Primary Classes of n-Color Compositions - Enumerations and Combinatorial Connections

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted for any degree or examination in any other University.

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Abstract

The study of compositions was initiated by MacMahon in 1893, and later extended to n-color compositions by Agarwal in 2000. In this dissertation we study standard compositions and n-color composition enumerations. Combinatorial proofs are studied using the graphical representation of compositions and n-color compositions with restricted parts. We also consider bounded part and color sizes of n-color compositions, and explore interesting combinatorial links and conclusions. The shape of n-color compositions was introduced by Munagi, where he dealt with general compositions. Here we establish the link between the Fibonacci sequence and restricted compositions, classified by part size and color size, by providing enumerative proofs.

Keywords— n-color composition, Fibonacci sequence, lattice paths, generating function

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Chapter 1

Introduction

Everything pertaining to n-color compositions stems from general compositions. We will start with some fundamentals of compositions and then look at n-color compositions.

1.1 Basics of Compositions

Definition 1.1.1. *A composition of a positive integer n is a sequence of positive integers with sum n, i.e., given a composition $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m)$ we have $\sum_{i=1}^{m} \sigma_i = n$. The σ_i's are called parts.*

Note that the difference between partitions and compositions is a matter of order. Thus for every partition with at least two different parts, there are more compositions than partitions, as we can reorder the partition in all possible ways to get the associated compositions.

Example 1.1.1. *For $n = 4$, we have five partitions, namely $(2, 1, 1)$, $(3, 1)$, $(1, 1, 1, 1)$, $(2, 2)$ and (4) itself. But we have 8 compositions of 4, namely (4), $(3, 1)$, $(1, 3)$, $(2, 2)$, $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$ and $(1, 1, 1, 1)$.*

Definition 1.1.2. *Let $k \geq 1$. We define a k-composition as a composition with k parts.*

Given a composition σ we will write $|\sigma| = n$ to indicate that σ is a composition of n, and the integer n will be referred to as the *weight* of σ.

1.1.1 The Line Graph

The weight n of a composition σ is depicted as a line divided into n equal segments and separated by $n - 1$ spaces. A composition $\sigma = (\sigma_1, \sigma_2, ..., \sigma_k)$ of n then corresponds to a choice of $k - 1$ from the $n - 1$ spaces, indicated with nodes, such that a node is placed after σ_1 segments, the next node is placed after σ_2 segments, and so on.

The **conjugate composition** of σ, denoted by σ', is found by placing nodes on the other $n - k$ spaces [24].

For instance the line graph of the composition $\sigma = (5, 3, 1, 2, 2)$ of 13 is

By considering the 8 free spaces in the graph we deduce that $\sigma' = (1, 1, 1, 1, 2, 1, 3, 2, 1)$.

Definition 1.1.3. *Two compositions of n are conjugate if in their respective line graphs, one has nodes exactly at the positions where the other one does not have nodes and vice versa.*

Theorem 1.1 (Heubach-Mansour)**.** *The number of compositions of n with exactly m parts is given by $\binom{n-1}{m-1}$ and the total number of compositions of n is 2^{n-1}.*

Proof. Let σ be an arbitrary composition of n with m parts. Then from the line graph of σ, there are $m-1$ choices out of $n-1$ spaces to specify σ. This gives $\binom{n-1}{m-1}$ possible ways to obtain σ. It follows that the number of compositions of n with exactly m parts is given by $\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}$. Hence the total number of compositions of n is given by

$$\sum_{m=1}^{\infty} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} = 2^{n-1}.$$

□

Definition 1.1.4. *If $\sigma = (\sigma_1, \sigma_2, ..., \sigma_k)$ is a composition then its* inverse *is give by $\overline{\sigma} = (\sigma_k, \sigma_{k-1}, ..., \sigma_1)$. If a composition can be read the same from left to right as from right to left then it is called* self-inverse or palindromic.

1.1.2 Four Basic Shapes of Compositions

We define a $1c1$-composition to be one with the first and last parts are both equal to 1, a $1c2$-composition to be one with first part equal to 1 and last part > 1, a $2c1$-composition to be one with first part > 1 and last part 1, and a $2c2$-composition to be one with first and last parts both > 1. The set of compositions of n splits naturally into four subsets of equal cardinality corresponding to the four types of compositions.

Theorem 1.2 (Munagi [23])**.** *Let n be a natural number > 1. Then the following classes of compositions are equinumerous:*

(i) $1c1$*-compositions of n*

(ii) $1c2$*-compositions of n*

(iii) $2c1$*-compositions of n*

(iv) $2c2$*-compositions of n*

The cardinality of each class of compositions is 2^{n-3}.

Proof. It is clear that $1c2$-compositions are inverses of $2c1$-compositions, and $1c1$-compositions are conjugates of $2c2$-compositions. Hence we have proved the respective bijections (ii) $\iff$ (iii) and (i) $\iff$ (iv). To complete the proof it remains to show the bijection (i) $\iff$ (ii). A composition in (ii) has the form $\sigma = (1, \sigma_2, \ldots, \sigma_k), \sigma_k > 1$. Deleting the initial 1 and subtracting 1 from σ_k gives $(\sigma_2, \ldots, \sigma_k - 1) = T$, which is an arbitrary composition of $n-2$. Now pre-pend and append 1 to obtain $(1, \sigma_2, \ldots, \sigma_k - 1, 1)$, which is a unique composition in (i). Hence the bijection (i) $\iff$ (ii) holds.

Lastly, note that the passage from σ to T is a bijection from (i) to the class of compositions of $n-2$. In other words the common number of compositions in each of the classes is $\sigma(n-2) = 2^{n-3}$. □

Example 1.1.2. *Given n=6, we have that each set will have cardinality* $2^3 = 8$*, as seen below,*

$$\begin{aligned}1c1 =&(1,1,1,1,1,1),(1,2,2,1),(1,4,1),(1,2,1,1,1),(1,1,2,1,1),(1,3,1,1),\\&(1,1,3,1);(1,1,1,2,1),\\2c1 =&(2,1,1,1,1),(2,2,1,1),(2,1,2,1),(2,3,1),(3,2,1),(4,1,1),\\&(5,1),(3,1,1,1),\\1c2 =&\textit{inverse of 2c1},\\2c2 =&(6),(3,3),(2,1,1,2),(2,2,2),(4,2),(2,4),(3,1,2,),(2,1,3).\end{aligned}$$

1.2 n-Color Compositions

In this section we consider a class of compositions in which several copies of each integer are used as parts, which was first introduced by Agarwal.

Colored compositions are generalized compositions in which a part size may come in a prescribed number of types or colors which are denoted with positive-integer subscripts, $1_1, 1_2, \ldots, 2_1, 2_2, \ldots$.

In other words, a colored composition $((\sigma_1)_{c_1}, (\sigma_2)_{c_2}, \ldots, (\sigma_k)_{c_k})$, $k \geq 1$, consists of an ordinary composition $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ and a corresponding sequence of colors $(c_1, c_2, \ldots, c_k)$ with $c_i \in \{1, 2, 3, \ldots\}$.

Definition 1.2.1. *An n-color composition is a colored composition of n in which a part of size m may come in m different colors:* $m_1, m_2, \ldots, m_m$, $m = 1, 2, 3, \ldots$.

Thus an n-color composition $((\sigma_1)_{c_1}, (\sigma_2)_{c_2}, \ldots, (\sigma_k)_{c_k})$, $k \geq 1$, is a colored composition in which $1 \leq c_i \leq \sigma_i$.

Example 1.2.1. *For n=3, we have 8 n-color compositions. These are:-*

$$(3_1),(3_2),(3_3),(2_1,1_1),(2_2,1_1),(1_1,2_1),(1_1,2_2),(1_1,1_1,1_1).$$

Note, for instance, that (3_4) *is not an n-color composition because* $3 < 4$*, and* $(2_2, 1_2)$ *is not an n-color composition because* $1 < 2$.

Definition 1.2.2. *An n-color composition whose sequence of parts from left to right is identical with the sequence of parts from right to left, is called an n-color* self-inverse composition *or an n-color* palindromic composition.

Example 1.2.2. *For n=3, we have 4 n-color palindromic compositions. These are:-*

$$(3_1),(3_2),(3_3),(1_1,1_1,1_1)$$

Define $C(n)$ and $C(n,m)$ as the number of n-color compositions of n and number of n-color compositions of n into m parts respectively. Furthermore let us recall the definition of the well known Fibonacci Sequence.

Definition 1.2.3 (Fibonacci)**.** *Let* F_n *denote the nth Fibonacci number,* $n \geq 0$*. Then we have,*

$$F_n = F_{n-1} + F_{n-2},\ n > 1,$$

where $F_0 = 0, F_1 = 1$.

Theorem 1.3. *Given* F_n *to be the nth Fibonacci number, we have the following:*

$$\sum_{k=0}^{n} F_k \binom{n}{k} = F_{2n}$$

Proof. Firstly, F_{2n} counts the number of ways of tiling a strip of length $2n-1$ with tiles of length either 1 (squares) or 2 (dominos)(tilings are covered in more detail in [17]). Now, let us look at the number of tilings of that strip in which k of the first n tiles are squares. There are $\binom{n}{k}$ ways of arranging the first n tiles. A collection of k squares and $n-k$ dominos will have length $2n-k$. So the strip remaining after the first n tiles has length $k-1$, which means that it can be tiled F_k ways. So this implies that the number of tilings of this sort is $F_k\binom{n}{k}$.
Furthermore any tiling of the strip uses at least n tiles (or it would have length at most $2n-2$), and the first n tiles cannot all be dominos (or they would have length $2n$). So we have

$$F_{2n} = \sum_{k=1}^{n} F_n \binom{n}{k}.$$

Please note that the term 'domino' in this context refers to a tile with a dot in the middle(refer to Figure 1.1). □

We recall the Cauchy product of two generating functions rule, which will be useful for manipulating generating functions.

