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THE ACTION OF THE PICARD GROUP ON HYPERBOLIC 3-SPACE AND COMPLEX CONTINUED FRACTIONS

by

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ABSTRACT

Continued fractions have been extensively studied in number theoretic ways. We consider continued fractions with partial quotients that are in $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}, i^2 = -1\}$. These continued fractions are expressed as compositions of Möbius maps in the Picard group $PSL(2, \mathbb{C})$ that act, by Poincaré's extension, as isometries on $\mathbb{H}^3$. We investigate the Picard group with its generators and derive the fundamental domain using a direct method. From the fundamental domain, we produce an ideal octahedron, $\mathbb{O}_0$, that generates the Farey tessellation of $\mathbb{H}^3$. We explore the properties of Farey neighbours, Farey geodesics and Farey triangles that arise from the Farey tessellation and relate these to Ford spheres. We consider the Farey addition of two rationals in $\mathbb{R}$ as a subdivision of an interval and hence are able to generalise this notion to a subdivision of a Farey triangle with Gaussian Farey neighbour vertices. This Farey set allows us to revisit the Farey triangle subdivision given by Schmidt [44] and interpret it as a theorem about adjacent octahedra in the Farey tessellation of $\mathbb{H}^3$. We consider continued fraction algorithms with Gaussian integer coefficients. We introduce an analogue of Series [45] cutting sequence across $\mathbb{H}^2$ in $\mathbb{H}^3$. We derive a continued fraction expansion based on this cutting sequence generated by a geodesic Λ in $\mathbb{H}^3$ that ends at the point ζ in $\mathbb{C}$ that passes through $\mathbb{O}_0$.

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I would like to acknowledge my family and friends, my supervisor and my colleagues for all the support given during this endeavour.

DECLARATION

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

_____

Grant Paul Hayward

Signed on this the _____ day of _____, at Johannesburg, South Africa.

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Chapter 1

Introduction

1.1 Historical Background

In this thesis we consider general continued fractions of the form

$$b_0 + \mathrm{K}(a_n|b_n) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{1.1.1}$$

where the $\{a_i\}$ and $\{b_i\}$ are sequences of integers, real numbers or complex numbers [4], [5], [7], [11], [19], [31], [40], [45].

In this thesis we will consider continued fractions $b_0 + \mathbf{K}(a_i|b_i)$ where the sequences $\{b_i\}_{i\geq 0}$ and $\{a_i\}_{i\geq 1}$ are of complex numbers.

Continued fractions have been used in mathematics since the 16th century, mainly as a tool for evaluating or approximating real numbers. The first two people who used finite continued fractions explicitly in this way were R. Bombelli and P. Cataldi in 1572. They found approximations to $\sqrt{13}$ and $\sqrt{18}$.

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \cdots}}} \qquad \sqrt{18} = 4 + \cfrac{2}{8 + \cfrac{2}{8 + \cfrac{2}{8 + \cdots}}}$$

Lord Brouckner gave the first infinite continued fraction expansion for $\frac{4}{\pi}$ in 1659 [10].

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

The first recorded study of a general theory of continued fractions appeared in John Wallis' Opera Mathematica in 1695, and introduced the term 'continued fraction'. Many well known mathematicians have added their knowledge to the subject. In particular, these were Euler; in his exposition from 1737, 'De Fractionlous Continious', Gauss (1813), Jacobi (1843), Hermite (1843), Stieltjes (1884) and Ramanujan (1913).

Traditionally, number theorists studied continued fractions with $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 0}$ being sequences of integers. We may define continued fractions by recurrence relationships, with initial conditions $A_{-1} = 1, A_0 = b_0, B_{-1} = 0, B_0 = 1$ with $A_n = b_n A_{n-1} + a_n A_{n-2}$ and $B_n = b_n B_{n-1} + a_n B_{n-2}$ for $n \geq 1$. Thus $\frac{A_n}{B_n} = \frac{b_n A_{n-1} + a_n A_{n-2}}{b_n B_{n-1} + a_n B_{n-2}}$ where $\{a_i\}$ and $\{b_i\}$ are sequences of integers [40].

We examine the recurrence relations, where $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 0}$ are sequences of complex numbers. Assume again that $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = b_0$ and $B_0 = 1$.

Let $b_0 = \frac{A_0}{B_0}$, $b_0 + \frac{a_1}{b_1} = \frac{A_1}{B_1}$, $b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} = \frac{A_2}{B_2}$,...

We see that if $a_n = 1$ for all n then $\begin{pmatrix} A_{-1} & A_0 \\ B_{-1} & B_0 \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}$ and

$$A_{-1}B_0 - B_{-1}A_0 = 1 - 0 = 1 = det\begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix} = \begin{pmatrix} b_0 & b_0b_1 + 1 \\ 1 & b_1 \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}$$

and

$$A_0B_1 - B_0A_1 = 0 - 1 = -1 = det\begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}.$$

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = \begin{pmatrix} b_0b_1 + 1 & b_0b_1b_2 + b_0 + b_2 \\ b_1 & b_1b_2 + 1 \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix}$$

and

$$A_1B_2 - B_1A_2 = 0 - 1 = 1.$$

In general,

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$$

and $|A_{n-1}B_n - B_{n-1}A_n| = 1$.

We recall from [29], a Möbius map $g : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ is defined as follows:

$$g(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, with $g(\infty) = \frac{a}{c}$ and $g\left(-\frac{d}{c}\right) = \infty$ where $c \neq 0$. This group of Möbius maps is represented by $\mathcal{M}$ or $PGL(2, \mathbb{C})$ and is explored further in Section

1.2. By convention, there is a homomorphism from $GL(2,\mathbb{C})$ to $\mathcal{M}$, hence we may associate each Möbius map g with a matrix $M_g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or a scalar multiple of M_g, in $GL(2,\mathbb{C})$ [29].

In particular

$$t(z) = \frac{a}{z+b} \text{ with associated matrix } \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$$

and

$$s(z) = b + \frac{a}{z} \text{ with associated matrix } \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}$$

are Möbius maps, where a and b are non-zero complex numbers. These maps form the basis of a geometric representation of continued fractions.

The Möbius map $t_n : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is given as $t_n(z) = \dfrac{a_n}{b_n + z}$ where $t_0(z) = z + b_0$, $a_n, b_n \in \mathbb{C}$, $n \geq 1$ and corresponds to the matrix $\begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$. Certainly $\mathcal{M}$ is a group [29]. If $T_n(z) = t_0 t_1 \dots t_n(z)$ is the composition of these maps, then the matrix corresponding to T_n is $\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}$ with $T_n(z) = \dfrac{A_{n-1}z + A_n}{B_{n-1}z + B_n}$ where $|A_{n-1}B_n - B_{n-1}A_n| = 1$.

Definition 1.1.1. *The continued fraction* (1.1.1) *converges classically to a value α if the sequence*

$$b_0, \quad b_0 + \frac{a_1}{b_1}, \quad b_0 + \frac{a_1}{b_1 + \dfrac{a_2}{b_2}}, \quad b_0 + \frac{a_1}{b_1 + \dfrac{a_2}{b_2 + \dfrac{a_3}{b_3}}}, \quad \dots$$

of partial quotients converge to α, where α could be ∞ or any complex number. The terms of this sequence are called the convergents, or approximates, of the infinite continued fraction.

Thus the convergents of the continued fractions can be written in terms of partial quotients:

$$b_0 = \frac{A_0}{B_0}, \quad b_0 + \frac{a_1}{b_1} = \frac{A_1}{B_1}, \quad b_0 + \frac{a_1}{b_1 + \dfrac{a_2}{b_2}} = \frac{A_2}{B_2}, \quad b_0 + \frac{a_1}{b_1 + \dfrac{a_2}{b_2 + \dfrac{a_3}{b_3}}} = \frac{A_3}{B_3}, \ldots$$

The convergent, $\dfrac{A_n}{B_n}$, can be expressed in terms of T_n and T_{n+1} as $T_n(0)$ or $T_{n+1}(\infty)$. Since $\dfrac{A_{n-1}}{B_{n-1}}$ and $\dfrac{A_n}{B_n}$ are successive convergents of the continued fraction (1.1.1), we have

$$|A_{n-1}B_n - B_{n-1}A_n| = 1.$$

That is:

$$\begin{aligned} b_0 &= t_0(0) \\ b_0 + \frac{a_1}{b_1} &= t_0 t_1(0) \\ b_0 + \frac{a_1}{b_1 + \dfrac{a_2}{b_2}} &= t_0 t_1 t_2(0) \\ &\vdots \end{aligned}$$

This leads to a result that yields an alternative definition of convergence of continued fractions.

Lemma 1.1.2. *The continued fraction $b_0 + \mathbf{K}(a_n|b_n)$ converges classically to α if and only if*

$$\lim_{n\to\infty} T_{n+1}(\infty) = \lim_{n\to\infty} T_n(0) = \alpha$$

where $T_n(0) = T_n(t_{n+1}(\infty)) = T_{n+1}(\infty)$.

Definition 1.1.3. *A regular Gaussian integer continued fraction expression is of the form*

$$\zeta = \beta_0 + \cfrac{\epsilon_1}{\beta_1 + \cfrac{\epsilon_2}{\beta_2 + \cfrac{\epsilon_3}{\beta_3 + \cdots}}}$$

where $\epsilon_n = \pm 1, \pm i$ *and* $\beta_n \in \mathbb{Z}[i]$.

In this thesis, we will consider continued fractions with $a_i = 1$ for all i. The b_i will be elements of the set of Gaussian integers, $\mathbb{Z}[i] = \left\{a + ib : a, b \in \mathbb{Z}, i^2 = -1\right\}$. We will establish that each $z \in \mathbb{C}$ may be expressed as a Gaussian integer continued fraction expansion in many ways. The following discussions are informed by Ford, [15],[16], Schmidt [44], and Vulakh [49]. In the 1920's, L. R. Ford published several papers in which he considered continued fractions in the context of the geometry of the action of certain Möbius maps on two and three-dimensional hyperbolic space [15],[16]. Ford made the geometric connection between integer continued fractions and Ford circles, which are horocycles in $\mathbb{H}^2$ touching the real axis. Each Ford circle represents a fraction of integers. He also noted that Gaussian rationals may be represented by Ford spheres that touch the complex plane and lie in the upper half space. Ford further established that any vertical line $z = \alpha, \alpha \in \mathbb{R}$ or $\mathbb{C}$ will cut through a sequence of Ford circles or spheres, each of which represents a 'convergent' to α with respect to some continued fraction algorithm [15].

In 1975, A. L. Schmidt [44] noted that many attempts had been made over the past 100 years to develop an algorithm for the continued fraction of complex numbers, with properties that the simple continued fractions algorithm [31], [40], for integers, is known to possess. Schmidt developed two new kinds of algorithms for continued fractions with Gaussian integer coefficients. He based his development on what he defines as Farey sets and dual Farey sets taken together with the actions of the Picard group acting on the complex plane. He noted that the Farey sets are natural extensions of the Ford circles and mesh triangles introduced by Ford [15],[16].

In 1999, L. Y. Vulakh [49] introduced the notion of a 'v-cell'. He used the notion of a 'v-cell' to describe a tessellation of $\mathbb{H}^3$ by these v-cells under the Picard group. The analogous

tessellation of the hyperbolic plane $\mathbb{H}^2$ using the modular group was introduced by Caroline Series in 1985 [45]. Vulakh noted that Farey polytopes in $\mathbb{R}^2$ are the projection of the non-vertical faces of a v-cell in $\mathbb{H}^3$ from ∞ into $\mathbb{R}^2$. He established the basic properties of Farey polytopes and related these properties to definitions of Farey triangles used by Schmidt [44]. He developed an algorithm for the extraction of convergents of the continued fraction that is analogous, but different to the classical continued fraction algorithm and the algorithms presented by Schmidt in 1975.

1.2 Möbius Group

Möbius transformations acting on $\mathbb{R}^n$ are defined in terms of reflections in planes in $\mathbb{R}^n$ or inversions in spheres in $\mathbb{R}^n$. From [6], [9], we clarify these actions.

Let $S(a,r)$ be a sphere in $\mathbb{R}^n$ given by $S(a,r) = \{x \in \mathbb{R}^n : |x-a| = r\}$ where $a \in \mathbb{R}^n$ and $r > 0$. The reflection (or inversion) in $S(a,r)$ is the function ϕ defined by

$$\phi(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a), \tag{1.2.2}$$

where $x \neq a$ and with $\phi(a) = \infty$, $\phi^{-1} = \phi$ and $\phi(\infty) = a$. This map is a bijection of $\mathbb{R}^n_\infty = \mathbb{R}^n \cup \{\infty\}$, with $\phi(x) = x$ if and only if $x \in S(a,r)$.

Let $P(a,t)$ be a plane in $\mathbb{R}^n_\infty$ given by $P(a,t) = \{x \in \mathbb{R}^n : (x \cdot a) = t\} \cup \{\infty\}$ where $a \in \mathbb{R}^n$, $a \neq 0$, $(x \cdot a)$ is the usual scalar inner product $\sum x_j a_j$ and t is a real number. By definition ∞ lies on every plane. The reflection ϕ in $P(a,t)$ is defined by $\phi(x) = x + \lambda a$ where λ is chosen so that $\frac{1}{2}(x + \phi(x))$ lies on $P(a,t)$. This yields the explicit formula

$$\phi(x) = x - 2[(x \cdot a) - t]\frac{a}{|a|^2} \tag{1.2.3}$$

for $x \in \mathbb{R}^n$. Here $\phi(\infty) = \infty$, and $\phi(x) = x$, if and only if $x \in P(a,t)$. The map is once again a bijection of $\mathbb{R}^n_\infty$ onto itself.

We now define a general Möbius transformation on $\mathbb{R}^n_\infty$:

Definition 1.2.1. *A Möbius transformation acting on $\mathbb{R}^n_\infty$ is a finite composition of reflections or inversions in planes or spheres in $\mathbb{R}^n_\infty$.*

Definition 1.2.2. *The group of Möbius transformations acting on $\mathbb{R}^n_\infty$ is called the general Möbius group and is denoted by $GM(\mathbb{R}^n_\infty)$.*

Definition 1.2.3. *The Möbius group, $\mathcal{M}$ acting on $\mathbb{R}^n_\infty$ is the subgroup of $GM(\mathbb{R}^n_\infty)$ consisting of orientation preserving Möbius transformations. We refer to the elements of $\mathcal{M}$ as Möbius transformations and thus assume they preserve orientation.*

In our work, we will consider the actions of Möbius transformations acting in $\mathbb{R}^2_\infty$ and their extension to $\mathbb{R}^3_\infty$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and the point (x, y, t) in $\mathbb{R}^3$ with the quaternion $x + yi + tj = z + tj$ where $z = x + yi \in \mathbb{C}$. This identification allows us to express Poincaré extensions of Möbius transformations algebraically in terms of quaternions.

Möbius transformations are usually represented as mappings $z \mapsto \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We note that $\dfrac{-d}{c} \mapsto \infty$ for $c \neq 0$ while $\infty \mapsto \dfrac{a}{c}$ under this action. Thus $\mathcal{M}$ is given as:

$$\mathcal{M} = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}$$

is the Möbius group. Since $\mathcal{M}$ is a homomorphic image of the matrix group $GL(2,\mathbb{C})$, up to multiples and each matrix can be normalised, we write $\mathcal{M} = PGL(2,\mathbb{C}) = PSL(2,\mathbb{C})$ [29]. Every $g \in \mathcal{M}$ with $g(x) = \dfrac{az+b}{cz+d}, ad - bc \neq 0$ may be associated with a matrix $M_g \in GL(2,\mathbb{C})$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ [6], [29].

We note the following result on the decomposition of Möbius maps.

Proposition 1.2.4. *Each element $g \in \mathcal{M}$ can be written as a composition of orientation preserving elements in $\mathcal{M}$ given as follows:*

1. *Translations: $\tau_a(z) = z + a$, where $a \in \mathbb{C}$.*
2. *Dilations: $D_\lambda(z) = \lambda z$, where $\lambda \in \mathbb{R}, \lambda > 0, \lambda \neq 1$.*
3. *Rotations: $R_\theta(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$.*
4. *Complex Inversion: $J(z) = \dfrac{1}{z}$.*

Proof. [29, p.21].

Since $\lambda^{\frac{1}{2}} \in \mathbb{C}$ for each $\lambda \in \mathbb{C}$, each Möbius transformation can be written in a normalised form as $g(z) = \dfrac{az+b}{cz+d}$ with $ad - bc = 1$.

1. If $c = 0$, then $g(z) = \dfrac{az+b}{d}$ with $a, d \neq 0$. Let $\dfrac{a}{d} = \lambda e^{i\theta}$ and $\dfrac{b}{d} = p$, then $g(z) = \lambda e^{i\theta}z + p$, so $g = \tau_p D_\lambda R_\theta$.
2. If $c \neq 0$, then $g(z) = \dfrac{a}{c} - \dfrac{1}{c(cz+d)} = (\tau_{a/c}J)(-c^2z - cd)$. By the method used in the case $c = 0$, the transformation $z \mapsto -c^2z - cd$ can be expressed in terms of the given generators, and hence so can g.

Thus $g \in \mathcal{M}$ can be written as a composition of orientation preserving generators in $\mathcal{M}$. □

Since the group $\mathcal{M}$ acts on $\mathbb{R}^2_\infty$ (and in fact $\mathbb{R}^3_\infty$), we clarify the following ideas.

Definition 1.2.5. *Let G be a group acting on a set X, with $x_1, x_2 \in X$ and $g_1, g_2 \in G$.*

1. *Let $g_1, g_2 \in G$. The map g_1 is conjugate to g_2 if there exists $h \in G$ such that $hg_1h^{-1} = g_2$. In fact conjugacy is an equivalence relation in G. We write $g_1 \approx g_2$.*

2. *The orbit of $x \in X$ is the set $Gx = \{g(x) : g \in G\}$ in X.*

3. *The stabiliser of x in G is the subgroup of G given by $Stab(x, G) = \{g \in G : g(x) = x\}$.*

4. *Points x_1 and x_2 in X are equivalent if they have the same orbit. That is $Gx_1 = Gx_2$.*

These concepts are combined in the important result.

Proposition 1.2.6. *Let G act on X. Then for any $x \in X$ we have $|Gx| = |G : Stab(x, G)|$. That is the index of the stabilizer is the size of the orbit of x in X.*

Definition 1.2.7. *The function $tr^2 : PSL(2, \mathbb{C}) \to \mathbb{C}$ is given by $tr^2(g) = (trace\ M_g)^2 = (a + d)^2$. That is, $tr(g)$ is the trace of the matrix M_g, where M_g is associated with the normalised g.*

It can be seen that two Möbius maps are conjugate under $\mathcal{M}$ if and only if their respective tr^2 functions are equal [29]. We can thus use the trace function to classify all elements of $PSL(2, \mathbb{C})$ by their conjugacy classes.

Definition 1.2.8. *Let $g \neq 1_{map}$ be any map in $\mathcal{M}$. Then*

1. *g is parabolic if $tr^2(g) = 4$*

2. *g is elliptic if $tr^2(g) \in [0, 4)$*

3. *g is loxodromic if $tr^2(g) \notin [0, 4]$*

From [7] we have the following classifications:

Theorem 1.2.9. *Let g be a Möbius map other than the identity. Then the following are equivalent:*

(a1) *g is parabolic;*

(a2) *g is conjugate to a translation $z \mapsto z + 1$;*

(a3) *g has exactly one fixed point ζ and $g^n(z)$ converges pointwise to ζ on $\mathbb{C}_\infty$.*

The following are equivalent:

(b1) *g is elliptic;*

(b2) *g is conjugate to a Euclidean rotation $z \mapsto e^{i\theta}z$, where $e^{i\theta} \neq 1$;*

(b3) *g has two fixed points, and $g^n(z)$ converges if and only if z is a fixed point of g.*

The following are equivalent:

(c1) *g is loxodromic;*

(c2) *g is conjugate to a map $z \mapsto \lambda z$, where $|\lambda| \neq 0, 1$;*

(c3) *g has two fixed points, u and v, which can be chosen so that if $z \neq v$ then $g^n(z) \to u$ as $n \to \infty$.*

From [14] we have the following results:

Definition 1.2.10. *Let $g(z) = \dfrac{az+b}{cz+d}, c \neq 0, g \in \mathcal{M}$. The circle I_g given by*

$$|cz + d| = 1, c \neq 0$$

is called the isometric circle of the transformation g.

This circle is the complete locus of points in which lengths and areas are unaltered in magnitude by the transformation g. We see that the centre of I_g is $\dfrac{-d}{c} \in \mathbb{C}$, and the radius of I_g is $\dfrac{1}{|c|}$.

We note that if $c = 0$, ∞ is a fixed point of the map g and there is no unique circle with the property of the isometric circle. This is the case with rotations, $R_\theta(z) = e^{i\theta}z$, and

translations, $\tau_a(z) = z + a, a \in \mathbb{C}$, and compositions thereof.

Theorem 1.2.11. *Lengths and areas inside the isometric circle are increased in magnitude, and lengths and areas outside the isometric circle are decreased in magnitude, by the action of the transformation g.*

That is if $g \in \mathcal{M}$ and $g(\infty) \neq \infty$, then $|g'(z)| > 1$ for all z inside I_g, while $|g'(z)| < 1$ for all z outside I_g.

Theorem 1.2.12. *A transformation $g \in \mathcal{M}$, $g \notin 1_{map}$, carries its isometric circle into the isometric circle of the inverse transformation, so*

$$g(I_g) = I_{g^{-1}} \text{ and } g^{-1}(I_{g^{-1}}) = I_g.$$

1.3 Hyperbolic Space

Definition 1.3.1. *The upper-half space of $\mathbb{R}^3$ given by*

$$\mathbb{H}^3 = \{z + tj : z \in \mathbb{C}, t \in \mathbb{R}, t > 0\}$$

with the associated hyperbolic metric

$$ds^2 = \frac{|dz|^2 + |dt|^2}{t^2}$$

is a model of three dimensional hyperbolic space.

We note that $z + tj$ is a quaternion in $\mathbb{K}$ and that $\mathbb{C}_\infty$ is the boundary of $\mathbb{H}^3$. Also $z + tj$ has no k-component in $\mathbb{K}$, where $k = ij$.

Given two distinct points ω_1 and ω_2 in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$, let Λ be a piecewise continuously differentiable curve in $\mathbb{H}^3$ between ω_1 and ω_2.

Definition 1.3.2. *The hyperbolic length of* Λ *is defined as*

$$h(\Lambda) = \int_{\Lambda} \frac{|d\omega|}{t} \qquad \omega = z + tj$$

Definition 1.3.3. *The metric* $\rho : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}^+$ *is given by*

$$\rho(\omega_1, \omega_2) = \inf_{\Lambda} h(\Lambda)$$

where Λ *ranges over all piecewise continuously differentiable curves between* ω_1 *and* ω_2*.*

We note that there exists a path which realises $\rho(\omega_1, \omega_2)$. These paths are called the hyperbolic line segment or geodesic segment between ω_1 and ω_2. We may denote this path as $\Lambda_{\omega_1,\omega_2}$, where $\Lambda_{\omega_1,\omega_2} = \Lambda_{\omega_2,\omega_1}$ or as $[\omega_1 : \omega_2]$. If ω_1, ω_2 lie on $\mathbb{C}_\infty$ the paths are called geodesics.

It is seen that a geodesic is either a vertical line or a semicircle orthogonal to $\mathbb{C}_\infty$, the boundary of $\mathbb{H}^3$ and denoted by $[a : b]$ or $\Lambda_{a,b}$ where the distinct endpoints are $a, b \in \mathbb{C}_\infty$. The geodesic $\mathbb{I}_0 = \Lambda_{0,\infty} = [\infty : 0]$ is called the fundamental geodesic with end points 0 and ∞ on $\mathbb{C}_\infty$.

Definition 1.3.4. *Let* $n \in \mathbb{N}, n \geq 3$*. A hyperbolic n-gon in* $\mathbb{H}^3 \cup \partial\mathbb{H}^3$ *is a convex set bounded by hyperbolic line segments as its sides and lying on an hyperbolic surface. The line segments intersect in pairs at points and we call these points cusps or vertices of the n-gon. The line segments do not intersect other than at these points. The cusps or vertices will be used to represent the n-gon in* $\mathbb{H}^3$*.*

If the vertices of the n-gon all lie on $\partial\mathbb{H}^3$ *or* $\mathbb{C}_\infty$ *then the n-gon is said to be ideal. In particular if* $n = 3$ *then we have a hyperbolic triangle with vertices* v_1*,* v_2 *and* v_3 *and sides* Λ_{v_1,v_2}*,* Λ_{v_2,v_3} *and* Λ_{v_1,v_3}*. If* $\mathbb{T}$ *is a hyperbolic triangle, then* $\overline{\mathbb{T}}$ *is its closure. Thus* $\overline{\mathbb{T}} = \mathbb{T} \cup \Lambda_{\infty,0} \cup \Lambda_{0,1} \cup \Lambda_{1,\infty}$.

We note further that a hyperbolic plane in $\mathbb{H}^3$ is either a vertical plane $P(a,d)$, or a hemisphere $S(a,r)$, both orthogonal to the boundary $\mathbb{C}_\infty$. That is, for $a=(a_1,a_2,0)\in\mathbb{R}^3$, $d\in\mathbb{R}$ and $r>0$

$$P(a,d)=\{x\in\mathbb{H}^3:(x\cdot a)=d\}\cup\{\infty\},$$

and

$$S(a,r)=\{x\in\mathbb{H}^3:|x-a|=r\}.$$

We note that reflections in these hyperbolic surfaces are considered as inversion in the Euclidean spheres or planes and are given by the formulae

$$p(\omega)=\omega-2[(\omega\cdot a)-d]\frac{a}{|a|^2}$$

for inversion in the plane $(\omega\cdot a)=d$, where $2(\omega\cdot a)=\omega\bar{a}+a\bar{\omega}$, $d\in\mathbb{R}$ and $a=(a_1,a_2,0)\in\mathbb{R}^3$ is a normal to the plane, and

$$q(\omega)=a+\left(\frac{r}{|\omega-a|}\right)^2(\omega-a)$$

for inversion in the sphere with radius r and centre $a=(a_1,a_2,0)\in\mathbb{R}^3$.

It is seen that these definitions of inversions extend the definition of inversion in a line (1.2.3) and inversion in a circle (1.2.2) in $\mathbb{C}$.

The definition of the metric $\rho:\mathbb{H}^3\times\mathbb{H}^3\to\mathbb{R}^+$ is given an explicit formulation through the following theorem [6, p.35].

Theorem 1.3.5. *Let $\omega_i=\alpha_i+t_ij\in\mathbb{H}^3$, $i=1,2$, then*

$$\sinh^2\frac{1}{2}\rho(\omega_1,\omega_2)=\frac{|\alpha_1-\alpha_2|^2}{4t_1t_2}$$

Many other useful identities may be derived from this, as seen in [6].

Definition 1.3.6. *A horoball in $\mathbb{H}^3$ is an open Euclidean ball in $\mathbb{H}^3$ which is tangent to the boundary $\mathbb{C}_\infty$. If the point of tangency is ω, then we denote the horoball by $\mathcal{R}_\omega$.*

The boundary of the horoball is the horosphere. A horoball based at ∞ is a set of the form

$$\left\{z + tj \in \mathbb{H}^3 : t > k, k > 0\right\}.$$

In this case, a horosphere is a horizontal Euclidean surface in $\mathbb{H}^3$ which is orthogonal to all hyperbolic planes containing the point ∞. If $k = 1$, then this horosphere is denoted by $\mathcal{S}_\infty$ and is called the fundamental Ford sphere. In general from [15]

Definition 1.3.7. *Ford spheres are defined at all Gaussian rational points $\frac{\alpha}{\gamma}$ with radius $\frac{1}{2|\gamma|^2}$ denoted by $\mathcal{S}_{\frac{\alpha}{\gamma}}$.*

They are the boundaries of horoballs tangent at the point $\frac{\alpha}{\gamma}$. Each $\frac{\alpha}{\gamma}$ has the form $x+yi+tj$ where $x, y, t \in \mathbb{Q}$ (Section 3.2).

Definition 1.3.8. *An isometry of $\mathbb{H}^3$ is a distance preserving map. That is, a map $g : \mathbb{H}^3 \mapsto \mathbb{H}^3$ such that $\rho(g(\alpha), g(\beta)) = \rho(\alpha, \beta)$ where $\alpha, \beta \in \mathbb{H}^3$.*

It is seen that the composition of isometries is an isometry in $\mathbb{H}^3$.

