussian
26. $\mathbb{E}_{t,i}^{\mathcal{A}_{t+1,i}}$ Agent i's conditional expectation at time t over all random variables in $\mathcal{A}_{t+1,i}$, conditional on $\mathcal{I}_{t,s_t,i}$
27. $\mathbb{V}_{t,i}^{\mathcal{A}_{t+1,i}}$ Agent i's conditional variance estimate at time t over all random variables in $\mathcal{A}_{t+1,i}$, conditional on $\mathcal{I}_{t,s_t,i}$

28. $\mathbb{E}_{t,i}^{\{s_{t+1,i}\}}$ Agent i's conditional expectation at time t over the state s_{t+1}, conditional on $\mathcal{I}_{t,s_t,i}$

29. $\mathbb{V}_{t,i}^{\{s_{t+1,i}\}}$ Agent i's conditional variance estimate at time t over the random variable s_{t+1}, conditional on $\mathcal{I}_{t,s_t,i}$

30. $\sim$ Distributed

31. $\mu_{t,s_t,i}$ Agent i's conditional expectation at time t of one period ahead prices plus dividends, over all variables except the future state variable

32. $\sigma_{t,s_t,i}$ Agent i's conditional variance estimate at time t of one period ahead prices plus dividends, over all variables except the future state variable

33. $\mu_{t,,i}$ Agent i's conditional expectation at time t for agent i of one period ahead prices plus dividends, over all variables

34. $U(W_{t,s_t,i})$ Utility of agent i at time t and in state s_t given wealth $W_{t,s_t,i}$

35. a_i Factor loading in a linear forecast of prices plus dividends for agent i (not rule specific)

36. b_i Constant parameter in a linear vector forecast of prices plus dividends for agent i and rule j (not rule specific)

37. $a_{i,j}$ Factor loading in a linear vector forecast of prices plus dividends for agent i and rule j

38. $b_{i,j}$ Constant parameter in a linear vector forecast of prices plus dividends for agent i and rule j

39. ${\sigma^2}_{i,j}$ Agent i's variance-covariance estimate of their rule j linear vector forecast of prices plus dividends

40. A_{re} Factor loading in a linear REE forecast of future prices plus dividends

41. B_{re} Constant parameter in a linear REE forecast of future prices plus dividends

42. MA_v v-period prior moving average

43. τ Parameter governing degree of smoothing in an EWMA

44. K Time period after which agent's re-learn

45. A_{range} Admissible parameter range for A_{re}.

46. B_{range} Admissible parameter range for B_{re}.

47. T Number of time periods after which price estimates are recorded

48. N Number of agents

49. J Number of states

50. $\sigma^2_{P+D,i}$ Agent i's estimate of the variance of one period ahead prices plus dividends
51. $\overset{law}{=}$ Equal to in probability law

Chapter 5.

1. t Time
2. s_t Market state at time t
3. n Number of stocks
4. $\mathbf{P}_t$ Price vector of stocks at time t
5. $\mathbf{P}_{t,s_t}$ Price vector of stocks at time t and in state s_t
6. $\mathbf{D}_{t,s_t}$ Dividend vector of stocks at time t and in state s_t
7. r_f Interest rate per period of the risk-free asset
8. $Q(j)$ Probability of state j occuring
9. MG Mixed Gaussian
10. $\bar{\mathbf{D}}_{s_t}$ Constant mean of $\mathbf{D}_{t,s_t}$
11. $\mathbf{u}_t$ Innovation process of the process $\mathbf{D}_t - \bar{\mathbf{D}}$
12. $\mathbf{\Sigma}_{\mathbf{u}_t}$ Variance-covariance matrix of the multivariate innovation process $\mathbf{u}_t$
13. $\mathbf{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{u_t}})$ Multivariate Gaussian distribution with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{\Sigma}_{\mathbf{u}_t}$
14. ρ First order autocorrelation coefficient of the process $\mathbf{D}_t - \bar{\mathbf{D}}$
15. $\overset{\text{law}}{=}$ Equal to in probability law
16. $\mathcal{I}_t$ Information set of publicly available information at time t
17. N Number of smart traders/investors and number of stocks.
18. λ Constant absolute risk aversion coefficient
19. $W_{t,i}$ Wealth of agent i at time t
20. $\mathbf{X}_{t,i}$ Stock holdings all stocks of agent i at time t
21. $\mathbf{X}^*_{t,i}$ Optimal stock holdings of all stocks of agent i at time t

22. $\mathcal{I}_{t,i}$ Information set of agent i at time t which includes the entire history of prices, dividends and *past* states

23. $\mathcal{I}_{t,i,s_t}$ Information set off agent i at time t which includes the entire history of prices, dividends and states (including the current state)

24. $\mathbb{E}_{t,i}$ Agent i's conditional expectation at time t over all random variables, i.e. conditional on $\mathcal{I}_{t,i,s_t}$

25. $\mathbb{V}_{t,i}$ Agent i's conditional variance estimate at time t over all random variables, i.e. conditional on $\mathcal{I}_{t,i,s_t}$

26. $\tilde{\mathbb{E}}_{t,i}$ Agent i's conditional expectation at time t over all random variables except s_t, i.e. conditional on $\mathcal{I}_{t,i}$

27. $\tilde{\mathbb{V}}_{t,i}$ Agent i's conditional variance estimate at time t over all random variables except s_t, i.e. conditional on $\mathcal{I}_{t,i}$

28. MG Mixed Gaussian

29. $\mathbb{E}_{t,i}^{\mathcal{A}_{t+1,i}}$ Agent i's conditional expectation at time t over all random variables in $\mathcal{A}_{t+1,i}$, conditional on $\mathcal{I}_{t,i}$

30. $\mathbb{V}_{t,i}^{\mathcal{A}_{t+1,i}}$ Agent i's conditional variance estimate at time t over all random variables in $\mathcal{A}_{t+1,i}$, conditional on $\mathcal{I}_{t,i}$

31. $\mathbb{E}_{t,i}^{\{s_{t+1,i}\}}$ Agent i's conditional expectation at time t over the state s_{t+1}, conditional on $\mathcal{I}_{t,i}$

32. $\mathbb{V}_{t,i}^{\{s_{t+1,i}\}}$ Agent i's conditional variance estimate at time t over the random variable s_{t+1}, conditional on $\mathcal{I}_{t,i}$

33. $\sim$ Distributed

34. $\mu_{t,s'_{t+1},i}$ Agent i's conditional expectation at time t of one period ahead vector prices plus dividends, for all stocks, over all variables except the future state variable

35. $\mathbf{\Sigma}_{t,s'_{t+1},i}$ Agent i's conditional variance-covariance estimate at time t of one period ahead vector prices plus dividends, for all stocks, over all variables except the future state variable

36. $\mu_{t,,i}$ Agent i's conditional expectation at time t for agent i of one period ahead prices plus dividends, for all stocks, over all variables

37. $U(W_{t,i})$ Utility of agent i at time t given wealth $W_{t,i}$

38. $a_{s,i,j}$ Factor loading in a linear forecast of prices plus dividends for agent i, rule j and stock s

39. $b_{s,i,j}$ Constant parameter in a linear forecast of prices plus dividends for agent i, rule j and stock s

40. $\sigma^2{}_{s,i,j}$ Agent i's variance estimate of their rule j linear vector forecast of prices plus dividends for stock s

41. $a_{i,j}$ Factor loading in a linear vector forecast of prices plus dividends for agent i and rule j

42. $\mathbf{b}_{i,j}$ Constant parameter in a linear vector forecast of prices plus dividends for agent i and rule j

43. $\mathbf{\Sigma}_{i,j}$ Agent i's variance-covariance estimate of their rule j linear vector forecast of prices plus dividends

44. $\mathbf{a}_i$ Factor loading in a linear vector forecast of prices plus dividends for agent i (not rule specific)

45. $\mathbf{b}_i$ Constant parameter in a linear vector forecast of prices plus dividends for agent i and rule j (not rule specific)

46. a_i^* Agent i's optimal vector factor loading in a linear vector forecast of prices plus dividends for agent i

47. b_i^* Agent i's optimal vector constant parameter loading in a linear vector forecast of prices plus dividends for agent i

48. a_{re} Factor loading in a linear REE forecast of future prices plus dividends

49. $\mathbf{b}_{re}$ Constant parameter in a linear REE forecast of future prices plus dividends

50. MA_v v-period prior moving average

51. τ Parameter governing degree of smoothing in an EWMA

52. K Time period after which agent's re-learn

53. $\mathcal{M}(\theta)$ A computerised agent-based model with parameters θ

54. θ Parameters of a computerised agent-based model

55. A range Admissible parameter range for A_{re}.

56. $\mathbf{B}$ range Admissible parameter range for $\mathbf{B}_{re}$.

57. J Number of states

58. T Number of time periods after which price estimates are recorded

59. N Number of agents (i.e. traders/investors)

List of Abbreviations

1.	SFASM	Santa-Fe Artificial Stock Market
2.	S&P 500	Standard and Poor's 500
3.	REE	Rational Expectations Equilibrium
4.	ARCH	Autoregressive Conditional Heteroscedasticity
5.	GARCH	Generalised Autoregressive Conditional Heteroscedasticity
6.	DCC	Dynamic Conditional Correlation
7.	CARA	Constant Absolute Risk Aversion
8.	EWMA	Exponentially Weighted Moving Average
9.	*a.s.*	Almost Surely
10.	i.i.d	Independent and Identically Distributed
11.	law	Probability Law
12.	MSE	Mean Squared Error
13.	KDE	Kernel Density Estimate
14.	*MG*	Mixed Gaussian
15.	MLE	Maximum Likelihood Estimation

Chapter 1

Introduction

1.1 Statement of Problem Investigated and Justificiation

This thesis is concerned with developing novel computerised agent-based models of the stock market in order to examine a range of issues. Agent-based models in finance are defined as models of financial markets where the interacting agents are heterogeneous and boundedly rational, as discussed in LeBaron (2006). Typically, in such models, significant computational power and numerical methods are required as no analytic solutions exist, see LeBaron (2006). Mathematics is also required to derive some of the underlying dynamics of the system.

The range of issues explored in this thesis are broadly summarised as follows (and are described in detail in later sections of this Introductory Chapter): (1) the development of more accurate models of the stock market in an agent-based modelling framework, (2) the development of methods to analyse parameter impacts on agent-based models, (3) the exploration and estimation of the (time-varying) kernel distribution properties of real-world stock market returns and (4) the development of models and methods that can explain these probabilistic and distributional phenomena.

The focus on developing new agent-based models is important for the following reasons. Given the recent global financial crisis, it is believed (Geanakoplos *et al.*, 2012) that conventional macroeconomic and financial economic methods failed to explain or forecast the crisis due in large part to overly simplistic assumptions such as agent homogeneity and the existence of purely rational agents which implied that market prices coincided with rational expectations equilbria. Such assumptions allowed for analytic tractability at the expense, it seems, of accuracy. Agent-based models allow for agent heterogeneity and bounded rationality and the majority use the power of a computer to run simulations from which models and forecasts can be derived. Market prices need not match up with rationally expected prices. As computerised agent-based models make more realistic assumptions regarding the agent type space and rationality, they can potentially offer significant benefits relative to conventional approaches. This explains why point (1) above is important. Point (2) is important

as it is mentioned in LeBaron (2006) that "the most common criticism is that the (agent-based) models have far too many parameters, and the impact of many of these parameters is not well understood". One can plausibly deduce that computerised agent-based models will be more widely used only when a criticism such as this is adequately addressed. Point (3) is important since asset prices are highly probablistic and their kernel distributions may be time-varying so exploring the dynamics of this is important from both a policy perspective and practitioner perspective since this is concerned with issues like market risk, tail-risk, asset price prediction and so forth. In fact, it is documented in this thesis that various stock return distributions are mixed Gaussian, both in a time unconditional and time conditional perspective. Point (4) is important since it allows us to understand what causes the phenomena documented in (3), namely why do stock market returns behave in the manner documented in this thesis?

1.2 Aims and Objectives

As noted above, it is mentioned in LeBaron (2006) that "the most common criticism is that the (agent-based) models have far too many parameters, and the impact of many of these parameters is not well understood". One can plausibly deduce that computerised agent-based models will be more widely used only when a criticism such as this is adequately addressed. In Chapter 2, the following research questions are addressed: (i) can a well-defined theoretical method be developed to examine the impact of parameters on an agent-based model, (ii) can a statistical model (in the form of (multivariate) regression) be used to examine the impact of parameters on an agent-based model, and (iii) how can such methods be applied reasonably to agent-based models? Moreover, our aim is to develop a multiple stock agent-based model that significantly generalises the model of Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997) to multiple stocks. Such a generalisation is necessary as the stock market consists of multiple stocks. In fact, it is mentioned in Palmer *et al.* (1999) that introducing multiple stocks into their framework would be a useful research pursuit.

In Chapter 3, a multiple stock agent-based model is developed that allows one to address the following specific questions through computational experiments: (i) what are the precise relationships, in a computational multi-asset market, between salient stock market parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices, (ii) what is the relationship between traders beliefs of the structure of the market on asset mispricing and (iii) how do stock price volatility and cross-correlations depend on agent beliefs?

The aim in Chapter 4 and Chapter 5 is to (i) estimate the unconditional and conditional kernel densities of stock market returns and (ii) to develop new computational agent-based models that can produce the estimated kernel densities of stock market returns and that can explain these

empirical phenomena that we discover. Two novel agent-based models will be provided to explain the estimated densities. These are show to explain the estimated densities which are shown to be a mixture of distributions, both unconditionally and conditionally (including during various important financial economic events such as the Great Depression, the Dot-Com bubble and the recent Great Recession).

1.3 Literature Review

There are numerous seminal computerised agent-based models of financial markets, see the surveys by LeBaron (2006), Hommes (2006), Chiarella *et al.* (2009), Hommes & Wagener (2009) or Lux (2009) for example.

One of the earliest works within agent-based computational finance is the single stock model of Day & Huang (1990) where demand exceeds supply. Whilst the simulated price series of the Day & Huang (1990) model does not appear to correspond to price series that we observe in real-world markets, their model is capable of producing switching bull and bear markets.

Arifovic (1991) showed that in a single stock market, based on the model of Bray (1982), if agents use genetic algorithms to learn about the structure of the stock market then prices and quantities converge to REE quantities and prices. A paper with a similar research agenda, in some sense, is Routledge (2001). In the single stock model of Routledge (2001), agent types are represented by specific linear forecasts of future dividends as well as by the decision of the agent to acquire a costly informative signal. The primary result is that some parameters will lead to convergence of the REE solution whereas others will not. The model of Routledge (2001) has connections with the Grossman-Stiglitz model (Grossman & Stiglitz, 1980) which is itself linked to the informational efficiency of asset prices.

There has been some research undertaken to understand the optimality of agent trading within a simulation agent-based environment. In particular, Lettau (1997) presented a simple one stock, repeated two-period model of optimal trading where the optimal portfolio holding is modelled as a univariate learning exercise. Lettau assumed traders learn via a genetic algorithm (mutation and crossover) and he showed that traders' ability to learn the optimal trading positions depends on the frequency with which they learn. A lower frequency of learning leads to a lower bias in traders' trading positions relative to the optimal value since the sample size for learning and updating purposes is larger. While the result that agents can learn the optimal solution of the trading problem through a genetic algorithm is appealing, the set-up is trivial as the dividend and thus price process dynamics, which are important, are abstracted away in a repeated two period model.

There exist agent-based models that produce simulated price series whose returns resemble the Autoregressive Conditional Heteroscedasticity (ARCH) and the Generalised Autoregressive

Conditional Heteroscedasticity (GARCH) volatility effects of Engle (1982) and Bollerslev (1986). For example, Youssefmir & Huberman (1997) presented a multi-agent model of (random) clustered volatility which arises from traders' switching strategies. In addition, Lux & Marchesi (1999) presented a multi-agent, single stock agent-based model of the stock market and showed that scaling and GARCH effects can arise out of agent interactions.

There have also been attempts to model minimally intelligent agents and to understand what types of phenomena can still emerge if agents are minimally intelligent. Farmer *et al.* (2005) presented a minimally intelligent agent model with a unique method of stock price determination, namely a continuous double auction. They modelled the limit order book directly where agents post bids and offers randomly within some constrained set. They showed that their model can explain a sizable portion of the variance between the best buying and selling prices in a cross-section of almost a dozen stocks.

Furthermore, Raberto *et al.* (2001) presented a random order generation single stock model similar to Farmer *et al.* (2005) in the sense that the model assumes that agents randomly choose whether to buy or sell. While such assumptions may seem at odds with our perception that a significant portion of stock market trading volume arises from the actions of large institutional investors, each of whom base their trading and investing decisions on their (non-empty) information set, their model can produce a leptokurtic shaped probability density of log price returns as well as volatility clustering. However, their model does not produce an autocorrelogram of returns that resembles real-world markets.

There has been some work to model other financial markets, instead of stock markets, within a simulation agent-based framework. Most of this work has focused on exchange rate markets. One such example is the model of Arifovic (1996) that presented a two period model of univariate exchange rate dynamics in an economy populated by agents which vary regarding their rules for first-period consumption and first-period relative currency investments, each of which is represented by bit-strings. The primary and interesting finding of Arifovic (1996) is that even if traders employ genetic algorithms, the exchange rate will never settle down to a stable exchange rate. Marengo & Tordjman (1996) also presented a simulation model of univariate exchange rate dynamics. By using a classifier system to model evolutionary learning, they showed that their model can produce periods of market stability and periods of market turbulence. Geanakoplos *et al.* (2012) presented a model where they show that leverage, not interest rates, were the underlying driver of the housing boom and bust, using counterfactual reasoning with insights from an agent-based model of the housing market.

Boer *et al.* (2007) illustrated that there are important differences between discrete-time and continuous-time computational agent-based models of financial markets. Aloud & Fasli (2017) presented a computational agent-based model of the foreign exchange market and compared and

contrasted the differences in effects of zero intelligent agents (of both constraint strategy and directional change event strategy types) and agents with a genetic programming approach. Aloud *et al.* (2017) developed a computational agent-based model of the high frequency foreign exchange market and presented a set of sufficient conditions that eventually result in a model with stylised facts that are similar to the real-world foreign exchange market.

The view of the stock market presented in Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997) is one of a market that can be approximated by an agent-based model with many different types of agents, each of whom behaves according to a different decision making rule. In these models, jointly referred to as the Santa Fe model, agents are represented by the parameters governing their linear forecasts of prices plus dividends and associated estimated variances. A classifier forecasting system similar to Holland (1983) is incorporated and dynamic evolutionary learning is modelled using a genetic algorithm that combines mutation and crossover. Their analysis showed that their computational (single-stock) agent-based model can replicate numerous phenomena documented to occur in real-world financial and stock systems such as fat-tails of the return distribution, volatility persistence and significant trading volumes. Their model is a computational model of the stock market with a single stock and with agent types that are represented by the parameters governing their linear forecasts of prices plus dividends and their associated estimated variances. Their results suggest that if learning is sufficiently slow, convergence to the linear homogeneous REE is achieved. There has been some research developed to understand the trading strategies of the model of Palmer *et al.* (1999). Joshi *et al.* (2000) provided a very interesting Nash characterisation of these trading strategies where it is shown that technical trading can arise due to a multi-person prisoner's dilemma since inclusion of technical trading strategy rules to a single agent's trading strategy set is a dominant strategy. Chen & Yeh (2001) developed a model with a single risky stock and where a genetic programming approach is used to model agent learning (specifically future expectations of prices plus dividends depends on a genetic programme) and prices adjust in response to excess demands.

Multivariate stock computerised agent-based models of the stock market are less common and a brief overview is also presented here. The model of Westerhoff (2004) provided reasons for the high degrees of comovement between stocks in financial systems. Westerhoff & Dieci (2006) provided a computational system of two stocks and showed that price volatility falls if a Tobin tax is introduced.[1] Chiarella *et al.* (2007) established conditions for the fundamental stability case in a system with a market maker and Dieci & Westerhoff (2010) studied finance systems that oscillate between bull and bear states.

[1] A Tobin tax is a tax on international financial transactions.

1.4 Thesis Outline

In Chapter 2, it is shown how two methods can be used to examine parameter impacts on an agent-based model. The first method is developed in this Chapter and is a well-defined and consistent (almost surely) method to examine parameter impacts on an agent-based model and is developed using an application of the law of large numbers pointwise to Monte Carlo simulation paths. The second method is a statistical model - in the form of (multivariate) regression using MLE - applied to an agent-based model to estimate parameter impacts. The second method - (multivariate) regression with MLE - allows us to (a) consider sample variance effects, (b) account for the fact that prices have many response variables (and to account for confounding variables) and (c) offer a systematic approach to model selection. Furthermore, MLE (regression) is used as it has attractive asymptotic properties, namely consistency (convergence in probability to the true value) and asymptotic normality. Both methods are then applied to the well known Santa Fe model of LeBaron *et al.* (1999) where it is shown that parameters have reasonable impacts on stock prices. It is also shown that risk-on and risk-off behaviour exists and that agent beliefs impact stock prices. One new theorem is established and proved. Furthermore, 43200 agent-based model Monte Carlo simulations are produced.

In Chapter 3, a multiple stock agent-based model of the stock model is developed which significantly generalises the model of Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997) to multiple stocks. Such a generalisation is necessary as the stock market consists of multiple stocks. In fact, it is mentioned in Palmer *et al.* (1999) that introducing multiple stocks into their framework would be a useful research pursuit. We achieve this in Chapter 3. In the model of Chapter 3, algorithmic traders and investors are assumed to learn by a multistock genetic algorithm and classifier system. In this multiple stock model, it is shown, by computational experiments and by applying the parameter variation Monte Carlo method and statistical model method (in the form of (multivariate) regression) that the relationships between salient stock market parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices are sensible. It is also shown, by Monte Carlo simulations and by the estimation of a DCC (1,1) model (Engle, 2002) that mispricing, stock price volatility and cross correlations are each a function of agent beliefs. Two new theorems are established and proved and 43200 agent-based model Monte Carlo simulations are produced.

In Chapter 4 it is shown that various unconditional and conditional kernel distributions of stock market Standard and Poor (S&P) 500 returns, including during important financial economic events such as the Great Depression and the recent Great Recession, are a mixture of distributions. Furthermore we find that markets possess time-varying kernel densities. To the best of our knowledge, these findings are novel. Thus, a single stock agent-based model is developed where smart (algorithmic) traders and portfolio managers trade and invest in an equally weighted average of stocks that underlie an equity index such as the S&P 500 and condition on the observed probabilistic properties

of the market as part of their trading activities. Mixture modelling is incorporated into the developed agent-based model. Optimal stock holdings of (algorithmic) traders and investors are computed by deriving the expectation of a log mixed random variable and agents are assumed to learn by a multistate, single stock, genetic algorithm and classifier system. Agent learning is conditional on the prevailing market state. It is shown, by kernel density estimation on simulated prices, that the agent-based model developed can produce a distribution of stock returns that resemble the mixture distributions of real-world stock returns. The model developed significantly generalises the model of LeBaron *et al.* (1999) to multiple states. It is also shown that the model of LeBaron *et al.* (1999) cannot produce a distribution of stock returns that resemble real-world stock returns. In this agent-based model, it is shown, by computational experiments and by applying the parameter variation Monte Carlo method and statistical model method (in the form of (multivariate) regression) that the relationships between salient stock market parameters such as state probabilities, agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices are sensible. Four new theorems are established and proved and 43200 agent-based model Monte Carlo simulations are produced.

In Chapter 5, further evidence is provided that various kernel distributions of stock market S&P 500 returns are a mixture of distributions. A multivariate agent-based model is then developed to explain this aggregated stock market feature where smart (algorithmic) traders and portfolio managers trade and invest in the *underlying equities that constitute an equity market index* and *optimise* their holdings in their portfolio optimisation making assumptions that are consistent with market dynamics as part of their trading activities. Stocks are modelled as a claim to a *multivariate mixed Gaussian* dividend stochastic process. It is shown, by kernel density estimation on simulated prices, that the agent-based model developed can produce a distribution of stock returns that resemble the mixture distributions of real-world stock returns. The model developed significantly generalises the models of LeBaron *et al.* (1999) and those developed in Chapters 3 and 4 to, jointly, multiple stocks and multiple states. In this multiple stock model, it is shown, by computational experiments and by applying the parameter variation Monte Carlo method and statistical model method (in the form of (multivariate) regression) that the relationships between salient stock market parameters such as state probabilities, agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices are sensible. Four new theorems are established and proved and 43200 agent-based model Monte Carlo simulations are produced.

Chapter 6 highlights the results obtained from the study as well as conclusions and intended future work.

1.5 Statement of Sources of Data

In this thesis and its associated underlying research papers, stock market data of the S&P 500 were obtained from Robert Shiller's website (www.econ.yale.edu/∼shiller).

Chapter 2

Examining Parameter Impacts on the Computerised Agent-Based Santa Fe Artificial Stock Market Model

Conventional methods failed to foresee the recent global financial crisis. An alternative to conventional methods is agent-based modelling which allows researchers to relax assumptions such as agent homogeneity and rationality. A criticism of computerised agent-based models of financial markets is that the impact of parameters on these models is not well understood. The focus of this Chapter is to address this by this by developing a methodology that uses a combination of (multivariate) regression (in the form of MLE) to synthetic financial data and by developing a parameter variation Monte Carlo simulation technique (by applying the law of large numbers pointwise to Monte Carlo simulation paths). These two methods allow researchers to examine parameter impacts on an agent-based model. These two new methods are then applied to examine the impact of parameters on the Santa Fe Artificial Stock Market model of LeBaron *et al.* (1999). It is shown how simulated prices vary with structural parameters. These relationships between parameters and simulated stock prices are shown to be sensible in a financial economic sense. Furthermore, evidence is provided that agent risk-aversion (i.e. risk-on and risk-off) exists and that agent beliefs impact stock prices. One new theorem is established and proved, and 43200 agent-based model Monte Carlo simulations are produced as part of this analysis.

This Chapter has been submitted to the journal **Macroeconomic Dynamics**, is currently under review and is structured according to the journal's format. The research paper was written by the student under the supervision of Prof S Abelman.

Examining Parameter Impacts on the Computerised Agent-Based Santa Fe Artificial Stock Market Model

Shaheen Seedat* and Shirley Abelman

School of Computer Science and Applied Mathematics

University of the Witwatersrand, Johannesburg

Private Bag 3, Wits 2050, South Africa

Abstract

In this paper, we aim to understand whether the Santa Fe Artificial Stock Market model of LeBaron *et al.* (1999) produces stock prices that vary in a plausible manner with various underlying important parameters such as agent risk aversion, agent beliefs, the underlying stock dividend process and returns to risk-free assets. We achieve this by using a combination of (multivariate) regression (in the form of MLE) to synthetic financial data and by developing a parameter variation Monte Carlo simulation technique and by producing a sequence of 72 computerised agent-based models and 43200 agent-based model simulations. We present evidence that the Santa Fe Artificial Stock Market model is plausible regarding the relationship between Santa Fe Artificial Stock Market model simulated stock prices and the parameters aforementioned. We also provide evidence that stock mispricing is reduced if agents' beliefs regarding the true parameters of the stock market have lower variance.

Keywords: Sante Fe Model, Monte Carlo Simulations, Agent-Based Modeling, Regression.

*Corresponding author. Email: seedatshaheen@gmail.com. Tel: +27 116461152.

1 Introduction

One of the failures of the macroeconomic and financial economic professions was the inability to predict the recent financial crisis. It is believed that this is due to shortcomings of conventional models that were used, see Geanakoplos *et al.* (2012), where agents were assumed homogeneous and rational in order to solve models analytically. This resulted, perhaps, in a lack of realism of these models. Computerised agent-based modelling offers an alternate approach. Computational agent-based models in finance are defined as models of financial markets where the interacting agents are heterogeneous and boundedly rational, and where the power of a computer is required, as discussed in LeBaron (2006).

However, computerised agent-based models still remain at the periphery of modern finance research. It is mentioned in LeBaron (2006) that "the most common criticism is that the (agent-based) models have far too many parameters, and the impact of many of these parameters is not well understood". One can plausibly deduce that computerised agent-based models will be more widely used only when a criticism such as this is adequately addressed. We show that it is in fact possible to effectively examine the effects of parameters on computerised agent-based models. We achieve this in two ways. In our first method, we develop a well-defined methodology that allows one to examine parameter impacts on an agent-based model, parameter by parameter. We then apply our method to examine the effects of parameters on a particular computerised agent-based model of the stock market. In our second method, we use the method of (multivariate) regression which allows us to (a) consider sample variance effects, (b) account for the fact that prices have many response variables and (c) offers a systematic approach to model selection. In particular, we achieve this by using MLE. We provide details on attractive features of MLE, namely that it is consistent (converges in probability to the true value) and is asymptotically normal with an asymptotic covariance-variance matrix that is known. This thus allows for statistical inference to take place as we can compute standard errors, sample variances and p-values. We then apply MLE to a (multivariate) regression of our model of interest, detailed below.

These two methods can be used on any computerised agent-based model of any dynamical system, irrespective of the dimension of the parameter space. There are numerous seminal computerised

agent-based models of financial markets, see the surveys by LeBaron (2006), Hommes (2006), Chiarella *et al.* (2009), Hommes & Wagener (2009) or Lux (2009) for example. In particular, we choose to examine the impact of parameters on what is commonly referred to as the Santa Fe Artificial Stock Market (SFASM) model of LeBaron *et al.* (1999), Palmer *et al.* (1994), Palmer *et al.* (1999) and Arthur *et al.* (1997). We choose to apply our methodology to the SFASM model because there is substantial pre-existing research suggesting that the SFASM is a reasonable approximation of real-world financial markets in the sense that it can replicate numerous financial phenomena we observe.

In Section 2, we describe the computerised agent-based SFASM. In Section 3, we use the two methods described above to examine parameter impacts on computerised agent-based models and we apply these two methods to the SFASM model. In section 4 we conclude our paper.

2 Stock Market Model

In this section, we provide some of the pertinent details of the computerised agent-based SFASM model. Further details may be found in LeBaron *et al.* (1999) and Palmer *et al.* (1999).

The stock-market consists of one risk-free asset and one dividend paying stock. The price of the stock at time t is denoted by p_t. The dividend paid at time t for the stock is d_t. The dividend process is of the form

$$d_t = \bar{d} + \rho(d_{t-1} - \bar{d}) + u_t \tag{1}$$

where $\bar{d}$ is a constant mean and u_t is assumed to be Gaussian $N(0, \sigma^2_{u_t})$ and independent over time.

Agents are assumed to be one-period myopic and to have constant absolute risk averse (CARA) preferences with risk-aversion parameter λ. Agents assume future prices plus dividends are conditionally Gaussian which, combined with CARA preferences, allows them to compute optimal holdings of the stock as per standard portfolio theory. Optimal holdings of the stock, for each agent i and each

time t satisfy:

$$X^*_{t,i} = \frac{\mu_{t,i} - (1 + r_f)p_t}{\lambda \sigma^2{}_{p+d,i}} \quad t = 1, 2, ... \tag{2}$$

where $\mu_{t,i}$ is the agent's forecasts of one-period ahead prices plus dividends, $\sigma^2{}_{p+d,i}$ the agent's estimate of the variances of these and r_f is the interest rate of the risk-free asset.

Agents forecast one-period ahead prices plus dividends and estimate the variances of these by using a linear classifier system. This allows agents to form linear forecasting equations of prices plus dividends, of the form

$$\mathbb{E}_{t,i,j}(p_{t+1} + d_{t+1}) = a_{i,j}(p_t + d_t) + b_{i,j} \tag{3}$$

with associated variance estimate $\sigma^2_{i,j}$ where j denotes the associated "rule". Given the set up of this model, the only source of heterogeneity for agents is their forecasts of futures prices and dividends (i.e. $\mathbb{E}_{t,i,j}(p_{t+1} + d_{t+1})$) and their estimates of the variances of these ($\sigma^2{}_{t,i,j}$). Agents compare different rules by using the following penalised criterion

$$\nu^2_{t,i,j} = \Big(1 - \frac{1}{\tau}\Big)\nu^2_{t-1,i,j} + \frac{1}{\tau}\big(p_t + d_t - (a_{i,j}(p_{t-1} + d_{t-1}) + b_{i,j})\big)^2 \tag{4}$$

as discussed in Palmer *et al.* (1999) and where τ is usually set to 75.

A genetic algorithm is used to model the evolution of the rules by eliminating the weakest ones and replacing them with a crossover or mutation of the remaining ones. In this manner, agents can solve for their dynamic optimal stock holdings over time. To complete the computerised agent-based model, prices are set by a specialist to clear the market and are not, in general, the same as prices obtained from a Rational Expectations Equilibrium (REE). Prices obtained from a computerised agent-based model of the stock market are obtained by *computer simulations* and not by the conventional method of REE. This also applies to the Santa Fe model.

In our paper, we will examine the impacts of the various parameters on this computerised agent-based model. We achieve this by a technique that we call parameter variation Monte Carlo simulation. We also examine the impact of a pertinent parameter, agent beliefs, on the degree

of stock mispricing that can arise in the Santa Fe model relative to a homogeneous linear REE. To assess the degree of mispricing, we need to compare simulated prices to those obtained in a homogeneous linear REE. By the no-trade theorem, the prices obtained from the homogeneous linear REE, of the Santa Fe model, satisfy:

$$\mathbb{E}_{t,i}(p_{t+1} + d_{t+1}) = \rho(p_t + d_t) + (1 - \rho)((1 + f)\bar{d} + e) \tag{5}$$

where f is the factor loading in the linear price function as a function of dividends and where e is the constant in a linear price function as a function of dividends.

We will compare simulated prices to this homogeneous linear REE solution to assess stock mispricing as a function of agent beliefs. We will denote $A_{re} = \rho$ and $B_{re} = (1 - \rho)((1 + f)\bar{d} + e)$.

3 Understanding Parameter Impacts on an Agent-Based Model by Parameter Variation Monte Carlo Simulation

3.1 Method 1: Parameter Variation Monte Carlo Simulation

We propose a method that allows us to understand the impact of parameters on any computerised agent-based model of the stock market. This method allows us to examine the impact of parameters on stock prices, *individually*, that is by holding other parameters constant. We will then apply this method to the Santa Fe model. It is also of interest to consider the *joint* effect parameters have on stock prices. This is achieved in method two.

We return to the first method. Let $\mathcal{M}(\theta)$ denote a (computerised) agent-based model of interest where $\theta \in \mathbb{R}^M$ are the parameters of the model. When we assign values to the parameters θ, we have a realisation of the agent-based model. An example of an agent-based model is the Santa Fe model without any parameters assigned to its parameters λ, ρ, etc. A realisation of the Santa Fe agent-based model is the Santa Fe model with particular parameters assigned to it, e.g. $\lambda = 0.1$, $\rho = 0.9$, etc.

To understand how our method of parameter variation Monte Carlo simulation proceeds, consider

again an agent-based model $\mathcal{M}(\theta)$ which may or may not be the Santa Fe model, with parameters $\theta \in \mathbb{R}^M$. Assume one of the parameters is $\alpha \in \mathbb{R}$. Assume we are interested in the impact of α on the agent-based model $\mathcal{M}(\theta)$. To do this, discretise the range of interest of α into intervals with points $\alpha_0, \alpha_1, ...$ Fix the remaining $M-1$ parameter values of the other parameters and denote the resulting M tuple parameter vector values by $\theta_0, \theta_1, ...$ Lastly, denote the associated realised agent-based models as $\mathcal{M}(\theta_{\alpha_0})$, $\mathcal{M}(\theta_{\alpha_1})$, ... Note that the only differences between θ_{α_i} and θ_{α_j} for $i \neq j$ are α_i and α_j. Equipped with this we can now compare how α affects the agent-based model $\mathcal{M}(\theta)$ by comparing $\mathcal{M}(\theta_{\alpha_i})$ and $\mathcal{M}(\theta_{\alpha_j})$ for all $i \neq j$.

Note that for each *realisation* of an agent-based model, we can simulate it many times. When we re-simulate a given *realisation* of an agent-based model (i.e. when we re-simulate an agent-based model with a fixed set of parameters) we obtain *different* outcomes. This is a natural consequence of simulation. For example, the stock prices obtained from one simulation of the realised Santa Fe model with parameters as set in table 1 will most likely be different from those stock prices of the next simulation of the realised Santa Fe model with parameters as set in the *same* table 1. This is true for all agent-based models and not just the Santa Fe model.

Table 1: Parameters of Agent-Based Model 1 (Santa Fe model).

Parameter	**Value**
λ	0.1
${\sigma^2}_{p+d}$	4
${\sigma^2}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
$\bar{d}$	10
A range	$[0.85, 1.05]$
B range	$[3.5, 5.5]$
N	25
T	2750
K	11

Table 2: Parameters of Agent-Based Model 2 (Santa Fe model).

Parameter	Base value
λ	0.5
${\sigma^2}_{p+d}$	4
${\sigma^2}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
$\bar{d}$	10
A range	$[0.85, 1.05]$
B range	$[3.5, 5.5]$
N	25
T	2750
K	11

3.2 Applying This Technique to the SFASM

We now consider how parameters impact a particular computerised agent-based model, the SFASM. Our method is general and we can apply it to any agent-based model.

Without loss of generality, let $\mathcal{M}(\theta)$ denote the Santa Fe agent-based model where our interest is in the structural parameters given by θ where $\theta = \big(\lambda, \sigma^2_{p+d}, \sigma^2_{u_t}, \rho, r_f, \bar{d}, A \text{ range}, B \text{ range}\big) \in \mathbb{R}^8$ are the structural parameters, where A range and B range will be defined later. Let $\alpha \in \mathbb{R}$ denote one of the parameters and assume we are interested in how it affects a given observable of $\mathcal{M}(\theta)$. Let us assume that the observable of interest is stock prices and, in particular, the time t-th price obtained from the model. We introduce some notation: let $\mathcal{M}(\theta_{\alpha_i})$ denote a realisation of the Santa Fe model which corresponds to the parameter values θ_{α_i} with corresponding parameter value for α given by α_i. For $j \neq i$, let $\mathcal{M}(\theta_{\alpha_j})$ denote a realisation of the Santa Fe model that has all parameter values the same as $\mathcal{M}(\theta_{\alpha_i})$ except the parameter value assigned to α where the new parameter value is α_j instead of α_i.

Let $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$ denote the price of a stock, at time t, obtained from a given realisation of the Santa Fe agent-based model, $\mathcal{M}(\theta_{\alpha_i})$ and from a given simulation j of this realised agent-based model. If we are interested in how α affects the time t-th stock price of the agent-based model $\mathcal{M}(\theta)$ then we want to know how α affects $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ where the expectation operator $\mathbb{E}$ is taken over all simulations j. This is because $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$, for a given simulation j, may contain significant noise. In an ideal world, we know $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$. In the real-world, we do not know $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ but we can form a reliable point estimate of it. The rational for this is provided

by the theorem below.

Theorem 1 *Estimate of stock price, for a given agent-based model and a given set of parameters. Let $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$ denote the price of a stock, at time t, obtained from a given realisation of the Santa Fe agent-based model, $\mathcal{M}(\theta_{\alpha_i})$ and from a given simulation j of this realised agent-based model. Then a reliable estimate of $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ is given by $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_i}, k))$ for N_1 large.*

Proof. By the pointwise (fixed t) law of large numbers for independent and identically distributed (i.i.d) random variables, we have that

$$\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_i}, k)) \overset{a.s.}{\to} \mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big) \tag{6}$$

where the expectation operator $\mathbb{E}$ is taken over all simulations j. The law of large numbers applies since $\mathcal{M}(\theta_{\alpha_i})$ is fixed in $\mathcal{M}(\theta_{\alpha_i}, k)$ and in $\mathcal{M}(\theta_{\alpha_i}, j)$.

■

We can thus define our method of parameter variation of an agent-based model as follows. We use stock prices as the simulated observable variable of interest.

Definition 2 Parameter Variation of an Agent-Based Model of the Stock Market.
Let $\mathcal{M}(\theta)$ denote an agent-based model of a stock market (which may or may not be the Santa Fe model) with parameters $\theta \in \mathbb{R}^M$ and where one parameter is $\alpha \in \mathbb{R}$. Let α_m and α_n be two distinct parameter *values* of α. The impact on stock prices of a parameter change from α_m to α_n is given by the change from $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_m}, j))\big)$ to $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_n}, j))\big)$ where $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$ denotes the price of a stock, at time t, obtained from a given realisation of the agent-based model, $\mathcal{M}(\theta_{\alpha_i})$ and from a given simulation j of this realised agent-based model.

Remark 3 By the above theorem, we have that the point sample estimates $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_m}, k))$ and $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_n}, k))$, for large N_1, are reliable estimates of $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_m}, j))\big)$ and $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_n}, j))\big)$. Thus, $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_m}, k))$ and $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_n}, k))$, for large N_1, can be used to understand the impact of α on stock prices of an agent-based model. Note that t is fixed in both sample estimates.

Note that our technique can be applied to any computerised agent-based model of any dynamical system.

3.3 Impact of Parameters on Simulated Stock Prices

We now computationally apply this first method to the Santa Fe model. To commence, let us examine the effect of the risk aversion parameter λ on stock prices of the model. We consider the interval $[0.1, 2.5]$ of λ and we discretise it into 12 subintervals. This yields the parameter values $\lambda_0 = 0.1$, $\lambda_1 = 0.3$ and so on. We now apply our theory. Let $\mathcal{M}(\theta)$ denote the Santa Fe model. Let $\mathcal{M}(\theta_{\lambda_i})$ denote a realisation of the Santa Fe model which corresponds to the parameter values θ_{λ_i} with corresponding parameter value of λ given by λ_i. For all $\mathcal{M}(\theta_{\lambda_i})$, we set the parameter values excluding the value of λ as the values in table 2 which are the values set in LeBaron *et al.* (1999). Consider a particular parameter value λ_i. First, we simulate the model with $N_1 = 600$. In each simulation, we obtain the $t = 2500$-th price observation. We are interested in this value of t as this allows for sufficient learning to take place. Hence, we have obtained $p(t, \mathcal{M}(\theta_{\lambda_i}, k))$, for each simulation $k \in \{1, 2, ..., N_1 = 600\}$. We plot each of these 600 prices in figure 1 (a) at the parameter value $\lambda = \lambda_i$. We then apply our law of large numbers result from the above theorem and can form a reliable estimate of $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ as depicted in figure 1 (b) at the value $\lambda = \lambda_i$. This is our estimate of $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\lambda_i}, j))\big)$ where $\mathbb{E}$ is the expectation over all simulations j. We do this for all λ_i under consideration. By analysing figure 1 (b), we can examine the impact of risk aversion on prices of the agent-based Santa Fe model. (Note that a given path in figure 1 (a) represents how the $t = 2500$-th price changes with λ for a *fixed random number generator*. Averaging these paths removes idiosyncratic noise, by the law of large numbers result above.) For exposition, if we were to consider a simple case where we have the interval $[0.1, 0.5]$ of λ and we discretise it into 1 sub-interval then θ_{λ_0} and θ_{λ_1} would now be as in table 1 and 2, respectively.

We repeat this process for each of the structural parameters, namely $\sigma^2_{p+d}, \sigma^2_{u_t}$, ρ, r_f and $\bar{d}$ and we also plot their respective graphs below. For each parameter, we discretise its range of interest into 12 subintervals. (We will examine the impact of agent beliefs, given by A range and B range, later on.)

We can now examine the impact of the six structural parameters on the Santa Fe model. It turns out that the impact of each parameter is sensible in a financial economic sense as we discuss below. Firstly, as shown in figure 1 (b), stock prices fall as agents' risk aversion coefficient rises. That is, stock prices fall during 'risk-off' periods. This is because agents have a greater preference for bonds to stocks when they are more risk averse: stocks are volatile and bonds are (generally) risk-free. Secondly, as shown in figures 1 (c) and 1 (d), stock prices fall as agents' variance beliefs about future prices plus dividends rise. That is, if agents believe stock prices and dividends will be more volatile then they will invest more in the bond than the stock. This is a natural consequence of the behaviour of risk-averse agents. Thirdly, as shown in figures 1 (e) and 1 (f), stock prices fall as idiosyncratic stock dividend variation rises. The reasoning is similar to the prior case. Specifically, since stocks are a claim to dividends, stock dividends are the underlying driver of stock prices, as postulated by the dividend discount model. If dividends are more volatile then agents will prefer an asset that has less volatile characteristics to the dividend-driven stock, namely the bond. Fourthly, as shown in figures 2 (a) and 2 (b), stock prices fall as the stock dividend process approaches an integrated non-stationary process. (ρ is the persistence parameter of the dividend process in the Santa Fe model.) The reasoning is that the dividend process will approach a random walk process and as it does so the volatility of dividends will rise. That is, in the limit, a Brownian Motion is reached where the variance will scale linearly with time. Fifthly, as shown in figures 2 (c) and 2 (d), stock prices are lower as the risk free-rate rises. This is because the interest earning risk-free bond will be more attractive. Lastly, as shown in figures 2 (e) and 2 (f), stock prices are higher if the mean of the dividend process rises. This is due to higher dividend payouts, in expectation.[1]

We have thus been able to examine the impact of the parameters on the agent-based model and we have noted that each relationship can be explained in a manner that is sensible in a financial economic sense. This is evidence to suggest that the agent-based Santa Fe model is plausible *in the manner in which it varies with its parameters*. In addition to the documented results in the literature on the Santa Fe model, our results are new and distinct and provide further support of

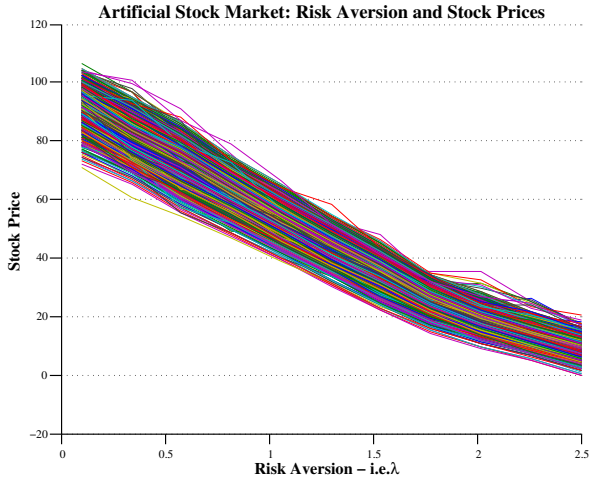
[1]Note that figures 1 (b), 1 (d), 1 (f), 2 (b), 2 (d) and 2 (f) are law of large number averages of (and thus derived from) figures 1 (a), 1 (c), 1 (e), 2 (a), 2 (c) and 2 (e), respectively. This is akin to the relationship between figure 1 (b) and 1 (a) as discussed above. These graphical relationships follow from Theorem 1, Definition 2 and Remark 3 and the manner in which Monte Carlo simulations were conducted. Hence, once derived, it is figures 1 (b), 1 (d), 1 (f), 2 (b), 2 (d) and 2 (f) which allow us to clearly examine the impact of the structural parameters of the model.

the Santa Fe model.

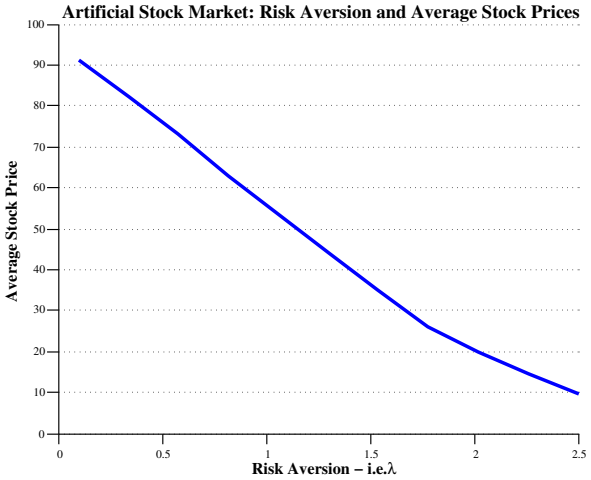
In total, we produced a total sequence of 72 (realised) agent-based models for our analysis and 43200 agent-based model simulations to conduct our parameter variation analysis.

Figure 1: Impact of Parameters on the agent-based SFASM: Parameters impact the model in a plausible manner (Panel A)

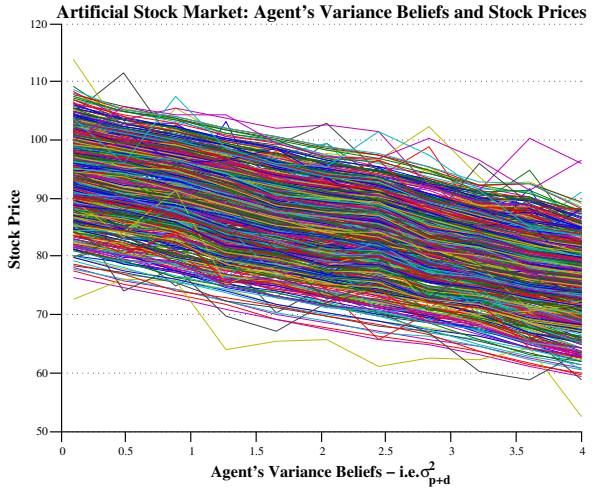
(a)



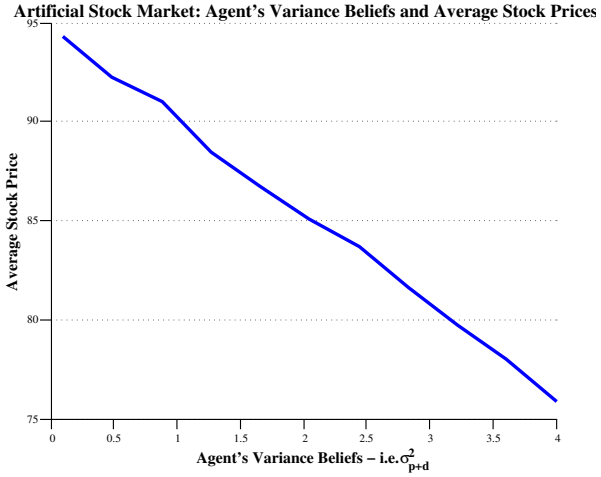
(b)



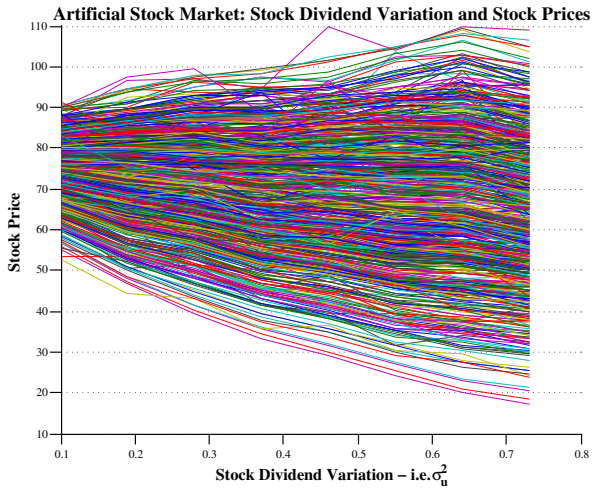
(c)



(d)



(e)



(f)

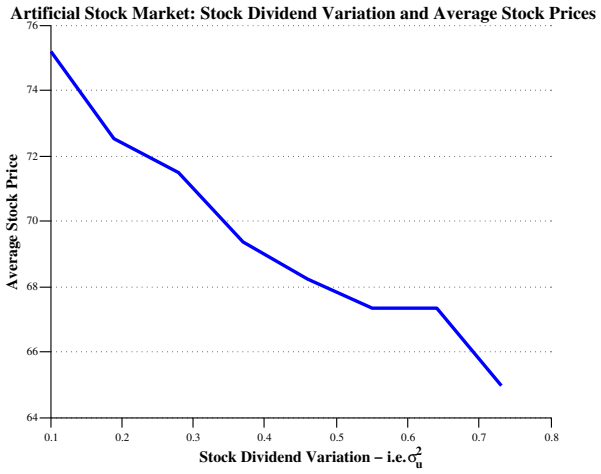
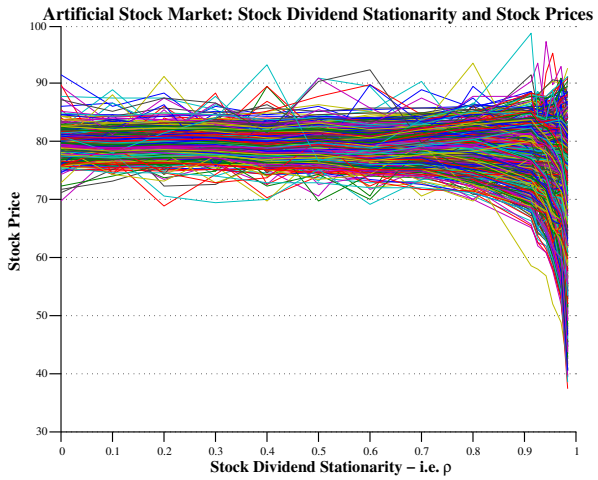
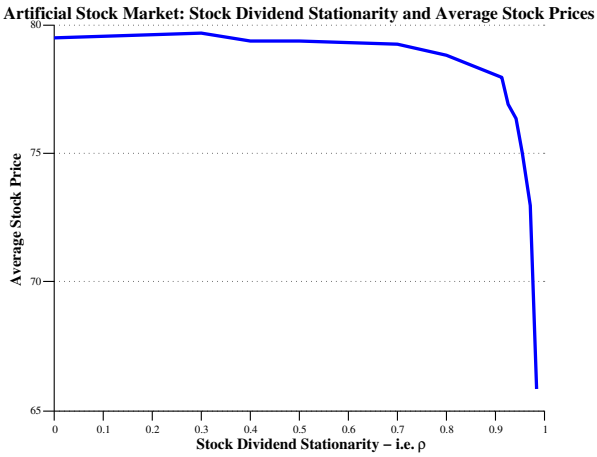


Figure 2: Impact of Parameters on the agent-based SFASM: Parameters impact the model in a plausible manner (Panel B)

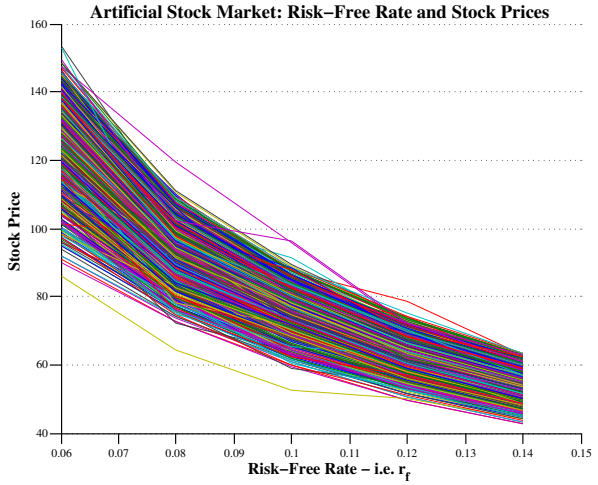
(a)



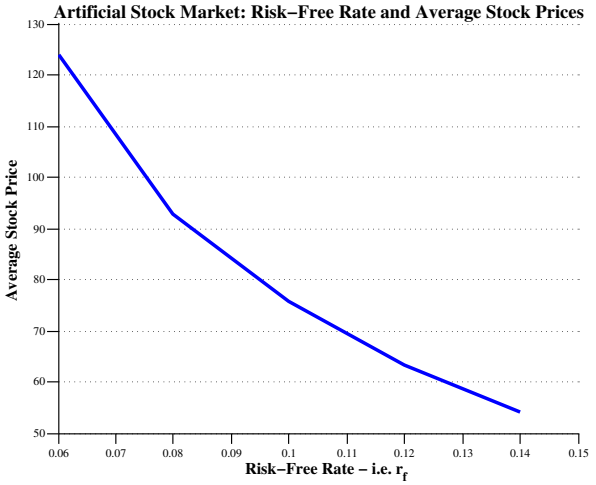
(b)



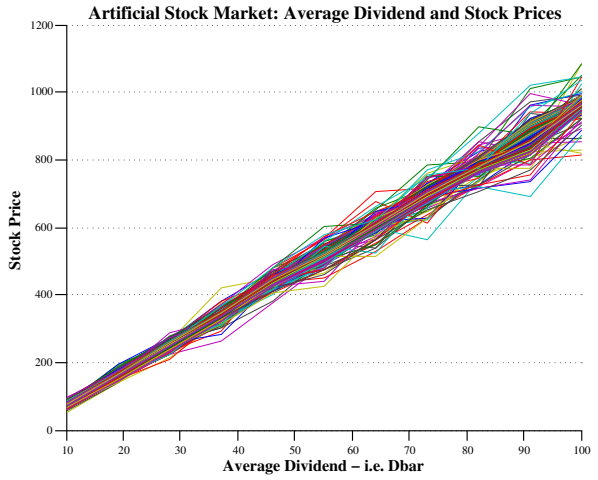
(c)



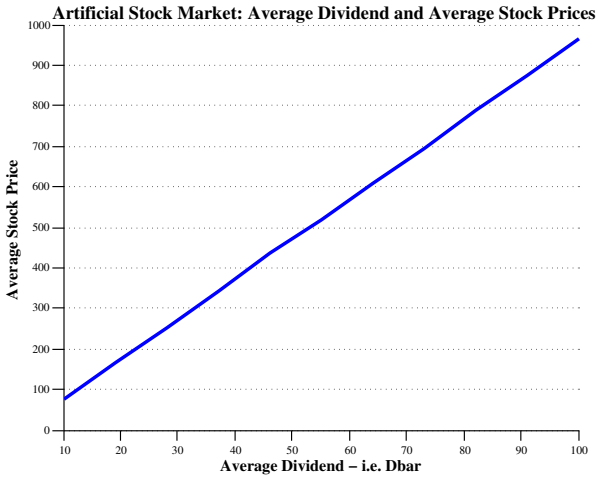
(d)



(e)



(f)



3.4 Method 2: Statistical Model (Regression)

Following on from method one, we are also interested in (a) including sample variance and statistical significance considerations, (b) accounting for multiple variables and for confounding variables and (c) offering a systematic approach to model selection and validation. This can be achieved by introducing a statistical model in the form of (multivariate) regression. We achieve this using the well-known Maximum Likelihood regression method.