Remark 1.2.1. *Given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, we have*

$$\begin{aligned} f(x)\cdot g(x) &= a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + ... \\ &= \sum_{n=0}^{\infty}\Big(\sum_{i=0}^{n} a_i b_{n-i}\Big)x^n. \end{aligned}$$

Theorem 1.4 (Agarwal [3])**.** *Let $C(m,x)$ and $C(x)$ denote the generating functions for $C(n,m)$ and $C(n)$ respectively. Then,*

$$C(m,x) = \frac{x^m}{(1-x)^{2m}}, \tag{1.1}$$

$$C(x) = \frac{x}{1-3x+x^2}, \tag{1.2}$$

$$C(n,m) = \binom{n+m-1}{2m-1}, \tag{1.3}$$

$$C(n) = F_{2n}, \tag{1.4}$$

where F_{2n} is the $(2n)^{th}$ Fibonacci number, i.e even-indexed Fibonacci numbers.

Proof. We have

$$\begin{aligned} C(m,x) &= \sum_{n=1}^{\infty} C(n,m)x^n, \\ &= (x + 2x^2 + 3x^3 + ...)^m, \\ &= \frac{x^m}{(1-x)^{2m}}. \end{aligned}$$

This proves equation (1.1).

$$\begin{aligned} C(x) &= \sum_{m=1}^{\infty} C(m,x), \\ &= \sum_{m=1}^{\infty} \frac{x^m}{(1-x)^{2m}}, \\ &= \frac{x}{1-3x+x^2}. \end{aligned}$$

This proves (1.2). Now if we look at (1.1) and sum over all m's to infinity, we have

$$\begin{aligned}
C(m,x) &= \sum_{m=1}^{\infty} \frac{x^m}{(1-x)^{2m}}, \\
&= \sum_{m=1}^{\infty} \frac{x^{2m}}{(1-x)^{2m}} \cdot \frac{1}{x^m}, \\
&= \sum_{m=1}^{\infty} \frac{x^{2m}}{(1-x)^{(2m)+1}} \cdot \frac{(1-x)}{x^m}, \\
&= (1-x)\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \binom{n}{2m} \cdot \frac{x^n}{x^m}, \\
&= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \binom{n+m}{2m} \cdot x^n - \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \binom{n+m-1}{2m} \cdot x^n.
\end{aligned}$$

And so

$$\begin{aligned}
[x^n]C(m,x) &= \binom{n+m}{2m} - \binom{n+m-1}{2m}, \\
&= \binom{n+m-1}{2m-1}.
\end{aligned}$$

This proves (1.3).
Finally, we know that (refer to Theorem 1.5)

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}.$$

Calculating the ordinary generating function from the left hand side we get

$$\begin{aligned}
\sum_{x\geq 0} x^n \sum_{k=0}^{n} \binom{n}{k} F_k &= \sum_{k\geq 0} F_k \sum_{n} \binom{n}{k} x^n, \\
&= \sum_{k\geq 0} F_k \sum_{n\geq k} (-1)^{n-k} \binom{-k-1}{n-k} x^n, \\
&= \sum_{k\geq 0} F_k x^k \sum_{n\geq k} \binom{-k-1}{n-k} (-x)^{n-k}, \\
&= \sum_{k\geq 0} F_k x^k \sum_{n\geq 0} \binom{-k-1}{n} (-x)^{n}, \\
&= \sum_{k\geq 0} F_k x^k (1-x)^{-k-1}, \\
&= (1-x)^{-1} \sum_{k\geq 0} F_k \cdot \left(\frac{x}{1-x}\right)^k.
\end{aligned}$$

Recalling the generating function for the Fibonacci numbers i.e

$$f(z) := \sum_{k\geq 0} F_k z^k = \frac{z}{1-z-z^2},$$

this implies that

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} \binom{n}{k} F_k = (1-x)^{-1} \cdot \frac{(\frac{x}{1-x})}{1-(\frac{x}{1-x})-(\frac{x}{1-x})^2}.$$

Which simplifies to

$$\frac{x}{1-3x-x^2},$$

i.e. (1.2). We see this proves (1.4). □

Remark 1.2.2. *We wish to highlight the minor difference between the above generating function, when it begins with $1x = x$ and when it begins with $0x^0 = 1$.*

We have just proved that

$$C(x) = \sum_{n=1}^{\infty} C(n)x^n = \frac{x}{1-3x+x^2} = x + 3x^2 + 8x^3 + 21x^4 + 55x^5 + 144x^6 + \cdots .$$

But alternatively we also have

$$C(x) = \sum_{n=0}^{\infty} C(n)x^n = \frac{(1-x)^2}{1-3x+x^2} = 1 + x + 3x^2 + 8x^3 + 21x^4 + 55x^5 + 144x^6 + \cdots .$$

1.3 Graphical Representation of n-Color Compositions

Though there are various ways of representing n-color compositions, we are only going to mention spotted tilings from Brian Hopkins in [17] and lattice paths from Agarwal in [7].

1.3.1 Spotted Tilings

Brian Hopkins in [17] described spotted tilings to provide a graphical representation of n-color compositions. A part k_i corresponds to a $1 \times k$ rectangle with a spot in position i. An example is shown below in Figure 1.1.

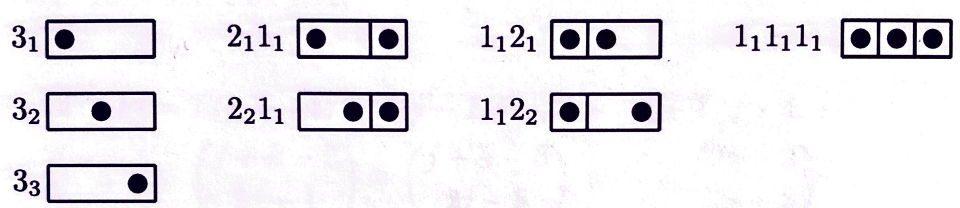


Figure 1.1: The 8 3-color composition representation

Theorem 1.5 (Brian Hopkins [17])**.** *An n-color composition is conjugable if its zig-zag graph(refer to [24]) has exactly one spot per column. This is explained algebraically by*

$$(2_1^a, 1_1, 2_2^b, (3_2, 2_1^a, 1_1, 2_2^b)^c).,$$

where exponents denote repetition, $a, b, c \geq 0$, and a, b may vary in each occurrence.

1.3.2 Lattice Paths

Another interesting way of representing n-color compositions is by lattice paths, defined by Agarwal in [7].

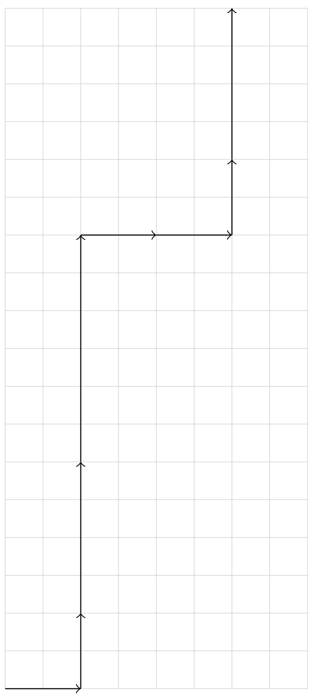
Definition 1.3.1 ([7]). *Our lattice paths will use the following terms.*

- *Let R denote a step defined by $(x, y) \rightarrow (x + 1, y + 1)$ i.e diagonal.*
- *Let S denote a step defined by $(x, y) \rightarrow (x + 1, y)$ i.e horizontal.*
- *Let $T_a, a \geq 1$ denote a step defined by $(x, y) \rightarrow (x, y + a)$ i.e vertical.*

We also define the $'destination'$ to be the end point of the lattice path.

Now using the bijection described in [7], we give an example.

Example 1.3.1. *Given an n-color composition* $(2_1, 1_1, 2_2, 3_2, 2_1, 1_1, 2_2)$ *which translates to* $ST_1T_2T_3SST_1T_2$ *the graph will look like ;*



Definition 1.3.2 (Van Lint and Wilson [12]). *Consider walks in the XY-plane where each step is $S : (x, y) \rightarrow (x + 1, y)$ or $T_a : (x, y) \rightarrow (x, y + a)$ with 'a' a positive integer. For $n \geq 1$, let A_n denote the number of walks from (0,0) that have the destination a point on the line $x + y = n$.*

Theorem 1.6 (Agarwal [7]). *For $n \geq 2$, let $B(n)$ denote the number of lattice paths first taking $'m'$ $(0 \leq m \leq n - 1)$ R steps followed by S or T_a steps such that the destination of the path lies on the line $x + y = n + m - 1$. Then $B(n) = C(n), \forall n \geq 1$, wehre $B(1) = 1$, the lone lattice path enumerated by $B(1)$ consists of the origin only.*

With this we can now prove the recurrence relation (1.4)

$$C(n) = 3 \cdot C(n - 1) - C(n - 2),$$

for C(1)=1 and C(2)=3.
We will count the relevant lattice paths counted or enumerated by $C(n) = B(n)$, with

R steps being at the centre of our argument. Using the above theorem, we can split our enumerations into different cases:
Case(i): When $m = n-1$. This implies we have $(n-1)R$-steps and our destinatinon is $x+y = 2(n-1)$. So our only relevant lattice path is $(n-1)R$-steps. Hence we have $B(n-2)$ relevant lattice paths.
Case(ii): When $m = 0$. This means we do not have any R-steps and instead we only have S/T_a. Hence we have two relevant lattice paths. Hence we have $B(n-1)$ lattice paths e.g. ST_a.
Case(iii): When $1 \leq m \leq n-2$, a relevant lattice path we have m number of R-steps and either of the following:
(a)all other steps are S. Hence our relevant paths will be $B(n-1)$ e.g. RS or
(b) all other steps are T_a. Hence our relevant paths will be $B(n-1)$ e.g. RT_a or
(c) there are S and T_a steps where either can not be 0. Hence our relevant paths will be $B(n)$ e.g. RST_a. Putting it all together, counting the number of R lattice paths we get

$$B(n) = B(n-1) + B(n-1) + B(n-1) - B(n-2).$$

We already know that $B(1) = 1$. If we have $B(2)$ this implies that $0 \leq m \leq 1$. So for $m = 0$, our relevant lattice path includes S and T. For $m = 1$, our relevant lattice path is only R. So hence we have $B(2) = 3$. This concludes the proof.