Definition 1.3.9. *If S_1 and S_2 are surfaces in a differentiable manifold and the map f is smooth, then we say that $f : S_1 \mapsto S_2$ is conformal if it preserves angles, that is, whenever curves Λ_1 and Λ_2 on S_1 meet at a point Q with an angle θ, then $f(\Lambda_1)$ and $f(\Lambda_2)$ meet at $f(Q)$ on S_2 with the same angle θ. A conformal map from an oriented surface to itself is directly or indirectly conformal if it preserves or reverses the orientation.*

It is seen that the composition of conformal maps is again conformal [29].

Definition 1.3.10. *Let G be the group of isometries of* $\mathbb{H}^3$ *generated by* τ_α, D_λ, R_θ, R *and* I *where*

1. *Translation:* $\tau_\alpha(z + tj) = z + \alpha + tj$, *where* $\alpha \in \mathbb{C}$.
2. *Dilation:* $D_\lambda(z + tj) = \lambda z + \lambda tj$, *where* $\lambda \in \mathbb{R}, \lambda > 0, \lambda \neq 1$.
3. *Rotation:* $R_\theta(z + tj) = e^{i\theta}z + tj$.
4. *Reflection:* $R(z + tj) = \bar{z} + tj$
5. *Inversion:* $I(z + tj) = \dfrac{z + tj}{|z|^2 + t^2}$.

We observe that translations, dilations and rotations are orientation preserving (direct) conformal isometries, while reflections and inversions are orientation reversing (indirect) conformal isometries. Following [35] we have:

Theorem 1.3.11. *G is a group and the elements in G are direct or indirect conformal isometries of hyperbolic space. Every* $g \in G$ *preserves* $\mathbb{H}^3$ *and* $\mathbb{C}_\infty$. *Further, every isometry of* $\mathbb{H}^3$ *is in G. Thus* $G = Isom(\mathbb{H}^3)$.

The proof of Theorem 1.3.11 can be found in [35, pp.54,61].

1.4 The Poincaré Extension

In order to show that $\mathcal{M}$ may be regarded as the group of orientation preserving isometries acting on $\mathbb{H}^3$, we need to extend the action of each $g \in \mathcal{M}$ acting on $\mathbb{C}$ to act in $\mathbb{H}^3$ [38]. It is noted [6], that each g acting on $\mathbb{R}^2_\infty$ can be uniquely extended to a map $\tilde{g}$ acting on $\mathbb{R}^3_\infty$.

Firstly, we note that $\mathbb{R}^2_\infty$ can be embedded in $\mathbb{R}^3_\infty$ in the following way:

$$x \mapsto \tilde{x} = (x_1, x_2, 0), \text{ where } x = (x_1, x_2)$$

Definition 1.4.1. *For each inversion g acting in $\mathbb{R}^2$, we define $\tilde{g}$ acting in $\mathbb{R}^3$ as follows:*

1. *If g is an inversion in the circle of radius r centred at $a = (a_1, a_2)$, written $S(a, r)$, then $\tilde{g}$ is the inversion in the sphere $S(\tilde{a}, r)$, $\tilde{a} = (a_1, a_2, 0)$.*

2. *If g is an inversion in the line $P(a, d) = \{x \in \mathbb{R}^2 : (x \cdot a) = d\} \cup \{\infty\}$, then $\tilde{g}$ is the inversion in the plane $P(\tilde{a}, d) = \{x \in \mathbb{R}^3 : (x \cdot \tilde{a}) = d\} \cup \{\infty\}$.*

We call $\tilde{g}$ the Poincaré extension of g.

We observe that if g_i and f_j are inversions in $\mathbb{C}_\infty$, then $\widetilde{g_i f_j} = \tilde{g}_i \tilde{f}_j$. Thus if g and f are in $\mathcal{M}$ with $g = g_1 g_2 \dots g_n$ and $f = f_1 f_2 \dots f_m$ where the g_i and f_j are inversions in $\mathbb{C}_\infty$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, then $\widetilde{gf} = (\tilde{g}_1 \tilde{g}_2 \dots \tilde{g}_n \tilde{f}_1 \tilde{f}_2 \dots \tilde{f}_m) = \tilde{g}\tilde{f}$.

Note that a circle in $\mathbb{C}_\infty$ is the equator to a sphere in $\mathbb{R}^3_\infty$ with centre on $\mathbb{C}$. In particular we extend the isometric circles I_g and $I_{g^{-1}}$ to corresponding isometric spheres in $\mathbb{H}^3$. We use I_g to represent the sphere and the circle corresponding to g. The proof of the following result can be found in [6].

Theorem 1.4.2. *If $\tilde{g}$ is a inversion in the sphere $S(\tilde{a}, r)$ with radius r and centre $\tilde{a}$, where $a \in \mathbb{R}^2$, then we have*

$$\frac{|\tilde{g}(y) - \tilde{g}(x)|}{|y - x|} = r^2 \left(\frac{1}{|y - a|^2} - \frac{2(x - a) \cdot (y - a)}{|x - a|^2 |y - a|^2} + \frac{1}{|x - a|^2} \right)^{\frac{1}{2}} = \frac{r^2}{|x - \tilde{a}||y - \tilde{a}|}, x, y \in \mathbb{R}^3$$

We note since $g \in \mathcal{M}$ is the composition of an even number of inversions in circles in $\mathbb{C}_\infty$, then $\tilde{g}$ is the composition of an even number of inversions in spheres in $\mathbb{R}^3_\infty$.

The transition of g acting on $\mathbb{C}_\infty$ to $\tilde{g}$ acting in $\mathbb{H}^3$ is made simple by the following result in [6].

Proposition 1.4.3. *Let $g \in \mathcal{M}$ and $z \in \mathbb{C}$. Then the Poincaré extension of g to $\tilde{g}$ is given by*

$$\tilde{g}(z + tj) = \frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2 + |ad - bc|tj}{|cz + d|^2 + |c|^2t^2}$$

where $z + tj = x + yi + tj$ is a quaternion in $\mathbb{K}$.

It is important to note that $\tilde{g}$ leaves the complex plane and the upper and lower half spaces respectively invariant. This invariance proves that a Poincaré extension exists and is unique [6]. Observe that when $t = 0$, this equation will simplify to the g acting in the complex plane.

Example 1.4.4. *The maps $\tau_\alpha(z) = z + \alpha, \alpha \in \mathbb{C}, \varphi(z) = \dfrac{-1}{z}, \phi(z) = \dfrac{i}{iz}, \psi(z) = \dfrac{iz}{-i}$, $\kappa(z) = iz$ act on $\mathbb{C}$. The Poincaré extensions of these maps act on $\mathbb{H}^3$ and are:*

$$\widetilde{\tau_\alpha}(z + tj) = z + \alpha + tj$$

$$\widetilde{\varphi}(z + tj) = \frac{-\bar{z} + tj}{|z|^2 + t^2}$$

$$\widetilde{\phi}(z + tj) = \frac{\bar{z} + tj}{|z|^2 + t^2}$$

$$\widetilde{\psi}(z + tj) = -z + tj$$

$$\widetilde{\kappa}(z + tj) = iz + tj$$

We will write $\tilde{g}(z + tj) = g(z + tj)$ and assume the extension is understood and we say that the elements of $\mathcal{M}$ act on $\mathbb{H}^3$ by the Poincaré extension. From [3] we have:

Theorem 1.4.5. *Every Möbius map acts on hyperbolic space $\mathbb{H}^3$ as a directly conformal hyperbolic isometry, and every isometry of $\mathbb{H}^3$ that preserves orientation is a Möbius map.*

Proof. From Proposition 1.2.4, g is a composition of $\tau_\alpha, D_\lambda, R_\theta, J$. Clearly $\tau_\alpha, D_\lambda, R_\theta, J \in G$, so $\mathcal{M} \subseteq G$.

Certainly the orientation-preserving generators τ_α, D_λ and R_θ of G are elements of $\mathcal{M}$. Further the composition of the orientation-reversing maps R and I, where $R(z + tj) = \overline{z} + tj$ and $I(z + tj) = \dfrac{z + tj}{|z|^2 + t^2}$, are orientation-preserving and are rotations or translations in $\mathbb{H}^3$. So the orientation-preserving elements of G are all Möbius maps. □

Thus it is seen that every Möbius transformation acting in $\mathbb{H}^3$ is a composition of a finite even number of inversions in $\mathbb{H}^3$ [7].

Chapter 2

The Picard Groups

2.1 Introduction

Our goal in this chapter is to discuss the properties of some subgroups of $\mathcal{M}$, in particular the Picard group, $\mathcal{P}$, and the extended Picard group, $\tilde{\mathcal{P}}$. We note that the modular group and the extended modular group are also subgroups of $\mathcal{M}$. In order to understand continued fractions with Gaussian integer coeffiecients, we need to explore the Picard groups. We note that while $\mathcal{P}$ and $\tilde{\mathcal{P}}$ act on $\mathbb{C}_\infty$, using the Poincaré extension we can find their actions in $\mathbb{H}^3$.

Definition 2.1.1. *The Picard group $\mathcal{P}$ is the subgroup of $\mathcal{M}$ given as follows:*

$$\mathcal{P} = \left\{ z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta} : \alpha, \beta, \gamma, \delta \in \mathbb{Z}[i], \alpha\delta - \beta\gamma = 1 \right\}$$

where $\mathbb{Z}[i] = \left\{ a + ib : a, b \in \mathbb{Z}, i^2 = -1 \right\}$ are the Gaussian integers.

To facilitate the discussion of $\mathcal{P}$, we define $\tau_1(z) = z + 1$, $\tau_i(z) = z + i$, $\phi(z) = \dfrac{i}{iz}$, $\varphi(z) = \dfrac{-1}{z}$, and $\psi(z) = \dfrac{iz}{-i}$ which are in $\mathcal{P}$. We note that $H = \{1, \varphi, \psi, \phi\}$, where $\phi = \varphi\psi$, is isomorphic to

D_2, the Klein 4-group and is a subgroup of $\mathcal{P}$. Let $\alpha = a + ib \in \mathbb{Z}[i]$. Then $\tau_\alpha = \tau_1^a \tau_i^b = \tau_i^b \tau_1^a$ where $a, b \in \mathbb{Z}$.

The maps τ_1, τ_i, ϕ and φ will be shown to generate $\mathcal{P}$ and can be used to express continued fractions with Gaussian integer coefficients.

The Picard group $\mathcal{P}$ consists of those Möbius maps which can be represented by matrices with Gaussian integer entries and whose determinant is ± 1. We note that $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ can be expressed as $M_g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Thus if $\alpha\delta - \beta\gamma = -1$, then we can write $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta} = \dfrac{\alpha i z + \beta i}{\gamma i z + \delta i}$ and $\alpha i \delta i - \beta i \gamma i = 1$.

Definition 2.1.2. *The extended Picard group $\tilde{\mathcal{P}}$ is the subgroup of $\mathcal{M}$ given as:*

$$\tilde{\mathcal{P}} = \left\{ z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta} : \alpha, \beta, \gamma, \delta \in \mathbb{Z}[i], |\alpha\delta - \beta\gamma| = 1 \right\}$$

It will be shown that $\tilde{\mathcal{P}}$ has generators $\tau_1(z) = z + 1$, $\tau_i(z) = z + i$, $\phi(z) = \dfrac{1}{z}$, and $\kappa(z) = iz$. We note that $\kappa^4 = 1$ and $\kappa^2 = \psi$. The group $\{1, \kappa, \kappa^2, \kappa^3\} = \{1, \kappa, \psi, \kappa\psi\}$ is a cyclic group of order 4 and a subgroup of $\tilde{\mathcal{P}}$. The extended Picard group consists of those maps which can be represented by matrices with Gaussian integer entries whose determinant is ± 1 or $\pm i$.

2.2 Gaussian Integers and Gaussian Rationals

Lemma 2.2.1. *Let $N : \mathbb{Z}[i] \to \mathbb{Z}^+ \cup \{0\}$ be given by $N(\alpha) = |\alpha|^2 = |a + bi|^2 = a^2 + b^2$. Then N is an Euclidean Norm and satisfies:*

1. *$N(\alpha) = 0$ if and only if $\alpha = 0$*

2. $N(\alpha\beta) = N(\alpha)N(\beta)$ *for* $\alpha, \beta \in \mathbb{Z}[i]$

3. $N(\alpha\beta) \geq N(\alpha)$ *and* $N(\beta)$ *for* $\alpha, \beta \in \mathbb{Z}[i]$, $\alpha, \beta \neq 0$

4. *For all* α *and* $\beta \neq 0$, *there exists* $\gamma, \delta \in \mathbb{Z}[i]$ *such that* $\alpha = \gamma\beta + \delta$ *and* $N(\delta) < N(\beta)$ *or* $\delta = 0$. *That is,* $0 \leq N(\delta) < N(\beta)$ *since* $N(\delta) = 0$ *if and only if* $\delta = 0$.

5. $N(\alpha) = 1$ *if and only if* $\alpha = \pm 1$ *or* $\pm i$. *That is,* α *is a unit in* $\mathbb{Z}[i]$.

Proof. The proof of parts 1, 2, 3 and 5 follow as seen in [17]. We consider part 4 only.

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{\alpha\bar{\beta}}{\beta\bar{\beta}} = \frac{\alpha\bar{\beta}}{|\beta|^2} \text{ where } |\beta|^2 \in \mathbb{Z} \\ &= r + si \text{ where } r, s \in \mathbb{Q} \end{aligned}$$

Let q_1 and q_2 be integers such that $|r - q_1| \leq \frac{1}{2}$ and $|s - q_2| \leq \frac{1}{2}$.

Then

$$\begin{aligned} \frac{\alpha}{\beta} &= (q_1 + iq_2) + (r - q_1) + i(s - q_2) \\ &= \gamma + (r - q_1) + i(s - q_2) \text{ where } \gamma = (q_1 + iq_2) \\ \text{then } \alpha &= \gamma\beta + \{(r - q_1) + i(s - q_2)\}\beta \end{aligned}$$

Thus $\frac{\alpha}{\beta} - \gamma = (r - q_1) + i(s - q_2)$. So

$$\begin{aligned} N\left(\frac{\alpha}{\beta} - \gamma\right) &= |(r - q_1) + i(s - q_2)|^2 \\ &= (r - q_1)^2 + (s - q_2)^2 \\ &\leq \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

Thus by parts 1-3, $N(\alpha - \gamma\beta) \leq N(\frac{\alpha}{\beta} - \gamma)N(\beta) \leq N(\beta)\frac{1}{2} < N(\beta)$.

Let $\delta = \alpha - \gamma\beta$, then $\alpha = \gamma\beta + \delta$ where $N(\beta) > N(\delta)$. □

From parts 3 and 4 we have that N is a Euclidean valuation map [17].

Thus $\mathbb{Z}[i]$ is a Euclidean domain and hence is a principal ideal domain and a unique factorisation domain. The elements of $\mathbb{Z}[i]$ form an integer lattice in $\mathbb{C}$. We note that the set $U = \{\pm 1, \pm i\}$ in $\mathbb{Z}[i]$ is the multiplicative group of units in $\mathbb{Z}[i]$.

The relationship $|\alpha\delta - \beta\gamma| = 1$ is closely related to the idea of greatest common divisor and co-prime numbers in $\mathbb{Z}[i]$.

Definition 2.2.2. *For non-zero $\alpha, \gamma \in \mathbb{Z}[i]$, a greatest common divisor (gcd) of α and γ is a common divisor with maximal Euclidean norm.*

Lemma 2.2.3. *Each gcd of α and γ can be written in the form $\mu\alpha + \nu\gamma$ for some $\mu, \nu \in \mathbb{Z}[i]$.*

Proof. Let W denote the set of positive norms of all linear combinations of α and γ. That is,

$$W = \{N(\mu\alpha + \nu\gamma) : N(\mu\alpha + \nu\gamma) > 0; \quad \mu, \nu \in \mathbb{Z}[i]\}.$$

Since $N(\mu\alpha + \nu\gamma) > 0$ for any $\mu, \nu \in \mathbb{Z}[i]$, $W \subseteq \mathbb{N}$, furthermore taking $\mu = \overline{\alpha}$ and $\nu = \overline{\gamma}$, $N(\mu\alpha + \nu\gamma) = N(N(\alpha) + N(\gamma)) = N(\alpha) + N(\gamma) > 0$, so $W \neq \emptyset$. Thus the Well Ordering Principle for non-empty subsets of $\mathbb{N}$ applies to W. So there exists an $a_0 = N(\mu_0\alpha + \nu_0\gamma)$, with $\mu_0, \nu_0 \in \mathbb{Z}[i]$ such that $a_0 \leq a$ for all $a \in W$.

Let $\delta = \mu_0\alpha + \nu_0\gamma$, so $a_0 = N(\delta)$, hence any common divisor of α and γ is a divisor of δ. We will show that $\delta = gcd(\alpha, \gamma)$. By part 4 of Lemma 2.2.1, for any $\alpha, \delta \in \mathbb{Z}[i], \delta \neq 0$ there exists η and $\rho \in \mathbb{Z}[i]$ such that $\alpha = \delta\eta + \rho$ and $N(\rho) < N(\delta) = a_0$ or $\rho = 0$. Let

$\rho = \alpha - \delta\eta = \alpha - (\mu_0\alpha + \nu_0\gamma)\eta = \alpha(1 - \mu_0\eta) - \gamma(\nu_0\eta)$, so $N(\rho) = N(\alpha - \delta\eta) \in W$. By the choice of α_0, this is not possible. So $\rho = 0$ and δ divides α. Similarly δ divides γ. Thus $\delta = gcd(\alpha, \gamma)$. □

Definition 2.2.4. *Two numbers $\alpha, \gamma \in \mathbb{Z}[i]$ are co-prime if and only if the only common divisors are the units. That is $gcd(\alpha, \gamma) = \pm 1, \pm i$.*

Lemma 2.2.5. *1. $\alpha, \gamma \in \mathbb{Z}[i]$ are co-prime if and only if there exists $\beta, \delta \in \mathbb{Z}[i]$ such that $|\alpha\delta - \beta\gamma| = 1$;*

2. If $\alpha, \gamma \in \mathbb{Z}[i]$ are co-prime then there exists $\beta, \delta \in \mathbb{Z}[i]$ such that $\alpha\delta - \beta\gamma = 1$.

Proof. Let $gcd(\alpha, \gamma) = \rho$. If α and γ are co-prime, then $\rho = \pm 1$ or $\pm i$ and the ideal generated by ρ is $\mathbb{Z}[i]$. If $\alpha\delta - \beta\gamma = \mu, \mu = \pm 1, \pm i$, then $\alpha\delta\mu^{-1} - \beta\gamma\mu^{-1} = 1$. Set $\delta' = \delta\mu^{-1}$ and $\gamma' = \gamma\mu^{-1}$ to complete the proof.

Thus we can find $\delta', \gamma' \in \mathbb{Z}[i]$ such that $\alpha\delta' - \beta\gamma' = 1$.

Conversely, if we can find δ and β in $\mathbb{Z}[i]$ with $|\alpha\delta - \beta\gamma| = 1$, then $\alpha\delta - \beta\gamma = \pm 1$ or $\pm i$. Say $gcd(\alpha, \gamma) = \rho$, then since ρ divides α and ρ divides γ, we have ρ divides ± 1 or $\pm i$. So ρ is ± 1 or $\pm i$ and α and γ are co-prime. □

We call the identity, $|\alpha\delta - \beta\gamma| = 1$, the condition for co-primeness of α and γ. Further if $\alpha\delta - \beta\gamma = \pm 1$, then $h(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ is in $\mathcal{P}$. If $\alpha\delta - \beta\gamma = \pm i$, then $h(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ is in $\tilde{\mathcal{P}}$. In this case, let $h(z) = \dfrac{\alpha i z + \beta}{\gamma i z + \delta}$ with $\alpha i\delta - \beta\gamma i = (\alpha\delta - \beta\gamma)i = \pm 1$. Then $h \in \mathcal{P}$. Thus we can always find $h \in \mathcal{P}$ with $h(\infty) = \frac{\alpha}{\gamma}$ and $h(0) = \frac{\beta}{\delta}$.

Definition 2.2.6. *A fraction $\dfrac{\alpha}{\gamma}$ is reduced if and only if there is β, $\delta \in \mathbb{Z}[i]$ such that $|\alpha\delta - \beta\gamma| = 1$. That is, α and γ are co-prime.*

Hence if $\frac{\alpha}{\gamma}$ is reduced, then there is a $g \in \mathcal{P}$ (and $\tilde{\mathcal{P}}$) such that $g(\infty) = \frac{\alpha}{\gamma}$. We denote the reduced form of ∞ to be $\frac{1}{0}$ where $\alpha = 1, \gamma = 0$ satisfy the condition for co-primeness.

Let $\mathbb{Q}(i)$ be the field of fractions of $\mathbb{Z}[i]$. Then $\mathbb{Q}(i) = \left\{\frac{\alpha}{\beta} : \alpha, \beta \in \mathbb{Z}[i], \beta \neq 0\right\}$ or $\mathbb{Q}(i) = \{a + ib : a, b \in \mathbb{Q}\}$ where $\frac{\alpha}{\beta} = \frac{\alpha\bar{\beta}}{|\beta|^2}$. We add ∞ to $\mathbb{Q}(i)$ in the usual way and denote $\mathbb{Q}(i) \cup \{\infty\}$ as $\mathbb{Q}_\infty(i)$. We refer to $\mathbb{Q}(i)$ as the Gaussian rationals. In what follows we will assume that a given rational $\frac{\alpha}{\gamma}$ is reduced.

We recall that each $g \in \mathcal{P}$ or $\tilde{\mathcal{P}}$ may have one or two fixed points. From Theorem 1.2.9, we see that if $g \in \mathcal{M}$ has exactly one fixed point, it is a parabolic map. We say its fixed point is a parabolic fixed point. The following theorem characterises the parabolic fixed points of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ in a most useful way.

Theorem 2.2.7. *$\mathbb{Q}_\infty(i)$ is precisely the set of parabolic fixed points for $\mathcal{P}$ and is the orbit of ∞ under $\mathcal{P}$.*

Proof. Let $\frac{\alpha}{\gamma} \in \mathbb{Q}_\infty(i)$. If $\alpha\delta - \beta\gamma = \pm 1$ then we can find $h(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ in $\mathcal{P}$ with $h(\infty) = \frac{\alpha}{\gamma}$. Similarly if $\alpha\delta - \beta\gamma = \pm i$, we can find $h \in \mathcal{P}$ with $h(z) = \frac{\alpha i z + \beta}{\gamma i z + \delta}$, $h(\infty) = \frac{\alpha}{\gamma}$. So $\frac{\alpha}{\gamma}$ is in the orbit of ∞ under $\mathcal{P}$. Consider $g = h\tau_1 h^{-1}$ where $\tau_1(z) = z + 1$ is parabolic. Then $g\left(\frac{\alpha}{\gamma}\right) = h\tau_1 h^{-1}\left(\frac{\alpha}{\gamma}\right) = h\tau_1(\infty) = h(\infty) = \frac{\alpha}{\gamma}$. Thus $\frac{\alpha}{\gamma}$ is a fixed point of a parabolic map g. Since h and τ_1 are in $\mathcal{P}$, we have $g \in \mathcal{P}$ and thus $\frac{\alpha}{\gamma}$ is the fixed point of $g \in \mathcal{P}$.

Conversely, say $g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ and $g \in \mathcal{P}$ and g is parabolic. Then $(\alpha + \delta)^2 = 4$ and $\alpha\delta - \beta\gamma = 1$.

Let z be the fixed point of g.

If $\gamma = 0$, then $g(z) = \frac{\alpha}{\delta}z + \frac{\beta}{\delta}$. Then $\frac{\alpha}{\delta}z - z = -\frac{\beta}{\delta}$ and $z = \frac{-\frac{\beta}{\delta}}{\frac{\alpha}{\delta} - 1} = \frac{-\beta}{\alpha - \delta} = \frac{\beta}{\delta - \alpha}$. If $\delta = \alpha$

then ∞ is the fixed point. Since $\alpha\delta - \beta\gamma = \alpha\delta = \pm 1$ and $\alpha \neq \delta$. The fixed point of g is $\frac{\beta}{\delta - \alpha}$ in $\mathbb{Q}(i)$.

If $\gamma \neq 0$ and $g(z) = z$ then $z = \dfrac{(\alpha - \delta) \pm \sqrt{(\delta - \alpha)^2 + 4\beta\gamma}}{2\gamma} = \dfrac{\alpha - \delta}{2\gamma} \in \mathbb{Q}(i)$ since $(\delta - \alpha)^2 + 4\beta\gamma = 0$. □

Definition 2.2.8. *Let $\alpha = a + ib \in \mathbb{C}$.*

1. *The floor or integer part of α, $\lfloor \alpha \rfloor$, is $\lfloor a \rfloor + \lfloor b \rfloor i$, where $\lfloor a \rfloor$ and $\lfloor b \rfloor$ are the integer parts of a and b respectively.*

2. *The nearest node or lattice point to $\alpha = a + ib$, $[\alpha]$, is the point $m + in$ in $\mathbb{Z}[i]$ such that $(a - m)^2 + (b - n)^2 \leq \dfrac{1}{2}$*

2.3 Generators of $\mathcal{P}$

The following theorem is an analogue of the result on the Modular group being generated by $\tau(z) = z + 1$ and $\varphi(z) = \dfrac{-1}{z}$ [1].

Theorem 2.3.1. *The Picard group $\mathcal{P}$ is generated by $\tau_1, \tau_i, \phi, \psi$. That is, $\mathcal{P} = \langle \tau_1, \tau_i, \phi, \psi \rangle$.*

Proof. Certainly, $\langle \tau_1, \tau_i, \phi, \psi \rangle \subseteq \mathcal{P}$

Take any g_0 in $\mathcal{P}$, and write

$$g_0(z) = \frac{\alpha_0 z + \beta_0}{\gamma_0 z + \delta_0}, \quad \alpha_0\delta_0 - \gamma_0\beta_0 = 1$$

If $\gamma_0 = 0$, then

$$g_0(z) = \frac{\alpha_0}{\delta_0} z + \frac{\beta_0}{\delta_0}, \quad \alpha_0\delta_0 = 1$$

Since $\alpha_0, \delta_0 \in \mathbb{Z}[i]$, $\alpha_0, \delta_0 = \pm 1$ or $\pm i$. We consider the four cases:

Let $\beta_0 = a_0 + ib_0$.

- If $\alpha_0 = \delta_0 = 1$, then $g_0(z) = z + \beta_0 = \tau^{\beta_0}(z) = \tau_1^{a_0}\tau_i^{b_0}(z)$
- If $\alpha_0 = \delta_0 = -1$, then $g_0(z) = z - \beta_0 = \tau^{-\beta_0}(z) = \tau_1^{-a_0}\tau_i^{-b_0}(z)$
- If $\alpha_0 = i, \delta_0 = -i$, then $g_0(z) = -z + i\beta_0 = \tau_i^{\beta_0}\psi(z) = \tau_1^{-b_0}\tau_i^{a_0}\psi(z)$
- If $\alpha_0 = -i, \delta_0 = i$, then $g_0(z) = -z - i\beta_0 = \tau_i^{-\beta_0}\psi(z) = \tau_1^{b_0}\tau_i^{-a_0}\psi(z)$

Hence the result holds.

Let $\gamma \neq 0$ and $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$.

For any Gaussian integer, μ, we have

$$\begin{aligned}(\tau_\mu\phi\psi)^{-1}g(z) &= \psi^{-1}\phi^{-1}\tau_{-\mu}g(z)\\ &= \psi\phi\tau_{-\mu}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)\\ &= \frac{\gamma z+\delta}{(\mu\gamma-\alpha)z+(\mu\delta-\beta)}.\end{aligned}$$

In the above, choose μ to be $\left[\dfrac{\alpha_0}{\gamma_0}\right]$ and g to be $g_0(z)$ with $\gamma_0 \neq 0$, then

$$\begin{aligned}\frac{\gamma_0 z+\delta_0}{(\mu\gamma_0-\alpha_0)z+(\mu\delta_0-\beta_0)} &= \frac{\gamma_0 z+\delta_0}{\left(\left[\dfrac{\alpha_0}{\gamma_0}\right]\gamma_0-\alpha_0\right)z+\left(\left[\dfrac{\alpha_0}{\gamma_0}\right]\delta_0-\beta_0\right)}\\ &= \frac{\alpha_1 z+\beta_1}{\gamma_1 z+\delta_1}\\ &= g_1(z)\end{aligned}$$

where $\alpha_1 = \gamma_0, \beta_1 = \delta_0, \gamma_1 = \left[\frac{\alpha_0}{\gamma_0}\right]\gamma_0 - \alpha_0$ and $\delta_1 = \left[\frac{\alpha_0}{\gamma_0}\right]\delta_0 - \beta_0$ and where $\alpha_1\delta_1 - \beta_1\gamma_1 = 1$. Further $N\left(\frac{\alpha_0}{\gamma_0} - \left[\frac{\alpha_0}{\gamma_0}\right]\right) \leq \frac{1}{2}$.