To commence we provide a review of MLE and its attractive properties. Maximum likelihood uses the distribution of the data to estimate unknown parameters and their sample variances by determining the parameter values that make the data as likely to have been observed, that is maximizing the likelihood. It proceeds as follows. Consider the joint distribution $f(y;\theta)$ of the data (either observed or simulated) y as a function of a k by 1 parameter vector θ. This is the joint density of the data. Then define the Likelihood of the data as $L(\theta;y) = f(y;\theta)$. The Maximum Likelihood estimator $\hat{\theta}$ is then defined as the solution to

$$\hat{\theta} = \arg\max_{\theta} L(\theta;y) \tag{7}$$

Since $L(\theta;y) = f(y;\theta)$ is strictly positive we can use the log-likelihood $l(\theta;y) = f(y;\theta) = lnL(\theta;y)$. The Maximum Likelihood Estimator is then obtained by solving the k by 1 score vector,

$$\frac{\partial l(\theta;y)}{\partial \theta} = 0 \tag{8}$$

The most well-known method of MLE is to assume that the data y_i is i.i.d normally (Gaussian) distributed with mean μ and variance σ^2. The probability density function (pdf) of a Gaussian random variable is given by

$$f(y;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(y_i-\mu)^2}{2\sigma^2}\Big) \tag{9}$$

The score vector is then given by the following

$$\frac{\partial l(\theta; y)}{\partial \mu} = \sum_{i=1}^{n} \frac{(y_i - \mu)}{\sigma^2} \tag{10}$$

$$\frac{\partial l(\theta; y)}{\partial \sigma^2} = \frac{n}{\sigma^2} + \frac{1}{2} \sum_{i=1}^{n} \frac{(y_i - \mu)^2}{\sigma^4} \tag{11}$$

Setting both of these equal to zero and solving the system leads to the following solution for the estimated mean and estimated sample variance, in the case of MLE regression:

$$\hat{\mu} = \frac{\sum_{i=1}^{n} y_i}{n} \tag{12}$$

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{n} (y_i - \mu)^2}{n} \tag{13}$$

To derive the p-value (and to thus be able to assess statistical significance and inference), the asymptotic properties need to be considered. Under the assumptions we have made above, the average scores will follow a central limit theorem and so

$$\sqrt{n} \nabla_\theta \bar{l}(\theta_0) = \sqrt{n} \left(n^{-1} \sum_{i=1} n \frac{\partial l(\theta; y)}{\partial \theta} \right) \Bigg|_{\theta = \theta_0} \tag{14}$$

where θ_0 is the true parameter value.

By a mean value expansion about θ_0,

$$\sqrt{n} \nabla_\theta \bar{l}(\hat{\theta}) = \sqrt{n} \nabla_\theta \bar{l}(\theta_0) + \sqrt{n} \nabla_{\theta\theta'} \bar{l}(\bar{\theta})(\hat{\theta} - \theta_0) \tag{15}$$

$$0 = \sqrt{n} \nabla_\theta \bar{l}(\theta_0) + \sqrt{n} \nabla_{\theta\theta'} \bar{l}(\bar{\theta})(\hat{\theta} - \theta_0) \tag{16}$$

$$-\sqrt{n} \nabla_{\theta\theta'} \bar{l}(\bar{\theta})(\hat{\theta} - \theta_0) = \sqrt{n} \nabla_\theta \bar{l}(\theta_0) \tag{17}$$

$$\sqrt{n}(\hat{\theta} - \theta_0) = \left(\nabla_{\theta\theta'} \bar{l}(\bar{\theta}) \right)^{-1} \sqrt{n} \nabla_\theta \bar{l}(\theta_0) \tag{18}$$

where

$$\nabla_{\theta\theta'}\bar{l}(\bar{\theta}) = n^{-1}\sum_{i=1}^{n}\frac{\partial^2 l(\theta;y)}{\partial\theta\partial\theta'}\bigg|_{\theta=\bar{\theta}} \tag{19}$$

where $\bar{\theta}$ lies between $\hat{\theta}$ and θ_0. Hence by the Central Limit Theorem, the asymptotic distribution of the Maximum Likelihood Estimator is

$$\sqrt{n}(\hat{\theta}-\theta_0) \xrightarrow{d} \left(0, \mathcal{I}^{-1}\mathcal{J}, \mathcal{I}^{-1}\right) \tag{20}$$

where

$$\mathcal{I} = -\mathbb{E}\left(\frac{\partial^2 l(\theta;y)}{\partial\theta\partial\theta'}\bigg|_{\theta=\theta_0}\right) \tag{21}$$

$$\mathcal{J} = -\mathbb{E}\left(\frac{\partial l(\theta;y)}{\partial\theta}\frac{\partial l(\theta;y)}{\partial\theta'}\bigg|_{\theta=\theta_0}\right) \tag{22}$$

Hence the Maximum Likelihood Estimator converges in probability to the true parameter (consistency) and we can compute the standard errors and sample variances using the variance-covariance matrix $\mathcal{I}^{-1}\mathcal{J},\mathcal{I}^{-1}$.

Now let us consider a multivariate regression model $y_i = \mathbf{x}_i\boldsymbol{\beta} + \epsilon_i$. Under the following standard regularity conditions, where $\mathbf{X}$ is a stacking of all the $\mathbf{x}_i$.

1. Conditional mean of errors: $\mathbb{E}(\epsilon_i|\mathbf{X}) = 0$ for all $i = 1, ..., n$.

2. Rank: The rank of $\mathbf{X}$ is k with probability 1, where k is the number of regressors.

3. Conditional homoscedasticity:$\mathbb{V}(\epsilon_i|\mathbf{X}) = \sigma^2$.

4. Conditional correlation: $\mathbb{E}(\epsilon_i \epsilon_j | \mathbf{X}) = 0$ for all $i = 1, ..., n$ and for all $j = i + 1, ..., n$.

5. Conditional normality: $\epsilon | \mathbf{X} \sim N(\mathbf{0}, \mathbf{\Sigma})$

Hence the conditional likelihood is normal (Gaussian) and can be written as

$$f(\mathbf{y}|\mathbf{X}; \boldsymbol{\beta}, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(- \frac{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{2\sigma^2} \right) \tag{23}$$

and taking logs we have that the log likelihood can now be expressed as

$$l(\boldsymbol{\beta}, \sigma^2; \mathbf{y}|\mathbf{X}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \tag{24}$$

As before we maximise the log likelihood to obtain the first-order conditions as follows

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2; \mathbf{y}|\mathbf{X})}{\partial \boldsymbol{\beta}} = \frac{\mathbf{X}'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})}{\sigma^2} = 0 \tag{25}$$

and

$$\frac{\partial l(\boldsymbol{\beta}, \sigma^2; \mathbf{y}|\mathbf{X})}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \left(\frac{(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})}{2\hat{\sigma}^4} \right) \tag{26}$$

The estimated parameters and variance can now be obtained as:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X}^{-1})\mathbf{X}'\mathbf{y} \tag{27}$$

$$\hat{\sigma}^2 = n^{-1}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \tag{28}$$

The estimated parameters, variances, p-values and confidence intervals can now be obtained. Inference can thus be carried out.

To assess the impact of the variables, we consider our agent based-model and we generate synthetic data $\mathbf{x}_i = \left(\lambda_i, \sigma^2_{p+d,i}, \sigma^2_{u_t,i}, \rho_i, r_{f,i}, \bar{d}_i\right)$ and correponding prices $y_i = p_i$. We are interested in the statistical model given by $y_i = \mathbf{x}_i\boldsymbol{\beta} + \epsilon_i$. By the theory above, we can estimate the coefficient-vector (our desired output) $\boldsymbol{\beta}$ and the sample variance $\hat{\sigma}^2$ (from which we can obtain the associated p-values). This leads to the following results:

Table 3: Statistical model: regression using MLE (Santa Fe model).
Estimates of the impacts of the various parameters on stock prices with associated p-values and 99% confidence intervals.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{d}$
Estimated parameter	-39.23	-6.87	-14.14	-14.86	-812.43	9.89
p-value	0.00	0.00	0.06	0.00	0.00	0.00
Confidence intervals	(-51.35, -27.11)	(-8.99, -4.75)	(-35.12, 6.84)	(-18.99, -10.73)	(-919.66, -705.20)	(7.76, 12.02)

The R^2 of the above model is 82.5% indicating that the model is an excellent fit and 0 does not fall within the confidence intervals (except for stock dividend variation), thus explaining the obtained p-values.

The results of the statistical model above confirm the results discussed above, namely (a) there is a negative relationship between risk-aversion and stock prices, (b) a negative relationship between agent's variance beliefs and stock prices, (c) a negative relationship between stock dividend variation and stock prices, (d) a negative relationship between stock dividend stationarity and stock prices, (e) a negative relationship between the risk free rate and stock prices and (f) a positive relationship between average dividends and stock prices. By inspection of the p-values, all these results are statistically significant at the 1% level apart from the result on stock-dividend variation (although it does have the correct sign and is thus also plausible). The finance intuition for these results were discussed above.

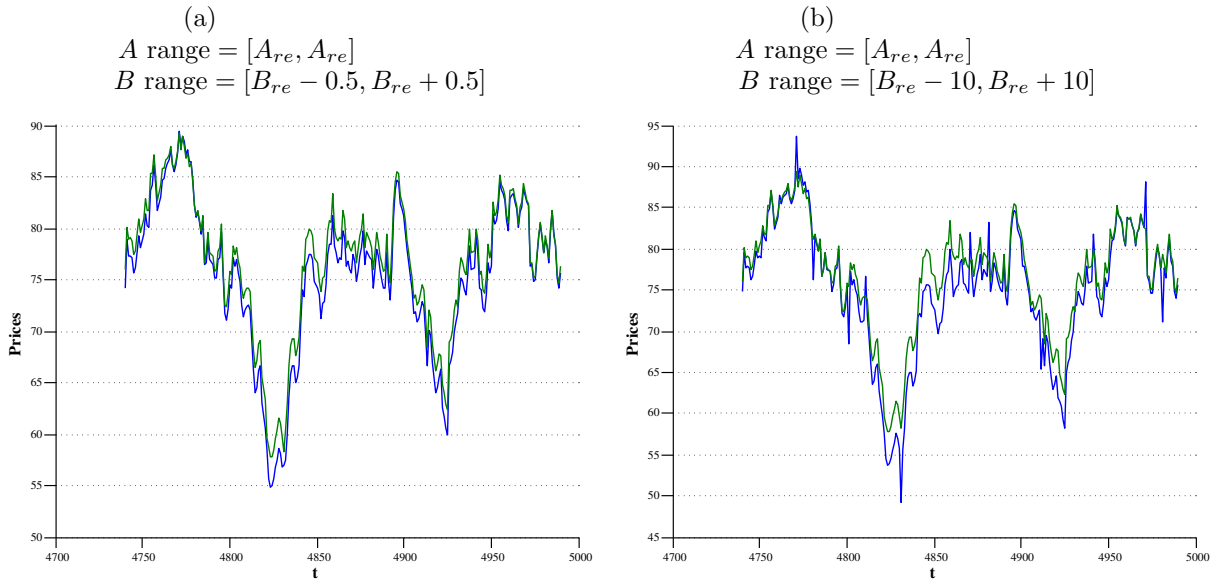
These results provide further evidence on the plausibillity of our model.

3.5 Impact of Agent Beliefs on Simulated Stock Prices

We now consider the impact of agent beliefs regarding various parameters of the computerised stock market on stock prices, using our methodology. We focus on the parameter B range which is defined as the support of the uniform distribution from which the agents' estimate of the parameter

B_{re} is obtained. If agents have more uncertain beliefs about the rationally expected value B_{re}, then B range is wider. Consider figure 3 where we vary B range. In each sub-figure, we plot the last 250 price observations of a single simulation of a realisation of the agent-based model with parameters as set in table 2, but with A range and B range as in the figures below. The green line represents the REE price and the blue line represents the computerised simulated price. We see that mispricing is wider when agents have more uncertain beliefs about the stock market. We see this as the two lines now have a greater deviation from each other when B range is wider. (Note the defferent scales of the graphs.) This is consistent with the empirical finance observation that real-world prices deviate from REE prices. Our result illustrates that agent beliefs is a key driving factor behind stock mispricing. Our parameter variation Monte Carlo technique has enabled us to examine the impact of agent beliefs on stock prices and on stock mispricing, in particular.

Figure 3: Agent-Based Model Stock Prices and Theoretical Stock Prices: mispricing is higher if agents have more uncertain beliefs regarding the stock market (Santa Fe model).



4 Conclusion

Conventional macroeconomic and financial economic models failed to foresee the recent financial crisis. Computerised agent-based models offer an alternate approach. A criticism of computerised agent-based models of financial markets is that the impact of parameters on these models is not well understood. In this paper, we aimed to address this by examining the effect parameters have on the computerised agent-based SFASM model. We achieved this by two methods namely (multivariate) regression using MLE and a parameter variation Monte Carlo simulation technique and by producing a sequence of 72 computerised-agent models and 43200 agent-based model simulations. We showed how simulated prices vary with structural parameters and we estimated these relationships rigorously using (multivariate) MLE regression. We also showed that these relationships between parameters and simulated stock prices are sensible in a financial economic sense. Furthermore, we provided evidence that agent risk-aversion affects stock prices (i.e. risk-on and risk-off) and that agent beliefs impact stock prices.

In future work, we aim to examine the effect parameters have on other computerised agent-based models of financial markets and of other dynamical systems.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of interest: none.

References

Arthur, W.B., Holland, J.H., LeBaron, B., Palmer, R.G. & Tayler, P. (1997). Asset pricing under endogenous expectations in an artificial stock market. *The Economy as an Evolving Complex System II*, 31–60.

Chiarella, C., Dieci, R. & He, X. (2009). Heterogeneity, market mechanisms and asset price dynamics. In *Handbook of Financial Markets: Dynamics and Evolution*, 277–344, Elsevier.

Geanakoplos, J., Axtell, R., Farmer, J.D., Howitt, P., Conlee, B., Goldstein, J., Hendrey, M., Palmer, N.M. & Yang, C.Y. (2012). Getting at systemic risk via an agent-based model of the housing market. *American Economic Review*, **102**, 53–58.

Hommes, C. & Wagener, F. (2009). Complex evolutionary systems in behavioral finance. In *Handbook of Financial Markets: Dynamics and Evolution*, 217–276, Elsevier.

Hommes, C.H. (2006). Heterogeneous agent models in economics and finance. In *Handbook of Computational Economics*, vol. 2, 1109–1186, Elsevier.

LeBaron, B. (2006). Agent-based computational finance. In *Handbook of Computational Economics*, vol. 2, 1187–1233, Elsevier.

LeBaron, B., Arthur, W.B. & Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dyamics and Control*, **23**, 1487–1516.

Lux, T. (2009). Stochastic behavioral asset-pricing models and the stylized facts. In *Handbook of Financial Markets: Dynamics and Evolution*, 161–215, North-Holland.

Palmer, R., Arthur, W.B., Holland, J.H. & LeBaron, B. (1999). An artificial stock market. *Artificial Life and Robotics*, **3**, 27–31.

Palmer, R.G., Arthur, W.B., Holland, J.H., LeBaron, B. & Tayler, P. (1994). Artificial economic life: a simple model of a stockmarket. *Physica D: Complex Phenomena*, **75**, 264–274.

Chapter 3

A Computational Multivariate Agent-Based Model of the Stock Market

In this Chapter, a multivariate (i.e. multiple stock) agent-based model is developed by taking the model of Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997) as a base case, and significantly generalising it. The model of Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997) is a univariate case of this new developed multiple stock model. Our research pursuit is interesting because stock markets consist of multple stocks. In fact, it is mentioned in Palmer *et al.* (1999) that introducing multiple stocks into their framework would be a useful research pursuit.

In the model of this Chapter, the market consists of multiple stocks, a risk-free bond as well as heterogeneous, computationally intelligent agents who trade using algorithms and learn via a multi-stock genetic algorithm and classifier system. To assess the impact of parameters on the multivariate agent-based model developed, the Monte Carlo method of parameter variation developed in Chapter 2 is applied. The impact of parameters such as agent risk aversion, agent beliefs, the underlying stock dividend process and returns to risk-free assets on simulated aggregate stock prices is shown to be plausible, both at an individual stock level as well as at the aggregated stock market level. This is achieved by applying (multivariate) regression, using MLE, and Monte Carlo simulations, to the multivariate agent-based model. By Monte Carlo simulations, stock bubbles are shown to exist and the relationship between agent beliefs and stock mispricing is studied. Furthermore, by the estimation of DCC (1,1) models, the relationship between agent beliefs and stock volatility and cross-correlations is studied. Two new theorems are established and proved, and 43200 agent-based model Monte Carlo simulations are produced as part of this analysis.

This Chapter has been submitted to the journal **Computational Intelligence**, is currently under review and is structured according to the journal's format. The research paper was written by the student under the supervision of Prof S Abelman.

A Computational Multivariate Agent-Based Model of the Stock Market

Shaheen Seedat* and Shirley Abelman

School of Computer Science and Applied Mathematics
University of the Witwatersrand, Johannesburg
Private Bag 3, Wits 2050, South Africa

Abstract

We develop a multivariate computational agent-based model of the equities market. The market comprises computationally intelligent agents, who are heterogeneous and boundedly rational, and a cross-section of publicly listed equities. Agents are computationally intelligent in the sense that they learn over time using a genetic algorithmic method of learning, by combining crossover and mutation, which we model by extending the method of Holland (1983). Stocks are modelled as a claim to a multivariate stochastic dividend process. Our model allows us to examine, *in a multi-asset market*, the impact of parameters such as agent risk aversion, agent beliefs, the underlying stock dividend process and returns to risk-free assets on simulated aggregate stock prices. We achieve this by applying (multivariate) regression, using MLE, and Monte Carlo simulations, to our model. In addition, our agent-based model allows us to analyse the impact of agent beliefs on stock mispricing and the strength of bubbles and to understand, via the estimation of a Dynamic Conditional Correlation (DCC) models, the impact of agent beliefs on stock market volatility and cross-corelations between equity returns. In total, we conduct 43200 model simulations.

Keywords: agent-based modelling, computational finance, computational experiments, (multivariate) regression, Monte-Carlo simulations, evolutionary algorithms

*Corresponding author: seedatshaheen@gmail.com

1 Introduction

One of the recent failures of the modelling of financial markets was the inability to predict the recent financial crisis. It is believed that this is due to shortcomings of conventional methods that were used, see Geanakoplos *et al.* (2012), where agents were assumed homogeneous and rational in order to solve models analytically. This resulted in a lack of realism of these models. *Computerised* agent-based modelling, which falls within the scope of computational agent-based finance, offers an alternate approach.

The literature on agent-based models of financial markets is vast and we refer the interested reader to LeBaron *et al.* (1999), Farmer *et al.* (2005), Raberto *et al.* (2001), Lux & Marchesi (1999), Arifovic (1996), Marengo & Tordjman (1996), Routledge (2001) and Day & Huang (1990). For multivariate models, we specifically refer the interested reader to Westerhoff (2004), Westerhoff & Dieci (2006), Chiarella *et al.* (2007) and Dieci & Westerhoff (2010).

In this journal (Computational intelligence), the following three agent-based models of financial markets have been published. Boer *et al.* (2007) illustrated that there are important differences between discrete-time and continuous-time computational agent-based models of financial markets. Aloud & Fasli (2017) presented a computational agent-based model of the foreign exchange market and compared and contrasted the differences in effects of zero intelligent agents (of both constraint strategy and directional change event strategy types) and agents with a genetic programming approach. Aloud *et al.* (2017) developed a computational agent-based model of the high frequency foreign exchange market and presented a set of sufficient conditions that eventually result in a model with stylised facts that are similar to the real-world foreign exchange market.

In our paper we aim to address the following specific questions through computational experiments: (i) what are the precise relationships, *in a multi-asset market*, between salient stock market parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices, (ii) what is the relationship, *in a multi-asset market*, between traders beliefs of the structure of the market on asset mispricing and (iii) how do stock price volatility and cross-correlations depend on agent beliefs, *in a multi-asset market*? We answer these research questions by developing a multi-asset computational agent-based model of the stock market.

The computational agent-based model we develop is as follows. We consider the single-stock model of Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997), and then generalise and extend it to a multivariate computational agent-based model. In fact, it is mentioned in Palmer *et al.* (1999) that introducing multiple stocks into their framework would be a useful research pursuit.

The view of the stock market presented in LeBaron *et al.* (1999) is one of a market that can be approximated by a computational agent-based model with many different types of agents, each of whom behaves according to a different decision making rule. Their analysis showed that their

computational (single-stock) agent-based model can replicate numerous phenomena documented to occur in real-world financial and stock systems such as fat-tails of the return distribution, volatility persistence and significant trading volumes. Their model is a computational model of the stock market with a single stock and with agent types that are represented by the parameters governing their linear forecasts of prices plus dividends and their associated estimated variances. They incorporated a univariate classifier forecasting system similar to Holland (1983). Dynamic evolutionary learning is modelled using a univariate genetic algorithm that combines mutation and crossover. The stock is modelled as a claim to a dividend process. Prices in this model are set by a specialist who observes the order flow and sets prices to (numerically) clear the market. One of the many fundamental research questions being pursued by LeBaron *et al.* (1999) is whether a genetic algorithmic method of learning combined with a classifier forecasting system can lead to agents learning the linear homogeneous rational expectations equilibrium (REE) over time. Their results suggest that if learning is sufficiently slow, convergence to the linear homogeneous REE is achieved. Extending the single stock model of Palmer *et al.* (1999), LeBaron *et al.* (1999), Palmer *et al.* (1994) and Arthur *et al.* (1997) to a multiple stock setting is interesting and worthwhile as is noted by the authors themselves in Palmer *et al.* (1999).

While the model of LeBaron *et al.* (1999) is multi-agent, it remains single stock (univariate) whereas real-world stock markets are multivariate as they are comprised of multiple stocks. In this paper, we develop a multivariate computational agent-based model of the stock market by modelling the stock market as a cross-section of stocks and thus as a claim to a multivariate stock dividend stochastic process. We model computational agent learning by developing a multivariate genetic algorithm and a multivariate classifier system.

The contribution of our article is five-fold. Firstly, we develop a multivariate computational agent-based model of the stock market that generalises the models of LeBaron *et al.* (1999) and Palmer *et al.* (1999) to a multiasset setting.

Secondly, we show, by (1) using (parameter variation) Monte Carlo simulations and (2) using a statistical model (namely (multivariate) regression with MLE), that our multivariate computational agent-based model is plausible in the sense that the relationships between stock prices and parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk free rate and the underlying stochastic dividend process are sensible in a financial economic sense. The first method (parameter variation) is developed in Seedat & Abelman (2019) and details of the method are contained in that paper and the interested reader is referred to the paper. The second method - (multivariate) regression with MLE - allows us to (a) consider sample variance effects, (b) account for the fact that prices have many response variables (and to account for confounding variables) and (c) offer a systematic approach to model selection. Furthermore, MLE (regression) is used as it has attractive asymptotic properties, namely consistency (convergence in probability to the true value) and asymptotic normality. Theoretical details of the second method are contained in Seedat & Abelman (2019) and the interested reader is referred to the paper.

Thirdly, we show that stock mispricing (i.e. positive and negative bubbles) is greater if agents' beliefs regarding parameters of the stock market have higher variation. These results hold for individual stocks and for the aggregated stock market. Fourthly, we show that stock market volatility appears to increase when agents' stock market beliefs are more likely to have a greater deviation from their rationally expected values and, lastly, we show that different agent beliefs regarding the stock market support different pair-wise correlations between stock returns.

The paper is structured as follows. Section 2 provides an outline of the multivariate computational agent-based model of the stock market that we develop including, *inter alia*, details of the stock market system, the multivariate dividend process, computationally intelligent agents, multiple stock forecasting rules, the multivariate classifier system, price determination and the multivariate genetic algorithm. In Section 3, show by (multivariate) regression and parameter variation Monte Carlo that our model is sensible in a financial economic sense. We also assess stock mispricing, stock market volatility and correlations between stocks. Details of our computational Monte Carlo simulation results are also presented. Finally, Section 4 concludes the paper.

2 Computational Agent-Based Stock Market

In this section, we provide details of our multivariate computational agent-based model.

2.1 Stock market

The stock market consists of $n + 1$ assets including n risky dividend-paying stocks and one risk-free bond. The model is of discrete time and infinite period. The price of stock s at time t is denoted by $P_{s,t}$ and the price vector of all stocks at time t is denoted by $\mathbf{P}_t := (P_{1,t}, ..., P_{n,t})$. The stocks each pay a stream of dividends over the life of the asset and the dividend of each stock paid at time t is given by the dividend vector $\mathbf{D}_t := (D_{1,t}, ..., D_{n,t})$. The risky assets are in fixed finite supply of N shares each. The risk-free asset pays interest at a rate r_f per period.

2.2 Dividend process

We model the dividend process of the stocks as a multivariate Gaussian first-order autoregressive process given by

$$\mathbf{D}_t = \bar{\mathbf{D}} + \rho(\mathbf{D}_{t-1} - \bar{\mathbf{D}}) + \mathbf{u}_t,$$

where $\bar{\mathbf{D}}$ is a constant mean vector and $\mathbf{u}_t$ is assumed to be multivariate Gaussian $\mathbf{N}(\mathbf{0}, \Sigma^2_{u_t})$ and independent over time. $\mathbf{u}_t$ is also assumed to be an innovation or shock term and is thus independent of all other variables. $\mathbf{\Sigma_{u_t}}$ need not be diagonal and thus allows for contemporaneous stochastic dividend effects between stocks.

2.3 Computationally Intelligent Agents

There are N *computationally intelligent* (software) agents, or traders, in this model. Each of them is assumed one-period myopic with Constant Absolute Risk Aversion (CARA) preferences over their one-period ahead investable invests or wealth with constant absolute risk aversion coefficient λ. Each agent i has wealth $W_{t,i}$ at the start of t and must then divide their wealth between the $2n+1$ assets through trading in the stock market. Their asset holdings of the stocks at time t are denoted $\mathbf{X}_{t,i} := (X_{t,i,1}, ..., X_{t,i,n})$ and variation over time in holdings of each asset is achieved by trades in the stock market. Each agent assumes that discounted one-period ahead gross returns are multivariate Gaussian, conditional on their information set $\mathcal{I}_{t,i}$ at time t. That is, each agent i assumes that

$$\frac{1}{1+r_f}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) - \mathbf{P}_t \;\Big|\; \mathcal{I}_{t,i}$$

is multivariate Gaussian. This is one way that agents are computationally intelligent in the sense that they make assumptions consistent with the underlying stock market structure. This assumption, combined with CARA preferences, allows agents to solve for optimal holdings of the assets in terms of the first two moments of the price plus dividend distribution, conditional on $\mathcal{I}_{t,i}$. Since we also assume that the prices at t and the risk-free rate are known to each agent i at t (i.e. that $\mathbf{P}_t \in \mathcal{I}_{t,i}$ and $r_f \in \mathcal{I}_{t,i}$), agents can deduce that

$$\mathbf{P}_{t+1} + \mathbf{D}_{t+1} \;\Big|\; \mathcal{I}_{t,i} \;\sim\; \mathbf{N}(\mu_{\mathbf{t,i}}, \mathbf{\Sigma_{t,i}}),$$

where $\mu_{\mathbf{t,i}} := \mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1})$ and $\mathbf{\Sigma}_{t,i} := \mathbb{V}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1})$ where $\mathbb{E}_{t,i}$ and $\mathbb{V}_{t,i}$ denote agent i's conditional expectation and variance at time t. Each agent is also assumed to know the dividend structure of the economy and hence, $\mathbb{E}_{t,i}(\mathbf{D}_{t+1}) = \bar{\mathbf{D}} + \rho(\mathbf{D}_t - \bar{\mathbf{D}})$ and $\mathbb{V}_{t,i}(\mathbf{D}_{t+1}) = \mathbf{\Sigma_{u_{t+1}}}$.

Since agents have one-period ahead CARA utility over their wealth, their utility functions over their wealth at time $t+1$ has the form $U(W_{t+1,i}) = -\exp(-\lambda W_{t+1,i})$. Given the agents' portfolio choice and trading optimisation problem discussed above, their wealth at $t+1$ satisfies:

$$W_{t+1,i} = \mathbf{X}_{t,i}^{\intercal}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) + (1+r_f)(W_{t,i} - \mathbf{X}_{t,i}^{\intercal}\mathbf{P}_t).$$

Finally, we can thus state each agent i's trading optimisation problem at each time t concisely as:

$$\max_{\mathbf{X}_{t,i}} \mathbb{E}_{t,i}\Bigg(-\exp\Big\{ -\lambda\Big(\mathbf{X}_{t,i}^{\intercal}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) + (1+r_f)(W_{t,i} - \mathbf{X}_{t,i}^{\intercal}\mathbf{P}_t)\Big)\Big\}\Bigg).$$

Now define:

$$\mathbf{X}^*_{t,i} \equiv \arg\max_{\mathbf{X}_{t,i}} \mathbb{E}_{t,i}\Bigg(-\exp\Big\{ -\lambda\Big(\mathbf{X}^{\intercal}_{t,i}(\mathbf{P}_{t+1}+\mathbf{D}_{t+1}) + (1+r_f)(W_{t,i} - \mathbf{X}^{\intercal}_{t,i}\mathbf{P}_t)\Big)\Big\}\Bigg).$$

This has the solution given by

$$\mathbf{X}^*_{t,i} = \frac{\mathbf{\Sigma}^{-1}_{t,i}}{\lambda}\Bigg(\mu_{\mathbf{t},\mathbf{i}} - (1+r_f)\mathbf{P}_t\Bigg) \quad t = 1, 2, ..., T-1. \tag{1}$$

which follows as each agent conjectures that one period ahead wealth is univariate Gaussian condition on their current information set which means that, as utility is CARA, one period ahead utility is conditionally log-Gaussian. As each agent can compute their expected (log-normal) utility by the moment generating function of a univariate Gaussian random variable, they can thus compute optimal stock holdings. This yields the above result.

Each agent is assumed to initially have 1 share of each stock. Given the set up of this system, the only source of heterogeneity for agents is their forecasts of futures prices and dividends (i.e. $\mu_{t,i}$) and their estimates of the variance matrix of these ($\mathbf{\Sigma}_{t,i}$). That is, agents differ in their expectation formation and their assessment of risk in the market.

In sections 2.5 and 2.7 , we provide additional details regarding the computational intelligence of the agents.

2.4 Homogeneous linear rational expectations equilibrium

In some of the simulation work in this paper, we will compare the time series of prices produced by the multivariate computational agent-based model to theoretical (i.e. homogeneous rational expectations) prices.

In a homogeneous equilibrium of the stock market, all stock market agents are perfectly identical. If all stock market agents are identical, then by the no-trade theorem there is no trade in a stock market system *for each of the N stocks* and thus agents will, in every period, hold their initial stock allocation (i.e 1 share). A linear rational expectations equilibrium is one where agents are optimizing and the price function as a function of underlying variables (here dividends) is linear.

To solve for a linear REE in our multiple stock model, let us conjecture that a linear solution exists,

$$\mathbf{P}_t = \mathbf{A}\mathbf{D}_t + \mathbf{B}$$

then enforce agent optimisation and the no trade theorem constraint ($\mathbf{X}^*_{t,i} = \mathbf{1}$ for all agents and times). Solving this system yields the solution

Theorem 1 Under the maintained assumptions above, the linear REE solution satisfies:

$$\mathbf{P}_t = \mathbf{A}\mathbf{D}_t + \mathbf{B} \tag{2}$$

where

$$\mathbf{A} = \frac{\rho \mathbf{I}}{1 + r_f - \rho} \tag{3}$$

$$\mathbf{B} = \frac{(1-\rho)(\mathbf{A} - \mathbf{I})\bar{\mathbf{D}} - \lambda \mathbf{\Sigma}_{t,i}^{-1}\mathbf{1}}{r_f}. \tag{4}$$

Proof. *Recall that the dividend process satisfies*

$$\mathbf{D}_t = \bar{\mathbf{D}} + \rho(\mathbf{D}_{t-1} - \bar{\mathbf{D}}) + \mathbf{u}_t,$$

We conjecture that a linear solution of the form below exists:

$$\mathbf{P}_t = \mathbf{A}\mathbf{D}_t + \mathbf{B}$$

Recall, we have that optimal stock holdings satisfy:

$$\mathbf{X}_{t,i}^* = \frac{\mathbf{\Sigma}_{t,i}^{-1}}{\lambda}\left(\mu_{\mathbf{t,i}} - (1 + r_f)\mathbf{P}_t\right)$$

where

$$\mu_{\mathbf{t,i}} := \mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1})$$
$$\mathbf{\Sigma}_{t,i} := \mathbb{V}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1})$$

By the no-trade condition we have that $\mathbf{X}_{t,i}^* = 1 \ \forall i, t$. *Now*

$$\begin{aligned}
\mu_{\mathbf{t,i}} &= \mathbb{E}_{t,i}(\mathbf{P_{t+1}} + \mathbf{D_{t+1}}) \\
&= \mathbb{E}_{t,i}(\mathbf{A}\mathbf{D}_{t+1} + \mathbf{B} + \mathbf{D}_{t+1}) \\
&= \mathbb{E}_{t,i}((\mathbf{A} + \mathbf{I})\mathbf{D}_{t+1} + \mathbf{B}) \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}(\mathbf{D}_{t+1}) + \mathbf{B}
\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}_{t,i}(\mathbf{D}_{t+1}) &= \mathbb{E}_{t,i}(\bar{\mathbf{D}} + \rho(\mathbf{D}_t - \bar{\mathbf{D}}) + \mathbf{u}_{t+1}) \\ &= \bar{\mathbf{D}} + \rho(\mathbf{D}_t - \bar{\mathbf{D}}) \\ &= (1-\rho)\bar{\mathbf{D}} + \rho\mathbf{D}_t\end{aligned}$$

Hence

$$\mu_{t,i} = (\mathbf{A} + \mathbf{I})((1-\rho)\bar{\mathbf{D}} + \rho\mathbf{D}_t) + \mathbf{B}$$

Optimal holdings of stocks and the no-trade theorem result imply that

$$\begin{aligned}\lambda\boldsymbol{\Sigma}_{t,i}\mathbf{I} &= \mu_{t,i} - (1+r_f)\mathbf{P}_t && (5)\\ &= (\mathbf{A}+\mathbf{I})((1-\rho)\bar{\mathbf{D}} + \rho\mathbf{D}_t) + \mathbf{B} - (1+r_f)\mathbf{P}_t && (6)\\ &= (\mathbf{A}+\mathbf{I})((1-\rho)\bar{\mathbf{D}} + \rho\mathbf{D}_t) + (\mathbf{P_t} - \mathbf{AD_t}) - (1+r_f)\mathbf{P}_t && (7)\\ &= (\mathbf{A}+\mathbf{I})((1-\rho)\bar{\mathbf{D}} + \rho\mathbf{D}_t) - \mathbf{AD}_t - r_f\mathbf{P}_t && (8)\end{aligned}$$

Solving for prices yields

$$\mathbf{P}_t = \frac{(\mathbf{A}+\mathbf{I})\Big(((1-\rho)\bar{D}\Big) - \lambda\boldsymbol{\Sigma}_{t,i}}{r_f} + \frac{\rho(\mathbf{A}+\mathbf{I}) - \mathbf{A}}{r_f}\mathbf{D}_t$$

Hence the solution for $\mathbf{B}$ *is*

$$\mathbf{B} = \frac{(\mathbf{A}+\mathbf{I})\Big(((1-\rho)\bar{\mathbf{D}}\Big) - \lambda\boldsymbol{\Sigma}_{t,i}}{r_f}$$

and the solution for $\mathbf{A}$ *is obtained from*

$$\mathbf{A} = \frac{\rho(\mathbf{A}+\mathbf{I}) - \mathbf{A}}{r_f}$$

from which the solution for $\mathbf{A}$ *is*

$$\mathbf{A} = \frac{\rho \mathbf{I}}{1 + r_f - \rho}$$

and

$$\mathbf{P}_t = \mathbf{A}\mathbf{D}_t + \mathbf{B}$$

■

This gives us a process of theoretical stock prices in our multivariate world based on a homogeneous linear rational expectations equilibrium (REE). Note that the homogeneous linear REE that we derive in our paper is a significant generalisation of the one derived in LeBaron *et al.* (1999). This is due to our multiple stock representation. In computational experiments, we use the linear homogeneous pricing solution to obtain theoretical prices from which the simulated prices obtained from the computational agent-based model prices are compared to obtain an estimate of mispricing and, thus, the strength of bubbles (negative and positive).

Based on this REE solution, we can compute optimal mean squared error (MSE) forecasts of prices plus dividends in the class of linear forecasts. This has the solution given by the following result:

Theorem 2 Under the maintained assumptions above, the optimal mean squared error forecasts of prices plus dividends in the class of linear forecasts satisfy:

$$\mathbb{E}_{t,i}\big(\mathbf{P_{t+1}} + \mathbf{D_{t+1}}\big) = \rho\big(\mathbf{P_t} + \mathbf{D_t}\big) + (1-\rho)\Big((\mathbf{A} + \mathbf{I})\bar{\mathbf{D}} + \mathbf{B}\Big)$$

Proof.

$$\mathbf{P}_t = \mathbf{A}\mathbf{D}_t + \mathbf{B}$$

with $\mathbf{A}$ *and* $\mathbf{B}$ *computed as above.*

The MSE optimal forecast is the conditional expectation, conditional on everything inside the information set of $\mathbb{E}_{t,i}$.

Hence, the MSE optimal forecast of $\mathbf{P}_{t+1} + \mathbf{D}_{t+1}$ *is given by*

$$\begin{aligned}
\mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) &= \mathbf{E}_{t,i}(\mathbf{A}\mathbf{D}_{t+1} + \mathbf{B} + \mathbf{D}_{t+1}) \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}(\mathbf{D}_{t+1,s_{t+1}}) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}(\bar{\mathbf{D}} + \rho(\mathbf{D}_t - \bar{\mathbf{D}}) + \mathbf{u}_{t+1}) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}(\bar{\mathbf{D}} + \rho(\mathbf{D}_t - \bar{\mathbf{D}})) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})((1-\rho)\bar{\mathbf{D}} + \rho\mathbf{D}_t) + \mathbf{B} \\
&= \rho(\mathbf{A} + \mathbf{1})\mathbf{D}_t + \Big((\mathbf{A} + \mathbf{I})((1-\rho)\bar{\mathbf{D}} + \mathbf{B}\Big)
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{A} + \mathbf{I})\mathbf{D}_t &= \mathbf{D}_t + \mathbf{A}\mathbf{D}_t \\
&= \mathbf{D}_t + \mathbf{P}_t - \mathbf{B} \\
&= (\mathbf{P}_t + \mathbf{D}_t) - \mathbf{B}
\end{aligned}$$

Hence,

$$\rho(\mathbf{A} + \mathbf{I})\mathbf{D}_t = \rho(\mathbf{P}_t + \mathbf{D}_t) - \rho\mathbf{B}$$

Thus

$$\mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) = \rho(\mathbf{P}_t + \mathbf{D}_t) + (1-\rho)\left((\mathbf{A} + \mathbf{I})\bar{\mathbf{D}} + \mathbf{B}\right)$$

■

We will denote:

$$a_{re} = \rho$$

and

$$\mathbf{b}_{re} = (1-\rho)\left((\mathbf{A}+\mathbf{I})\bar{\mathbf{D}}+\mathbf{B}\right)$$

Linear forecasts will be of the form

$$\hat{\mathbb{E}}_{t,i,j}(\mathbf{P}_{t+1}+\mathbf{D}_{t+1}) = a_{i,j}(\mathbf{P}_t+\mathbf{D}_t)+\mathbf{b}_{i.j}$$

with variance estimate $\Sigma_{t,i,j}$ and where the index j will denote "rules" which we shall describe later.

Note that the optimal mean squared error forecasts of prices plus dividends in the class of linear forecasts that we derive in our paper is a significant generalisation of the one derived in LeBaron *et al.* (1999). This is due to our multiple stock representation.

2.5 Multivariate stock forecasting rules and multivariate classifier system

This section provides further details regarding the computational intelligence of agents in our model. We generalise the method of LeBaron *et al.* (1999) to a multivariate system.

In our *multivariate* computational agent-based, a type of agent refers to a specific way of computing the *multivariate* stock mean $\mu_{t,i}$ and the *multivariate* stock covariance matrix $\mathbf{\Sigma}_{t,i}$. Each agent i is assumed to have an information set at t that consists of various publicly available information at t about the state of the market system. Such information at t can include the multivariate stock price vector $\mathbf{P}(t)$ at t, the multivariate dividend vector $\mathbf{D}(t)$ at t, moving averages of the multivariate stock price series and so on. *For each asset* s, each computationally intelligent agent condenses this information into world bits and each agent is assumed to have the following twelve world bits, *for each stock* s:

1. $r_f P_{s,t}/D_{s,t} > 1/4$
2. $r_f P_{s,t}/D_{s,t} > 1/2$
3. $r_f P_{s,t}/D_{s,t} > 3/4$
4. $r_f P_{s,t}/D_{s,t} > 7/8$
5. $r_f P_{s,t}/D_{s,t} > 1$
6. $r_f P_{s,t}/D_{s,t} > 9/8$
7. $P(s,t) > \mathrm{MA}_5(P_{s,t})$
8. $P(s,t) > \mathrm{MA}_{10}(P_{s,t})$

9. $P(s,t) > \text{MA}_{100}(P_{s,t})$
10. $P(s,t) > \text{MA}_{500}(P_{s,t})$
11. On: 1
12. Off: 0

where $\text{MA}_v(P_{s,t})$ denotes the v-period prior moving average of the price series of stock s. At each point in time, each world bit corresponding to each stock s is either true or false. Bits that involve just the price series are referred to as technical bits; 7-10 are thus technical bits. Bits involving dividends are referred to as fundamental bits; 1-6 are fundamental bits, while 11 and 12 are control bits and used as experimental controls.

Each agent will now make *multivariate* linear forecasts of the form

$$\mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) = \mathbf{a}_i(\mathbf{P}_t + \mathbf{D}_t) + \mathbf{b}_i.$$

Note that this is more general than the linear REE solution we computed before.

Furthermore, agents use a *multivariate* classifier forecasting system to obtain their forecast of future stock prices plus dividends for the cross-section of stocks. This means forecasts at t are state-dependent where the state is described by whichever bits the agent i decides is important for each respective stock. For each stock s, each agent is assumed to have 100 such states that they feel are salient. A state for an agent i and for a given stock s is a ternary twelve string of the symbols $\{0, 1, \#\}$. The k^{th} component of the ternary twelve string of a state corresponds to the k^{th} world bit for that stock. A value of 0 means that the world bit for that stock must be false for the condition to match. A value of 1 means the world bit for that stock must be true and a value of # means indifference to that world bit. So, if we were in a world of only five world bits, a ternary string of #10#1 for stock 1 would match the world bit string 11001 for stock 1 but not 10001. For each stock and for each agent, rules consist of ternary strings, as described, as well as a forecasting equation of the future stock price plus dividend (and an estimated variance) connected with that rule. From now on, we shall denote rules by the subscript j. Rule j for stock s will refer to a particular world state and, given the form of the linear REE, the following forecasting equation (which activates when that world state is activated for stock s)

$$\mathbb{E}_{t,i,j}(P_{s,t+1} + D_{s,t+1}) = a_{s,i,j}(P_{s,t} + D_{s,t}) + b_{s,i,j}$$

with associated covariance estimate $\sigma^2_{s,i,j}$. This is the essence of the multivariate forecasting rules and multivariate classifier system that computationally intelligent agents implement.

For each rule j (and for each agent i and stock s) its associated parameters $a_{s,i,j}$ and $b_{s,i,j}$ are initially set to random values uniformly distributed in the one dimensional intervals A range and the s^{th} component of $\mathbf{B}$ range, respectively, where A range and the $\mathbf{B}$ range are intervals (one

dimensional and multi-dimensional, respectively) centered about the mean-squared error optimal solutions in the homogeneous linear rational expectations equilibrium. Combined with this uniform distribution, A range and $\mathbf{B}$ range thus represent the beliefs of the agents regarding two stock market parameters, namely A_{re} and $\mathbf{B}_{re}$.

At each point in time t, and for each stock s, agent i may have several rules that are activated by the world bit string at t for asset s. Agent i chooses for stock s the rule that has least variance, a measure of forecasting accuracy and from this optimal rule the agent can now compute their optimal forecast of prices plus dividends (i.e. their optimal computation of $\mu_{\mathbf{t},\mathbf{i}} = \mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1})$) and also has an estimate of the variance $\mathbf{\Sigma}_{t,i}$ associated with this forecast. This optimal forecast will have the form:

$$\mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1}) = \mathbf{a}_i^*(\mathbf{P}_t + \mathbf{D}_t) + \mathbf{b}_i^*$$

where $\mathbf{a}_i^*$ and $\mathbf{b}_i^*$ are each a stacking of the optimal a's and b's across the cross-section of stocks.

Hence, agents can compute their optimal trading positions using the above result and agent optimisation given by:

$$\mathbf{X}_{t,i}^* = \frac{\mathbf{\Sigma}_{t,i}^{-1}}{\lambda}\left(\mu_{\mathbf{t},\mathbf{i}} - (1 + r_f)\mathbf{P}_t\right) \quad t = 1, 2, ..., T-1. \tag{9}$$

2.6 Trading and price determination

Prices are set by a *specialist* who observes the demands (order flow) of the stocks and sets $\mathbf{P}_t$ such that all the n stock markets clear (demand equals supply for each stock). That is, the specialist sets prices such that $\sum_{i=1}^{N} \mathbf{X}_{t,i}^* = \mathbf{1}$.

After trading, each agent i updates the accuracy of their forecast rules for each asset s according to an exponentially weighted average of squared forecast error,

$$\nu_{s,t,i,j}^2 = \left(1 - \frac{1}{\tau}\right)\nu_{s,t-1,i,j}^2 + \frac{1}{\tau}\left(P_{s,t} + D_{s,t} - (a_{i,j}(P_{s,t-1} + D_{s,t-1}) + b_{s,i,j})\right)^2$$

and where τ is usually set to 75.

2.7 Evolutionary learning and multivariate genetic algorithm

To allow for evolutionary learning, agents implement a multivariate genetic algorithmic learning approach. This is an additional manner in which agents, in our model, are computationally intelligent. We again generalise the univariate method of Palmer *et al.* (1999) to a multivariate system. Insofar as we are aware, this is the first attempt to develop a multivariate version of the

univariate genetic algorithm developed by Palmer *et al.* (1999). The details of the multivariate genetic algorithm we use are summarised neatly as follows. After each K periods, agents update their rules according to the following genetic algorithm. For each agent i, *and for each stock* s, 20 of their worst rules are removed according to a penalised variance criterion and exactly 20 new rules are added. These 20 new rules are obtained by the set of surviving 81 rules and are added through one of two approaches: crossover or mutation. Crossover occurs with probability 0.1 and mutation with probability 0.9. Crossover means selecting two parent rules, according to tournament selection and uniformly crossing the two rules - bit by bit for the ternary string and real number by real number for the vectors $(a_{s,i,j}, b_{s,i,j}, \sigma^2_{s,i,j})$. Mutation means selecting a single parent by tournament selection and modifying that parent's string randomly bit by bit and modifying that parent's $(a_{s,i,j}, b_{s,i,j}, \sigma^2_{s,i,j})$ by random amounts. The probabilities we use, here, are kept as in Palmer *et al.* (1999) to maintain some degree of uniformity.

3 Simulation Results

Section 2 provided details of the various pertinent features of the multivariate computational agent-based developed in this paper. The incorporation of multivariate stock forecasting rules, multivariate classifier system and a multivariate genetic algorithm mean that our model is computational in nature leading to the necessity of computational experiments and simulations to understand the dynamics of the financial market that we are modelling.

In this section, we apply two methods to understand whether the model simulated stock prices vary with salient parameters (such as agent risk aversion) in a plausible manner. We also use Monte Carlo simulations to understand whether mispricing of stocks depends on agent beliefs. We conduct such analyses for individual stocks in a cross-section of stocks as well as by aggregating to the market level.

3.1 Method 1: Parameter Variation Monte Carlo Simulation

One of the criticisms leveled against computational agent-based modeling is that it does not allow a researcher to examine the impact of parameters on the model, see LeBaron (2006) and Aloud *et al.* (2017). We now summarise the method developed in Seedat & Abelman (2019) that allows one to examine the impact of parameters on an agent-based model. The method is general and can be applied to any computerised agent-based model. This is one of two methods that we will use. The second method, (multivariate) regression, is used later on.

Without loss of generality, let $\mathcal{M}(\theta)$ denote an computational agent-based model of the stock market where our interest is in the structural parameters given by θ where $\theta \in \mathbb{R}^M$ are the structural parameters. Let $\alpha \in \mathbb{R}$ denote one of the parameters, for example λ, and assume we are interested in how it affects a given observable of $\mathcal{M}(\theta)$. Let us assume that the observable of interest is stock

prices and, in particular, the time t-th price obtained from the model. We introduce some notation: let $\mathcal{M}(\theta_{\alpha_i})$ denote a realisation of the computational agent-based model which corresponds to the parameter values θ_{α_i} with corresponding parameter value for α given by α_i. For $j \neq i$, let $\mathcal{M}(\theta_{\alpha_j})$ denote a realisation of the model that has all parameter values the same as $\mathcal{M}(\theta_{\alpha_i})$ except the parameter value assigned to α where the new parameter value is α_j instead of α_i.

Let $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$ denote the price of a stock, at time t, obtained from a given realisation of the model, $\mathcal{M}(\theta_{\alpha_i})$ and from a given simulation j of this realised computational agent-based model. If we are interested in how α affects the time t-th stock price of the computational agent-based model $\mathcal{M}(\theta)$ then we want to know how α affects $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ where the expectation operator $\mathbb{E}$ is taken over all simulations j. This is because $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$, for a given simulation j, may contain significant noise. In an ideal world, we know $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$. In the real-world, we do not know $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ but we can form a reliable point estimate of it, given by $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_i}, k))$ for N_1 large. The rational for this is provided by the theorem below and the proof is contained in Seedat & Abelman (2019) .

Theorem 3 *Estimate of stock price, for a given agent-based model and a given set of parameters. Let $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$ denote the price of a stock, at time t, obtained from a given realisation of the Santa Fe agent-based model, $\mathcal{M}(\theta_{\alpha_i})$ and from a given simulation j of this realised agent-based model. Then a reliable estimate of $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_i}, j))\big)$ is given by $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_i}, k))$ for N_1 large.*

Seedat & Abelman (2019) thus defines the method of parameter variation of an agent-based model as follows where they use stock prices as the simulated observable variable of interest.

Definition 4 Parameter Variation of an Agent-Based Model of the Stock Market.
Let $\mathcal{M}(\theta)$ denote a computational agent-based model of a stock market with parameters $\theta \in \mathbb{R}^M$ and where one parameter is $\alpha \in \mathbb{R}$. Let α_m and α_n be two distinct parameter *values* of α. The impact on stock prices of a parameter change from α_m to α_n is given by the change from $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_m}, j))\big)$ to $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_n}, j))\big)$ where $p(t, \mathcal{M}(\theta_{\alpha_i}, j))$ denotes the price of a stock, at time t, obtained from a given realisation of the agent-based model, $\mathcal{M}(\theta_{\alpha_i})$ and from a given simulation j of this realised agent-based model.

Remark 5 By the above theorem, we can observe that the sample estimates $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_m}, k))$ and $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_n}, k))$, for large N_1, are reliable estimates of $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_m}, j))\big)$ and $\mathbb{E}\big(p(t, \mathcal{M}(\theta_{\alpha_n}, j))\big)$. Thus, $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_m}, k))$ and $\frac{1}{N_1}\sum_{k=1}^{N_1} p(t, \mathcal{M}(\theta_{\alpha_n}, k))$, for large N_1, can be used to understand the impact of α on stock prices of an agent-based model. t is fixed in both sample estimates.

Variation of simulated stock market prices with salient parameters

We now apply our theory above to our multivariate computational agent-based model to understand whether simulated prices of our model vary plausibly with important parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the first moment of the underlying stochastic dividend process. We will restrict consideration to the case where each asset has the same variance of stochastic dividend shocks and where agent's covariation beliefs are consistent with this. Hence, let σ_u^2 denote the variance of individual dividends and let ${\sigma^2}_{P+D}$ denote each agents' belief regarding the variance of each stock's future price plus dividend. Our results do not change if these considerations are relaxed.

We analyse the relationship between stock prices and salient underlying parameters, within our simulated multivariate computational agent-based environment. Key parameters that determine stock prices in our model are agents' risk aversion coefficient (λ), agents' beliefs regarding the variance of future prices plus dividends (${\sigma^2}_{P+D}$), the variance of the underlying stock dividend stochastic process (σ_u^2), the persistence of the stochastic dividend process (ρ), the risk-free rate (r_f) and the mean of the stock dividend stochastic process ($\bar{\mathbf{D}}$). We study the effects of these parameters on the agent-based model simulated price series. We find that two other parameters have important effects on stock prices, in this artificial stock market context, namely A range and $\mathbf{B}$ range which are the parameters that represent the beliefs of the agents regarding the stock market parameters a_{re} and $\mathbf{b}_{re}$. The effects of A range and $\mathbf{B}$ range on simulated stock prices will be examined in the following subsection.

To analyse the relationship between stock prices and our first group of salient underlying parameters, we start by fixing a base case (see Table 1) of the simulated model with the following parameters. To keep the analysis feasible, we simulate a two-stock version of the model.

Table 1: Parameters: Base Case (Multivariate model).

Parameter	**Base value**
λ	0.5
$\mathbf{\Sigma^2_{P+D}}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
$\mathbf{\Sigma^2_{u_t}}$	$\begin{pmatrix} 0.07429 & 0.037145 \\ 0.037145 & 0.07429 \end{pmatrix}$
ρ	0.95
r_f	0.10
$\bar{\mathbf{D}}$	$\begin{pmatrix} 30 \\ 20 \end{pmatrix}$
A range	$[A_{re} - 0.1, A_{re} + 0.1]$
$\mathbf{B}$ range	$[\mathbf{B}_{re} - \mathbf{1}, \mathbf{B}_{re} + \mathbf{1}]$
n	2
N	25
T	2750
K	11

The base values chosen for λ, ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, ρ and r_f are the same as those that were used by (LeBaron *et al.*, 1999) to maintain some degree of consistency but we extend to a multivariate

setting. Note that in our paper we allow for correlated shocks among the cross section of stocks and we also allow for correlated prices plus dividends in the cross section. We assume all covariation terms are half the respective variation terms although any other assumption can be made here. To ensure agents do not make systematic errors in their decision making, A range and $\mathbf{B}$ range are centred about our multivariate REE solutions, namely a_{re} and $\mathbf{b}_{re}$. T and K are chosen to ensure $T/K = 250$, as in LeBaron *et al.* (1999), whilst ensuring that all models simulated in this section can be simulated in a reasonable amount of time. Lastly, we assume that each stock has a heterogeneous stochastic dividend mean.

Given the set of base case parameters, we then simulate the model numerous times by varying, one at a time, each of λ, ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, ρ, r_f and, for each stock s, $\bar{D}_s$. In particular, we individually discretise the parameter spaces of each of λ, ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, r_f and $\bar{D}_s$ into equispaced sub-intervals over plausible ranges as given in Figures 1 to 6. Since ρ measures the degree of stock dividend persistence, as $\rho \to 1$, we approach a non-stationary dividend process driven by a Brownian motion. This leads to highly non stationary prices for values of ρ very close to 1. For this reason, as ρ approaches 1, we divide its parameter space even more finely to obtain a clearer impact of ρ on stock prices. For none of the other parameters is it necessary to divide any of its subspaces more finely.

To illustrate how we conducted our analysis, consider λ as our parameter being varied with a parameter range space of $[0.1, 2.5]$ which contains its base value. First consider $\lambda = 0.1$ and set all the other parameters to their base case values as in Table 1. Then simulate the model over the time span 0 to T and obtain the simulated stock prices for stock 1 and 2 as well as the market (defined as the equally-weighted average price of the two) at time T. We then conduct 599 additional simulations and obtain each of these prices which we plot at the value $\lambda = 0.1$ in Figure 1 (a). The reason we simulate the model 600 times is because we can then apply a pointwise law of large numbers to obtain a precise estimate of the stock price corresponding to $\lambda = 0.1$. (The rational for this is given by our parameter variation Monte Carlo method which is outlined in the prior section.) We plot this price estimate in Figure 1 (b) at the value $\lambda = 0.1$. By continuing in this fashion we can complete the graphs 1 (a) and 1 (b). We then repeat this exercise for stock 2 and the stock market which gives the plots contained in Figures 3 (a) and (b) as well as Figures 5 (a) and (b), respectively.

We repeat this entire exercise for each of the parameters of interest, namely ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, ρ, r_f and $\bar{D}_s$. This now allows us to assess the relationship between simulated stock prices and salient underlying parameters in our multiple stock agent-based model.