Chapter 2

n-Color Compositions with Bounded Part and Color sizes

Now let us investigate n-color compositions with bounded part sizes and color sizes. We present their generating functions, and demonstrate how the corresponding recurrence relations are obtained from the generating functions.

The following notation will be used in this chapter and the next. Let $C(n, m \mid P)$ denote the number of n-color compositions of n with m parts that satisfy property P. We can then define $C(n \mid P) := C(n, 1 \mid P) + C(n, 2 \mid P) + \cdots + C(n, n \mid P)$.

2.1 Part sizes Bounded from Above

Let $C(n, m \mid \text{parts} \leq t)$ denote the number of n-color compositions of n into m-parts with no part $> t$. Then we can generate one part by

$$x + 2x^2 + ... + tx^t = \frac{x}{(1-x)^2} - \frac{x^{t+1}(1+t-tx)}{(1-x)^2}.$$

So, for $m \geq 0$ parts, we have

$$\sum_{n=0}^{\infty} C(n, m \mid \text{parts} \leq t)x^n = \Big(\frac{x - x^{t+1}(1+t-tx)}{(1-x)^2}\Big)^m, \tag{2.1}$$

$$\implies \sum_{n=0}^{\infty} C(n \mid \text{parts} \leq t)x^n = \sum_{m=0}^{\infty} \Big(\frac{x - x^{t+1}(1+t-tx)}{(1-x)^2}\Big)^m, \tag{2.2}$$

$$= \frac{(1-x)^2}{1 - 3x + x^2 + (t+1)x^{t+1} - tx^{t+2}}. \tag{2.3}$$

Observe that

$$\lim_{t\to\infty} \frac{(1-x)^2}{1 - 3x + x^2 + (t+1)x^{t+1} - tx^{t+2}} = \frac{(1-x)^2}{1 - 3x + x^2}$$

since the two terms $(t+1)x^{t+1} - tx^{t+2}$ tend to $\infty - \infty$ as t grows without bound. So our limit simply recovers $C(n)$. This makes sense since as we increase upper bound of the part sizes t we eventually negate the condition since t becomes large enough.

In order to obtain a recurrence, we restate Equation (2.3) below, where $C(n \mid \text{parts} \leq t) \equiv c_t(n)$, with $t \geq 1$ and $c_t(0) = 0$.

$$\sum_{m=0}^{\infty} c_t(n)x^n = \frac{(1-x)^2}{1 - 3x + x^2 + (t+1)x^{t+1} - tx^{t+2}}. \tag{2.4}$$

When $t = 1$ we have:

$$\sum_{n=0}^{\infty} c_1(n)x^n = \frac{1}{1-x}. \tag{2.5}$$

This confirms $c_1(n) = 1$, where the enumerated object is $(1_1, 1_1, 1_1, ..., 1_1)$, n-copies of 1_1. Equation (2.4) is equivalent to

$$\sum_{n=0}^{\infty} c_t(n)x^n - 3\sum_{n=1}^{\infty} c_t(n-1)x^n + \sum_{n=2}^{\infty} c_t(n-2)x^n + (t+1)\sum_{n=t+1}^{\infty} c_t(n-t-1)x^n$$
$$-t\sum_{n=t+2}^{\infty} c_t(n-t-2)x^n = (1-x)^2$$

which implies

$$\sum_{n=t+2}^{\infty} (c_t(n) - 3c_t(n-1) + c_t(n-2) + (t+1)c_t(n-t-1) - tc_t(n-t-2))x^n = \sum_{n=t+2}^{\infty} 0x^n.$$

Equating coefficients of x^n on both sides we obtain the following recurrence relation.

Theorem 2.1. *Let n,t be positive integers with $t \geq 2$ and $n \geq t+3$. Then*

$$c_t(n) = 3c_t(n-1) - c_t(n-2) - (t+1)c_t(n-1-t) + tc_t(n-2-t),$$

$$c_1(n) = 1, n \geq 1; c_t(n) = F_{2n}, n \leq t; c_t(t+1) = F_{2(t+1)} - t - 1, c_t(t+2) = F_{2(t+2)} - 3t + 8.$$

Proof. It remains to explain the initial conditions. Equation (2.5) implies $c_1(n) = 1$ for all $n \geq 1$.

If $n \leq t$, then all the parts in every n-color composition of n is less than or equal to t; so $c_t(n)$ equals $C(n) = F_{2n}$.

If $n = t+1$, then all the parts in every n-color composition of n is less than or equal to t except the n compositions $(n_1), (n_2), \ldots, (n_n)$; so $c_t(n) = C(n) - n = F_{2n} - n$ or $c_t(t+1) = F_{2(t+1)} - t - 1$.

If $n = t+2$ (or $t = n-2$), then all the parts in every n-color composition of n is less than or equal to t except two classes of objects, namely, (1) the n compositions $(n_1), (n_2), \ldots, (n_n)$; and (2) the $n-1$ compositions $(1, (n-1)_1, (n-1)_2, \ldots, (n-1)_{n-1})$ and their inverses, giving $2(n-1)$ objects. So the total number of exceptions is $n+2(n-1) = 3n-2$. Hence $c_t(n) = C(n) - 3n + 2 = F_{2n} - 3n + 2$ or $c_t(t+2) = F_{2(t+2)} - 3t + 8$. □

2.2 Color Sizes Bounded from Above

Consider $C(n, m \mid \text{colors} \leq r)$. Then we can generate one part by

$$x + 2x^2 + ... + rx^r + rx^{r+1} + ... = \frac{x - x^{r+1}(1 + r - rx)}{(1-x)^2} + \frac{rx^{r+1}}{1-x} = \frac{x - x^{r+1}}{(1-x)^2}.$$

So, for $m \geq 0$, we have

$$\sum_{n=0}^{\infty} C(n, m \mid \text{colors} \leq r)x^n = \left(\frac{x - x^{r+1}}{(1-x)^2}\right)^m$$

which implies

$$\sum_{n=0}^{\infty} C(n \mid \text{colors} \leq r)x^n = \sum_{m=0}^{\infty} \Big(\frac{x - x^{r+1}}{(1-x)^2}\Big)^m, \tag{2.6}$$

$$= \frac{(1-x)^2}{1 - 3x + x^2 + x^{r+1}}. \tag{2.7}$$

Notice that (2.7) implies that $\lim\limits_{t\to\infty} C(n|\text{colors} \leq r) = 0$, since the term $x^{r+1} \to \infty$ and causes the denominator to increase without bound. (The recurrence, below, supports this because $d_r(n-r-1)$ tends to an undefined quantity (i.e., negative subscript) as $r \to \infty$).

In order to obtain the recurrence corresponding to Equation (2.7), let $C(n \mid \text{colors} \leq r) \equiv d_r(n)$ with $r \geq 1$ and $d_r(0) = 0$.

When $r = 1$ we obtain

$$\sum_{n=0}^{\infty} d_1(n)x^n = \frac{1-x}{1-2x}, \tag{2.8}$$

which is the generating function for the number of all ordinary compositions, that is, $d_1(n) = 2^{n-1}$, $n \geq 1$. But Equation (2.7) is equivalent to

$$\sum_{n=0}^{\infty} d_r(n)x^n - 3\sum_{n=1}^{\infty} d_r(n-1)x^n + \sum_{n=2}^{\infty} d_r(n-2)x^n +$$

$$+ \sum_{n=r+1}^{\infty} d_r(n-r-1)x^n = (1-x)^2$$

which implies

$$\sum_{n=r+2}^{\infty} (d_r(n) - 3d_r(n-1) + d_r(n-2) + d_r(n-r-1))x^n = \sum_{n=r+2}^{\infty} 0x^n.$$

Equating coefficients of x^n on both sides we obtain the following recurrence relation.

Theorem 2.2. *Let n, r be positive integers with $r \geq 1$ and $n \geq r+2$. Then*

$$d_r(n) = 3d_r(n-1) - d_r(n-2) - d_r(n-r-1),$$

where

$$d_1(n) = 2^{n-1}, n \geq 1;\ d_r(n) = F_{2n}, n \leq r;\ d_r(r+1) = F_{2(r+1)} - 1.$$

Proof. In order to complete the proof the initial conditions can be found as follows.

If $r = 1$ then we are refer to Equation (2.8). Note that if we restrict all subscripts to be 1, then we are simply counting our n-color compositions as normal compositions. This is confirmed by the fact that the number of ordinary compositions of n is equal to 2^{n-1}.

If $n \leq r$ then all parts in every n-color composition of n is less than or equal to r; so $d_r(n) = F_{2n}$.

Lastly, if $n = r+1$ then from all our n-color compositions, only one is excluded i.e., $(1+r)_{1+r}$. Hence we have $d_r(r+1) = F_{2(r+1)} - 1$. □

Example 2.2.1. *Let $n = 4$ and $t = 3$. Then $C(4|\text{parts} \leq 3)$ equals the number of colored compositions of 4 in which parts of size $x \in \{1, 2\}$ can be colored in 1 or 2 possible ways,*

$(1_1, 1_1, 1_1, 1_1)$, $(1_1, 1_1, 2_1)$, $(1_1, 1_1, 2_2)$, $(2_1, 1_1, 1_1)$, $(2_2, 1_1, 1_1)$, $(1_1, 2_1, 1_1)$, $(1_1, 2_2, 1_1)$, $(2_1, 2_2)$, $(2_2, 2_2)$, $(2_1, 2_2)$, $(2_2, 2_1)$

and every other part size ≥ 3 bears exactly 3 possible colors,

$(1_1, 3_1)$, $(1_1, 3_2)$, $(1_1, 3_3)$, $(3_1, 1_1)$, $(3_2, 1_1)$, $(3_3, 1_1)$, (4_1), (4_2), (4_3).