Now $0 \leq N(\gamma_1) = N\left(\left[\frac{\alpha_0}{\gamma_0}\right]\gamma_0 - \alpha_0\right) = N(\gamma_0)\, N\left(\left[\frac{\alpha_0}{\gamma_0}\right] - \frac{\alpha_0}{\gamma_0}\right) \leq \frac{1}{2}N(\gamma_0) < N(\gamma_0)$. Choosing $\mu_1 = \left[\frac{\alpha_1}{\gamma_1}\right]$ and finding $(\tau^{\mu_1}\psi\phi)^{-1}g_1$, we produce $g_2(z) = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2}$ in $\mathcal{P}$ with $0 \leq N(\gamma_2) < N(\gamma_1) < N(\gamma_0)$. Since $N(\gamma_i)$ are all positive integers or zero, the process must stop in a finite number of steps.

Thus there exists k such that $0 = N(\gamma_k)$ and hence $\gamma_k = 0$.

Then we have:

$$(\tau_{\mu_k}\psi\phi)^{-1} \ldots (\tau_{\mu_0}\psi\phi)^{-1} g(z) = g_k(z) = \frac{\alpha_k z + \beta_k}{\delta_k}$$

By the above case for $\gamma_k = 0$ the result is proved. □

Theorem 2.3.2. *The group $\tilde{\mathcal{P}}$ is generated by $\tau_1(z) = z + 1$, $\tau_i(z) = z + i$, $\phi(z) = \frac{1}{z}$, and $\kappa(z) = iz$ and $\mathcal{P}$ is a subgroup of $\tilde{\mathcal{P}}$ of index 2.*

Proof. Certainly $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ and $\kappa^2 = \psi$. So $\mathcal{P} \triangleleft \tilde{\mathcal{P}}$ and $\left|\tilde{\mathcal{P}} : \mathcal{P}\right| = 2$ with $\tilde{\mathcal{P}} = \mathcal{P} \cup \kappa\mathcal{P}$ and hence $\tilde{\mathcal{P}} = \langle \tau_1, \tau_i, \phi, \kappa \rangle$.

If $g \in \tilde{\mathcal{P}}/\mathcal{P}$ then $g\kappa \in \mathcal{P}$. So $\tilde{\mathcal{P}} = \mathcal{P} \cup \mathcal{P}\kappa$ or $\mathcal{P} \cup \kappa\mathcal{P}$. □

We have seen in Example 1.4.4 that these generators can be extended to act on $\mathbb{H}^3$ as follows. $\tau_1(\omega) = z + tj + 1$, $\tau_i(\omega) = z + tj + i$, $\phi(\omega) = \frac{\bar{z} + tj}{|z|^2 + t^2}$, $\psi(\omega) = -z + tj$, $\varphi(\omega) = \frac{-\bar{z} + tj}{|z|^2 + t^2}$ and $\kappa(\omega) = iz + tj$ for $\omega = z + tj \in \mathbb{H}^3$, $t > 0$.

We further note the geometric interpretations of the action of the generators on $\mathbb{H}^3$. $\tau_1(\omega) = z + tj + 1$ is a translation along the real axis by 1; $\tau_i(\omega) = z + tj + i$ is a translation along

the imaginary axis by 1; $\phi(\omega) = \dfrac{\bar{z} + tj}{|z|^2 + t^2}$ is the inversion in the unit sphere centred at the origin followed by the inversion through the vertical plane on the real axis; $\psi(\omega) = -z + tj$ is a rotation through π radians about the geodesic $\mathbb{I}_0$; $\varphi(\omega) = \dfrac{-\bar{z} + tj}{|z|^2 + t^2}$ is an inversion in the unit sphere followed by the reflection through the vertical plane through the imaginary axis; $\kappa(\omega) = iz + tj$ is the rotation through $\frac{\pi}{2}$ radians anticlockwise about the geodesic $\mathbb{I}_0$. τ_1, τ_i, ψ and κ leave the j component of ω in $\mathbb{H}^3$ invariant. Hence $\tau_1, \tau_i, \psi \in Stab(\infty, \mathcal{P})$ and $\tau_1, \tau_i, \psi, \kappa \in Stab(\infty, \tilde{\mathcal{P}})$. Since ϕ and φ involve inversion in the unit sphere, 0 and ∞ are interchanged as are the inside and outside of the unit sphere. The point $j = (0, 0, 1)$ is invariant under ϕ and φ. In particular the j component of $\varphi(\omega)$ or $\phi(\omega)$ is less than the j component of ω, if ω is outside the unit circle.

2.4 Discontinuous Groups

From [6], [27] and [48], we summarize important results which are used in establishing a fundamental region for $\mathcal{P}$ on $\mathbb{H}^3$.

Definition 2.4.1. *Let X be any topological space, a subspace of $\mathbb{R}^3_\infty$ with the usual topology. Let G be a group of homeomorphisms of X onto itself. We say G acts discontinuously on X if and only if for every compact subset K of X, $g(K) \cap K = \emptyset$ except for a finite number of g in G.*

We will usually take X to be $\mathbb{H}^3$. We know that if $x \in \mathbb{H}^3$, then the stabilizer of x, $Stab(x, G)$ for G a discontinuous group, is finite [6]. Also, it can be established that a subgroup G of $\mathcal{M}$ is discrete if and only if it acts discontinuously in $\mathbb{H}^3$. We must stress that if G acts discontinuously in some non-empty subset of $\mathbb{C}_\infty$, then G is discrete. The converse, however, is false. That is, it is possible for G to be discrete yet not act discontinuously on

any subset of $\mathbb{C}_\infty$. In fact, the Picard group is such a group. To explore this example, we note the following result [14].

Lemma 2.4.2. *Let G be any subgroup of $\mathcal{M}$ and let $\mathcal{D}$ be an open subset of $\mathbb{C}_\infty$ which contains a fixed point v of some parabolic or loxodromic element $g \in G$. Then G does not act discontinuously in $\mathcal{D}$.*

Consider the Picard group acting on $\mathbb{C}_\infty$. By Lemma 2.4.2 it is sufficient to show that the parabolic fixed points of $\mathcal{P}$ are dense in $\mathbb{C}_\infty$. We have already shown that $\mathbb{Q}_\infty(i)$ is the set of parabolic fixed points of $\mathcal{P}$. Since $\frac{\alpha}{\gamma}$ is a reduced rational in $\mathbb{Q}_\infty(i)$, then we can find δ, γ such that $|\alpha\delta - \beta\gamma| = 1$. Then if $h(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ we see that $h\tau h^{-1}$ fixes $\frac{\alpha}{\gamma}$ and is parabolic in $\mathcal{P}$. Thus $\mathcal{P}$ does not act discontinuously in any open subset of $\mathbb{C}_\infty$. However $\mathcal{P}$ does act discontinuously in $\mathbb{H}^3$ as it is a discrete subgroup of $\mathcal{M}$. A similar result holds for $h(z) = \frac{\alpha i z + \beta}{\gamma i z + \delta}$.

We define a fundamental domain of a group G acting on X. We note that if $E \subset X$, then $\overline{E}$ is the closure of E with respect to X.

Definition 2.4.3. *Let G be a group of homeomorphisms acting on a topological space X and let $\mathcal{D}$ be an open subset of X. If $g \in G$ and $g \neq 1_{map}$ implies*

1. *$g(\mathcal{D}) \cap \mathcal{D} = \emptyset$, and*
2. *$X = \bigcup_{g \in G} g(\overline{\mathcal{D}})$*

then $\mathcal{D}$ is a fundamental domain for G. X is tessellated by the images of $\overline{\mathcal{D}}$ under G.

Equivalently, if we let G be a group acting on a topological space X, then an open subset $\mathcal{D}$ of X is a fundamental domain for G if every point of X is equivalent to at most one point in $\mathcal{D}$, and at least one point in the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ [6]. From [35, p.67], we have:

Theorem 2.4.4. *Let x be a point of $\mathbb{H}^3$, and let G be a subgroup of $\mathcal{M}$. G acts discontinuously at x if and only if G is a discrete subgroup of $\mathcal{M}$.*

2.5 Fundamental region of the Picard group

In this section we introduce a proof that

$$\mathcal{D} = \left\{ \omega = z + tj \in \mathbb{H}^3 : \left|\Re(z)\right| < \frac{1}{2}, 0 \leq \Im(z) < \frac{1}{2}, |\omega| > 1 \right\}$$

is a fundamental domain of $\mathcal{P}$ acting on $\mathbb{H}^3$. This proof is based on a result, that establishes a fundamental domain of the modular group acting on $\mathbb{H}^2$.

Definition 2.5.1. *A convex polyhedron in $\mathbb{H}^3$ is the intersection of countably many open half-spaces in $\mathbb{H}^3$ created by finitely many planes or spheres in $\mathbb{H}^3$. [35]*

Following [24] and [50], we have:

Theorem 2.5.2. *The domain*

$$\mathcal{D} = \left\{ \omega = z + tj \in \mathbb{H}^3 : \left|\Re(z)\right| < \frac{1}{2}, 0 \leq \Im(z) < \frac{1}{2}, |\omega| > 1 \right\}$$

is a fundamental domain for $\mathcal{P}$ acting on $\mathbb{H}^3$.

Proof. Let $g \in \mathcal{P}$ with $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma = 1$. We wish to show that the two conditions in Definition 2.4.3 are satisfied.

If $g = 1_{map}$ then $\gamma = \beta = 0$ and $\alpha = \delta$. If $g \neq 1_{map}$ then either $\gamma = 0$ or $\gamma \neq 0$ and $|\gamma| \geq 1$, $\gamma \in \mathbb{Z}[i]$.

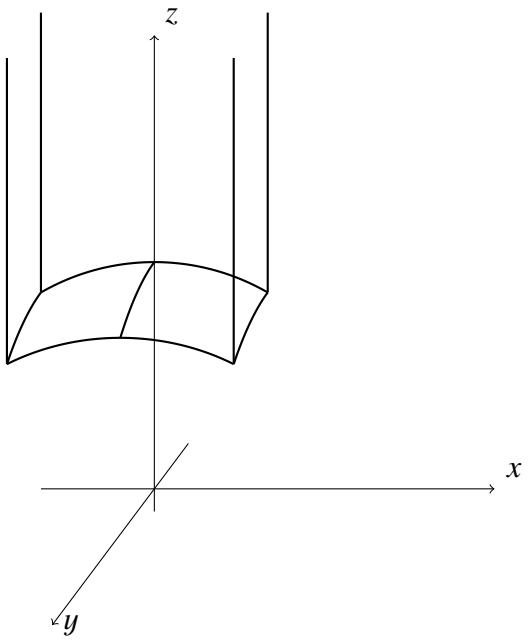


Figure 2.1: A fundamental domain $\mathcal{D}$ of $\mathcal{P}$ acting on $\mathbb{H}^3$.

Condition 1: Suppose that for $g \in \mathcal{P}$, we have $\mathcal{D} \cap g(\mathcal{D}) \neq \emptyset$, where $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i]$ and $\alpha\delta - \beta\gamma = 1$. Let $\omega = z + tj$, $t > 0$ be in $\mathbb{H}^3$. Then by Proposition 1.4.3,

$$g(\omega) = g(z + tj) = \frac{(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha\overline{\gamma}t^2 + tj}{|\gamma z + \delta|^2 + |\gamma|^2 t^2}.$$

Since $\mathcal{D} \cap g(\mathcal{D}) \neq \emptyset$, there is $\omega \in \mathcal{D}$ with $g(\omega)$ also in $\mathcal{D}$. Without loss of generality, replacing g with g^{-1} if necessary, we may assume that the

$$j \text{ component of } \omega \leq j \text{ component of } g(\omega)$$

or equivalently that

$$|\gamma z + \delta|^2 + |\gamma|^2 t^2 \leq 1.$$

Since $\omega \in \mathcal{D}$, we note that if $\omega = z + tj$, then $t \geq \frac{1}{\sqrt{2}}$.

We consider the following cases:

1. Let $\gamma = 0$. Then $g(\omega) = \dfrac{(\alpha z + \beta)\overline{\delta} + tj}{|\delta|^2}$ and $\alpha\delta = 1$. Then α and δ are units in $\mathbb{Z}[i]$ and $\delta^{-1} = \alpha$ and $|\delta| = 1$. Thus $g(\omega) = \alpha\overline{\delta}z + \beta\overline{\delta} + tj = \pm z + \beta' + tj$.

   - If $\alpha = \delta = \pm 1$, then $g(\omega) = z + \beta' + tj$. If $\beta' = 0$, then $g = 1_{map}$, which is excluded by Definition 2.4.3. Thus $\beta' \neq 0$ and so $\beta \neq 0$, hence $g(\omega) \notin \mathcal{D}$.
   - If $\alpha = \pm i$ and $\delta = \mp i$, then $g(\omega) = -z + \beta' + tj$. If $\beta' = 0$ then $g(\omega) = -z + tj$. For $\omega = z + tj$ with $|\Re(z)| < \frac{1}{2}$ and $0 < \Im(z) < \frac{1}{2}$, then $-\frac{1}{2} < \Im(g(z)) < 0$ and $g(\omega) \notin \mathcal{D}$. Since $\beta' \in \mathbb{Z}[i]$, we also have $g(\omega) \notin \mathcal{D}$.

   Thus if ω and $g(\omega) \in \mathcal{D}$, $\gamma \neq 0$.

2. Let $|\gamma| > 1$. Since $\gamma \in \mathbb{Z}[i]$, we may thus assume that $|\gamma|^2 \geq 2$. Then

$$\begin{aligned} |\gamma z + \delta|^2 + |\gamma|^2 t^2 &\geq |\gamma z + \delta|^2 + 2\left(\frac{1}{2}\right) \\ &= |\gamma z + \delta|^2 + 1 \\ &> 1 \end{aligned}$$

   If $\delta = 0$ and $z = 0$ then $t \geq 1$ and $|\gamma z + \delta|^2 + |\gamma|^2 t^2 > 1$. Thus j component of $g(\omega) < j$ component of ω, a contradiction.

3. Let $|\gamma| = 1$. Then we know that γ is a unit.

   We write $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta} = \dfrac{\alpha}{\gamma} - \dfrac{1}{\gamma(\gamma z + \delta)} = \alpha' - \dfrac{1}{\pm z + \delta'}$ where $\alpha' = \alpha\gamma^{-1}$ and $\delta' = \gamma\delta \in \mathbb{Z}[i]$. Thus $g = \tau_{\alpha'}\phi\tau_{\delta'}$ or $\tau_{\alpha'}\varphi\tau_{\delta'}$. Thus the j component of $g(\omega) < j$ component of ω, again a contradiction to the assumption.

Hence for all cases, if $g \neq 1_{map}$, $\mathcal{D} \cap g(\mathcal{D}) = \emptyset$ and Condition 1 of Definition 2.4.3 is satisfied.

Condition 2: To verify this condition, consider $\omega_0 = z_0 + t_0 j = x_0 + y_0 i + t_0 j \in \mathbb{H}^3$. We need to find an element $z + tj \in \overline{\mathcal{D}}$, in the same orbit as $z_0 + t_0 j$ under $\mathcal{P}$.

Consider all pairs of co-prime Gaussian integers γ and δ. For each co-prime pair (γ, δ) we can find a co-prime pair (α, β) such that $\alpha\delta - \gamma\beta = 1$ (Lemma 2.2.5). For any $K > 0$ and $\gamma' \in \mathbb{Z}[i]$, consider the sphere centred at ω_0 with radius $\frac{K}{|\gamma'|}$. This sphere may intersect $\mathbb{C}$ and contain at most a finite number of points with denominators being $|\gamma'|$. Thus the set of numbers $|\gamma'\omega_0 + \delta'|$ taken over all co-prime pairs of Gaussian integers (γ', δ') attains a positive minimum. Let this minimum be attained at γ_0 and δ_0. Thus we can find α_0 and β_0 such that $g(z) = \dfrac{\alpha_0 z + \beta_0}{\gamma_0 z + \delta_0}$ and $g \in \mathcal{P}$.

The j component of $g(\omega)$ is $\dfrac{t}{|\gamma_0 z + \delta_0|^2 + |\gamma_0|^2 t^2}$

By the choice of the pair (γ_0, δ_0) we know that for all $g \in \mathcal{P}$, $|\gamma_0\omega_0 + \delta_0|$ is a minimum. Now

$$\begin{aligned}
|\gamma_0 z + \delta_0|^2 + |\gamma_0|^2 t^2 &= |\gamma_0|^2 \left\{ \left| z + \frac{\delta_0}{\gamma_0} \right|^2 + t^2 \right\} \\
&= |\gamma_0|^2 \left| \omega_0 + \frac{\delta_0}{\gamma_0} \right|^2 \\
&= |\gamma_0\omega_0 + \delta_0|^2
\end{aligned}$$

Therefore for $g \in \mathcal{P}$, since the j component of $g(\omega)$ is $\dfrac{t}{|\gamma_0 z + \delta_0|^2 + |\gamma_0|^2 t^2}$, then $g(\omega_0)$ has the largest j component among all $\mathcal{P}$-images of ω_0. By composing g with a suitable element, $g' \in \langle \tau_1, \tau_i, \psi \rangle$, we may assume that $g'g(\omega_0) = z + tj$ is such that $-\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}$; $0 \leq \Im(z) \leq \frac{1}{2}$ and the j component of $g'g(\omega_0) = j$ component of $g(\omega_0)$. then $|g'g(\omega_0)| \geq 1$ where $|g'g(\omega_0)|^2 = |z|^2 + t^2$. Thus we have found an element in $\overline{\mathcal{D}}$ in the orbit of ω_0 under $\mathcal{P}$. Thus condition 2 of Definition 2.4.3 is satisfied, and so $\mathcal{D}$ is a fundamental domain of $\mathcal{P}$ acting on $\mathbb{H}^3$. □

Ford [16] uses a fundamental domain of $\mathcal{P}$ acting on $\mathbb{H}^3$ is given as

$$\mathcal{D} = \quad \left\{ z + tj \in \mathbb{H}^3 : \left| \Im(z) \right| < \frac{1}{2}, 0 \leq \Re(z) < \frac{1}{2}, |\omega| > 1 \right\}$$

$$= \quad \left\{ x + yi + tj \in \mathbb{H}^3 : |y| < \frac{1}{2}, 0 \leq x < \frac{1}{2}, |\omega| > 1 \right\}$$

where points in $\mathbb{H}^3$ are ordered triples (x, y, t) where $\omega = z + tj = x + yi + tj$ in $\mathbb{C}$ and $t > 0$.

Ford notes that this is a pentahedron lying above the sphere $x^2 + y^2 + t^2 = 1$ where $z = x + yi$ and enclosed by the four planes $x = 0$, $x = \frac{1}{2}$, $y = \frac{-1}{2}$ and $y = \frac{1}{2}$.

Ford notes that for convenience we may double the fundamental domain by reflecting it in the plane $x = 0$. This double pentahedron is bounded by $x = \pm\frac{1}{2}$, $y = \pm\frac{1}{2}$ and is referred to as **P**, a double fundamental domain of $\mathcal{P}$. The four faces are called the lateral faces of the pentahedron and the remaining face, on the sphere $x^2 + y^2 + t^2 = 1$ is called the base. The vertex ∞ is called the peak of the pentahedron **P**. Ford further notes that since $\mathbb{H}^3$ is tessellated by **P** under $\mathcal{P}$, the orbit of ∞ under $\mathcal{P}$ forms the complete set of peaks of the tessellation. The other vertices of **P** lie in the plane $t = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$. The peaks of the pentahedra **P** correspond to the orbit of ∞ under $\mathcal{P}$.

In the next chapter we introduce the Farey tessellation of $\mathbb{H}^3$. This is a tessellation of $\mathbb{H}^3$ by an ideal octahedron under the action of $\mathcal{P}$.

Chapter 3

Tessellations of $\mathbb{H}^3$ under $\mathcal{P}$.

We have seen in Definition 2.4.3 that if group G acts on the space $\mathbb{H}^3$ with fundamental domain $\mathcal{D}$, then $\left\{g(\overline{\mathcal{D}})|g \in G\right\}$ is a tessellation of $\mathbb{H}^3$.

3.1 The Farey tessellation of $\mathbb{H}^3$

We now consider alternate tessellations of $\mathbb{H}^3$ under $\mathcal{P}$ [9], [34]. Recall the Picard Group is given by $\mathcal{P} = \langle \tau_1, \tau_i, \phi, \psi \rangle$ where $\tau_1(z) = z + 1$, $\tau_i(z) = z + i$, $\phi(z) = \dfrac{i}{iz}$ and $\psi(z) = \dfrac{iz}{-i}$.

While we have established that a fundamental domain of $\mathcal{P}$ is given as

$$\mathcal{D} = \left\{\omega = z + tj \in \mathbb{H}^3 : |\omega| > 1, |\Re(z)| < \frac{1}{2}, 0 < \Im(z) < \frac{1}{2}\right\},$$

we choose an equivalent domain as $\tau_1(\mathcal{D}_1) \cup \mathcal{D}_2$ where $\mathcal{D}_1 = \left\{\omega \in \mathcal{D} : -\frac{1}{2} < \Re(z) < 0\right\}$ and $\mathcal{D}_2 = \left\{\omega \in \mathcal{D} : 0 < \Re(z) < \frac{1}{2}\right\}$ and $\tau_1(z) = z + 1$. Since these are all fundamental domains and are all equivalents, in the sequel we will use $\mathcal{D}$ to denote $\tau_1(\mathcal{D}_1) \cup \mathcal{D}_2$ where $\tau_1(z) = z + 1$.

In this form the fundamental domain is bounded by planes $x = 0$, $x = 1$, $y = 0$, $y = \frac{1}{2}$, and spheres $|\omega| = 1$ and $|\omega - 1| = 1$. We also refer to this fundamental domain as $\mathcal{D}$.

We follow Vulakh [49] in the construction of the v-cell, $\mathcal{N}(v)$. We write $\overline{\mathcal{D}}$ for the closure of $\mathcal{D}$ in $\mathbb{H}^3$. Let $Stab(\infty, \mathcal{P}) = \langle \tau_1, \tau_i, \psi \rangle$. Set $K = K(\infty) = \{g(\overline{\mathcal{D}}) | g \in Stab(\infty, \mathcal{P})\}$ with boundary ∂K. Since $g(\infty) = \infty$ for all $g \in Stab(\infty, \mathcal{P})$, $K(\infty)$ is the union of images of $\overline{\mathcal{D}}$ each with a vertex at ∞.

The intersection, $\partial K \cap \overline{\mathcal{D}}$, is called the floor of $\mathcal{D}$. If $g \in \mathcal{P}$, then we have shown that $g(\infty)$ is in $\mathbb{Q}_\infty(i)$. We note $g(K(\infty)) := K(g(\infty))$, the union of images of $\overline{\mathcal{D}}$ with vertices at $g(\infty)$ for $g \in \mathcal{P}$.

We note that ∂K is composed of components of dimension of 0, called vertices of K; of dimension 1, called edges of K; and dimension 2, called faces of K. The vertices, edges and faces of K that belong to $\overline{\mathcal{D}}$ will be called vertices, edges and faces of $\mathcal{D}$. We now choose $\mathcal{D}_2 \cup \tau_1(\mathcal{D}_1)$ as a fundamental domain and without loss of generality denote it again by $\mathcal{D}$. We note that $v = \frac{1}{2} + \frac{1}{2}i + \frac{1}{\sqrt{2}}j$ is the only vertex of $\mathcal{D}$. The points j, $\frac{1}{2} + \frac{\sqrt{3}}{2}j$, $1 + j$, $1 + \frac{1}{2}i + \frac{\sqrt{3}}{2}j$ and $\frac{1}{2}i + \frac{\sqrt{3}}{2}j$ are not considered to be vertices of $\mathcal{D}$, since they are not vertices of K even though multiple planes meet at these points.

We note that a face, ϕ_t (component of dimension 2) of K lies on the isometric sphere of $g_t \in \mathcal{P}$ with centre at $g_t(\infty)$ in $\mathbb{Q}_\infty(i)$, $g_t \in Stab(\infty, \mathcal{P})$. The non-vertical faces of $\mathcal{D}$ lie on the isometric spheres centred at 0 and 1 with unit radius.

The projection, p, of the face ϕ_t from ∞ onto $\mathbb{C}$ is the polytope with $p(g_t(\infty)) = p(u_t)$, $u_t = g_t(\infty)$. These squares tessellate $\mathbb{C}$ where $\mathbb{C}_\infty = \partial\mathbb{H}^3$. Certainly the unit square with vertices $\{0, 1, i, 1 + i\}$ and centre at $\frac{1}{1-i}$ tessellate $\mathbb{C}$ under $Stab(\infty, \mathcal{P})$ to form a centred lattice of $\mathbb{C}$.

The vertex v of $\mathcal{D}$ and $Stab(v,\mathcal{P})$ play a central role in establishing the Farey tessellation of $\mathbb{H}^3$. It is shown [18] that $Stab(v,\mathcal{P})$ is isomorphic to A_4, the alternating group on 4 elements. In fact $Stab(v,\mathcal{P})$ has a presentation of the form

$$Stab(v,\mathcal{P}) = \left\langle \sigma_i, \sigma_1 : \sigma_i^3 = \sigma_1^3 = (\sigma_i\sigma_1)^2 = 1 \right\rangle$$

where $\sigma_i = \phi\tau_i^{-1}$ and $\sigma_1 = \tau_1\varphi$ and $\sigma = \sigma_i\sigma_1$. That is, $\sigma_i(z) = \dfrac{i}{iz+1}$ and $\sigma_1(z) = \dfrac{-1}{z} + 1$ and $\sigma(z) = \dfrac{iz}{(1+i)z - i}$ [12], [13], [37]. These twelve elements of $Stab(v,\mathcal{P})$ act on $\mathcal{D} = \mathcal{D}_2 \cup \tau_1(\mathcal{D}_1)$.

The union, $\bigcup_{g\in Stab(v,\mathcal{P})} g(\overline{\mathcal{D}})$ is a closed ideal octahedron in $\mathbb{H}^3$ with vertices at ∞, 0, 1, i, $1+i$ and $\frac{1}{1-i}$ and is called the v-cell $\mathcal{N}(v)$.

Since $\mathbb{H}^3$ is tessellated by $\mathcal{D}$ under $\mathcal{P}$, so too is $\mathbb{H}^3$ tessellated by $\mathcal{N}(v)$ under $\mathcal{P}$. This tessellation is invariant under $\mathcal{P}$. Hence we have:

Theorem 3.1.1. *$\mathbb{H}^3$ is tessellated by $\mathcal{N}(v)$ under $\mathcal{P}$. In fact each octahedron in this tessellation is in the orbit of $\mathcal{N}(v)$ under $\mathcal{P}$.*

Definition 3.1.2. *This tessellation of $\mathbb{H}^3$ by $\mathcal{N}(v)$ under $\mathcal{P}$ is called the Farey tessellation of $\mathbb{H}^3$.*

Vulakh [49] notes that when the fundamental domain $\mathcal{D} = \mathcal{D}_2 \cup \tau_1(\mathcal{D}_1)$ has only one vertex, $v \neq \infty$ and v not on a lateral face of $\mathcal{N}(v)$, then $\mathcal{N}(v)$ is a fundamental domain of a subgroup of $\mathcal{P}$ whose index in $\mathcal{P}$ is equal to $|Stab(v,\mathcal{P})|$. We recall that vertices on vertical and lateral faces are not considered as vertices.

Before examining the Farey tessellation in more detail, we will show that we can find a unique normal subgroup of $\mathcal{P}$ whose fundamental region is the interior of $\mathcal{N}(v)$.

Following Fine and Newman [13], we can construct this normal subgroup from the commutator subgroups of a group. We recall that if G is a group, then the commutator subgroup G' of G is the subgroup of G generated by all commutators of G. That is, if x and y are in G, then $xyx^{-1}y^{-1} = [x, y]$ is a commutator and G' is the smallest subgroup of G containing all the commutators.

We write $G' = [G : G]$ and note that $G' \triangleleft G$, G/G' is abelian and isomorphic to G^{abl} where G^{abl} is the maximal abelian subgroup of G. In our case we have $\mathcal{P}' = [\mathcal{P}, \mathcal{P}]$ and $\mathcal{P}' \triangleleft \mathcal{P}$ with $\mathcal{P}/\mathcal{P}' \cong D_2$ (Klein 4-group). That is, $|\mathcal{P} : \mathcal{P}'| = 4$, where $\mathcal{P}^{abl} = D_2$ [13].