We can see, from Figures 1 (a)-(b) and Figures 3 (a)-(b), that as the level of risk aversion of agents rises, stock prices for each stock fall monotonically. This is sensible in a financial economic sense since higher risk aversion implies that agents possessing risk averse CARA utility have greater preference for, *ceteris paribus*, risk-free assets to stocks. This then averages through to the aggregated stock market level as depicted in Figures 5 (a)-(b). We can also see, from Figures

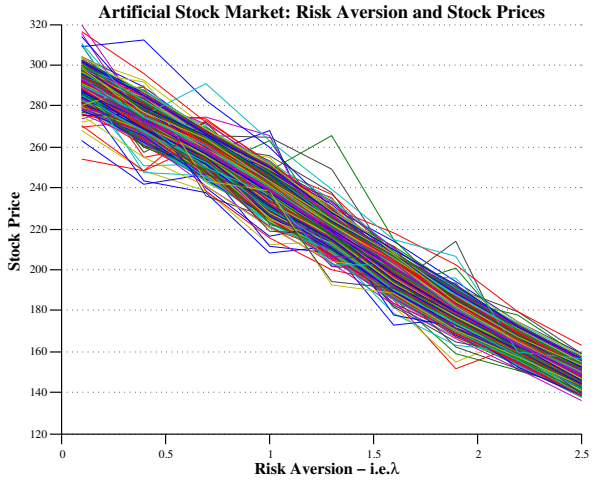
1 (c)-(d) and Figures 3 (c)-(d), that as agents' variance beliefs of the future price plus dividend of stocks increase, the stock price falls for each individual stock. This is also reasonable given the assumption of risk averse CARA utility; if agents expect the stock to be more volatile (higher ${\sigma^2}_{P+D}$), they have reduced preference for stocks as opposed to risk free assets such as the bond in this model. In our multi-stock world, this relationship also averages through to the aggregated stock market level as depicted in Figures 5 (c)-(d).

For fairly similar reasons relating to the structure of CARA utility of investors and traders, we should expect a negative relationship between the risk-free rate and stock prices and a positive relationship between the mean of the dividend process of each respective stock and the stock price of that stock. This model is also consistent with both these expectations as depicted in Figures 2 (c)-(f), 4 (c)-(f) and 6 (c)-(f).

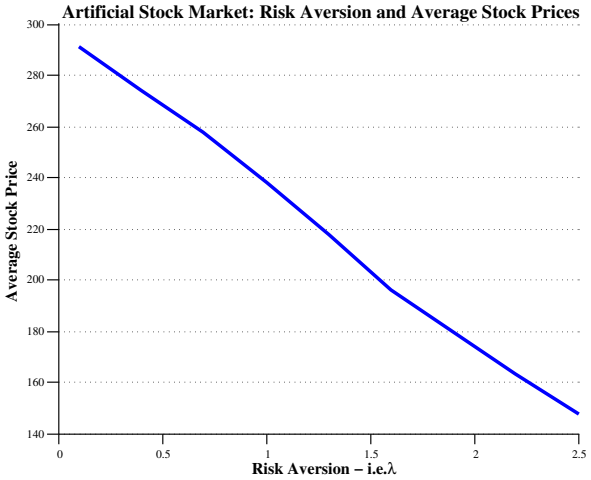
To the best of our knowledge, finance theory does not provide strong postulates regarding the relationship between stock dividend variation and stock prices and between stock dividend persistence and stock prices. Nevertheless, we believe it is useful to provide some insights into this, within the multivariate computational agent-based model we have developed. From Figures 1 (e), 3 (e) and 5 (e), higher stock dividend variation seems to lead to lower stock prices (for individual stocks in the cross-section and for the aggregated market) although this effect is not as strong as it was for agents' variance beliefs and there is some noticeably substantial stochastic variation. More insightful, perhaps, is the fact that stock dividend variation (and not variance beliefs on the part of agents) is the key parameter that determines the *variation* of stock prices themselves, as can be seen from Figures 1 (f), 3 (f) and 5 (f). As the stock dividend process approaches non-stationarity, we see, from Figures 2 (a) - (b), 4 (a) - (b) and 6 (a) - (b) that stock prices fall dramatically.

Figure 1: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel A.

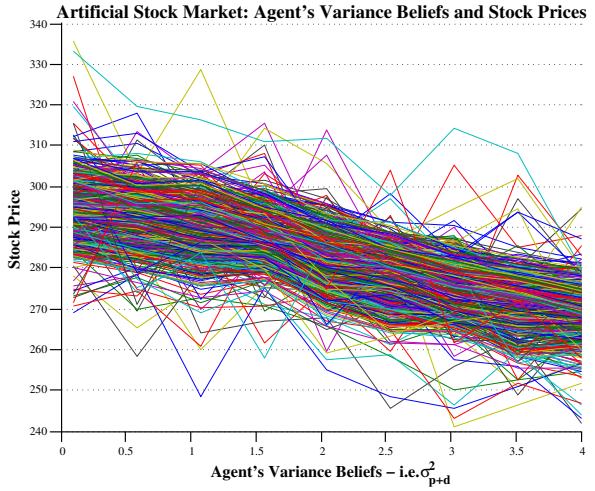
(a)



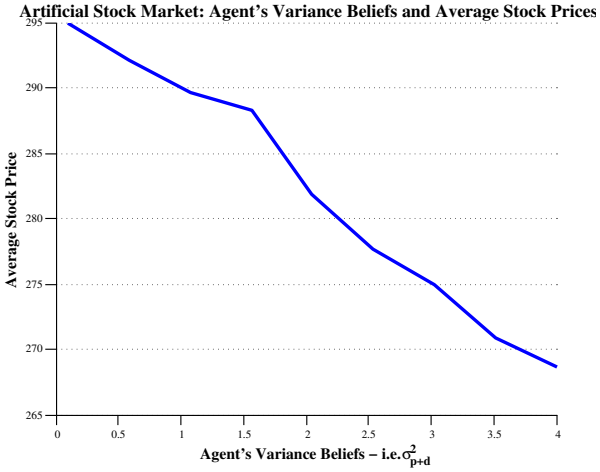
(b)



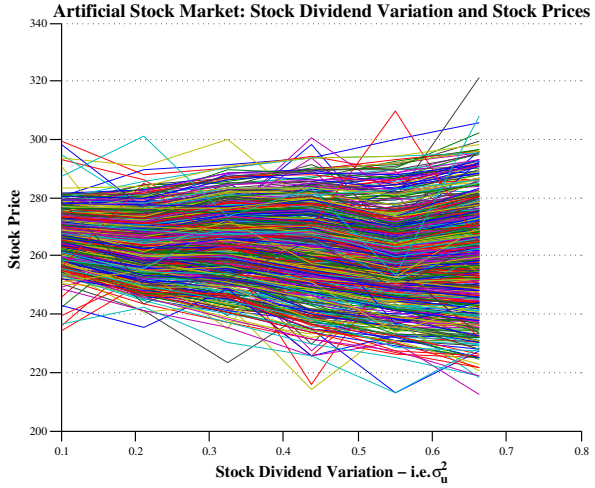
(c)



(d)



(e)



(f)

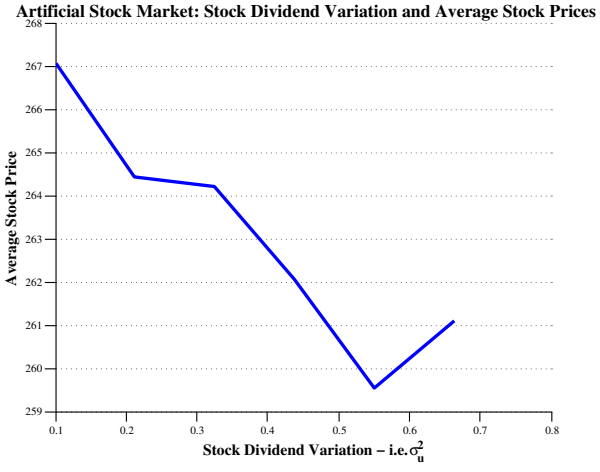
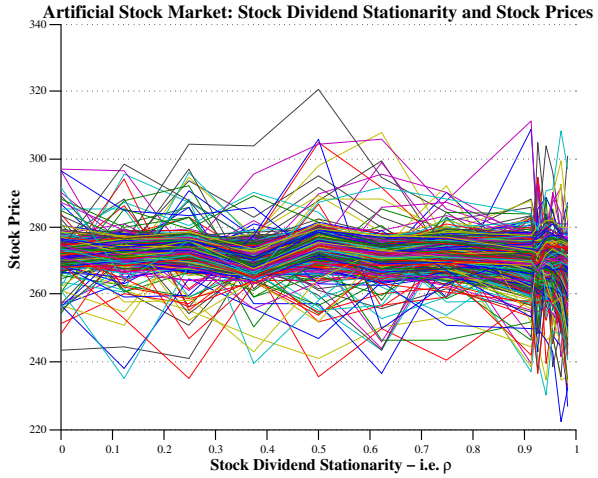
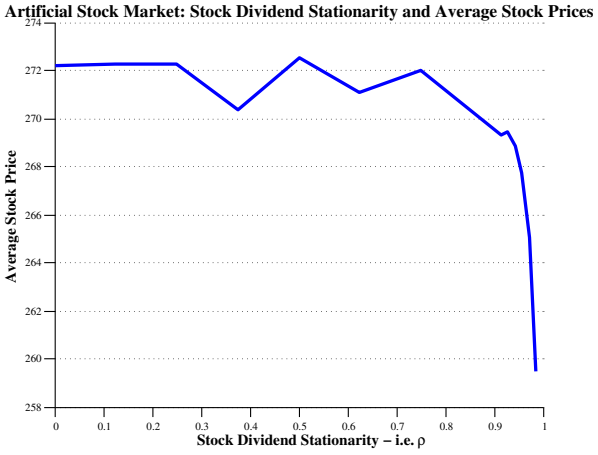


Figure 2: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel B.

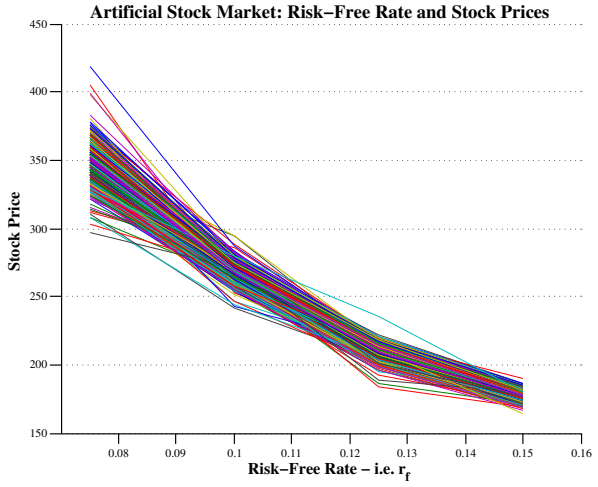
(a)



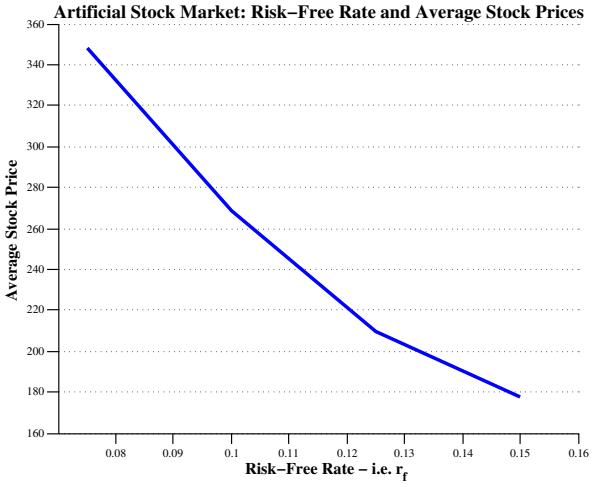
(b)



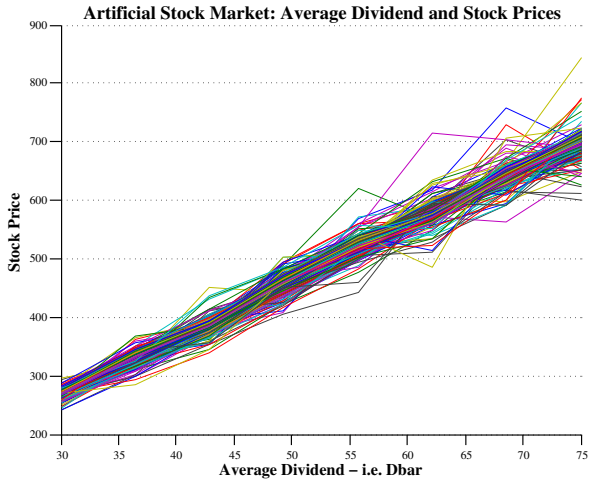
(c)



(d)



(e)



(f)

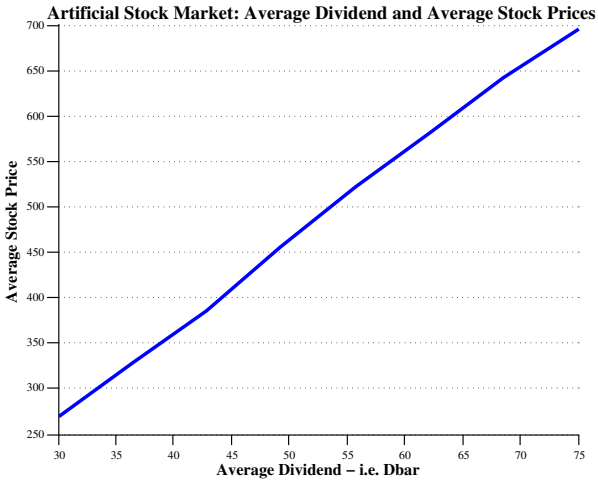
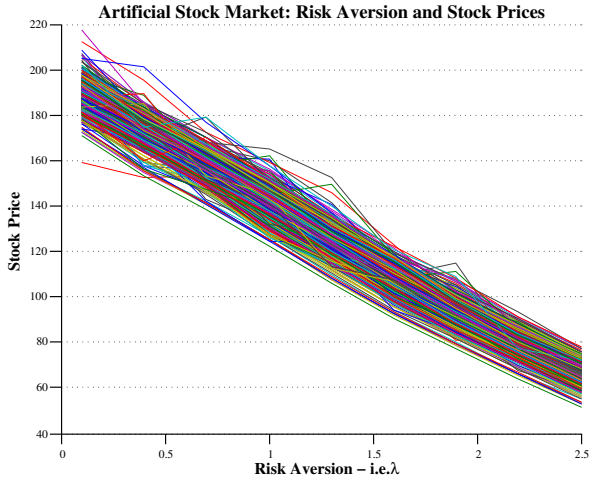
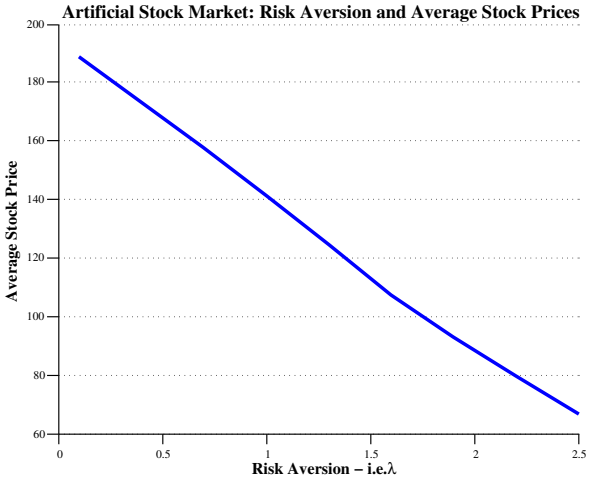


Figure 3: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel A.

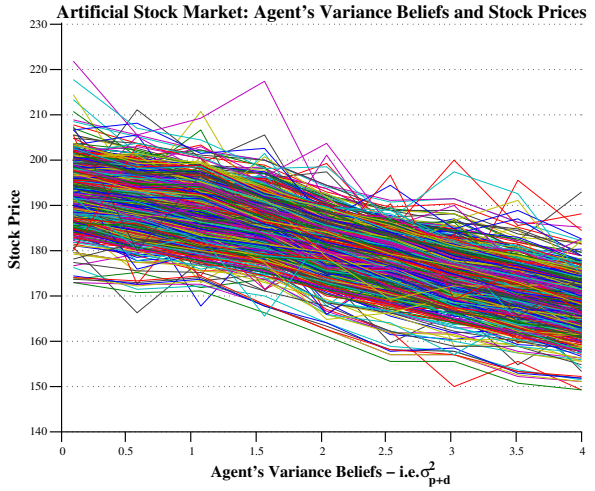
(a)



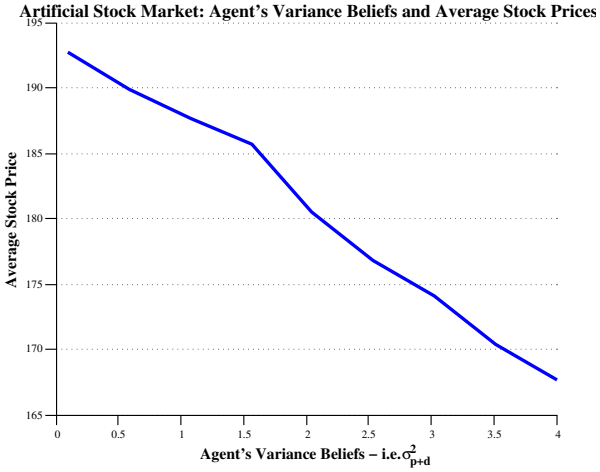
(b)



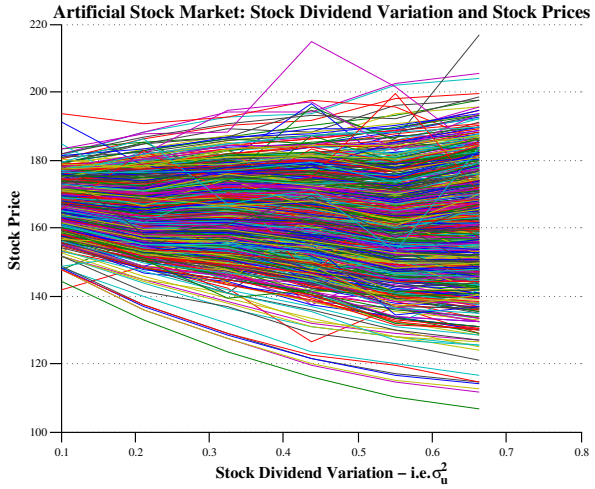
(c)



(d)



(e)



(f)

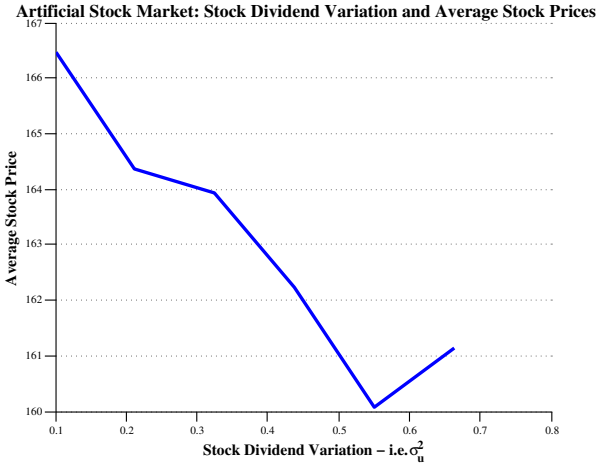
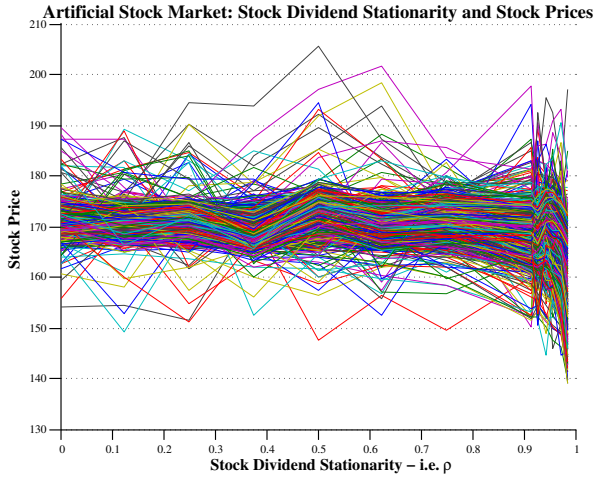
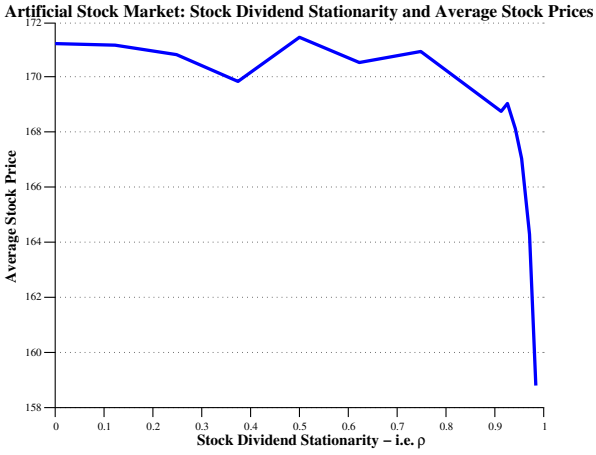


Figure 4: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel B.

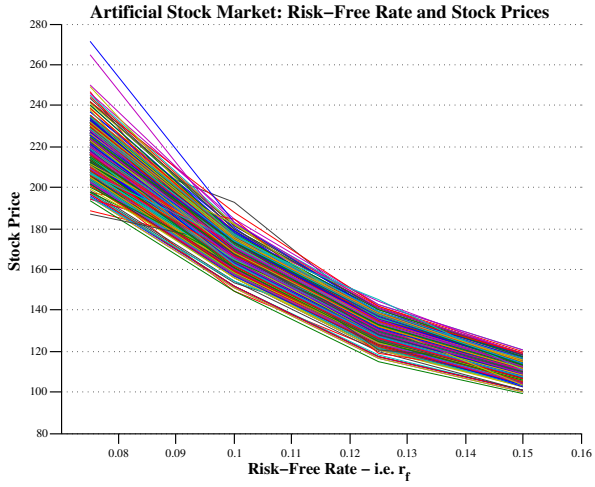
(a)



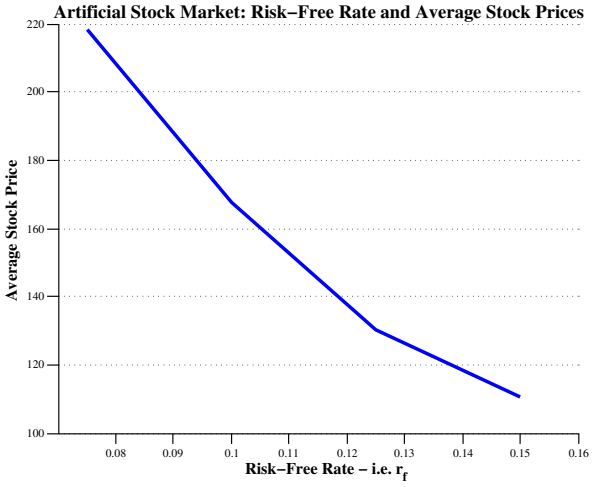
(b)



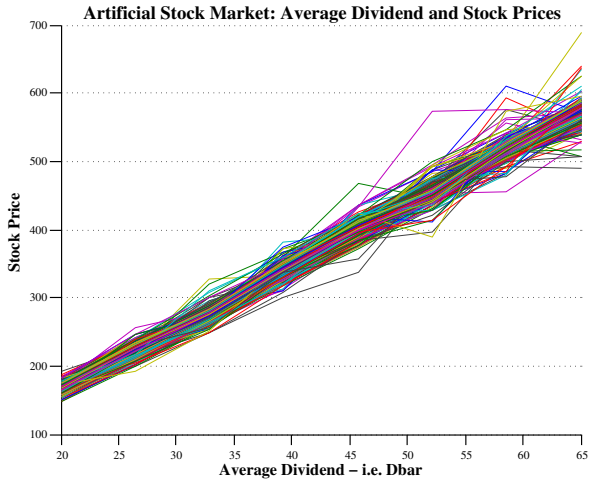
(c)



(d)



(e)



(f)

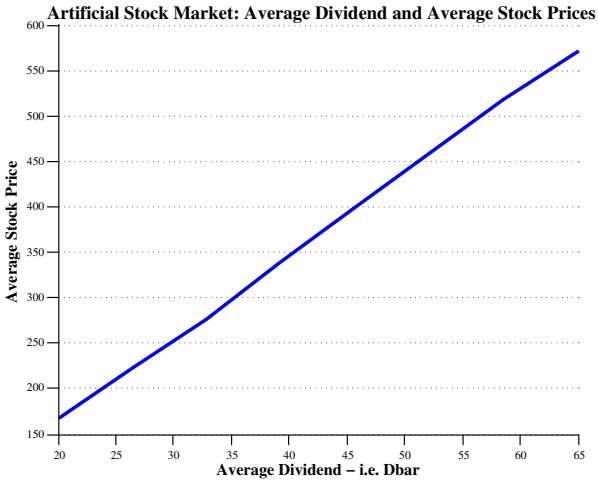
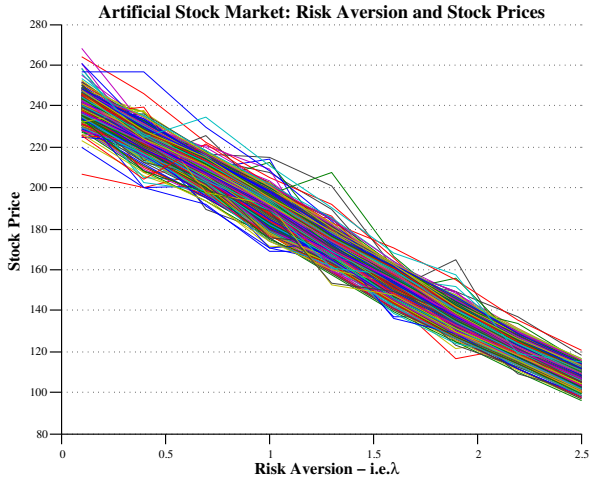
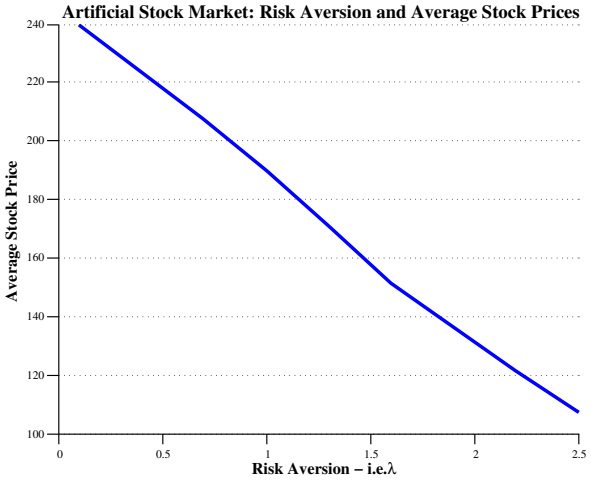


Figure 5: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel A.

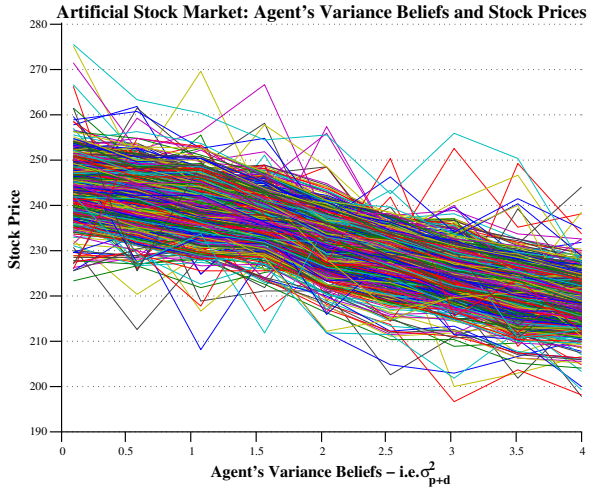
(a)



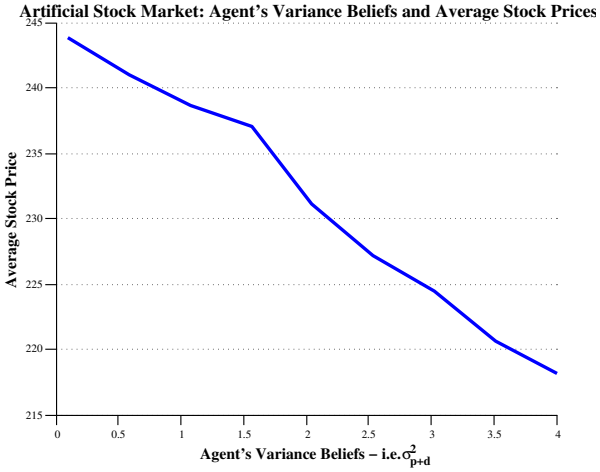
(b)



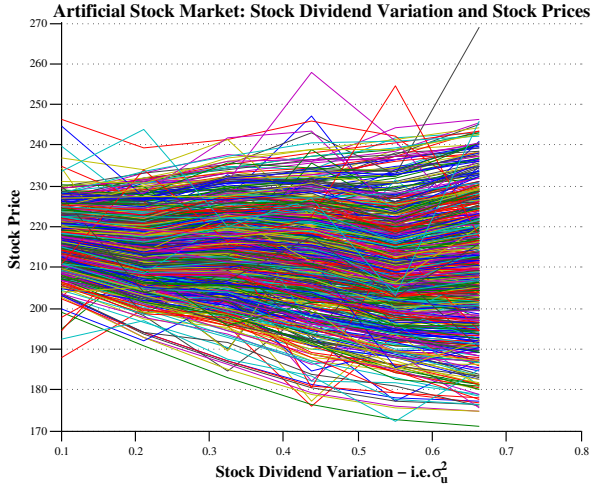
(c)



(d)



(e)



(f)

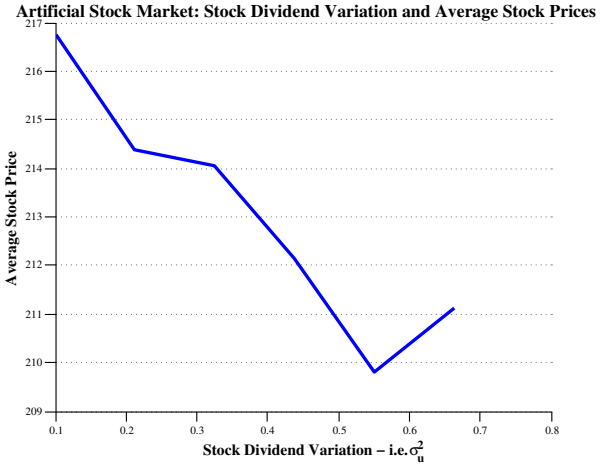
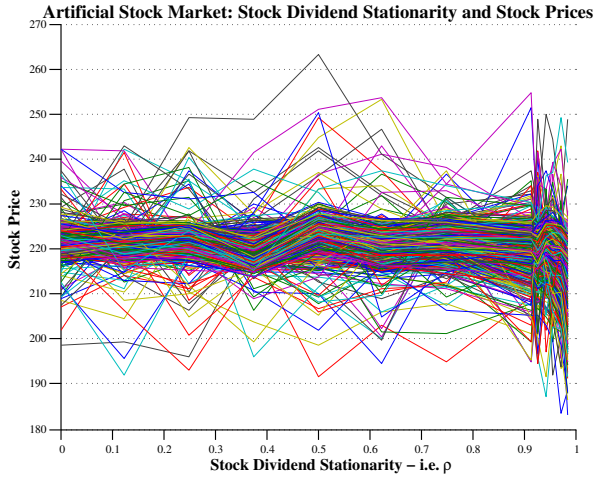
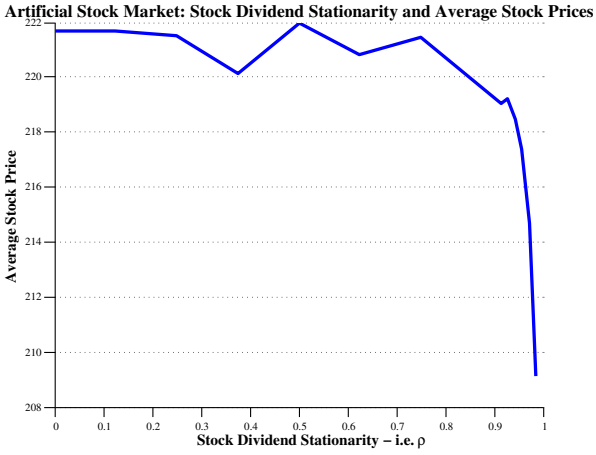


Figure 6: Multivariate Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel B.

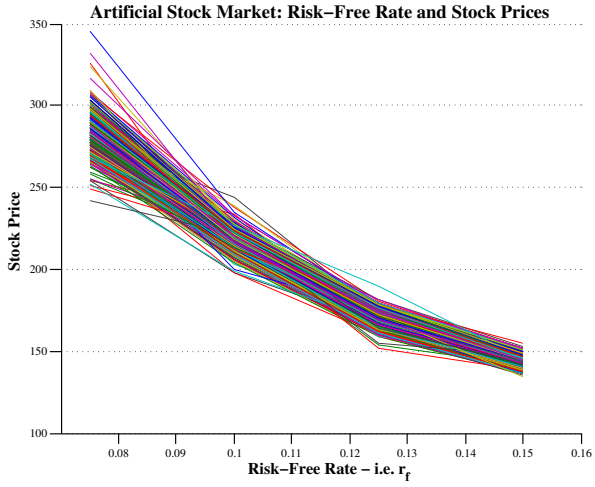
(a)



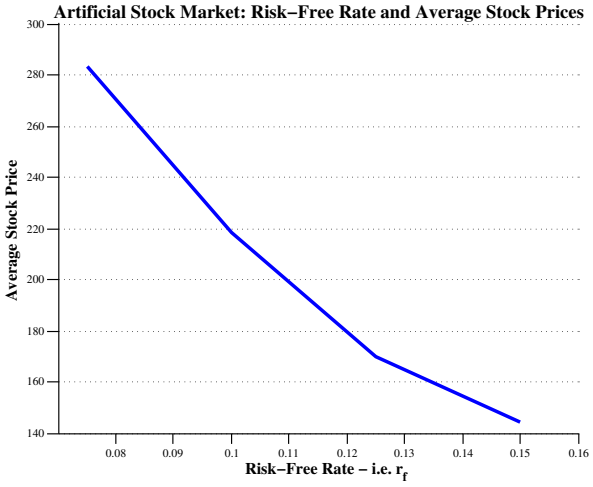
(b)



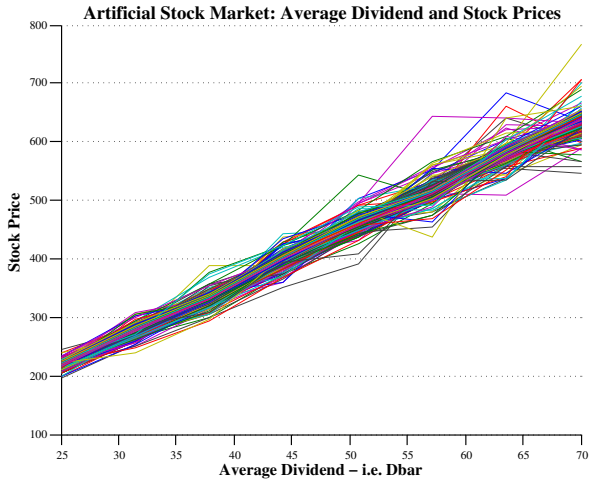
(c)



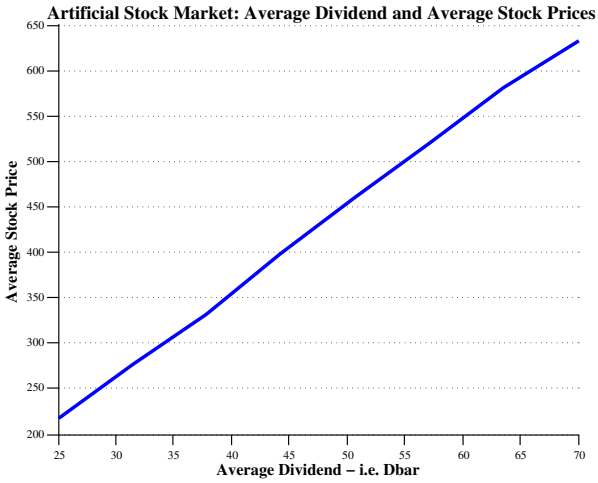
(d)



(e)



(f)



3.2 Method 2: Statistical Model (Regression)

In our second method, we aim to also (a) include sample variance and statistical significance considerations, (b) account for multiple variables and for confounding variables and (c) offer a systematic approach to model selection and validation. The method that achieves this is a statistical model in the form of (multivariate) regression. We achieve this using the well-known Maximum Likelihood regression method which has attractive properties, namely consistency (convergence in probability to true values) and asymptotic normality.

The details of MLE and the statistical model (regression) that we use is detailed in Seedat & Abelman (2019). We refer the interested reader to the relevant section in that paper for the necessary derivations.

To assess the impact of the variables, we consider our agent based-model and we generate synthetic data $\mathbf{x}_i = \left(\lambda_i, \sigma^2_{p+d,i}, \sigma^2_{u_t,i}, \rho_i, r_{f,i}, \bar{d}_i\right)$ and correponding prices $y_i = p_i$, where p_i can represent the price of stock 1, stock 2 or the stock market. We are interested in the statistical model given by $y_i = \mathbf{x}_i\boldsymbol{\beta} + \epsilon_i$. By the theory in Seedat & Abelman (2019), we can estimate the coefficient-vector (our desired output) $\boldsymbol{\beta}$ and the sample variance $\hat{\sigma}^2$ (from which we can obtain the associated p-values). This leads to the following results, for stock 1, stock 2 and the stock market.

Table 2: Statistical model: regression using MLE (Multivariate model) - stock 1.
Estimates of the impacts of the various parameters on stock prices with associated p-values.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{d}$
Estimated parameter	-55.27	-6.47	-8.43	-13.56	-2062.51	9.56
p**-value**	0.00	0.00	0.09	0.00	0.00	0.00
Confidence interval	(-65.39, -45.15)	(-7.79, -5.15)	(-20.55, 3.69)	(-18.45, -8.67)	(-2491.64, -1633.38)	(8.09, 11.03)

Table 3: Statistical model: regression using MLE (Multivariate model) - stock 2.
Estimates of the impacts of the various parameters on stock prices with associated p-values.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{d}$
Estimated parameter	-59.87	-6.63	-9.29	-13.12	-673.92	9.07
p**-value**	0.00	0.00	0.08	0.00	0.00	0.00
Confidence interval	(-72.02, -47.72)	(-7.77, -5.49)	(-24.11, 4.27)	(-17.83, -8.14)	(-777.17, -570.67)	(7.94, 10.20)

Table 4: Statistical model: regression using MLE (Multivariate model) - stock market.
Estimates of the impacts of the various parameters on stock prices with associated p-values.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{d}$
Estimated parameter	-50.79	-6.87	-9.14	-13.32	-1745.34	7.19
p**-value**	0.00	0.00	0.07	0.00	0.00	0.00
Confidence interval	(-60.97, -40.61)	(-9.59, -4.15)	(-24.49, 6.21)	(-17.46, -9.18)	(-2712.58, -778.10)	(4.75, 9.63)

The R^2 of the above models are 86.2%, 87.9% and 91.1% indicating that each model is an excellent fit and 0 does not fall within the confidence intervals (except for stock dividend variation), thus explaining the obtained p-values.

The results of the statistical model above confirm the results discussed above, namely (a) there is a negative relationship between risk-aversion and stock prices, (b) a negative relationship between agent's variance beliefs and stock prices, (c) a negative relationship between stock dividend variation and stock prices, (d) a negative relationship between stock dividend stationarity and stock prices, (e) a negative relationship between the risk free rate and stock prices and (f) a positive relationship between average dividends and stock prices. By inspection of the p-values, all these results are statistically significant at the 1% level apart from the result on stock-dividend variation (although it does have the correct sign and is thus also plausilble). The finance intuition for these results were discussed above. These results are true for each stock and for the stock market.

These results provide further evidence on the plausibillity of our model.

3.3 Stock mispricing depends on agent beliefs

We provide evidence that stock mispricing is possible in a multiasset market and that the degree of such mispricing is larger if agents' beliefs of the stock market are more likely to have a greater deviation from their rationally expected values.

We simulate four different versions of our computational model. Each of these respective four model simulations is given a different range for the draws of $a_{i,j}$ and $b_{i,j}$. The remaining parameters are each fixed at the following values:

Each model simulation is run for 2500 time periods to ensure sufficient learning with $K = 10$ and is then allowed to run for a further 2500 periods after which the last 250 simulated price observations are extracted and plotted. (Alternate ranges for T and K produce similar overall results.) Figure 7 presents the simulated dividend series from the model.

Figure 8 contains the time series plots for each of the model simulations. The corresponding ranges for the $a_{i,j}$ and $b_{i,j}$ parameters are included alongside the time series plots. The plots in blue are the simulated price series. The plots in green are the theoretical price series obtained using the linear homogeneous rational expectations solution from Section 2.4.

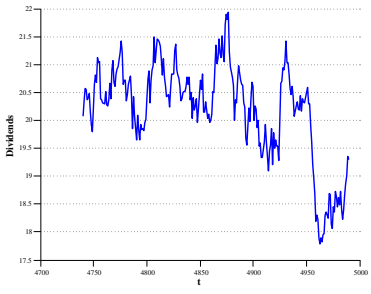
We observe that mispricing tends to increase when agents' stock market beliefs are more likely to have a greater deviation from their rationally expected values (given by a greater A range and/or B range). There are some periods when, in the case of A range $= [a_{re} - 0.05, a_{re} + 0.05]$ and B range $= [\mathbf{b}_{re}, \mathbf{b}_{re}]$ mispricing spikes noticeably although these are very short-lived. We notice that in the case when agents' beliefs regarding a_{re} and $\mathbf{B}_{re}$ have the highest variance, mispricing tends be a persistent phenomenon.

Figure 7: Agent-Based Model Stock Dividend Process (Multivariate model).

(a) Asset 1



(b) Asset 2



(c) Market

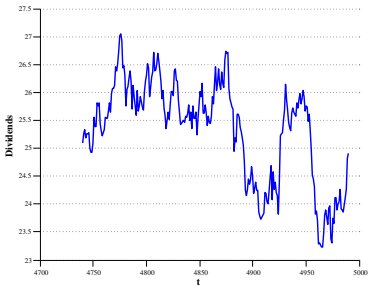
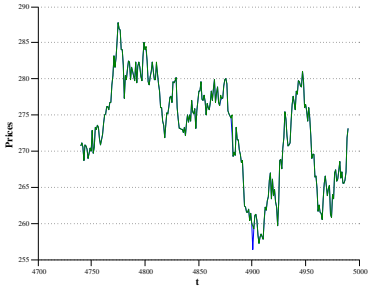
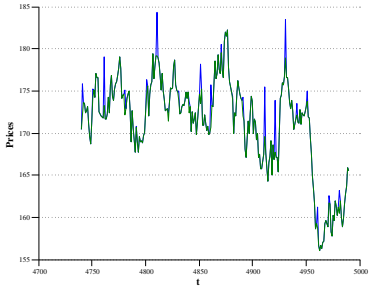


Figure 8: Agent-Based Model Stock Prices and Theoretical Stock Prices (Multivariate model).

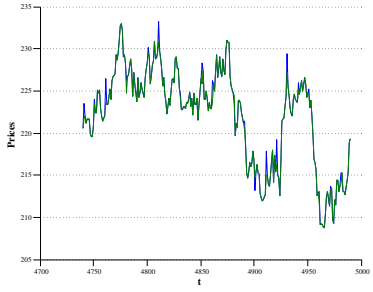
(a) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



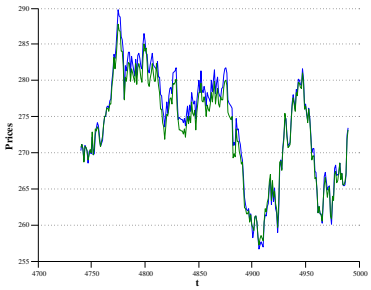
(b) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



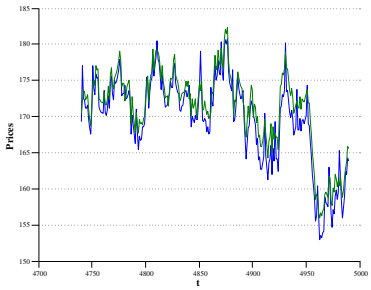
(c) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



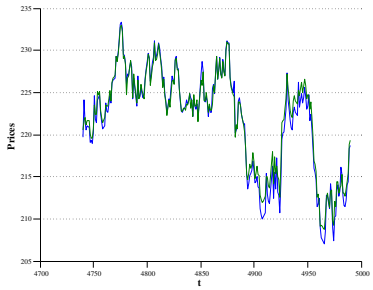
(d) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



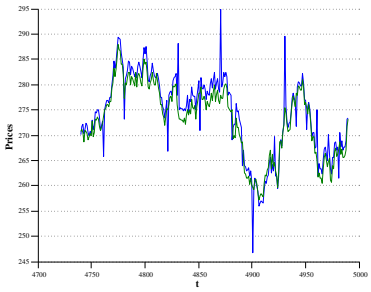
(e) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



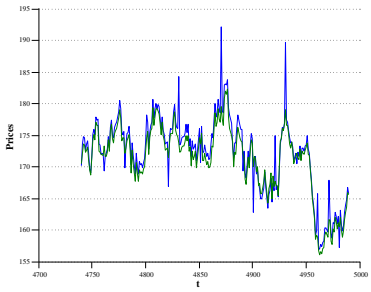
(f) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



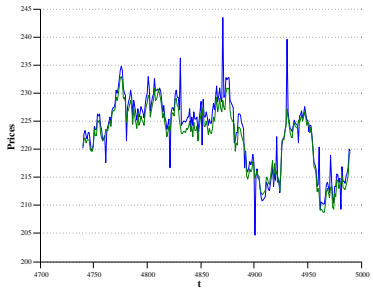
(g) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



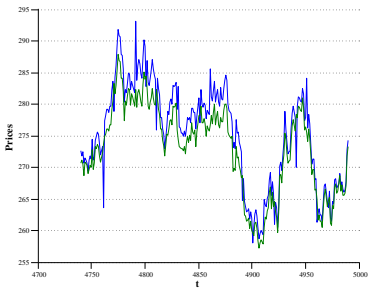
(h) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



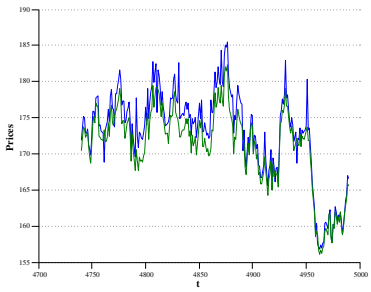
(i) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



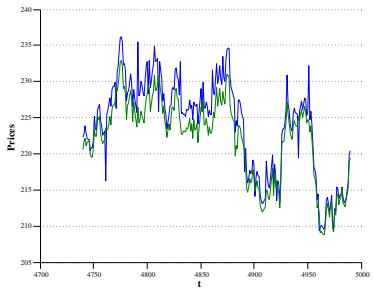
(j) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(k) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(l) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



3.4 Stock volatility and covariation depends on agent beliefs

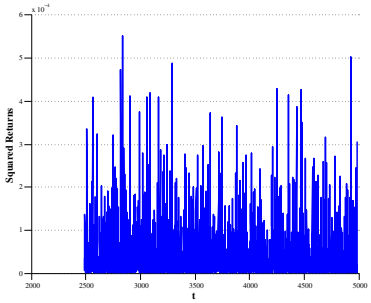
We conclude this paper by examining the impact on stock volatility and covariation by agent beliefs, within our developed multivariate computational agent-based model. We simulate the same four models as in the previous subsection.

As illustrated by the time series plots of squared stock returns (for each asset and for the equally-weighted market) in Figure 9, stock volatility appears to increase when agents' stock market beliefs (i.e. their beliefs regarding a_{re} and $\mathbf{b}_{re}$) are more likely to have a greater deviation from their rationally expected values (note the scale on the y-axis). This appears to hold for each individual stock and for the equally-weighted stock market.

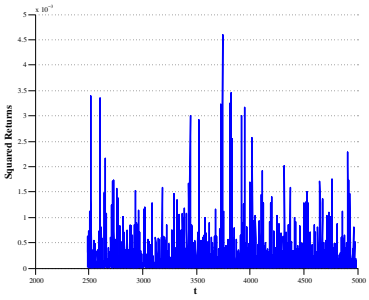
We assess this more rigorously by estimating a DCC (1,1) model (Engle, 2002) and plotting the time series of individual stock volatilities. We also extract and plot the stock correlation estimates from this model. These are contained in Figure 10. We also notice that stock volatilities tend to increase as agents' beliefs regarding a_{re} and $\mathbf{b}_{re}$ are more likely to have a greater deviation from their rationally expected values. This seems to be plausible. If agents' parameters, which are periodically chosen (and then sequentially updated using some learning mechanism), governing their portfolio optimisation choices have a greater variance then updating of such beliefs will, in expectation, be of a greater magnitude which in turn leads to more volatile stocks. Lastly, we note that different belief structures can lead to different correlations between stocks. Certain beliefs, such as those underpinning Figure 10 (f), depress correlations between stocks whereas others, such as those underpinning Figure 10 (i), lead to heightened correlations between stocks.

Figure 9: Multivariate Agent-Based Model Squared Stock Returns.

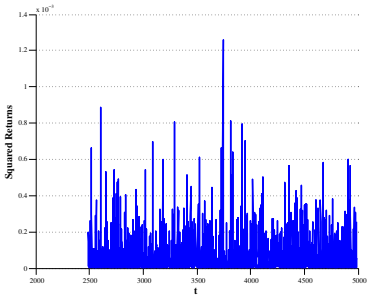
(a) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



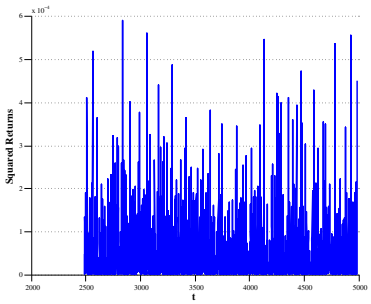
(b) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



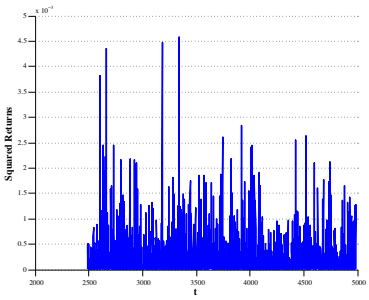
(c) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



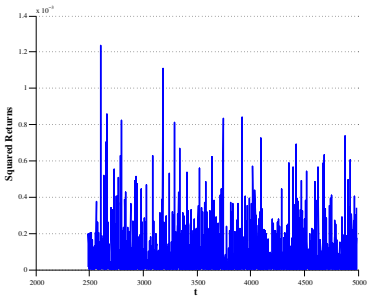
(d) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



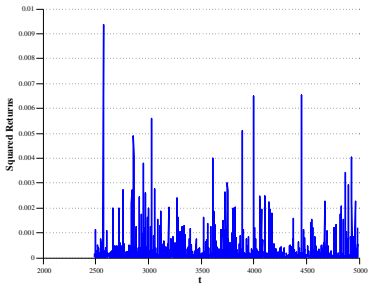
(e) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



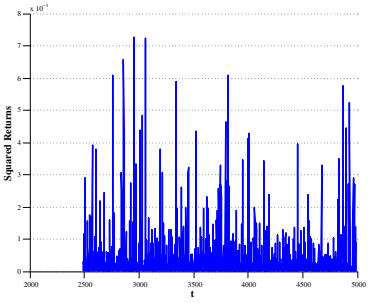
(f) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



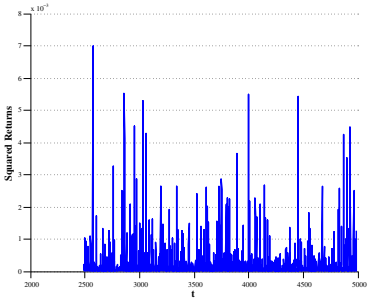
(g) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



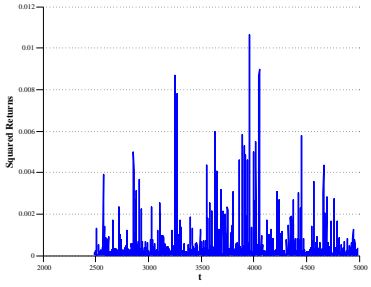
(h) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



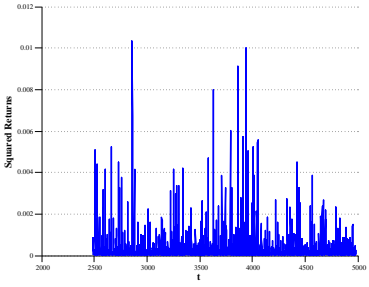
(i) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



(j) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(k) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(l) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$

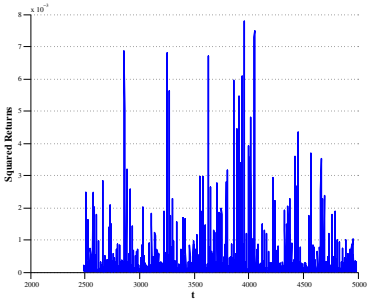
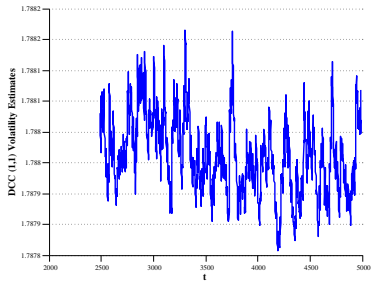
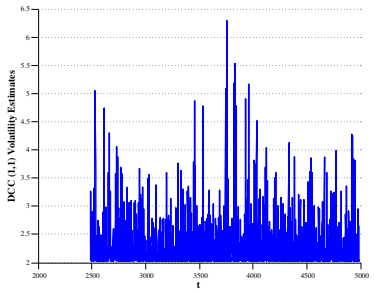


Figure 10: Multivariate Agent-Based Model Volatility and Covariation.

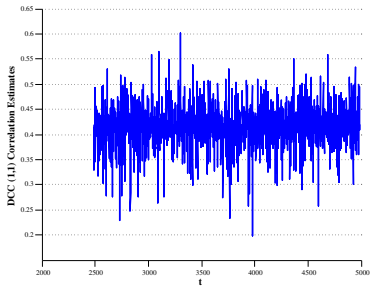
(a) Asset 1 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



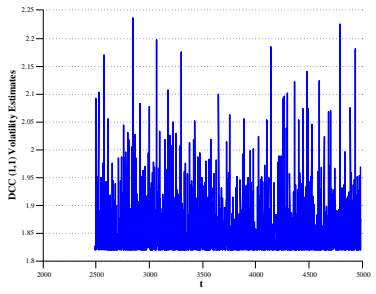
(b) Asset 2 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



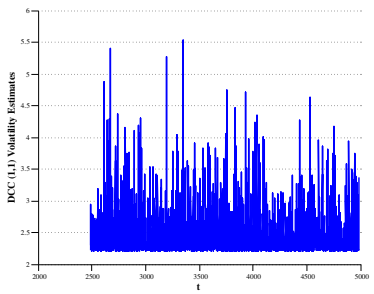
(c) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



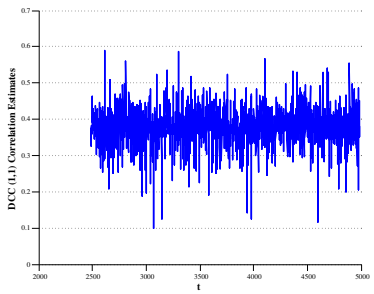
(d) Asset 1 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



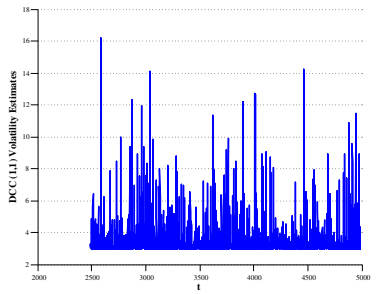
(e) Asset 2 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



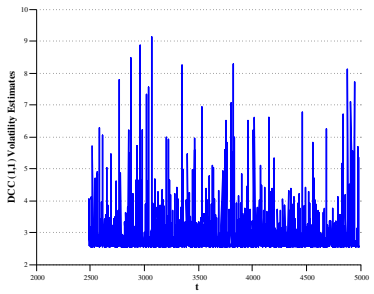
(f) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



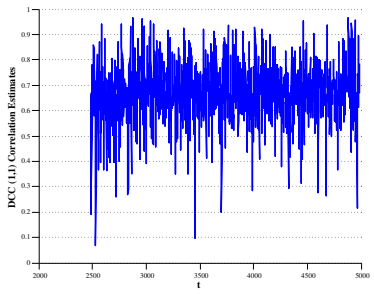
(g) Asset 1 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



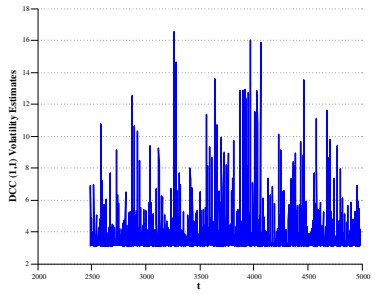
(h) Asset 2 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



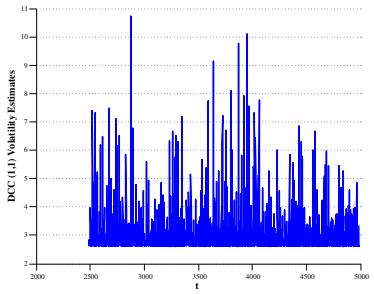
(i) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



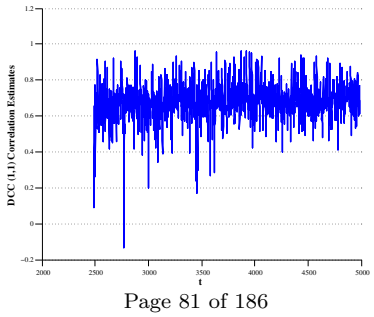
(j) Asset 1 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(k) Asset 2 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(l) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



4 Conclusion

In this paper, we developed a multivariate computational agent-based model by significantly generalising the model of LeBaron *et al.* (1999) which is a degenerate univariate case of our multivariate model. To achieve this, we included a multiasset market with n risky stocks and we assumed that computationally intelligent agents trade and invest the cross section of stocks. We also developed multivariate forecasting rules, a multivariate classifier system and a multivariate genetic algorithm which agents can use to form expectations of stock returns. We applied the methodology developed in Seedat & Abelman (2019) to our multivariate financial market and showed that a number of appealing relationships and results exist in our general model. This was established by analysing the plausibility of our model by simulating 43200 variants of the model. We examined the impact on stock prices by parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk free rate and the first moment of the underlying stochastic dividend process. We have shown that the impact of these parameters on the simulated multivariate model stock prices are plausible and sensible in a financial economic sense. We achieved this by using a statistical model (namely (multivariate) regression with MLE) and by Monte Carlo simulations. These analyses were conducted at the individual stock level as well as at the aggregated equally-weighted stock market level. We also provided insights, through computational experiments, into the relationship between stock prices and two other pertinent parameters, namely stock dividend persistence and stock dividend volatility. We applied the method of Monte Carlo simulations to show that if the variance of agents' beliefs regarding key stock market parameters is lower, then, *ceteris paribus*, the degree of stock price mispricing, at the individual stock level and at the stock market level, is reduced. Lastly, we conducted computational experiments to examine the impact on stock volatility and covariation by agent beliefs, and showed that stock volatility appears to increase when agents' stock market beliefs are more likely to have a greater deviation from their rationally expected values. We also showed that different stock market beliefs on the part of agents support different pairwise stock correlations.