2.3 Part and Color Sizes Bounded from Below

Let $C(n, m \mid \text{parts} \geq t)$ denote the number of n-color compositions of n into m-parts with no part $< t$. Then we can generate one part by

$$tx^t + (t+1)x^{t+1} + (t+2)x^{t+2} + ... = \frac{tx^t + x^{t+1} - tx^{t+1}}{(1-x)^2}.$$

So, for $m \geq 0$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} C(n, m \mid \text{parts} \geq t)x^n &= \Big(\frac{tx^t + x^{t+1} - tx^{t+1}}{(1-x)^2}\Big)^m, \\
\implies \sum_{n=0}^{\infty} C(n \mid \text{parts} \geq t)x^n &= \sum_{m=0}^{\infty} \Big(\frac{tx^t + x^{t+1} - tx^{t+1}}{(1-x)^2}\Big)^m, \\
&\frac{(1-x)^2}{1 - 2x + x^2 - tx^t - x^{1+t} + tx^{1+t}}.
\end{aligned}$$

Using a similar analysis as above, we can derive the following recursive theorem, where $C(n \mid \text{parts} \geq t) \equiv e_t(n)$.

Theorem 2.3. *Let n, t be positive integers with $r \geq 2$ and $n \geq t + 2$. Then*

$$e_t(n) = 2e_t(n-1) - e_t(n-2) + te_t(n-t) - (t-1)e_t(n-t-1),$$

$$e_1(n) = F_{2n}, n > 0;\ e_t(n) = 0, n < t;\ e_t(t) = t;\ e_t(t+1) = 3t + 1.$$

Let $C(n, m \mid \text{colors} \geq r)$ denote the number of n-color compositions of n into m-parts and with no color $< r$. Then one is generated by

$$x^r + 2x^{r+1} + 3x^{r+2} + ... = \frac{x^r}{(1-x)^2}.$$

So, for $m \geq 0$ we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} C(n, m \mid \text{colors} \geq r)x^n &= \Big(\frac{x^r}{(1-x)^2}\Big)^m, \\
\rightarrow \sum_{n=0}^{\infty} C(n \mid \text{colors} \geq r) &= \sum_{m=0}^{\infty} \Big(\frac{x^r}{(1-x)^2}\Big)^m, \\
&= \frac{(1-x)^2}{1 - 2x + x^2 - x^r}
\end{aligned}$$

As before, we find that the following recurrence relation for $C(n \mid \text{colors} \geq r) \equiv f_r(n)$ holds for $n \geq r + 1$, provided that $f_1(n) = F_{2n}, n \geq 1$; $f_r(n) = 0, n < r$; $f_r(r) = 1$:

$$f_r(n) = 2f_r(n-1) - r_r(n-2) + f_r(n-r),\ r > 1,\ n > r.$$

2.4 The 'missing part' and 'missing color' Functions

Let us consider the 'missing part' function, to be denoted by $C(n, m \mid \text{no part}\, t)$, i.e., the number of n-color compositions of n into m-parts and in which we do not have any part $= t$. We can generate one part by

$$(x + 2x^2 + 3x^3 + \cdots) - tx^t = \frac{x - tx^t(1-x)^2}{(1-x)^2}.$$

So, for $m \geq 0$ we have,

$$\sum_{n=0}^{\infty} C(n,m \mid \text{no part}\, t)x^n = \Big(\frac{x - tx^t(1-x)^2}{(1-x)^2}\Big)^m,$$

$$\implies \sum_{n=0}^{\infty} C(n \mid \text{no part}\, t) = \sum_{m=0}^{\infty} \Big(\frac{x - tx^t(1-x)^2}{(1-x)^2}\Big)^m,$$

$$= \frac{(1-x)^2}{1 - 3x + x^2 + tx^t(1-x)^2}.$$

As a check, we compare the generating functions for $C(n \mid \text{no part}\, t)$ and $C(n \mid \text{parts} \geq t)$ when $t = 1$ and $t = 2$, respectively, and obtain

$$\sum_{n=0}^{\infty} C(n \mid \text{no part}\, 1) = \frac{(1-x)^2}{1 - 2x - x^2 + x^3} = C(n \mid \text{parts} \geq 2).$$

Finally, let us consider the missing color function to be denoted by $c(n,m \mid \text{no color}\, r)$ i.e. number of n-color compositions of n into m-parts in which a color of size r is forbidden. We can generate one part by

$$x + 2x^2 + \cdots + (r-1)x^{r-1} + (r-1)x^r + rx^{r+1} + (r+1)x^{r+2} + (r+2)x^{r+3} + \cdots$$

So, for $m \geq 0$ we have

$$\sum_{n=0}^{\infty} C(n,m \mid \text{no color}\, r)x^n = \left(\frac{x - x^r + x^{r+1}}{(1-x)^2}\right)^m$$

$$\implies \sum_{n=0}^{\infty} C(n \mid \text{no color}\, r)x^n = \left(\frac{x - x^r + x^{r+1}}{(1-x)^2}\right)^m$$

$$= \frac{(1-x)^2}{1 - 3x + x^2 + x^r - x^{r+1}}.$$

As a check, we compare the generating functions for $C(n \mid \text{no color}\, r)$ and $C(n \mid \text{colors} \geq r)$ when $r = 1$ and $r = 2$, respectively, and obtain

$$\sum_{n=0}^{\infty} C(n \mid \text{no color}\, 1) = \frac{(1-x)^2}{1-2x} = C(n \mid \text{colors} \geq 2).$$

Hence we deduce the combinatorial identity:

Corollary 2.3.1. *Let n be a positive integer. Then*
The number of n-color compositions of n without the color 1 is equal to the number of ordinary compositions of $n-1$.

For example, we have $C(5 \mid \text{no color}\, 1) = 8$:

$$(2_2, 3_3), (3_3, 2_2), (2_2, 3_2), (3_2, 2_2), (5_2), (5_3), (5_4), (5_5).$$

The corresponding 8 ordinary compositions of 4 are:

$$(4), (1,3), (2,2), (3,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,1).$$

Chapter 3

n-Color Compositions with Parts and Colors in a Residue Class

Let us now have a look at some interesting enumerations involving n-color residue classes including even parts, odd parts and prime numbers. We will use similar terminology adapted in Chapter 2.

3.1 Part Sizes belonging to an Even Residue Class

Let $C(n, m \mid \text{parts even})$ denote the number of n-color compositions of n into m even parts. Then, according to Yu-hong Guo in [13], we can generate one part by

$$2x^2 + 4x^4 + 6x^6 + \ldots = \frac{2x^2}{(1-x^2)^2}.$$

So, for $m > 0$ parts, we have

$$\sum_{n=0}^{\infty} C(n, m \mid \text{parts even}) x^n = \Big(\frac{2x^2}{(1-x^2)^2}\Big)^m, \tag{3.1}$$

$$\implies \sum_{n=0}^{\infty} C(n \mid \text{parts even}) x^n = \sum_{m=1}^{\infty} \Big(\frac{2x^2}{(1-x^2)^2}\Big)^m, \tag{3.2}$$

$$= \frac{2x^2}{1-4x^2+x^4}. \tag{3.3}$$

From this we can get the recurrence formula. We restate the above equation, (3.3), where $C(n \mid \text{parts even}) \equiv c_e(n)$, with $c_e(0) = 0$.

$$\sum_{n=0}^{\infty} c_e(n) x^n = \frac{2x^2}{1-4x^2+x^4}.$$

This is equal to,

$$\sum_{n=0}^{\infty} c_e(n) x^n - 4\sum_{n=2}^{\infty} c_e(n-2) x^n + \sum_{n=4}^{\infty} c_e(n-4) x^n = 2x^2. \tag{3.4}$$

Equating coefficients of x^n we obtain that $c_e(n) = 0$ if n is odd, and if n is even we have

$$c_e(n) = 4c_e(n-2) - c_e(n-4), \;\; n \geq 4$$

with our initial conditions being $c_e(0) = 0$ and $c_e(2) = 2$, which are found through general reasoning.
We can obtain an explicit formula for $C(n, m \mid \text{parts even})$ as follows.

$$\begin{aligned}\Big(\frac{2x^2}{(1-x^2)^2}\Big)^m &= 2^m x^{2m}\sum_{v=0}^{\infty}\binom{v+2m-1}{2m-1}x^{2v}\\ &= 2^m\sum_{n=2m}^{\infty}\binom{\frac{n}{2}+m-1}{2m-1}x^n, \quad \text{where } n = 2v+2m \geq 2m.\end{aligned}$$

Hence

$$C(n, m \mid \text{parts even}) = 2^m \cdot \binom{\frac{n}{2}+m-1}{2m-1},$$

where $\binom{N}{k} > 0$ if and only if both N and k are nonnegative integers with $N \geq k$.

Now going further, let $C(n, m \mid \text{parts even} \leq \text{t})$ be the number of n-color compositions with m even parts and with no even part greater than $2t$, for $t \geq 1$. Then we can generate one part using the result of section 2.1:

$$\begin{aligned}2x^2 + 4x^4 + ... + 2tx^{2t} &= 2(x + 2x^2 + 3x^3 + \cdots + tx^t)\Big|_{x=x^2}\\ &= 2\cdot\frac{x - x^{t+1}(1+t-tx)}{(1-x)^2}\Big|_{x=x^2}\\ &= \frac{2x^2 - 2x^{2t+2}(1+t-tx^2)}{(1-x^2)^2}.\end{aligned}$$

So, for $m \geq 1$ parts, we have

$$\begin{aligned}\sum_{n=0}^{\infty} C(n, m \mid \text{parts even} \leq \text{t})x^n &= \Big(\frac{2x^2 - 2x^{2t+2}(1+t-tx^2)}{(1-x^2)^2}\Big)^m,\\ \Longrightarrow \sum_{n=0}^{\infty} C(n \mid \text{parts even} \leq \text{t})x^n &= \sum_{m=1}^{\infty}\Big(\frac{2x^2 - 2x^{2t+2}(1+t-tx^2)}{(1-x^2)^2}\Big)^m,\\ &= \frac{2x^2(1 - x^{2t}(1+t-tx^2))}{1 - 4x^2 + x^4 + 2(1+t)x^{2t+2} - 2tx^{2t+4}}. \qquad (3.5)\end{aligned}$$

It can be shown that the limit, as $t \to \infty$, of equation (3.5) is equal to equation (3.3).