We then consider $\mathcal{P}'' = [\mathcal{P}', \mathcal{P}']$ so that $\mathcal{P}'' \triangleleft \mathcal{P}'$ and $\mathcal{P}'/\mathcal{P}''$ is abelian and isomorphic to $\mathcal{P}'^{abl}$ or C_3 the cyclic group of order 3 [13].

So $|\mathcal{P}' : \mathcal{P}''| = 3$. Since $\mathcal{P}'' \triangleleft \mathcal{P}' \triangleleft \mathcal{P}$ we have that $|\mathcal{P} : \mathcal{P}''| = |\mathcal{P} : \mathcal{P}'||\mathcal{P}' : \mathcal{P}''| = 4 \cdot 3 = 12$.

Fine and Newman, [13], further establish that $\mathcal{P}''$ is the unique normal subgroup of $\mathcal{P}$ of index 12. In fact $\mathcal{P}/\mathcal{P}'' \cong S_3 \times C_2$. It is also established in [13] that $\mathcal{P}''$ is the smallest normal subgroup of $\mathcal{P}$ containing π where $\pi = -z + (1 + i) = \sigma_1\sigma_i$ and $\pi^2 = 1$.

This subgroup of $\mathcal{P}$ contains all conjugates of π by elements in $\mathcal{P}$. In particular π, $\tau_1\pi\tau_1^{-1}$, $\tau_i\pi\tau_i^{-1}$, $\varphi_1\pi\varphi^{-1}$, $\psi\pi\psi^{-1}$ and their inverses are in this $\mathcal{P}''$.

Fine and Newman [13] give a presentation for $\mathcal{P}'' = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ with the relators $x_1^2 = x_2^2 = x_3^2 = x_4^2 = 1$ and

$$(x_1x_2)^2 = (x_2x_5)^2 = (x_1x_6)^2 = (x_4x_5)^2 = (x_3x_6)^2 = (x_1x_6x_5x_2)^2 = (x_6x_5x_4x_3)^2 = 1$$

where the elements x_1 to x_6 are as follows:

$$x_1 = \tau_{1+i}\psi$$

$$x_2 = \varphi\tau_{i-1}\phi$$

$$\begin{aligned} x_3 &= \tau_{1-i}\psi \\ x_4 &= \varphi\tau_{1+i}^{-1} \\ x_5 &= \varphi\tau_1^2\varphi \\ x_6 &= \tau_1^2 \end{aligned}$$

Certainly the generators of $\mathcal{P}''$ associate the edges of $\mathcal{N}(v)$, satisfying the Poincaré Theorem [6].

Then $\mathcal{P}''$ has the interior $\mathcal{N}(v)$ as a fundamental domain.

As seen, the non-vertical faces of the v-cell $\mathcal{N}(v)$ are surfaces of isometric spheres with radii $\frac{1}{2}$ with centres at $\frac{1}{2}$, $\frac{i}{2}$, $\frac{1}{2}+i$, $1+\frac{i}{2}$. These spheres meet at the point $\frac{1}{1-i}$ and $\sigma(\infty) = \frac{1}{1-i}$.

Since $Stab(v,\mathcal{P}) \cong A_4$, then $Stab(v,\infty;\mathcal{P}) = \{1, \tau_{1+i}\psi\} = Stab(v,\mathcal{P}) \cap Stab(\infty,\mathcal{P})$ and $|Stab(v,\mathcal{P}) : Stab(v,\infty;\mathcal{P})| = \frac{12}{2} = 6$. Thus $Stab(v,\mathcal{P})$ acting on ∞ yields the six vertices of the v-cell $\mathcal{N}(v)$ as $V = \left\{\infty, 0, 1, 1+i, i, \dfrac{1}{1-i}\right\}$.

For any $g \in \mathcal{P}$, the set $g(\mathcal{N}(v)) := \mathcal{N}(g(v))$ will also be called a v-cell. We show that all possible images of $\mathcal{N}(v)$ are ideal octahedra.

Lemma 3.1.3. *If $g \in \mathcal{P}$ and $\mathcal{N}(v)$ is the v-cell (ideal octahedron with vertices V) then $g(\mathcal{N}(v))$ is an ideal octahedron with vertices $g(V)$.*

Proof. $\mathcal{N}(v)$ is an ideal octahedron with all six vertices on $\mathbb{C}$. These vertices are ∞, 0, 1, i, $1+i$ and $\frac{1}{1-i}$.

Since $g \in \mathcal{P}$ with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i]$ and the vertices are also elements in $\mathbb{C}$, the images of the vertices will be complex numbers also. □

3.2 The Ford Spheres in $\mathbb{H}^3$

Let $g \in \mathcal{P}$ where $g(\infty) \neq \infty$. Let a Ford sphere, $\mathcal{S}_{g(\infty)}$, be the boundary of the horoball, or open Euclidean ball in $\mathbb{H}^3$ tangent to $\mathbb{C}_\infty$ at $g(\infty)$ having radius $\frac{r^2}{2}$ where $r = r(g)$ is the radius of the isometric sphere $I(g)$.

If $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\gamma \neq 0$ then $I(g) : \left| z - \left(\dfrac{-\delta}{\gamma}\right)\right| = \dfrac{1}{|\gamma|}$ and $r = \dfrac{1}{|\gamma|}$.

Recall that $\mathcal{S}_\infty$, the Ford sphere at ∞ is given by the plane $t = 1$. So the radius of $\mathcal{S}_{g(\infty)}$, $g(\infty) \neq \infty$ is $\dfrac{1}{2|\gamma|}$. We recall that $g(\infty)$ is in $\mathbb{Q}_\infty(i)$ and is a reduced Gaussian rational [39], [47].

We follow Ford [15] in the next result.

Theorem 3.2.1. *A Ford sphere is either adjacent or wholly external from another Ford sphere. In general, if the Ford spheres are based on $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta} \in \mathbb{C}$, then the spheres are adjacent if $|\alpha\delta - \beta\gamma| = 1$ and wholly external if $|\alpha\delta - \beta\gamma| > 1$.*

Proof. To prove that a Ford sphere is wholly external, adjacent or intersecting another Ford sphere, we consider the centres and radii of the two spheres. If the distance between the centers of the sphere is greater than the sum of the radii, then the spheres are wholly external. For adjacent spheres, the distance equals the sum of the radii, and if the distance is less than the sum of radii, then the spheres intersect.

$\mathcal{S}_{\frac{\alpha}{\gamma}}$ has centre $\dfrac{\alpha}{\gamma} + \dfrac{1}{2|\gamma|^2} j$ and radius $\dfrac{1}{2|\gamma|^2}$, similarly for $\mathcal{S}_{\frac{\beta}{\delta}}$, the centre is $\dfrac{\beta}{\delta} + \dfrac{1}{2|\delta|^2} j$ and radius is $\dfrac{1}{2|\delta|^2}$.

Then

$$\left|\left(\frac{\beta}{\delta} + \frac{1}{2|\delta|^2} j\right) - \left(\frac{\alpha}{\gamma} + \frac{1}{2|\gamma|^2} j\right)\right|^2 = \left(\frac{1}{2|\delta|^2} + \frac{1}{2|\gamma|^2}\right)^2 + \frac{|\alpha\delta - \beta\gamma|^2 - 1}{|\gamma\delta|^2}$$

The expression $|\alpha\delta - \beta\gamma|^2$ decides the interaction of the spheres. Since $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i]$ we know that $|\alpha\delta - \beta\gamma|^2$ can be 0, 1 or greater than 1. For $|\alpha\delta - \beta\gamma|^2 = 0$, $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ are the same fraction. For adjacent spheres, then $|\alpha\delta - \beta\gamma|^2 = 1$. $|\alpha\delta - \beta\gamma|^2 > 1$ results in wholly external spheres. □

We recall that if $\gamma = 0$, then $\mathcal{S}_\infty$ or $\mathcal{S}_{\frac{1}{0}}$ is defined to be the plane: $\left\{z + tj \in \mathbb{H}^3 : t = 1\right\}$. Thus $\mathcal{S}_\infty$ is adjacent to all $\mathcal{S}_{\frac{\alpha}{1}} = \mathcal{S}_\alpha$, where $\alpha \in \mathbb{Z}[i]$, since $|1 \cdot 1 - \alpha \cdot 0| = 1$ for all $\alpha \in \mathbb{Z}[i]$. Further, if $\mathcal{S}_{\frac{\alpha}{\gamma}}$ is adjacent to $\mathcal{S}_{\frac{1}{0}}$, then $|\gamma| = 1$ so γ is a unit and $\frac{\alpha}{\gamma} = \frac{\alpha'}{1}$ where $\alpha' \in \mathbb{Z}[i]$.

The open Euclidean horoballs are denoted by $\mathcal{R}_{g(\infty)}$ or $\mathcal{R}_\infty$ where $\mathcal{R}_\infty = \left\{z + tj \in \mathbb{H}^3 : t > 1\right\}$. The exterior of all the open Euclidean horoballs in $\mathbb{H}^3 \cup \mathbb{Q}_\infty(i)$ for $g \in \mathcal{P}$ is called the mesh.

The theorems on the octahedra have natural duals with respect to the Ford spheres. As such the following Theorem is a dual to the Lemma 4.3.1.

Theorem 3.2.2. *All Ford Spheres are in the orbit of the Ford sphere $\mathcal{S}_\infty$ under the action of the Picard group.*

Proof. Let $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ be adjacent Ford spheres touching $\mathbb{C}_\infty$ at $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ respectively. By Theorem 3.2.1 we have $|\alpha\delta - \beta\gamma| = 1$. Thus we can find $h \in \mathcal{P}$ with $h(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$; $h(\infty) = \frac{\alpha}{\gamma}$ and $h(0) = \frac{\beta}{\delta}$ or $h(z) = \dfrac{\beta z + \alpha}{\delta z + \gamma}$ with $h(\infty) = \frac{\beta}{\delta}$ and $h(0) = \frac{\alpha}{\gamma}$.

Then $h^{-1}\left(\mathcal{S}_{\frac{\alpha}{\gamma}}\right) = \mathcal{S}_\infty$ is adjacent to $h^{-1}\left(\mathcal{S}_{\frac{\beta}{\delta}}\right) = \mathcal{S}_0$ touching $\mathbb{C}_\infty$ at $h^{-1}\left(\frac{\alpha}{\gamma}\right) = \infty$ and $h^{-1}\left(\frac{\beta}{\delta}\right) = 0$ respectively. Since $h \in \mathcal{M}$, and $\mathcal{S}_\infty$ is adjacent to $\mathcal{S}_0$ at the point j, the two spheres $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ are adjacent at $h(j)$.

Certainly for any sphere $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\frac{\alpha}{\gamma} \in \mathbb{Q}_\infty(i)$, we can find infinitely many $\frac{\beta}{\delta}$ in $\mathbb{Q}_\infty(i)$ such that $|\alpha\delta - \beta\gamma| = 1$. Thus we can find $h \in \mathcal{P}$ with $h(\mathcal{S}_\infty) = \mathcal{S}_{\frac{\alpha}{\gamma}}$. Thus each Ford sphere is in the orbit of $\mathcal{S}_\infty$ under $\mathcal{P}$. □

Theorem 3.2.3. *Let $\mathcal{S}_{\frac{\alpha}{\gamma}}$ be the Ford sphere at $\frac{\alpha}{\gamma} \in \mathbb{Q}_\infty(i)$.*

1. *At least one $\frac{\beta}{\delta} \in \mathbb{Q}_\infty(i)$ can be found such that $|\alpha\delta - \beta\gamma| = 1$. So $\mathcal{S}_{\frac{\beta}{\delta}}$ is adjacent to $\mathcal{S}_{\frac{\alpha}{\gamma}}$.*

2. *If $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ are adjacent Ford spheres, then all adjacent Ford spheres to $\mathcal{S}_{\frac{\alpha}{\gamma}}$ are $\mathcal{S}_{\frac{\beta_n}{\delta_n}}$ where $\frac{\beta_n}{\delta_n} = \frac{\epsilon_n\alpha+\beta}{\epsilon_n\gamma+\delta}$ and $\epsilon_n \in \mathbb{Z}[i]$.*

Proof. 1. Given that α and $\gamma \in \mathbb{Z}[i]$ are reduced and co-prime, by Lemma 2.2.5 we can find β and $\delta \in \mathbb{Z}[i]$ such that $|\alpha\delta - \beta\gamma| = 1$. Thus by Theorem 3.2.1, $\mathcal{S}_{\frac{\alpha}{\gamma}}$ is adjacent to $\mathcal{S}_{\frac{\beta}{\delta}}$.

2. Let $\frac{\beta_n}{\delta_n} = \frac{\epsilon_n\alpha+\beta}{\epsilon_n\gamma+\delta}$. Then $|\alpha(\epsilon_n\gamma + \delta) - (\epsilon_n\alpha + \beta)\gamma| = |\alpha\delta - \beta\gamma| = 1$, since $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ are adjacent Ford spheres. Assume $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\upsilon}{\omega}}$ are adjacent. By Theorem 3.2.2, we have $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ with $g \in \mathcal{P}$ and g^{-1} maps $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ to $\mathcal{S}_\infty$ and $\mathcal{S}_{\frac{0}{1}}$ respectively. Thus $\mathcal{S}_{g^{-1}(\frac{\upsilon}{\omega})}$ is adjacent to $\mathcal{S}_\infty$. Let $g^{-1}\left(\frac{\upsilon}{\omega}\right) = \frac{\upsilon'}{\omega'}$. Since $\mathcal{S}_{\frac{\upsilon'}{\omega'}}$ is adjacent to $\mathcal{S}_{\frac{1}{0}}$, $|\omega'| = 1$ and ω' is a unit. Thus $\frac{\upsilon'}{\omega'} = \frac{\epsilon}{1}$ where $\epsilon \in \mathbb{Z}[i]$. Thus $\frac{\upsilon}{\omega} = g(\epsilon) = \dfrac{\alpha\epsilon + \beta}{\gamma\epsilon + \delta}$ as required.

□

Ford [16] notes that since $\mathcal{S}_\infty$ is the Ford sphere given as $t = 1$, the image $h(\mathcal{S}_\infty) = \mathcal{S}_{h(\infty)}$ is the sphere tangent to $\mathbb{C}$ at $h(\infty) = \dfrac{\alpha}{\gamma}$ where $h(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ and $h \in \mathcal{P}$.

Ford also notes that the part of $\mathbb{H}^3$ above the plane $\mathcal{S}_\infty$ is transformed to the interior of $h(\mathcal{S}_\infty)$, while the part of $\mathbb{H}^3$ below $\mathcal{S}_\infty$ is mapped to the exterior of $h(\mathcal{S}_\infty) = \mathcal{S}_{h(\infty)}$. Since $\mathcal{S}_\infty$ is tangential to the base of $\mathcal{D}$, the image of $\mathcal{S}_\infty$ is tangential to the bases of all the pentahedron with peaks at $\dfrac{\alpha}{\gamma}$ where $h(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $h \in \mathcal{P}$.

Ford further notes that the sphere $\mathcal{S}_{\frac{\alpha'}{\gamma'}}$ will be adjacent to a sphere $\mathcal{S}_{\frac{\alpha}{\gamma}}$ if a pentahedron with peak at $\dfrac{\alpha'}{\gamma'}$ has a base in common with a pentahedron with peak at $\dfrac{\alpha}{\gamma}$. In other cases the

spheres are entirely exterior to one another.

Ford proves the following result:

Theorem 3.2.4. *The necessary and sufficient condition that a pentahedron with peak at $\frac{\alpha'}{\gamma'}$ and a pentahedron with peak at $\frac{\alpha}{\gamma}$ have a common base is that $\alpha\gamma' - \gamma\alpha' = \pm 1, \pm i$.*

This result is recorded as [15].

We have from Ford [15]:

Theorem 3.2.5. *For any pair of adjacent Ford spheres, there are 4 distinct Ford spheres that are simultaneously adjacent to each sphere.*

Proof. Consider $\mathcal{S}_\infty$ and $\mathcal{S}_0$ adjacent at j. Then $\mathcal{S}_1$, $\mathcal{S}_i$, $\mathcal{S}_{-1}$, $\mathcal{S}_{-i}$ are adjacent to both $\mathcal{S}_\infty$ and $\mathcal{S}_0$. By Theorem 3.2.2, since any pair of adjacent spheres can be mapped back to $\mathcal{S}_\infty$ and $\mathcal{S}_0$ by $g^{-1} \in \mathcal{P}$, we have $\mathcal{S}_{g(1)}$, $\mathcal{S}_{g(i)}$, $\mathcal{S}_{g(-1)}$, $\mathcal{S}_{g(-i)}$ will be adjacent to the given pair of Ford spheres. □

Definition 3.2.6. *The right side of the vertical plane in $\mathbb{C}_\infty$ formed by 0, i and ∞ is defined to be the inside of the generalized sphere circumscribing ∞, 0 and i.*

Theorem 3.2.7. *If we have three Ford spheres that are mutually adjacent, then we can find exactly three spheres, each adjacent to two of the original three spheres, that are also mutually adjacent and whose points of tangency to $\mathbb{C}_\infty$ lie inside the circumscribed circle of the original three tangency points.*

Proof. Since Theorem 3.2.2 shows that all Ford spheres are in the orbit of $\mathcal{S}_\infty$, we can look at a general situation with $\mathcal{S}_\infty$ and map the general situation to the specific case using $g \in \mathcal{P}$.

Consider $\mathcal{S}_\infty$, $\mathcal{S}_0$ and $\mathcal{S}_i$ which are mutually adjacent. Taking each pair of Ford spheres, we find the set of mutually adjacent Ford spheres of which there are four in each set.

$$\mathcal{S}_\infty \text{ and } \mathcal{S}_0 \text{ have } \{\mathcal{S}_1, \mathcal{S}_i, \mathcal{S}_{-1}, \mathcal{S}_{-i}\} \text{ as an adjacent set,}$$

$$\mathcal{S}_\infty \text{ and } \mathcal{S}_i \text{ have } \{\mathcal{S}_0, \mathcal{S}_{1+i}, \mathcal{S}_{i-1}, \mathcal{S}_{2i}\} \text{ as an adjacent set and}$$

$$\mathcal{S}_0 \text{ and } \mathcal{S}_i \text{ have } \left\{\mathcal{S}_\infty, \mathcal{S}_{\frac{-1}{1+i}}, \mathcal{S}_{\frac{i}{2}}, \mathcal{S}_{\frac{1}{1-i}}\right\} \text{ as an adjacent set.}$$

Then $\mathcal{S}_1$, $\mathcal{S}_{\frac{1}{1-i}}$ and $\mathcal{S}_{1+i}$ are three spheres with tangency points inside this circumscribing sphere. These spheres $\mathcal{S}_1$, $\mathcal{S}_{\frac{1}{1-i}}$, $\mathcal{S}_{1+i}$ are adjacent to two of the original Ford spheres, $\mathcal{S}_\infty$, $\mathcal{S}_0$, $\mathcal{S}_i$, and are adjacent to the other two spheres. □

Chapter 4

Farey Triangles and the Farey Tessellation

We know that geodesics in $\mathbb{H}^3$ are arcs of semicircles in $\mathbb{H}^3$ orthogonal to the boundary $\mathbb{C}_\infty$ of $\mathbb{H}^3$ including vertical rays in $\mathbb{H}^3$ orthogonal to $\mathbb{C}$. In what follows, we distinguish a particular class of geodesics in $\mathbb{H}^3$ and hence a particular class of triangles in $\mathbb{H}^3$ too.

4.1 Farey neighbours and Farey geodesics

We know that $\mathbb{Q}(i) = \left\{ \frac{a+ib}{r} : a, b, r \in \mathbb{Z}, r \neq 0 \right\}$ is the field of quotients of $\mathbb{Z}[i]$. We extend this field to include ∞ in the usual way and write $\mathbb{Q}_\infty(i) = \mathbb{Q}(i) \cup \{\infty\}$. Let $U = \{\pm 1, \pm i\}$ be the multiplicative group of units in $\mathbb{Z}[i]$. Further, $N(\mu) = 1$ if and only if μ is a unit. We also recall that α and β in $\mathbb{Z}[i]$ are co-prime if $gcd(\alpha, \beta) = \mu$, where μ is a unit in $\mathbb{Z}[i]$. We know from Lemma 2.2.5 that $gcd(\alpha, \beta) = \mu$, μ a unit if and only if we can find γ, δ in $\mathbb{Z}[i]$

such that $|\alpha\delta - \beta\gamma| = 1$. From Definition 2.2.6 we know that a rational, $\frac{\alpha}{\gamma}$, $\gamma \neq 0$ in $\mathbb{Q}(i)$ is said to be reduced if $gcd(\alpha, \gamma) = \mu$, where μ is a unit or equivalently we can find δ, β in $\mathbb{Z}[i]$ such that $|\alpha\delta - \beta\gamma| = 1$.

We note that ∞ has a reduced form as $\frac{1}{0}$. Further, we note that if $\frac{\alpha}{\gamma}$ is reduced, then $\frac{\mu\alpha}{\mu\gamma}$ is reduced for any unit μ in $\mathbb{Z}[i]$. If $\frac{\alpha}{\gamma} \in \mathbb{Q}_\infty(i)$ and $N(\gamma) = 1$, then γ is a unit and we may reduce $\frac{\alpha}{\gamma}$ to $\frac{\alpha\overline{\gamma}}{1}$ where $\alpha\overline{\gamma} \in \mathbb{Z}[i]$ with $N(\gamma) = \gamma\overline{\gamma} = 1$.

Definition 4.1.1. *Two reduced rationals, $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ in $\mathbb{Q}_\infty(i)$ are said to be Farey neighbours if $|\alpha\delta - \beta\gamma| = 1$. We write $\frac{\alpha}{\gamma} \sim \frac{\beta}{\delta}$. In fact each reduced rational $\frac{\alpha}{\gamma}$ has infinitely many Farey neighbours.*

We note that $\frac{1}{0} \sim \frac{\beta}{1}$ for all $\beta \in \mathbb{Z}[i]$. It is clear that if $\frac{\alpha}{\gamma} \sim \frac{\beta}{\delta}$, we can find $g \in \tilde{\mathcal{P}}$ such that $g(\infty) = \frac{\alpha}{\gamma}$ and $g(0) = \frac{\beta}{\delta}$. Actually if $\frac{\alpha}{\gamma} \sim \frac{\beta}{\delta}$, then either $g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ or $g(z) = \frac{\alpha i z + \beta}{\gamma i z + \delta}$ are in $\mathcal{P}$, where $g(\infty) = \frac{\alpha}{\gamma}$ and $g(0) = \frac{\beta}{\delta}$ in either case. Hence we can find $g \in \mathcal{P}$ such that $g(\infty) = \frac{\alpha}{\gamma}$ and $g(0) = \frac{\beta}{\delta}$. Further, it is noted that if $\frac{\alpha}{\gamma} \sim \frac{\beta}{\delta}$, then $|\alpha\delta - \beta\gamma| = 1$.

The following useful result about Farey neighbours can be established.

Lemma 4.1.2. *If $\frac{\alpha}{\gamma} \sim \frac{\beta}{\delta}$ then $h\left(\frac{\alpha}{\gamma}\right) \sim h\left(\frac{\beta}{\delta}\right)$ for $h \in \mathcal{P}$.*

Proof. Since $\frac{\alpha}{\gamma} \sim \frac{\beta}{\delta}$, we know $|\alpha\delta - \beta\gamma| = 1$ and $\frac{\alpha}{\gamma} = g(\infty)$, $\frac{\beta}{\delta} = g(0)$, $g \in \mathcal{P}$.

Thus $h\left(\frac{\alpha}{\gamma}\right) = hg(\infty)$ and $h\left(\frac{\beta}{\delta}\right) = hg(0)$ with $hg \in \mathcal{P}$. That is, we can find $\alpha', \beta', \gamma', \delta'$ with $hg(z) = \frac{\alpha' z + \beta'}{\gamma' z + \delta'}$, $hg \in \mathcal{P}$ and $hg(\infty) = \frac{\alpha'}{\gamma'}$, $hg(0) = \frac{\beta'}{\delta'}$.

Since $|\alpha'\delta' - \beta'\gamma'| = 1$, we have $\frac{\alpha'}{\gamma'} \sim \frac{\beta'}{\delta'}$ or $h\left(\frac{\alpha}{\gamma}\right) \sim h\left(\frac{\beta}{\delta}\right)$ as required. □

Definition 4.1.3. *A geodesic in $\mathbb{H}^3$ is called a Farey geodesic if its endpoints on $\mathbb{C}_\infty$ are Farey neighbours.*

If $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ are the endpoints of a geodesic we may write the geodesic as $\left[\frac{\alpha}{\gamma} : \frac{\beta}{\delta}\right]$ or $\Lambda_{\frac{\alpha}{\gamma},\frac{\beta}{\delta}}$. In particular, $\Lambda_{\frac{1}{0},\frac{0}{1}} = \left[\frac{1}{0} : \frac{0}{1}\right]$ is a Farey geodesic. We refer to it as the fundamental Farey geodesic, denoted by $\mathbb{I}_0$.

We know that any Möbius transformation g maps circles and spheres conformally to circles and spheres, and hence g maps geodesics to geodesics in $\mathbb{H}^3$. We note that $g \in \mathcal{M}$ may map the interior of a circle or sphere to either the interior or exterior of its image. It follows that if Λ is a Farey geodesic with endpoints $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ in $\mathbb{C}$ and $h \in \mathcal{P} \subseteq \mathcal{M}$, then $h(\Lambda)$ has endpoints $h\left(\frac{\alpha}{\gamma}\right)$ and $h\left(\frac{\beta}{\delta}\right)$, which are Farey neighbours, and thus $h(\Lambda)$ is a Farey geodesic.

In fact we have:

Lemma 4.1.4. *The set of all Farey geodesics in $\mathbb{H}^3$ is exactly the orbit of $\mathbb{I}_0$ under $\mathcal{P}$.*

We note the 12 geodesic edges of the octahedron $\mathcal{N}(v)$ with vertices $\left\{\infty, 0, 1, i, 1+i, \frac{1}{1-i}\right\}$ that tessellates $\mathbb{H}^3$ under $\mathcal{P}$ are all Farey geodesics.

We notice that adjacent Ford spheres and Farey neighbours have the same condition of $|\alpha\delta - \beta\gamma| = 1$. Thus we make the link between the two:

Corollary 4.1.5. *Two Ford spheres $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ are adjacent if and only if $|\alpha\delta - \beta\gamma| = 1$ if and only if $\Lambda_{\frac{\alpha}{\gamma},\frac{\beta}{\delta}}$ is a Farey geodesic.*

We note that there is a correspondence between the Ford sphere $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and the parabolic fixed point $\frac{\alpha}{\gamma}$ on $\mathbb{C}$. Further, there is a correspondence between the Farey geodesic $\Lambda_{\frac{\alpha}{\gamma},\frac{\beta}{\delta}}$ and the

adjacency point between $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$. So the adjacency points between $\mathcal{S}_{\frac{\alpha}{\gamma}}$ and $\mathcal{S}_{\frac{\beta}{\delta}}$ are in the orbit of j under $\mathcal{P}$.

Hence we note Ford spheres are in the orbit of $\mathcal{S}_{\frac{1}{0}}$ under $\mathcal{P}$, while the Farey geodesics are in the orbit of $\mathbb{I}_0$ under $\mathcal{P}$. This duality results in the extension of theorems on Farey geodesics to theorems on Ford spheres and vice versa.

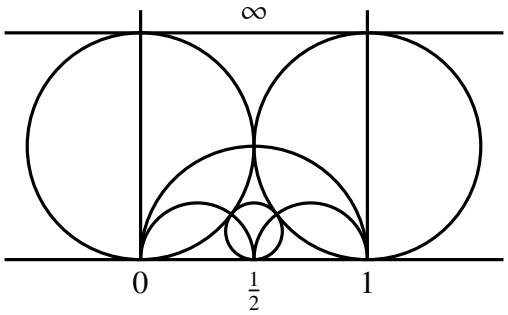


Figure 4.1: The duality of Farey geodesics and Ford spheres in $\mathbb{H}^{\perp}$.

4.2 Hyperbolic triangles and Farey triangles in $\mathbb{H}^3$

The ideal triangle with vertices ∞, 0 and 1 on $\mathbb{C}_\infty$ is called the fundamental triangle and denoted by $\mathbb{T}_0$. The sides of $\mathbb{T}_0$ are $\Lambda_{\infty,0}$, $\Lambda_{0,1}$ and $\Lambda_{1,\infty}$. These geodesics are all Farey geodesics and together with $\mathbb{T}_0$ play a central role in the development of the Farey tessellation of $\mathbb{H}^3$ by the ideal octahedron, $\mathcal{N}(v)$.