References

Aloud, M. & Fasli, M. (2017). Exploring trading strategies and their effects in the foreign exchange market. *Computational Intelligence*, **33**, 280–307.

Aloud, M., Fasli, M., Tsang, E., Dupuis, A. & Olsen, R. (2017). Modeling the high-frequency fx market: An agent-based approach. *Computational Intelligence*, **33**, 771–825.

Arifovic, J. (1996). The behavior of the exchange rate in the genetic algorithm and experimental economies. *Journal of Political Economy*, 510–541.

Arthur, W.B., Holland, J.H., LeBaron, B., Palmer, R.G. & Tayler, P. (1997). Asset pricing under endogenous expectations in an artificial stock market. *In W. B. Arthur, S. Durlauf & D. Lane, eds, The Economy as an Evolving Complex System II*.

Boer, K., Kaymak, U. & Spiering, J. (2007). From discrete-time models to continuous-time, asynchronous modeling of financial markets. *Computational Intelligence*, **23**, 142–161.

Chiarella, C., Dieci, R. & He, X.Z. (2007). Heterogeneous expectations and speculative behavior in a dynamic multi-asset framework. *Journal of Economic Behavior and Organization*, **62**, 408–427.

Day, R.H. & Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, **14**, 299–329.

Dieci, R. & Westerhoff, F. (2010). Heterogeneous speculators, endogenous fluctuations and interacting markets: a model of stock prices and exchange rates. *Journal of Economic Dynamics and Control*, **34**, 743–764.

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, **20**, 339–350.

Farmer, J.D., Patelli, P. & Zovko, I.I. (2005). The predictive power of zero intelligence in financial markets. *Proceedings of the National Academy of Sciences of the United States of America*, **102**, 2254–2259.

Geanakoplos, J., Axtell, R., Farmer, J.D., Howitt, P., Conlee, B., Goldstein, J., Hendrey, M., Palmer, N.M. & Yang, C.Y. (2012). Getting at systemic risk via an agent-based model of the housing market. *American Economic Review*, **102**, 53–58.

Holland, J.H. (1983). Escaping brittleness. In *Proceedings Second International Workshop on Machine Learning*, 92–95, Citeseer.

LeBaron, B. (2006). Agent-based computational finance. *Handbook of Computational Economics*, **2**, 1187–1233.

LeBaron, B., Arthur, W.B. & Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, **23**, 1487–1516.

Lux, T. & Marchesi, M. (1999). Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, **397**, 498–500.

Marengo, L. & Tordjman, H. (1996). Speculation, heterogeneity and learning: A simulation model of exchange rates dynamics. *Kyklos*, **49**, 407–438.

Palmer, R., Arthur, W.B., Holland, J.H. & LeBaron, B. (1999). An artificial stock market. *Artificial Life and Robotics*, **3**, 27–31.

Palmer, R.G., Arthur, W.B., Holland, J.H., LeBaron, B. & Tayler, P. (1994). Artificial economic life: a simple model of a stockmarket. *Physica D: Nonlinear Phenomena*, **75**, 264–274.

Raberto, M., Cincotti, S., Focardi, S.M. & Marchesi, M. (2001). Agent-based simulation of a financial market. *Physica A: Statistical Mechanics and its Applications*, **299**, 319–327.

Routledge, B.R. (2001). Genetic algorithm learning to choose and use information. *Macroeconomic Dynamics*, **5**, 303–325.

Seedat, S. & Abelman, S. (2019). Examining parameter impacts on the computerised agent-based santa fe artificial stock market model. *Submitted Manuscript*.

Westerhoff, F.H. (2004). Multiasset market dynamics. *Macroeconomic Dynamics*, **8**, 596–616.

Westerhoff, F.H. & Dieci, R. (2006). The effectiveness of keynes–tobin transaction taxes when heterogeneous agents can trade in different markets: a behavioral finance approach. *Journal of Economic Dynamics and Control*, **30**, 293–322.

Chapter 4

Explaining Stock Return Distributions via an Agent-Based Model

This Chapter provides evidence that various time unconditional and conditional stock return kernel distributions are a mixture of distributions. Periods include numerous decades worth of stock returns as well as salient and noteworthy financial economic periods such as the Great Depression and the Great Recession. To explain these observed phenomena, a new computational agent-based model is developed by modelling the equity market index as a claim to a mixed Gaussian dividend stochastic process and by assuming that smart (algorithmic) traders and investors condition on this phenomenon regarding the observed probabilistic properties of the market. Smart (algorithmic) traders and portfolio managers trade and invest in an equally weighted average of stocks that underlie an equity index such as the S&P 500. A new computerised agent-based model had to be developed since, due to the assumptions made, there is considerable heterogeneity. This heterogeneity enters because optimal stock demands are a function of agents' state dependent forecast of future returns. By computing optimal trader demand functions, which requires the derivation of a new probability theory result (regarding the expectation of a log *mixed Gaussian* random variable), and by Monte Carlo simulations as well as kernel density estimations, it is shown that the model can produce mixed Gaussian distributions as evidenced in the real-world. Furthermore, evidence is provided that the model of LeBaron *et al.* (1999), which is a degenerate case of the model developed in this Chapter, cannot produce such distributional properties. The impact of parameters such as agent risk aversion, agent beliefs, the underlying stock dividend process, returns to risk-free assets and state probabilities on simulated aggregate stock prices is shown to be plausible. This is achieved by applying (multivariate) regression, using MLE, and Monte Carlo simulations, to the agent-based model. The impact of parameters such as mixture probabilities, agent risk aversion, agent beliefs, the underlying stock dividend process and returns to risk-free assets on simulated aggregate stock prices is shown to be plausible. Four new theorems

are established and proved, and 43200 agent-based model Monte Carlo simulations are produced as part of this analysis.

This Chapter has been submitted to the journal **Physica A**, is currently under review and is structured according to the journal's format. The research paper was written by the student under the supervision of Prof S Abelman.

Explaining Stock Return Distributions via an Agent-Based Model

Shaheen Seedat* and Shirley Abelman

School of Computer Science and Applied Mathematics
University of the Witwatersrand, Johannesburg
Private Bag 3, Wits 2050, South Africa

Abstract

We document that various stock return kernel distributions are a mixture of distributions, over various time periods and frequencies. To explain this, we develop a "single-stock" agent-based model by modelling stocks as a claim to a mixed Gaussian dividend stochastic process and by assuming that univariate smart traders condition on this phenomenon. By Monte Carlo simulations, we show that the model is capable of generating stock return distributions that are mixed, similar to the shape of Standard and Poor (S&P) 500 stock return distributions. We show that the simulated stock price series generated by the model varies in a plausible manner with various underlying important parameters such as dividend state probabilities, agent risk aversion, agent beliefs, the underlying stock dividend process and returns to risk-free assets. We achieve this by applying (multivariate) regression, using MLE, and Monte Carlo simulations, to our model. We provide evidence that stock mispricing is higher if agents' beliefs regarding the true parameters of the stock market have higher variation. In total we conduct 43200 Monte Carlo simulations as part of our analysis.

Declaration of interest: none.

Keywords: agent-based modeling, mixed Gaussian models, stock return distributions, genetic algorithm, evolutionary learning, Monte Carlo simulations

*Corresponding author. Email: seedatshaheen@gmail.com

1 Introduction

The literature on stock return distributions provides us with remarkable insights. We know that stock return distributions at the daily to monthly frequency exhibit a slight left skewness with fat tails and that, on average, the mean annual return to an equity market index such as the S&P 500, is about 8% per annum, gross dividends. While the literature is deep, we uncover some hidden facts regarding the entire *shape* of the distribution. By appling a kernel density regression estimation to stock return data, we show that numerous stock return distributions, over various periods and frequencies, are a mixture of distributions. Insofar as we know, this is a novel finding.

To explain this new empirical finding, we develop a computational agent-based asset pricing model. We utilise a computational agent-based approach due to its numerous advantages. Given the recent financial crisis, we know that conventional macroeconomic and financial economic methods failed to explain or forecast the crisis, see Geanakoplos *et al.* (2012). This was, in large part, due to assumptions such as agent homogeneity and the existence of purely rational agents which implied that market prices coincided with rational expectations equilbria, see see Geanakoplos *et al.* (2012). Such assumptional drawbacks of traditional methods are also outlined in LeBaron (2006). Such assumptions allowed for analytic tractability at the expense, it seems, of accuracy. Agent-based models allow for agent heterogeneity and the majority use the power of a computer to run simulations from which models and forecasts can be derived. Market prices need not match up with rationally expected prices and this can thus allow for asset mispricing as observed in real-world markets.

The literature on agent-based models of financial markets is vast and we refer the interested reader to LeBaron *et al.* (1999), Farmer *et al.* (2005), Raberto *et al.* (2001), Lux & Marchesi (1999), Arifovic (1996), Marengo & Tordjman (1996), Routledge (2001) and Day & Huang (1990). While such models have noticeably advanced our understanding of various facets of financial markets they do not explicitly model or simulate the *entire shape of the stock return distribution* and neither do they produce the distribution shapes that we empirically document in this paper. In this paper, we provide a model that addresses this void. We take the model of LeBaron *et al.* (1999) as a base case and modify it significantly. We assume that the dividend process is not state independent but, instead, a state dependent process. This is entirely reasonable, since during some states, *expected* dividends can be reasonably assumed to be higher than in other states. We include univariate smart traders in our model who trade the equally-weighted average of all stocks in some equity index, such as the S&P500 index, and the risk free bond and who make conjectures of future returns that are consistent with the structure of the stock market. These alterations lead to a significantly richer model. We show that our model can explain and produce a stock return distribution that resembles the real-world distributions we empirically document.

We also show, by (1) using (parameter variation) Monte Carlo simulations and (2) using a statistical model (namely (multivariate) regression with MLE), that our agent-based model is

plausible in the sense that the relationships between stock prices and parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk free rate, the underlying stochastic dividend process and state probabilities are sensible in a financial economic sense. The first method (parameter variation) is developed in Seedat & Abelman (2019) and details of the method are contained in that paper and the interested reader is referred to the paper. The second method - (multivariate) regression with MLE - allows us to (a) consider sample variance effects, (b) account for the fact that prices have many response variables (and to account for confounding variables) and (c) offer a systematic approach to model selection. Furthermore, MLE (regression) is used as it has attractive asymptotic properties, namely consistency (convergence in probability to the true value) and asymptotic normality. Theoretical details of the second method are contained in Seedat & Abelman (2019) and the interested reader is referred to the paper.

Furthermore, we are able to assess the degree of asset mispricing (positive and negative "bubbles") that occurs and we show that this increases if agents have more uncertain beliefs regarding the true parameters of the stock market.

Our paper is structured as follows. Section 2 provides a motivation for our work by providing empirical evidence that real-world stock returns have a distribution that is a mixture of distributions. Section 3 provides our agent-based asset pricing model including, *inter alia*, details of the stock market, the dividend process (and the mixture probabilistic description of the model), agents, forecasting rules and classifier system, price determination and the genetic algorithm. Section 4 contains extensive simulation results of the model including evidence that it can produce stock return distributions that match real-world distributions. Furthermore, evidence is provided regarding the plausibility of the model. Asset mispricing as a function of agent beliefs is also studied. Finally, Section 5 concludes the paper.

2 Motivation: Stock Return Distributions

We document some new empirical observations which are the motivation for our work. Using monthly stock market data of the S&P 500 from Robert Shiller's website (www.econ.yale.edu/∼shiller), we show that various stock return distributions, gross with dividends, are a mixture of distributions. For example, during the post crisis period between June 2010 - June 2015 it can be seen that the monthly gross stock return distribution is mixed Gaussian as illustrated by the kernel density estimate in Figure 1. We estimate the density using a nonparameteric kernel regression with an Epanechnikov kernel, which is mean square optimal. Our empirical observation is robust to various time periods and various frequencies. This evidence is contained in Figures 2 and 3 where we notice that each also represent a mixture of distributions. In particular, from Figures 2 and 3, we note that various important financial economic periods appear here, notably the Great Depression and the recent financial crisis and post crisis period.

Figure 1: Density Estimate of S&P 500 Stock Market Returns (June 2010 - June 2015) - Single-Stock Santa Fe Artificial Stock Market

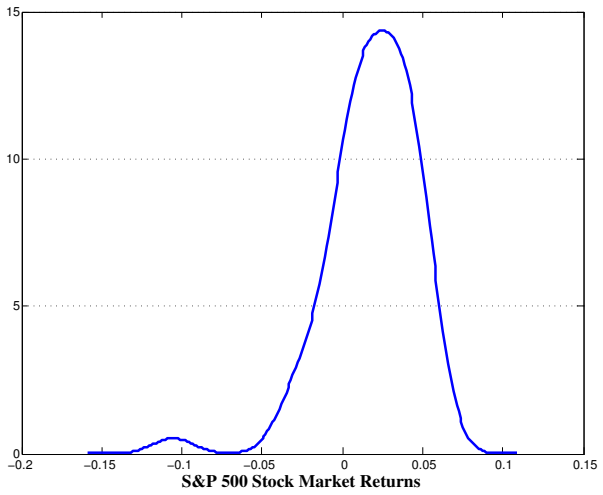


Figure 2: Kernel Density Estimates of S&P 500 Stock Market Returns (Single-Stock Santa Fe Artificial Stock Market) - Panel A.

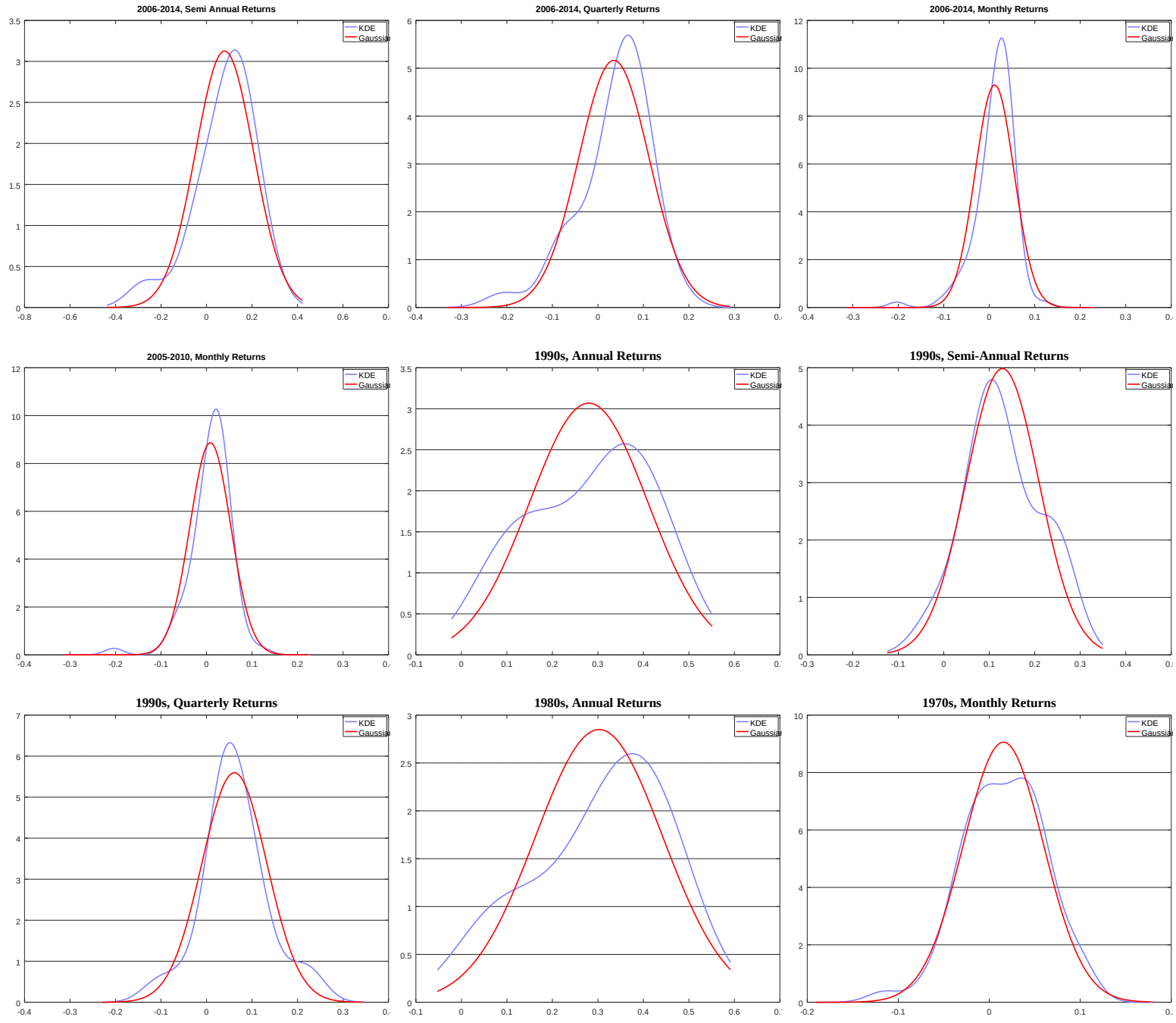
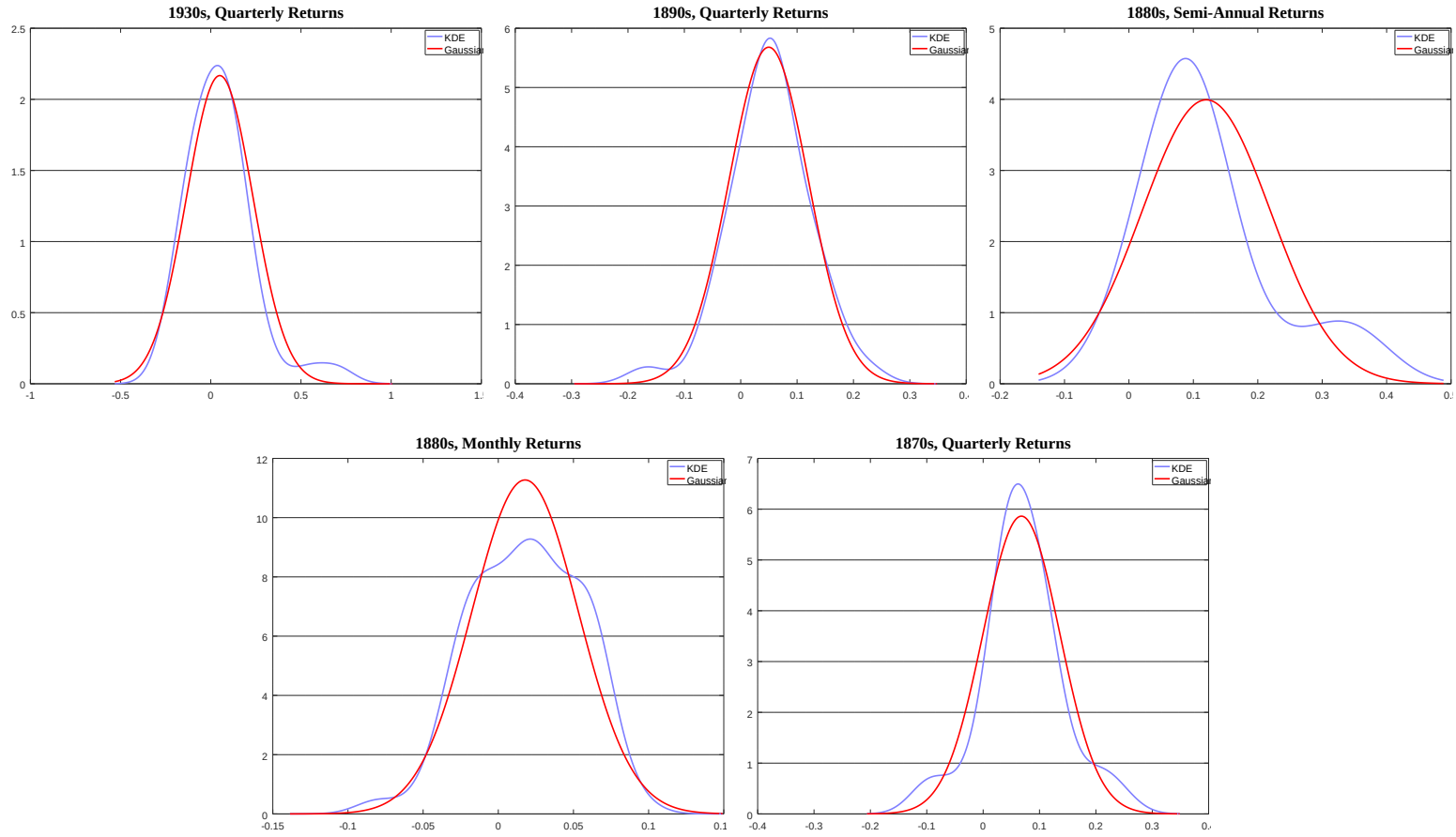


Figure 3: Kernel Density Estimates of S&P 500 Stock Market Returns (Single-Stock Santa Fe Artificial Stock Market) - Panel B.



3 Stock Market Model

3.1 Stock market

The stock market consists of two assets including one risky dividend-paying stock and one risk-free bond. The model is of discrete time and infinite period. Let s_t denote the state variable at time t; there are one of J states possible at each time t. The price of the stock at time t is denoted by P_{t,s_t}. The stock pays a stream of dividends over its life. We will denote the dividend paid at time t for the stock as D_{t,s_t}. The risky asset is in fixed finite supply of N units. The risk-free asset pays interest at a rate r_f per period. We think of the stock price P_{t,s_t} as being the equally-weighted average of all stock prices in some equity index, such as the S&P500 index, at time t and state s_t. We think of the dividend D_{t,s_t} as being the equally-weighted average of all stock dividends paid in some equity index, such as the S&P500 index, at time t and state s_t.

3.2 Dividend process

The dividend process is a *mixed* Gaussian first-order autoregressive process. This assumption regarding the stochastic dividend process drastically alters some of the key properties of existing artificial stock market models including the process for theoretical prices, the manner in which agents will form beliefs about the stock market and their trading actions as well as the resulting distribution of stock returns. We will provide details of each of these in subsequent sections. Before describing our dividend process more formally, we need to introduce the following definitions.

Definition 1 Mixed Gaussian random variable.
If T is a random variable satisfying the following probability law

$$T \overset{\text{law}}{=} \sum_{j=1}^{J} Q(j) T_j$$

where $\{Q(j)\}_{j=1}^{J}$ is a set of probabilities satisfying Kolmogorov's three axioms of a probability measure and where each $T_j \sim N(\mu_j, \sigma_j)$ then we say T is mixed Gaussian of size J and we write $T \sim MG(J, \{\mu_j, \sigma_j\}_{j=1}^{J}, \{Q(j)\}_{j=1}^{J})$.

Definition 2 Mixed Gaussian stochastic process.
If $\{T_t\}_t$ is a stochastic process such that each $T_t \sim MG(J, \{\mu_{t,j}, \sigma_{t,j}\}_{j=1}^{J}, \{Q_t(j)\}_{j=1}^{J})$ then we say $\{T_t\}_t$ is a mixed Gaussian stochastic process of size J.

We model the univariate dividend process as a mixed Gaussian stochastic process as follows

$$D_{t,s_t} = \bar{D}_{s_t} + \rho(D_{t-1,s_{t-1}} - \bar{D}_{s_t}) + u_t$$

where s_t is a state variable that can take on one of finitely many values $1, ..., J$ at each time t, where $\bar{D}_{s_t}$ is a constant for each s_t and where u_t is assumed to be a random variable such that $u_t|\mathcal{I}_{t-1,s_{t-1}} \overset{law}{=} N(0, \sigma^2_{u_t})$ and independent over time, where $\mathcal{I}_{t,s_t}$ denotes the set of *publicly available information* at time t and state s_t. u_t is also assumed to be an innovation or shock term and is thus independent of all other variables. Let $Q(j)$ denote the probability that $s_t = j$. Hence, at time t, the dividend process has the following probabilistic structure

$$D_{t,s_t} = \begin{cases} \bar{D}_1 + \rho(D_{t-1,s_{t-1}} - \bar{D}_1) + u_t & \text{with probability } Q(1) \\ \bar{D}_2 + \rho(D_{t-1,s_{t-1}} - \bar{D}_2) + u_t & \text{with probability } Q(2) \\ \vdots & \vdots \\ \bar{D}_J + \rho(D_{t-1,s_{t-1}} - \bar{D}_J) + u_t & \text{with probability } Q(J) \end{cases} \tag{1}$$

We can thus express the probability law of the dividend process as

$$D_{t,s_t}\Big|\mathcal{I}_{t-1,s_{t-1}} \overset{law}{=} \sum_{s'_t=1}^{J} Q(s'_t) N\Big(\bar{D}_{s'_t} + \rho(D_{t-1,s_{t-1}} - \bar{D}_{s'_t}), \sigma_{u_t}\Big),$$

where $\mathcal{I}_{t-1,s_{t-1}}$ denotes the time $t-1$ information set of *publicly available information*.

We thus have that our dividend process $\{D_{t,s_t}\}_t$ is a mixed Gaussian stochastic process of size J, conditional on $\mathcal{I}_{t-1,s_{t-1}}$, and where each D_{t,s_t} is mixed Gaussian such that $D_{t,s_t}|\mathcal{I}_{t-1s_{t-1}} \sim MG\big(J, \{\bar{D}_{s_t} + \rho(D_{t-1,s_{t-1}} - \bar{D}_{s_t}), \sigma_{u_t}\}_{j=1}^J, \{Q(j)\}_{j=1}^J\big)$. We observe that when $J = 1$, the model of LeBaron *et al.* (1999) is a degenarate case of our much more general model. In our model here, at each point in time, our dividend *process* can take on one of J Gaussian stochastic *processes* each of which has a different mean $\bar{D}_j$. The motivation for this is that the stock market may have different states, each corresponding to different average dividend payouts and each of which may have important implications for the underlying dividend process.

3.3 Agents

There are N smart traders in this model indexed by i. Each has one unit of the stock at the start of trading. We assume that these traders are univariate in that they only invest in one risky "stock", such as the equally weighted average of all stocks underlying the S&P 500 index. Each agent is assumed one-period myopic with constant absolute risk aversion (CARA) preferences over their one-period ahead wealth with CARA coefficient λ. Each agent i has state-dependent wealth $W_{t,s_t,i}$ at the start of time t and must then divide their wealth between the two assets through trading in the stock market. Their state-dependent asset holdings of the stock at time t is denoted $X_{t,s_t,i}$ and variation over time in holdings of each asset is achieved by trades in the stock market.

We now proceed to the trading (i.e. portfolio) optimisation problem faced by each smart agent i. In the literature on computational agent-based models as well as in portfolio optimisation,

a standard assumption used to solve portfolio optimisation problems where agents have CARA preferences is to assume that each agent assumes that future (gross discounted) returns are Gaussian, conditional on that agent's information set $\mathcal{I}_{t,s_t,i}$ at time t which we will assume will contain all publicly available information at time t. Since agents may also hold private information it is true that $\mathcal{I}_{t,s_t} \subseteq \mathcal{I}_{t,s_t,i}$. We can plausibly justify making such Gaussian assumptions when our view of the world is that the underlying source of uncertainty, which in this model is the univariate dividend process, is univariate Gaussian conditional on $\mathcal{I}_{t,i}$. If this is true, and if agents make the conditional Gaussian assumption (conditional on their information set), their beliefs regarding the distribution of future variables is consistent with the structure of the model. In this paper, our structure of the model and the world is of a mixed Gaussian one where the univariate dividend process is mixed Gaussian of size J, conditional on the set of publicly available information $\mathcal{I}_{t,s_t}$. We assume that traders in our model are smart in the sense that they continue to have beliefs consistent with the stock market, even when the conditional dividend process is not a simple Gaussian process. Given the structure of dividends in our model, smart traders now assume that future (gross discounted) returns are mixed Gaussian of size J, conditional on their information set, $\mathcal{I}_{t,s_t,i}$. As we describe below, one period ahead wealth will also be mixed (univariate) Gaussian, conditional on $\mathcal{I}_{t,s_t,i}$. Hence, the resulting portfolio optimisation problem is non-standard and we need to derive a probability theory result regarding log mixed Gaussian random variables that will allow us to solve the optimisation problem faced by each trader. As we shall see, the portfolio optimisation that agents carry out in our model results in a new form of expectation formation (agent heterogeneity), much different to that of the model of LeBaron *et al.* (1999). Due to this difference, and the other differences above such as the structure on the dividend process, we develop a new computational agent-based model as the model of LeBaron *et al.* (1999) cannot account for these differences. We will also illustrate that the model of LeBaron *et al.* (1999) is insufficient from an alternate perspective: specifically, it cannot produce the distributions of real-world stock returns that we document empirically in this paper.

Let agent i's information set at time t and state s_t, $\mathcal{I}_{t,s_t,i}$, contain all prior and current prices, dividends and states. We introduce the following notation to simplify statements of formal results. Denote i's conditional expectation and variance at time t over all random variables by $\mathbb{E}_{t,i}$ and $\mathbb{V}_{t,i}$, respectively., conditional on $\mathcal{I}_{t,s_t,i}$. Denote i's conditional expectation and variance at time t with respect to all random variables except the state variable s_{t+1} by $\tilde{\mathbb{E}}_{t,i}$ and $\tilde{\mathbb{V}}_{t,i}$, respectively, conditional on $\mathcal{I}_{t,s_t,i}$. Denote i's conditional expectation and variance at time t over the variables in the set $\mathcal{A}_{t+1,i} \subset \mathcal{I}_{i,t}$ by $\mathbb{E}_{t,i}^{\mathcal{A}_{t+1,i}}$ and $\mathbb{V}_{t,i}^{\mathcal{A}_{t+1,i}}$, respectively, conditional on $\mathcal{I}_{t,s_t,i}$. We now state the above discussion in more formal terms. Each agent makes consistent beliefs about the market structure, that is they assume that discounted one-period ahead gross returns are univariate mixed Gaussian of size J, conditional on their information set $\mathcal{I}_{t,s_t,i}$ at time t. As we shall see, this assumption, combined with CARA preferences and lemma 3, allows agents to solve for optimal holdings of the assets in terms of the first two moments of the price plus dividend distribution, conditional on

$\mathcal{I}_{t,s_t,i}$, as well as the mixture probabilities $Q(1), ..., Q(J)$. Given that agents are assumed to form consistent beliefs about the structure of the market, they assume that

$$P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}} \;\Big|\; \mathcal{I}_{t,s_t,i} \;\sim\; \sum_{k=1}^{J} Q(k) N\Big(\mu^k_{t+1,s_{t+1},i}, (\sigma^k_{t+1,s_{t+1},i})^2\Big).$$

where $\mu^k_{t+1,s_{t+1},i} := \tilde{\mathbb{E}}_{t,i}(P^k_{t+1,s_{t+1}} + D^k_{t+1,s_{t+1}})$ and $(\sigma^k_{t+1,s_{t+1},i})^2 := \tilde{\mathbb{V}}_{t,i}(P^k_{t+1,s_{t+1}} + D^k_{t+1,s_{t+1}})$ for the state dependent prices $P^k_{t+1,s_{t+1}}$ and dividends $D^k_{t+1,s_{t+1}}$, $k = 1 : J$. Each agent is also assumed to know the dividend structure of the economy and hence

$$\tilde{\mathbb{E}}_{t,i}(D_{t+1,s_{t+1}}) = \tilde{\mathbb{E}}_{t,i}(\bar{D}_{s_{t+1}} + \rho(D_{t,s_t} - \bar{D}_{s'_{t+1}})).$$

and $\tilde{\mathbb{V}}_{t,i}(D_{t+1,s'_{t+1}}) = \sigma_{u_{t+1}}$. Given the structure of the model, it is clear that $\sigma_{t,s'_{t+1},i} = \sigma_{t,s_{t+1},i}$ for all $s'_{t+1}, s_{t+1} \in \{1, 2, ..., J\}$. We can thus define $\sigma_{t,i} = \sigma_{t,s'_{t+1},i}$. We also define $\mu_{t,s_t,i} := \mathbb{E}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}})$. Lastly, all the covariation at time t comes from the innovation u_t - that is, the state variable s_t only ever affects first moments. Hence, $\mathbb{V}$ and $\tilde{\mathbb{V}}$ are the same.

Given that prices and dividends are state-dependent (i.e. they they depend on the state s_t), wealth is also state dependent (and depends on the state s_t). Let $W_{t,s_t,i}$ denote agent i's wealth at time t and in state s_t by $W_{t,s_t,i}$. Since agents have one-period ahead CARA utility over their state-dependent wealth $W_{t+1,s_{t+1},i}$, their utility functions over their wealth at time $t+1$ has the state-dependent form $U(W_{t+1,s_{t+1},i}) = -\exp(-\lambda W_{t+1,s_{t+1},i})$. Given each agent's portfolio choice and trading optimisation problem discussed above, their wealth at $t+1$ satisfies:

$$W_{t+1,s_{t+1},i} = X_{t,s_t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) + (1 + r_f)(W_{t,s_t,i} - X_{t,i}P_{t,s_t}).$$

Finally, we can thus state each agent i's state-dependent trading optimisation problem at each time t concisely as:

$$\max_{X_{t,s_t,i}} \mathbb{E}_{t,i}\Bigg(-\exp\Big\{ -\lambda\Big(X_{t,s_t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) + (1 + r_f)(W_{t,s_t,i} - X_{t,s_t,i}P_{t,s_t})\Big)\Big\}\Bigg)$$

Now define state-dependent optimal stock holdings as follows:

$$X^*_{t,s_t,i} \equiv \arg\max_{X_{t,s_t,i}} \mathbb{E}_{t,i}\Bigg(-\exp\Big\{ -\lambda\Big(X_{t,s_t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) + (1 + r_f)(W_{t,s_t,i} - X_{t,i}P_{t,s_t})\Big)\Big\}\Bigg).$$

To solve for optimal stock holdings, we need a probability theory result regarding the expectation of a log *mixed Gaussian* random variable. We present this result, which is novel insofar as we know.

Theorem 3 *Expectation of a log mixed-normal random variable.*
Let V be a mixed Gaussian random variable of size J such that $V \sim MG(J, \{\mu_j, \sigma_j\}_{j=1}^J, \{Q(j)\}_{j=1}^J)$.

That is, V has the following probability law:

$$V \overset{\text{law}}{=} \sum_{j=1}^{J} Q(j)V_j$$

where $\{Q(j)\}_{j=1}^{J}$ is a set of probabilities satisfying Kolmogorov's three axioms of a probability measure and where each $V_j \sim N(\mu_j, \sigma_j)$. Suppose $(V_1, V_2, ..., V_J) \sim \mathbf{N}(\mu, \Sigma)$, then V is Gaussian with $V \sim N\big(\mathbb{E}(V), \text{Var}(V)\big)$, $\exp(V)$ is log-normal with

$$\mathbb{E}(\exp(V)) = \exp\left(\sum_{j=1}^{J} Q(j)\mu_j + \frac{1}{2}\left(\sum_{j=1}^{J}\big(Q(j)\big)^2 \text{var}\big(V_j\big) + \sum_{j=1, i\neq j} Q(i)Q(j)\text{Cov}\big(V_i, V_j\big)\right)\right)$$

Proof. Since $V \overset{\text{law}}{=} \sum_{j=1}^{J} Q(j)V_j$ and $(V_1, V_2, ..., V_J) \sim \mathbf{N}(\mu, \Sigma)$ then V is Gaussian. Hence, $\exp(V)$ is log-normal and it follows that $\mathbb{E}(\exp(V)) = \exp(\mathbb{E}(V) + \frac{1}{2}\text{Var}(V))$. Furthermore, computing the first two moments:

$$\begin{aligned}
\mathbb{E}\big(V\big) &= \mathbb{E}\left(\sum_{j=1}^{J} Q(j)V_j\right) \\
&= \sum_{j=1}^{J} Q(j)\mathbb{E}\big(V_j\big) \\
&= \sum_{j=1}^{J} Q(j)\mu_j
\end{aligned}$$

and

$$\begin{aligned}
\text{var}\big(V\big) &= \text{var}\left(\sum_{j=1}^{J} Q(j)V_j\right) \\
&= \sum_{j=1}^{J} \text{var}\left(Q(j)V_j\right) + \sum_{j=1, i\neq j} \text{Cov}\left(Q(i)V_i, Q(j)V_j\right) \\
&= \sum_{j=1}^{J}\big(Q(j)\big)^2 \text{var}\big(V_j\big) + \sum_{j=1, i\neq j} Q(i)Q(j)\text{Cov}\big(V_i, V_j\big)
\end{aligned}$$

Thus the result follows. ■

We can now solve for traders' optimal holdings which are given by the following theorem.

Theorem 4 *Portfolio optimisation (single-stock) under the mixed-Gaussian assumption*
Suppose agent i has mypoic, CARA preferences over their one-period ahead wealth with risk aversion

parameter λ. Suppose there are two assets, a risk-free asset with risk-free rate r_f and a risky stock with price P_{t,s_t} and dividend D_{t,s_t} where t denotes time and s_t denotes the "state" at time t. For each state s_t, P_{t,s_t} is a random variable. Let $\mathcal{I}_{t,s_t}$ be the set of publicly available information at time t and in state s_t. Suppose i assumes that conditional one period ahead prices plus dividends are mixed Gaussian of size J with $P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}|\mathcal{I}_{t,i,s_t} \sim MG(J, \{\mu^j_{t+1,s_{t+1},i}, \sigma^j_{t+1,s_{t+1},i}\}_{j=1}^J, \{Q(j)\}_{j=1}^J)$ where $\mathcal{I}_{t,i,s_t}$ denotes i's information set that contains the entire history of prices, dividends and states. Suppose that i knows the entire history of prices, dividends and states that have been realised. Assume that agents assume the *joint* distribution of conditional prices plus dividends is multivariate Gaussian given by

$$\begin{pmatrix} P_{t+1,1} + D_{t+1,1} \\ P_{t+1,2} + D_{t+1,2} \\ \cdot \\ \cdot \\ P_{t+1,J} + D_{t+1,J} \end{pmatrix} \Bigg| \mathcal{I}_{t,s_t,i} \sim \mathbf{N}\big(\mu, \Sigma\big)$$

where Σ is the conditional variance-covariance matrix of prices plus dividends, which allows for correlated prices and dividends across states. Then optimal holdings of stocks at time t for agent i, $X^*_{t,s_t,i}$ satisfies

$$X^*_{t,s_t,i} = \frac{\lambda^{-1}\big(\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t+1,s_{t+1},i} - (1+r_f)P_{t,s_t}\big)}{\sum_{v=1}^J Q^2(v)\mathrm{Var}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}|\mathcal{F}\big) + \sum_{v,w\neq v}^J Q(v)Q(w)\mathrm{Cov}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}|\mathcal{F}\big)}$$

where $\mathcal{F} = \mathcal{I}_{t,s_t,i}$ and where $P^v_{t+1,s_{t+1}}$ and $D^v_{t+1,s_{t+1}}$ are the state dependent prices and dividends at time $t+1$, for $v = 1:J$.

Proof. Since conditional one-period ahead prices plus dividends are mixed Gaussian of size J with $P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} \sim MG(J, \{\mu^j_{t+1,s_{t+1},i}, \sigma^j_{t+1,s_{t+1},i}\}_{j=1}^J, \{Q(j)\}_{j=1}^J)$, there exists J random variables $P^k_{t+1,s_{t+1}} + D^k_{t,s_{t+1}}$, $k = 1, ..., J$, such that

$$P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} \overset{\text{law}}{=} \sum_{k=1}^J Q(k)(P^k_{t+1,s_{t+1}} + D^k_{t+1,s_{t+1}})$$

where each $P^k_{t+1,s_{t+1}} + D^k_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} \sim N(\mu^k_{t+1,s_{t+1},i}, (\sigma^k_{t+1,s_{t+1},i})^2)$. Let

$$\sigma_{k_1,k_2} := \mathrm{Cov}\Big(P^{k_1}_{t+1,s_{t+1}} + D^{k_1}_{t+1,s_{t+1}}, P^{k_2}_{t+1,s_{t+1}} + D^{k_2}_{t+1,s_{t+1}}\Big|\mathcal{I}_{t,s_t,i}\Big)$$

Note that wealth of agents is now *state-dependent* and at time $t+1$ is given by

$$W_{t+1,s_{t+1},i} = X_{t,s_t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) + (1+r_f)(W_{t,s_t,i} - X_{t,s_t,i}P_{t,s_t})$$

where $W_{t+1,s_{t+1},i}$ is the wealth at time $t+1$ and state s_{t+1} and $X_{t,s_t,i}$ is the stock holdings of i at

t given the state at t was s_t. Conditional on $\mathcal{I}_{t,s_t,i}$, the only unknown random variables that are components of $W_{t+1,s_{t+1},i}$ at time t are $P_{t+1,s_{t+1}}$ and $D_{t+1,s_{t+1}}$. Since agents are assumed to know the entire history of prices, dividends and states then, at time $t+1$ their optimal stock holdings at time t are *state-dependent* and satisfy,

$$X^*_{t,s_t,i} := \underset{X_{t,s_t,i}}{\arg\max}\, \mathbb{E}^{\{s_t\}}_{t,i}\Big(- \exp\big(- \lambda W_{t+1,s_{t+1},i}\big)\Big)$$

where $\mathbb{E}^{\{s_t\}}_{t,i}(\cdot)$ denotes $\mathbb{E}(\cdot|\mathcal{I}_{t,s_t,i})$.
Now as

$$\begin{aligned} P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)(P^k_{t+1,s_{t+1}} + D^k_{t+1,s_{t+1}}) \\ &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k) N(\mu^k_{t,s_{t+1},i}, \sigma^k_{t,s_{t+1},i}) \end{aligned}$$

then

$$W_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} \overset{\text{law}}{=} \sum_{k=1}^{J} Q(k) N\big(\mu^k_{W_{t+1,s_{t+1},i}}, (\sigma^k_{W_{t+1,s_{t+1},i}})^2\big)$$

That is, $W_{t+1,s_{t+1},i}$ is a univariate mixed Gaussian random variable of the form $W_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} \sim MG(J, \{\mu^k_{W_{t+1,s_{t+1},i}}, \sigma^k_{W_{t+1},s_{t+1},i}\}_{k=1}^J, \{Q(k)\}_{k=1}^J)$ and where

$$\begin{aligned} \mu_{W_{t+1,s_{t+1},i}} &:= \mathbb{E}\big(W_{t+1,s_{t+1},i}\big|\mathcal{I}_{t,,s_t,i}\big) \\ &= X_{t,s_t,i}\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t+1,s_{t+1},i} + (1+r_f)(W_{t,s_t,i} - X_{t,s_t,i}P_{t,s_t}) \end{aligned}$$

where $\mu_{t+1,s_{t+1},i} = \tilde{\mathbb{E}}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}})$ and where the last line follows since the right hand side is equal to the left hand side. Also,

$$\begin{aligned} \sigma^2_{W_{t+1,s_t+1,i}} &:= \mathrm{var}\big(W_{t+1,s_{t+1},i}\big|\mathcal{I}_{t,s_t,i}\big) \\ &= \mathrm{var}\big(X_{t,s_t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}})\big|\mathcal{I}_{t,s_t,i}\big) \\ &= X^2_{t,s_t,i}\mathrm{var}\big((P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}})\big|\mathcal{I}_{t,s_t,i}\big) \end{aligned}$$

Now as

$$\begin{aligned} P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}|\mathcal{I}_{t,i,s_t} &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)(P^k_{t+1,s_{t+1}} + D^k_{t+1,s_{t+1}}) \\ &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k) N(\mu^k_{t,s_{t+1},i}, \sigma^k_{t,s_{t+1},i}) \end{aligned}$$

then

$$\begin{aligned}\text{var}\big((P_{t+1,s_{t+1}}+D_{t+1,s_{t+1}})\big|\mathcal{I}_{t,s_t,i}\big) &= \text{Var}\Big(\sum_{k=1}^{J}Q(k)(P^k_{t+1,s_{t+1}}+D^k_{t+1,s_{t+1}})\Big)\\ &= \sum_{v=1}^{J}Q^2(v)\text{Var}\big(P^v_{t+1,s_{t+1}}+D^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\\ &+ \sum_{v,w\neq v}^{J}Q(v)Q(w)\text{Cov}\big(P^v_{t+1,s_{t+1}}+D^v_{t+1,s_{t+1}},P^w_{t+1,s_{t+1}}+D^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\end{aligned}$$

Hence

$$\begin{aligned}\sigma^2_{W_{t+1,s_t+1,i}} = X^2_{t,s_t,i}\Bigg(&\sum_{v=1}^{J}Q^2(v)\text{Var}\big(P^v_{t+1,s_{t+1}}+D^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\\ &+\sum_{v,w\neq v}^{J}Q(v)Q(w)\text{Cov}\big(P^v_{t+1,s_{t+1}}+D^v_{t+1,s_{t+1}},P^w_{t+1,s_{t+1}}+D^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\end{aligned}$$

Since $W_{t+1,s_{t+1},i}$ is mixed Gaussian with $W_{t+1,s_{t+1},i}|\mathcal{I}_{t,i,s_t}\sim MG(J,\{\mu^k_{W_{t+1,s_{t+1},i}},\sigma^k_{W_{t+1,s_{t+1},i}}\}_{k=1}^J,\{Q(k)\}_{k=1}^J)$, and since

$$W_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} = X_{t,s_t,i}(P_{t+1,s_{t+1}}+D_{t+1,s_{t+1}})+(1+r_f)(W_{t,s_t,i}-X_{t,s_t,i}P_{t,s_t})|\mathcal{I}_{t,s_t,i}$$

there exists J random variables $W^k_{t+1,s_{t+1},i}$, $k=1,...J$ such that

$$W_{t+1,s_{t+1},i}|\mathcal{I}_{t,i,s_t}\overset{\text{law}}{=}\sum_{k=1}^{J}Q(k)W^k_{t+1,s_{t+1},i}$$

where each $W^k_{t+1,s_{t+1},i}|\mathcal{I}_{t,i,s_t}\sim N(\mu^k_{W_{t,s_{t+1},i}},\sigma^k_{W_{t,s_{t+1},i}})$ and where

$$W^k_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} = X_{t,s_t,i}(P^k_{t+1,s_{t+1}}+D^k_{t+1,s_{t+1}})+(1+r_f)(W_{t,s_t,i}-X_{t,s_t,i}P_{t,s_t})|\mathcal{I}_{t,s_t,i}$$

Moreover, since

$$\begin{pmatrix} P^1_{t+1,s_{t+1}} + D^1_{t+1,s_{t+1}} \\ P^2_{t+1,s_{t+1}} + D^2_{t+1,s_{t+1}} \\ \cdot \\ \cdot \\ P^J_{t+1,s_{t+1}} + D^J_{t+1,s_{t+1}} \end{pmatrix}$$

conditional on $\mathcal{I}_{t,s_t,i}$ is multivariate Gaussian we have that

$$\begin{pmatrix} W^1_{t+1,s_{t+1},i} \\ W^2_{t+1,s_{t+1},i} \\ \cdot \\ \cdot \\ W^J_{t+1,s_{t+1},i} \end{pmatrix}$$

conditional on $\mathcal{I}_{t,s_t,i}$ is multivariate Gaussian. Thus, the conditions for the above theorem apply to $W_{t+1,s_{t+1},i}$ and to $-\lambda W_{t+1,s_{t+1},i}$ and we conclude that $-\lambda W_{t+1,s_{t+1},i}$ is log-normal and that

$$\begin{aligned}
\mathbb{E}\Big(\exp\big(-\lambda W_{t+1,s_{t+1},i}\big)\big|\mathcal{I}_{t,s_t,i}\Big) &= \exp\left(\mathbb{E}\big(-\lambda W_{t+1,s_{t+1},i}\big|\mathcal{I}_{t,s_t,i}\big) + \frac{1}{2}\mathrm{Var}\big(-\lambda W_{t+1,s_{t+1},i}\big|\mathcal{I}_{t,s_t,i}\big)\right) \\
&= \exp\left(-\lambda\mu_{W_{t+1,s_{t+1},i}} + \frac{1}{2}\lambda^2\sigma^2_{W_{t+1,s_{t+1},i}}\right) \\
&= \exp\Bigg(-\lambda\Big(X_{t,s_t,i}\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} + (1+r_f)(W_{t,s_t,i} - X_{t,s_t,i}P_{t,s_t})\Big) + \\
&\quad \frac{1}{2}\lambda^2 X^2_{t,s_t,i}\Bigg(\sum_{v=1}^{J} P^2(v)\mathrm{Var}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big) \\
&\quad + \sum_{v,w\neq v}^{J} P(v)P(w)\mathrm{Cov}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\Bigg)
\end{aligned}$$

Now by state-dependent optimisation

$$
\begin{aligned}
X^*_{t,s_t,i} &:= \arg\max_{X_{t,s_t,i}} \mathbb{E}^{\{s_t\}}_{t,i}\Big(-\exp\big(-\lambda W_{t+1,s_t,i}\big)\Big) \\
&= \arg\min_{X_{t,s_t,i}} \mathbb{E}\Big(\exp\big(-\lambda W_{t+1,s_{t+1},i}\big)\Big|\mathcal{I}_{t,i,s_t}\Big) \\
&= \arg\min_{X_{t,s_t,i}} \Bigg(-\lambda\Big(X_{t,s_t,i}\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} + (1+r_f)(W_{t,s_t,i} - X_{t,s_t,i}P_{t,s_t})\Big) + \\
&\quad \frac{1}{2}\lambda^2(X_{t,s_t,i})^2\Bigg(\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big) \\
&\quad + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\Bigg)
\end{aligned}
$$

We can solve the above by state-dependent optimisation to yield:

$$
\begin{aligned}
&\lambda\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} + \lambda(1+r_f)P_{t,s_t} + \lambda^2 X_{t,s_t,i}\Bigg(\sum_{v=1}^{J} P^2(v)\mathrm{Var}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big) \\
&\quad + \sum_{v,w\neq v}^{J} P(v)P(w)\mathrm{Cov}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg) = 0
\end{aligned}
$$

Rearranging yields the result:

$$
X^*_{t,s_t,i} = \frac{\lambda^{-1}\big(\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} - (1+r_f)P_{ts_t}\big)}{\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}|\mathcal{F}\big) + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}|\mathcal{F}\big)}
$$

where $\mathcal{F} = \mathcal{I}_{t,s_t,i}$. ■

Remark 5 Note that optimal stock holdings dependent on states in two ways: (1) they depend on the current state s_t and (2) they depend on future state dependent prices $P_{t+1,s_{t+1}}$, future state dependent dividends $D_{t+1,s_{t+1}}$ and state probabilities $Q(s_{t+1})$.

Each agent is assumed to have 1 share of the stock. Given the arrangement of this model, the only source of heterogeneity for agents is their forecasts of future state-dependent prices and dividends and their estimates of the variances and covariances of these. Agent heterogeneity enters in the form that expectation formation for agents are a function of each agent's state dependent forecast of future returns. Since the structure of the stock market is different in our model compared to pre-existing models, we require a new computational agent-based model that accounts for the new methods of expectation formation, learning and stock price dynamics. Expectation formation is now very different to prior models and is state dependent as well as being dependent on the future probability measure of dividends. Each agent will form expectations in a different manner to existing models, forming forecasts around a new price equation and carrying this out in a state-

dependent manner. In later sections, we will discuss how the learning mechanism is different in our model.

3.4 Homogeneous linear rational expectations equilibrium (REE)

In some of the simulation work in our model, we will compare the time series of prices produced by the computational agent-based model to theoretical prices. This will allow us to determine the degree of mispricing (positive and negative "bubbles") that can arise in our computational agent-based model, as is observed in the real-world. Before we proceed, we thus need to solve for the homogeneous linear rational expectations equilibrium in our new stock model which is now driven by a mixed Gaussian dividend process. In a homogeneous equilibrium of the stock market, all stock market agents are perfectly identical. If all stock market agents are identical, then by the no-trade theorem there is no trade in the stock market and thus agents will, in every period, hold their initial stock allocation (i.e 1 share). A linear rational expectations equilibrium is one where agents are optimizing and the price function as a function of underlying variables (here dividends) is linear.

To solve for a linear REE in our single stock model driven by mixed Gaussian dividends, let us conjecture that a linear solution exists,

$$P_{t,s_t} = AD_{t,s_t} + B$$

then enforce agent optimisation (Theorem 4) and the no trade theorem constraint ($X^*_{t,s_t,i} = 1$ for all agents and times). Solving this system yields the solution given by the following result.