3.2 Part Sizes belonging to an Odd Residue Class

Let us take a look at n-color into odd parts. Consider $C(n, m \mid \text{parts odd})$. Then, according to Yu-hong Guo in [13], we can enumerate one part by

$$x + 3x^3 + 5x^5 + ... = \frac{x(1+x^2)}{(1-x^2)^2}.$$

So, for $m > 0$, we have

$$\sum_{n=1}^{\infty} C(n, m \mid \text{parts odd})x^n = \Big(\frac{x(1+x^2)}{(1-x^2)^2}\Big)^m, \qquad (3.6)$$

$$\Longrightarrow \sum_{n=1}^{\infty} C(n \mid \text{parts odd})x^n = \sum_{m=1}^{\infty}\Big(\frac{x(1+x^2)}{(1-x^2)^2}\Big)^m, \qquad (3.7)$$

$$= \frac{x + x^3}{1 - x - 2x^2 - x^3 + x^4}. \qquad (3.8)$$

We can get the recurrence formula by observing equation (3.8) and letting $C(n, m \mid \text{parts odd}) \equiv c_o(n)$. Then we have

$$\sum_{n=0}^{\infty} c_o(n)x^n = \frac{x + x^3}{1 - x - 2x^2 - x^3 + x^4}.$$

Which is equal to

$$\begin{aligned}\sum_{n=0}^{\infty} c_o(n)x^n &= \sum_{n=1}^{\infty} c_o(n-1)x^n + 2\sum_{n=2}^{\infty} c_o(n-2)x^n \\ &+ \sum_{n=3}^{\infty} c_o(n-3)x^n - \sum_{n=4}^{\infty} c_o(n-4)x^n + x + x^3.\end{aligned}$$

After equating coefficients of x^n we get

$$c_o(n) = c_o(n-1) + 2c_o(n-2) + c_o(n-3) - c_o(n-4),\ n \geq 4,$$

with initial conditions being $c_0(0) = 0$, $c_o(1) = 1$, $c_o(2) = 1$ and $c_o(3) = 4$.

Going a bit further, we can count the number of n-color compositions of n with only odd parts not exceeding $2t - 1$, $t \geq 1$. We observe that we can count a single part with the function $S_t(x)$ as follows.

$$S_t(x) = x + 3x^3 + 5x^5 + \cdots + (2t-1)x^{2t-1} \tag{3.9}$$

$$x^2 S_t(x) = x^3 + 3x^5 + \cdots + (2t-3)x^{2t-1} + (2t-1)x^{2t+1}. \tag{3.10}$$

Then (3.9) - (3.10) gives us

$$(1 - x^2)S_t(x) = x + 2x^3 + 2x^5 + \cdots + 2x^{2t-1} - (2t-1)x^{2t+1} \tag{3.11}$$

$$v^2(1 - x^2)S_t(x) = x^3 + 2x^5 + 2x^7 + \cdots + 2x^{2t+1} - (2t-1)x^{2t+3}. \tag{3.12}$$

Then (3.11) - (3.12) gives us

$$(1 - x^2)^2 S_t(x) = x + x^3 - (2t+1)x^{2t+1} + (2t-1)x^{2t+3}.$$

Therefore,

$$x + 3x^3 + 5x^5 + \cdots + (2t-1)x^{2t-1} = \frac{x + x^3 - (2t+1)x^{2t+1} + (2t-1)x^{2t+3}}{(1 - x^2)^2},\ t \geq 1.$$

Then, for $m > 0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} C(n, m \mid \text{parts odd} \leq \text{t})x^n = \Big(\frac{x + x^3 - (2t+1)x^{2t+1} + (2t-1)x^{2t+3}}{(1 - x^2)^2}\Big)^m, \\ \implies &\sum_{n=0}^{\infty} C(n \mid \text{parts odd} \leq \text{t})x^n = \sum_{m=1}^{\infty} \Big(\frac{x + x^3 - (2t+1)x^{2t+1} + (2t-1)x^{2t+3}}{(1 - x^2)^2}\Big)^m, \\ &= \frac{x(1 + x^2 - x^{2t} - 2tx^{2t} - x^{2+2t} + 2tx^{2+2t}}{1 - x - 2x^2 - x^3 + x^4 + x^{1+2t} + 2tx^{1+2t} + x^{3+2t} - 2tx^{3+2t}}.\end{aligned}$$

And once gain, we can see that the limit of the above equation will retain equation (3.8), as $t \to \infty$.

Now both residue classes, odd and even, can be summarised better by using modulo arithmetic as shown in the theorem below.

Theorem 3.1 (Dedrickson III, [11]). *Let $C_{r,k}(n,m \mid parts \equiv r \pmod{k})$ denote the number of n-color compositions of n into m parts congruent to r $\pmod{k}$. Then the generating function is*

$$\Big(\frac{rx^r + (k-r)x^{k+r}}{(1-x^k)^2}\Big)^m.$$

And furthermore,

$$\sum_{n=0}^{\infty} C(n \mid parts \equiv r \pmod{k})x^n = \frac{rx^r + (k-r)x^{k-r}}{(1-x^k)^2 - (rx^r + (k-r)x^{k-r})}$$

Proof. Let $C_{r,k}(n,m \mid \text{parts} \equiv r \pmod{k}) = c_{r,k}(n,m)$. Then the set of admissible integers has generating function

$$\sum_{n=0}^{\infty} c_{r,k}(n,m)x^n = (rx^r + (k+r)x^{k+r} + (2k+r)x^{2k+r} + ...,)^m \tag{3.13}$$

$$= \Big(x\frac{d}{dx}\Big(\frac{x^r}{1-x^k}\Big)\Big)^m, \tag{3.14}$$

$$= \Big(x \cdot \Big(\frac{rx^{r-1}(1-x^k) + kx^{r+k-1}}{(1-x^k)^2}\Big)\Big)^m, \tag{3.15}$$

$$= \Big(\frac{rx^r + (k-r)x^{k+r}}{(1-x^k)^2}\Big)^m. \tag{3.16}$$

And furthermore,

$$\sum_{n=0}^{\infty} c_{r,k}(n)x^n = \sum_{m=1}^{\infty} \Big(\frac{rx^r + (k-r)x^{k+r}}{(1-x^k)^2}\Big)^m,$$
$$= \frac{rx^r + (k-r)x^{k-r}}{(1-x^k)^2 - (rx^r + (k-r)x^{k-r})}.$$

□

With this theorem above we can make an attempt to make a bridge between prime numbers and n-color compositions.

Proposition 3.2.1. *Let $p \in$ {Prime Numbers}.*
Then p^2 is congruent to 1 modulo 24 i.e.,

$$p^2 \equiv 1 \pmod{24}.$$

Proof. For every prime number p we can simplify each into two cases:

$$\text{Case one} : p = 6k + 1, \forall k \in N/0,$$
$$\text{Case two} : p = 6k - 1, \forall k \in N/0.$$

Case one:
We have two scenarios. k can either be even or odd i.e. $k = 2m$ or $k = 2m + 1$ $\forall m \in N$.
If k is even then we have

$$(12m + 1)^2 = 144m^2 + 24m + 1,$$
$$= 24(6m^2 + 1) + 1.$$

If k is odd then we have

$$\begin{aligned}(12m+7)^2 &= 144m^2 + 168m + 49,\\ &= 24(6m^2 + 7m + 1) + 1.\end{aligned}$$

Case two:
We have two scenarios. k can also be either even or odd. If k is even then we have

$$\begin{aligned}(12m-1)^2 &= 144m^2 - 24m + 1,\\ &= 24(6m^2 + 7m + 1) + 1.\end{aligned}$$

And finally, if k is odd then we have

$$\begin{aligned}(12m+5)^2 &= 144m^2 - 120m + 25,\\ &= 24(6m^2 - 5m + 1) + 1.\end{aligned}$$

In each case we have shown that

$$p^2 = 24m + 1, \forall m \in N$$

□

Corollary 3.1.1. *Let $C(n, m \mid \text{parts congruent to 1 modulo 24})$ denote the number of n-color compositions with parts congruent to* 1 *modulo* 24*, which is equivalent to p^2 where p is a prime number. Then,*

$$\sum_{n=0}^{\infty} C(n, m \mid \textit{parts congruent to 1 modulo 24}) x^n = \Big(\frac{x(1+23x^{24})}{1-x^{24}}\Big)^m.$$

Proof. The proof follows from directly substituting $p^2 \equiv 1 \pmod{24}$ into Theorem 3.1. □

3.3 Color sizes Bound to Even Residue Class

Let us investigate some interesting color size enumerations involving even and odd residue classes.
Let $C(n, m \mid \text{colors even})$ denote the number of n-color compositions of n into m parts with even color sizes, so that

$$C(n \mid \text{colors even}) = \sum_{m \geq 1} C(n, m \mid \text{colors even}).$$

As examples we have the following:
$C(1 \mid \text{colors even}) = 0 : \emptyset$,
$C(2 \mid \text{colors even}) = 1 : (2_2)$,
$C(3 \mid \text{colors even}) = 1 : (3_2)$,
$C(4 \mid \text{colors even}) = 2 : (4_2), (4_4), (2_2, 2_2)$,
$C(4 \mid \text{colors even}) = 2 : (5_2), (5_4), (2_2, 3_2), (3_2, 2_2)$,
……… and so on.