Definition 4.2.1. *A triangle in $\mathbb{H}^3$ is a Farey triangle if the bounding geodesics or sides of the triangle are all Farey geodesics. Thus Farey triangles are ideal triangles whose vertices, in pairs, are Farey neighbours.*

In particular $\mathbb{T}_0$ is a Farey triangle. If ∞ is not a vertex of the triangle $\mathbb{T}$, then the projection from ∞ onto $\mathbb{C}$ of $\mathbb{T}$ is an Euclidean triangle with the same vertices as $\mathbb{T}$. We may call this

triangle in $\mathbb{C}$ the polytope of $\mathbb{T}$. In this case, we will use the vertices of $\mathbb{T}$ to refer to either the Farey triangle or its polytope. If ∞ is a vertex of a Farey triangle then its polytope in $\mathbb{C}_\infty$ is a straight line. That is, the projection of $\mathbb{T}_0$ can be regarded as the extended Real axis, while the projection of $\mathbb{T} = \left\{\frac{1}{0}, \frac{\alpha}{1}, \frac{\beta}{1}\right\}$ where $\beta = \alpha \pm 1$ or $\alpha \pm i$, is a vertical or horizontal line in $\mathbb{C}_\infty$. Such triangles are called degenerate triangles with straight line polytopes.

4.3 Generalised Farey sets of Farey triangles

Lemma 4.3.1. *Let $\mathbb{T}_0 = \{\infty, 0, 1\}$ be the fundamental Farey triangle. Then every Farey triangle $\mathbb{T}$ in $\mathbb{H}^3$ can be written as $g(\mathbb{T}_0)$ for some $g \in \mathcal{P}$.*

Proof. Consider any Farey triangle in $\mathbb{H}^3$ with vertices $\frac{\alpha_1}{\gamma_1}$, $\frac{\alpha_2}{\gamma_2}$ and $\frac{\alpha_3}{\gamma_3}$ where

$$|\alpha_1\gamma_2 - \alpha_2\gamma_1| = |\alpha_2\gamma_3 - \alpha_3\gamma_2| = |\alpha_1\gamma_3 - \alpha_3\gamma_1| = 1.$$

Let $\alpha_1\gamma_2 - \alpha_2\gamma_1 = u$, $\alpha_2\gamma_3 - \alpha_3\gamma_2 = v$ and $\alpha_1\gamma_3 - \alpha_3\gamma_1 = w$ where u, v and $w \in U = \{\pm 1, \pm i\}$.

Let $g(z) = \dfrac{\alpha_1 z + \alpha_2}{\gamma_1 z + \gamma_2}$ with $g^{-1}(z) = \dfrac{\gamma_2 z - \alpha_2}{-\gamma_1 z + \alpha_1}$.

Then $g^{-1}\left(\dfrac{\alpha_3}{\gamma_3}\right) = \dfrac{\gamma_2\alpha_3 - \alpha_2\gamma_3}{-\gamma_1\alpha_3 + \alpha_1\gamma_3} = \dfrac{-v}{w} = \dfrac{\pi}{\rho}$ where $\pi = -v$ and $\rho = w$ are units in $\mathbb{Z}[i]$ and clearly $\dfrac{\pi}{\rho}$ is a unit in $\mathbb{Z}[i]$. Thus $g\left(\dfrac{\pi}{\rho}\right) = \dfrac{\alpha_3}{\gamma_3}$.

Put $g_0(z) = \dfrac{\alpha_1\left(\frac{\pi}{\rho}\right)z + \alpha_2}{\gamma_1\left(\frac{\pi}{\rho}\right)z + \gamma_2} = \dfrac{\alpha_1\pi z + \alpha_2\rho}{\gamma_1\pi z + \gamma_2\rho}$, then $g_0(\infty) = \dfrac{\alpha_1}{\gamma_1}$, $g_0(0) = \dfrac{\alpha_2}{\gamma_2}$ and $g_0(1) = \dfrac{\alpha_1\pi + \alpha_2\rho}{\gamma_1\pi + \gamma_2\rho} = g\left(\frac{\pi}{\rho}\right) = \dfrac{\alpha_3}{\gamma_3}$.

We note that $|\pi\alpha_1\rho\gamma_2 - \rho\gamma_1\pi\alpha_2| = |\pi\rho||\alpha_1\gamma_2 - \gamma_1\alpha_2| = 1$ since π and ρ are units and $g \in \tilde{\mathcal{P}}$. If $g_0 \notin \mathcal{P}$ we may consider $g_0(z) = \dfrac{\alpha_1 i\pi z + \alpha_2\rho}{\gamma_1 i\pi z + \gamma_2\rho}$ and then $g_0 \in \mathcal{P}$. □

Definition 4.3.2. *Given a Farey triangle $\mathbb{T}$ in $\mathbb{H}^3$ with vertices $\frac{\alpha_t}{\gamma_t}$, $t = 1, 2, 3$, then each vertex can be written as a Farey sum of the other two vertices with respect to some unit μ in U. That is, there exists $\mu \in U$ such that $\frac{\alpha_i}{\gamma_i} = \frac{\mu\alpha_j + \alpha_k}{\mu\gamma_j + \gamma_k}$ where i, j, k are distinct and may be $1, 2$ or 3.*

Definition 4.3.3. *The norm of a Farey triangle is given by $N(\mathbb{T}) = N(\gamma_1) + N(\gamma_2) + N(\gamma_3)$ where $\mathbb{T}$ has vertices $\frac{\alpha_1}{\gamma_1}$, $\frac{\alpha_2}{\gamma_2}$, $\frac{\alpha_3}{\gamma_3}$ and $\mathbb{T}$ is a Farey triangle.*

We note that $N(\mathbb{T}_0) = 2$. In fact, $N(\mathbb{T}) = 2$ only for triangles $\mathbb{T}$ with ∞ as a vertex and that $N(\mathbb{T}) \geq 4$ if ∞ is not a vertex as in $\mathbb{T} = \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{1-i}\right\}$. We may order the vertices of $\mathbb{T}$ where $\frac{\alpha_i}{\gamma_i}$, $i = 1, 2, 3$ with $N(\gamma_1) \geq N(\gamma_2) \geq N(\gamma_3)$.

In what follows we will write $\mathbb{T}$ with $N(\gamma_1) \geq N(\gamma_2) \geq N(\gamma_3)$, and $g : \mathbb{T}_0 \to \mathbb{T}$ with $g(z) = \frac{\alpha_1\mu z + \alpha_2}{\gamma_1\mu z + \gamma_2}$, μ any unit. We make the following important observations about the ideal octahedron, $\mathcal{N}(v)$. These observations concur with the representation formulated by Schmidt [43, p.4].

Example 4.3.4. The octahedron $\mathcal{N}(v)$, $v = \frac{1}{2} + \frac{1}{2}i + \frac{1}{\sqrt{2}}j$ has 8 faces, each being a Farey triangle. Further we note that each face can be written as $g(\mathbb{T}_0)$ for some $g \in \tilde{\mathcal{P}}$. These maps $g \in \tilde{\mathcal{P}}$ correspond to the matrices given by Schmidt [43] as V_1, V_2, V_3, E_1, E_2, E_3, and C. We note that the maps g may involve $\kappa(z) = iz$, where $\kappa \notin \mathcal{P}$.

Let τ_1, τ_i, φ, ϕ be as before. Recall $\sigma_1 = \tau_1\varphi$ and $\sigma_i = \phi\tau_i^{-1}$ then:

$$\begin{aligned} \text{Let } c(z) =& \frac{z + (-1 + i)}{(1 - i)z + i} \\ =& \frac{1}{\frac{(1 - i)z + i}{z + (-1 + i)}} \\ =& \frac{1}{(1 - i) - \frac{i}{z + (-1 + i)}} \end{aligned}$$

$$= \quad \phi\tau_{1-i}\kappa^3\phi\tau_{1-i}^{-1}(z)$$

With corresponding matrix $M_c = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}$.

Then

$$\mathbb{T}_0 = \{\infty, 0, 1\} \text{ with } M_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tau_i(\mathbb{T}_0) = \{\infty, i, 1+1\}, \quad M_{\tau_i} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1^{-1}\tau_i\sigma_1(\mathbb{T}_0) = \left\{i, 0, \frac{1}{1-i}\right\}, \quad M_{\sigma_1^{-1}\tau_i\sigma_1} = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$$

$$\sigma_1\tau_i\sigma_1^{-1}(\mathbb{T}_0) = \left\{1+i, \frac{1}{1-i}, 1\right\}, \quad M_{\sigma_1\tau_i\sigma_1^{-1}} = \begin{pmatrix} 1-i & i \\ -i & 1+i \end{pmatrix}$$

$$\kappa(\mathbb{T}_0) = \{\infty, 0, i\}, \quad M_\kappa = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1^{-1}\kappa\sigma_1(\mathbb{T}_0) = \left\{\frac{1}{1-i}, 0, 1\right\}, \quad M_{\sigma_1^{-1}\kappa\sigma_1} = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}$$

$$\sigma_1\kappa\sigma_1^{-1}(\mathbb{T}_0) = \{\infty, 1+i, 1\}, \quad M_{\sigma_1\kappa\sigma_1^{-1}} = \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix}$$

$$\sigma_1^{-1}(\mathbb{T}_0) = \{0, 1, \infty\}, \quad M_{\sigma_1^{-1}} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$c(\mathbb{T}_0) = \left\{\frac{1}{1-i}, 1+i, i\right\}, \quad M_c = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}$$

The triangle $\mathbb{T}_0$ can be considered to be circumscribed by the plane

$$\mathbb{H}^{\perp} = \left\{ z + tj = x + yi + tj \in \mathbb{H}^3 : y = 0 \right\}.$$

For $g \in \mathcal{P} \subseteq \mathcal{M}$, the triangle $\mathbb{T} = g(\mathbb{T}_0)$ is thus circumscribed by a sphere $g(\mathbb{H}^{\perp})$. The plane $\mathbb{H}^{\perp}$ separates $\mathbb{H}^3$ into parts $\{z \in \mathbb{H}^3 : y > 0\}$ and $\{z \in \mathbb{H}^3 : y < 0\}$. The former is called the inside of the sphere $\mathbb{H}^{\perp}$, while the latter is called the outside of $\mathbb{H}^{\perp}$. Similarly $g(\mathbb{H}^{\perp})$ separates $\mathbb{H}^3$ into two parts inside the sphere and outside the sphere. We extend the notion of Farey addition [22] to a concept of a Farey set [25].

Definition 4.3.5. *The Farey set of the pair* $\frac{\alpha_1}{\gamma_1}$ *and* $\frac{\alpha_2}{\gamma_2}$ *is defined as follows:*

Let $\frac{\alpha_1}{\gamma_1}$ *and* $\frac{\alpha_2}{\gamma_2}$ *be Farey neighbours in* $\mathbb{Q}_{\infty}(i)$*. Then*

$$\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2} = \left\{ \frac{\mu\alpha_1 + \alpha_2}{\mu\gamma_1 + \gamma_2} : \mu \in U \right\} = \left\{ g(\mu) : \mu \in U, g(z) = \frac{\alpha_1 z + \alpha_2}{\gamma_1 z + \gamma_2} \right\}$$

is the Farey set corresponding to the neighbours $\frac{\alpha_1}{\gamma_1}$ *and* $\frac{\alpha_2}{\gamma_2}$*. Each element in the Farey set is called a Farey sum of the neighbours. Subsets of this Farey set are called Farey subsets.*

Since $|U| = 4$, the Farey set of a pair of Farey neighbours consists of four distinct points.

Example 4.3.6. Considering the points ∞, 0, 1 and the maps $\phi(z) = \frac{1}{z}$, $\tau_1(z) = z + 1$ and $\phi\tau_1(z) = \frac{1}{z+1}$, we see the Farey sets as follows:

$$\frac{1}{0} \oplus \frac{0}{1} = \left\{ \frac{1}{1}; \frac{-1}{1}; \frac{i}{1}; \frac{-i}{1} \right\} = U = \{\phi(\mu) : \mu \in U\}$$
$$\frac{1}{0} \oplus \frac{1}{1} = \{2; 0; 1 + i; 1 - i\} = \{\tau_1(\mu) : \mu \in U\}$$
$$\frac{0}{1} \oplus \frac{1}{1} = \left\{ \frac{1}{2}; \infty; \frac{1}{1+i}; \frac{1}{1-i} \right\} = \{\phi\tau_1(\mu) : \mu \in U\}$$

The union of the Farey sets of the vertices, taken pairwise, has 12 elements and can be written as the union of 3 disjoint sets as follows:

$$\left\{ -1; 0; \frac{1}{2}; 1; 2; \infty \right\} \cup \left\{ i; \frac{1}{1-i}; 1 + i \right\} \cup \left\{ -i; 1 - i; \frac{1}{1+i} \right\} = A_1 \cup A_2 \cup A_3$$

We see that the elements in set A_1 all lie on the plane $\mathbb{H}^\perp$ and include the given vertices 0, 1, ∞, and the additional 3 points -1, 2 and $\frac{1}{2}$. The set A_2 lies 'inside' the sphere $\mathbb{H}^\perp$ while the points in A_3 lie outside the sphere $\mathbb{H}^\perp$. This example displays a duality between results of Farey geodesics and Ford spheres as seen in Theorem 3.2.7.

Theorem 4.3.7. *Let Farey triangle,* $\mathbb{T} = \left\{\frac{\alpha_1}{\gamma_1}, \frac{\alpha_2}{\gamma_2}, \frac{\alpha_3}{\gamma_3}\right\}$ *in* $\mathbb{H}^3$ *is circumscribed by* $g(\mathbb{H}^\perp)$ *where* $\mathbb{T} = g(\mathbb{T}_0)$, $g \in \mathcal{P}$. *The Farey sums of the vertices, taken pairwise, result in 12 distinct points, 6 of which lie on* $g(\mathbb{H}^\perp)$ *(including the given vertices), 3 lie inside* $g(\mathbb{H}^\perp)$ *while 3 lie outside of* $g(\mathbb{H}^\perp)$.

To establish this result we need the following Lemma.

Lemma 4.3.8. *For* $h \in \mathcal{P}$, $h\left(\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}\right) = h\left(\frac{\alpha_1}{\gamma_1}\right) \oplus h\left(\frac{\alpha_2}{\gamma_2}\right)$.

Proof. $\frac{\alpha_1}{\gamma_1} \sim \frac{\alpha_2}{\gamma_2}$ and so $\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2} = \left\{\frac{\mu\alpha_1 + \alpha_2}{\mu\gamma_1 + \gamma_2} : \mu \in U\right\}$. Assume without loss of generality that $g(z) = \frac{\alpha_1 z + \alpha_2}{\gamma_1 z + \gamma_2}$, $g \in \mathcal{P}$ with $g(\infty) = \frac{\alpha_1}{\gamma_1}$, $g(0) = \frac{\alpha_2}{\gamma_2}$. So $\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2} = \{g(\mu) : \mu \in U\}$. We have $\frac{1}{0} \oplus \frac{0}{1} = \{\mu : \mu \in U\} = \{\pm 1, \pm i\}$. Thus $g\left(\frac{1}{0} \oplus \frac{0}{1}\right) = \frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}$.

Let $h(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ be in $\mathcal{P}$. Then $h\left(\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}\right) = hg\left(\frac{1}{0} \oplus \frac{0}{1}\right)$, $hg \in \mathcal{P}$ where $hg\left(\frac{1}{0}\right) = h\left(\frac{\alpha_1}{\gamma_1}\right)$ and $hg\left(\frac{0}{1}\right) = h\left(\frac{\alpha_2}{\gamma_2}\right)$.

Further

$$\begin{aligned} hg(\mu) = h(g(\mu)) = & \quad h\left\{\frac{\alpha_1\mu + \alpha_2}{\gamma_1\mu + \gamma_2}\right\} \\ = & \quad \frac{\alpha\frac{\alpha_1\mu + \alpha_2}{\gamma_1\mu + \gamma_2} + \beta}{\gamma\frac{\alpha_1\mu + \alpha_2}{\gamma_1\mu + \gamma_2} + \delta} \\ = & \quad \frac{\alpha(\alpha_1\mu + \alpha_2) + \beta(\gamma_1\mu + \gamma_2)}{\gamma(\alpha_1\mu + \alpha_2) + \delta(\gamma_1\mu + \gamma_2)} \end{aligned}$$

$$= \frac{\mu(\alpha\alpha_1 + \beta\gamma_1) + (\alpha\alpha_2 + \beta\gamma_2)}{\mu(\gamma\alpha_1 + \delta\gamma_1) + (\gamma\alpha_2 + \delta\gamma_2)}$$

Thus $h\left(\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}\right) = h\left(\frac{\alpha_1}{\gamma_1}\right) \oplus h\left(\frac{\alpha_2}{\gamma_2}\right)$. □

Proof. (Theorem 4.3.7) This result follows since $g \in \mathcal{P} \subseteq \mathcal{M}$ and g preserves spheres while perhaps interchanging the interior and exterior of spheres and Lemma 4.3.8. □

Theorem 4.3.9. *If* $\frac{\alpha_1}{\gamma_1} \sim \frac{\alpha_2}{\gamma_2}$ *in* $\mathbb{Q}_\infty(i)$ *then each element in the Farey set,* $\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}$*, is a Farey neighbour of each of the neighbours* $\frac{\alpha_i}{\gamma_i}$*,* $i = 1, 2$*. We write* $\left\{\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}\right\} \sim \frac{\alpha_i}{\gamma_i}$ *for* $i = 1, 2$*.*

Proof. $\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2} = \{g(\mu) : \mu \in U\}$ and $g(z) = \frac{\alpha_1 z + \alpha_2}{\gamma_1 z + \gamma_2}$. Now $g(\mu) = \frac{\alpha_1\mu + \alpha_2}{\gamma_1\mu + \gamma_2} \sim \frac{\alpha_i}{\gamma_i}$ since $|\gamma_i(\alpha_1\mu + \alpha_2) - \alpha_i(\gamma_1\mu + \gamma_2)| = |\gamma_i\alpha_1\mu + \gamma_i\alpha_2 - \alpha_i\gamma_1\mu - \alpha_i\gamma_2| = 1$ for $i = 1, 2$ since μ is a unit. Thus $\left\{\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}\right\} \sim \frac{\alpha_i}{\gamma_i}$ for $i = 1, 2$. □

The next result follows immediately.

Corollary 4.3.10. *If* $\frac{\alpha_1}{\gamma_1} \sim \frac{\alpha_2}{\gamma_2}$*, then* $\frac{\alpha_1}{\gamma_1}$*,* $\frac{\alpha_2}{\gamma_2}$ *and each element of* $\frac{\alpha_1}{\gamma_1} \oplus \frac{\alpha_2}{\gamma_2}$ *form Farey triangles.*

Example 4.3.11. Consider the fundamental triangle $\mathbb{T}_0 = \left\{\frac{1}{0}, \frac{0}{1}, \frac{1}{1}\right\}$ and the subset of Farey sums that lie in the interior of $\mathbb{H}^\perp$, namely $\left\{i, 1 + i, \frac{1}{1-i}\right\}$. Using these three vertices together with the vertices in the set $\left\{\frac{1}{0}, \frac{0}{1}, \frac{1}{1}\right\}$, we can form 7 new Farey triangles as follows:

$\left\{\frac{1}{0}, 0, i\right\}$; $\left\{\frac{1}{0}, 1, 1 + i\right\}$; $\left\{\frac{1}{0}, i, 1 + i\right\}$; with $\infty = \frac{1}{0}$ as a vertex and $\left\{0, 1, \frac{1}{1-i}\right\}$; $\left\{0, i, \frac{1}{1-i}\right\}$; $\left\{1, 1 + i, \frac{1}{1-i}\right\}$; $\left\{i, 1 + i, \frac{1}{1-i}\right\}$; with $\frac{1}{1-i}$ as a vertex.

These triangles together with the fundamental triangle, $\mathbb{T}_0$, form the 8 faces of the octahedron, $\mathcal{N}(v)$, $v = \frac{1}{2} + \frac{1}{2}i + \frac{1}{\sqrt{2}}j$.

If we consider the Farey subset, $\left\{-i, 1-i, \frac{1}{1+i}\right\}$ lying outside of $\mathbb{H}^{\perp}$ we may form 7 Farey triangles outside of $\mathbb{H}^{\perp}$ and hence an octahedron with the 8 triangular faces. The fundamental triangle, $\mathbb{T}_0$, is the common face of these two octahedra that are inverse of each other with respect to $\mathbb{H}^{\perp}$.

Theorem 4.3.12. *Every Farey triangle in $\mathbb{H}^3$ is the common face between exactly two ideal octahedra in $\mathbb{H}^3$. These octahedra are inverse to each other with respect to the circumscribed sphere of the Farey triangle. The internal and external vertices are Farey subsets of the vertices of the given triangle.*

Proof. This result follows by Example 4.3.11 and the fact that $g \in \mathcal{P}$ maps spheres to spheres and Farey neighbours to Farey neighbours, possibly inverting the interior and exterior of spheres. □

Definition 4.3.13. *The internal Farey triangles in Theorem 4.3.12 form the inner Farey subdivision of the given Farey triangle. The reflection of the Inner Farey subdivision in the circumscribed sphere is the outer Farey subdivision of the given Farey triangle.*

4.4 Fundamental properties of Farey triangles in $\mathbb{H}^3$

Following Schmidt [44] we establish the theorem.

Theorem 4.4.1. *Let $\zeta \in \mathbb{C}$ satisfy $\left|\zeta - \frac{\alpha_0}{\gamma_0}\right| < \frac{1}{\sqrt{2}|\gamma_0|^2}$, $\gamma_0 \neq 0$, then ζ lies inside a Farey polytope with vertices in $\mathbb{Q}(i)$ and with $\frac{\alpha_0}{\gamma_0}$ as a vertex. Further, $\sqrt{2}$ is the smallest constant such that this holds.*

Recall if $\mathbb{T} = \left\{\frac{\alpha_0}{\gamma_0}, \frac{\alpha_1}{\gamma_1}, \frac{\alpha_2}{\gamma_2}\right\}$ is a Farey triangle in $\mathbb{H}^3$ where $\gamma_i \neq 0$, $i = 0, 1, 2$, then the Farey polytope is a Euclidean triangle in $\mathbb{C}$ with the same vertices as $\mathbb{T}$.

We establish the following Lemma to prove Theorem 4.4.1.

Lemma 4.4.2. *Let* Γ *be a circle in* $\mathbb{C}$ *with radius* $\sqrt{2}$ *and arbitrary centre O. Then one can select 4, 6 or 8 lattice points in* $\mathbb{C}$ *from* $\mathbb{Z}[i]$*, all different from O and lying inside or on the boundary of* Γ *such that the selected points* $z_1 = z_{n+1}, z_2, z_3, \cdots z_n$*,* $n = 4, 6$ *or* 8 *satisfy the following conditions:*

1. $|z_{j+1} - z_j| = 1$ *or equivalently* $z_j \sim z_{j+1}$ *for* $1 \leq j \leq n+1$*;*
2. *the polygon* $z_1 z_2 \cdots z_n z_1$ *is a rectangle with O as an interior point; and*
3. *circles through* O, z_j, z_{j+1}*,* $1 \leq j \leq n$ *are wholly inside or touch the boundary of* Γ*.*

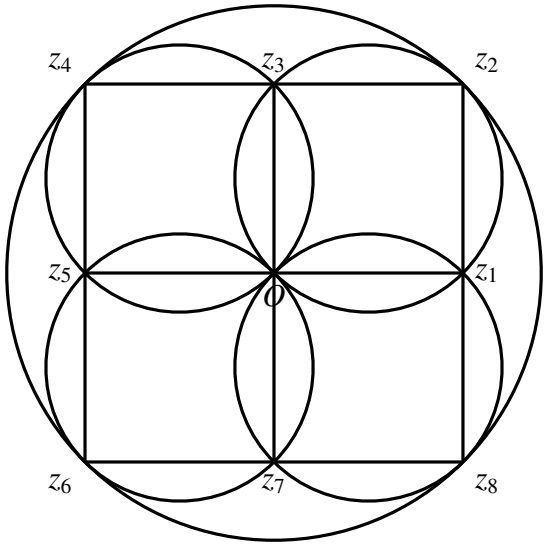


Figure 4.2: $n = 8$

Proof. Since the radius of Γ is $\sqrt{2}$, there are definitely lattice points contained inside Γ. First we consider the implications of the third condition. Let C_j be a circle through O, z_j

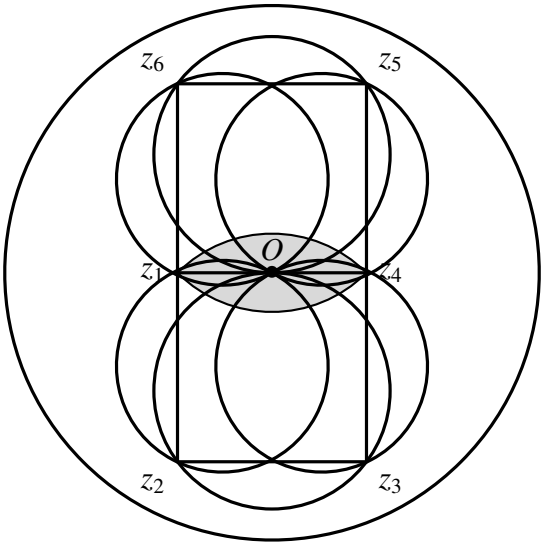


Figure 4.3: $n = 6$

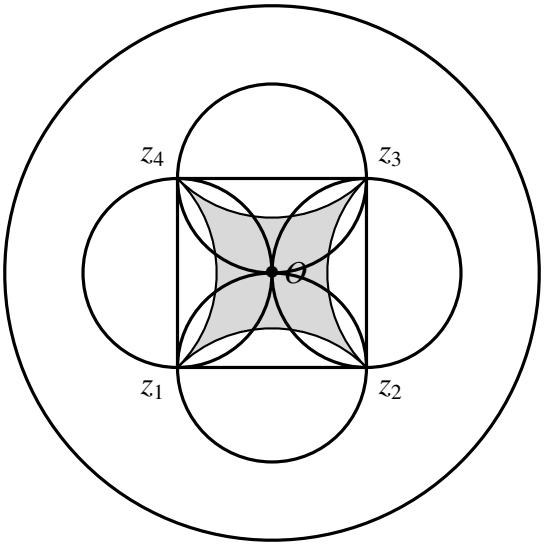


Figure 4.4: $n = 4$

and z_{j+1} that has diameter d_j with angle α_j at $z_j\hat{O}z_{j+1}$. Since O, z_j and z_{j+1} lie in or on the boundary of Γ, we know that the diameter d_j of C_j is less than or equal to $\sqrt{2}$, equality occurring if C_j is tangential to Γ.

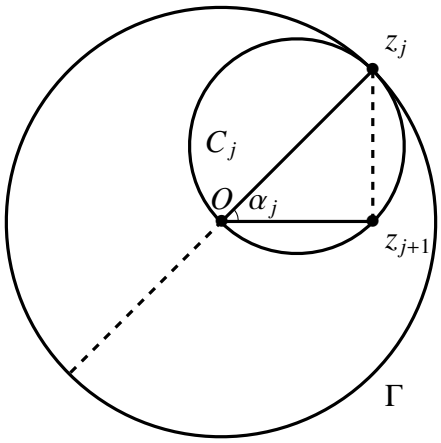


Figure 4.5: z_j lying on Γ

We note that if z_j is on the boundary of Γ and C_j, then z_{j+1} cannot lie on the boundary of Γ, since Γ and C_j can touch exactly once if C_j lies inside or on the boundary of Γ.

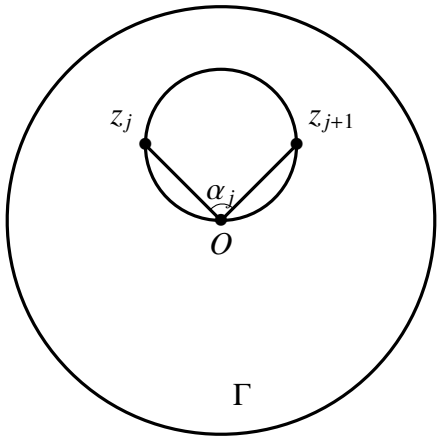


Figure 4.6: z_j lying inside Γ

The angle α_j is the angle $z_j\hat{O}z_{j+1}$ and is less than π. Then using any diameter d_j of C_j and by the sine rule, we have $\dfrac{\sin\alpha_j}{|z_j - z_{j+1}|} = \dfrac{\sin\frac{\pi}{2}}{d_j}$ and so $d_j = \dfrac{|z_j - z_{j+1}|}{\sin\alpha_j}$. If $|z_j - z_{j+1}| = 1$ (first condition), then $d_j = \dfrac{1}{\sin\alpha_j} \leq \sqrt{2}$, so $\sin\alpha_j = \dfrac{1}{d_j} \geq \dfrac{1}{\sqrt{2}}$. Using the sine graph we see that

$\frac{\pi}{4} \leq \alpha_j \leq \frac{3\pi}{4}$ for $1 \leq j \leq n$.