Theorem 6 Under the maintained assumptions above, the linear REE solution satisfies:

$$P_{t,s_t} = AD_{t,s_t} + B$$

where

$$A = \frac{\rho}{1 + r_f - \rho}$$
$$B = \frac{(1-\rho)(A-1)\sum_{j=1}^{J} Q(j)\bar{D}_j - \lambda\sigma_{t,s_t,i}}{r_f}.$$

where

$\sigma_{t+1,s_{t+1},i} = \sum_{v=1}^{J} Q^2(v)\, Var\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}|\mathcal{F}\big) + \sum_{v,w\neq v}^{J} Q(v)Q(w)\, Cov\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}|\mathcal{F}\big)$ *and where* $\mathcal{F} = \mathcal{I}_{t,s_t,i}$

Proof. *Recall that the dividend process satisfies*

$$D_{t,s_t} = \bar{D}_{s_t} + \rho(D_{t-1,s_{t-1}} - \bar{D}_{s_t}) + u_t$$

Now conjecture that a linear solution of the form below exists:

$$P_{t,s_t} = AD_{t,s_t} + B$$

Now by the above theorem, optimal stock holdings satisfy:

$$X^*_{t,s_t,i} = \frac{\mathbb{E}^{\{s_{t+1}\}}_{t,i}(\mu_{t+1,s_{t+1},i}) - (1+r_f)P_{t,s_t}}{\lambda\sigma_{t+1,s_{t+1},i}}$$

where

$$\mu_{t+1,s_{t+1},i} = \tilde{\mathbb{E}}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}})$$

$$\sigma_{t+1,s_{t+1},i} = \sum_{v=1}^{J} Q^2(v)\, Var\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}\big|\mathcal{F}\big) + \sum_{v,w\neq v}^{J} Q(v)Q(w)\, Cov\big(P^v_{t+1,s_{t+1}} + D^v_{t+1,s_{t+1}}, P^w_{t+1,s_{t+1}} + D^w_{t+1,s_{t+1}}\big|\mathcal{F}\big)$$

and where $\tilde{\mathbb{E}}$ is the expectation over all variables except the state s_t. By the no-trade condition it is true that $X^*_{t,s_t,i} = 1\ \forall i,t$. Note that as agents are smart and know the market structure,

$$\begin{aligned}
\mathbb{E}^{\{s_{t+1}\}}_{t,i}\tilde{\mathbb{E}}_{t,i}(D_{t+1,s_{t+1}}) &= \mathbb{E}^{\{s_{t+1}\}}_{t,i}\tilde{\mathbb{E}}_{t,i}\big(\bar{D}_{s_{t+1}} + \rho(D_{t,s_t} - \bar{D}_{s_{t+1}}) + u_t\big) \\
&= \sum_{j=1}^{J} Q(j)\Big(\bar{D}_j + \rho D_{t,s_t} - \bar{D}_j\Big) \\
&= (1-\rho)\sum_{j=1}^{J} Q(j)\bar{D}_j + \rho D_{t,s_t}
\end{aligned}$$

Optimal holdings of stocks and the no-trade theorem result imply that

$$\lambda\sigma_{t+1,s_{t+1},i} = \mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t+1,s_{t+1},i} - (1+r_f)P_t$$

Now

$$\begin{aligned}
\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t+1,s_{t+1},i} &= \mathbb{E}^{\{s_{t+1}\}}_{t,i}\tilde{\mathbb{E}}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) \\
&= \mathbb{E}^{\{s_{t+1}\}}_{t,i}\tilde{\mathbb{E}}_{t,i}(AD_{t+1,s_{t+1}} + B + D_{t+1,s_{t+1}}) \\
&= (A+1)\mathbb{E}^{\{s_{t+1}\}}_{t,i}\tilde{\mathbb{E}}_{t,i}D_{t+1,s_{t+1}} + B \\
&= (A+1)\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{D}_j + \rho D_{s_t})\Big) + B
\end{aligned}$$

Hence

$$\lambda\sigma_{t+1,s_{t+1},i} = (A+1)\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{D}_j + \rho D_{s_t})\Big) + B - (1+r_f)P_{t,s_t}$$
$$= (A+1)\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{D}_j + \rho D_{s_t})\Big) - AD_{t,s_t} - r_f P_{t,s_t}$$

Solving for prices yields

$$P_{t,s_t} = \frac{(A+1)\Big(((1-\rho)\sum_{j=1}^{J} P(j)\bar{D}_j\Big) - \lambda\sigma_{t+1,s_{t+1},i}}{r_f} + \frac{\rho(A+1)-A}{r_f} D_{t,s_t}$$

Hence the solution for B is

$$B = \frac{(A+1)\Big(((1-\rho)\sum_{j=1}^{J} P(j)\bar{D}_j\Big) - \lambda\sigma_{t+1,s_{t+1},i}}{r_f}$$

from which the solution for A is

$$A = \frac{\rho}{1+r_f-\rho}$$

■

This provides us with a process of theoretical stock prices in our stock model based on a homogeneous linear REE. Note that the homogeneous linear REE that we derive in our paper is a significant generalisation of the one derived in LeBaron *et al.* (1999). This is due to our mixed state representation. In computations, we use the linear homogeneous pricing solution to obtain theoretical prices from which the simulated prices obtained from the agent-based model are compared. Based on this REE solution, we can compute optimal mean squared error forecasts of prices plus dividends in the class of linear forecasts. This has the solution given by the following theorem:

Theorem 7 Under the maintained assumptions above, each agent's optimal mean squared error (MSE) forecasts of prices plus dividends in the class of linear forecasts satisfies:

$$\mathbb{E}_{t,i}\big(P_{t+1} + D_{t+1,s_{t+1}}\big) = \rho\big(P_t + D_{t,s_t}\big) + (1-\rho)\Big((A+1)\sum_{j=1}^{J} Q(j)\bar{D}_j + B\Big)$$

Proof.

$$P_{t,s_t} = AD_{t,s_t} + B$$

The MSE optimal forecast is the conditional expectation, conditional on everything inside the information set of $\mathbb{E}_{t,s_t,i}$. Hence, the MSE optimal forecast, in the class of linear forecasts, of $P_{t+1,s_t} + D_{t+1,s_t+1}$ is given by

$$\begin{aligned}
\mathbb{E}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_t+1}) &= \mathbb{E}_{t,s_t,i}(AD_{t+1,s_{t+1}} + B + D_{t+1,s_t+1}) \\
&= (A+1)\mathbb{E}_{t,i}(D_{t+1,s_{t+1}}) + B \\
&= (A+1)\mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}(D_{t+1,s_{t+1}}) + B \\
&= (A+1)\mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}(\bar{D}_{s_{t+1}} + \rho(D_{t,s_t} - \bar{D}_{s_{t+1}}) + u_{t+1}) + B \\
&= (A+1)\mathbb{E}_{t,i}^{\{s_{t+1}\}}(\bar{D}_{s_{t+1}} + \rho(D_{t,s_t} - \bar{D}_{s_{t+1}})) + B \\
&= (A+1)\mathbb{E}_{t,i}^{\{s_{t+1}\}}((1-\rho)\bar{D}_{s_{t+1}} + \rho D_{t,s_t}) + B \\
&= (A+1)((1-\rho)\sum_{j=1}^{J} Q(j)\bar{D}_j + \rho D_{t,s_t}) + B \\
&= \rho(A+1)D_{t,s_t} + \left((A+1)((1-\rho)\sum_{j=1}^{J} Q(j)\bar{D}_j + B\right)
\end{aligned}$$

and

$$\begin{aligned}
(1+A)D_{t,s_t} &= D_{t,s_t} + AD_{t,s_t} \\
&= D_{t,s_t} + P_{t,s_t} - B \\
&= (P_{t,s_t} + D_{t,s_t}) - B
\end{aligned}$$

Hence,

$$\rho(1+A)D_{t,s_t} = \rho(P_{t,s_t} + D_{t,s_t}) - \rho B$$

Thus

$$\mathbb{E}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_t+1}) = \rho(P_t + D_{t,s_t}) + (1-\rho)\left((A+1)\sum_{j=1}^{J} Q(j)D_j + B\right)$$

■

These are used when constructing rules for agents, as described fully in Section 3.5. We will

denote $A_{re} = \rho$ and $B_{re} = (1-\rho)\Big((A+1)\sum_{j=1}^{J} Q(j)\bar{D}_j + B\Big)$.

Note that the optimal mean squared error forecasts of prices plus dividends in the class of linear forecasts that we derive in our paper is a significant generalisation of the one derived in LeBaron *et al.* (1999). This is due to our mixed state representation.

3.5 Forecasting rules and classifier system

In our paper, the agent-based model developed is a significant generalisation of the model of LeBaron *et al.* (1999). LeBaron *et al.* (1999) incorporated the use of forecasting rules and the classifier system first developed by Holland (1983). In this paper, we modify the forecasting rules and classifier system of LeBaron *et al.* (1999) so that it is consistent with the structure of the stock market model of our paper given the assumptions made on the underlying stochastic dividend process. The forecasting rules and classifier systems need to be modified because they have to be based on the homogeneous linear REE solution that applies to the particular model which, as we noted above, is different between that of LeBaron *et al.* (1999) and our more general model. Furthermore, new forecasting rules and a new classifier system are required because agents now consider multiple states.The forecasting rules and classifier system used by LeBaron *et al.* (1999) are valid for a univariate system with no state dependence whereas we have explicit state dependence. Agent expectation formation in our model occurs relative to a new market structure which has a different probabilistic description. The expectation formation of our model can generate stock return distributions that resemble real world distributions that we documented empirically above.

Each agent i is assumed to have an information set about the state of the market at time t that consists of various publicly available information. Such information can include the state-dependent price P_{t,s_t} at t and state s_t, the state-dependent dividend D_{t,s_t} paid at t and state s_t, moving averages of the price series and so on. Agents condense this information into world bits and each agent is assumed to have twelve world bits of price-dividend rations and price moving averages. At each point in time, each world bit is either true or false. Bits that involve just the price series are referred to as technical bits. Bits involving dividends are referred to as fundamental bits. We also include two control bits. Each agent is assumed to make linear forecasts. This means forecasts will be of the form

$$\mathbb{E}_{t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) = a_i(P_{t,s_t} + D_{t,s_t}) + b_i.$$

Furthermore, a classifier forecasting system is used to obtain agents' forecasts. This means forecasts at t are state-dependent where the state is described by whichever bits the agent i decides is important. Each agent is assumed to have 100 such states that they feel are salient. A state for an agent is a ternary twelve string of the symbols $\{0, 1, \#\}$. The k^{th} component of the ternary twelve string of a state corresponds to the k^{th} world bit. A value of 0 means that the world bit must

be false for the condition to match. A value of 1 means the world bit must be true and a value of # means indifference to that world bit. So, if we were in a world of only five world bits, a ternary string of #10#1 would match the world bit string 11001 but not 10001. Rules consist of ternary strings, as described, as well as a forecasting equation of prices plus dividends (and an estimated variance) connected with that rule. From now on, we shall denote rules by the subscript j. Rule j for example could refer to the state 01101#1##10 and the following forecasting equation (which activates when the state 01101#1##10 is activated)

$$\mathbb{E}_{t,i,j}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}) = a_{i,j}(P_{t,s_t} + D_{t,s_t}) + b_{i,j}$$

with associated variance estimate $\sigma^2_{i,j}$. For each rule j (and for each agent i) its associated parameters $a_{i,j}$ and $b_{i,j}$ are initially set to random values uniformly distributed in the intervals A range and B range, respectively, where A range and B range are intervals centred about the mean-squared error optimal solutions in the homogeneous linear REE of this mixed Gaussian Artificial Stock Model, see below.

$$\mathbb{E}_{t,i}\big(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}}\big) = \rho\big(P_{t,s_t} + D_{t,s_t}\big) + (1-\rho)\Big((A+1)\sum_{j=1}^{J} Q(j)\bar{D}_j + B\Big)$$

where we will denote $A_{re} = \rho$ and $B_{re} = (1-\rho)\Big((A+1)\sum_{j=1}^{J} Q(j)\bar{D}_j + B\Big)$.

Combined with this uniform distribution, A range and B range thus represent the beliefs of the agents regarding A_{re} and B_{re}. At each point in time t, agent i may have several rules that are activated by the world bit string at t. Agent i chooses the rule that has least variance and from this optimal rule the agent can now compute their optimal forecast of prices plus dividends (i.e. their optimal computation of $\mathbb{E}_{t,s_t,i}(P_{t+1,s_{t+1}} + D_{t+1,s_{t+1}})$). Each agent also has an estimate of the variance $\sigma^2_{i,j}$ associated with this forecast. Given these, optimal trading positions can be computationally computed using our novel result of optimal stock holdings in our stock market driven by mixed Gaussian dividends.

3.6 Trading and price determination

Prices are set by a *specialist* who observes the demands (order flow) of the stock and sets P_{t,s_t} such that the market clears (demand equals supply). That is, the specialist sets prices such that $\sum_{i=1}^{N} X^*_{t,s_t,i} = N$. After trading, each agent i updates the accuracy of forecast rules according to an exponentially weighted average of squared forecast error,

$$\nu^2_{t,i,j} = \Big(1 - \frac{1}{\tau}\Big)\nu^2_{t-1,i,j} + \frac{1}{\tau}\big(P_{t,s_t} + D_{t,s_t} - (a_{i,j}(P_{t-1,s_{t-1}} + D_{t-1,s_{t-1}}) + b_{i,j})\big)^2;$$

where τ is usually set to 75.

3.7 Learning genetic algorithm

To allow for evolutionary learning, a genetic algorithmic learning approach is incorporated into the agent-based model. The genetic algorithm is the same as that discussed in Palmer *et al.* (1999) to ensure that we can analyse the effects of altering the structure of the stock model and keeping other aspects constant. An overview of the genetic algorithm used by Palmer *et al.* (1999) is provided here. After each K periods, agents update their rules according to the following genetic algorithm. For each agent i, 20 rules are removed according to a penalised variance criterion and exactly 20 new rules are added. These 20 new rules are obtained by the set of surviving 81 rules and are added through one of two approaches: crossover or mutation. Crossover occurs with probability 0.1 and mutation with probability 0.9. Crossover means selecting two parent rules, according to tournament selection and uniformly crossing the two rules - bit by bit for the ternary string and real number by real number for the vectors $(a_{i,j}, b_{i,j}, \sigma^2_{i,j})$. Mutation means selecting a single parent by tournament selection and modifying that parent's string randomly bit by bit and modifying that parent's $(a_{i,j}, b_{i,j}, \sigma^2_{i,j})$ by random amounts. We use the probabilities as given in Palmer *et al.* (1999) to maintain some degree of uniformity and for comparison of results.

4 Simulation Results

In this section, we simulate the behavior of agents who interact in our new model, in a world of multiple states, and where markets have a different probabilistic feature.

In particular, we simulate our computational agent-based model and show that our model can produce stock return distributions that are a mixture of distributions, similar to the shape of various distribution of real-world stock returns, as documented in Section 2. We also simulate the computational agent-based model of Palmer *et al.* (1999) and show that it does *not* produce a distribution of real-world stock returns that we documented empirically to exist in real-world markets (see Section 2). We then proceed to simulate many variants of the new model where we apply the method of parameter variation Monte Carlo simulation developed in Seedat & Abelman (2019). This allows us to understand whether the model simulated prices vary with salient parameters (such as state probabilities and agent risk aversion) in a plausible manner and to understand how, through the use of a computational agent-based model, mispricing of stocks depends on agent beliefs.

4.1 Stock return distributions

One of the motivations behind our paper, as mentioned in Section 2, is to develop an artificial stock market model that is capable of producing a distribution of stock returns that is a mixture

of distributions, similar to the shape of various real-world stock return distributions empirically documented in our paper. In this subsection, we show that we have achieved this. Furthermore, in this subsection, we show that the Santa Fe artificial stock market model, with parameter values as used in Palmer *et al.* (1999), does not produce a distribution whose shape is similar to that observed in real-world stock markets during the periods and frequencies we document. We then illustrate that the model we have developed in this paper produces a distribution whose shape resembles the stock return distribution of real world (in particular, S&P 500) stock returns during the periods and frequencies we document.

We start by first simulating the single-stock Santa Fe artificial stock market as given in LeBaron *et al.* (1999) with the parameter values as used in their paper. This is a simple degenerate case of the model developed here with parameter values given by Table 1. Note that the Santa Fe artificial stock market is univariate (single-stock) and with no mixed state representation ($J = 1$). We simulate a market with 25 traders each of which has a CARA coefficient of 0.5. Note that A range $= [A_{re} - 0.25, A_{re} + 0.25]$ and B range $= [B_{re} - 14.5, B_{re} + 14.5]$ and thus contain A_{re} and B_{re}, respectively. The model simulation is run for 2500 time periods to ensure sufficient learning with $K = 10$ and is then allowed to run for a further 2500 periods after which the last 2500 simulated price observations are extracted and the kernel density estimate for the corresponding stock returns are plotted in Figure 4 (a). T and K are chosen to ensure $T/K = 250$, as in LeBaron *et al.* (1999). (Alternate values for T and K can be used and they produce similar results.) The kernel density estimate of the simulated stock returns produced by the degenerate Santa Fe model has a shape that is symmetric and non-mixture in shape.

Table 1: Parameters: Single-Stock Santa Fe artificial stock market.

Parameter	**Base value**
λ	0.5
${\sigma^2}_{P+D}$	4
${\sigma^2}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
$\bar{D}$	10
A range	$[0.7, 1.2]$
B range	$[-10, 19]$
N	25
T	5000
K	10

Following on from this exercise, we conduct a similar simulation for our computational agent-based model to illustrate that out model can produce distributions whose shape is similar to that observed in real-world stock markets during the periods and frequencies we document. Using our result on the optimal stock holdings of agents and the parameters including the mixture probabilities as in Table 2, we simulate the mixture model for 2500 time periods to ensure sufficient learning with $K = 10$. We simulate a market with 25 traders each of which has a CARA coefficient of 0.5.

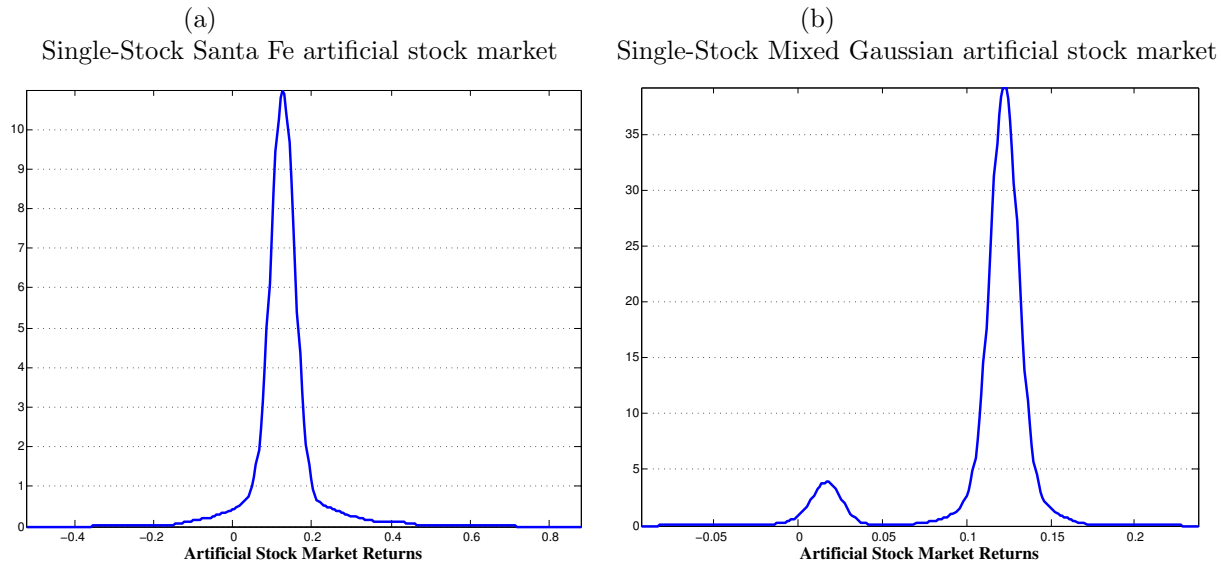
We introduce two dividend states, a high state which occurs with probability 90% and a low state which occurs with probability 10%. After running the simulation for 2500 time periods, we then allow it to run for a further 2500 periods after which the last 2500 simulated price observations are extracted and the density estimate for the corresponding stock returns are plotted in Figure 4 (b). (Alternate ranges for T and K can be used and they produce similar results.)

Table 2: Parameters: Single-Stock Mixed Gaussian artificial stock market.

Parameter	**Base value**
λ	0.5
${\sigma^2}_{P+D}$	4
${\sigma^2}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
J	2
$\bar{D}_j$	$\bar{D}_1 = 20$ and $\bar{D}_2 = 1$
$Q(j)$	$Q(1) = 0.9$ and $Q(2) = 0.1$ i.e. state 1 and 2 occur with probabilities 0.9 and 0.1, respectively
A range	$[A_{re} - 0.25, A_{re} + 0.25]$
B range	$[B_{re} - 14.5, B_{re} + 14.5]$
N	25
T	5000
K	10

The kernel density estimate of the simulated stock returns produced by our model has a shape that resembles the mixed Gaussian shape of the real-world stock return distribution over the period June 2010 - June 2015, illustrated in Section 2.

Figure 4: Density Estimate of Artificial Stock Market returns (Single-Stock Mixed Gaussian Artificial Stock Market)



By altering the parameter values, we are able to replicate each of the real-world stock return

distributions we document in our paper.

4.2 Method 1: Parameter Variation Monte Carlo Simulation

In this subsection, we present the first method that will allows us to analyse the relationship between stock prices and salient underlying parameters, within our simulated environment. (The second method will follow in a subsequent section.) Key parameters that determine stock prices in the computational artificial stock market model are agents' risk aversion coefficient (λ), agents' beliefs regarding the variance of future prices plus dividends ($\sigma^2{}_{P+D}$), the variance of the underlying stock dividend stochastic process (σ_u^2), the persistence of the stochastic dividend process (ρ), the risk-free rate (r_f), the mean of the stock dividend stochastic process ($\bar{D}$) and the mixture probabilities $Q(j)$. The effects of A range and B range on simulated stock prices will be examined in the following subsection. LeBaron (2006) mentions that one of the existing drawbacks of computational agent-based models is that there is no currently existing method for one to examine the impact of parameters on the model. By analysing the variation of simulated prices with salient parameters, we also illustrate the method developed in Seedat & Abelman (2019) which addresses this criticism of agent-based models. To analyse the relationship between stock prices and our first group of salient underlying parameters, we start by fixing a base case of the simulated model with the following parameters. For ease of tractability, we simulate a two-state version of the model.

Table 3: Parameters: Base Case (Single-Stock Mixed Gaussian Artificial Stock Market).

Parameter	**Base value**
λ	0.5
$\sigma^2{}_{P+D}$	4
$\sigma^2{}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
J	2
$\bar{D}_j$	$\bar{D}_1 = 30$ and $\bar{D}_2 = 20$
$Q(j)$	$Q(1) = 0.95$ and $Q(2) = 0.05$ i.e. state 1 and 2 occur with probabilities 0.95 and 0.05, respectively
A range	$[A_{re} - 0.1, A_{re} + 0.1]$
B range	$[B_{re} - 1, B_{re} + 1]$
N	25
T	2750
K	11

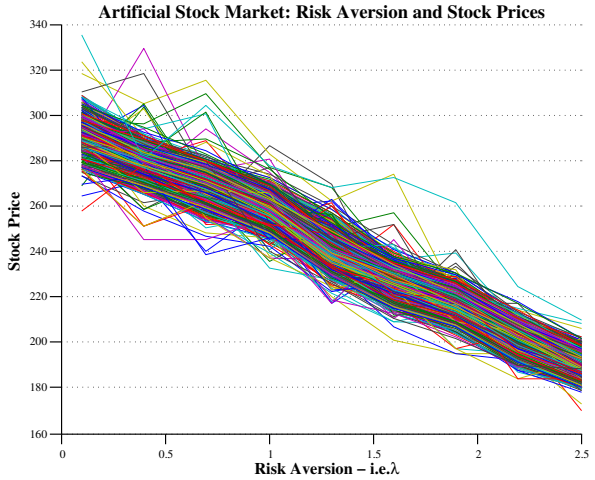
We centre A range and B range about the new REE solutions, namely A_{re} and B_{re}. Finally, we now have state dependent mean levels of the stochastic dividend process, and probabilities associated for each of these states. These parameters are initially set as shown in Table 3. Given the set of base case parameters, we then simulate the model numerous times by varying, one at a time, each of λ, $\sigma^2{}_{P+D}$, $\sigma^2{}_{u_t}$, ρ, r_f, $\bar{D}_j$ and $Q(j)$ whilst holding the remaining parameters fixed at their base case values. In particular, we individually discretise the parameter spaces of each of λ, $\sigma^2{}_{P+D}$, $\sigma^2{}_{u_t}$, r_f, $\bar{D}_j$ and $Q(j)$ into equispaced sub-intervals over plausible ranges as given in the

graphs in Figures 5 and 6. To illustrate how we conducted our analysis, consider λ as our parameter being varied with a parameter range space of $[0.1, 2.5]$ which contains its base value. First consider $\lambda = 0.1$ and set all the other parameters to their base case values as shown in Table 3. Then simulate the model over the time span 0 to $T = 2750$ and obtain the simulated stock prices over this time span which will correspond to $\lambda = 0.1$. Extract the T^{th} price observation. Repeat this exercise 600 times and plot each of these values at the value $\lambda = 0.1$ in Figure 5 (a). By averaging these paths, we remove idiosyncratic noise and we now have a reliable estimate, by a pointwise law of large numbers, of the price at the value $\lambda = 0.1$ which we plot in Figure 5 (b). We then repeat this exercise for each of the λ values in our discretised set and complete the price paths. We repeat this entire exercise for each of the parameters of interest, namely ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, ρ, r_f, $\bar{D}_j$ and $Q(j)$. This now allows us to assess the relationship between our model's simulated stock prices and salient underlying parameters.

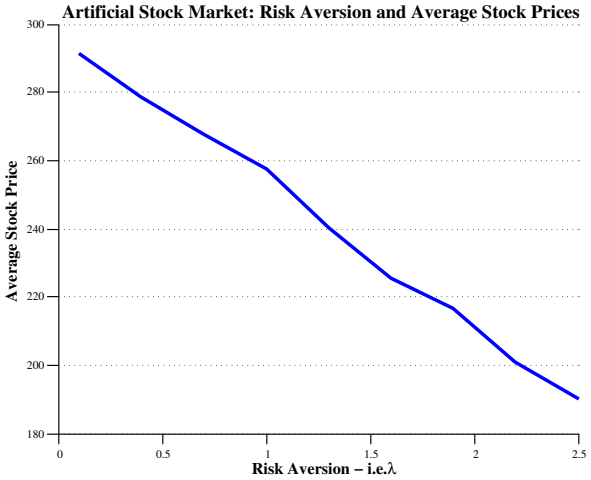
We can see, from Figure 5 (b), that as the level of risk aversion of agents rises, stock prices fall monotonically (since agents are risk averse). We can also see, from Figure 5 (d), that the stock price falls as the agents' variance beliefs of the future price plus dividend of the stock increase (since agents are risk-averse). To the best of our knowledge, finance theory does not provide strong postulates regarding the relationship between stock dividend variation and stock prices and between stock dividend persistence and stock prices. Higher stock dividend variation seems to lead to lower stock prices although this effect is not as strong as it was for agents' variance beliefs and there is some noticeably significant stochastic variation. More insightful, perhaps, is the fact that stock dividend variation (and not variance beliefs on the part of agents) is the key parameter that determines the *variation* of stock prices themselves, as can be seen from Figure 5 (f). For reasons relating to the structure of CARA utility of investors and traders, we should expect a negative relationship between the risk-free rate and stock prices and a positive relationship between the mean of the dividend process and stock prices. This model is also consistent with both these expectations as depicted in Figures 6 (d) and (f). As the stock dividend process approaches non-stationarity, we see, from Figure 6 (b) that stock prices fall dramatically. Lastly, the higher the probability of the mean dividend state, the higher the stock price (since stocks are a claim to dividends) as shown in Figure 7.

Figure 5: Single Stock Mixed Gaussian artificial stock market: Relationship Between Stock Prices and Underlying Variables - Panel A.

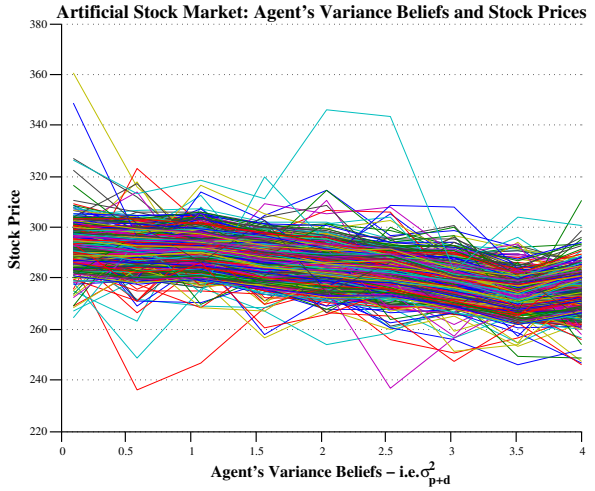
(a)



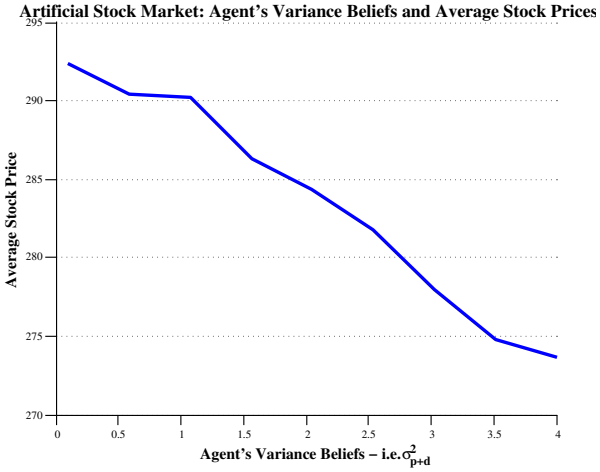
(b)



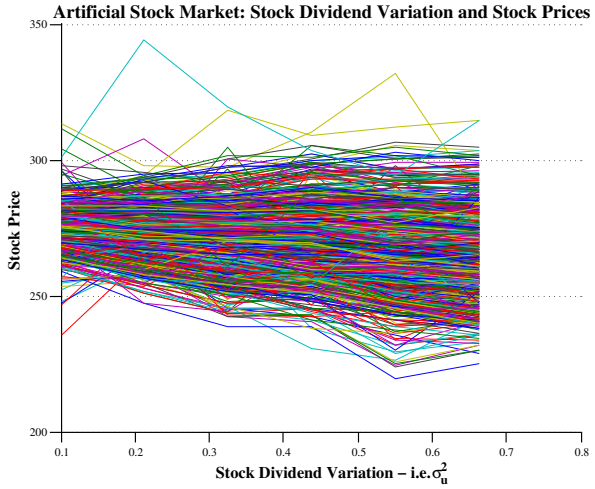
(c)



(d)



(e)



(f)

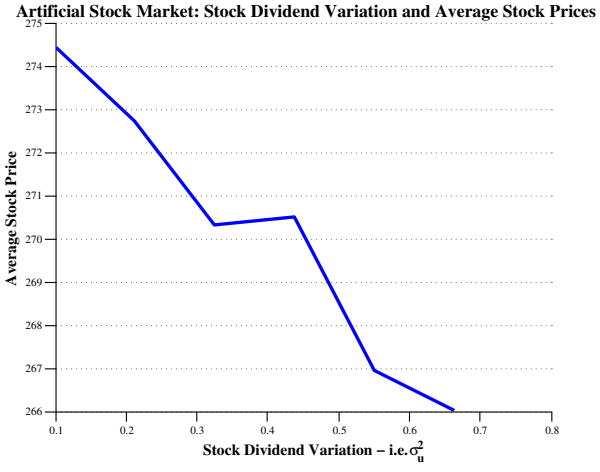
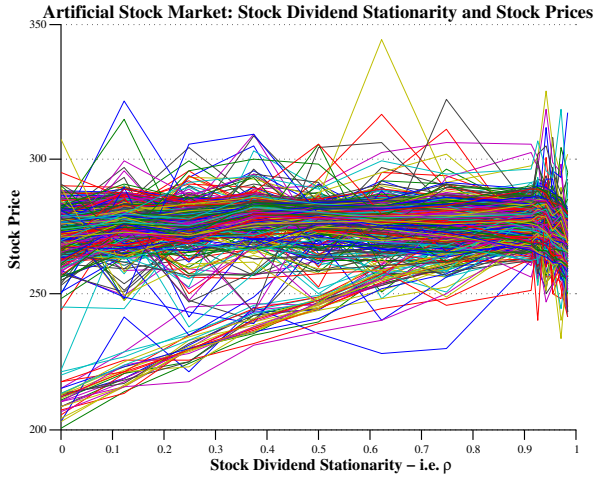
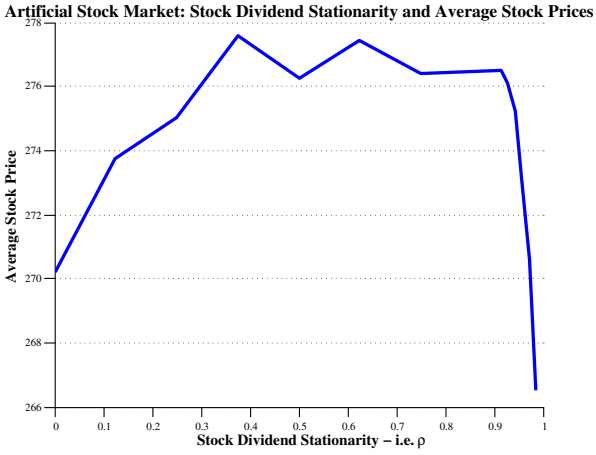


Figure 6: Single Stock Mixed Gaussian artificial stock market: Relationship Between Stock Prices and Underlying Variables - Panel B.

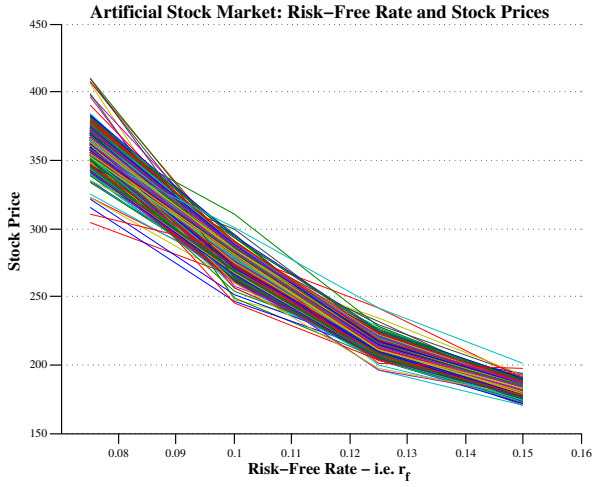
(a)



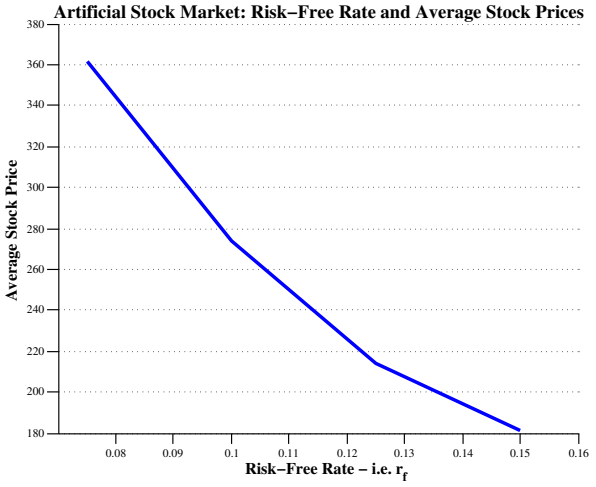
(b)



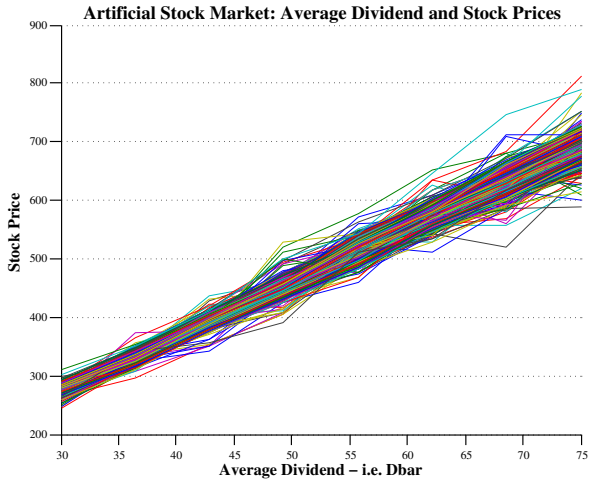
(c)



(d)



(e)



(f)

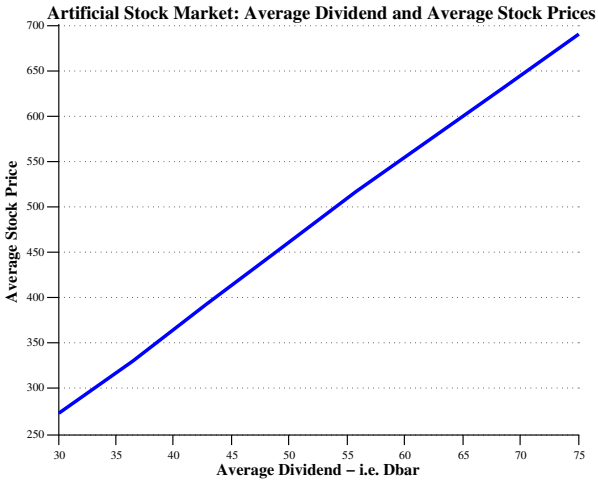
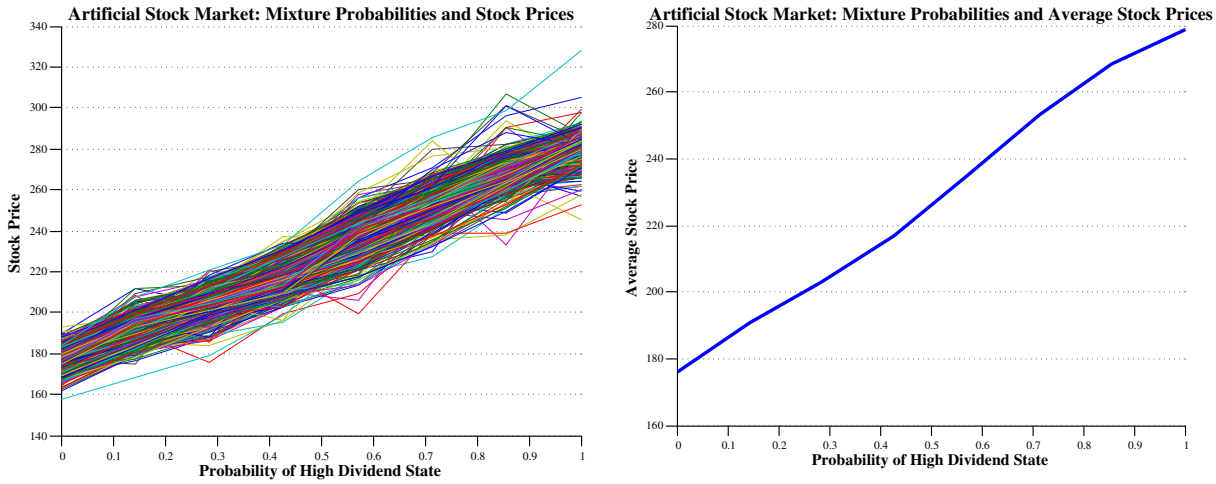


Figure 7: Single Stock Mixed Gaussian artificial stock market: Relationship Between Stock Prices and Underlying Variables - Panel C.

(a) (b)



4.3 Method 2: Statistical Model (Regression)

In our second method, we aim to also (a) include sample variance and statistical significance considerations, (b) account for multiple variables and for confounding variables and (c) offer a systematic approach to model selection and validation. The method that achieves this is a statistical model in the form of (multivariate) regression. We achieve this using the well-known Maximum Likelihood regression method which has attractive properties, namely consistency (convergence in probability to true values) and asymptotic normality.

The details of MLE and the statistical model (regression) that we use is detailed in Seedat & Abelman (2019). We refer the interested reader to the relevant section in that paper for the necessary derivations.

To assess the impact of the variables, we consider our agent based-model and we generate synthetic data $\mathbf{x}_i = \left(\lambda_i, \sigma^2_{p+d,i}, \sigma^2_{u_t,i}, \rho_i, r_{f,i}, \bar{D}_{1i}, Q(1)_i\right)$ and correponding prices $y_i = p_i$, where p_i can represent the price of stock 1, stock 2 or the stock market. We are interested in the statistical model given by $y_i = \mathbf{x}_i\boldsymbol{\beta} + \epsilon_i$. By the theory in Seedat & Abelman (2019), we can estimate the coefficient-vector (our desired output) $\boldsymbol{\beta}$ and the sample variance $\hat{\sigma}^2$ (from which we can obtain the associated p-values). This leads to the following results.

The R^2 of the above model is 91.3% indicating that the model is an excellent fit and 0 does not fall within the confidence intervals (except for stock dividend variation and stationarity), thus explaining the obtained p-values.

The results of the statistical model above confirm the results discussed above, namely (a) there is a negative relationship between risk-aversion and stock prices, (b) a negative relationship between

agent's variance beliefs and stock prices, (c) a negative relationship between stock dividend variation and stock prices, (d) a negative relationship between stock dividend stationarity and stock prices, (e) a negative relationship between the risk free rate and stock prices, (f) a positive relationship between average dividends and stock prices and (g) a positive relationship between the probability of a high dividend state and stock prices. By inspection of the p-values, all these results are statistically significant at the 5% level apart from the result on stock-dividend variation (although it does have the correct sign and is thus also plausible). The finance intuition for these results were discussed above.

These results provide further evidence on the plausibillity of our model.

Table 4: Statistical model: regression using MLE (Single Stock Mixed Gaussian artificial stock market).
Estimates of the impacts of the various parameters on stock prices with associated p-values and 99% confidence intervals.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	D_1	Q(1)
Estimated parameter	-39.58	-3.68	-15.45	-3.12	-2275.04	9.38	104.21
p**-value**	0.00	0.00	0.03	0.12	0.00	0.00	0.00
Confidence interval	(-45.34, -33.82)	(-4.35, -3.01)	(-33.61, 2.71)	(-10.10, 3.86)	(-3289.43, -1260.65)	(7.94, 10.20)	(69.30, 139.12)

4.4 Stock mispricing depends on agent beliefs

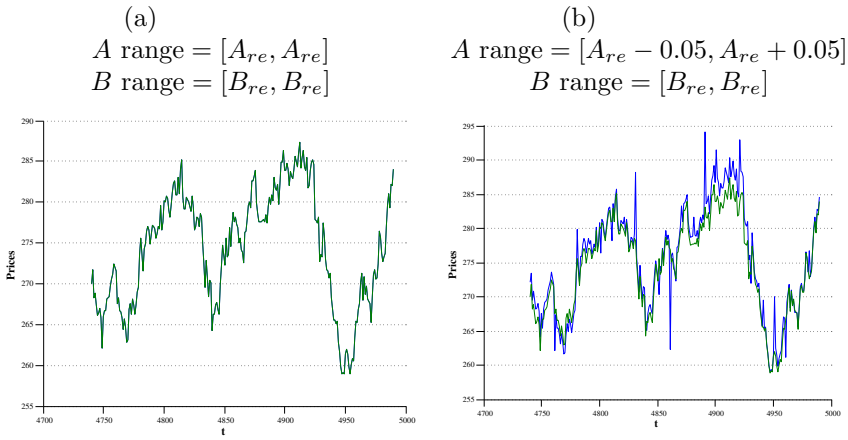
In this subsection, we provide evidence that stock mispricing (bubbles and crashes) is possible in our computational agent-based model, similar to that observed in real-world markets, and that the degree of such mispricing is larger if agents' beliefs of the stock market (i.e. of A_{re} and B_{re}) are more likely to have a greater deviation from their rationally expected values. We simulate two different versions of our model, each of which is given a different range for the draws of $a_{i,j}$ and $b_{i,j}$. The remaining parameters are each fixed as shown is Table 5.

Table 5: Stock Mispricing Parameters: Base Case (Single Stock Mixed Gaussian artificial stock market).

Parameter	**Base value**
λ	0.5
${\sigma^2}_{P+D}$	4
${\sigma^2}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
J	2
$\bar{D}_j$	$\bar{D}_1 = 30$ and $\bar{D}_2 = 20$
$Q(j)$	$Q(1) = 0.95$ and $Q(2) = 0.05$ i.e. state 1 and 2 occur with probabilities 0.95 and 0.05, respectively
N	25
T	5000
K	10

Each model simulation is run for 2500 time periods to ensure sufficient learning with $K = 10$ and is then allowed to run for a further 2500 periods after which the last 250 simulated price observations are extracted and plotted. Figures 8 (a)-(b) contain the time series plots. The corresponding ranges for the $a_{i,j}$ and $b_{i,j}$ parameters are included alongside the time series plots. The plots in blue are the simulated price series. The plots in green are the theoretical price series obtained using the linear homogeneous rational expectations solution from Section 3.4. In Figure 8 (a), since A range $= [A_{re}, A_{re}]$ and B range $= [B_{re}, B_{re}]]$, we are in a REE and the two lines coincide and we have no mispricing. In Figure 8 (b), there is uncertain agent beliefs and hence the lines do not coincide and we have mispricing. Noticeably, we have crashes and bubbles. We observe that when agents' stock market beliefs are more likely to have a greater deviation from their rationally expected values (given by a greater A range and/or B range), the larger is mispricing relative to theoretical prices. This can be seen by, for example, comparing Figure 8(a) with Figure 8(b).

Figure 8: Single-Stock Mixed Gaussian Agent-Based Model Stock Prices and Theoretical Stock Prices.



5 Conclusion

In this paper, we documented that stock return distributions, over various periods and time frequencies, are a mixture of distributions. To explain this, we developed a computational agent-based asset pricing model where smart traders have beliefs that are consistent with the structure of the stock market. By computing optimal trader demands and by Monte Carlo simulations, we were able to show that our model can replicate the observed real-world stock return distributions. Furthermore, we showed that the price series generated by the simulated model varies in a plausible way with salient financial economic variables. We achieved this by two methods namely (multivariate) regression using MLE and a parameter variation Monte Carlo simulation technique. We also showed that asset mispricing is larger the more uncertain agents' beliefs of the stock market. In total we

conducted 43200 Monte Carlo simulations as part of our analysis.

References

Arifovic, J. (1996). The behavior of the exchange rate in the genetic algorithm and experimental economies. *Journal of Political Economy*, 510–541.

Day, R.H. & Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, **14**, 299–329.

Farmer, J.D., Patelli, P. & Zovko, I.I. (2005). The predictive power of zero intelligence in financial markets. *Proceedings of the National Academy of Sciences of the United States of America*, **102**, 2254–2259.

Geanakoplos, J., Axtell, R., Farmer, J.D., Howitt, P., Conlee, B., Goldstein, J., Hendrey, M., Palmer, N.M. & Yang, C.Y. (2012). Getting at systemic risk via an agent-based model of the housing market. *American Economic Review*, **102**, 53–58.

Holland, J.H. (1983). Escaping brittleness. In *Proceedings Second International Workshop on Machine Learning*, 92–95, Citeseer.

LeBaron, B. (2006). Agent-based computational finance. *Handbook of Computational Economics*, **2**, 1187–1233.

LeBaron, B., Arthur, W.B. & Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, **23**, 1487–1516.

Lux, T. & Marchesi, M. (1999). Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, **397**, 498–500.

Marengo, L. & Tordjman, H. (1996). Speculation, heterogeneity and learning: A simulation model of exchange rates dynamics. *Kyklos*, **49**, 407–438.

Palmer, R., Arthur, W.B., Holland, J.H. & LeBaron, B. (1999). An artificial stock market. *Artificial Life and Robotics*, **3**, 27–31.

Raberto, M., Cincotti, S., Focardi, S.M. & Marchesi, M. (2001). Agent-based simulation of a financial market. *Physica A: Statistical Mechanics and its Applications*, **299**, 319–327.

Routledge, B.R. (2001). Genetic algorithm learning to choose and use information. *Macroeconomic Dynamics*, **5**, 303–325.

Seedat, S. & Abelman, S. (2019). Examining parameter impacts on the computerised agent-based santa fe artificial stock market model. *Submitted Manuscript*.

Chapter 5

Explaining Aggregated Stock Return Distributions via a Multivariate Agent-Based Model

In this Chapter, further evidence is provided that kernel distributions of aggregated stock market returns, over various frequencies and periods, are a mixture of distributions. To explain this phenomenon, a *multivariate* agent-based asset pricing model is developed by modelling stocks as a claim to a new *multivariate mixed Gaussian* dividend stochastic process. The stock market is assumed as being comprised of multiple stocks that are a claim to a multivariate mixed Gaussian dividend stochastic process and where it is assumed that heterogeneous interacting smart (algorithmic) traders and portfolio managers trade and invest in the *underlying equities that constitute an equity market index* and *optimise* their holdings in their portfolio optimisation, have consistent beliefs about stock market dynamics and learn over time using a multi-stock, multi-state genetic algorithm and classifier system. The models in Chapters 3 and 4 as well as the model of LeBaron *et al.* (1999) are degenerate cases of this much more general model. A new computerised agent-based model had to be developed since, due to the assumptions made, there is considerable heterogeneity (and multiple stocks). This heterogeneity enters because optimal stock demands are a function of agents' state dependent forecast of future returns. By computing optimal trader demands and by Monte Carlo simulations, it is shown that the model developed in this Chapter can replicate the observed real-world aggregated stock return distributions. The impact of parameters such as agent risk aversion, agent beliefs, the underlying stock dividend process, returns to risk-free assets and state probabilities on simulated aggregate stock prices is shown to be plausible, both at an individual stock level as well as at the aggregated stock market level. This is achieved by applying (multivariate) regression, using MLE, and Monte Carlo simulations, to the agent-based model. The impact of parameters such as mixture probabilities, agent risk aversion, agent beliefs, the underlying stock

dividend process and returns to risk-free assets on simulated aggregate stock prices is shown to be plausible.

It is shown, by Monte Carlo simulations and the estimation of DCC (1,1) models, that asset mispricing and stock cross-correlations are higher the more uncertain agents' beliefs of the stock market are. Four new theorems are established and proved, and 43200 agent-based model Monte Carlo simulations are produced as part of this analysis.

This Chapter has been submitted to the journal **Journal of Economic Coordination and Interaction**, is currently under review and is structured according to the journal's format. The research paper was written by the student under the supervision of Prof S Abelman.

Explaining Aggregated Stock Return Distributions via a Multivariate Agent-Based Model

Shaheen Seedat* and Shirley Abelman

School of Computer Science and Applied Mathematics
University of the Witwatersrand, Johannesburg
Private Bag 3, Wits 2050, South Africa

Abstract

We develop a multivariate agent-based model of the stock market to explain the empirical observations that we document, namely that various aggregated Standard and Poor (S&P) 500 stock return distributions are a mixture of distributions, over various time periods and frequencies. We model the stock market as being comprised of multiple stocks that are a claim to a multivariate mixed Gaussian dividend stochastic process and we assume that heterogeneous interacting smart traders, who make consistent beliefs, *optimally* invest in the cross section of underlying stocks in an equity index. By simulations, we show that the model is capable of generating stock return distributions that resemble real-world S&P 500 stock return distributions. We show that simulated stock price series generated by the model vary in a plausible manner with important underlying parameters. We achieve this by applying (multivariate) regression, using MLE, and Monte Carlo simulations, to our model. We provide evidence that stock mispricing ("bubbles" and "crashes") is higher ("stronger") if agents' beliefs regarding the true parameters of the stock market are more volatile. By simulating multivariate Dynamic Conditional Correlation (DCC) models, we also show that our model can produce Generalised Autoregressive Conditional Heteroscedastic (GARCH) effects and that different agent beliefs support different pairwise correlations between risky stocks. Our results hold true for each individual stock and for the aggregated stock market as a whole. In total we conduct 43200 Monte Carlo simulations as part of our analysis.

Keywords: agent-based modeling, mixed Gaussian models, stock return distributions, genetic algorithm, evolutionary learning, Monte Carlo simulations

JEL classification codes: C02, C30, C63, G12, G17, G40

*Corresponding author. Email: seedatshaheen@gmail.com. Telephone: +27 11 646 1152

1 Introduction

In this paper, we provide further evidence that returns of stock return distributions, over various periods and frequencies, are a mixture of distributions. To explain this empirical finding, we develop a *multiple-stock (i.e. multivariate)* agent-based model. We utilise a computational agent-based approach due to its numerous advantages. Computational agent-based models allow for agent heterogeneity and the majority use the power of a computer to run simulations from which models and forecasts can be derived. Market prices need not match up with rationally expected prices.

The literature on agent-based models of financial markets is vast and we refer the interested reader to LeBaron *et al.* (1999), Farmer *et al.* (2005), Raberto *et al.* (2001), Lux & Marchesi (1999), Arifovic (1996), Marengo & Tordjman (1996), Routledge (2001) and Day & Huang (1990). For multivariate models, we specifically refer the interested reader to Westerhoff (2004), Westerhoff & Dieci (2006), Chiarella *et al.* (2007) and Dieci & Westerhoff (2010).

While agent-based models of financial markets have noticeably advanced our understanding of various facets of financial markets, they do not explicitly model the entire shape of the stock return distribution and neither do they produce the distribution shapes of real-world stock returns documented in our paper. In our paper, we provide a model that addresses this void. We take the model of LeBaron *et al.* (1999) as a base case and modify it significantly. We assume that the dividend process is not state independent but, instead, a state dependent process. We include *multivariate* interacting heterogeneous smart agents (traders) in our model who trade the cross-section of underlying stocks and the risk free bond in an optimal manner and who make conjectures of future returns that are consistent with the structure of the stock market. Agents in our model do not utilise a simple trading rule of just investing in the equally weighted average of underlying stocks. Instead, they weight their investments depending on factors idiosyncratic and particular to each stock such as its average return, its average dividend payment and its expected volatility. These alterations lead to a significantly richer model. We develop new theoretical results regarding the portfolio optimisation of agents in an agent-based model. We show that our model can explain and produce a stock return distribution that resembles the real-world distributions we document. Most importantly, our model shows how trading behavior can aggregate individual stock weighting such that the market index is a mixed distribution.

We also we show, by (1) using (parameter variation) Monte Carlo simulations and (2) using a statistical model (namely (multivariate) regression with MLE), that our agent-based model is plausible in the sense that the relationships between stock prices and parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk free rate, the underlying stochastic dividend process and state probabilities are sensible in a financial economic sense. The first method (parameter variation) is developed in Seedat & Abelman (2019a) and details of the method are contained in that paper and the interested reader is referred to the paper. The second method - (multivariate) regression with MLE - allows us to (a) consider sample variance effects,

(b) account for the fact that prices have many response variables (and to account for confounding variables) and (c) offer a systematic approach to model selection. Furthermore, MLE (regression) is used as it has attractive asymptotic properties, namely consistency (convergence in probability to the true value) and asymptotic normality. Theoretical details of the second method are contained in Seedat & Abelman (2019a) and the interested reader is referred to the paper.

Furthermore, we are able to assess the degree of asset mispricing that occurs and we show that this increases if agents have more uncertain beliefs regarding the true parameters of the stock market. Lastly, by estimating multivariate DCC models, we illustrate that our model produces GARCH effects and that cross-sectional correlations are a function of agents' beliefs.

One can also explain our empirical distribution findings by developing a univariate model where smart traders trade just the equally weighted average of stocks in an equity index. (This is achieved in Seedat & Abelman (2019b).) However, in our paper we are cognisant of the fact that some real world traders do weight their asset holdings optimally. We then show that even in this case, it is possible for mixed distributions of stock returns to emerge.

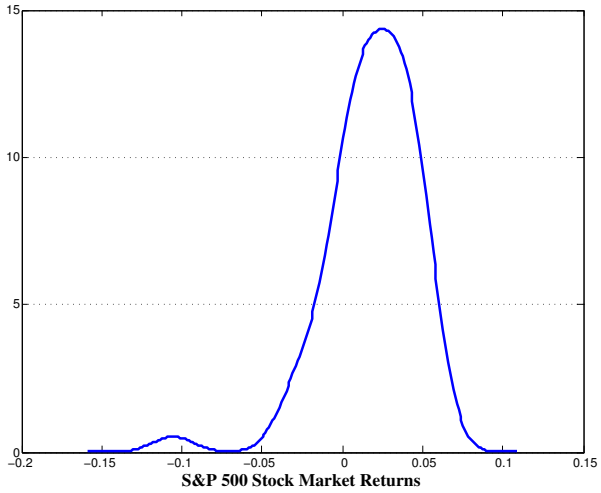
Our paper is structured as follows. Section 2 provides a motivation for our work by providing empirical evidence that real-world stock returns have a distribution that is a mixture of distributions. Section 3 provides our agent-based asset pricing model including, *inter alia*, details of the multiple-stock stock market, the multivariate dividend process (and the mixture probabilistic description of the model), heterogeneous interacting agents, forecasting rules and classifier system, price determination and the genetic algorithm. Section 4 contains extensive simulation results of the model including evidence that it can produce stock return distributions that match real-world distributions. Section 4 also provides evidence regarding the plausibility of the model and studies asset mispricing. Finally, Section 5 concludes the paper.

2 Motivation: Stock Return Distributions

Initial evidence was presented in Seedat & Abelman (2019b) that stock return distributions are, time-unconditionally and time-conditionally, a mixture of distributions. In this paper we provide considerably more evidence in this regard. We provide some new empirical observations which are the motivation for our work. Using monthly stock market data of the S&P 500 from Robert Shiller's website (www.econ.yale.edu/∼shiller), we show that various stock return distributions, gross with dividends, are a mixture of distributions. For example, during the post crisis period between June 2010 - June 2015, we have that the monthly gross stock return distribution is mixed as illustrated in Figure 1 were we plot the density estimate of monthly S&P 500 stock returns over the period June 2010 – June 2015 using data obtained from Robert Shiller's website. We estimate the density using a nonparameteric kernel regression with an Epanechnikov kernel, which is mean square optimal.

Our empirical observation is robust to various time periods and various frequencies. This evidence is contained in Figures 2 to 9 which illustrate that various aggregated stock return

Figure 1: Density Estimate of Monthly S&P 500 Stock Market Returns (June 2010 - June 2015) - Multivariate Mixed Model



distributions produce mixed effects. We note that various important financial economic periods appear here, notably the Great Depression, the dot-com bubble, the recent financial crisis and the post crisis period. Some of these graphs were included in Seedat & Abelman (2019b) and we include them here for completeness.