Thus according to Sachdeva in [25], we can generate one arbitrary part by

$$\begin{aligned}x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + ... &= (x^2 + 2x^4 + 3x^6 + 4x^8 + ...) + (x^3 + 2x^5 + 3x^7 + ...),\\ &= \frac{x^2}{(1-x^2)^2} + \frac{x^3}{(1-x^2)^2},\\ &= \frac{x^2}{(1+x)(1-x)^2}.\end{aligned}$$

. So, for $m > 0$ parts, we have

$$\sum_{n=0}^{\infty} C(n, m \mid \text{colors even}) x^n = \Big(\frac{x^2}{(1+x)(1-x)^2}\Big)^m,$$

$$\Longrightarrow \sum_{n=0}^{\infty} C(n \mid \text{colors even}) x^n = \sum_{m=1}^{\infty} \Big(\frac{x^2}{(1+x)(1-x)^2}\Big)^m,$$

$$= \frac{x^2}{1-x-2x^2+x^3}. \tag{3.17}$$

If we equate the coefficients of x^m in equation (3.17), we get

$$C(n, m \mid \text{colors even}) = \sum_{i=0} \binom{m-i-1}{m-1}\binom{n-2i-m-1}{m-1}.$$

The recurrence formula follows simply from equation (3.17) i.e., let $C(n \mid \text{colors even}) \equiv c_{e^c}(n)$. Then, for $c_{e^c}(1) = 0$ and $c_{e^c}(2) = c_{e^c}(3) = 1$, we have

$$c_{e^c}(n) = c_{e^c}(n-1) + 2c_{e^c}(n-2) - c_{e^c}(n-3).$$

3.4 Color sizes Bound to Odd Residue Class

Now let us take a look at odd color sizes. Let $C(n, m \mid \text{colors odd})$ denote the number of n-color compositions of n into m-parts with strictly odd color sizes. Then according to Sachdeva in [25], we can generate one part by

$$x + x^2 + 2x^3 + 2x^4 + 3x^5 + 3x^6 + ... = (x + 2x^3 + 3x^5 + ...) + (x^2 + 2x^4 + 3x^6 + ...),$$

$$= \frac{x}{(1-x^2)^2} + \frac{x^2}{(1-x^2)^2},$$

$$= \frac{x}{(1+x)(1-x)^2}.$$

So, for $m > 0$ parts, we have

$$\sum_{n=0}^{\infty} C(n, m \mid \text{colors odd}) x^n = \Big(\frac{x}{(1+x)(1-x)^2}\Big)^m,$$

$$\Longrightarrow \sum_{n=0}^{\infty} C(n \mid \text{colors odd}) x^n = \sum_{m=1}^{\infty} \Big(\frac{x}{(1+x)(1-x)^2}\Big)^m,$$

$$= \frac{x}{1-2x-x^2+x^3}. \tag{3.18}$$

On equating coefficients of x^m in equation (3.18), as we did in section 3.1, we get

$$C(n, m \mid \text{colors odd} = \sum_{i+j=n-m} (-1)^i \binom{m+i-1}{m-1}\binom{2m+j-1}{2m-1}.$$

Once again, the recurrence formula follows simply from equation (3.18) i.e., let $C(n \mid \text{colors odd}) \equiv c_{o^c}(n)$ to get

$$c_{o^c}(n) = 2c_{o^c}(n-1) + c_{o^c}(n-2) - c_{o^c}(n-3),$$

for n>3, $c_{o^c}(3) = 5$, $c_{o^c}(2) = 2$ and $c_{o^c}(1) = 1$.

Both enumerations, in section 3.3 and 3.4, can be summarised well using generalized modulo arithmetic as we explore below.
Given positive integers a and b, let $C_{ak+b}(n)$ denote the number of n-color compositions of n into parts with colors congruent to $a \pmod b$, for some integer $k \geq 0$. Also, let $C_{ak+b}(m,n)$ be the number of n-color compositions of n into m-parts with each color congruent to $a \pmod b$.

Example 3.4.1. *For $n = 4$, parts with colors congruent to $1 \pmod 3$ i.e., $C_{3k+1}(4)$, are 9. These are*

$$(4_1), (4_4), (3_1, 1_1), (1_1, 3_1), (2_1, 2_1), (2_1, 1_1, 1_1), (1_1, 2_1, 1_1), (1_1, 1_1, 2_1), (1_1, 1_1, 1_1, 1_1).$$

We can enumerate as follows:

Theorem 3.2 (Acosta, Caicedo and Poveda [1]). *Let $GC_{ak+b}(m,x)$ and $GC_{ak+b}(x)$ denote the generating functions for the sequences $C_{ak+b}(m,n)$ and $C_{ak+b}(n)$, respectively. Then we have*

$$GC_{ak+b}(m,x) = \Big(\frac{x^b}{(1-x)(1-x^a)}\Big)^m, \tag{3.19}$$

$$GC_{ak+b}(x) = \frac{x^b}{1 - x - x^a + x^{a+1} - x^b}. \tag{3.20}$$

Proof. Let $\sigma = (\sigma_1, ..., \sigma_m)$ be a non-empty n-color composition having m parts where each color is congruent to $a \pmod b$ i.e. is of the form $ak + b$ for $k \geq 0$. If $\sigma_j = i$ with $i \geq b$, then σ_j contributes to the generating function the term $w_i x^i$, where

$$w_i = \Big\lfloor \frac{i - b + a}{a} \Big\rfloor,$$

while if $i < b$, then if fails to contribute.
Note that the generating function of the sequence

$$\{w_i\}_{i\geq 0} = \Big\{ \underbrace{0, ..., 0}_{b}, \underbrace{1, ..., 1}_{a}, \underbrace{2, ..., 2}_{a}, ... \Big\}$$

is given by

$$\frac{x^b}{(1-x)(1-x^a)}.$$

Therefore,

$$GC_{ak+b}(m,x) = \Big(\sum_{i\geq 0} w_i x^i\Big)^m = \Big(\frac{x^b}{(1-x)(1-x^a)}\Big)^m.$$

Hence,

$$GC_{ak+b}(x) = \frac{x^b}{1 - x - x^a + x^{a+1} x^b}.$$

$\square$

Remark 3.4.1. *We know that even and odd numbers can be tabulated using $0 \pmod 2$ and $1 \pmod 2$, respectively. So, in Theorem 3.3 we can set $a = 2$ and $b = 2$ and recover equation (3.17). Similarly, we also set $a = 2$ and $b = 1$ and recover equation (3.18). Interestingly enough, if we set $a = 1$ and $b = 1$ i.e., $1 \pmod 1$, then we recover equation (1.1) and equation (1.2).*

Chapter 4

Four Basic Shapes of n-Color Compositions

We have seen that we can classify ordinary compositions of n into four basic shapes, namely $1c1, 1c2, 2c1$ and $2c2$, such that the cardinality of each shape set is 2^{n-3} (see Theorem 1.2). In this chapter we study a similar classification of the set of n-color compositions, with respect to part sizes, and with respect to color sizes. By our running notation, $C(n \mid 1c1)$ is the number of n-color compositions of n in which the sequence of part sizes take on the shape $1c1$, e.g., $(1_1, 3_2, 1_1)$. Similarly $C(n \mid 1c1\text{-color})$ denotes the number of n-color compositions of n in which the sequence of color-sizes take on the shape $1c1$, e.g., $(2_1, 3_2, 3_1)$. (Note that $(2_1, 3_2, 3_1)$ is a $2c2$-composition by the sequence of part sizes).

The contents of this chapter are appearing here for the first time; so they constitute a new contribution to the literature.

4.1 Classification by Part-Size

Earlier on, we established that the number of n-color compositions is equal to the even-numbered Fibonacci numbers, i.e., $C(n) = F_{2n}$. If we break down the set of n-color compositions of n into the classes $1c1, 1c2, 2c1$ and $2c2$, we find that the numbers of n-color compositions of n in the classes, by part sizes, are given by $F_{2(n-2)}, F_{2(n-2)+1}, F_{2(n-2)+1}$ and $F_{2(n-1)}$, respectively. This is summarized in the following theorem.

Theorem 4.1. *We have the following:*

$$C(n \mid 1c1) = F_{2(n-2)}, \tag{4.1}$$

$$C(n \mid 1c2) = F_{2(n-2)+1}, \tag{4.2}$$

$$C(n \mid 2c1) = F_{2(n-2)+1}, \tag{4.3}$$

$$C(n \mid 2c2) = F_{2(n-1)}. \tag{4.4}$$

Example 4.1.1. *For n=4, we have a total of 21 n-color compositions, which is* $F_8 = 21$: *1,1,2,3,5,8,13,21,34,55,...*

The classification of these compositions, according to the theorem, is given as

$$F_4 = 3: \ 1c1 = (1_1,1_1,1_1,1_1), (1_1,2_1,1_1), (1_1,2_2,1_1)$$
$$F_5 = 5: \ 1c2 = (1_1,3_1), (1_1,3_2), (1_1,3_3), (1_1,1_1,2_1), (1_1,1_1,2_2)$$
$$F_5 = 5: \ 2c1 = (3_1,1_1), (3_2,1_1), (3_3,1_1), (2_1,1_1,1_1), (2_2,1_1,1_1)$$
$$F_6 = 8: \ 2c2 = (2_1,2_1), (2_2,2_2), (2_1,2_2), (2_2,2_1), (4_1), (4_2), (4_3), (4_4).$$

Proof of Theorem 4.1. We begin with some notation we will use throughout the proof:

$$C_{ij}(k,x) := \sum_{n=1}^{\infty} C(n,k \mid icj)x^n, i,j \geq 1; \tag{4.5}$$

$$C_{ij}(x) := \sum_{n=1}^{\infty} C(n \mid icj)x^n = \sum_{k} C_{ij}(k,x). \tag{4.6}$$

Now for (4.1), we observe that $C(1,1 \mid 1c1)$ and $C(2,2 \mid 1c1)$ are both equal to 1, where the corresponding compositions are (1_1) and $(1_1,1_1)$. This implies for $k=1,2$ we have the generating function $x+x^2$.
For $k \geq 3$, any composition must begin and end with (1_1), i.e.,

$$(1_1) \times (\text{any } n\text{-color composition}) \times (1_1)$$

.
This implies

$$C_{11}(k,x) = (x)(x+2x^2+3x^3+...)^{k-2}(x),\ \ k \geq 3.$$

The middle part is $\big(\frac{x}{(1-x)^2}\big)^{k-2}$, from the proof of Theorem 1.6. Thus

$$C_{11}(k,x) = x^2\Big(\frac{x}{(1-x)^2}\Big)^{k-2}, k \geq 3.$$

Therefore,

$$\begin{aligned} C_{11}(x) = \sum_{n=3}^{\infty} C(n \mid 1c1)x^n &= x^2 \sum_{k=3}^{\infty}\Big(\frac{x}{(1-x)^2}\Big)^{k-2}, \\ &= x^2 \cdot \frac{x}{(1-x)^2-x}, \\ &= \frac{x^3}{1-3x+x^2}. \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{n=3}^{\infty} C(n \mid 1c1)x^n &= \frac{x^3}{1-3x+x^2}, \\ &= x^2\sum_{n=1}^{\infty} F_{2n}x^n, \\ &= \sum_{n=3}^{\infty} F_{2(n-2)}x^n. \end{aligned}$$