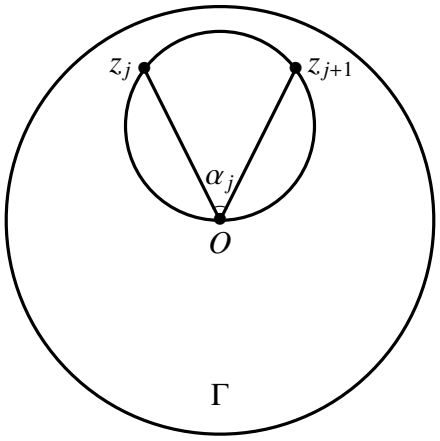


Figure 4.7: z_j lying inside Γ with different α_j

Thus if the first condition holds, then the third condition reduces to showing that $\frac{\pi}{4} \leq \alpha_j \leq \frac{3\pi}{4}$ for $1 \leq j \leq n$.

To prove the Lemma we distinguish three general cases according to the position of O in the lattice $\mathbb{Z}[i]$. All other possible cases are proven using the same methods in the three shown here.

Case (i) O is a lattice point, so $O \in \mathbb{Z}[i]$ or $O = \frac{\alpha}{\beta}$, $\alpha \in \mathbb{Z}[i], \beta = \pm 1, \pm i$.

Case (ii) O is at the midpoint of a lattice line, thus $O = \frac{\alpha}{\beta} \notin \mathbb{Z}[i]$ where $\beta = 2$, $\alpha \in \mathbb{Z}[i]$.

Case (iii) O is the centre of a lattice square, thus $O = \frac{\alpha}{\beta}$ and $|\beta|^2 = 2$, and $\frac{\alpha}{\beta} \notin \mathbb{Z}[i]$.

Thus by congruence modulo $\tilde{\mathcal{P}}$ we have O the origin, O the point $\frac{1}{2}$ and O the point $\frac{1}{1-i}$ respectively.

Case (i). The centre O is a lattice point and Γ has radius $\sqrt{2}$ as shown in Figure 4.2. Without

loss of generality, regard O to be the origin in $\mathbb{C}$. Then Γ includes the points $\pm 1, \pm i$ inside Γ and $\pm 1 \pm i$ on the boundary of Γ. The points $\{1, 1+i, i, -1+i, -1, -1-i, -i, 1-i\} = \left\{z_j : j = 1, \cdots 8\right\}$ all lie on a square in Γ (or on boundary) and O is at its centre. Each is a unit distance from every neighbour. Finally $z_j \hat{O} z_{j+1}$ is $\frac{\pi}{4}$ for $1 \leq j \leq 8$. Let C_j be a circle through z_j, O and z_{j+1} for $1 \leq j \leq 8$. The centre of C_j is at $\frac{\pm 1 \pm i}{2} = \frac{1}{\pm 1 \pm i}$. Thus C_j also passes through either z_{j+2} or z_{j-1}, and has diameter $\sqrt{2}$, the radius of Γ. Thus C_j and Γ are tangential at z_j or z_{j+1}. Let Ω_8 be the region enclosed by the circles C_j, $1 \leq j \leq 8$. Thus Ω_8 will have vertices $\left\{z_j : j = 1, \cdots 8\right\}$. The same result holds using any lattice point in $\mathbb{Z}[i]$.

Case (ii) Without loss of generality, assume O is point $\frac{1}{2}$ as in Figure 4.3. We note that $\left|\frac{1}{2} - (1+i)\right| = \left|-\frac{1}{2} - i\right| = \frac{\sqrt{3}}{2}$ and $\left|\frac{1}{2} - i\right| = \frac{\sqrt{3}}{2}$.

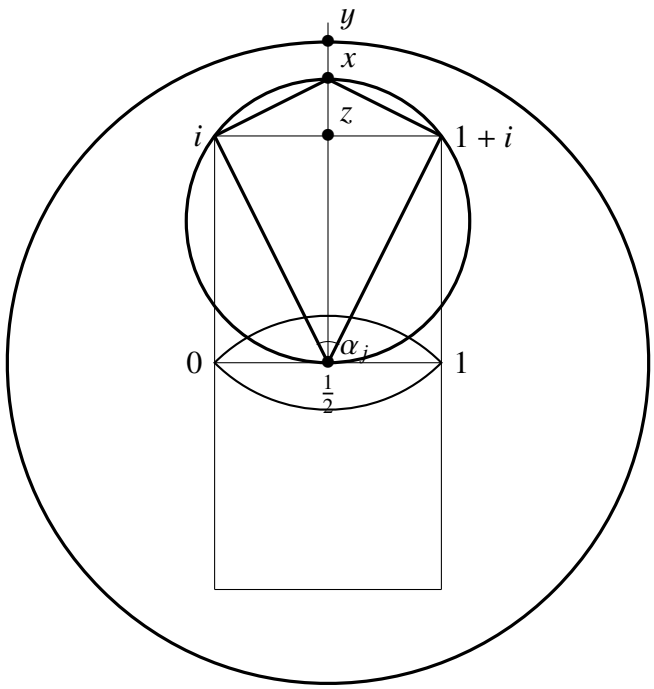


Figure 4.8: Case (ii)

A circle centre $\frac{1}{2}$ radius $\sqrt{2}$ will include lattice points

$$\{0, -i, 1-i, 1, 1+i, i\} = \left\{z_j : j = 1, \cdots 6\right\},$$

all in the interior of Γ. Certainly conditions (i) and (ii) of the Lemma are satisfied. Let C_j pass through $\frac{1}{2}$, z_j and z_{j+1} for $1 \leq j \leq 6$. Draw a diameter of C_j from $\frac{1}{2}$ to cut the segment $z_j z_{j+1}$ at z and C_j at x and Γ at y. Using Pythagoras' theorem, we see that $\left|z - \frac{1}{2}\right|^2 = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{2}$.

Let $\alpha = \dfrac{\pi - \alpha_j}{2}$, then $\cos\alpha = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$, while $\cot\alpha = \dfrac{|x-z|}{\frac{1}{2}}$.

Thus $\left|\dfrac{1}{2} - x\right| = \left|\dfrac{1}{2} - z\right| + |z - x| = \dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\cot\alpha = \dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\dfrac{\sqrt{2}}{2} = \dfrac{3\sqrt{2}}{4} < \sqrt{2}.$

So C_j is interior to Γ, and $\dfrac{\pi}{4} < \alpha_j < \dfrac{3\pi}{4}$, $1 \leq j \leq 6$.

Let Ω_6 be the region bounded by the circular arcs of circles centred at $\dfrac{1}{1-i}$ and $\dfrac{1}{1+i}$ of radius $\dfrac{1}{\sqrt{2}}$. The vertices $\left\{z_j : j = 1, \cdots 6\right\}$ lie on the boundary of Ω_6. The result holds for all points in Ω_6. Similar results will hold for circles taken as $\frac{i}{2}$, $1 + \frac{i}{2}$ and $\frac{1}{2} + i$ and in fact at any midpoint of a lattice line.

Case (iii) Without loss of generality, let O be the point $\dfrac{1}{1-i} = \dfrac{1+i}{2}$ and Γ be the circle centre O radius $\sqrt{2}$ as in Figure 4.4.

The points $\{0, 1, 1+i, i\} = \left\{z_j : j = 1, \cdots 4\right\}$ all lie inside of Γ. Let C_j be a circle through O, z_j, z_{j+1} for $1 \leq j \leq 4$.

We see that $|z_j - z_{j+1}| = 1$ for $1 \leq j \leq 4$ where $z_1 = z_5$ and $0, 1, 1+i, i$ is a square with $\dfrac{1}{1-i}$ as centre. Angle α_j is $\dfrac{\pi}{2}$ and so $z_j z_{j+1}$ is a diameter of C_j for $1 \leq j \leq 4$. Draw a diameter from $\dfrac{1}{1-i}$, perpendicular to $z_j z_{j+1}$ to cut C_j at x, $z_j z_{j+1}$ at z and Γ at y. Then $\left|\dfrac{1}{1-i} - z\right| =$

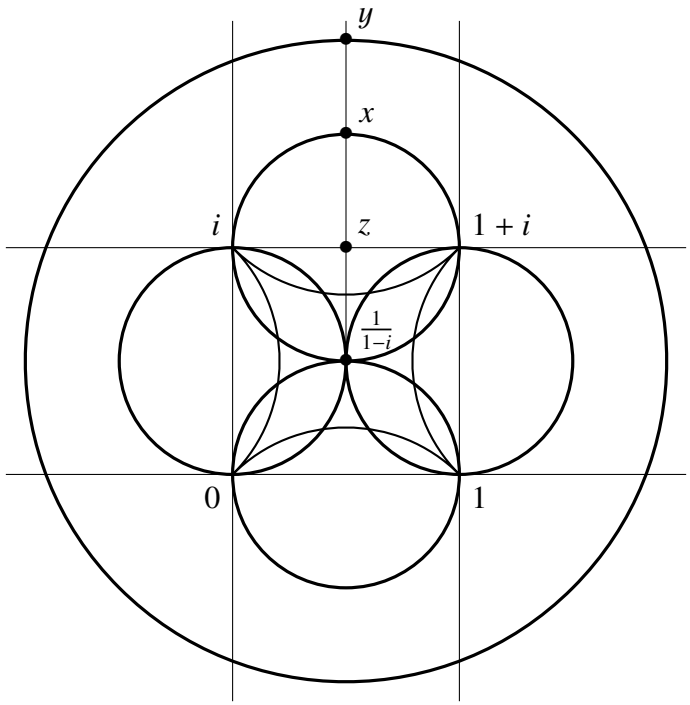


Figure 4.9: Case (iii)

$|z - x| = \frac{1}{2}d_j = \frac{1}{2}\left|z_j - z_{j+1}\right| = \frac{1}{2}$ and $\left|\frac{1}{1-i} - y\right| = \sqrt{2}$. Certainly $\left|\frac{1}{1-i} - y\right| \geq \left|\frac{1}{1-i} - z\right|$ and so C_j lies inside Γ with $\frac{1}{1-i}, x, y$ and z are collinear. Consider circle centre at x and radius $\frac{1}{\sqrt{2}}$ for each $1 \leq j \leq 4$. Let Ω_4 be the region between these circular arcs of radius $\frac{1}{\sqrt{2}}$ then the result holds for each point in Ω_4. The vertices $\left\{z_j : j = 1, \cdots 4\right\}$ lie on the boundary of Ω_4. Similarly we have the same result with the centre point of a lattice square in $\mathbb{Z}[i]$.

We note that the union of Ω_8, Ω_6 and Ω_4 will cover the whole of the unit square $\{0, 1, 1 + i, i\}$ in the lattice of $\mathbb{Z}[i]$ on $\mathbb{C}$. The same result holds for any unit square in the lattice $\mathbb{Z}[i]$ in $\mathbb{C}$.

Hence for any point O in $\mathbb{C}$, the region containing O is congruent to the square $\left\{0, \frac{1}{2}, \frac{1}{1-i}, \frac{i}{2}\right\}$ modulo $Stab(\infty, \tilde{\mathcal{P}})$. A circle Γ centre O radius $\sqrt{2}$ is thus congruent to the union of the regions Ω_8, Ω_6 and Ω_4. Thus the result holds in general. □

Proof. (Theorem 4.4.1) Let C be a circle centred at $\frac{\alpha_0}{\gamma_0}$ with radius $\frac{1}{\sqrt{2}|\gamma_0|^2}$. The Theorem claims that the closed disc of C is covered by the set of all Farey polytopes ($\mathbb{T}$) with $\frac{\alpha_0}{\gamma_0}$ as a vertex. Using Lemma 4.4.2 we shall prove that 4, 6 or 8 such Farey polytopes will suffice to cover the bounded disc of C (Figures 4.10, 4.11 and 4.12).

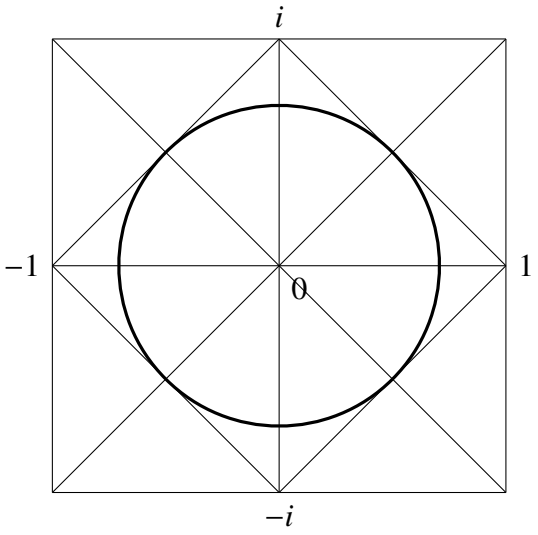


Figure 4.10: The covering of the bounded disc of C by 8 triangles.

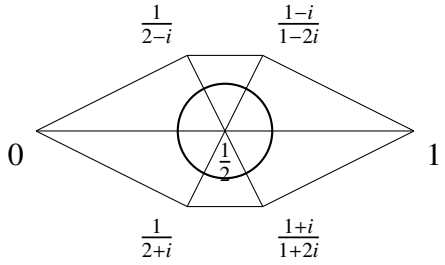


Figure 4.11: The covering of the bounded disc of C by 6 triangles.

Since α_0 and $\gamma_0 \neq 0$ are in $\mathbb{Z}[i]$ and we assume $\frac{\alpha_0}{\gamma_0}$ is reduced, we can find α, γ in $\mathbb{Z}[i]$ such that $\alpha_0\gamma - \gamma_0\alpha = 1$ from Definition 2.2.2. Consider the map $g(z) = \frac{\alpha_0 z + \alpha}{\gamma_0 z + \gamma}$ where $g \in \mathcal{P}$.

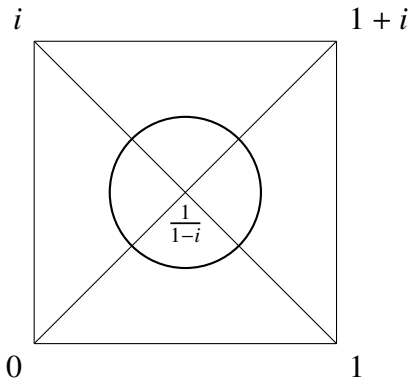


Figure 4.12: The covering of the bounded disc of C by 4 triangles.

Since $\gamma_0 \neq 0$, the isometric sphere of g, $I_g : |\gamma_0 z + \gamma| = 1$ is a sphere in $\mathbb{R}^3$ centre at $-\frac{\gamma}{\gamma_0}$ of radius $\frac{1}{|\gamma_0|}$. g maps sphere I_g to the isometric sphere of g^{-1} with the interior of I_g going to the exterior of $I_{g^{-1}}$ and exterior of I_g mapping to the interior of $I_{g^{-1}}$, where $g^{-1}(z) = \frac{\gamma z - \alpha}{-\gamma_0 z + \alpha_0}$ [6].

Our given circle C has centre $\frac{\alpha_0}{\gamma_0}$ and radius $\frac{1}{\sqrt{2}|\gamma_0|^2}$. The intersection of $I_{g^{-1}}$ with $\mathbb{C}$ is the isometric circle $I_{g^{-1}}$, centre $\frac{\alpha_0}{\gamma_0}$ and radius $\frac{1}{|\gamma_0|^2}$. So C lies inside $I_{g^{-1}}$. Thus $g^{-1}C$ lies outside of the isometric circle I_g, the intersection of the isometric sphere I_g with $\mathbb{C}$ as in Figure 4.13.

If Γ is a circle centre $\frac{-\gamma}{\gamma_0}$ with radius $\sqrt{2}$, then Γ has the same centre as I_g and is exterior to I_g. Thus $g(\Gamma)$ lies inside of $I_{g^{-1}}$.

Let $\frac{\alpha_0}{\gamma_0} \pm \frac{1}{\sqrt{2}|\gamma_0|^2}$ and $\frac{\alpha_0}{\gamma_0} \pm \frac{1}{\sqrt{2}|\gamma_0|^2} i$ lie on C.

It is seen that $g^{-1}\left(\frac{\alpha_0}{\gamma_0} \pm \frac{1}{\sqrt{2}|\gamma_0|^2}\right)$ and $g^{-1}\left(\frac{\alpha_0}{\gamma_0} \pm \frac{i}{\sqrt{2}|\gamma_0|^2}\right)$ lie on Γ, and thus $g^{-1}(C) = \Gamma$, while $g(\Gamma) = C$.

From Lemma 4.4.2 the points z_j, z_{j+1} and ∞ are pairwise Farey neighbours and thus $\mathbb{T}_j =$

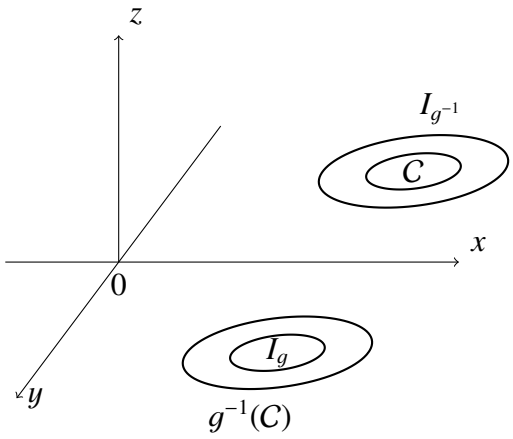


Figure 4.13: C, I_g and their images.

$\left\{z_j, z_{j+1}, \infty\right\}$ is a Farey triangle in $\mathbb{H}^3$ for $1 \leq j \leq n$ and $n = 4, 6, 8$. By Lemma 4.3.1, the image of a Farey triangle under $\mathcal{P}$ is a Farey triangle. The Farey triangles $\mathbb{T}_j$, $j = 1 \cdots m$, together with the region Ω_m, $m = 4, 6, 8$ form ideal polyhedrons in $\mathbb{H}^3$ with vertex at ∞ and with faces (excluding the base) being Farey triangles. The vertices of Ω_m, $m = 4, 6, 8$ lie inside or on the boundary of Γ. Now $g(\infty) = \frac{\alpha_0}{\gamma_0}$, the centre of C and $g(z_j)$ all lie either on the boundary or outside of C. Then $g(\mathbb{T}_j)$ is the Farey triangle $\left\{g(z_j), g(z_{j+1}), \frac{\alpha_0}{\gamma_0} = g(\infty)\right\}$ and the union of the Farey polytopes covers the whole of C, as required. □

4.5 Subdivision of Farey triangles

Lemma 4.5.1. *Let $\zeta \in \mathbb{C}$ belong to a non-degenerate Farey triangle $\mathbb{T}$. Then there is a non-degenerate Farey triangle $\mathbb{T}'$ among the Farey triangles in the inner subdivision of $\mathbb{T}$ such that $\zeta \in \mathbb{T}'$ and $N(\mathbb{T}') > N(\mathbb{T})$*

Proof. Let $\zeta \in \mathbb{T}$ where $\mathbb{T} = \left\{\frac{\alpha_t}{\gamma_t} : t = 1, 2, 3\right\}$ and $\gamma_t \neq 0$. Assume $N(\gamma_1) \geq N(\gamma_2) \geq N(\gamma_3)$. Since $\mathbb{T}$ is non-degenerate we have $N(\gamma_1) > 2$. Let $g(z) = \dfrac{\alpha_1 \mu z + \alpha_2}{\gamma_1 \mu z + \gamma_2}$ where $g \in \mathcal{P}$, μ a unit

and $g(\infty) = \dfrac{\alpha_1}{\gamma_1} = A$, $g(0) = \dfrac{\alpha_2}{\gamma_2} = B$ and $g(1) = \dfrac{\alpha_1\mu + \alpha_2}{\gamma_1\mu + \gamma_2} = \dfrac{\alpha_3}{\gamma_3} = C$.

We assume g maps the interior of $\mathbb{H}^\perp$ to the interior of $g(\mathbb{H}^\perp)$. If not, we may consider the map $g\phi \in \mathcal{P}$ where $\phi(z) = \frac{1}{z}$. We note that $g(\mathbb{R}_\infty)$ bounds $g(\mathbb{H}^\perp)$ as $\mathbb{R}_\infty$ bounds $\mathbb{H}^\perp$.

The new points that yield the inner Farey subdivision of $\mathbb{T}$ are $g\left(\dfrac{1}{1-i}\right) = A' = \dfrac{\alpha_1'}{\gamma_1'}$, $g(1+i) = B' = \dfrac{\alpha_2'}{\gamma_2'}$ and $g(i) = C' = \dfrac{\alpha_3'}{\gamma_3'}$.

Now $g\left(\dfrac{-\gamma_2}{\mu\gamma_1}\right) = \infty$ and ∞ lies outside of $g(\mathbb{H}^\perp)$. Thus $\dfrac{-\gamma_2}{\mu\gamma_1}$ lies in the lower half plane of $\mathbb{C}_\infty$ bounded by $\mathbb{R}_\infty$ and not including i, $1+i$ and $\dfrac{1}{1-i}$. That is, $\mathbb{R}_\infty$ separates $\dfrac{-\gamma_2}{\mu\gamma_1}$ from i, $1+i$ and $\dfrac{1}{1-i}$. By the above, $\dfrac{\alpha_1'}{\gamma_1'} = A' = g\left(\dfrac{1}{1-i}\right) = \dfrac{\alpha_1\mu + \alpha_2(1-i)}{\gamma_1\mu + \gamma_2(1-i)}$. So $N(\gamma_1') = N(\gamma_1\mu + \gamma_2(1-i)) = N(\gamma_1)N(1-i)N\left(\dfrac{1}{1-i} + \dfrac{\gamma_2}{\gamma_1\mu}\right)$. Thus

$$
\begin{aligned}
N(\gamma_1') \le N(\gamma_1) \iff\ & N(\gamma_1)N(1-i)N\left(\frac{1}{1-i} + \frac{\gamma_2}{\gamma_1\mu}\right) \le N(\gamma_1), N(1-i) = 2. \\
\iff\ & N\left(\frac{1}{1-i} + \frac{\gamma_2}{\gamma_1\mu}\right) \le \frac{1}{2}. \\
\iff\ & \left|\frac{1}{1-i} - \left(\frac{-\gamma_2}{\gamma_1\mu}\right)\right| \le \frac{1}{\sqrt{2}}. \\
\iff\ & \frac{-\gamma_2}{\gamma_1\mu} \text{ lies in the circle } K \text{ (or on the boundary) centre } \frac{1}{1-i}, \text{ radius } \\
& \frac{1}{\sqrt{2}} \text{ that passes through } 0, 1, i \text{ and } 1+i. \\
\iff\ & \frac{-\gamma_2}{\gamma_1\mu} \text{ lies in the sector } \mathbb{T}_1 \text{ of } K \text{ containing } \mathbb{T}_1 = \left\{0, 1, \frac{1}{1-i}\right\} \text{ but in } \\
& \text{the lower half plane.} \\
\iff\ & \text{Circle } K_1 \text{ through } \frac{-\gamma_2}{\gamma_1\mu}, 0 \text{ and } 1 \text{ contains } \frac{1}{1-i}, i \text{ and } 1+i \text{ in its } \\
& \text{interior and } \infty \text{ outside.} \\
\iff\ & \text{Circle } g(K_1) \text{ through } g(0), g(1) \text{ and } \infty = g\left(\frac{-\gamma_2}{\gamma_1\mu}\right) \text{ (line BC) has}
\end{aligned}
$$

$g\left(\frac{1}{1-i}\right) = A', g(i) = C'$ and $g(1+i) = B'$ on the same side and opposite the side containing $g(\infty) = A$.

$\iff$ A', B' and C' lie in the half plane bounded by line BC and not including A but inside $g(\mathbb{R}_\infty)$.

Consequently, none of the Farey triangles in the subdivision of $\mathbb{T}$ that have $A' = \frac{\alpha_1'}{\gamma_1'}$ as a vertex, can contain ζ and these triangles can be ignored.

That is, if $N(\gamma_1') \leq N(\gamma_1)$, we need not consider vertex $A' = \frac{\alpha_1'}{\gamma_1'}$.

We have assumed that $N(\gamma_1) \geq N(\gamma_2) \geq N(\gamma_3)$. We know that $B = g(0)$ and $C = g(1)$ are Farey neighbours and $|BC|^2 = \left|\frac{\alpha_2}{\gamma_2} - \frac{\alpha_3}{\gamma_3}\right|^2 = \left|\frac{\alpha_2\gamma_3 - \gamma_2\alpha_3}{\gamma_2\gamma_3}\right|^2 = \frac{1}{N(\gamma_2)N(\gamma_3)}$ and so $|BC| \geq |AB|$ and $|BC| \geq |AC|$. □

Definition 4.5.2. *The diameter of a triangle $\mathbb{T}$ is the longest side of $\mathbb{T}$ denoted by Diam($\mathbb{T}$).*

By Lemma 4.3.1, we can choose $g \in \mathcal{P}$ so that $g(0)$, $g(1)$ and $g(\infty)$ may be any permutation of A, B and C. We know that if $N(\gamma_t') \leq N(\gamma_1)$, $t = 1, 2, 3$, we can ignore a Farey triangle with vertex $\frac{\alpha_t'}{\gamma_t'}$ in the subdivision of the triangle $\mathbb{T}$.

Thus in all the allowable Farey triangles in the inner subdivision of $\mathbb{T}$, we have $N(\gamma_t') > N(\gamma_1)$. Consequently, the given Farey triangle $\mathbb{T}$ is covered by non-degenerate Farey triangles in the inner subdivision of $\mathbb{T}$ with norms strictly greater than the norm of $\mathbb{T}$.

Theorem 4.5.3. *Every complex number ζ is contained in a chain of Farey triangles arising from the faces of the octahedron, where $\mathbb{T}_0, \mathbb{T}_1, \mathbb{T}_2 \cdots$ is the chain of Farey triangles and $\mathbb{T}_{i+1}$ is in the inner subdivision of $\mathbb{T}_i$, and where $N(\mathbb{T}_{i+1}) > N(\mathbb{T}_i)$. We thus have $N(\mathbb{T}_n) \longrightarrow \infty$ as $n \longrightarrow \infty$, the diameter of $\mathbb{T}_n \longrightarrow 0$ as $n \longrightarrow \infty$ and the vertices $\frac{\alpha_t^n}{\gamma_t^n} \longrightarrow \zeta$ as $n \longrightarrow \infty$, $t = 1, 2, 3$.*

Proof. By Lemma 4.5.1 we can find the chain of Farey triangles with strictly increasing norms, so $N(\mathbb{T}_n) \to \infty$ as $n \longrightarrow \infty$. $Diam(\mathbb{T}_n)$ is inversely proportional to $N(\gamma_t^n), t = 1, 2, 3$, so $Diam(\mathbb{T}_n) \to 0$ as $N(T_n) \to \infty$. Thus the radii of the circumcircles of $\mathbb{T}_n$ tend to 0, and thus the vertices converge to ζ. We say that $\mathbb{T}_n$ converges to ζ. □

Corollary 4.5.4. *For every $\zeta \in \mathbb{C}$ we can find a chain of adjacent octahedra in $\mathbb{H}^3$ that converges to ζ.*

Proof. From Theorem 4.5.3 we have a chain of Farey triangles $\mathbb{T}_0, \mathbb{T}_1, ...$ where $\mathbb{T}_{j+1}$ is in the Farey subdivision of $\mathbb{T}_j$ and $N(\mathbb{T}_{j+1}) > N(\mathbb{T}_j)$. Consecutive Farey triangle in this chain are faces of common octahedra since $\mathbb{T}_{j+1}$ is in the subdivision of $\mathbb{T}_j$. Hence this chain of Farey triangles gives rise to a chain of octahedra, each adjacent to its successor and predecessor. These octahedra converge to ζ, since the Farey triangles converge to ζ. □

Chapter 5

Gaussian Integer Continued Fraction Algorithms

5.1 Introduction

In this section we introduce four Gaussian integer continued fraction algorithms. Firstly, we consider the Hurwitz complex continued fraction algorithm. The Hurwitz algorithm does not yield a unique continued fraction expansion, since a multivalued choice for every complex number may be made at some stages in the algorithm.

Secondly, we will consider the "nearest node" continued fraction expansion of any complex number ζ. Following Hensley [20] we will choose an option that creates a unique expansion for each complex number.

Thirdly, we consider the floor continued fraction algorithm. Dani and Nogueira [11] gives a description of an algorithm that is reminiscent of the simple continued fraction algorithm

for the reals discussed by Khintchine [31] and Rockett and Szüz [40].