Figure 2: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel A

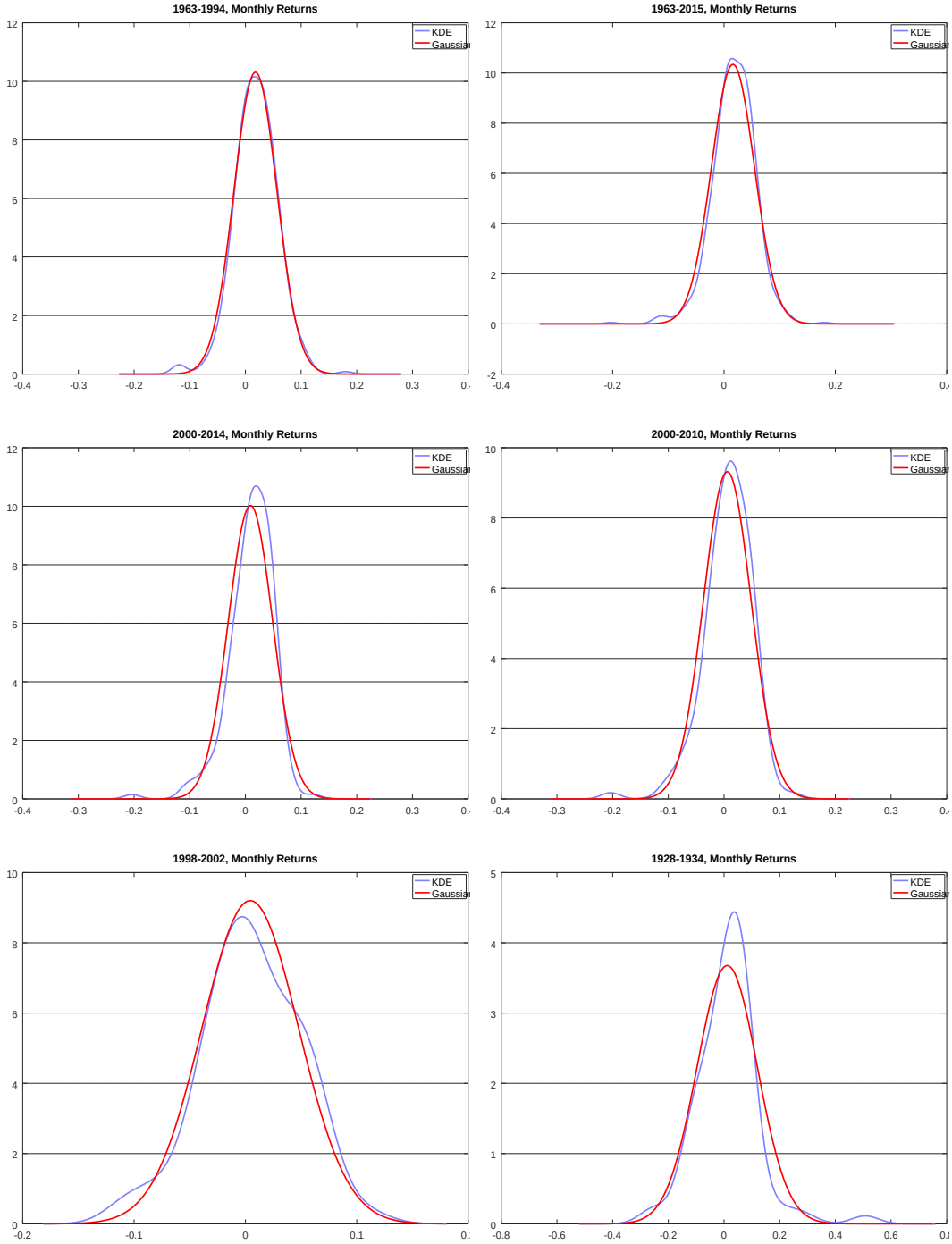


Figure 3: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel B

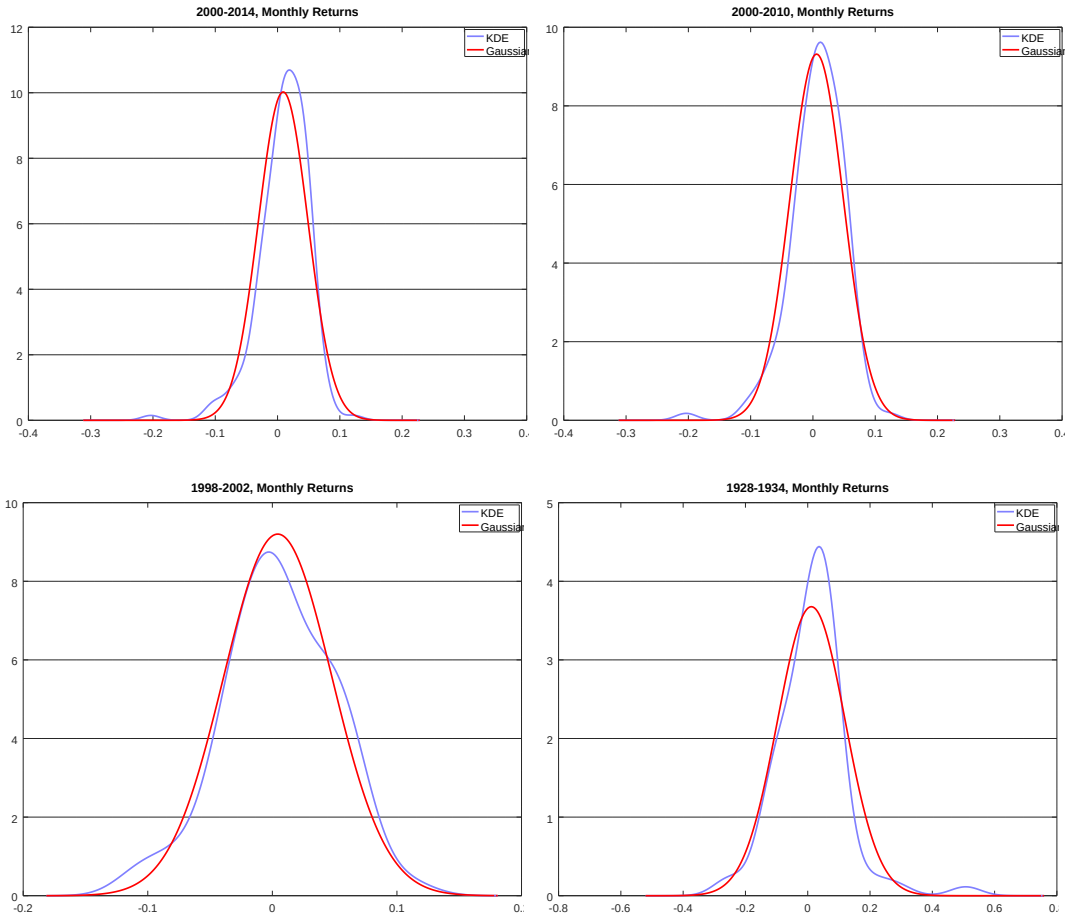


Figure 4: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel C

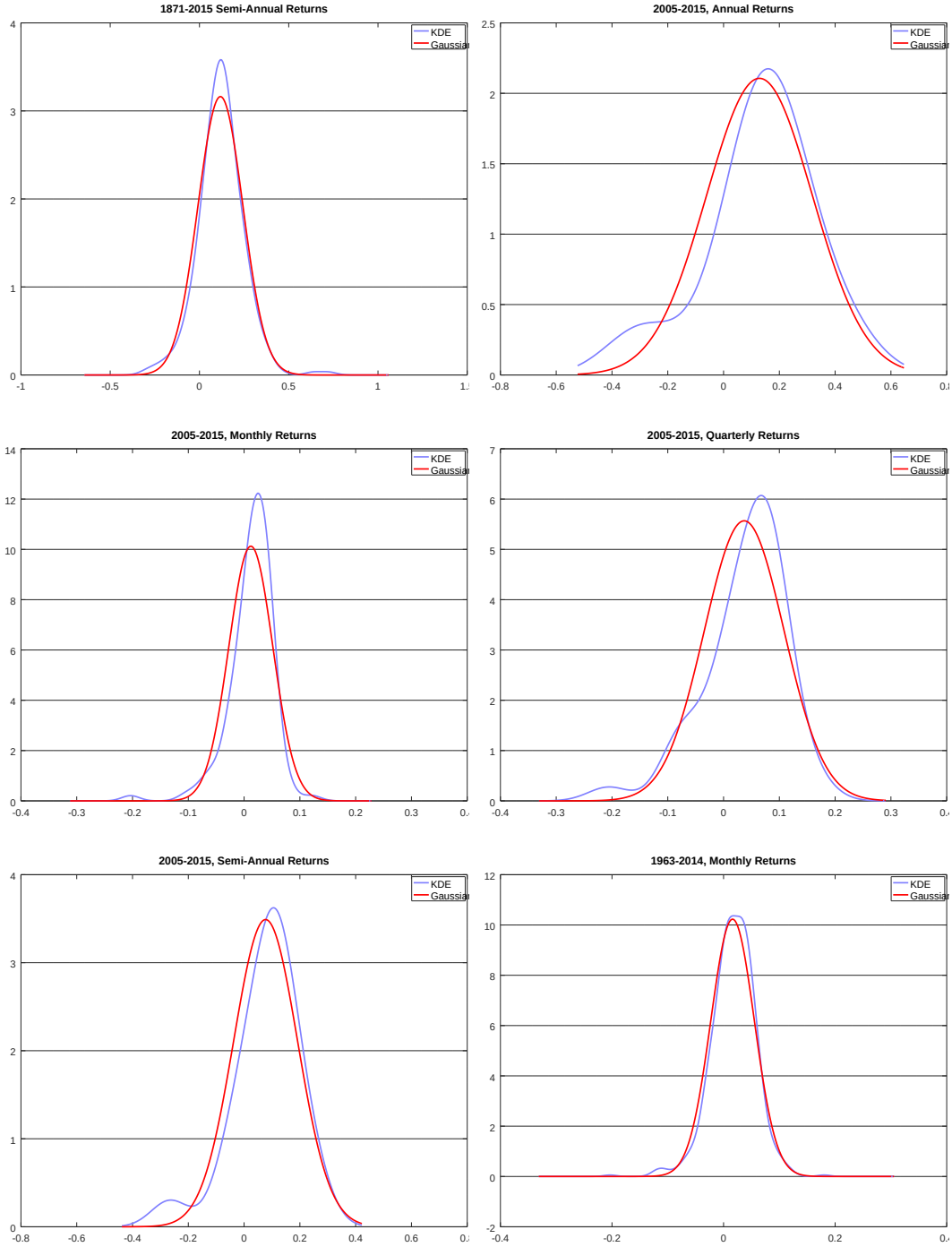


Figure 5: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel D

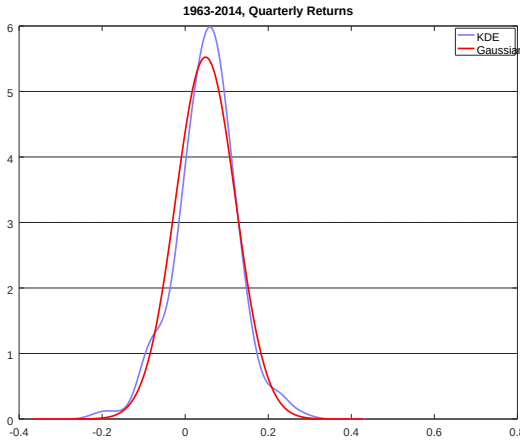


Figure 6: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel E

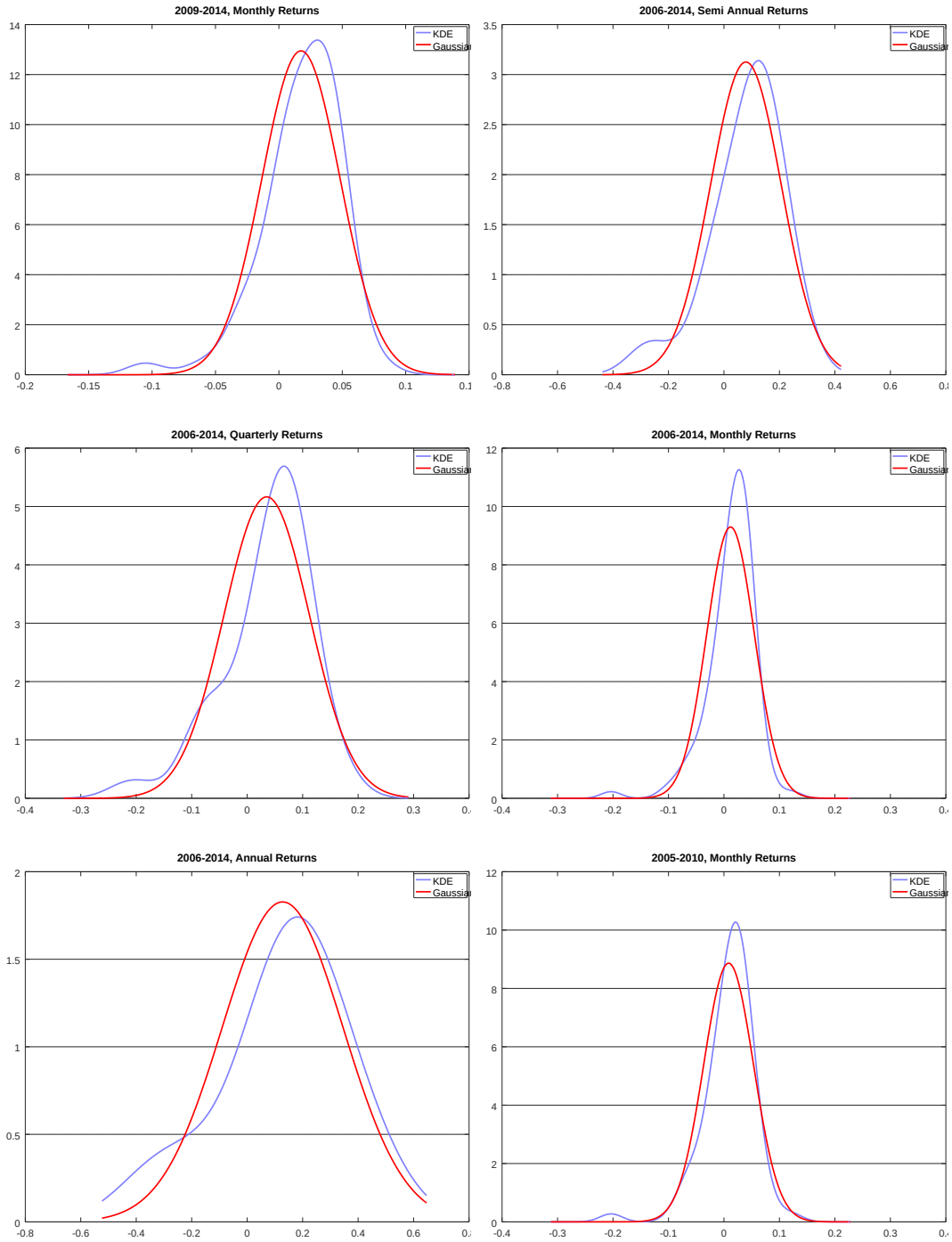


Figure 7: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel F

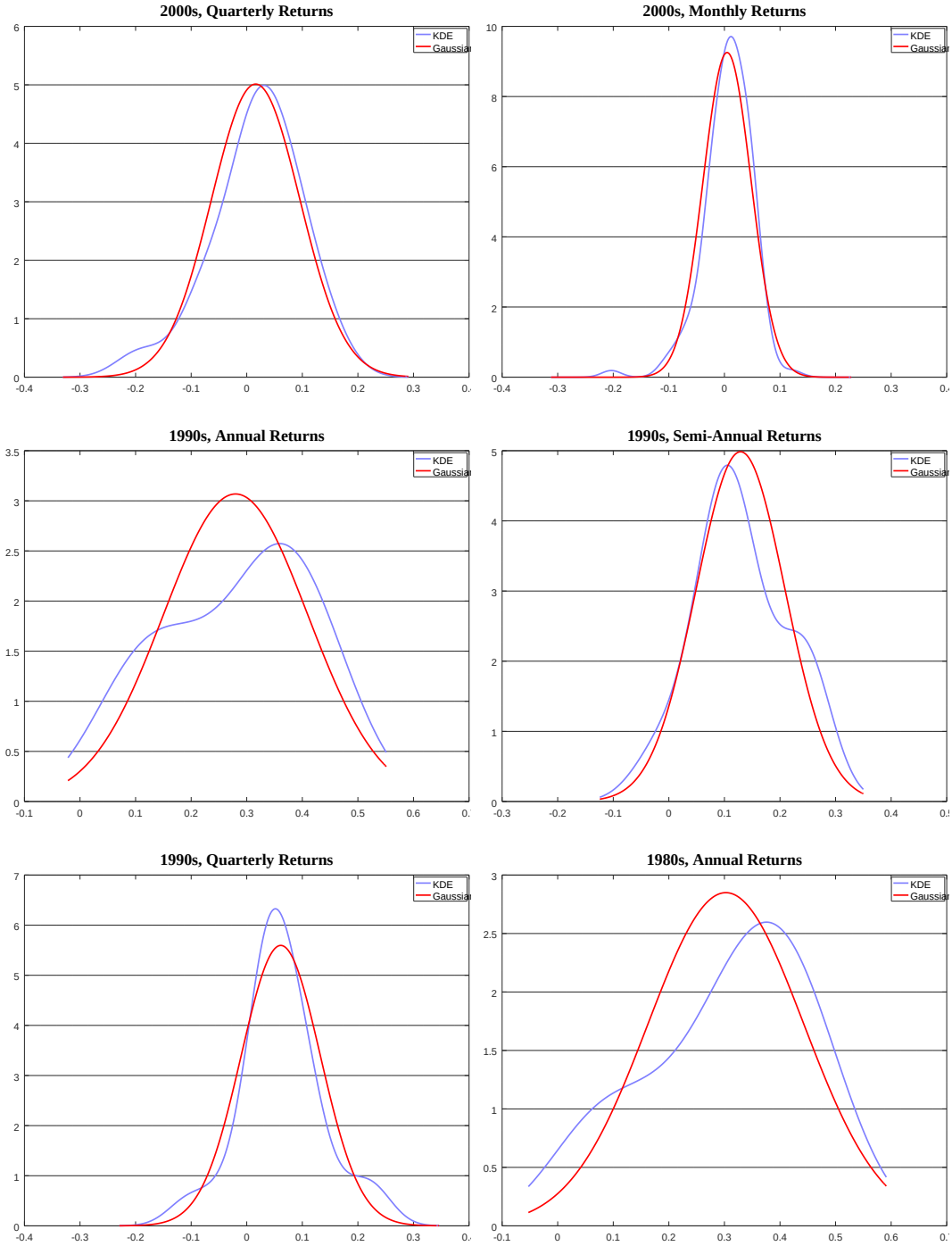


Figure 8: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel G

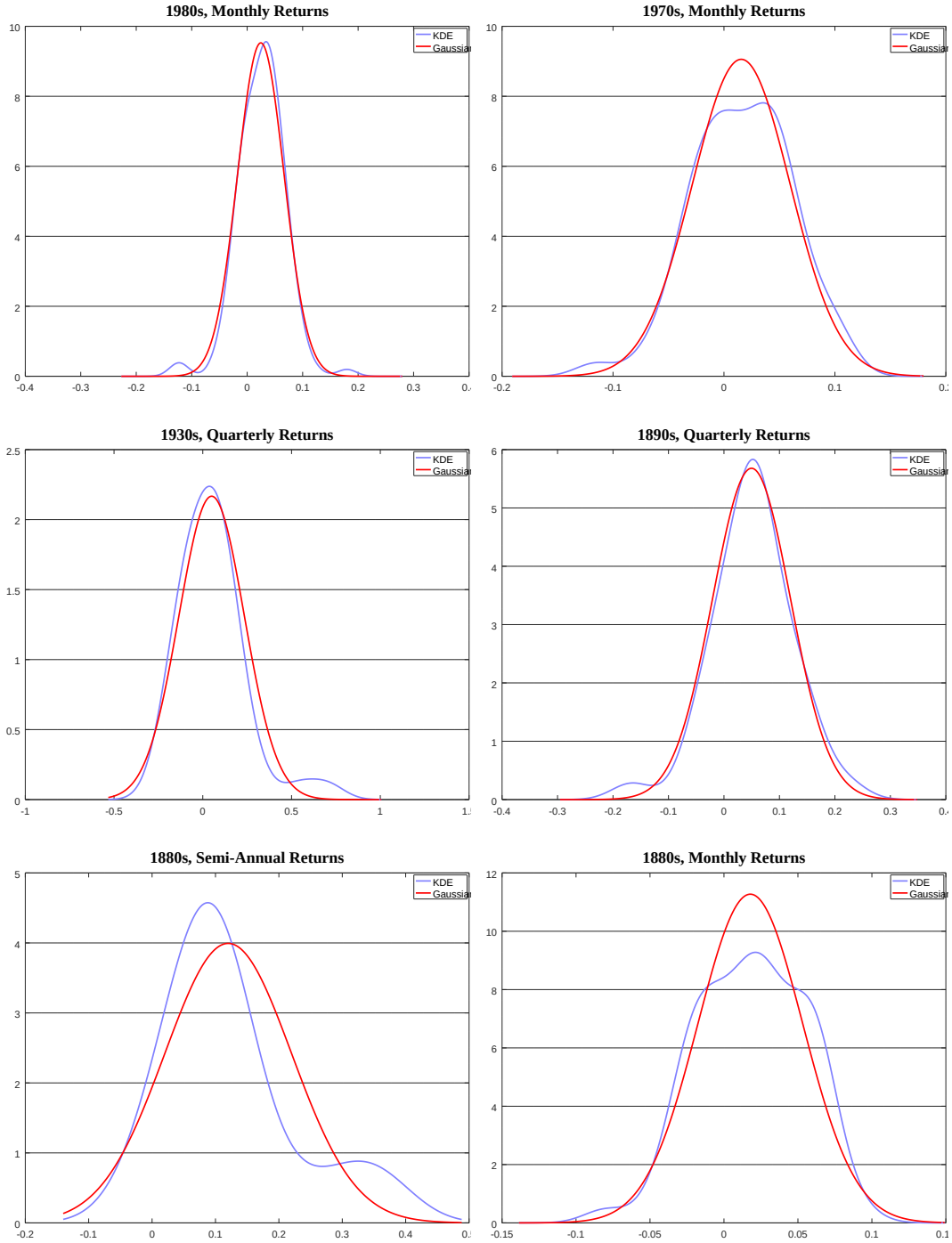
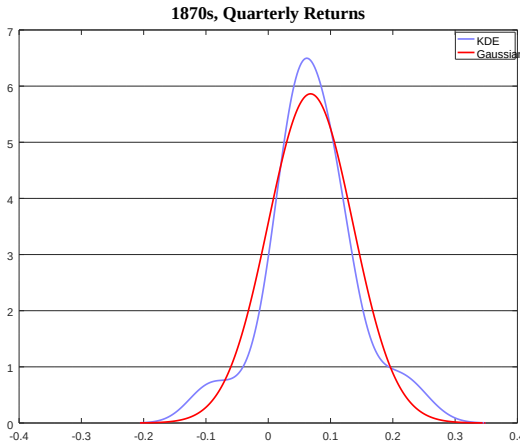


Figure 9: Kernel Density Estimates of S&P 500 Stock Market Returns (Multivariate Mixed Model) - Panel H



3 Stock Market Model

In this section, we discuss the various features of our model.

3.1 Stock market

The stock market consists of $n+1$ assets including n risky dividend-paying stocks and one risk-free bond. The model is of discrete time and infinite period. Let s_t denote the state variable at time t; there are one of J states possible at each time t. The price of the stocks at time t is denoted by the vector $\mathbf{P}_{t,s_t}$. The stocks pay a stream of dividends over their life spans. The price of the stock s is given by $P_{t,s_t}(s)$. We will denote the dividends paid at time t for the stocks by the vector $\mathbf{D}_{t,s_t}$. The risky asset is in fixed finite supply of N units. The risk-free asset pays interest at a rate r_f per period.

3.2 Dividend process

The dividend process is a *multivariate mixed* Gaussian first-order autoregressive process. This new assumption regarding the stochastic dividend process drastically alters some of the key properties of the process for theoretical prices, the manner in which agents will form beliefs about the stock market and their trading actions as well as the resulting distribution of stock returns, relative to exisiting agent-based models in finance. We will provide details of each of these in subsequent sections.

Before describing our dividend process more formally, we need to introduce the following definitions.

Definition 1 Multivariate Mixed Gaussian random variable.
If $\mathbf{T}$ is a random variable satisfying the following probability law

$$\mathbf{T} \overset{\text{law}}{=} \sum_{j=1}^{J} Q(j)\mathbf{T}_j$$

where $\{Q(j)\}_{j=1}^{J}$ is a set of probabilities satisfying Kolmogorov's three axioms of a probability measure and where each $\mathbf{T}_j \sim N(\mu_{\mathbf{j}}, \mathbf{\Sigma_j})$, then we say $\mathbf{T}$ is mixed Gaussian of size J and we write $\mathbf{T} \sim MG(J, \{\mu_{\mathbf{j}}, \mathbf{\Sigma_j}\}_{j=1}^{J}, \{Q(j)\}_{j=1}^{J})$.

Definition 2 Multivariate Mixed Gaussian stochastic process.
If $\{\mathbf{T}_t\}_t$ is a stochastic process such that each $\mathbf{T}_t \sim MG(J, \{\mu_{\mathbf{t,j}}, \mathbf{\Sigma_{t,j}}\}_{j=1}^{J}, \{Q_t(j)\}_{j=1}^{J})$, then we say $\{\mathbf{T}_t\}_t$ is a mixed Gaussian stochastic process of size J.

We model the multivariate dividend process as a multivariate mixed Gaussian stochastic process of the following form

$$\mathbf{D}_{t,s_t} = \bar{\mathbf{D}}_{s_t} + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_{s_t}) + \mathbf{u}_t$$

where s_t is a state variable that can take on one of finitely many values $1, ..., J$ at each time t, where $\bar{\mathbf{D}}_{s_t}$ is a constant for each s_t and where $\mathbf{u}_t$ is assumed to be a random variable such that $\mathbf{u}_t|\mathcal{I}_{t-1,s_{t-1}} \overset{law}{=} \mathbf{N}(\mathbf{0}, \mathbf{\Sigma}^2_{\mathbf{u}_t})$ and independent over time, where $\mathcal{I}_{t,s_t}$ denotes the set of *publicly available information* at time t and state s_t. $\mathbf{u}_t$ is also assumed to be an innovation or shock term and is thus independent of all other variables. Let $Q(j)$ denote the probability that $s_t = j$. Hence, at time t, the dividend process has the following probabilistic structure

$$\mathbf{D}_{t,s_t} = \begin{cases} \bar{\mathbf{D}}_1 + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_1) + \mathbf{u}_t & \text{with probability } Q(1) \\ \bar{\mathbf{D}}_2 + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_2) + \mathbf{u}_t & \text{with probability } Q(2) \\ \vdots & \vdots \\ \bar{\mathbf{D}}_J + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_J) + \mathbf{u}_t & \text{with probability } Q(J) \end{cases} \tag{1}$$

We can thus express the probability law of the dividend process as

$$\mathbf{D}_{t,s_t}\Big|\mathcal{I}_{t-1,s_{t-1}} \overset{law}{=} \sum_{s'_t=1}^{J} Q(s'_t)\mathbf{N}\Big(\bar{\mathbf{D}}_{s'_t} + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_{s'_t}), \mathbf{\Sigma}_{u_t}\Big)$$

where $\mathcal{I}_{t-1,s_{t-1}}$ denotes the time $t-1$ information set of publicly available information.

We thus have that our dividend process $\{\mathbf{D}_{t,s_t}\}_t$ is conditionally a mixed Gaussian stochastic process of size J where each $\mathbf{D}_{t,s_t}$ is mixed Gaussian such that

$$\mathbf{D}_{t,s_t}|\mathcal{I}_{t-1,s_{t-1}} \sim MG\big(J, \{\bar{\mathbf{D}}_{s_t} + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_{s_t}), \mathbf{\Sigma}_{u_t}\}_{j=1}^{J}, \{P(j)\}_{j=1}^{J}\big).$$

We can thus see that, when $J = 1$ and $n = 1$, the model of LeBaron *et al.* (1999), which is single stock and with a state independent process, is a degenerate case of our much more general model. At each point in time our model contains multiple stocks and the dividend process is explicitly state dependent and can take on one of J Gaussian stochastic processes each of which has a different mean $\bar{\mathbf{D}}_j$. The motivation for this is that the stock market may have different states, each corresponding to different average dividend payouts and each of which may have important implications for the underlying dividend process.

3.3 Agents

There are N smart traders in this model indexed by i. Each has one unit of each stock at the start of trading. We assume that these traders are multivariate in that they do not neccessarily invest in just the equally weighted average of all stocks underlying the S&P 500 index. Each agent is assumed one-period myopic with constant absolute risk aversion (CARA) preferences over their one-period ahead wealth with CARA coefficient λ. Each agent i has state-dependent wealth $W_{t,s_t,i}$ at the start of time t and must then divide their wealth between the assets through trading in the stock market. Their state-dependent asset holdings of the stock at time t is denoted $\mathbf{X}_{t,s_t,i}$ and variation over time in holdings of each asset is achieved by trades in the stock market.

We now proceed to the trading (i.e. portfolio) optimisation problem faced by each smart agent i. In the literature on computational agent-based models as well as in portfolio optimisation, a standard assumption used to solve portfolio optimisation problems where agents have CARA preferences is to assume that each agent assumes that future (gross discounted) returns are Gaussian, conditional on that agent's information set $\mathcal{I}_{t,s_t,i}$ at time t which we will assume will contain all publicly available information at time t. Since agents may also hold private information it is true that $\mathcal{I}_{t,s_t} \subseteq \mathcal{I}_{t,s_t,i}$. We can plausibly justify making such Gaussian assumptions when our view of the world is that the underlying source of uncertainty, which in this model is the multivariate dividend process, is multivariate Gaussian conditional on $\mathcal{I}_{t,i}$. If this is true, and if agents make the conditional Gaussian assumption (conditional on their information set), their beliefs regarding the distribution of future variables is consistent with the structure of the model. In this paper, our structure of the model and the world is of a mixed Gaussian one where the multivariate dividend process is mixed Gaussian of size J, conditional on the set of publicly available information $\mathcal{I}_{t,s_t}$. We assume that traders in our model are smart in the sense that they continue to have beliefs consistent with the stock market, even when the conditional dividend process is not a simple Gaussian process. Given the structure of dividends in our model, smart traders now assume that future (gross discounted) returns are mixed Gaussian of size J, conditional on their information set, $\mathcal{I}_{t,s_t,i}$. As we describe below, one period ahead wealth will also be mixed (*univariate*) Gaussian, conditional on $\mathcal{I}_{t,s_t,i}$. Hence, the resulting portfolio optimisation problem is non-standard and we need to derive a probability theory result regarding log mixed Gaussian random variables that will allow us to solve the optimisation problem faced by each trader. As we shall see, the portfolio optimisation that agents carry out in our model results in a new form of expectation formation (agent heterogeneity), much different to that of the model of LeBaron *et al.* (1999). Due to this difference, and the other differences above such as the structure on the dividend process, we develop a new computational agent-based model as the model of LeBaron *et al.* (1999) cannot account for these differences. We will also illustrate that the model of LeBaron *et al.* (1999) is insufficient from an alternate perspective: specifically, it cannot produce the distributions of real-world stock returns that we document empirically in this paper.

Let agent i's information set at time t and state s_t, $\mathcal{I}_{t,s_t,i}$, contain all prior and current prices,

dividends and states. We introduce the following notation to simplify statements of formal results. Denote i's conditional expectation and variance at time t over all random variables by $\mathbb{E}_{t,i}$ and $\mathbb{V}_{t,i}$, respectively., conditional on $\mathcal{I}_{t,s_t,i}$. Denote i's conditional expectation and variance at time t with respect to all random variables except the state variable s_{t+1} by $\tilde{\mathbb{E}}_{t,i}$ and $\tilde{\mathbb{V}}_{t,i}$, respectively, conditional on $\mathcal{I}_{t,s_t,i}$. Denote i's conditional expectation and variance at time t over the variables in the set $\mathcal{A}_{t+1,i} \subset \mathcal{I}_{i,t}$ by $\mathbb{E}_{t,i}^{\mathcal{A}_{t+1,i}}$ and $\mathbb{V}_{t,i}^{\mathcal{A}_{t+1,i}}$, respectively, conditional on $\mathcal{I}_{t,s_t,i}$. We now state the above discussion in more formal terms. Each agent makes consistent beliefs about the market structure, that is they assume that discounted one-period ahead gross returns are univariate mixed Gaussian of size J, conditional on their information set $\mathcal{I}_{t,s_t,i}$ at time t. As we shall see, this assumption, combined with CARA preferences and lemma 3, allows agents to solve for optimal holdings of the assets in terms of the first two moments of the price plus dividend distribution, conditional on $\mathcal{I}_{t,s_t,i}$, as well as the mixture probabilities $Q(1), ..., Q(J)$. Given that agents are assumed to form consistent beliefs about the structure of the market, they assume that

$$\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}} \;\Big|\; \mathcal{I}_{t,s_t,i} \;\sim\; \sum_{k=1}^{J} Q(k)\mathbf{N}\Big(\mu^k_{t+1,s_{t+1},i}, (\mathbf{\Sigma}^k_{t+1,s_{t+1},i})\Big).$$

where $\mu^k_{t+1,s_{t+1},i} := \tilde{\mathbb{E}}_{t,i}(\mathbf{P}^k_{t+1,s_{t+1}} + \mathbf{D}^k_{t+1,s_{t+1}})$ and $(\mathbf{\Sigma}^k_{t+1,s_{t+1},i}) := \tilde{\mathbb{V}}_{t,i}(\mathbf{P}^k_{t+1,s_{t+1}} + \mathbf{D}^k_{t+1,s_{t+1}})$ for the state dependent prices $\mathbf{P}^k_{t+1,s_{t+1}}$ and dividends $\mathbf{D}^k_{t+1,s_{t+1}}$, $k = 1 : J$. Each agent is also assumed to know the dividend structure of the economy and hence

$$\tilde{\mathbb{E}}_{t,i}(\mathbf{D}_{t+1,s_{t+1}}) = \tilde{\mathbb{E}}_{t,i}(\bar{\mathbf{D}}_{s_{t+1}} + \rho(\mathbf{D}_{t,s_t} - \bar{\mathbf{D}}_{s'_{t+1}})).$$

and $\tilde{\mathbb{V}}_{t,i}(\mathbf{D}_{t+1,s'_{t+1}}) = \mathbf{\Sigma}_{\mathbf{u}_{t+1}}$. Given the structure of the model, it is clear that $\mathbf{\Sigma}_{t,s'_{t+1},i} = \mathbf{\Sigma}_{t,s_{t+1},i}$ for all $s'_{t+1}, s_{t+1} \in \{1, 2, ..., J\}$. We can thus define $\mathbf{\Sigma}_{t,i} = \mathbf{\Sigma}_{t,s'_{t+1},i}$. We also define $\mu_{t,s_t,i} := \mathbb{E}_{t,i}(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}})$. Lastly, all the covariation at time t comes from the innovation $\mathbf{u}_t$ - that is, the state variable s_t only ever affects first moments. Hence, $\mathbb{V}$ and $\tilde{\mathbb{V}}$ are the same.

Given that prices and dividends are state-dependent (i.e. they they depend on the state s_t), wealth is also state dependent (and depends on the state s_t). Let $W_{t,s_t,i}$ denote agent i's wealth at time t and in state s_t by $W_{t,s_t,i}$. Since agents have one-period ahead CARA utility over their state-dependent wealth $W_{t+1,s_{t+1},i}$, their utility functions over their wealth at time $t+1$ has the state-dependent form $U(W_{t+1,s_{t+1},i}) = -\exp(-\lambda W_{t+1,s_{t+1},i})$. Given each agent's portfolio choice and trading optimisation problem discussed above, their wealth at $t+1$ satisfies:

$$W_{t+1,s_{t+1},i} = \mathbf{X}^\intercal_{t,s_t,i}(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}) + (1 + r_f)(W_{t,s_t,i} - \mathbf{X}^\intercal_{t,i}\mathbf{P}_{t,s_t}).$$

Finally, we can thus state each agent i's state-dependent trading optimisation problem at each time

t concisely as:

$$\max_{\mathbf{X}_{t,s_t,i}} \mathbb{E}_{t,i}\Bigg(-\exp\Big\{ -\lambda\Big(\mathbf{X}_{t,s_t,i}^{\intercal}(\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})+(1+r_f)(W_{t,s_t,i}-\mathbf{X}_{t,s_t,i}^{\intercal}\mathbf{P}_{t,s_t})\Big)\Big\}\Bigg)$$

Now define state-dependent optimal stock holdings as follows:

$$\mathbf{X}_{t,s_t,i}^{*} \equiv \arg\max_{\mathbf{X}_{t,s_t,i}} \mathbb{E}_{t,i}\Bigg(-\exp\Big\{ -\lambda\Big(\mathbf{X}_{t,s_t,i}^{\intercal}(\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})+(1+r_f)(W_{t,s_t,i}-\mathbf{X}_{t,i}^{\intercal}\mathbf{P}_{t,s_t})\Big)\Big\}\Bigg).$$

To solve for optimal stock holdings, we need a probability theory result regarding the expectation of a log *mixed Gaussian* random variable. This result is presented below and the proof is contained in Seedat & Abelman (2019b).

Theorem 3 *Expectation of a log mixed-normal random variable.*
Let V be a mixed Gaussian random variable of size J such that $V \sim MG(J, \{\mu_j, \sigma_j\}_{j=1}^{J}, \{Q(j)\}_{j=1}^{J})$. That is, V has the following probability law:

$$V \overset{\text{law}}{=} \sum_{j=1}^{J} Q(j)V_j$$

where $\{Q(j)\}_{j=1}^{J}$ is a set of probabilities satisfying Kolmogorov's three axioms of a probability measure and where each $V_j \sim N(\mu_j, \sigma_j)$. Suppose $(V_1, V_2, ..., V_J) \sim \mathbf{N}(\mu, \Sigma)$, then V is Gaussian with $V \sim N\big(\mathbb{E}(V), \text{Var}(V)\big)$, $\exp(V)$ is log-normal with

$$\mathbb{E}(\exp(V)) = \exp\left(\sum_{j=1}^{J} Q(j)\mu_j + \frac{1}{2}\left(\sum_{j=1}^{J}\big(Q(j)\big)^2 \text{var}\big(V_j\big) + \sum_{j=1, i\neq j} Q(i)Q(j)\text{Cov}\big(V_i, V_j\big)\right)\right)$$

Proof. Contained in Seedat & Abelman (2019b). ■

Theorem 4 *Portfolio optimisation (multiple-stock) under the mixed-Gaussian assumption*
Suppose agent i has mypoic, CARA preferences over their one-period ahead wealth with risk aversion parameter λ. Suppose there are $n+1$ assets, a risk-free asset with risk-free rate r_f and n risky stocks with price vector $\mathbf{P}_{t,s_t}$ and dividend vector $\mathbf{D}_{t,s_t}$ where t denotes time and s_t denotes the state at time t. For each state s_t, $\mathbf{P}_{t,s_t}$ is a random variable. Let $\mathcal{I}_{t,s_t}$ be the set of publicly available information at time t and in state s_t. Suppose i assumes that conditional one period ahead prices plus dividends are mixed Gaussian of size J with $\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}} | \mathcal{I}_{t,i,s_t} \sim MG(J, \{\mu^j_{t+1,s_{t+1},i}, \mathbf{\Sigma}^j_{t+1,s_{t+1},i}\}_{j=1}^{J}, \{Q(j)\}_{j=1}^{J})$ where $\mathcal{I}_{t,i,s_t}$ denotes i's information set that contains the entire history of prices, dividends and states. Suppose that i knows the entire history of prices, dividends and states that have been realised. Assume that agents assume the *joint* distribution of

conditional prices plus dividends is multivariate Gaussian given by

$$\begin{pmatrix} \mathbf{P}_{t+1,1} + \mathbf{D}_{t+1,1} \\ \mathbf{P}_{t+1,2} + \mathbf{D}_{t+1,2} \\ \cdot \\ \cdot \\ \mathbf{P}_{t+1,J} + \mathbf{D}_{t+1,J} \end{pmatrix} \Bigg| \mathcal{I}_{t,s_t,i} \sim \mathbf{N}\big(\mu, \mathbf{\Sigma}\big)$$

where $\mathbf{\Sigma}$ is the conditional variance-covariance matrix of prices plus dividends, which allows for correlated prices and dividends across states. Then optimal holdings of stocks at time t for agent i, $\mathbf{X}^*_{t,s_t,i}$ satisfies

$$\mathbf{X}^*_{t,s_t,i} = \frac{\lambda^{-1}\big(\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} - (1+r_f)\mathbf{P}_{t,s_t}\big)}{\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}|\mathcal{F}\big) + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}}|\mathcal{F}\big)}$$

where $\mathcal{F} = \mathcal{I}_{t,s_t,i}$ and where, by division, we mean matrix inversion.

Proof. Let P^1_{t,s_t} denote the price of stock 1 at time t and state s_t. Let $\mathbf{P}_{t,s_t} = \{P^1_{t,s_t}, P^2_{t,s_t}, ..., P^n_{t,s_t}\}$ denote the price vector at time t. Since conditional one-period ahead prices plus dividends are mixed Gaussian of size J with $\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} \sim MG(J, \{\mu^j_{t,s_{t+1},i}, \mathbf{\Sigma}^j_{t,s_{t+1},i}\}_{j=1}^J, \{Q(j)\}_{j=1}^J)$, there exists J random variables $\mathbf{P}^k_{t+1,s_{t+1}} + \mathbf{D}^k_{t+1,s_{t+1}}$, $k = 1, ..., J$, such that

$$\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}|\mathcal{I}_{t,i,s_t} \overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)(\mathbf{P}^k_{t+1,s_{t+1}} + \mathbf{D}^k_{t+1,s_{t+1}})$$

where each $\mathbf{P}^k_{t+1,s_{t+1}} + \mathbf{D}^k_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} \sim \mathbf{N}(\mu^k_{t,s_{t+1},i}, \mathbf{\Sigma}^k_{t,s_{t+1},i})$. Let

$$\mathbf{\Sigma}_{k_1,k_2} := \mathrm{Cov}\Big(\mathbf{P}^{k_1}_{t+1,s_{t+1}} + \mathbf{D}^{k_1}_{t+1,s_{t+1}}, \mathbf{P}^{k_2}_{t+1,s_{t+1}} + \mathbf{D}^{k_2}_{t+1,s_{t+1}} \Big| \mathcal{I}_{t,s_t,i}\Big)$$

Note that wealth of agents is now *state-dependent* and at time $t+1$ is given by

$$W_{t+1,s_{t+1},i} = \mathbf{X}^{\intercal}_{t,s_t,i}(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}) + (1+r_f)(W_{t,s_t,i} - \mathbf{X}^{\intercal}_{t,s_t,i}\mathbf{P}_{t,s_t})$$

where $W_{t+1,s_{t+1},i}$ is the wealth at time $t+1$ and state s_{t+1} and $\mathbf{X}_{t,s_t,i}$ is the stock holdings of i at t given the state at t was s_t. Conditional on $\mathcal{I}_{t,s_t,i}$, the only unknown random variables that are components of $W_{t+1,s_{t+1},i}$ at time t are $\mathbf{P}_{t+1,s_{t+1}}$ and $\mathbf{D}_{t+1,s_{t+1}}$. Since agents are assumed to know the entire history of prices, dividends and states then, at time $t+1$ their optimal stock holdings at time t are *state-dependent* and satisfy,

$$\mathbf{X}^*_{t,s_t,i} := \underset{\mathbf{X}_{t,s_t,i}}{\arg\max}\, \mathbb{E}^{\{s_t\}}_{t,i}\Big(-\exp\big(-\lambda W_{t+1,s_{t+1},i}\big)\Big)$$

where $\mathbb{E}^{\{s_t\}}_{t,i}(\cdot)$ denotes $\mathbb{E}(\cdot|\mathcal{I}_{t,s_t,i})$.

Now as

$$\begin{aligned}\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}}|\mathcal{I}_{t,s_t,i} &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)(\mathbf{P}^k_{t+1,s_{t+1}}+\mathbf{D}^k_{t+1,s_{t+1}}) \\ &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)\mathbf{N}(\mu^k_{t+1,s_{t+1},i},\boldsymbol{\Sigma}^{\mathbf{k}}_{\mathbf{t+1},\mathbf{s_{t+1}},\mathbf{i}})\end{aligned}$$

then

$$W_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} \overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)N\big(\mu^k_{W_{t+1},s_{t+1},i},(\sigma^k_{W_{t+1},s_{t+1},i})^2\big)$$

That is, $W_{t+1,s_{t+1},i}$ is a univariate mixed Gaussian random variable of the form $W_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} \sim MG(J,\{\mu^k_{W_{t+1,s_{t+1}},i},\sigma^k_{W_{t+1},s_{t+1},i}\}_{k=1}^J,\{Q(k)\}_{k=1}^J)$ and where

$$\begin{aligned}\mu_{W_{t+1,s_{t+1},i}} &:= \mathbb{E}\big(W_{t+1,s_{t+1},i}\big|\mathcal{I}_{t,,s_t,i}\big) \\ &= \mathbf{X}^{\intercal}_{t,s_t,i}\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t+1,s_{t+1},i}+(1+r_f)(W_{t,s_t,i}-\mathbf{X}^{\intercal}_{t,s_t,i}\mathbf{P}_{t,s_t})\end{aligned}$$

where $\mu_{t+1,s_{t+1},i}=\tilde{\mathbb{E}}_{t,i}(\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})$ and where the last line follows since the right hand side is equal to the left hand side. Also,

$$\begin{aligned}\boldsymbol{\Sigma}_{W_{t+1,s_t+1,i}} &:= \mathrm{var}\big(W_{t+1,s_{t+1},i}\big|\mathcal{I}_{t,s_t,i}\big) \\ &= \mathrm{var}\big(\mathbf{X}^{\intercal}_{t,s_t,i}(\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})\big|\mathcal{I}_{t,s_t,i}\big) \\ &= \mathbf{X}^{\intercal}_{t,s_t,i}\mathrm{var}\big((\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})\big|\mathcal{I}_{t,s_t,i}\big)\mathbf{X}_{t,s_t,i}\end{aligned}$$

Now as

$$\begin{aligned}\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}}|\mathcal{I}_{t,i,s_t} &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)(\mathbf{P}^k_{t+1,s_{t+1}}+\mathbf{D}^k_{t+1,s_{t+1}}) \\ &\overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)\mathbf{N}(\mu^k_{t,s_{t+1},i},\boldsymbol{\Sigma}^{\mathbf{k}}_{\mathbf{t},\mathbf{s_{t+1}},\mathbf{i}})\end{aligned}$$

then

$$\begin{aligned}\text{var}\big((\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})\big|\mathcal{I}_{t,s_t,i}\big) &= \text{Var}\Big(\sum_{k=1}^{J} Q(k)(\mathbf{P}^k_{t+1,s_{t+1}}+\mathbf{D}^k_{t+1,s_{t+1}})\Big)\\ &= \sum_{v=1}^{J} Q^2(v)\text{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}}+\mathbf{D}^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\\ &+ \sum_{v,w\neq v}^{J} Q(v)Q(w)\text{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}}+\mathbf{D}^v_{t+1,s_{t+1}},\mathbf{P}^w_{t+1,s_{t+1}}+\mathbf{D}^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\end{aligned}$$

Hence

$$\begin{aligned}\sigma^2_{W_{t+1,s_t+1,i}} = \mathbf{X}^{\intercal}_{t,s_t,i}\Bigg(&\sum_{v=1}^{J} Q^2(v)\text{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}}+\mathbf{D}^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\\ &+\sum_{v,w\neq v}^{J} Q(v)Q(w)\text{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}}+\mathbf{D}^v_{t+1,s_{t+1}},\mathbf{P}^w_{t+1,s_{t+1}}+\mathbf{D}^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\mathbf{X}_{t,s_t,i}\end{aligned}$$

Since $W_{t+1,s_{t+1},i}$ is mixed Gaussian with $W_{t+1,s_{t+1},i}|\mathcal{I}_{t,i,s_t} \sim MG(J,\{\mu^k_{W_{t+1,s_{t+1},i}},\sigma^k_{W_{t+1},s_{t+1},i}\}_{k=1}^J,\{Q(k)\}_{k=1}^J)$, and since

$$W_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} = \mathbf{X}^{\intercal}_{t,s_t,i}(\mathbf{P}_{t+1,s_{t+1}}+\mathbf{D}_{t+1,s_{t+1}})+(1+r_f)(W_{t,s_t,i}-\mathbf{X}^{\intercal}_{t,s_t,i}\mathbf{P}_{t,s_t})|\mathcal{I}_{t,s_t,i}$$

there exists J random variables $W^k_{t+1,s_{t+1},i}$, $k=1,...J$ such that

$$W_{t+1,s_{t+1},i}|\mathcal{I}_{t,i,s_t} \overset{\text{law}}{=} \sum_{k=1}^{J} Q(k)W^k_{t+1,s_{t+1},i}$$

where each $W^k_{t+1,s_{t+1},i}|\mathcal{I}_{t,i,s_t} \sim N(\mu^k_{W_{t,s_{t+1},i}},\sigma^k_{W_{t,s_{t+1},i}})$ and where

$$W^k_{t+1,s_{t+1},i}|\mathcal{I}_{t,s_t,i} = \mathbf{X}^{\intercal}_{t,s_t,i}(\mathbf{P}^k_{t+1,s_{t+1}}+\mathbf{D}^k_{t+1,s_{t+1}})+(1+r_f)(W_{t,s_t,i}-\mathbf{X}^{\intercal}_{t,s_t,i}\mathbf{P}_{t,s_t})|\mathcal{I}_{t,s_t,i}$$

Moreover, since

$$\begin{pmatrix} \mathbf{P}_{t+1,1} + \mathbf{D}_{t+1,1} \\ \mathbf{P}_{t+1,2} + \mathbf{D}_{t+1,2} \\ \cdot \\ \cdot \\ \mathbf{P}_{t+1,J} + \mathbf{D}_{t+1,J} \end{pmatrix} \Bigg| \mathcal{I}_{t,s_t,i}$$

is multivariate Gaussian we have that

$$\begin{pmatrix} W_{t+1,1,i} \\ W_{t+1,2,i} \\ \cdot \\ \cdot \\ W_{t+1,J,i} \end{pmatrix} \Bigg| \mathcal{I}_{t,s_t,i}$$

is multivariate Gaussian. Thus, the conditions for the above lemma apply to $W_{t+1,s_{t+1},i}$ and to $-\lambda W_{t+1,s_{t+1},i}$ and we conclude that $-\lambda W_{t+1,s_{t+1},i}$ is log mixed-normal and that

$$\begin{aligned}
\mathbb{E}\Big(\exp\big(-\lambda W_{t+1,s_{t+1},i}\big) \big| \mathcal{I}_{t,s_t,i}\Big) &= \exp\left(\mathbb{E}\big(-\lambda W_{t+1,s_{t+1},i} \big| \mathcal{I}_{t,s_t,i}\big) + \frac{1}{2}\mathrm{Var}\big(-\lambda W_{t+1,s_{t+1},i} \big| \mathcal{I}_{t,s_t,i}\big) \right) \\
&= \exp\left(-\lambda \mu_{W_{t+1,s_{t+1},i}} + \frac{1}{2}\lambda^2 \sigma^2_{W_{t+1,s_{t+1},i}} \right) \\
&= \exp\Bigg(-\lambda \Bigg(\mathbf{X}^{\intercal}_{t,s_t,i} \mathbb{E}^{\{s_{t+1}\}}_{t,i} \mu_{t+1,s_{t+1},i} + (1+r_f)(W_{t,s_t,i} - \mathbf{X}^{\intercal}_{t,s_t,i}\mathbf{P}_{t,s_t}) \Bigg) + \\
&\quad \frac{1}{2}\lambda^2 \mathbf{X}^{\intercal}_{t,s_t,i} \Bigg(\sum_{v=1}^{J} Q^2(v) \mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}} \big| \mathcal{I}_{t,s_t,i}\big) \\
&\quad + \sum_{v,w \neq v}^{J} Q(v)Q(w) \mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}} \big| \mathcal{I}_{t,s_t,i}\big) \Bigg) \mathbf{X}_{t,s_t,i} \Bigg)
\end{aligned}$$

Now by state-dependent optimisation

$$\begin{aligned}
\mathbf{X}^*_{t,s_t,i} &:= \arg\max_{X_{t,s_t,i}} \mathbb{E}^{\{s_t\}}_{t,i}\Big(-\exp\big(-\lambda W_{t+1,s_t,i}\big)\Big) \\
&= \arg\max_{X_{t,s_t,i}} \mathbb{E}\Big(-\exp\big(-\lambda W_{t+1,s_{t+1},i}\big)\Big|\mathcal{I}_{t,i,s_t}\Big) \\
&= \arg\min_{X_{t,s_t,i}} \mathbb{E}\Big(\exp\big(-\lambda W_{t+1,s_{t+1},i}\big)\Big|\mathcal{I}_{t,i,s_t}\Big) \\
&= \arg\min_{X_{t,s_t,i}} \exp\Bigg(-\lambda\Bigg(\mathbf{X}^\intercal_{t,s_t,i}\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} + (1+r_f)(W_{t,s_t,i} - \mathbf{X}^\intercal_{t,s_t,i}\mathbf{P}_{t,s_t})\Bigg) + \\
&\quad \frac{1}{2}\lambda^2\mathbf{X}^\intercal_{t,s_t,i}\Bigg(\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big) \\
&\quad + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\mathbf{X}_{t,s_t,i}\Bigg) \\
&= \arg\min_{X_{t,s_t,i}} \Bigg(-\lambda\Bigg(\mathbf{X}^\intercal_{t,s_t,i}\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} + (1+r_f)(W_{t,s_t,i} - \mathbf{X}^\intercal_{t,s_t,i}\mathbf{P}_{t,s_t})\Bigg) + \\
&\quad \frac{1}{2}\lambda^2\mathbf{X}^\intercal_{t,s_t,i}\Bigg(\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big) \\
&\quad + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\mathbf{X}_{t,s_t,i}\Bigg)
\end{aligned}$$

We can solve the above by state-dependent optimisation to yield:

$$\begin{aligned}
&\lambda\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} + \lambda(1+r_f)\mathbf{P}_{t,s_t} + \lambda^2\mathbf{X}^\intercal_{t,s_t,i}\Bigg(\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big) + \\
&\quad \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}}\big|\mathcal{I}_{t,s_t,i}\big)\Bigg)\mathbf{X}_{t,s_t,i} = 0
\end{aligned}$$

Rearranging yields the result:

$$\mathbf{X}^*_{t,s_t,i} = \frac{\lambda^{-1}\big(\mathbb{E}^{\{s_{t+1}\}}_{t,i}\mu_{t,s_{t+1},i} - (1+r_f)\mathbf{P}_{ts_t}\big)}{\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}|\mathcal{F}\big) + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}}|\mathcal{F}\big)}$$

where $\mathcal{F} = \mathcal{I}_{t,s_t,i}$ and where, by division, we mean matrix inversion.

■

Each agent is assumed to have 1 share of each stock.

Given the set up of this model, agents are heterogeneous as they have varying forecasts of futures prices and dividends (i.e. $\mu_{t,i}$) and estimates of the variances and covariances of these. This

expectation formation, which is state-dependent and dependent on the future state probability measure, is new to the literature and adds an additional level of richness to pre-existing models in the literature. State dependence as we have introduced it is the reason a new computational-agent based model needs to be developed. Each agent will form their expectations in different manners, forming forecasts about a new linear pricing equation. This new method of expectation formation that we introduce is novel to the literature. Learning and expectation formation occurs in our paper occurs relative to a new market structure which has a different probabilistic description to existing models.

3.4 Homogeneous linear rational expectations equilibrium (REE)

In some of the simulation work in our model, we will compare the time series of prices produced by the computational agent-based model to theoretical prices. This will allow us to determine the degree of mispricing (positive and negative "bubbles") that can arise in our computational agent-based model, as is observed in the real-world. Before we proceed, we thus need to solve for the homogeneous linear rational expectations equilibrium in our new stock model which is now driven by a mixed Gaussian dividend process. In a homogeneous equilibrium of the stock market, all stock market agents are perfectly identical. If all stock market agents are identical, then by the no-trade theorem there is no trade in the stock market and thus agents will, in every period, hold their initial stock allocation (i.e 1 share of each stock). A linear rational expectations equilibrium is one where agents are optimizing and the price function as a function of underlying variables (here dividends) is linear.