Hence after extracting the coefficient of x^n we have $C_{11}(n \mid 1c1) = F_{2(n-2)}, n \geq 3$.
To complete the proof, we need to account for all n. Thus we have,

$$\begin{aligned} C_{11}(x) &= C_{11}(1,x) + C_{11}(2,x) + C_{11}(k,x), \\ &= x + x^2 + \sum_{k=3}^{\infty}\Big(\frac{x}{(1-x)^2}\Big)^{k-2}. \end{aligned}$$

Thus

$$[x^n]C_{11}(x) = F_{2(n-2)}, \quad n \geq 3.$$

Now lets look at equation (4.2). Note that

$$C(n,1 \mid 1c2) = C(1,1 \mid 1c2) = C(2,2 \mid 1c2) = 0.$$

But $C(3,2 \mid 1c2) = 2$ because of the compositions $(1_1, 2_2)$ and $(1_1, 2_1)$. So for $k \geq 2$ any compositions in this class must begin with (1_1) but must end with the any part > 1, i.e.,

$$(1_1) \times (\text{any } n\text{-color composition}) \times (\text{a part} > 1).$$

This leads to

$$C_{12}(k,x) = (x)(x + 2x^2 + 3x^3 + ...)^{k-2}(2x^2 + 3x^3 + 4x^4 + ...), \quad k \geq 2.$$

We already know that $x + 2x^2 + 3x^3 + ... = \frac{x}{(1-x)^2}$ and $2x^2 + 3x^3 + ... = \frac{x^2(2-x)}{(1-x)^2}$. So we get

$$C_{12}(k,x) = \frac{x^3(2-x)}{(1-x)^2} \cdot \Big(\frac{x}{(1-x)^2}\Big)^{k-2}, k \geq 2.$$

Therefore,

$$\begin{aligned} C_{12}(x) = \sum_{n=3}^{\infty} C(n \mid 1c2)x^n &= \frac{x^3(2-x)}{(1-x)^2} \sum_{k=2}^{\infty} \Big(\frac{x}{(1-x)^2}\Big)^{k-2}, \\ &= \frac{x^3(2-x)}{(1-x)^2} \cdot \frac{(1-x)^2}{1-3x+x^2}, \\ &= \frac{x^3(2-x)}{1-3x+x^2}. \end{aligned}$$

Observe that,

$$\begin{aligned} \sum_{n=3}^{\infty} C(n \mid 1c2)x^n &= \frac{x^3(2-x)}{1-3x+x^2} \\ &= \Big(\frac{x}{1-3x+x^2}\Big)(2-x)x^2 \\ &= (\sum_{n=1}^{\infty} F_{2n}x^n)(2x^2 - x^3) \\ &= 2x^2 \sum_{n=1}^{\infty} F_{2n}x^n - x^3 \sum_{n=1}^{\infty} F_{2n}x^n \\ &= 2\sum_{n=3}^{\infty} F_{2(n-2)}x^n - \sum_{n=4}^{\infty} F_{2(n-3)}x^n \\ &= \sum_{n=3}^{\infty}(2F_{2(n-2)} - F_{2(n-3)})x^n, (F_{2(3-3)} = F_0 = 0) \\ &= \sum_{n=3}^{\infty} F_{2(n-2)+1}x^n, \end{aligned}$$

since $2F_{2(n-2)} - F_{2(n-3)} = F_{2(n-2)} + F_{2(n-2)-1} = F_{2(n-2)+1}$.
Hence,

$$C(n \mid 1c2) = F_{2(n-2)+1}, \quad n \geq 3.$$

We also obtain the proof of equation (4.3) at once, that is,

$$C(n \mid 2c1) = F_{2(n-2)+1}, \quad n \geq 3.$$

This is so because σ is counted by $C(n \mid 1c2)$ if and only if $\bar{\sigma}$ is counted by $C(n \mid 2c1)$, by simple inversion of compositions.

Lastly lets prove equation (4.4).
For n=1, k=1 we have $C(1,1 \mid 2c2) = 0$.
For $n \geq 2$ and k=1 we have

$$\begin{aligned} C_{22}(1,x) &= \sum_{n\geq 2} C(n,1 \mid 2c2)x^n \\ &= 2x^2 + 3x^3 + ... \\ &= \frac{x^2(2-x)}{(1-x)^2}. \end{aligned}$$

So for $k \geq 2$, we need to we fix the first and last parts to be any part-size greater than 1 i.e
(a part > 1) $\times$ (any n-color composition) $\times$ (a part > 1).
This leads to

$$(2x^2 + 3x^3 + ...)(x + 2x^2 + 3x^3 + ...)^{k-2}(2x^2 + 3x^3 + ...).$$

This is implies that

$$C_{22}(k,x) = \frac{x^2(2-x)}{(1-x)^2} \cdot \Big(\frac{x}{(1-x)^2}\Big)^{k-2} \cdot \frac{x^2(2-x)}{(1-x)^2}, k \geq 2.$$

Hence we have,

$$\begin{aligned} C_{22}(x) &= \frac{x^4(2-x)^2}{(1-x)^4} \cdot \sum_{k=2}^{\infty} \Big(\frac{x}{(1-x)^2}\Big)^{k-2}, \\ &= \frac{x^4(2-x)^2}{(1-x)^4} \cdot \frac{(1-x)^4}{(1-x)^2(1-3x+x^2)}, \\ &= \frac{x^4(2-x)^2}{(1-x)^2(1-3x+x^2)}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \sum_{n=2}^{\infty} C(n \mid 2c2)x^n &= \frac{x^4(2-x)^2}{(1-x)^2(1-3x+x^2)} + \frac{x^2(2-x)}{(1-x)^2}, \\ &= \frac{(2-x)(1-x)x^2}{1-3x+x^2}, \\ &= \sum_{n\geq 1} F_{2n}x^{n+1}(2-3x+x^2), \\ &= 2\sum_{n\geq 2} F_{2(n-1)}x^n - 3\sum_{n\geq 3} F_{2(n-2)}x^n + \sum_{n\geq 4} F_{2(n-3)}x^n, \\ &= 2\sum_{n\geq 2} F_{2(n-1)}x^n - \Big(\sum_{n\geq 3}(3F_{2(n-2)} - F_{2(n-3)})x^n\Big), \\ &= 2\sum_{n\geq 2} F_{2(n-1)}x^n - \sum_{n\geq 3} F_{2(n-1)}x^n, \\ &= 2F_2x^2 + \sum_{n\geq 3} F_{2(n-1)}x^n, \\ &= 2x^2 + \sum_{n\geq 3} F_{2(n-1)}x^n. \end{aligned}$$

After extraction of the coefficient of x^n, we see that the formula holds for $n \geq 3$ but not for $n = 2$ since $C(2 \mid 2c2) = 2$, i.e., $C(2 \mid 2c2)$ counts $(2_1), (2_2)$. So we have

$$C(n \mid 2c2) = F_{2(n-1)}, \; n \geq 3.$$

This finishes the proof after we extract the coefficient. □

4.2 Classification by Color-Size

Theorem 4.2. *We have the following:*

$$C(n \mid 1c1\text{-}color) = F_{2(n-1)} + 1, \tag{4.7}$$
$$C(n \mid 1c2\text{-}color) = F_{2(n-2)+1} - 1, \tag{4.8}$$
$$C(n \mid 2c1\text{-}color) = F_{2(n-2)+1} - 1, \tag{4.9}$$
$$C(n \mid 2c2\text{-}color) = F_{2(n-2)} + 1. \tag{4.10}$$

Example 4.2.1. *For n=4, we have,*

$$\begin{aligned}
1c1 &= (1_1, 1_1, 1_1, 1_1), (1_1, 2_1, 1_1), (1_1, 2_2, 1_1), (1_1, 3_1), (1_1, 1_1, 2_1), (3_1, 1_1),\\
&\quad (2_1, 1_1, 1_1), (2_1, 2_1), (4_1),\\
1c2 &= (1_1, 3_2), (1_1, 3_3), (2_1, 2_2), (1_1, 1_1, 2_2),\\
2c1 &= (3_2, 1_1), (3_3, 1_1), (2_2, 1_1, 1_1), (2_2, 2_1),\\
2c2 &= (2_2, 2_2), (4_2), (4_3), (4_4).
\end{aligned}$$

Hence the theorem is verified by the following direct counts:

$$\begin{aligned}
C(4 \mid 1c1\text{-}color) &= F_6 + 1 = 9,\\
C(4 \mid 1c2\text{-}color) &= F_5 - 1 = 4,\\
C(4 \mid 2c1\text{-}color) &= F_5 - 1 = 4,\\
C(4 \mid 2c2\text{-}color) &= F_4 + 1 = 4.
\end{aligned}$$

Proof of Theorem 4.2. For $i, j \geq 1$, denote

$$\begin{aligned}
R_{ij}(k, x) &= \sum_{n=1}^{\infty} C(n, k \mid icj\text{-color}) x^n;\\
R_{ij}(x) &= \sum_{n=1}^{\infty} C(n \mid icj\text{-color}) x^n = \sum_k R_{ij}(k, x).
\end{aligned}$$

Now to prove equation (4.7), we observe that

$$\begin{aligned}
R_{11}(1, x) &= \sum_{n=1}^{\infty} c(n, 1 \mid 1c1\text{-}color) x^n,\\
&= x + x^2 + x^3 + ...,\\
&= \frac{x}{1-x}.
\end{aligned}$$

But for $k \geq 2$ the first part is any part with specifically 1 as the color and last part is a part with specifically 1 as the color i.e

$$R_{11}(k, x) = (x + x^2 + x^3 + ...)(x + 2x^2 + 3x^3 + ...)^{k-2}(x + x^2 + x^3 + ...), k \geq 2.$$

Evaluating further we get,

$$R_{11}(k,x) = \frac{x^2}{(1-x)^2} \cdot \Big(\frac{x}{(1-x)^2}\Big)^{k-2}, k \geq 2.$$