By analogy to the work of Katok [30] on "minus" or backward Gaussian integer continued fractions, we show that every complex number ζ can be expressed as an infinite continued fraction.

Each of the algorithms gives a general continued fraction expansion that is regular as defined below.

Definition 5.1.1. *A regular Gaussian integer continued fraction expression is of the form*

$$\zeta = \beta_0 + \cfrac{\epsilon_1}{\beta_1 + \cfrac{\epsilon_2}{\beta_2 + \cfrac{\epsilon_3}{\beta_3 + \cdots}}}$$

where $\epsilon_n = \pm 1, \pm i$ *and* $\beta_n \in \mathbb{Z}[i]$.

Definition 5.1.2. *Let* ζ *be a regular Gaussian integer continued fraction as above. Let* $\tau^{\varsigma_j} = \tau_1^{a_j}\tau_i^{b_j}$, $\varsigma_j = a_j + ib_j$, *with* $\zeta_n = \dfrac{\alpha_n}{\gamma_n} = \tau^{\varsigma_0}\varphi\tau^{\varsigma_1}\varphi\tau^{\varsigma_2}\varphi\ldots\tau^{\varsigma_n}\varphi(\infty) = \tau^{\varsigma_0}\varphi\tau^{\varsigma_1}\varphi\tau^{\varsigma_2}\varphi\ldots\tau^{\varsigma_n}(0)$ *being the* n^{th} *convergent to* ζ. *Then* $\lim\limits_{n\to\infty}\zeta_n = \zeta$.

5.2 Hurwitz continued fraction algorithm

One of the well-known algorithms is the Hurwitz algorithm, introduced by Hurwitz [28]. It consists of assigning to $\zeta \in \mathbb{C}$ a Gaussian integer nearest to it. This may be viewed as a multivalued choice function $X : \mathbb{C} \mapsto \mathbb{Z}[i]$ defined by

$$X(z) = \{\alpha \in \mathbb{Z}[i] : |\zeta - \alpha| \leq |\zeta - \gamma|, \gamma \in \mathbb{Z}[i]\}.$$

Thus a nearest node or lattice point to $\zeta = x + iy$ is a point $m + in$ in $\mathbb{Z}[i]$ such that $(x-m)^2 + (y-n)^2 \leq \dfrac{1}{2}$. We denote $m + in$ by $[\zeta]$. We note that $\zeta = \frac{1}{1-i}$ has four nearest nodes; 0, 1, i and $1 + i$.

From [20], we note for completeness the following properties of the convergents ζ_n in the Hurwitz continued fraction.

Theorem 5.2.1. *Let $\zeta \in \mathbb{C}$ have a Hurwitz continued fraction expansion with $\frac{\alpha_n}{\gamma_n}$ as the n^{th} convergent to ζ with $\alpha_n, \gamma_n \in \mathbb{Z}[i]$ and $n \geq 0$, then*

$$\left|\frac{\gamma_{n+2}}{\gamma_n}\right| \geq \frac{3}{2}.$$

Theorem 5.2.2. *Let $\zeta_n = \dfrac{\alpha_n}{\gamma_n}$ be the n^{th} convergent to ζ. Then*

1. *The sequence $\{\gamma_n\}$ increases exponentially.*
2. *The sequence $\left\{\dfrac{\alpha_n}{\gamma_n}\right\}$ terminates at ζ and is finite if ζ is a Gaussian rational for some n.*
3. $\left|\zeta - \dfrac{\alpha_n}{\gamma_n}\right| \leq \dfrac{1}{|\gamma_n|^2}$

Example 5.2.3. Let $\zeta = \dfrac{-10}{9} + \dfrac{11i}{2}$, a Hurwitz continued fraction expansion for ζ is

$$-1 + 5i + \cfrac{1}{-2i + \cfrac{1}{-2 + \cfrac{1}{-1 + 2i + \cfrac{1}{1 + 2i}}}}$$

or

$$-1 + 5i + \cfrac{1}{-2i + \cfrac{1}{-2 - i + \cfrac{1}{-1 - 2i + \cfrac{1}{1 - 2i}}}}.$$

We note from Ford [16] that the convergents of a Hurwitz continued fraction expansion can be interpreted as a chain of Ford spheres converging to ζ or as a sequence of Hermitian fractions approximating ζ.

Ford [16] uses a chain of adjacent Ford spheres to approximate the complex number ζ. Once again we notice that there may be multiple chains of adjacent Ford spheres leading from $\mathcal{S}_\infty$ to a point $\zeta \in \mathbb{C}$. This choice relates to the multiplicity of choosing a nearest node to $\zeta \in \mathbb{C}$. For completeness we note the following result from Ford.

Theorem 5.2.4. *A regular continued fraction (definition 5.1.1) has convergents,* ∞, $\frac{\alpha_1}{1}$, $\frac{\alpha_2}{\gamma_2}$, $\frac{\alpha_3}{\gamma_3}$, $\cdots$, *which determine a suite of Ford spheres,* $\mathcal{S}(\infty)$, $\mathcal{S}\left(\frac{\alpha_1}{1}\right)$, $\mathcal{S}\left(\frac{\alpha_2}{\gamma_2}\right)$, $\cdots$, *where the first is the plane* $t = 1$ $(\mathcal{S}_\infty)$*, and such that each sphere of the suite is adjacent to that which precedes it.*

Conversely, if $\mathcal{S}(\infty)$, $\mathcal{S}\left(\frac{\alpha_1}{1}\right)$, $\mathcal{S}\left(\frac{\alpha_2}{\gamma_2}\right)$, $\cdots$ *is any suite of Ford spheres (the first being the plane* $t = 1$*) such that each is adjacent to that which precedes it, then the points of tangency of the spheres* ∞, $\frac{\alpha_1}{1}$, $\frac{\alpha_2}{\gamma_2}$, $\cdots$, *are the convergents of a regular continued fraction for some* $\zeta \in \mathbb{C}$.

Ford further relates the condition $|\alpha\gamma' - \alpha'\gamma| = 1$ (Farey neighbours) to successive fractions $\dfrac{\alpha}{\gamma}$ and $\dfrac{\alpha'}{\gamma'}$ in the suite of fractions approximating ζ developed by Hermite [21].

Ford [16, pp.4,6] notes that given the positive definite Hermitian form,

$$F(x, y) = (x - \zeta y)(\overline{x} - \overline{\zeta}\,\overline{y}) + k^2 y\overline{y}$$

if we set $\frac{x}{y} = \omega$ and equate the form to zero, we obtain the equation of an imaginary circle in $\mathbb{C}$ given by

$$a\omega\overline{\omega} + b\omega + \overline{b}\,\overline{\omega} + c = 0.$$

The 'real point sphere' in the upper half space, being $x = \frac{-b_1}{a}$, $y = \frac{b_2}{a}$ and $t = +\frac{\sqrt{D}}{a}$, is chosen as the representative point of the form, where $b = b_1 + ib_2$, $D = b\overline{b} - ac$. The

representative point of the form is found to be $\omega = \zeta$, $t = k$. The fraction of Hermite for a given value of k is $\frac{\alpha}{\gamma}$, where $x = \alpha$, $y = \gamma$ gives the minimum of the form. That is, it is the z co-ordinate in $\mathbb{C}$ of the peak of the pentahedron in which the representative point of the form lies. As k decreases from ∞ to 0 the representative point traces the half-line $z = \zeta$ from ∞ to the complex plane. We have then the following interpretation:

Theorem 5.2.5. *Let a moving point trace the half of the line $z = \zeta$ lying in the upper half-space, passing from ∞ to the complex plane. The z co-ordinates of the peaks, lying on $\mathbb{C}$, of the successive pentahedra through which this point passes, are the fractions of Hermite tending toward the value ζ.*

From this interpretation it is obvious that if ζ is a Gaussian rational there are only a finite number of fraction approximates of ζ. The moving point eventually enters and remains within pentahedra with peaks at ζ, and the suite terminates.

5.3 Nearest Gaussian integer continued fraction expansion

As seen in Section 5.2, the nearest node for the Hurwitz continued fraction is not necessarily unique. Following Hensley [20], we give a formal definition that will yield a unique nearest node. Let $[\zeta]$ be the Gaussian integer nearest to the complex number ζ rounding up, in both

the real and imaginary components, to break ties for uniqueness as follows.

$$[\zeta] = [x + iy] = \begin{cases} \lfloor x \rfloor + i \lfloor y \rfloor \text{ if } x - \lfloor x \rfloor < \frac{1}{2}, \quad y - \lfloor y \rfloor < \frac{1}{2} & (1) \\ \lfloor x \rfloor + i \lceil y \rceil \text{ if } x - \lfloor x \rfloor < \frac{1}{2}, \quad \lceil y \rceil - y \leq \frac{1}{2} & (2) \\ \lceil x \rceil + i \lfloor y \rfloor \text{ if } \lceil x \rceil - x \leq \frac{1}{2}, \quad y - \lfloor y \rfloor < \frac{1}{2} & (3) \\ \lceil x \rceil + i \lceil y \rceil \text{ if } \lceil x \rceil - x \leq \frac{1}{2}, \quad \lceil y \rceil - y \leq \frac{1}{2} & (4) \end{cases}$$

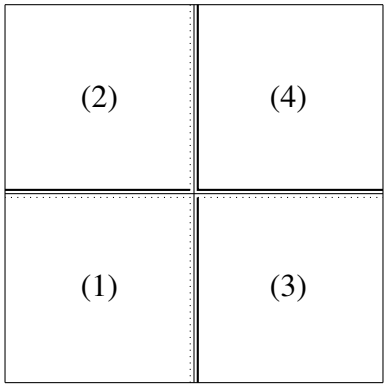


Figure 5.1: Geometric interpretation of $[\zeta]$

In each case we see that $|\zeta - [\zeta]|^2 \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Example 5.3.1. $\left[\frac{1}{4} + \frac{1}{2}i\right] = i$, $\left[\frac{1}{2} + \frac{i}{2}\right] = 1 + i$ and $\left[\frac{1+\sqrt{3}i}{2}\right] = 1 + i$.

We introduce an algorithm for a continued fraction based on this nearest node choice for any complex number. The algorithm is called the nearest node continued fraction algorithm and is a special case of the Hurwitz continued fraction algorithm. So a nearest node continued fraction expansion is also a Hurwitz continued fraction expansion.

Let $\zeta = \zeta_0$. Choose $\beta_0 = [\zeta_0] \in \mathbb{Z}[i]$. Then $\tau^{-\beta_0}(\zeta_0) = \zeta_0 - \beta_0$, a translation of ζ_0 through $-\beta_0$. We note that $\zeta_0 - \beta_0$ lies in the domain given as $B = \left\{x + iy : -\frac{1}{2} \leq x < \frac{1}{2}, -\frac{1}{2} \leq y < \frac{1}{2}\right\}$. Thus B is bounded by the lines $x = \pm\frac{1}{2}$; $y = \pm\frac{1}{2}$ and lies inside the unit circle centred at

0. Under the action of $\phi(z) = \frac{1}{z}$ these lines are mapped to circles passing through 0 and the points ± 2 and $\pm 2i$. We note that $\phi(B)$ is the region shaded in Figure 5.2 as $\frac{1}{B}$. Thus if $\phi(\zeta_0 - \beta_0) = \phi\tau^{-\beta_0}(\zeta_0) = \zeta_1$ then ζ_1 lies in the region $\frac{1}{B}$ and hence outside the unit circle. $\zeta_0 = \tau^{\beta_0}\phi(\zeta_1)$. Consider ζ_1, the algorithm can be repeated as necessary.

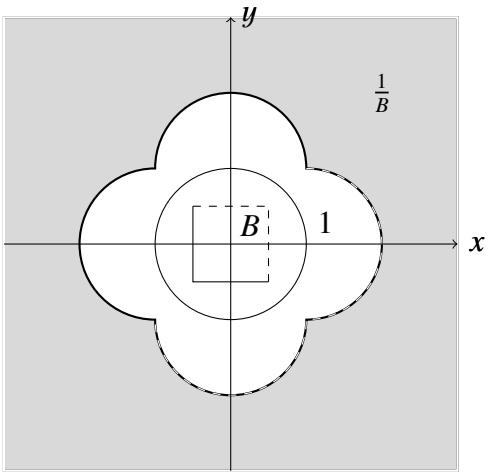


Figure 5.2: The region B and $\phi(B)$

Thus we can find a sequence of Gaussian integers $\beta_0, \beta_1, \ldots \beta_n$ such that

$$\zeta_0 = \tau^{\beta_0}\phi\tau^{\beta_1}\phi \ldots \tau^{\beta_n}\phi(\zeta_n).$$

The convergents to ζ_0 are given by $\tau^{\beta_0}\phi\tau^{\beta_1}\phi \ldots \tau^{\beta_n}\phi(\infty) = \dfrac{\alpha_n}{\gamma_n}$ for $n = 0, 1, \ldots$.

Example 5.3.2. Let $\zeta = \dfrac{-10}{9} + \dfrac{11i}{2}$. The nearest node continued fraction expansion is given as

$$-1 + 6i + \cfrac{1}{2i + \cfrac{1}{-2 + i + \cfrac{1}{-1 + 2i + \cfrac{1}{1 + 2i}}}}.$$

5.4 The floor continued fraction algorithm

The theory of "plus" continued fractions for real numbers is known as simple or regular continued fractions [31], [40]. Let α be any real number. We define a sequence of integers $\{a_i\}$, $i = 0, 1, 2, \ldots$ and a sequence of real numbers $\{\alpha_i\}$, $i =, 1, 2, \ldots$ by $a_0 = \lfloor \alpha \rfloor$, $\alpha_1 = \frac{1}{\alpha - a_0}$, and inductively,

$$a_n = \lfloor \alpha_n \rfloor, \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n}.$$

A sequence of rational real numbers is defined as

$$\begin{aligned} r_n = & \quad (a_0, a_1, \ldots, a_{n-1}, a_n) \\ = & \quad a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}, \quad n \geq 0 \\ = & \quad \tau^{a_0} \phi \tau^{a_1} \phi \ldots \tau^{a_n} \phi(\infty) \end{aligned}$$

where $\phi(z) = \frac{1}{z}$ is in $\mathcal{P}$ but not the modular group.

To extend the simple continued fraction algorithm to an algorithm with Gaussian integer coefficients, we use an algorithm given by Dani and Nogueira [11].

A map $f : \mathbb{C} \mapsto \mathbb{Z}[i]$ is defined as follows: Let $\zeta_0 \in \mathbb{C}$ and $\beta_0 \in \mathbb{Z}[i]$ be such that $\zeta_0 - \beta_0 = x + iy$ lies in the unit square with $0 \leq x < 1$ and $0 \leq y < 1$. If $x^2 + y^2 < 1$ we define $f(\zeta)$ to be β_0. If $x^2 + y^2 \geq 1$, $f(\zeta)$ is defined to be $\beta_0 + 1$ or $\beta_0 + i$, whichever is nearer to ζ (choosing

say the former if they are equidistant). Let α_n be the n^{th} choice by the function f. Then

$$\alpha_0 = \begin{cases} \beta_0 & = \lfloor \zeta_0 \rfloor \\ \beta_0 + 1 & = \lfloor \zeta_0 \rfloor + 1 \\ \beta_0 + i & = \lfloor \zeta_0 \rfloor + i \end{cases}$$

Consider $\phi(\zeta_0 - \alpha_0) = \phi\tau_{\alpha_0}^{-1}(\zeta_0) = \zeta_1$. Since $\zeta_0 - \alpha_0$ lies inside the unit sphere at 0, ζ_1 lies outside the unit sphere, so $\alpha_1 \neq 0$. Repeating the process we can find a sequence of Gaussian integers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that

$$\zeta_n = \tau_{\alpha_{n-1}}^{-1}\phi\tau_{\alpha_{n-2}}^{-1}\phi \ldots \phi\tau_{\alpha_0}^{-1}(\zeta_0).$$

Thus

$$\zeta_0 = \tau_{\alpha_0}\phi\tau_{\alpha_1}\phi \ldots \phi\tau_{\alpha_{n-1}}(\zeta_n).$$

The process terminates when $\zeta_n \in \mathbb{Z}[i]$. The Figure 5.3 shows all possible positions of $\zeta_n - \beta_n$.

Example 5.4.1. $f\left(\frac{1}{4} + \frac{1}{2}i\right) = 0$, $f\left(\frac{3}{4} + \frac{3i}{4}\right) = 1$ and $f\left(\frac{1+\sqrt{3}i}{2}\right) = i$.

It may be seen that with respect to this algorithm, if $\zeta \in \mathbb{C}$ and $\{\alpha_n\}$, $n = 0, 1, 2 \ldots$, is the corresponding sequence of partial quotients, then for $n \geq 1$ we have $\Re(\alpha_n) \geq -1$ and $\Im(\alpha_n) \geq -1$. Dani suggests that this may be compared with the simple continued fractions for real numbers where the later partial quotients are positive.

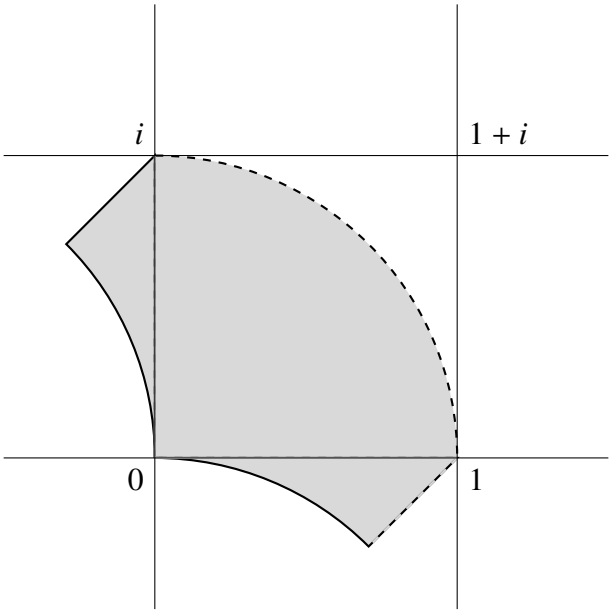


Figure 5.3: The region $\zeta_n - \beta_n = \tau_{\beta_n}^{-1}(\zeta_n)$

Example 5.4.2. Let $\zeta = \dfrac{-10}{9} + \dfrac{11i}{2}$. The floor continued fraction expansion is given as

$$-1 + 5i + \cfrac{1}{-2 - 8i + \cfrac{1}{-i + \cfrac{1}{1 + \cfrac{1}{1 - 2i + \cfrac{1}{-i + \cfrac{1}{1 + \cfrac{1}{1 - i}}}}}}}.$$

5.5 Minus or backward Gaussian integer continued fractions

Katok [30] introduces the theory of "minus" continued fractions for real numbers. Let α be any real number. We define a sequence of integers $\{a_i\}$, $i = 0, 1, 2, \ldots$ and a sequence of real numbers $\{\alpha_i\}$, $i =, 1, 2, \ldots$ by $a_0 = \lfloor \alpha \rfloor + 1$, $\alpha_1 = \dfrac{1}{a_0 - \alpha}$, and inductively,

$$a_n = \lfloor \alpha_n \rfloor + 1, \quad \alpha_{n+1} = \frac{1}{a_n - \alpha_n}.$$

A sequence $\{r_n\}$ of real integer rationals is defined as:

$$r_n = (a_0, a_1, \ldots, a_{n-1}, a_n) = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}}, n \geq 0.$$

We note that this algorithm gives the following two theorems [30]:

Theorem 5.5.1. *α is represented as an infinite continued fraction. We can find a sequence of real integer rationals $\{r_n\}$ such that $\lim_{n\to\infty} r_n = \alpha$. That is:*

$$\alpha = (a_0, a_1, \ldots, a_{n-1}, \ldots) = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{\cdots}}}}$$

Theorem 5.5.2. *$\alpha \in \mathbb{Q}$ if and only if from some point in the expression all partial quotients a_i are equal to 2. (i.e. if there exists $k > 0$ such that $\alpha_k = 2$ for all $k \geq n$).*

We apply the backward algorithm to any $\zeta \in \mathbb{C}$. Let $\lceil \zeta \rceil = \lfloor \zeta \rfloor + 1 + i$. We call this the ceiling of ζ in $\mathbb{C}$. The point $\zeta - \lceil \zeta \rceil = x + yi$ will have $-1 \leq x < 0$ and $-1 \leq y < 0$.

We can expand any complex number ζ into a unique continued fraction according to the following "backward" continued fraction algorithm.

Let ζ be any complex number. We define a sequence of Gaussian integers $\{\alpha_i\}$, $i = 0, 1, 2, \ldots$ and a sequence of complex numbers $\{\zeta_i\}$, $i =, 1, 2, \ldots$ by $\alpha_0 = \lceil \zeta_0 \rceil$, $\zeta_1 = \frac{1}{\alpha_0 - \zeta_0}$, and inductively,

$$\alpha_n = \lceil \zeta_n \rceil, \quad \zeta_{n+1} = \frac{1}{\alpha_n - \zeta_n}.$$

A sequence $\{\beta_n\}$ of Gaussian rational numbers is defined as

$$\beta_n = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = \alpha_0 - \cfrac{1}{\alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_n}}}}, n \geq 0.$$

Example 5.5.3.

$$\frac{-10}{9} + \frac{11}{2}i = -1 + 6i - \cfrac{1}{1 - i - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \ldots}}}}}.$$

Chapter 6

The Cutting sequence of Gaussian Integer Continued Fractions

6.1 Introduction

Caroline Series [45] introduces a connection between directed geodesics cutting across the tessellated hyperbolic plane $\mathbb{H}^2$, the modular group $\Gamma = PSL(2,\mathbb{Z})$ and simple continued fractions. She considers the Farey tessellation $\mathcal{F}$ of $\mathbb{H}^2$ by the fundamental ideal triangle $\mathbb{T}_0$, with cusps $\{\infty, 0, 1\}$ under Γ. The maps τ and ρ are given as follows: $\tau(z) = z + 1$, $\rho(z) = \frac{z}{z+1} = \varphi\tau^{-1}\varphi(z) = \phi\tau\phi(z)$ and $\phi(z) = \frac{1}{z}$ and $\varphi(z) = \frac{-1}{z}$. In [23] it is established that any non rational real number can be expressed as

$$\alpha = \tau^{a_0}\rho^{a_1}\tau^{a_2}\rho^{a_3}\cdots\tau^{a_{k-1}}\rho^{a_k}(\alpha_{k+1})$$

where $a_t \in \mathbb{Z}^+$ for $t \geq 1$ and where a_0 may be any integer and α_{k+1} is the $k + 1^{th}$ complete quotient. It is known that the modular group $\Gamma = PSL(2,\mathbb{Z})$ is generated by τ and φ and the fundamental domain of Γ has a cusp at $\frac{1}{2} + \frac{\sqrt{3}i}{2}$. The stabilizer of this point is the group of

order 3 generated by $\tau\varphi$ with

$$(\tau\varphi)^3 = \tau\varphi\tau\varphi\tau\varphi = 1$$

and so $\tau\rho^{-1}\tau = \varphi$. Thus Γ is generated by τ and ρ. When representing continued fractions as compositions of Möbius maps, it is natural to use the map ϕ rather than φ even though ϕ is not in the modular group. This problem is resolved by considering the hyperbolic plane $\mathbb{H}^2$, as a vertical plane $\mathbb{H}^{\perp}$ of $\mathbb{H}^3$ and by considering the modular group as the subgroup of the Picard group that leaves $\mathbb{R}_{\infty}$ invariant. The action of ϕ on $\mathbb{H}^{\perp}$ corresponds to the action of the inversion map in the unit sphere on $\mathbb{H}^{\perp}$. Certainly $\phi(z) = \frac{i}{iz}$ is in the Picard group. It is the special role of ϕ in the expression of the continued fractions derived from the cutting sequences and of the simple continued fractions that gives an indication of the analogue of this situation in $\mathbb{H}^3$ [36].

Firstly, we have seen in [23] that the cutting sequence generated by a geodesic which passes through the fundamental triangle $\mathbb{T}_0 = \{\infty, 0, 1\}$ and the fundamental geodesic $[0, \infty]$ and ends at a real number α, can be regarded as a path on a graph whose vertices are the Farey triangles in the orbit of $\mathbb{T}_0$ under Γ and whose edges are the Farey geodesics. As such we can express $\alpha \in \mathbb{R}$ as follows:

$$\alpha = \lim_{k\to\infty} \tau^{a_0}\rho^{a_1}\tau^{a_2}\rho^{a_3}\cdots\tau^{a_{k-1}}\rho^{a_k}(\mathbb{T}_0)$$

or

$$\alpha = \lim_{k\to\infty} \tau^{a_0}\phi\tau^{a_1}\phi\tau^{a_2}\phi\tau^{a_3}\cdots\tau^{a_{k-1}}\phi\tau^{a_k}(\mathbb{T}_0).$$

Next we note that ϕ interchanges ∞ and 0 while leaving 1 and hence $\mathbb{T}_0$ fixed. Using the expansion for α given above we can find the successive convergents to α and also a nested chain of Farey intervals that converge to α. That is:

$$\mathbb{I}_0 = [0, \infty], \mathbb{I}_1 = \tau^{a_0}\phi\mathbb{I}_0 = [a_0, \infty], \mathbb{I}_2 = \tau^{a_0}\phi\tau^{a_1}\phi\mathbb{I}_0 = \tau^{a_0}\phi[a_1, \infty] = \left[a_0, a_0 + \frac{1}{a_1}\right], \cdots,$$

with

$$\mathbb{I}_1 \supset \mathbb{I}_2 \supset \mathbb{I}_3 \supset \cdots$$

and

$$\lim_{k\to\infty} \mathbb{I}_k = \alpha.$$

Each interval $\mathbb{I}_k$ is bounded by consecutive convergents of α.

We know from Schmidt [44] that, given any complex number $\alpha \in \mathbb{C}$, we can find a chain of Farey triangles in $\mathbb{C}$ that converge to α.

In the sequel following [26] we wish to build an analogue of the above situation in $\mathbb{H}^3$. We consider the Picard group, a *σ-regular* continued fraction for any $\alpha \in \mathbb{C}$, the tessellation of $\mathbb{H}^3$ by the fundamental octahedron $\mathbb{O}_0$ with cusps $\left\{\infty, 0, 1, i, 1+i, \frac{1}{1-i}\right\}$ and the cutting sequence generated by a geodesic γ in $\mathbb{H}^3$ that ends at the point α and passes through $\mathbb{O}_0$. In what follows, we will use the fact that the Picard group is generated by maps $\tau_i(z) = z + i$, $\tau_1(z) = z + 1$, $\phi(z) = \frac{i}{iz}$ and $\varphi(z) = \frac{-1}{z}$ where we may write $\phi\varphi = \varphi\phi = \psi = \kappa^2$.

6.2 The Fundamental Octahedron $\mathbb{O}_0$ and its Stabilizing Group

The fundamental octahedron $\mathbb{O}_0$ is given as the octahedron whose vertices are orbit of ∞ under the stabilizer of the cusp $v = \frac{1+i+\sqrt{2}j}{2}$ of a fundamental region of the Picard group $\mathcal{P}$. That is

$$\mathbb{O}_0 = \{g(\infty) : g \in Stab(v, \mathcal{P})\} = \left\{\infty, 0, 1, 1+i, i, \frac{1}{1-i}\right\}$$

where we have from [18]

$$Stab(v, \mathcal{P}) = \left\langle \sigma_i, \sigma_1 : \sigma_i^3 = \sigma_1^3 = \sigma^2 = 1 \right\rangle$$

where $\sigma_i(z) = \phi\tau_i^{-1}(z) = \frac{i}{iz+1}$, $\sigma_1(z) = \tau_1\varphi(z) = \frac{z-1}{z}$ and $\sigma_i\sigma_1(z) = \sigma(z) = \frac{iz}{(1+i)z-i}$.

We note that the map σ can be associated with the matrix $\begin{pmatrix} i & 0 \\ 1+i & -i \end{pmatrix}$.

Further we see that $\sigma(\infty) = \frac{1}{1-i}, \sigma\left(\frac{1}{1-i}\right) = \infty, \sigma(i) = 1, \sigma(1) = i$ and that σ fixes both 0 and $1 + i$. In fact σ is a rotation through π radians in the geodesic with endpoints 0 and $1 + i$. Thus σ fixes the fundamental octahedron $\mathbb{O}_0$.

We note that σ is not unique in fixing $\mathbb{O}_0$ and that in fact the $Stab(v, \mathcal{P})$ fixes $\mathbb{O}_0$ and is isomorphic to A_4.