To solve for a linear REE in our single stock model driven by mixed Gaussian dividends, let us conjecture that a linear solution exists,

$$\mathbf{P}_{t,s_t} = \mathbf{A}\mathbf{D}_{t,s_t} + \mathbf{B}$$

then enforce agent optimisation (theorem 4) and the no trade theorem constraint ($\mathbf{X}^*_{t,s_t,i} = \mathbf{1}$ for all agents and times). Solving this system yields the solution

Theorem 5 Under the maintained assumptions above, the linear REE solution satisfies:

$$\mathbf{P}_{t,s_t} = \mathbf{A}\mathbf{D}_{t,s_t} + \mathbf{B}$$

where

$$\mathbf{A} = \frac{\rho \mathbf{I}}{1 + r_f - \rho}$$
$$\mathbf{B} = \frac{(1-\rho)(\mathbf{A} - \mathbf{I}) \sum_{j=1}^{J} Q(j) \bar{\mathbf{D}}_j - \lambda \mathbf{\Sigma}_{t,s_t,i}}{r_f}.$$

and where

$$\mathbf{\Sigma}_{t+1,s_{t+1},i} = \left(\sum_{v=1}^{J} Q^2(v)\, Var\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}} \big| \mathcal{F}\big) + \sum_{v,w \neq v}^{J} Q(v)Q(w)\, Cov\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}} \big| \mathcal{F}\big) \right) \tag{2}$$

and where $\mathcal{F} = \mathcal{I}_{t,s_t,i}$

Proof. *Recall that the dividend process satisfies*

$$\mathbf{D}_{t,s_t} = \bar{\mathbf{D}}_{s_t} + \rho(\mathbf{D}_{t-1,s_{t-1}} - \bar{\mathbf{D}}_{s_t}) + \mathbf{u}_t$$

We conjecture that a linear solution of the form below exists:

$$\mathbf{P}_{t,s_t} = \mathbf{A}\mathbf{D}_{t,s_t} + \mathbf{B}$$

Now by the above theorem, we have that optimal stock holdings satisfy:

$$X^*_{t,i} = \frac{\mathbb{E}^{\{s_{t+1}\}}_{t,i}(\mu_{t+1,s_{t+1},i}) - (1+r_f)\mathbf{P}_{t,s_t}}{\lambda \mathbf{\Sigma}_{t+1,s_{t+1},i}} \tag{3}$$

where

$$\mu_{t+1,s_{t+1},i} := \tilde{\mathbb{E}}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1,s_{t+1}}) \tag{4}$$
$$\mathbf{\Sigma}_{t+1,s_{t+1},i} := \tilde{\mathbb{V}}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1,s_{t+1}}) \tag{5}$$
$$= \left(\sum_{v=1}^{J} Q^2(v)\, Var\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}} \big| \mathcal{F}\big) + \sum_{v,w \neq v}^{J} Q(v)Q(w)\, Cov\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}} \big| \mathcal{F}\big) \right) \tag{6}$$

where $\mathcal{F} = \mathcal{I}_{t,s_t,i}$ *and where, by division, we mean matrix inversion and where* $\tilde{\mathbb{E}}$ *and* $\tilde{\mathbb{V}}$ *are the expectation and variance over all variables except the state* s_t*, respectively.*

By the no-trade condition we have that $\mathbf{X}^*_{t,s_t,i} = \mathbf{1}\ \forall i, t.$

Note that as agents are smart and know the market structure,

$$\begin{aligned}\mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}(\mathbf{D}_{t+1,s_{t+1}}) &= \mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}\big(\bar{\mathbf{D}}_{s_{t+1}} + \rho(\mathbf{D}_{t,s_t} - \bar{\mathbf{D}}_{s_{t+1}}) + \mathbf{u}_t\big) \\ &= \sum_{j=1}^{J} Q(j)\Big(\bar{\mathbf{D}}_j + \rho\mathbf{D}_{t,s_t} - \bar{\mathbf{D}}_j\Big) \\ &= (1-\rho)\sum_{j=1}^{J} Q(j)\bar{\mathbf{D}}_j + \rho\mathbf{D}_{t,s_t}\end{aligned}$$

Optimal holdings of stocks and the no-trade theorem result imply that

$$\lambda\boldsymbol{\Sigma}_{t+1,s_{t+1},i} = \mathbb{E}_{t,i}^{\{s_{t+1}\}}\mu_{t+1,s_{t+1},i} - (1+r_f)\mathbf{P}_{t,s_t}$$

Now

$$\begin{aligned}\mathbb{E}_{t,i}^{\{s_{t+1}\}}\mu_{t+1,s_{t+1},i} &= \mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}) \\ &= \mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}(\mathbf{A}\mathbf{D}_{t+1,s_{t+1}} + \mathbf{B} + \mathbf{D}_{t+1,s_{t+1}}) \\ &= (\mathbf{A}+\mathbf{1})\mathbb{E}_{t,i}^{\{s_{t+1}\}}\tilde{\mathbb{E}}_{t,i}\mathbf{D}_{t+1,s_{t+1}} + \mathbf{B} \\ &= (\mathbf{A}+\mathbf{1})\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{\mathbf{D}}_j + \rho\mathbf{D}_{s_t})\Big) + \mathbf{B}\end{aligned}$$

Hence

$$\begin{aligned}\lambda\boldsymbol{\Sigma}_{t+1,s_{t+1},i} &= (\mathbf{A}+\mathbf{1})\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{\mathbf{D}}_j + \rho\mathbf{D}_{s_t})\Big) + \mathbf{B} - (1+r_f)\mathbf{P}_{t,s_t} \\ &= (\mathbf{A}+\mathbf{1})\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{\mathbf{D}}_j + \rho\mathbf{D}_{s_t})\Big) - \mathbf{A}\mathbf{D}_{t,s_t} - r_f\mathbf{P}_{t,s_t}\end{aligned}$$

Solving for prices yields

$$\mathbf{P}_{t,s_t} = \frac{(\mathbf{A}+\mathbf{1})\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{\mathbf{D}}_j\Big) - \lambda\boldsymbol{\Sigma}_{t+1,s_{t+1},i}}{r_f} + \frac{\rho(\mathbf{A}+\mathbf{1}) - \mathbf{A}}{r_f}\mathbf{D}_{t,s_t}$$

Hence the solution for $\mathbf{B}$ *is*

$$\mathbf{B} = \frac{(\mathbf{A}+\mathbf{1})\Big(((1-\rho)\sum_{j=1}^{J} Q(j)\bar{\mathbf{D}}_j\Big) - \lambda\boldsymbol{\Sigma}_{t+1,s_{t+1},i}}{r_f}$$

from which the solution for $\mathbf{A}$ *is*

$$\mathbf{A} = \frac{\rho \mathbf{I}}{1 + r_f - \rho}$$

■

This provides us with a process of theoretical stock prices in our stock model based on a homogeneous linear REE. Note that the homogeneous linear REE that we derive in our paper is a significant generalisation of the one derived in LeBaron *et al.* (1999). This is due to our mixed state representation. In computations, we must use the linear homogeneous pricing solution to obtain theoretical prices from which the simulated prices obtained from our agent-based model are compared. Based on this REE solution, we can compute optimal mean squared error forecasts of prices plus dividends in the class of linear forecasts. This has the solution given by the following theorem:

Theorem 6 Under the maintained assumptions above, the optimal mean squared error (MSE) forecasts of prices plus dividends in the class of linear forecasts satisfies:

$$\mathbb{E}_{t,i}\big(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}\big) = \rho\big(\mathbf{P}_{t,s_t} + \mathbf{D}_{t,s_t}\big) + (1-\rho)\Big((\mathbf{A}+\mathbf{I})\sum_{j=1}^{J} P(j)\bar{\mathbf{D}}_j + \mathbf{B}\Big)$$

Proof.

$$\mathbf{P}_{t,s_t} = \mathbf{A}\mathbf{D}_{t,s_t} + \mathbf{B}$$

with $\mathbf{A}$ *and* $\mathbf{B}$ *computed as above.*

The MSE optimal forecast is the conditional expectation, conditional on everything inside the information set of $\mathbb{E}_{t,i}$.

Hence, the MSE optimal forecast of $\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_t+1}$ *is given by*

$$
\begin{aligned}
\mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1,s_t+1}) &= \mathbb{E}_{t,i}(\mathbf{A}\mathbf{D}_{t+1,s_{t+1}} + \mathbf{B} + \mathbf{D}_{t+1,s_t+1}) \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}(\mathbf{D}_{t+1,s_{t+1}}) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}^{s_{t+1}}\tilde{\mathbb{E}}_{t,i}(\mathbf{D}_{t+1,s_{t+1}}) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}^{s_{t+1}}\tilde{\mathbb{E}}_{t,i}(\bar{\mathbf{D}}_{s_{t+1}} + \rho(\mathbf{D}_{t,s_t} - \bar{\mathbf{D}}_{s_{t+1}}) + \mathbf{u}_{t+1}) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}^{s_{t+1}}(\bar{\mathbf{D}}_{s_{t+1}} + \rho(\mathbf{D}_{t,s_t} - \bar{\mathbf{D}}_{s_{t+1}})) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})\mathbb{E}_{t,i}^{s_{t+1}}((1-\rho)\bar{\mathbf{D}}_{s_{t+1}} + \rho\mathbf{D}_{t,s_t}) + \mathbf{B} \\
&= (\mathbf{A} + \mathbf{I})((1-\rho)\sum_{j=1}^{J}\bar{\mathbf{D}}_j + \rho\mathbf{D}_{t,s_t}) + \mathbf{B} \\
&= \rho(\mathbf{A} + \mathbf{I})\mathbf{D}_{t,s_t} + \Big((\mathbf{A} + \mathbf{I})((1-\rho)\sum_{j=1}^{J}\bar{\mathbf{D}}_j + \mathbf{B}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathbf{I} + \mathbf{A})\mathbf{D}_{t,s_t} &= \mathbf{D}_{t,s_t} + \mathbf{A}\mathbf{D}_{t,s_t} \\
&= \mathbf{D}_{t,s_t} + \mathbf{P}_t - \mathbf{B} \\
&= (\mathbf{P}_t + \mathbf{D}_{t,s_t}) - \mathbf{B}
\end{aligned}
$$

Hence,

$$
\rho(\mathbf{I} + \mathbf{A})\mathbf{D}_{t,s_t} = \rho(\mathbf{P}_t + \mathbf{D}_{t,s_t}) - \rho\mathbf{B}
$$

Thus

$$
\mathbb{E}_{t,i}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1,s_t+1}) = \rho(\mathbf{P}_t + \mathbf{D}_{t,s_t}) + (1-\rho)\left((\mathbf{A} + \mathbf{I})\sum_{j=1}^{J}P(j)\mathbf{D}_j + \mathbf{B}\right)
$$

We will denote:

$$
a_{re} = \rho
$$

and

$$\mathbf{b}_{re} = (1-\rho)\left((\mathbf{A}+\mathbf{I})\sum_{j=1}^{J} P(j)\mathbf{D}_j + \mathbf{B}\right)$$

■

These are used when constructing rules for agents, as described fully in Section 3.5. We will denote:

$$a_{re} = \rho$$

and

$$\mathbf{b}_{re} = (1-\rho)\left((\mathbf{A}+\mathbf{I})\sum_{j=1}^{J} P(j)\mathbf{D}_j + \mathbf{B}\right).$$

Linear forecasts will be of the form

$$\hat{\mathbb{E}}_{t,i,j}(\mathbf{P}_{t+1} + \mathbf{D}_{t+1,s_{t+1}}) = a_{i,j}(\mathbf{P}_t + \mathbf{D}_{t,s_t}) + \mathbf{b}_{i.j}$$

with variance estimate

$$\mathbf{\Sigma}_{t,i} = \left(\sum_{v=1}^{J} Q^2(v)\mathrm{Var}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}|\mathcal{F}\big) + \sum_{v,w\neq v}^{J} Q(v)Q(w)\mathrm{Cov}\big(\mathbf{P}^v_{t+1,s_{t+1}} + \mathbf{D}^v_{t+1,s_{t+1}}, \mathbf{P}^w_{t+1,s_{t+1}} + \mathbf{D}^w_{t+1,s_{t+1}}|\mathcal{F}\big)\right). \tag{7}$$

Note that the optimal mean squared error forecasts of prices plus dividends in the class of linear forecasts that we derive in our paper is a significant generalisation of the one derived in LeBaron *et al.* (1999). This is due to our mixed state representation.

3.5 Forecasting rules and classifier system

In our paper, the agent-based model developed is a significant generalisation of the model of LeBaron *et al.* (1999). LeBaron *et al.* (1999) incorporated the use of forecasting rules and the classifier system first developed by Holland (1983). In this paper, we modify the forecasting rules and classifier system of LeBaron *et al.* (1999) so that it is consistent with the structure of the stock market model of our paper given the assumptions made on the underlying stochastic dividend process. The forecasting rules and classifier systems need to be modified because they have to

be based on the homogeneous linear REE solution that applies to the particular model which, as we noted above, is different between that of LeBaron *et al.* (1999) and our more general model. Furthermore, new forecasting rules and a new classifier system are required because agents now consider multiple states.The forecasting rules and classifier system used by LeBaron *et al.* (1999) are valid for a univariate system with no state dependence whereas we have explicit state dependence. Agent expectation formation in our model occurs relative to a new market structure which has a different probabilistic description. The expectation formation of our model can generate stock return distributions that resemble real world distributions that we documented empirically above.

Each agent i is assumed to have an information set at t that consists of various publicly available information at t about the state of the market. Such information at t can include the price $P_{t,s_t}(s)$ at time t and state s_t for stock s, the dividend $D_{t,s_t}(s)$ paid at time t and state s_t for stock s, moving averages of the price series and so on. Agents condense this information into world bits and each agent is assumed to have the following twelve world bits:

1. $r_f P_{t,s_t}(s)/D_{t,s_t}(s) > 1/4$
2. $r_f P_{t,s_t}(s)/D_{t,s_t}(s) > 1/2$
3. $r_f P_{t,s_t}(s)/D_{t,s_t}(s) > 3/4$
4. $r_f P_{t,s_t}(s)/D_{t,s_t}(s) > 7/8$
5. $r_f P_{t,s_t}(s)/D_{t,s_t}(s) > 1$
6. $r_f P_{t,s_t}(s)/D_{t,s_t}(s) > 9/8$
7. $P_{t,s_t}(s) > \mathrm{MA}_5(P_{t,s_t}(s))$
8. $P_{t,s_t}(s) > \mathrm{MA}_{10}(P_{t,s_t}(s))$
9. $P_{t,s_t}(s) > \mathrm{MA}_{100}(P_{t,s_t}(s))$
10. $P_{t,s_t}(s) > \mathrm{MA}_{500}(P_{t,s_t}(s))$
11. On: 1
12. Off: 0

where $\mathrm{MA}_v(P_{t,s_t}(s)$ denotes the v-period prior moving average of the price series. At each point in time, each world bit is either true or false. Bits that involve just the price series are referred to as technical bits; 7-10 are thus technical bits. Bits involving dividends are referred to as fundamental bits; 1-6 are fundamental bits, while 11 and 12 are control bits and used as experimental controls.

Each agent is assumed to make linear forecasts. This means forecasts will be of the form

$$\mathbb{E}_{t,i}(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}) = a_i(\mathbf{P}_{t,s_t} + \mathbf{D}_{t,s_t}) + \mathbf{b}_i.$$

Furthermore, a multivariate classifier forecasting system is used to obtain agents' forecasts of future stock prices plus dividends for the cross-section of stocks. This means forecasts at t are state-dependent where the state is described by whichever bits agent i decides is important for each respective stock. For each stock s, each agent is assumed to have 100 such states that they feel are salient. A state for an agent i and for a given stock s is a ternary twelve string of the symbols $\{0, 1, \#\}$. The k^{th} component of the ternary twelve string of a state corresponds to the k^{th} world bit for that stock. A value of 0 means that the world bit for that stock must be false for the condition to match. A value of 1 means the world bit for that stock must be true and a value of # means indifference to that world bit. So, if we were in a world of only five world bits, a ternary string of #10#1 for stock 1 would match the world bit string 11001 for stock 1 but not 10001. For each stock and for each agent, rules consist of ternary strings, as described, as well as a forecasting equation of the future stock price plus dividend (and an estimated variance) connected with that rule. From now on, we shall denote rules by the subscript j. Rule j for stock s will refer to a particular world state and, given the form of the linear REE, the following forecasting equation (which activates when that world state is activated for stock s)

$$\mathbb{E}_{t,i,j}(P_{t,s_t}(s) + D_{t,s_t}(s)) = a_{i,j,s}(P_{t,s_t}(s) + D_{t,s_t}(s)) + b_{i,j,s}$$

with associated variance estimate $\sigma^2_{i,j,s}$. This is the essence of the multiple stock forecasting rules and multivariate classifier system.

For each rule j (and for each agent i and stock s) its associated parameters $a_{i,j,s}$ and $b_{i,j,s}$ are initially set to random values uniformly distributed in the one dimensional intervals A range and the s^{th} component of $\mathbf{B}$ range, respectively, where A range and the $\mathbf{B}$ range are intervals (one dimensional and multi-dimensional, respectively) centered about the mean-squared error optimal solutions in the homogeneous linear REE. Combined with this uniform distribution, A range and $\mathbf{B}$ range thus represent the beliefs of the agents regarding two stock market parameters, namely a_{re} and $\mathbf{b}_{re}$, in this multiple stock state dependent mixed Gaussian stock market.

At each point in time t, and for each stock s, agent i may have several rules that are activated by the world bit string at t for asset s. Agent i chooses, for stock s, the rule that has least variance, a measure of forecasting accuracy and from this optimal rule the agent can now compute the optimal forecast of prices plus dividends (i.e.can carry out their expectation formation) and also has an estimate of the variance $\mathbf{\Sigma}^2_{i,j}$ associated with this forecast. This optimal forecast will have the form:

$$\mathbb{E}_{t,i}(\mathbf{P}_{t+1,s_{t+1}} + \mathbf{D}_{t+1,s_{t+1}}) = a^*_i(\mathbf{P}_{t,s_t} + \mathbf{D}_{t,s_t}) + \mathbf{b}^*_i$$

where $\mathbf{a}^*_i$ and $\mathbf{b}^*_i$ are each a stacking of the optimal a's and b's across the cross-section of stocks (i.e. over all the rules j).

Hence, optimal expectations can be formed using our prior results.

Recall that, as we mentioned above, our method of expectation formation, which is state-

dependent and dependent on the future state probability measure, is new to the literature and adds an additional level of richness. We have introduced a new market structure (with explicit state dependence) and new methods of learning centred around new pricing equations.

Explicit state dependence as we have introduced it is the primary reason a new computational-agent based model needed to be developed. Each agent will form their expectations in different manners, forming forecasts about a new linear pricing equation. This new method of expectation formation that we introduce is novel to the literature. Learning and expectation formation occurs in our paper occurs relative to a new market structure which has a different probabilistic description to existing models.

The structure of our market, as well as the state-dependent expectation formation and learning mechanism of agents, produces distributions of stock returns that are mixed, as we shall soon illustrate.

3.6 Trading and price determination

Prices are set by a *specialist* who observes the demands (order flow) of the stocks and sets $\mathbf{P}_t$ such that the markets clear (demand equals supply for each stock). That is, the specialist sets prices such that $\sum_{i=1}^{N} \mathbf{X}^*_{t,i} = \mathbf{Q}$.

After trading, each agent i updates the accuracy of their forecast rules for each asset according to an exponentially weighted average of squared forecast error,

$$\nu^2_{t,s_t,i,j,s} = \Big(1 - \frac{1}{\tau}\Big)\nu^2_{t,s_t,i,j,s} + \frac{1}{\tau}\big(P_{t,s_t}(s) + D_{t,s_t}(s) - (a_{i,j,s}(P_{t-1,s_{t-1}}(s) + D_{t-1,s_{t-1}}(s)) + b_{i.j.s})\big)^2$$

As in Palmer *et al.* (1999), τ is usually set to 75 and we maintain it for a sense of uniformity.

3.7 Learning and genetic algorithm

To allow for evolutionary learning, we develop a multivariate genetic algorithmic learning approach that we then incorporate into the agent-based model. The genetic algorithm we develop is as a multivariate extension of the algorithm used in Palmer *et al.* (1999).

The details of the multivariate genetic algorithm we develop are summarised neatly as follows. After each K periods, agents update their rules according to the following genetic algorithm. For each agent i, *and for each stock* s, 20 of their worst rules are removed according to a penalised variance criterion and exactly 20 new rules are added. These 20 new rules are obtained by the set of surviving 81 rules and are added through one of two approaches: crossover or mutation. Crossover occurs with probability 0.1 and mutation with probability 0.9. Crossover means selecting two parent rules, according to tournament selection and uniformly crossing the two rules - bit by bit for the ternary string and real number by real number for the vectors $(a_{i,j,s}, b_{i,j,s}, \sigma^2_{i,j,s})$. Mutation means

selecting a single parent by tournament selection and modifying that parent's string randomly bit by bit and modifying that parent's $(a_{i,j,s}, b_{s,i,j}, \sigma^2_{i,j,s})$ by random amounts. The probabilities we use, here, are kept as in Palmer *et al.* (1999) to maintain some sense of uniformity.

4 Simulation Results

Section 3 provided details of the various pertinent features of the model we develop. In this section we simulate the behavior of agents who interact in a new model, in a world of multiple markets, and where markets have a different probabilistic feature. This provides us with results regarding our new computational agent-based model.

In this section, we show that our model can produce stock return distributions that are a mixture of distributions, similar to the shape of various distribution of real-world stock returns, as documented in Section 2. We also show that the Santa Fe model of Palmer *et al.* (1999) produces a distribution that does not replicate the real-world distributional shape in stock returns that we documented in Section 2 .

We then proceed to simulate many variants of the new model where we apply our method of parameter variation Monte Carlo simulation. This allows us to understand whether the model simulated prices vary with salient parameters (such as dividend state probabilities) in a plausible manner and to understand how, through the use of an agent-based model, mispricing of stocks, volatility of stocks and correlation of stocks depend on agent beliefs.

4.1 Stock return distributions

One of the motivations behind our paper, as mentioned in Section 2, is to develop a multiple stock artificial stock market model that is capable of producing a distribution of aggregated stock returns that is a mixture of distributions, similar to the shape of various stock return distributions documented in our paper.

In this subsection, we show that the model of LeBaron *et al.* (1999) produces a distribution whose shape is distinctly different to what we have seen in real-world stock markets during the periods and frequencies we document. We then illustrate that our model produces a distribution whose shape resembles the stock return distribution of real world (in particular, S&P 500) stock returns during the periods and frequencies we document.

We start by first simulating the single-stock Santa Fe Artificial Stock Market as presented in LeBaron *et al.* (1999) with the parameter values as used in their paper. This is a degenerate case of the model developed here with parameter values given by Table 1. Note that A range $= [a_{re}-0.25, a_{re}+0.25]$ and $\mathbf{B}$ range $= [\mathbf{b}_{re}-14.5, \mathbf{b}_{re}+14.5]$ and thus contain a_{re} and $\mathbf{b}_{re}$, respectively. The model simulation is run for 2500 time periods to ensure sufficient learning with $K = 10$ and is then allowed to run for a further 2500 periods after which the last 2500 simulated price observations

are extracted and the density estimate for the corresponding stock returns are plotted in Figure 10 (a). T and K are chosen to ensure $T/K = 250$, as in LeBaron *et al.* (1999). (Alternate values for T and K can be used and they produce similar results.)

The density estimate of the simulated stock returns produced by the degenerate Santa Fe model has a shape that is symmetric and distinctly non-mixture in shape.

Table 1: Parameters: Multiple-Stock Santa Fe Artificial Stock Market for Density Estimation.

Parameter	**Base value**
λ	0.5
${\sigma^2}_{P+D}$	4
${\sigma^2}_{u_t}$	0.07429
ρ	0.95
r_f	0.10
$\bar{D}$	10
A range	$[0.7, 1.2]$
B range	$[-10, 19]$
N	25
T	5000
K	10

Following on from this exercise, we conduct a similar simulation for our model. Using the parameters including the mixture probabilities as in Table 2, we simulate the mixture model for 2500 time periods to ensure sufficient learning with $K = 10$. It is then allowed to run for a further 2500 periods after which the last 2500 simulated price observations are extracted. We then construct the stock market index as an equal-weighted average of the stock returns for each stock and we plot the density estimate for the stock market in Figure 10 (b). (Alternate values for T and K can be used and they produce similar results.)

The density estimate of the simulated stock returns produced by our model has a shape that resembles the mixed Gaussian shape of the real-world stock return distribution over the period June 2010 - June 2015.

By altering the parameter values, we are able to replicate each of the mixed real world stock return distributions we document in our paper.

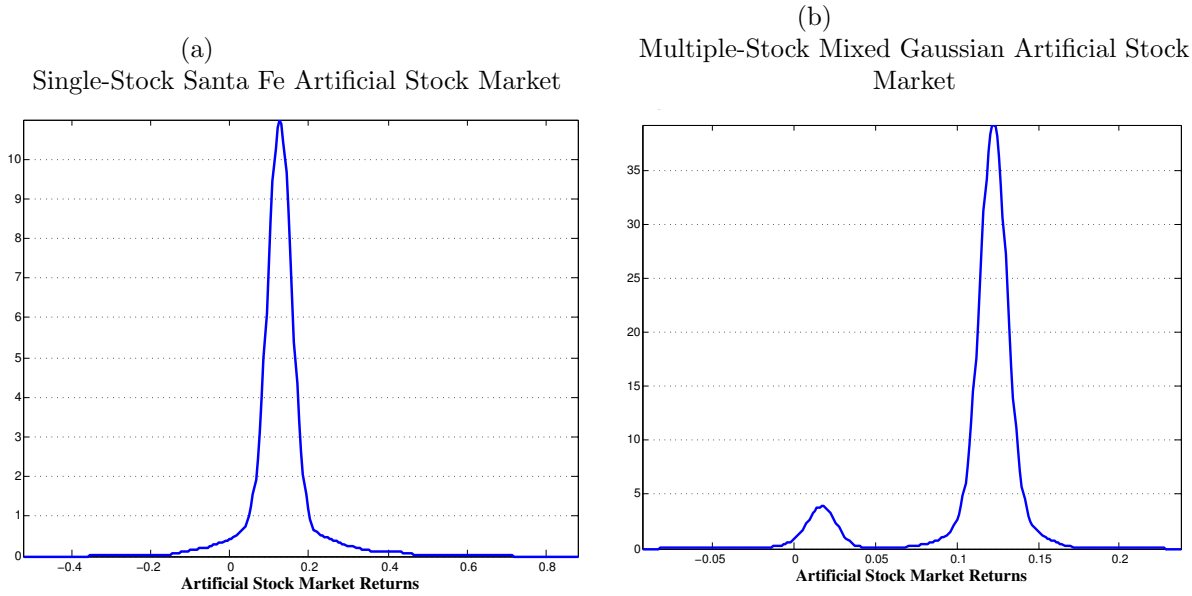
4.2 Method 1: Parameter Variation Monte Carlo Simulation

In this subsection, we apply our first method to analyse the relationship between stock prices and salient underlying parameters, within the simulated environment. Key parameters that determine stock prices in our multivariate computational Artificial Stock Market model are agents' risk aversion coefficient (λ), agents' beliefs regarding the variance of future prices plus dividends (${\sigma^2}_{P+D}$), the variance of the underlying stock dividend stochastic process (σ_u^2), the persistence of the stochastic dividend process (ρ), the risk-free rate (r_f), the mean of the stock dividend stochastic process ($\bar{\mathbf{D}}$) and the mixture probabilities $Q(j)$. We study the effects of these parameters on the agent-based

Table 2: Parameters: Multiple-Stock Mixed Gaussian Artificial Stock Market.

Parameter	Base value
λ	0.5
$\mathbf{\Sigma}^2_{\mathbf{P+D}}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
$\mathbf{\Sigma}^2_{\mathbf{u}_t}$	$\begin{pmatrix} 0.07429 & 0.037145 \\ 0.037145 & 0.07429 \end{pmatrix}$
ρ	0.5
r_f	0.10
J	2
$\bar{D}_j$	$\bar{D}_1 = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$ and $\bar{D}_2 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}$
$Q(j)$	$Q(1) = 0.95$ and $Q(2) = 0.05$ i.e. state 1 and 2 occur with probabilities 0.95 and 0.05, respectively
A range	$[A_{re} - 0.1, A_{re} + 0.1]$
$\mathbf{B}$ range	$[\mathbf{B}_{re} - \mathbf{1}, \mathbf{B}_{re} + \mathbf{1}]$
n	2
N	25
T	5000
K	10

Figure 10: Density Estimate of Artificial Stock Market Returns (Multiple-Stock Mixed Gaussian Artificial Stock Market)



model simulated price series. Two other parameters that have important effects on stock prices, in this artificial stock market context, are A range and $\mathbf{B}$ range which are the parameters that represent the beliefs of the agents regarding the stock market parameters a_{re} and $\mathbf{b}_{re}$. The effects of A range and $\mathbf{B}$ range on simulated stock prices will be examined in the following subsection. By doing so, we also provide a method for one to examine the impact of parameters on an agent-based model. LeBaron (2006) mentions that one of the existing drawbacks of agent-based models is that

there is no currently existing method for one to examine the impact of parameters on the model. By analysing the variation of simulated prices with salient parameters, we also provide a method to address this criticism of agent-based models.

To analyse the relationship between stock prices and our first group of salient underlying parameters, we start by fixing a base case (Table 3) of the simulated model with the following parameters. To keep the analysis feasible, we simulate a two-stock, two-state version of the model.

Table 3: Parameters: Base Case (Multiple-Stock Mixed Gaussian Artificial Stock Market).

Parameter	**Base value**
λ	0.5
$\mathbf{\Sigma^2_{P+D}}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
$\mathbf{\Sigma^2_{u_t}}$	$\begin{pmatrix} 0.07429 & 0.037145 \\ 0.037145 & 0.07429 \end{pmatrix}$
ρ	0.95
r_f	0.10
J	2
$\bar{D}_j$	$\bar{D}_1 = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$ and $\bar{D}_2 = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$
$Q(j)$	$Q(1) = 0.95$ and $Q(2) = 0.05$ i.e. state 1 and 2 occur with probabilities 0.95 and 0.05, respectively
A range	$[A_{re} - 0.1, A_{re} + 0.1]$
$\mathbf{B}$ range	$[\mathbf{B}_{re} - \mathbf{1}, \mathbf{B}_{re} + \mathbf{1}]$
n	2
N	25
T	2750
K	11

We centre A range and $\mathbf{B}$ range about the new REE solutions, namely a_{re} and $\mathbf{B}_{rbe}$. T and K are chosen to ensure $T/K = 250$. Finally, we now have state dependent mean levels of the stochastic dividend process, and probabilities associated for each of these states. These parameters are initially set as in Table 3 and are also varied in our computational work here.

Given the set of base case parameters, we then simulate the model numerous times by varying, one at a time, each of λ, ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, ρ, r_f, $\bar{D}_j$ and $Q(j)$ whilst holding the rest fixed at their base case parameter values. In particular, we individually discretise the parameter spaces of each of λ, ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, r_f, $\bar{D}_j$ and $Q(j)$ into equispaced sub-intervals over plausible ranges as shown in Figures 11 to 19. Since ρ measures the degree of stock dividend persistence, as $\rho \to 1$, we approach a non-stationary dividend process driven by a Brownian motion. This leads to highly non stationary prices for values of ρ very close to 1. For this reason, as ρ approaches 1, we divide its parameter space even more finely to obtain a clearer impact of ρ on stock prices. For none of the other parameters is it necessary to divide any of its subspaces more finely.

To illustrate how we conducted our analysis, consider λ as our parameter being varied with a parameter range space of $[0.1, 2.5]$ which contains its base value. First consider $\lambda = 0.1$ and set all the other parameters to their base case values as in Table 3. Then simulate the model over the time span 0 to $T = 2750$ and obtain the simulated stock prices over this time span which will correspond

to $\lambda = 0.1$. Extract the T^{th} price observation. Repeat this exercise 600 times and plot each of these values at the value $\lambda = 0.1$ in Figure 11 (a). We then average these 600 values and plot the single valued mean at the value $\lambda = 0.1$ in Figure 11 (b). This removes idiosyncratic noise and we now have a reliable estimate, by the pointwise law of large numbers, of the price at the value $\lambda = 0.1$. We then repeat this exercise for each of the λ values in our discretised set and complete the price paths.

We repeat this entire exercise for each of the parameters of interest, namely ${\sigma^2}_{P+D}$, ${\sigma^2}_{u_t}$, ρ, r_f, $\bar{D}_j$ and $Q(j)$. This now allows us to assess the relationship between our model's simulated stock prices and salient underlying parameters.

We can see, from Figures 11 (a) and (b), that as the level of risk aversion of agents rises, stock prices fall monotonically. This is economically sensible since higher risk aversion implies that agents possessing risk averse CARA utility have greater preference for, *ceteris paribus*, risk-free assets to stocks. We can also see, from Figures 11 (c) and (d), that as agents' variance beliefs of the future price plus dividend of the stock increase, the stock price falls. This is also reasonable given the assumption of risk averse CARA utility; if agents expect the stock to be more volatile (higher ${\sigma^2}_{P+D}$), they have reduced preference for stocks as opposed to risk free assets such as the bond in this model.

For fairly similar reasons relating to the structure of CARA utility of investors and traders, we should expect a negative relationship between the risk free rate and stock prices and a positive relationship between the mean of the dividend process and stock prices. This model is also consistent with both these expectations as depicted in Figures 12 (c)-(e).

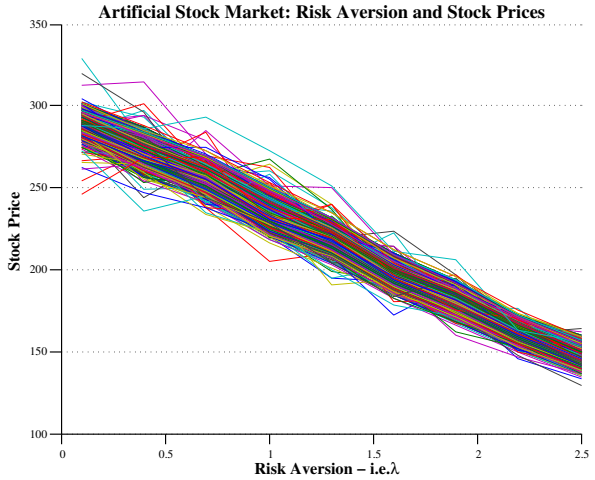
To the best of our knowledge, finance theory does not provide strong postulates regarding the relationship between stock dividend variation and stock prices and between stock dividend persistence and stock prices. Nevertheless, we believe it is useful to provide some insights into this, using the mixed Gaussian Artificial Stock Market model developed here. Higher stock dividend variation seems to lead to lower stock prices although this effect is not as strong as it was for agents' variance beliefs and there is some noticeably significant stochastic variation. More insightful, perhaps, is the fact that stock dividend variation (and not variance beliefs on the part of agents) is the key parameter that determines the *variation* of stock prices themselves, as can be seen from Figures 11 (e) and (f). Unlike for the original Santa Fe Artificial Stock Market model, stock dividend persistence now appears to have some significant impact on stock prices for values of $\rho < 0.9$ although the relationship is not monotone and there is significant stochastic variation. As the stock dividend process approaches non-stationarity, we see, from Figures 12 (a)-(b), that stock prices fall dramatically.

We are left to analyse the relationship between the mixture probabilities of our model and stock prices. This is contained in Figures 13 (a) and (b) and the relationship is rather clean, monotonic and as expected; the higher the probability of the high mean dividend state, the higher are stock prices and this is entirely plausible as stocks are claims to future dividends.

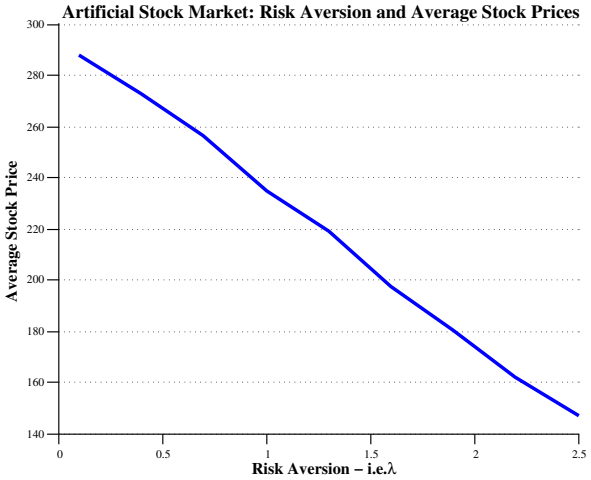
As can be seen from Figures 14 to 19, these plausible relationships, which provide strong additional support for our model, extend to all assets on the market and to the aggregated stock market itself.

Figure 11: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel A.

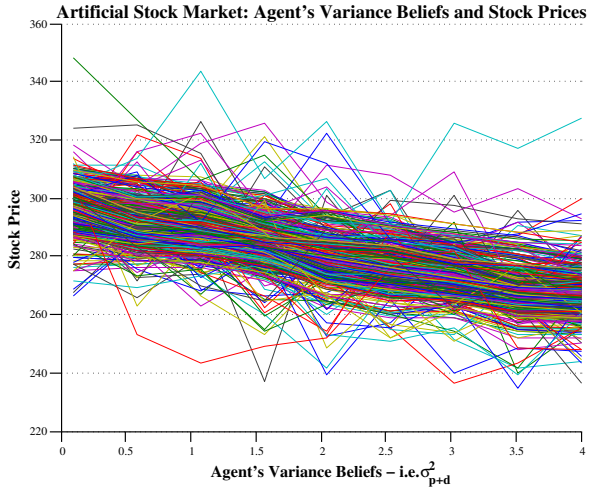
(a)



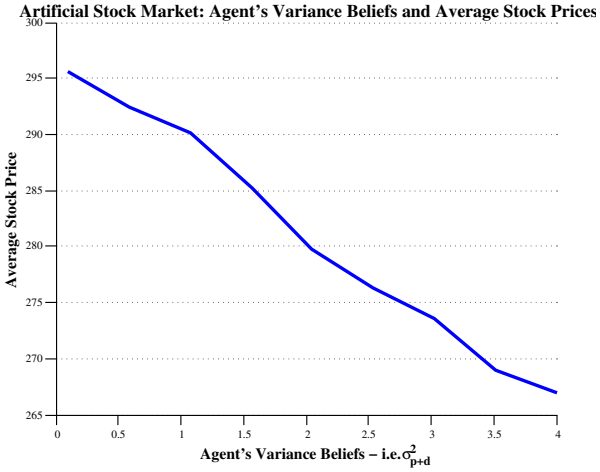
(b)



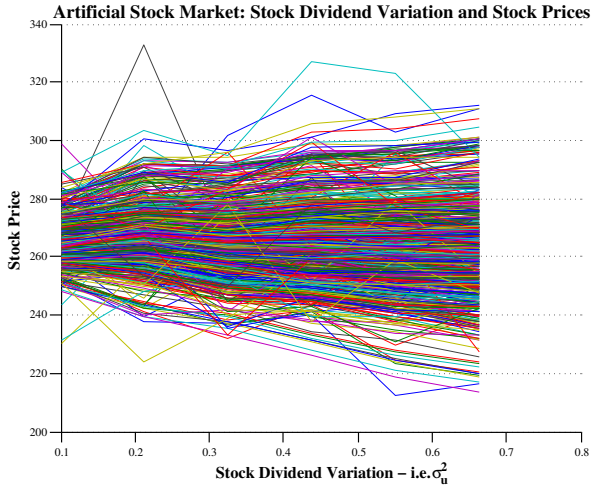
(c)



(d)



(e)



(f)

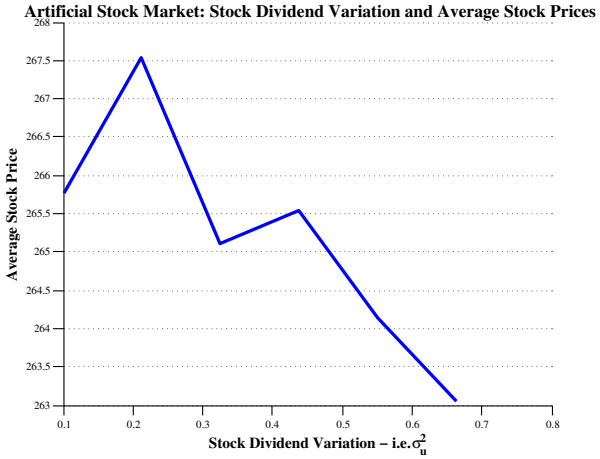
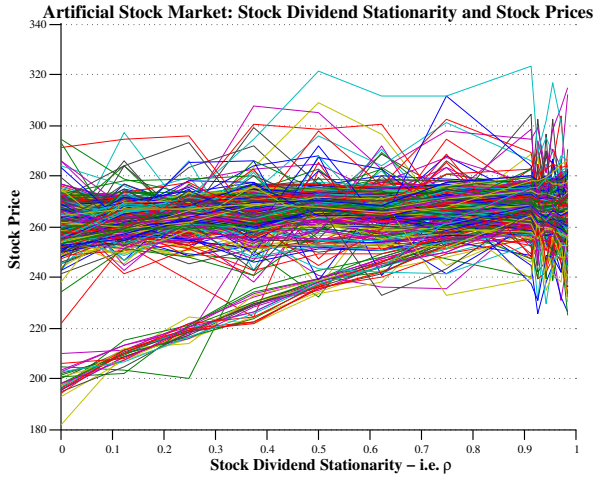
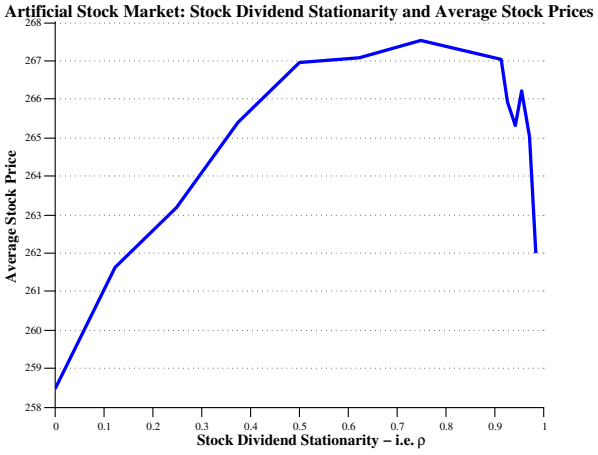


Figure 12: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel B.

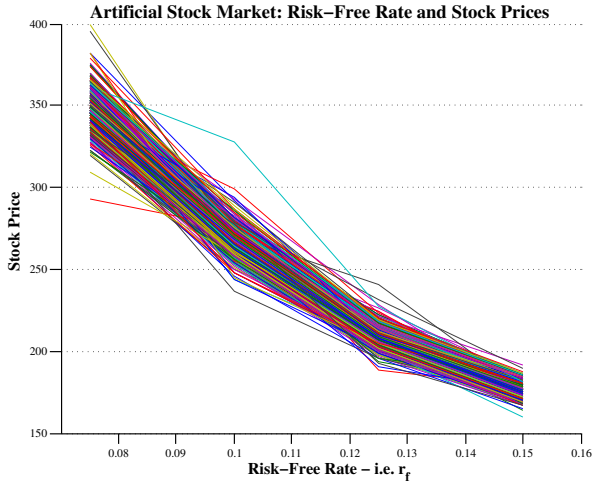
(a)



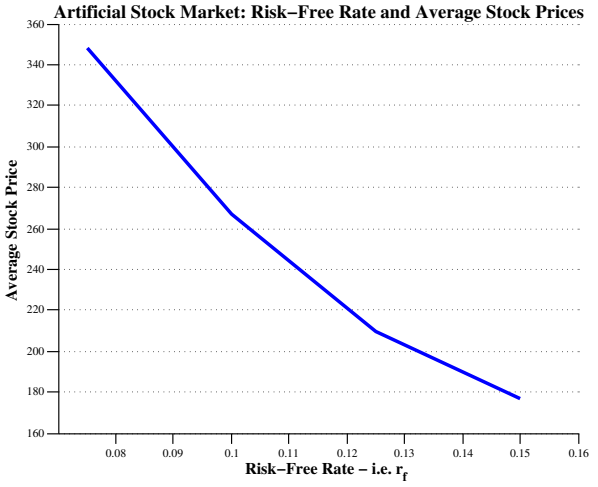
(b)



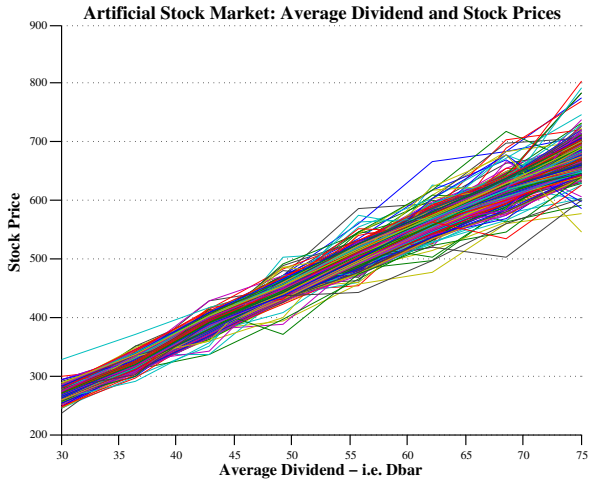
(c)



(d)



(e)



(f)

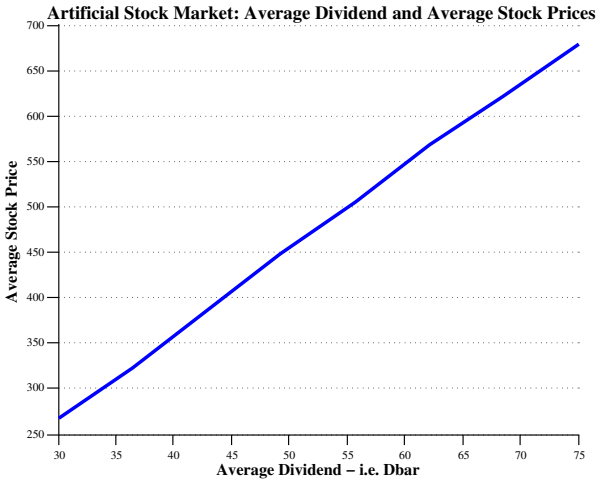
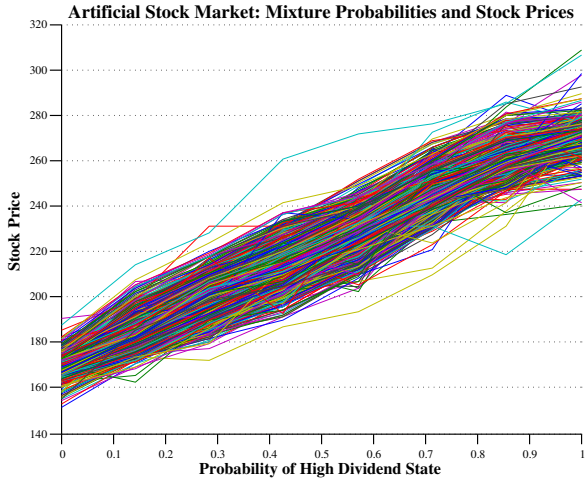


Figure 13: Multiple Stock Mixed Gaussian Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 1) - Panel C.

(a)



(b)

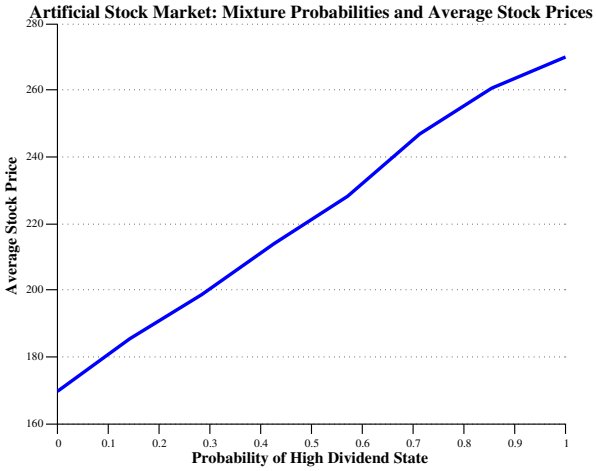
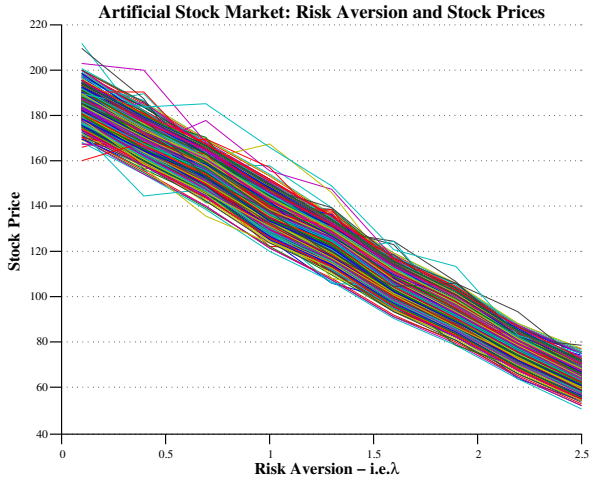
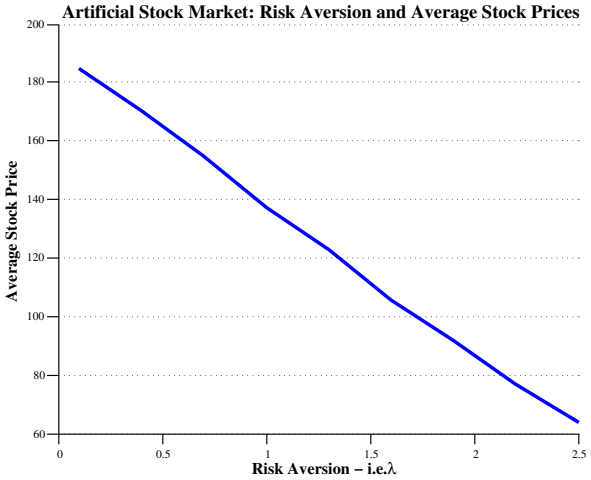


Figure 14: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel A.

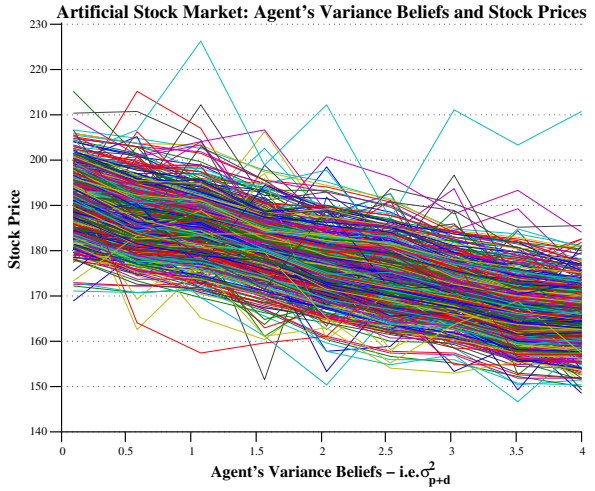
(a)



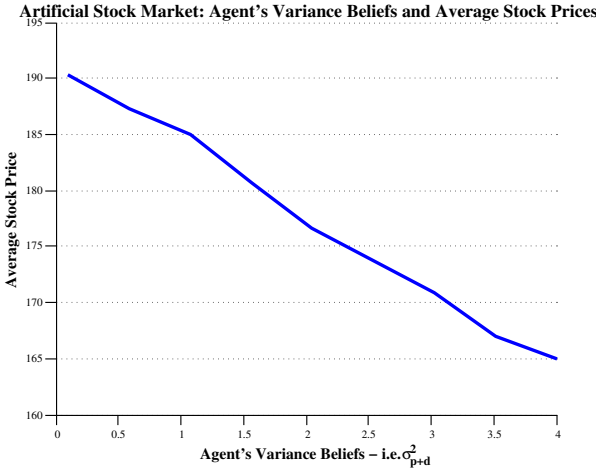
(b)



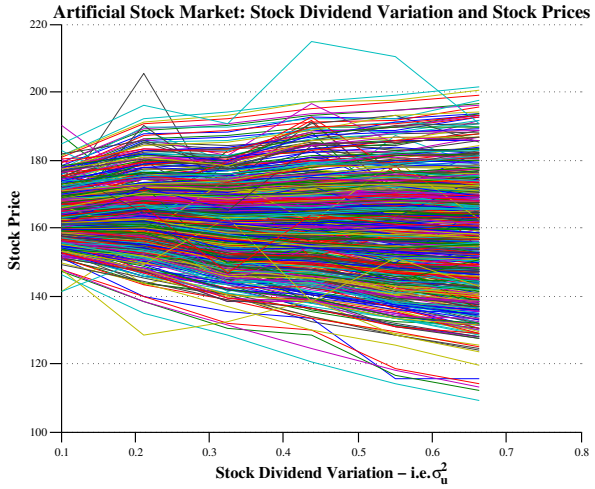
(c)



(d)



(e)



(f)

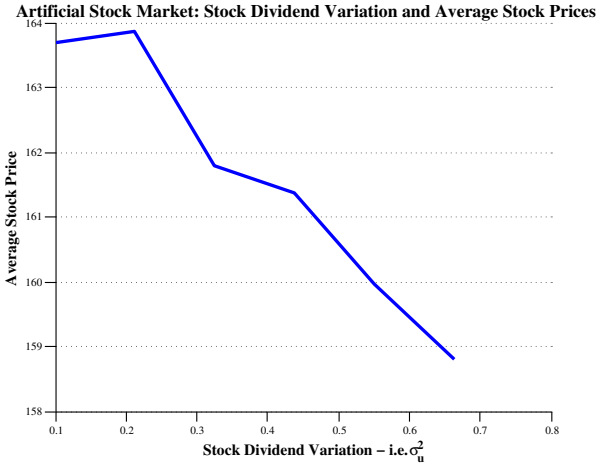
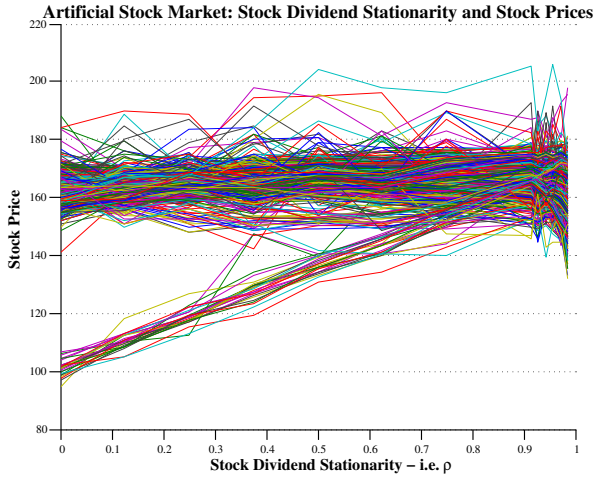
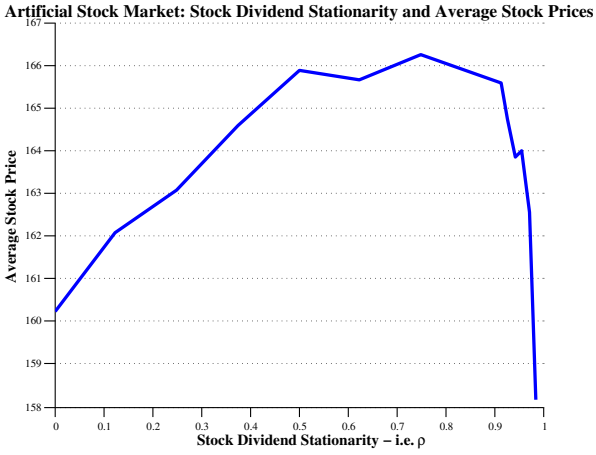


Figure 15: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel B.

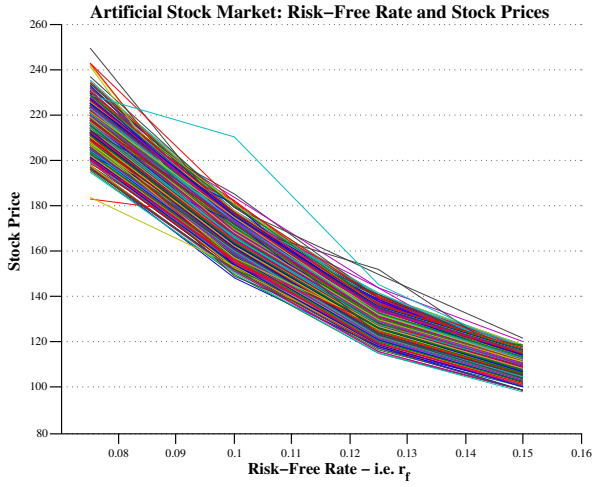
(a)



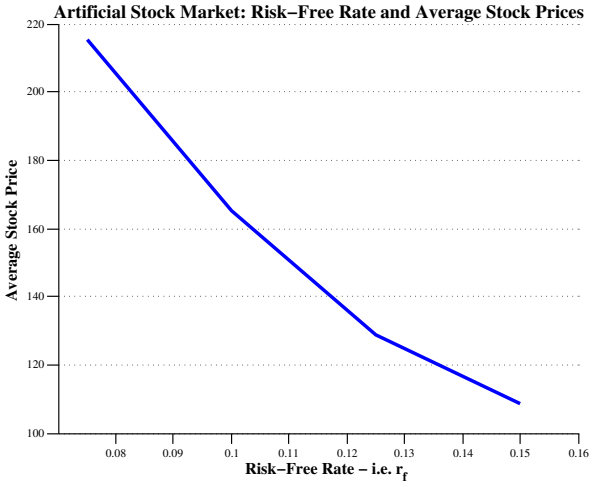
(b)



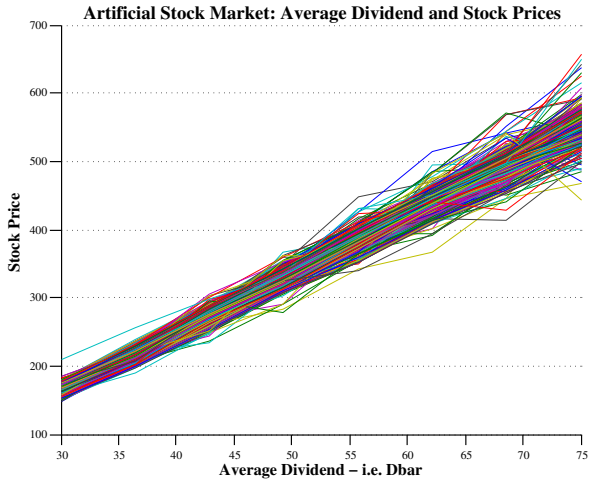
(c)



(d)



(e)



(f)

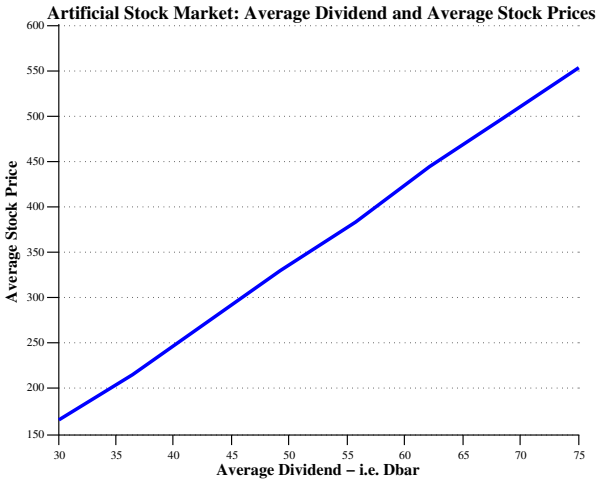
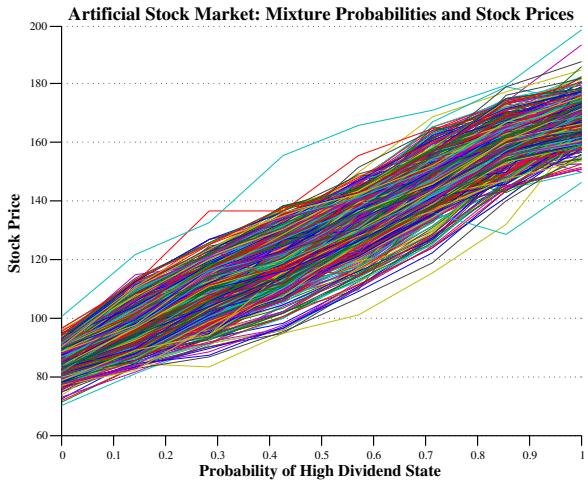


Figure 16: Multiple Stock Mixed Gaussian Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Asset 2) - Panel C.