So $\forall k$ we have,

$$\begin{aligned} R_{11}(x) &= \frac{x^2}{1-3x+x^2} + \frac{x}{1-x}, \\ &= \frac{x(1-2x)}{(1-x)(1-3x+x^2)}. \end{aligned}$$

Hence, we have (using Remark 1.2.1)

$$\begin{aligned} C(n \mid 1c1\text{-color}) &= \Big(\sum_{n=1}^{\infty} F_{2n}x^n\Big) \cdot \Big(\frac{1-2x}{1-x}\Big), \\ &= \Big(\sum_{n=0}^{\infty} F_{2n}x^n\Big) \cdot \Big(\sum_{n=0}^{\infty} x^n\Big) - \Big(\sum_{n=1}^{\infty} F_{2n}x^n\Big) \cdot \Big(\sum_{n=1}^{\infty} 2x^n\Big), (F_{2(0)} = 0), \\ &= \sum_{n=1}^{\infty} \Big(\sum_{i=1}^{n} F_{2i}\Big)x^n - 2\sum_{n=1}^{\infty} \Big(\sum_{i=1}^{n} F_{2i}\Big)x^{n+1}, \\ &= \sum_{n=1}^{\infty}(F_{2n+1}-1)x^n - 2\sum_{n=1}^{\infty}(F_{2n+1}-1)x^{n+1}, \\ &= \sum_{n=1}^{\infty}(F_{2n+1}-1)x^n - 2\sum_{n=2}^{\infty}(F_{2n-1}-1)x^n, \\ &= \sum_{n=1}^{\infty}(F_{2n+1}-1)x^n - 2\sum_{n=1}^{\infty}(F_{2n-1}-1)x^n, (F_1 - 1 = 0), \\ &= \sum_{n=1}^{\infty}(F_{2n+1} - 1 - 2F_{2n-1} + 2)x^n, \\ &= \sum_{n=1}^{\infty}(F_{2(n-1)} + 1)x^n. \end{aligned}$$

After we extract the coefficient of x^n we get our desired result.

For equation (4.8) and or (4.9), we observe that

$$R_{21}(1,x) = \sum_{n=2}^{\infty} C(n, 1 \mid 1c2\text{-color}) = 0.$$

But for $k \geq 2$, our decomposition is

(a part > 1 with color > 1)$\times$(any n-color composition)$\times$(a part with color 1).

This gives, for $k \geq 2$,

$$R_{21}(k,x) = (x^2 + 2x^3 + 3x^4 + ...)(x + 2x^2 + 3x^3 + ...)^{k-2}(x + x^2 + x^3 + ...).$$

Hence,

$$R_{21}(k,x) = \frac{x^2}{(1-x)^2}\Big(\frac{x}{(1-x)^2}\Big)^{k-2}\frac{x}{1-x}, k \geq 2.$$

Therefore,

$$
\begin{aligned}
R_{21}(x) = \sum_{n=3}^{\infty} C(n \mid 2c1\text{-color})x^n &= \frac{x^3}{(1-x)^3} \sum_{k=2}^{\infty} \Big(\frac{x}{(1-x)^2}\Big)^{k-2}, \\
&\frac{x^3}{(1-x)^3} \cdot \frac{(1-x)^4}{(1-x)^2(1-3x+x^2)}, \\
&= \Big(\frac{x}{1-3x+x^2}\Big) \cdot \Big(\frac{x^2}{1-x}\Big).
\end{aligned}
$$

To complete the proof, thus

$$
\begin{aligned}
\sum_{n=2}^{\infty} C(n \mid 1c2\text{-color})x^n &= \Big(\sum_{n=1}^{\infty} F_{2n}x^{n+1}\Big) \cdot \Big(\sum_{n=1}^{\infty} x^n\Big), \\
&= \Big(\sum_{n=2}^{\infty} F_{2(n-1)}x^n\Big) \cdot \Big(\sum_{n=1}^{\infty} x^n\Big), \\
&= \Big(\sum_{n=1}^{\infty} F_{2(n-1)}x^n\Big) \cdot \Big(\sum_{n=1}^{\infty} x^n\Big), (F_{2(1-1)} = 0), \\
&= \sum_{n=2}^{\infty} \Big(\sum_{i=1}^{n-1} F_{2i}\Big) x^{n+1}, \\
&= \sum_{n=2}^{\infty} (F_{2(n-1)+1} - 1)x^{n+1}, \\
&= \sum_{n=3}^{\infty} (F_{2(n-2)+1} - 1)x^{n}, \\
&= \sum_{n=2}^{\infty} (F_{2(n-2)+1} - 1)x^{n}, (F_{2(2-2)+1} - 1 = 0).
\end{aligned}
$$

□

Hence we get

$$C(n \mid 1c2\text{-color}) = C(n \mid 2c1\text{-color}) = F_{2(n-2)+1} - 1, n \geq 2.$$

Finally, we need to prove equation (4.10). We consider that for $k = 1$, we have

$$
\begin{aligned}
R_{22}(1, x) &= \sum_{n=2}^{\infty} C(n, 1 \mid 2c2\text{-color})x^n, \\
&= x^2 + 2x^3 + 3x^4 + ..., \\
&= \frac{x^2}{(1-x)^2}.
\end{aligned}
$$

For $k \geq 2$, our decomposition will be

(a part >1 with a color >1)$\times$(any n-color composition)$\times$(a part >1 with a color >1).

Thus

$$R_{22}(k, x) = (x^2 + 2x^3 + 3x^4 + ...)(x + 2x^2 + 3x^3 + ...)^{k-2}(x^2 + 2x^3 + 3x^4 + ...).$$

Therefore(for $k \geq 2$ which implies $n \geq 4$),

$$\begin{aligned}
R_{22}(x) &= \frac{x^4}{(1-x)^4}\sum_{k=2}^{\infty}\Big(\frac{x}{(1-x)^2}\Big)^{k-2},\\
&= \frac{x^4}{(1-x)^4}\cdot\frac{(1-x)^4}{(1-x)^2(1-3x+x^2)},\\
&= \frac{x^4}{(1-x)^2(1-3x+x^2)}.
\end{aligned}$$

So $\forall n \geq 1$,

$$\begin{aligned}
R_{22}(x) &= \frac{x^2}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-3x+x^2)},\\
&= \frac{x^2(1-2x)}{(1-x)(1-3x+x^2)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
C(n \mid 2c2\text{-color}) &= \Big(\sum_{n=1}^{\infty} F_{2n}x^{n+1}\Big)\cdot\Big(\sum_{n=0}^{\infty} x^n\Big) - 2\Big(\sum_{n=1}^{\infty} F_{2n}x^{n+1}\Big)\cdot\Big(\sum_{n=1}^{\infty} x^n\Big),\\
&= \Big(\sum_{n=0}^{\infty} F_{2n}x^{n+1}\Big)\cdot\Big(\sum_{n=0}^{\infty} x^n\Big) - 2\Big(\sum_{n=1}^{\infty} F_{2n}x^{n+1}\Big)\cdot\Big(\sum_{n=1}^{\infty} x^n\Big), (F_{2(0)} = 0),\\
&= \sum_{n=1}^{\infty}\Big(\sum_{i=1}^{n} F_{2i}\Big)x^{n+1} - 2\sum_{n=2}^{\infty}\Big(\sum_{i=1}^{n-1} F_{2i}\Big)x^{n+1},\\
&= \sum_{n=1}^{\infty}(F_{2n+1}-1)x^{n+1} - 2\sum_{n=2}^{\infty}(F_{2(n-1)+1}-1)x^{n+1},\\
&= \sum_{n=2}^{\infty}(F_{2(n-1)+1}-1)x^{n} - 2\sum_{n=3}^{\infty}(F_{2(n-2)+1}-1)x^{n},\\
&= \sum_{n=2}^{\infty}(F_{2(n-1)+1}-1)x^{n} - 2\sum_{n=2}^{\infty}(F_{2(n-2)+1}-1)x^{n}, (F_{2(2-2)+1}-1=0),\\
&= \sum_{n=2}^{\infty}(F_{2n-1}-2F_{2n-3}+1)x^{n},\\
&= \sum_{n=2}^{\infty}(F_{2(n-2)}+1)x^{n}.
\end{aligned}$$

Hence,

$$C(n \mid 2c2\text{-color}) = F_{2(n-2)} + 1, n \geq 2.$$

Thus concluding the proof of Theorem (4.2).

Lastly, from Theorem (4.1) and Theorem (4.2) we can deduce the following identities.

Theorem 4.3. *Let $n > 2$ be an integer. Then*

$$\begin{aligned}
C(n \mid 1c1) + 1 &= C(n \mid 2c2\textit{-color}) = F_{2(n-2)} + 1;\\
C(n \mid 1c2) - 1 &= C(n \mid 2c1\textit{-color}) = F_{2(n-2)+1} - 1;\\
C(n \mid 2c1) - 1 &= C(n \mid 1c2\textit{-color}) = F_{2(n-2)+1} - 1;\\
C(n \mid 2c2) + 1 &= C(n \mid 1c1\textit{-color}) = F_{2(n-1)} + 1.
\end{aligned}$$

Conclusion

This dissertation has succeeded in showing some enumerations of n-color compositions. Though touched on briefly, lattice paths have interested me greatly and evidence of that was seen at the end of Chapter 1, where we managed to connect lattice paths and the general recurrence relation of n-color compositions. Further investigations could stem from this, possibly covering combinatorial proofs of the basic shapes covered in Chapter 4. We then demonstrated our understanding of n-color compositions by showing enumerations in Chapter 2, and using generating functions we managed to come with new restricted functions that could possibly be expanded for further investigations. For instance, bounded n-color compositions can be used in odd or even enumerations or for palindromic n-color compositions for both part and color sizes. Furthermore, an interesting view point could be to investigate prime numbers and how that restriction to both part and color sizes could shape n-color compositions further. We managed to, without a doubt, describe the classification of n-color compositions; as described by Munagi in [24] for general compositions. Using Cauchy product and generating function techniques, we managed to distinctly enumerate and prove those different classes. Finally, Theorem 4.3 is interesting because it connects part sizes to color sizes of n-color compositions using Fibonacci sequences. A possible bijective proof could rise from this, which could tackle the combinatorical aspects of this discovery and making the enumerations clearer.

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