(a)



(b)

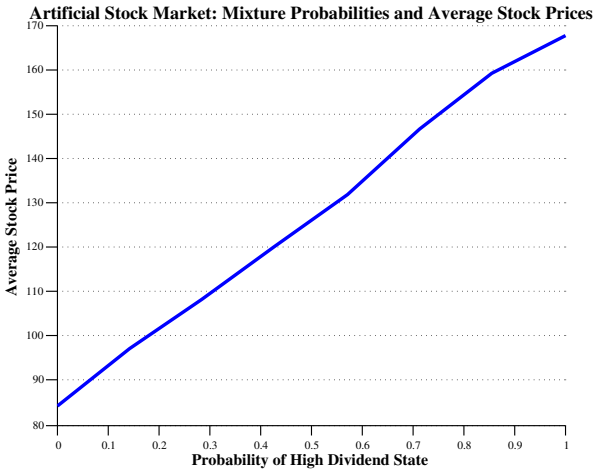
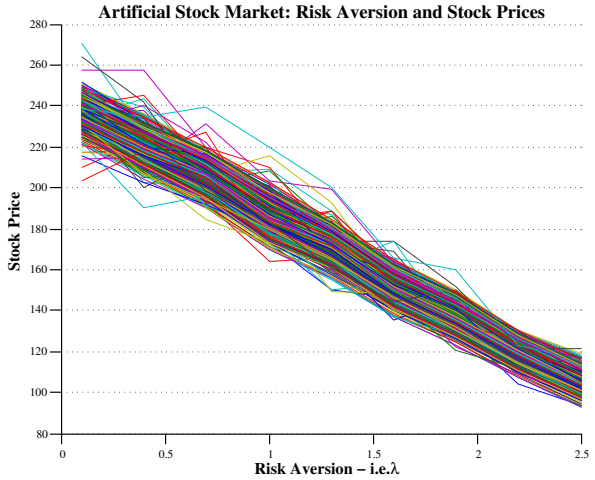
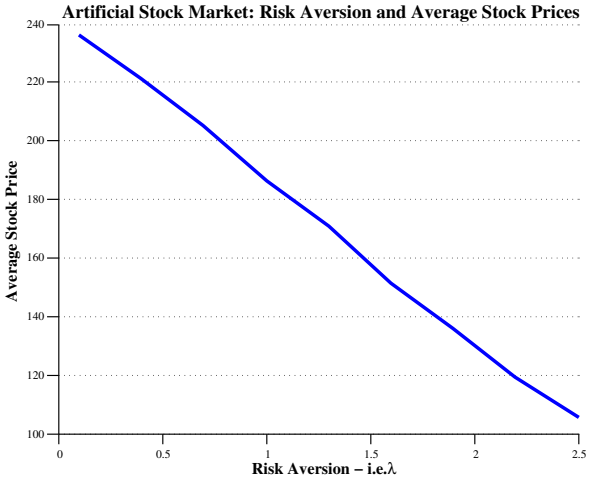


Figure 17: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel A.

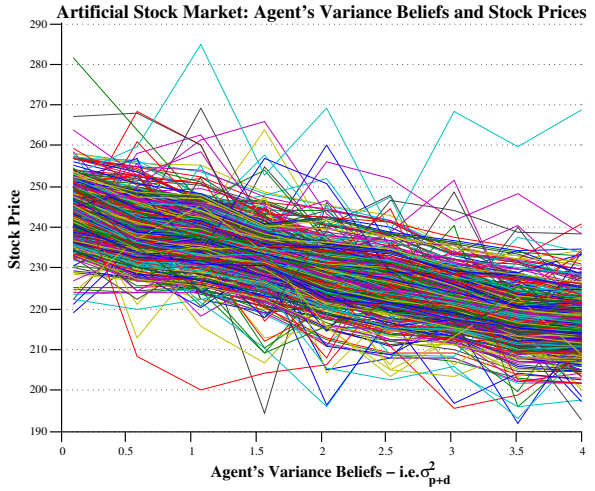
(a)



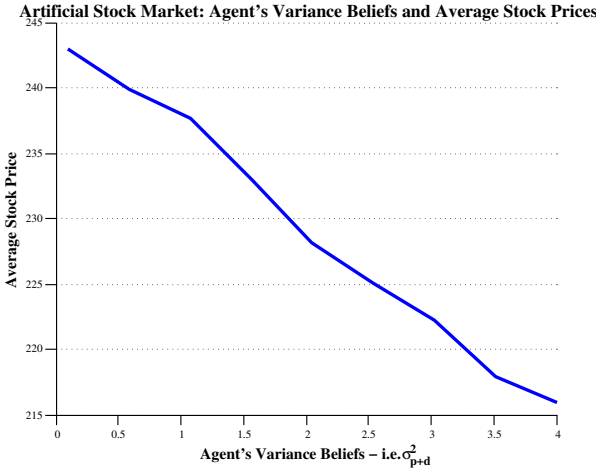
(b)



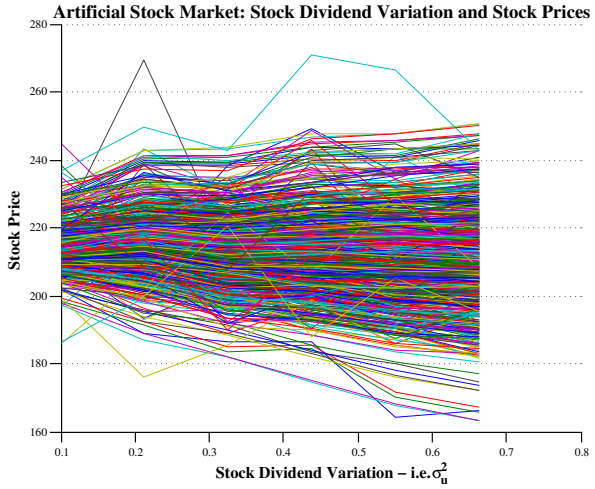
(c)



(d)



(e)



(f)

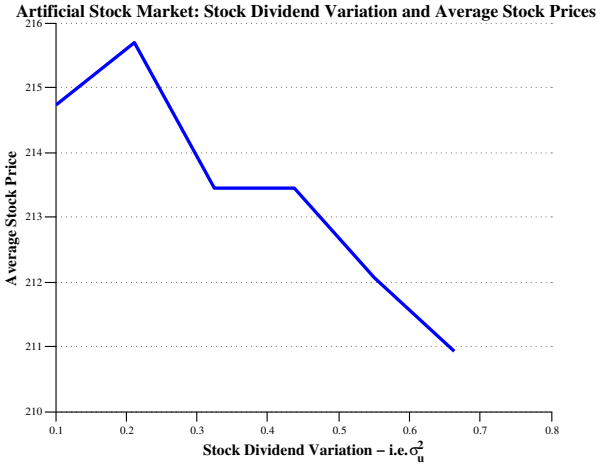
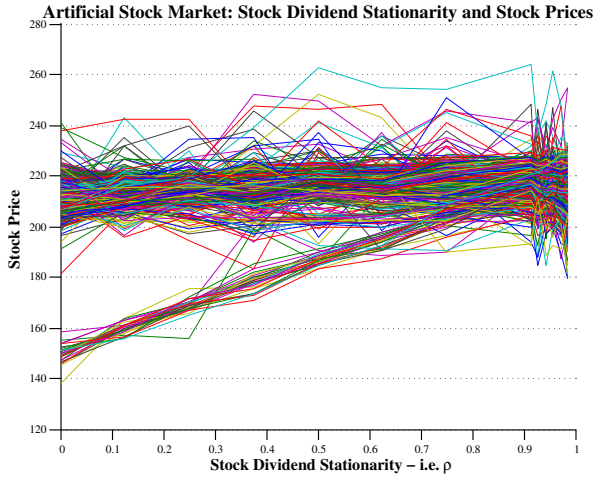
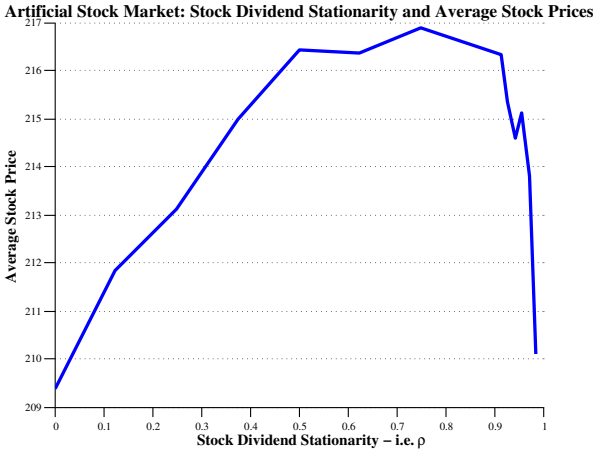


Figure 18: Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel B.

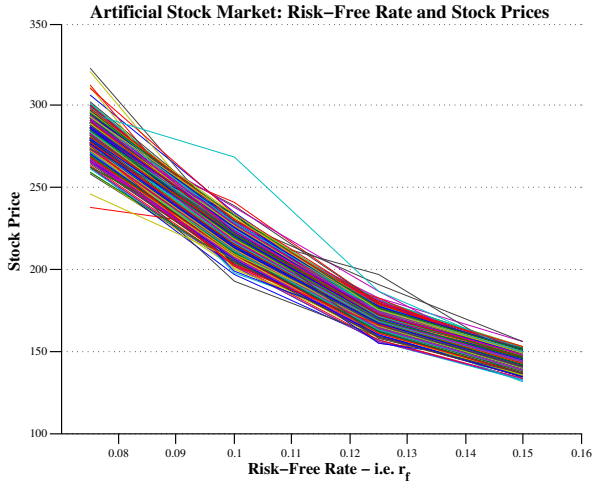
(a)



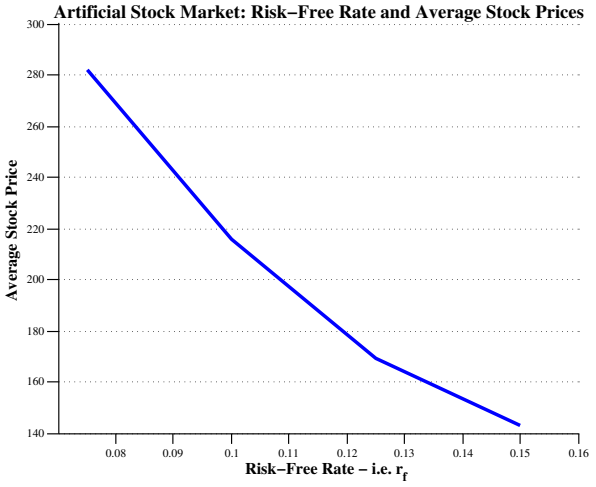
(b)



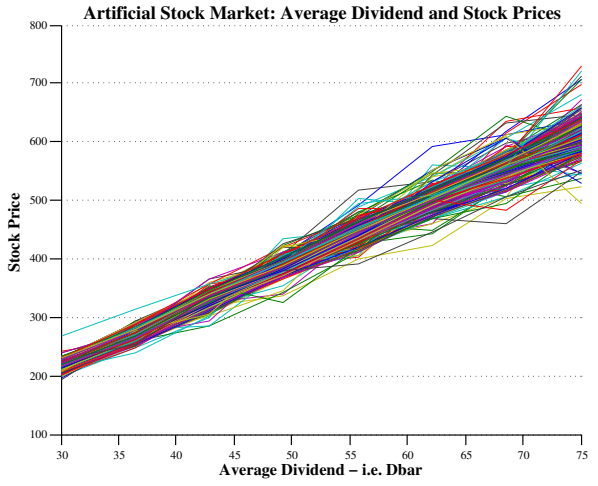
(c)



(d)



(e)



(f)

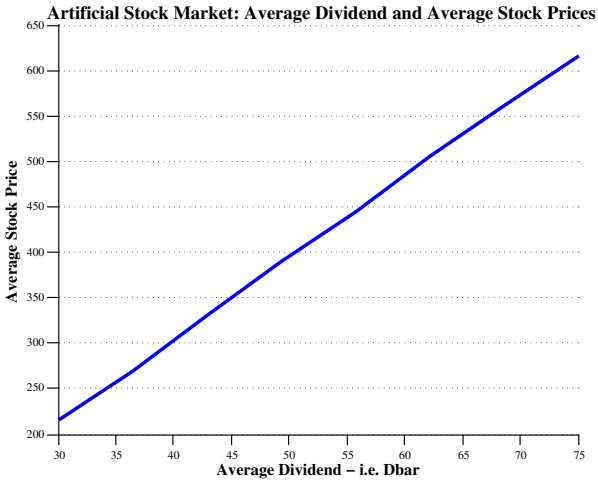
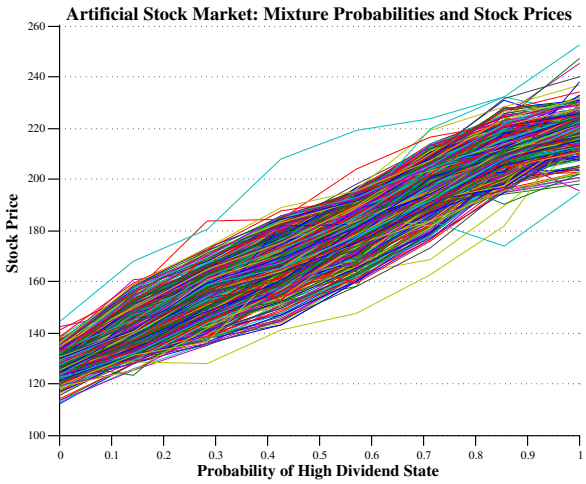
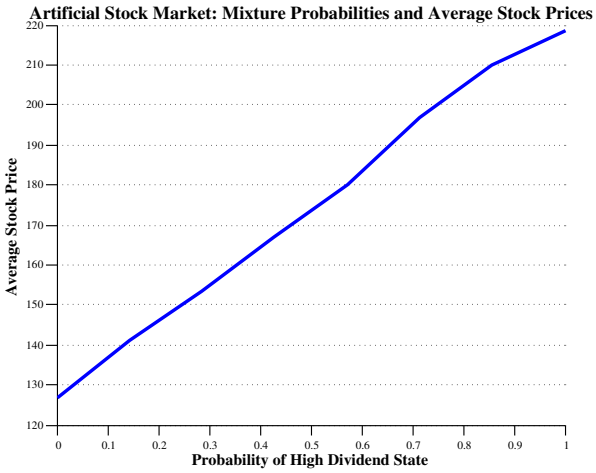


Figure 19: Multiple Stock Mixed Gaussian Multiple Stock Mixed Gaussian Artificial Stock Market: Relationship Between Stock Prices and Underlying Variables (Stock Market) - Panel C.

(a)



(b)



4.3 Method 2: Statistical Model (Regression)

In our second method, we aim to also (a) include sample variance and statistical significance considerations, (b) account for multiple variables and for confounding variables and (c) offer a systematic approach to model selection and validation. The method that achieves this is a statistical model in the form of (multivariate) regression. We achieve this using the well-known Maximum Likelihood regression method which has attractive properties, namely consistency (convergence in probability to true values) and asymptotic normality.

The details of MLE and the statistical model (regression) that we use is detailed in Seedat & Abelman (2019a). We refer the interested reader to the relevant section in that paper for the necessary derivations.

To assess the impact of the variables, we consider our agent based-model and we generate synthetic data $\mathbf{x}_i = \left(\lambda_i, \sigma^2_{p+d,i}, \sigma^2_{u_t,i}, \rho_i, r_{f,i}, \bar{D}_{1i}, Q(1)_i\right)$ and correponding prices $y_i = p_i$, where p_i can represent the price of stock 1, stock 2 or the stock market. We are interested in the statistical model given by $y_i = \mathbf{x}_i\boldsymbol{\beta} + \epsilon_i$. By the theory in Seedat & Abelman (2019a), we can estimate the coefficient-vector (our desired output) $\boldsymbol{\beta}$ and the sample variance $\hat{\sigma}^2$ (from which we can obtain the associated p-values). This leads to the following results.

Table 4: Statistical model: regression using MLE (Multiple Stock Mixed Gaussian artificial stock market) - stock 1.
Estimates of the impacts of the various parameters on stock prices with associated p-values and 99% confidence intervals.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{D}_1$	Q(1)
Estimated parameter	-80.23	-6.43	-5.21	-3.94	-2135.52	9.88	107.94
p**-value**	0.00	0.00	0.12	0.08	0.00	0.00	0.00
Confidence interval	(-90.68, -69.87)	(-7.55, -5.31)	(-16.19, 5.77)	(-9.56, 1.68)	(-2923.84, -1347.20)	(7.60, 12.16)	(74.99, 140.89)

Table 5: Statistical model: regression using MLE (Multiple Stock Mixed Gaussian artificial stock market) - stock 2.
Estimates of the impacts of the various parameters on stock prices with associated p-values and 99% confidence intervals.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{D}_1$	Q(1)
Estimated parameter	-38.74	-8.43	-9.46	-1.87	-1357.76	9.52	89.25
p**-value**	0.00	0.00	0.02	0.24	0.00	0.00	0.00
Confidence interval	(43.99, -33.75)	(-11.21, -5.65)	(-23.58, 4.66)	(-5.05, 1.31)	(-1780.59, -934.93)	(7.95, 11.09)	(73.70, 104.80)

The R^2 of the above models are 81.9%, 88.3% and 90.7% indicating that each model is an excellent fit and 0 does not fall within the confidence intervals (except for stock dividend variation and stationarity), thus explaining the obtained p-values.

The results of the statistical model above confirm the results discussed above, namely (a) there is a negative relationship between risk-aversion and stock prices, (b) a negative relationship

Table 6: Statistical model: regression using MLE (Multiple Stock Mixed Gaussian artificial stock market) - stock market.
Estimates of the impacts of the various parameters on stock prices with associated p-values and 99% confidence intervals.

	λ	σ^2_{p+d}	σ^2_u	ρ	r_f	$\bar{D}_1$	Q(1)
Estimated parameter	-47.26	-7.19	-6.69	0.00	-1728.51	9.79	81.25
p**-value**	0.00	0.00	0.08	0.84	0.00	0.00	0.00
Confidence interval	(-55.03, -39.49)	(-8.38, -6.00)	(-20.88, 7.50)	(-8.42, 8.42)	(-2270.90, -1186.12)	(8.46, 11.12)	(57.06, 105.44)

between agent's variance beliefs and stock prices, (c) a negative relationship between stock dividend variation and stock prices, (d) a negative relationship between stock dividend stationarity and stock prices (apart from the stock market which has statistically insignificant direction), (e) a negative relationship between the risk free rate and stock prices, (f) a positive relationship between average dividends and stock prices and (g) a positive relationship between the probability of the high dividend state and stock prices. By inspection of the p-values, all these results are statistically significant at the 1% level apart from the result on stock-dividend variation and stock dividend stationarity (neither of which has the wrong sign thus also providing support for the plausibility of the model). The finance intuition for these results were discussed above.

These results provide further evidence on the plausibillity of our model.

4.4 Stock mispricing depends on agent beliefs

We provide evidence that stock mispricing is possible in our multi-stock Artificial Stock Market model and that the degree of such mispricing is larger if agents' beliefs of the stock market (i.e. of a_{re} and $\mathbf{b}_{re}$) are more volatile (specifically, if their beliefs are likely to have a greater deviation from their rationally expected values).

We simulate four different versions of our model. Each of these respective model simulations is given a different range for the draws of $a_{i,j}$ and $\mathbf{b}_{i,j}$. The remaining parameters are each fixed at the values given in Table 7.

Each model simulation is run for 2500 time periods to ensure sufficient learning with $K = 10$ and is then allowed to run for a further 2500 periods after which the last 250 simulated price observations are extracted and plotted. (Alternate ranges for T and K produce similar overall results.) Figure 20 presents the simulated dividend series from the model.

Figure 21 contains the time series plots for the model simulations. The corresponding ranges for the $a_{i,j}$ and $\mathbf{b}_{i,j}$ parameters are included alongside the time series plots. The plots in blue are the simulated price series. The plots in green are the theoretical price series obtained using the linear homogeneous rational expectations solution from Section 3.4.

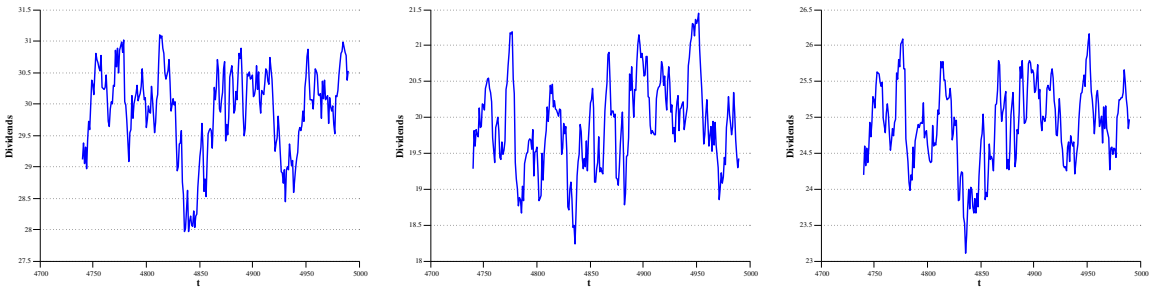
We observe that mispricing tends to increase when agents' stock market beliefs are more likely to have a greater deviation from their rationally expected values (given by a greater A range and/or

Table 7: Stock Mispricing Parameters: Base Case.

Parameter	Base value
λ	0.5
$\mathbf{\Sigma}^2_{\mathbf{P+D}}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
$\mathbf{\Sigma}^2_{\mathbf{u}_t}$	$\begin{pmatrix} 0.07429 & 0.037145 \\ 0.037145 & 0.07429 \end{pmatrix}$
ρ	0.95
r_f	0.10
J	2
$\bar{D}_j$	$\bar{D}_1 = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$ and $\bar{D}_2 = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$
$Q(j)$	$Q(1) = 0.95$ and $Q(2) = 0.05$ i.e. state 1 and 2 occur with probabilities 0.95 and 0.05, respectively
n	2
N	25
T	5000
K	10

Figure 20: Multiple Stock Mixed Gaussian Artificial Stock Market Agent-Based Model Stock Dividend Process.

(a) Asset 1 (b) Asset 2 (c) Market



B range).

4.5 Stock volatility and covariation depends on agent beliefs

We conclude by examining the impact on stock volatility and covariation by agent beliefs, within our multi-stock mixed Gaussian Artificial Stock Market model. We simulate the same four models as in the previous subsection.

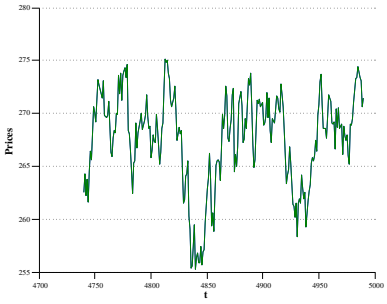
As illustrated by the time series plots of squared stock returns (for each asset and for the equally-weighted market) in Figure 22, stock volatility appears to increase when agents' stock market beliefs (i.e. their beliefs regarding a_{re} and $\mathbf{b}_{re}$) are volatile and specifically when they are more likely to have a greater deviation from their rationally expected values (note the scale on the y-axis). This holds for each individual stock and for the equally-weighted stock market.

We assess this more rigorously by estimating a DCC (1,1) model (Engle, 2002) on the simulated series produced by the multi-stock mixed Gaussian agent-based model. We then plot the time series

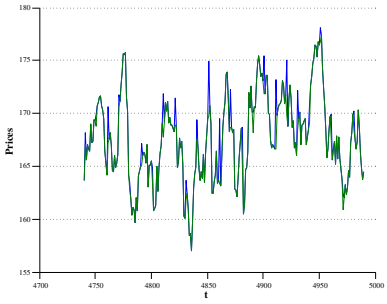
of individual stock volatilities. We also extract and plot the stock correlation estimates from this model. These are contained in Figure 23. We also notice that stock volatilities tend to increase as agents' beliefs regarding a_{re} and $\mathbf{b}_{re}$ are more volatile. This seems to be plausible. If agents' parameters, which are periodically chosen (and then sequentially updated using some learning mechanism), governing their portfolio optimisation choices have a greater variance then updating of such beliefs will, in expectation, be of a greater magnitude which in turn leads to more volatile stocks. Lastly, we note that different belief structures can lead to different correlations between stocks within our agent-based model. Certain beliefs, such as those underpinning Figure 23 (f), depress correlations between stocks whereas others, such as those underpinning Figure 23 (i), lead to heightened correlations between stocks.

Figure 21: Multiple Stock Mixed Gaussian Artificial Stock Market Stock Prices and Theoretical Stock Prices.

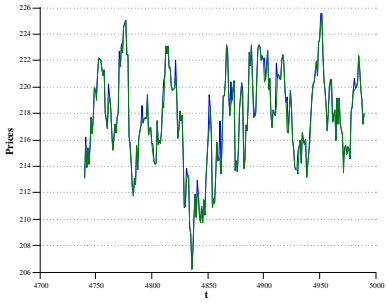
(a) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



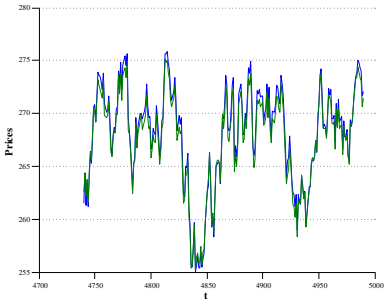
(b) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



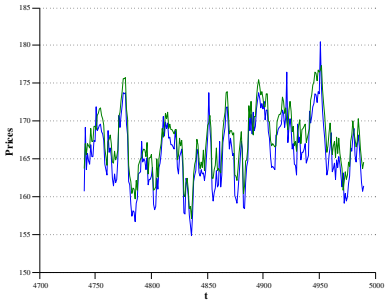
(c) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



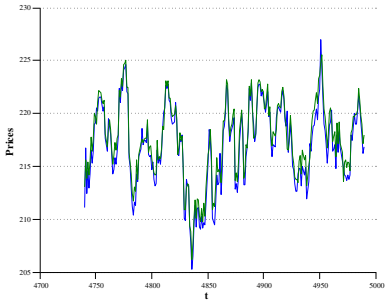
(d) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



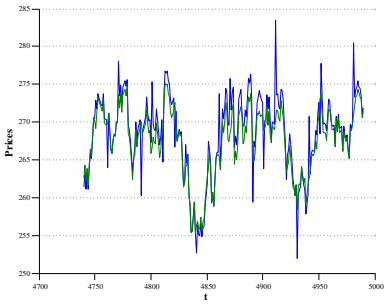
(e) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



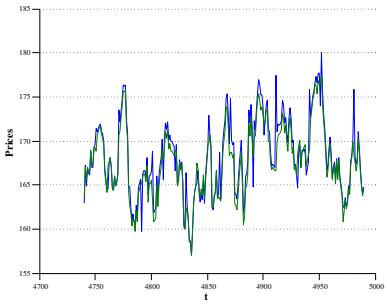
(f) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



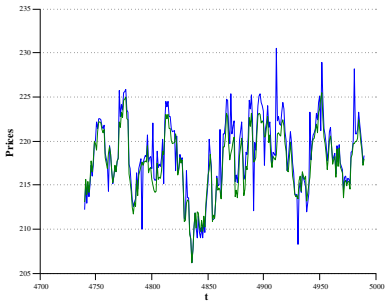
(g) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



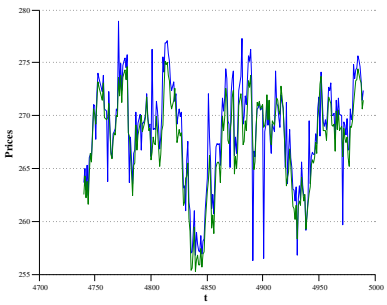
(h) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



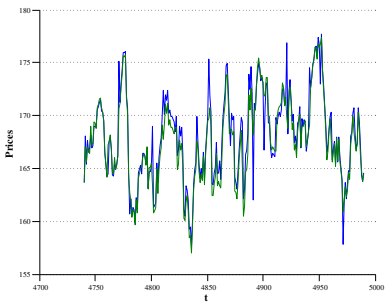
(i) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



(j) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(k) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(l) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$

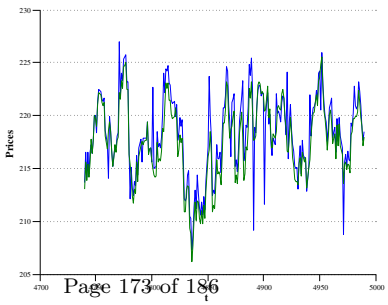
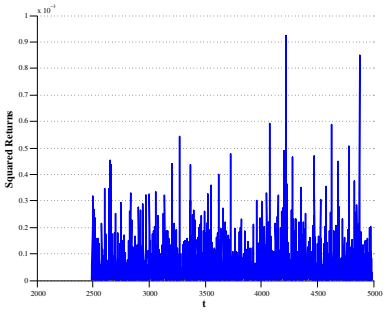
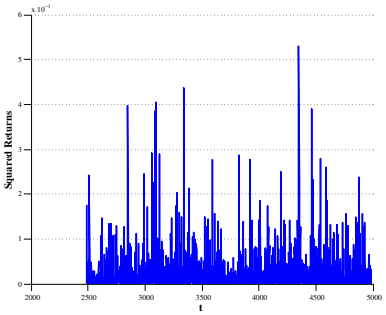


Figure 22: Multiple Stock Mixed Gaussian Artificial Stock Market Squared Stock Returns.

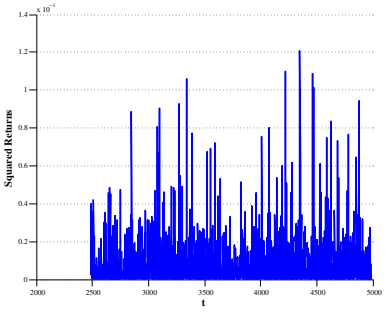
(a) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



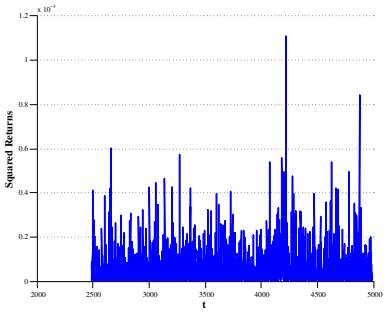
(b) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



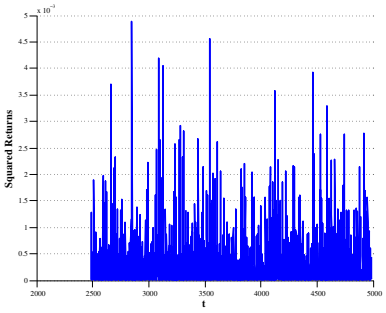
(c) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



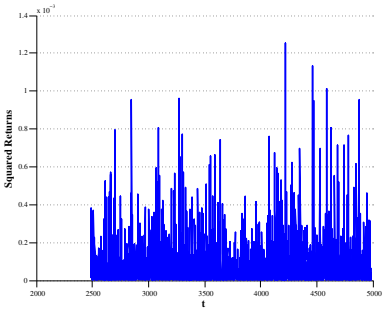
(d) Asset 1
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



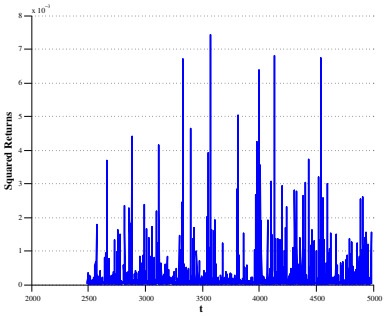
(e) Asset 2
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



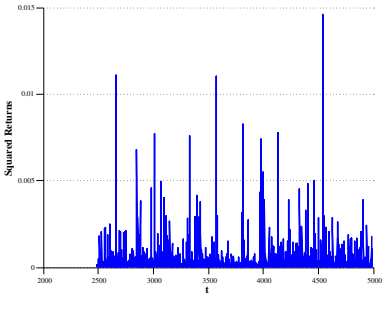
(f) Market
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



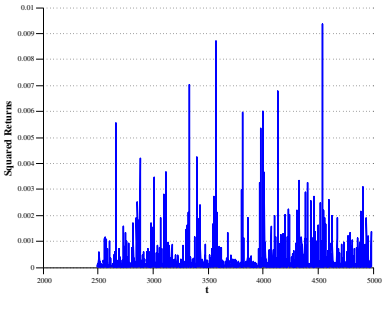
(g) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



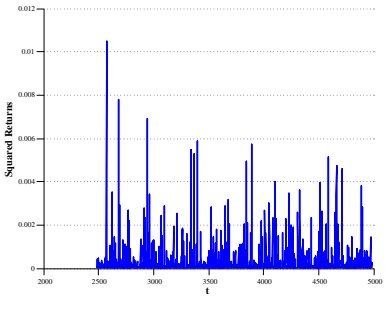
(h) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



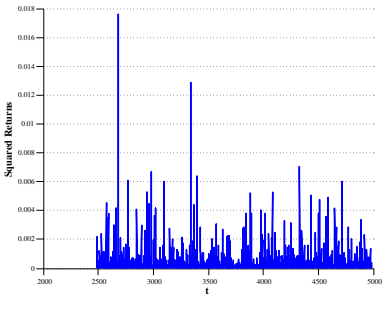
(i) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



(j) Asset 1
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(k) Asset 2
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(l) Market
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$

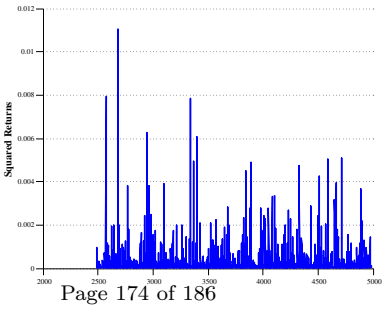
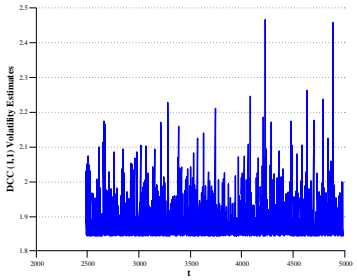
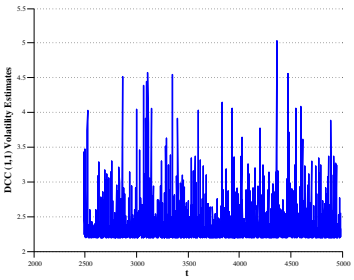


Figure 23: Multiple Stock Mixed Gaussian Artificial Stock Market Volatility and Covariation.

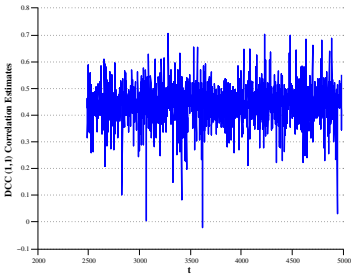
(a) Asset 1 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



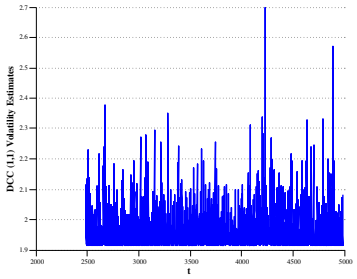
(b) Asset 2 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



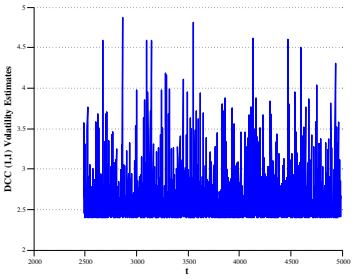
(c) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re}, A_{re}]$
B range $= [B_{re}, B_{re}]$



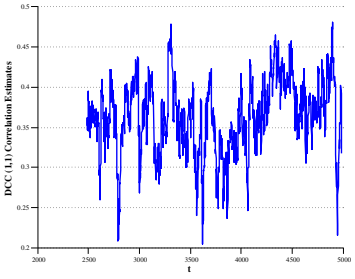
(d) Asset 1 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



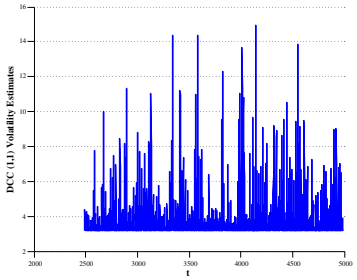
(e) Asset 2 DCC(1,1) Volatility
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



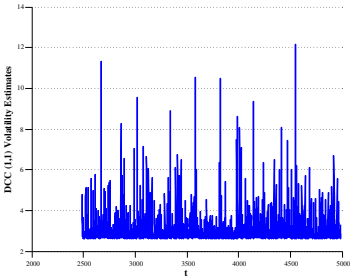
(f) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re}, A_{re}]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



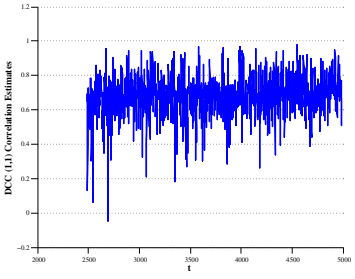
(g) Asset 1 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



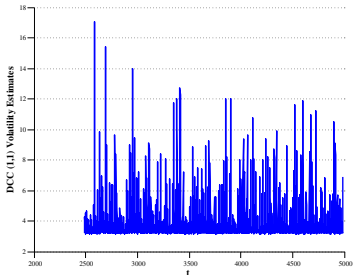
(h) Asset 2 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



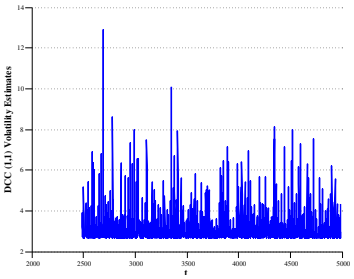
(i) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re}, B_{re}]$



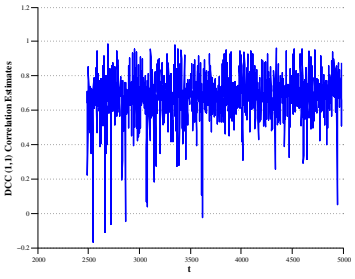
(j) Asset 1 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(k) Asset 2 DCC(1,1) Volatility
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



(l) Asset 1 and 2 DCC(1,1) Contemporaneous Correlation
A range $= [A_{re} - 0.05, A_{re} + 0.05]$
B range $= [B_{re} - 0.5, B_{re} + 0.5]$



5 Conclusion

In this paper, we documented that stock return distributions, over various periods and time frequencies, are a mixture of distributions. To explain this, we developed a multivariate agent-based asset pricing model where heterogeneous smart traders have beliefs that future gross returns are mixed Gaussian. By computing optimal trader demands and by Monte Carlo simulations, we were able to show that our model can replicate the observed real-world stock return distributions. Furthermore, we showed that the price series generated by the simulated model varies with salient financial economic variables in a plausible manner. We achieved this by using a statistical model (namely (multivariate) regression with MLE) and by Monte Carlo simulations. Furthermore we showed that asset mispricing and stock cross-correlations are higher the more uncertain agents' beliefs of the stock market. We showed that these observations hold for each individual stock and for the stock market index itself. In total we conducted 43200 Monte Carlo simulations as part of our analysis.

References

Arifovic, J. (1996). The behavior of the exchange rate in the genetic algorithm and experimental economies. *Journal of Political Economy*, 510–541.

Chiarella, C., Dieci, R. & He, X.Z. (2007). Heterogeneous expectations and speculative behavior in a dynamic multi-asset framework. *Journal of Economic Behavior & Organization*, **62**, 408–427.

Day, R.H. & Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, **14**, 299–329.

Dieci, R. & Westerhoff, F. (2010). Heterogeneous speculators, endogenous fluctuations and interacting markets: a model of stock prices and exchange rates. *Journal of Economic Dynamics and Control*, **34**, 743–764.

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, **20**, 339–350.

Farmer, J.D., Patelli, P. & Zovko, I.I. (2005). The predictive power of zero intelligence in financial markets. *Proceedings of the National Academy of Sciences of the United States of America*, **102**, 2254–2259.

Holland, J.H. (1983). Escaping brittleness. In *Proceedings Second International Workshop on Machine Learning*, 92–95, Citeseer.

LeBaron, B. (2006). Agent-based computational finance. *Handbook of Computational Economics*, **2**, 1187–1233.

LeBaron, B., Arthur, W.B. & Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, **23**, 1487–1516.

Lux, T. & Marchesi, M. (1999). Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, **397**, 498–500.

Marengo, L. & Tordjman, H. (1996). Speculation, heterogeneity and learning: A simulation model of exchange rates dynamics. *Kyklos*, **49**, 407–438.

Palmer, R., Arthur, W.B., Holland, J.H. & LeBaron, B. (1999). An artificial stock market. *Artificial Life and Robotics*, **3**, 27–31.

Raberto, M., Cincotti, S., Focardi, S.M. & Marchesi, M. (2001). Agent-based simulation of a financial market. *Physica A: Statistical Mechanics and its Applications*, **299**, 319–327.

Routledge, B.R. (2001). Genetic algorithm learning to choose and use information. *Macroeconomic Dynamics*, **5**, 303–325.

Seedat, S. & Abelman, S. (2019a). Examining parameter impacts on the computerised agent-based santa fe artificial stock market model. *Submitted Manuscript*.

Seedat, S. & Abelman, S. (2019b). Explaining stock return distributions via an agent-based model. *Submitted Manuscript*.

Westerhoff, F.H. (2004). Multiasset market dynamics. *Macroeconomic Dynamics*, **8**, 596–616.

Westerhoff, F.H. & Dieci, R. (2006). The effectiveness of keynes–tobin transaction taxes when heterogeneous agents can trade in different markets: a behavioral finance approach. *Journal of Economic Dynamics and Control*, **30**, 293–322.

Chapter 6

Results, Conclusions and Future Work

6.1 Results and Conclusions

This thesis, based on a series of four research papers, developed three new computational agent-based models of the stock market and provided a simulation-based technique, as well as a statistical model (regression) based technique, to assess the impact of parameters on an agent-based model. It also conducted a simulation-based analysis and a statistical model based analysis regarding the plausibility of a well-known pre-existing agent-based model of the stock market. The stock market agent-based models developed in Chapters 3, 4 and 5 are to the best of our knowledge novel models. This focus on developing new agent-based models is important for the following reasons. Given the recent global financial crisis, it is believed (Geanakoplos *et al.*, 2012) that conventional macroeconomic and financial economic methods failed to explain or forecast the crisis due in large part to overly simplistic assumptions such as agent homogeneity and the existence of purely rational agents which implied that market prices coincided with rational expectations equilbria.

In Chapter 2, it was noted that conventional macroeconomic and financial economic models failed to foresee the recent financial crisis and that computerised agent-based models offer an alternate approach. A criticism of computerised agent-based models of financial markets is that the impact of parameters on these models is not well understood. In Chapter 2, it was shown how two methods can be used to examine parameter impacts on an agent-based model. These two methods are (i) parameter variation Monte Carlo and (ii) a statistical model in the form of (multivariate) regression using MLE. By applying the two methods to the Santa Fe model, it was shown how simulated prices vary with structural parameters. It was shown that these relationships between parameters and simulated stock prices are sensible. Evidence was provided that agent risk-aversion (i.e. risk-on and risk-off) exists and that agent beliefs impact stock prices.

In Chapter 3, a computational multivariate (multiple stock) agent-based model was developed by significantly generalising the model of LeBaron *et al.* (1999) which is a degenerate univariate case

of the multivariate model developed. In this multiple stock model, it was shown, by computational experiments and by applying the parameter variation Monte Carlo method and statistical model method (in the form of (multivariate) regression) that the relationships between salient stock market parameters such as agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices are sensible. These analyses were conducted at the individual stock level as well as at the aggregated equally-weighted stock market level. Further insights were provided, through computational experiments, into the relationship between stock prices and two other pertinent parameters, namely stock dividend persistence and stock dividend volatility. Furthermore, Monte Carlo simulations were used to show that if the variance of agents' beliefs regarding key stock market parameters is lower, then, *ceteris paribus*, the degree of stock price mispricing, at the individual stock level and at the stock market level, is reduced. Lastly, computational experiments were conducted to examine the impact on stock volatility and covariation by agent beliefs, and it was shown that stock volatility appears to increase when agents' stock market beliefs are more likely to have a greater deviation from their rationally expected values. It was also shown that different stock market beliefs on the part of agents support different pairwise stock correlations.

In Chapter 4, the important empirical observation that various stock return kernel distributions of the S&P 500 index are a mixture of distributions was documented. This result holds unconditionally as well as during important financial economic events such as the Great Depression, the Dot-Com Bubble and the Great Recession. This finding, namely that stock returns have a mixed kernel distribution, and which was achieved by kernel density estimation, is to the best of our knowledge new and salient to the literature. To explain this, a mixed Gaussian agent-based asset pricing model was developed where univariate smart traders, who trade an equity market index, have beliefs that are consistent with the structure of the stock market. This model is, once again, a significant generalisation of the model of LeBaron *et al.* (1999) which is a degenerate unistate case of the multistate model developed. By Monte Carlo simulations, it was shown that the univariate and unistate model of LeBaron *et al.* (1999) does not produce the observed real-world stock return distributions. By computing optimal trader demands and by Monte Carlo simulations and by kernel density estimations, it was shown that the model developed can produce the observed real-world stock return distributions. This provided evidence that a mixed Gaussian agent-based asset pricing model is sufficient to produce the observed real-world distribution of stock returns. In this agent-based model model, it was shown, by computational experiments and by applying the parameter variation Monte Carlo method and statistical model method (in the form of (multivariate) regression) that the relationships between salient stock market parameters such as state probabilites, agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices are sensible. Furthermore, it was also shown that asset mispricing is larger the more uncertain agents' beliefs of the stock market. In total 43200 Monte Carlo simulations were conducted.

In Chapter 5, further evidence was provided that stock return kernel distributions of the S&P 500 index, over various periods and time frequencies, are a mixture of distributions. Following this observation, a computerised agent-based model was developed to explain the phenomenon that the real-world stock return distribution of an index such as the S&P 500 index is mixed given that portfolio managers and traders often invest and trade, respectively, the basket of assets underlying the S&P 500 equity index. To explain this emergent phenomenon, a multivariate computational agent-based model was developed where heterogeneous smart (algorithmic) traders and portfolio managers trade and invest in the *underlying equities that constitute an equity market index* and *optimise* their holdings in their portfolio optimisation. Traders are assumed to have consistent beliefs about stock market dynamics and learn over time. This model is a significant generalistion of the univariate, unistate model of LeBaron *et al.* (1999), of the multivariate, unistate model developed in Chapter 3 and of the univariate, multistate model of Chapter 4. By computing optimal trader demands and by Monte Carlo simulations and by kernel density estimations, it was shown that the agent-based model developed in Chapter 5 can replicate the observed real-world stock return distributions. In this multiple stock model, it was shown, by computational experiments and by applying the parameter variation Monte Carlo method and statistical model method (in the form of (multivariate) regression) that the relationships between salient stock market parameters such as state probabilities, agents' risk aversion, agents' beliefs regarding the volatility of stocks, the risk-free rate and the underlying stochastic dividend process on stock prices are sensible. It was also shown that asset mispricing and stock cross-correlations are higher the more uncertain agents' beliefs of the stock market are. The former was established by Monte Carlo simulations and the latter by the estimation of DCC (1,1) models. These observations hold for each individual stock and for the stock market index itself. In total 43200 Monte Carlo simulations were conducted.

To summarise, this thesis makes at least the following six contributions:

1. Firstly, it was shown how two methods can be used to examine parameter impacts on an agent-based model. These two methods are (i) parameter variation Monte Carlo and (ii) a statistical model in the form of (multivariate) regression using MLE. This methodology was applied to the Santa Fe model of Palmer *et al.* (1999), LeBaron *et al.* (1999) and Arthur *et al.* (1997). It was also applied to the three models developed in order to examine their plausibility.

2. Secondly, a multivariate computational agent-based model of the stock market was developed which provides a significant generalisation of the Santa Fe model. Such a research pursuit was described as worthwile and important by the authors themselves in Palmer *et al.* (1999). Analyses were conducted to examine the properties of this model.

3. Thirdly, and importantly, considerable empirical evidence was provided to show that kernel distributions of S&P 500 stock returns, unconditionally and conditionally, are a mixture of

distributions including during important and salient financial economic periods such as the Great Depression, the Dot-Com bubble and the Great Recession.

4. Fourthly, a single-stock mixed Gaussian agent-based model was developed that explains and produces these mixture of distributions discussed in point 3. In this model, smart (algorithmic) traders and portfolio managers trade and invest in an equally weighted average of stocks that underlie an equity index such as the S&P 500, have consistent beliefs about stock market dynamics and learn over time. Analyses were conducted to examine the properties of this model.

5. Fifthly, a multiple-stock mixed Gaussian agent-based model was developed, that explains and produces these mixture of distributions discussed in point 3. In this model, and unlike the model described above (in point 4), smart (algorithmic) traders and portfolio managers trade and invest in the *underlying equities that constitute an equity market index* and *optimise* their holdings in their portfolio optimisation, have consistent beliefs about stock market dynamics and learn over time. Analyses were conducted to examine the properties of this model. Importantly, we show that stock returns can aggregate to a mixture of distributions as we observed of the real world.

6. Sixthly, additional insights into stock mispricing ("bubbles") and into stock volatility and contemporaneous cross correlations were provided.

The validation techniques applied are a combination of theory (theorem-based and formal proofs), empirical (statistical and econometric) and computational (Monte Carlo simulation). A summary of these validation techniques is provided below.

In each of Chapters 2 to 5, new theorems and formal proofs that add significantly to the literature were developed. In Chapter 2, a proof using the law of large numbers pointwise was established that leads to a novel well-defined method to examine parameter impacts on an agent-based model. Statistical methods and models in the form of (multivariate) regression and MLE were incorporated. In each of Chapters 3 to 5, theoretical derivations were established regarding optimal stock holdings of traders and investors under different conditions. Some of these derivations require the use of probability theory results derived in these research papers (regarding a log mixed Gaussian random variable) as well as state-dependent mixed-Gaussian optimisation. In each of Chapters 4 and 5, the functional form of a linear rational expectations pricing function was derived which serves as a yardstick to compare with model prices to establish the magnitude of mispricing that prevails. Furthermore, the form of the mean square error optimal forecast for agents was proved which allows one to proceed with agent learning. All these results are new to the literature.

In Chapters 4 and 5, empirical analysis was conducted on S&P 500 stock market returns through the application of (time varying) kernel density estimation to real-world data. Estimation of a DCC (1,1) model was undertaken to estimate stock market volatilities and cross-correlations. All these

results are new to the literature.

In Chapters 2 to 5, Monte Carlo simulation techniques were utilised, primarily to examine the impact of stock market parameters on the agent-based model under consideration. All these results are new to the literature.

To the best of our knowledge, the research in Chapters 2 to 5 makes a substantial contribution to the literature on agent-based modelling of stock markets for the following reasons. It incorporates advanced mathematical and statistical methods to uncover new empirical properties regarding the probability distribution of stock market return distributions, which is a significance advance on existing research in the literature. The research conducted then develops more accurate computational agent-based models that can produce these distributional properties of stock market returns and this is particularly important given the failings exhibited by pre-existing models in the literature, especially during the recent global financial crisis of 2007-2009. Bias-free results were presented by strongly adhering to the scientific method by employing advanced mathematical and statistical methods to model a complex system. Rigorous methods such as kernel density estimation, mixture models, stochastic processes, genetic learning and (Monte Carlo) simulations were incorporated. Furthermore, eleven new formal results (lemmas and theorems) were established and proved.

The literature on agent-based models of the stock market has made noticeable progress in advancing our understanding of financial markets. The series of four research papers on which this thesis is based makes substantial contributions to this literature by explicitly developing artificial stock market models which are similar to real-world markets in that they are multi-stock and match the mixed distributions of real-world markets.

6.2 Future Work

In future work, the aim is to examine the effect parameters have on other computerised agent-based models of financial markets and of other dynamical systems using the method outlined in Chapter 2 which is itself based on applying the law of large numbers pointwise to Monte Carlo simulation paths and using statistical models such as (multivariate) regression.

An additional objective for future work is to establish the distributional shapes, through kernel density estimations similar to the technique used in Chapters 4 and 5, of other financial markets including global equities, bonds, commodity futures, currencies and so forth. Once these distributional shapes have been determined, the next aim would be to develop agent-based models, for each market, that can explain these distributional properties.

Bibliography

Aloud, M. & Fasli, M. (2017). Exploring trading strategies and their effects in the foreign exchange market. *Computational Intelligence*, **33**, 280–307.

Aloud, M., Fasli, M., Tsang, E., Dupuis, A. & Olsen, R. (2017). Modeling the high-frequency fx market: An agent-based approach. *Computational Intelligence*, **33**, 771–825.

Arifovic, J. (1991). *Learning by genetic algorithms in economic environments*. Ph.D. thesis, University of Chicago.

Arifovic, J. (1996). The behavior of the exchange rate in the genetic algorithm and experimental economies. *Journal of Political Economy*, **104**, 510–541.

Arthur, W.B., Holland, J.H., LeBaron, B., Palmer, R.G. & Tayler, P. (1997). Asset pricing under endogenous expectations in an artificial stock market. *In W. B. Arthur, S. Durlauf & D. Lane, eds, The Economy as an Evolving Complex System II*.

Boer, K., Kaymak, U. & Spiering, J. (2007). From discrete-time models to continuous-time, asynchronous modeling of financial markets. *Computational Intelligence*, **23**, 142–161.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**, 307–327.

Bray, M. (1982). Learning, estimation, and the stability of rational expectations. *Journal of Economic Theory*, **26**, 318–339.

Chen, S.H. & Yeh, C.H. (2001). Evolving traders and the business school with genetic programming: A new architecture of the agent-based artificial stock market. *Journal of Economic Dynamics and Control*, **25**, 363–393.

Chiarella, C., Dieci, R. & He, X.Z. (2007). Heterogeneous expectations and speculative behavior in a dynamic multi-asset framework. *Journal of Economic Behavior and Organization*, **62**, 408–427.

Chiarella, C., Dieci, R. & He, X. (2009). Heterogeneity, market mechanisms and asset price dynamics. In *Handbook of Financial Markets: Dynamics and Evolution*, 277–344, Elsevier.

Day, R.H. & Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, **14**, 299–329.

Dieci, R. & Westerhoff, F. (2010). Heterogeneous speculators, endogenous fluctuations and interacting markets: a model of stock prices and exchange rates. *Journal of Economic Dynamics and Control*, **34**, 743–764.

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, **20**, 339–350.

Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica: Journal of the Econometric Society*, 987–1007.

Farmer, J.D., Patelli, P. & Zovko, I.I. (2005). The predictive power of zero intelligence in financial markets. *Proceedings of the National Academy of Sciences of the United States of America*, **102**, 2254–2259.

Geanakoplos, J., Axtell, R., Farmer, J.D., Howitt, P., Conlee, B., Goldstein, J., Hendrey, M., Palmer, N.M. & Yang, C.Y. (2012). Getting at systemic risk via an agent-based model of the housing market. *American Economic Review*, **102**, 53–58.

Grossman, S.J. & Stiglitz, J.E. (1980). On the impossibility of informationally efficient markets. *American Economic Review*, **70**, 393–408.

Holland, J.H. (1983). Escaping brittleness. In *Proceedings Second International Workshop on Machine Learning*, 92–95, Citeseer.

Hommes, C. & Wagener, F. (2009). Complex evolutionary systems in behavioral finance. In *Handbook of Financial Markets: Dynamics and Evolution*, 217–276, Elsevier.

Hommes, C.H. (2006). Heterogeneous agent models in economics and finance. In *Handbook of Computational Economics*, vol. 2, 1109–1186, Elsevier.

Joshi, S., Parker, J. & Bedau, M.A. (2000). Technical trading creates a prisoner's dilemma: Results from an agent-based model. *Computational Finance*, **99**, 465–479.

LeBaron, B. (2006). Agent-based computational finance. *Handbook of Computational Economics*, **2**, 1187–1233.

LeBaron, B., Arthur, W.B. & Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, **23**, 1487–1516.

Lettau, M. (1997). Explaining the facts with adaptive agents: The case of mutual fund flows. *Journal of Economic Dynamics and Control*, **21**, 1117–1147.

Lux, T. (2009). Stochastic behavioral asset-pricing models and the stylized facts. In *Handbook of Financial Markets: Dynamics and Evolution*, 161–215, North-Holland.

Lux, T. & Marchesi, M. (1999). Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, **397**, 498–500.

Marengo, L. & Tordjman, H. (1996). Speculation, heterogeneity and learning: A simulation model of exchange rates dynamics. *Kyklos*, **49**, 407–438.

Palmer, R., Arthur, W.B., Holland, J.H. & LeBaron, B. (1999). An artificial stock market. *Artificial Life and Robotics*, **3**, 27–31.

Palmer, R.G., Arthur, W.B., Holland, J.H., LeBaron, B. & Tayler, P. (1994). Artificial economic life: a simple model of a stockmarket. *Physica D: Nonlinear Phenomena*, **75**, 264–274.

Raberto, M., Cincotti, S., Focardi, S.M. & Marchesi, M. (2001). Agent-based simulation of a financial market. *Physica A: Statistical Mechanics and its Applications*, **299**, 319–327.

Routledge, B.R. (2001). Genetic algorithm learning to choose and use information. *Macroeconomic Dynamics*, **5**, 303–325.

Westerhoff, F.H. (2004). Multiasset market dynamics. *Macroeconomic Dynamics*, **8**, 596–616.

Westerhoff, F.H. & Dieci, R. (2006). The effectiveness of keynes–tobin transaction taxes when heterogeneous agents can trade in different markets: a behavioral finance approach. *Journal of Economic Dynamics and Control*, **30**, 293–322.

Youssefmir, M. & Huberman, B.A. (1997). Clustered volatility in multiagent dynamics. *Journal of Economic Behavior & Organization*, **32**, 101–118.