We recall that a *Farey triangle* is defined as the image under any $h \in \mathcal{P}$ of the *fundamental Farey triangle* $\mathbb{T}_0$. The fundamental octahedron has 8 faces, each a Farey triangle, and each can thus be written as an image under $\mathcal{P}$ of the fundamental triangle $\mathbb{T}_0$, such as Example 4.3.4. The Farey tessellation $\mathcal{F}$ of $\mathbb{H}^3$ is thus the collection of these closed octahedra. Each Farey triangle in $\mathbb{C}$ can be circumscribed by a circle in $\mathbb{C}$, while a Farey triangle with ∞ as a cusp is inscribed by a line in $\mathbb{C}$ (see Theorem 4.3.7). Following Schmidt [44] we have established that each Farey triangle in $\mathbb{C}$ is in two different ways subdivisible into 7 Farey triangles, inverse of each other with respect to the circumscribed circle. We have extended this result to include Farey triangles with ∞ as a vertex. The subdivisions in this case are thus inverse with respect to the circumscribing line in $\mathbb{C}$.

In particular, the fundamental triangle $\mathbb{T}_0$ is circumscribed by the real axis and its inner Farey subdivision is given by the triangles $\{\infty, 0, i\}$, $\{\infty, 1, 1 + i\}$, $\{\infty, i, 1 + i\}$ all with ∞ as a cusp and $\left\{0, 1, \frac{1}{1-i}\right\}$, $\left\{1, 1 + i, \frac{1}{1-i}\right\}$, $\left\{i, 1 + i, \frac{1}{1-i}\right\}$ and $\left\{0, 1, \frac{1}{1-i}\right\}$, each with $\frac{1}{1-i}$ as a cusp. The outer Farey subdivision of the fundamental triangle is the inverse of the inner Farey subdivision with respect to the real axis. Since the elements of the Picard group preserve the

Farey neighbour condition (Lemma 4.1.2), the same result will hold for all Farey triangle in $\mathbb{H}^3$. This result is equivalent to saying that each face of the fundamental octahedron, and hence each face of any octahedron in the tessellation, is adjacent to exactly two octahedra in the tessellation. The one octahedron corresponds to the internal subdivision of a given Farey triangle and the other corresponds to the external subdivision of the Farey triangle, and these are inverse of each other with respect to the circumscribing sphere. In particular the Farey triangle $\mathbb{T}_0$ is the common face to $\mathbb{O}_0$ and $\phi(\mathbb{O}_0) = \tau_i^{-1}(\mathbb{O}_0)$. Thus $\mathbb{O}_0$, and hence each octahedron in the tessellation, is adjacent to 8 octahedra.

Thus we can represent the 8 octahedra adjacent to $\mathbb{O}_0$ in two distinct classes. The first 4 all have ∞ as a cusp and are of the form $\tau_1(\mathbb{O}_0), \tau_i(\mathbb{O}_0), \tau_1^{-1}(\mathbb{O}_0)$ and $\tau_i^{-1}(\mathbb{O}_0)$, while the second four, referred to as the floor of $\mathbb{O}_0$ are given as $\sigma\tau_1\sigma(\mathbb{O}_0), \sigma\tau_i\sigma(\mathbb{O}_0), \sigma\tau_1^{-1}\sigma(\mathbb{O}_0)$ and $\sigma\tau_i^{-1}\sigma(\mathbb{O}_0)$ and all have $\frac{1}{1-i}$ as a cusp where $\sigma = \sigma^{-1}$ and $\sigma = \sigma_i\sigma_1$.

6.3 The $\mathbb{O}_0$ - Farey Octahedron Graph

We have stated that $\mathcal{P} = \langle \tau_1, \tau_i, \phi, \varphi \rangle$ where $\phi\varphi = \varphi\phi = \psi = \kappa^2$, $\phi(z) = \frac{1}{z}$, $\varphi(z) = \frac{-1}{z}$ and $\kappa(z) = iz$. Since $\sigma_i = \phi\tau_i^{-1}$ and $\sigma_1 = \tau_1\varphi$, we may write $\mathcal{P} = \langle \sigma_1, \sigma_i, \phi, \varphi \rangle$ and since $\sigma = \sigma_i\sigma_1$, we have $\mathcal{P} = \langle \tau_1, \tau_i, \sigma, \varphi \rangle$ where $\sigma = \phi\tau_1\tau_{-i}\varphi = \phi\tau_{1-i}\kappa^2\phi$ and $\sigma^2 = \varphi^2 = 1$.

Let $T = \langle \tau_i, \tau_1 \rangle$ and $G = \langle \tau_i, \tau_1, \sigma \rangle$, then $T \leq G \leq \mathcal{P}$.

Thus each $h \in \mathcal{P}$ can be represented as $h_1\varphi h_2\varphi \cdots \varphi h_k$, where h_k may be 1 and $h_t \in G$ for all t. Further each $h_t \in G$ can be written as $\tau_{\alpha_1}\sigma\tau_{\alpha_2}\sigma \cdots \sigma\tau_{\alpha_m}$, where τ_{α_m} may be 1 and $\alpha_j \in \mathbb{Z}[i]$ with $\tau_{\alpha_j} = \tau_{a_j+ib_j} = \tau_1^{a_j}\tau_i^{b_j}$.

Consider the graph $\mathcal{G}$ with vertices the orbit of $\mathbb{O}_0$ under $\mathcal{P}$ and whose edges are Farey

triangles common to adjacent octahedra. We have noted above that each octahedron in the orbit is adjacent to 8 octahedra and thus the graph is a regular graph. Further we noted that the adjacent octahedra of $\mathbb{O}_0$ can be written as two distinct classes, those with ∞ as a cusp and those with $\frac{1}{1-i}$ as a cusp.

For any $\alpha \in \mathbb{Z}[i]$ with $\alpha = a+ib$, let $\tau_\alpha = \tau_1^a \tau_i^b$ and $\rho_\alpha = \sigma\tau_\alpha\sigma$. We will use the maps τ_α and ρ_α to define a *cutting sequences* of a geodesic in $\mathbb{H}^3$ in terms of a path on a graph whose vertices are Farey octahedra. In what follows we will use the term "*vertices*" to represent the elements (Farey octahedra) of the $\mathcal{G}$, the Farey octahedron graph, and the term "*cusps*" to refer to the vertices of the individual octahedra.

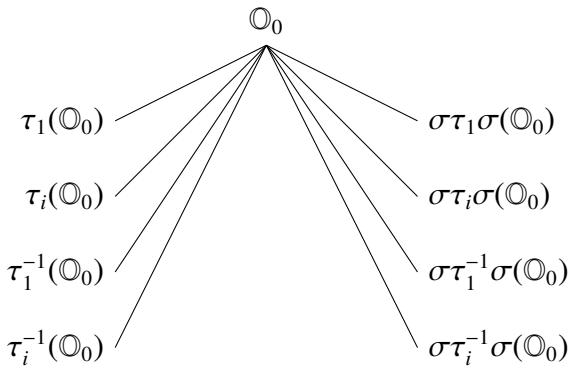


Figure 6.1: The adjacency graph of $\mathbb{O}_0$

Theorem 6.3.1. *Every vertex $\mathbb{O}_k$ of $\mathcal{G}$ can be expressed as $\mathbb{O}_k = \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\rho_{\alpha_3}...\tau_{\alpha_n}(\mathbb{O}_0)$ where $\alpha_k \in \mathbb{Z}[i]$ and α_0 and α_n may be zero. Further any $t \in \mathcal{P}$ may be written as $\tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\rho_{\alpha_3}...\tau_{\alpha_n}h$ with $h \in Stab(v, \mathcal{P})$.*

Proof. We prove the result by induction on $k \geq 0$.

Let $\mathbb{O}_0$ be the fundamental octahedron with 8 adjacent octahedra $\mathbb{O}_{0,i}$, $i = 1 \cdots 8$ given by $\tau_\alpha(\mathbb{O}_0)$ or $\sigma\tau_\alpha\sigma(\mathbb{O}_0)$, $\alpha = \pm 1, \pm i$. Thus

$$\mathbb{O}_{0,i} = \quad \tau_\alpha(\mathbb{O}_0) \text{ or } \sigma\tau_\alpha\sigma(\mathbb{O}_0)$$

$$= \quad \tau_\alpha(\mathbb{O}_0) \text{ or } \rho_\alpha(\mathbb{O}_0), \alpha = \pm 1, \pm i.$$

Thus the result holds for the adjacent octahedra to $\mathbb{O}_0$.

Assume the result holds for $\mathbb{O}_k$, $k \geq 0$. That is, $\mathbb{O}_k = \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}(\mathbb{O}_0) = t(\mathbb{O}_0)$, where $\alpha_i \in \mathbb{Z}[i]$ and only α_0, α_n may be zero. Here $t = \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}$ is in $\mathcal{P}$. Thus $\mathbb{O}_0 = t^{-1}(\mathbb{O}_k)$. The adjacent octahedra to $\mathbb{O}_0$ are $\tau_\alpha(\mathbb{O}_0)$ or $\sigma\tau_\alpha\sigma(\mathbb{O}_0)$, $\alpha = \pm 1, \pm i$ as above. Thus $\mathbb{O}_{0,i} = \tau_\alpha t^{-1}(\mathbb{O}_k)$ or $\sigma\tau_\alpha\sigma t^{-1}(\mathbb{O}_k)$. The adjacent octahedra of $\mathbb{O}_k$, $\mathbb{O}_{k,i}$, $i = 1\cdots 8$ are given as $t(\mathbb{O}_{0,i}) = t\tau_\alpha t^{-1}(\mathbb{O}_k)$ or $t\sigma\tau_\alpha\sigma t^{-1}(\mathbb{O}_k)$.

That is

$$\begin{aligned}
\mathbb{O}_{k,i} = & \begin{cases} t\tau_\alpha t^{-1}(\mathbb{O}_k) \\ t\sigma\tau_\alpha\sigma t^{-1}(\mathbb{O}_k) \end{cases} \quad i = 1\cdots 8 \\
= & \begin{cases} t\tau_\alpha(\mathbb{O}_0) \\ t\sigma\tau_\alpha\sigma(\mathbb{O}_0) \end{cases} \\
= & \begin{cases} \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}\tau_\alpha(\mathbb{O}_0) \\ \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}\sigma\tau_\alpha\sigma(\mathbb{O}_0) \end{cases} \\
= & \begin{cases} \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n+\alpha}(\mathbb{O}_0) \\ \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}\rho_{\alpha_{n+1}}(\mathbb{O}_0) \end{cases}
\end{aligned}$$

as required. Hence the result follows by induction.

It follows from Theorem 3.1.1 that each $t \in \mathcal{P}$ may be written as $\tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}h$ where $h \in Stab(v, \mathcal{P}) = Stab(\mathbb{O}_0, \mathcal{P})$ and $h(\mathbb{O}_0) = \mathbb{O}_0$. □

It is noted that while $\mathcal{G}$ is not a tree, by collecting the branches of the graph by common cusps ∞ and $\dfrac{1}{1-i}$ and then considering their images under $g \in \mathcal{P}$, we may form a graph $\mathcal{G}_0$

that bifurcates into either $\tau_\alpha(\mathbb{O}_k)$ or $\rho_\alpha(\mathbb{O}_k)$ at each vertex $\mathbb{O}_k$ of the graph. In Figure 6.2 we have $\tau_{\alpha_{ij}}$ and $\rho_{\alpha_{ij}}$ initially lying in the i^{th} row and the j^{th} position in the row where $\tau_{\alpha_{ij}} \in T$ and $\rho_{\alpha_{ij}} = \sigma\tau_{\alpha_{ij}}\sigma$.

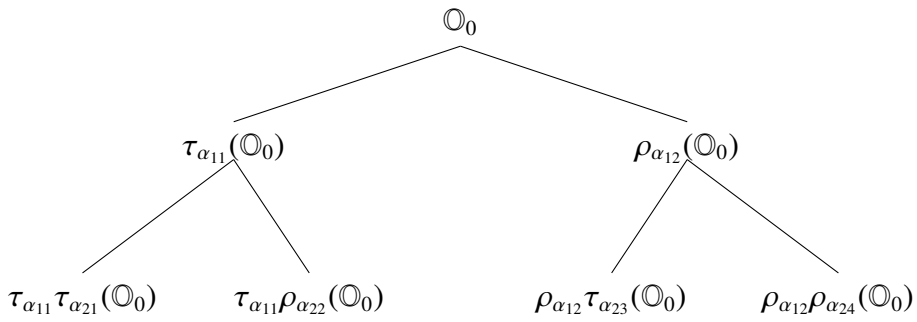


Figure 6.2: Graph $\mathcal{G}_0$

6.4 The σ-regular continued fractions

Let $\zeta_0 \in \mathbb{C}$. Define $\lfloor\zeta_0\rfloor = \alpha_0$ where $\alpha_0 = \lfloor\Re(\zeta_0)\rfloor + i\lfloor\Im(\zeta_0)\rfloor$ and where $\lfloor x\rfloor$ is the integer part of x. Then $\tau_{\alpha_0}^{-1}(\zeta_0) = \zeta_0 - \alpha_0$ lies in the unit square with cusps given as $\{0, 1, i, 1+i\}$. Now $\sigma(z) = \dfrac{z}{(1-i)z-1}$ leaves the cusps of the unit square invariant as a set. This unit square can be regarded as the convex hull of the region bounded by the lines $\Re(z) = 0$, $\Re(z) = 1$, $\Im(z) = 0$, $\Im(z) = 1$. The images of these lines under σ are circles through $\frac{1}{1-i}$. This σ maps the interior of the square to the region exterior to these circles, as seen in Figure 6.3. This region is outside the region of the unit square. Hence $\zeta_1 = \sigma\tau_{\alpha_0}{}^{-1}(\zeta_0)$ lies outside the unit square. Repeating the process by letting $\alpha_1 = \lfloor\zeta_1\rfloor$, we have $\sigma\tau_{\alpha_1}{}^{-1}\sigma\tau_{\alpha_0}^{-1}(\zeta_0) = \zeta_2$ lies outside the unit square. Continuing in this way we find ζ_{k+1} outside the unit square with

$$\zeta_{k+1} = \sigma\tau_{\alpha_k}^{-1}\sigma\tau_{\alpha_{k-1}}^{-1}\cdots\sigma\tau_{\alpha_1}^{-1}\sigma\tau_{\alpha_0}^{-1}(\zeta_0).$$

Thus

$$\zeta_0 = \tau_{\alpha_0}\sigma\tau_{\alpha_1}\cdots\sigma\tau_{\alpha_k}\sigma(\zeta_{k+1}) = \tau_{\alpha_0}\rho_{\alpha_1}\tau_{\alpha_2}\rho_{\alpha_3}\cdots\rho_{\alpha_k}(\zeta_{k+1}).$$

This expansion of ζ_0 in terms of τ and σ is called a σ-regular continued fraction expansion.

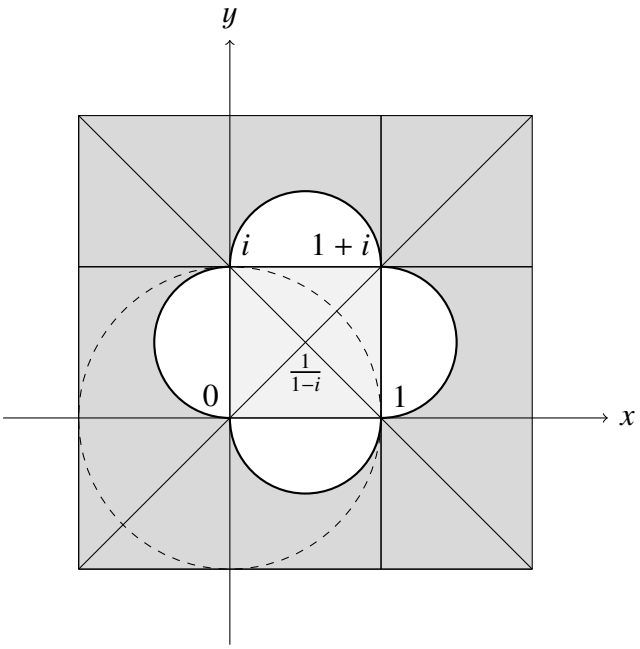


Figure 6.3: The unit square and its image under σ

If we now insert $\sigma = \phi\tau_{1-i}\varphi$ into the σ-regular continued fraction expansion for ζ_0, we have the regular Gaussian integer continued fraction (Definition 5.1.1).

$$\zeta_0 = \tau_{\alpha_0}\phi\tau_{1-i}\varphi\tau_{\alpha_1}\cdots\phi\tau_{1-i}\varphi\tau_{\alpha_k}\phi\tau_{1-i}\varphi(\zeta_{k+1}).$$

6.5 The σ-regular continued fraction and the cutting sequences

Let Λ_0 be a geodesic that emanates from the inside of $\mathbb{O}_0$ to meet $\mathbb{C}$ at a non-Gaussian rational ζ_0. We may divide the geodesic Λ_0 into an infinite sequence of segments, each determined by the sections within the octahedra crossed. That is

$$\Lambda_0 = \Lambda_{(0,1)} \cup \Lambda_{(0,2)} \cup \Lambda_{(0,3)} \cdots \cup \Lambda_{(0,t-1)} \cup \Lambda_{(0,t)} \cup \Lambda_{(0,t+1)} \cdots$$

Let $\Lambda_{(0,t)}$ be the first segment of Λ_0 that lies within a octahedron, $\mathbb{O}_{t_0}$, that does not have ∞ as a cusp. The previous octahedron $\mathbb{O}_{t_0-1}$ cut by Λ_0 and corresponding to the segment $\Lambda_{(0,t-1)}$ has ∞ as a cusp. This octahedron $\mathbb{O}_{t_0-1}$ projects onto a unit square in $\mathbb{C}$ with vertices $\{a_0 + ib_0, (a_0 + 1) + ib_0, a_0 + i(b_0 + 1), (a_0 + 1) + i(b_0 + 1)\}$ which contains ζ_0 and where $\alpha_0 = a_0 + ib_0 = \lfloor \zeta_0 \rfloor$. Thus $\tau_{\alpha_0}^{-1}(\zeta_0)$ lies within the unit square centre 0 that is the projection onto $\mathbb{C}$ of the fundamental octahedron, while $\zeta_1 = \sigma\tau_{\alpha_0}^{-1}(\zeta_0)$ lies outside this latter unit square. As noted before, the cusp set $\{0, 1, i, 1 + i\}$ is left invariant by σ, but since σ interchanges ∞ and $\frac{1}{1-i}$, the inside of the square is mapped outside of the region bounded by the circular images of the sides of the unit square as in Figure 6.3.

Now let $\sigma\tau_{\alpha_0}^{-1}(\Lambda_0) = \Lambda_1$ be the geodesic that emanates from $\mathbb{O}_0$ and whose initial segments are

$$\sigma\tau_{\alpha_0}^{-1}(\Lambda_{(0,t)}), \sigma\tau_{\alpha_0}^{-1}(\Lambda_{(0,t+1)}), \sigma\tau_{\alpha_0}^{-1}(\Lambda_{(0,t+2)}), \cdots \text{ or } \Lambda_{(1,1)}, \Lambda_{(1,2)}, \Lambda_{(1,3)}, \cdots$$

where $\sigma\tau_{\alpha_0}^{-1}(\Lambda_{(0,t)}) = \Lambda_{(1,1)}$ emanates from $\mathbb{O}_0$ and the sequence ends at ζ_1 cutting through copies of $\mathbb{O}_0$. The number of copies of $\mathbb{O}_0$ that have ∞ as a cusp corresponds to the number of octahedrons that are cut by Λ_0 and have $\tau_{\alpha_0}\sigma(\infty) = \tau_{\alpha_0}\left(\frac{1}{1-i}\right)$ as a cusp. The above process can now be repeated with the geodesic Λ_1 emanating from $\mathbb{O}_0$ and ending at ζ_1. Since the initial point ζ_0 is not a cusp of any octahedron in the tessellation, the process will

continue without end.

Thus at some point k we have

$$\zeta_k = \sigma\tau_{\alpha_k}^{-1} \cdots \sigma\tau_{\alpha_2}^{-1}\sigma\tau_{\alpha_1}^{-1}\sigma\tau_{\alpha_0}^{-1}(\zeta_0)$$

and so

$$\zeta_0 = \tau_{\alpha_0}\sigma \cdots \sigma\tau_{\alpha_{k-2}}\sigma\tau_{\alpha_{k-1}}\sigma\tau_{\alpha_k}(\zeta_k).$$

The points $\tau_{\alpha_0}\sigma \cdots \sigma\tau_{\alpha_{k-2}}\sigma\tau_{\alpha_{k-1}}\sigma\tau_{\alpha_k}(\infty)$ are successive (but not consecutive) convergents to ζ_0 for $k = 0, 1, 2, \cdots$. The sequence of triangles $\mathbb{T}_k = \tau_{\alpha_0}\sigma \cdots \sigma\tau_{\alpha_{k-2}}\sigma\tau_{\alpha_{k-1}}\sigma\tau_{\alpha_k}(\mathbb{T}_0)$ form a chain of nested Farey triangles $\mathbb{T}_0 \supset \mathbb{T}_1 \supset \mathbb{T}_2 \supset \cdots \supset \mathbb{T}_k \cdots$ and $\lim_{k\to\infty} \mathbb{T}_k = \zeta_0$ (Theorem 4.5.3). Hence the coefficients $\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ of the cutting sequence formed as the geodesic Λ_0 ending at ζ_0 cuts across the tessellation, yields the coefficients of the σ-regular continued fraction. The coefficients also give rise to a nested sequence of Farey triangles that converge to ζ_0.

Thus we have established:

Theorem 6.5.1. *Let $\zeta = \zeta_0 \in \mathbb{C}$ be any non rational complex number. Then*

$$\zeta = \zeta_0 = \tau_{\alpha_0}\sigma\tau_{\alpha_1} \cdots \tau_{\alpha_{k-1}}\sigma\tau_{\alpha_k}(\zeta_k)$$

where $\alpha_i \in \mathbb{Z}[i]$ and where only α_0 may be zero and ζ_k is the tail of the σ-regular continued fraction. Then ζ_0 has a Gaussian integer continued fraction expansion derived from the cutting sequence formed by a geodesic emanating from $\mathbb{O}_0$ and ending at ζ_0. This derived continued fraction also yields a nested chain of Farey triangles that converge to ζ_0.

Definition 6.5.2. *The σ-regular continued fraction expansion*

$$\zeta_0 = \tau_{\alpha_0}\phi\tau_{1-i}\varphi\tau_{\alpha_1}\phi\tau_{1-i}\varphi \cdots \tau_{\alpha_{k-1}}\phi\tau_{1-i}\varphi\tau_{\alpha_k} \cdots$$

with $\sigma = \phi\tau_{1-i}\varphi$ is called the derived continued fraction from a geodesic cutting the Farey tessellation emanating in $\mathbb{O}_0$ and ending at ζ_0.

Corollary 6.5.3. *The successive convergents given by the derived continued fraction expansion of ζ are Farey neighbours.*

Example 6.5.4. The σ-regular continued fraction for $\frac{-10}{9} + \frac{11}{2}i$ is given as:

$$\frac{-10}{9} + \frac{11}{2}i = \tau_{-2+5i}\sigma\tau_i\sigma\tau_2\sigma\tau_{-2i}\sigma(\infty) = \tau_{-2+5i}\rho_i\tau_2\rho_{-2i}(\infty), \tag{6.5.1}$$

while the derived continued fraction expansion is given as:

$$\frac{-10}{9} + \frac{11}{2}i = \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi\tau_{1-i}\varphi\tau_2\phi\tau_{1-i}\varphi\tau_{-2i}\phi\tau_{1-i}\varphi(\infty). \tag{6.5.2}$$

In (6.5.1), the sequence of nested triangles corresponding to the expansion are:

$$\begin{aligned}
\mathbb{T}_0 &= \{\infty, 0, 1\} \\
\mathbb{T}_1 &= \tau_{-2+5i}(\mathbb{T}_0) = \{\infty, -2+5i, -1+5i\} \\
\mathbb{T}_2 &= \tau_{-2+5i}\rho_i(\mathbb{T}_0) = \left\{-1 + \frac{11}{2}i, -1+5i, \frac{-6}{5} + \frac{27}{5}i\right\} \\
\mathbb{T}_3 &= \tau_{-2+5i}\rho_i\tau_2(\mathbb{T}_0) = \left\{-1 + \frac{11}{2}i, \frac{-19}{17} + \frac{93}{17}i, \frac{-40}{37} + \frac{203}{37}i\right\} \\
\mathbb{T}_4 &= \tau_{-2+5i}\rho_i\tau_2\rho_{-2i}(\mathbb{T}_0) = \left\{\frac{-10}{9} + \frac{11}{2}i, \frac{-307}{277} + \frac{1523}{277}i, \frac{-108}{97} + \frac{533}{97}i\right\}.
\end{aligned}$$

The sequence $\tau_{\alpha_0}\phi(\infty), \tau_{\alpha_0}\phi\tau_{1-i}\varphi(\infty), \tau_{\alpha_0}\phi\tau_{1-i}\varphi\tau_{\alpha_1}\phi(\infty), \cdots$, are convergents to ζ_0 in the derived continued fraction expansion. $\frac{\alpha_k}{\gamma_k} = g_k(\infty)$, $g_k \in \mathcal{P}$ and hence are peaks of pentahedrons [16].

We find the convergents to $\frac{-10}{9} + \frac{11}{2}i$ from (6.5.2) as follows:

$$\frac{\alpha_0}{\gamma_0} = \infty$$

$$\begin{aligned}
\frac{\alpha_1}{\gamma_1} &= \tau_{-2+5i}\phi(\infty) = -2+5i \\
\frac{\alpha_2}{\gamma_2} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi(\infty) = \frac{-3}{2}+\frac{11}{2}i \\
\frac{\alpha_3}{\gamma_3} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi(\infty) = -1+5i \\
\frac{\alpha_4}{\gamma_4} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi\tau_{1-i}\varphi(\infty) = -1+\frac{11}{2}i \\
\frac{\alpha_5}{\gamma_5} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi\tau_{1-i}\varphi\tau_2\phi(\infty) = \frac{-19}{17}+\frac{93}{17}i \\
\frac{\alpha_6}{\gamma_6} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi\tau_{1-i}\varphi\tau_2\phi\tau_{1-i}\varphi(\infty) = \frac{-11}{10}+\frac{11}{2}i \\
\frac{\alpha_7}{\gamma_7} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi\tau_{1-i}\varphi\tau_2\phi\tau_{1-i}\varphi\tau_{-2i}\phi(\infty) = \frac{-307}{277}+\frac{1523}{277}i \\
\frac{\alpha_8}{\gamma_8} &= \tau_{-2+5i}\phi\tau_{1-i}\varphi\tau_i\phi\tau_{1-i}\varphi\tau_2\phi\tau_{1-i}\varphi\tau_{-2i}\phi\tau_{1-i}\varphi(\infty) = \frac{-10}{9}+\frac{11}{2}i
\end{aligned}$$

Using (6.5.2), we generate the sequence of octahedrons corresponding to the expansion. We note that these octahedrons are not necessarily adjacent. They are as follows:

$$\begin{aligned}
\mathbb{O}_0 &= \left\{\infty, 0, 1, 1+i, i, \frac{1}{1-i}\right\} \\
\mathbb{O}_1 &= \tau_{-2+5i}(\mathbb{O}_0) = \left\{\infty, -2+5i, -1+5i, -1+6i, -2+6i, \frac{-3}{2}+\frac{11}{2}i\right\} \\
\mathbb{O}_2 &= \tau_{-2+5i}\sigma(\mathbb{O}_0) = \left\{\frac{-3}{2}+\frac{11}{2}i, -2+5i, -2+6i, -1+6i, -1+5i, \infty\right\} \\
\mathbb{O}_3 &= \tau_{-2+5i}\sigma\tau_i(\mathbb{O}_0) \\
&= \left\{\frac{-3}{2}+\frac{11}{2}i, -1+5i, -1+6i, \frac{-6}{5}+\frac{28}{5}i, \frac{-6}{5}+\frac{27}{5}i, -1+\frac{11}{2}i\right\} \\
\mathbb{O}_4 &= \tau_{-2+5i}\sigma\tau_i\sigma(\mathbb{O}_0) \\
&= \left\{-1+\frac{11}{2}i, -1+5i, \frac{-6}{5}+\frac{27}{5}i, \frac{-6}{5}+\frac{28}{5}i, -1+6i, \frac{-3}{2}+\frac{11}{2}i\right\} \\
\mathbb{O}_5 &= \tau_{-2+5i}\sigma\tau_i\sigma\tau_2(\mathbb{O}_0) \\
&= \left\{-1+\frac{11}{2}i, \frac{-19}{17}+\frac{93}{17}i, \frac{-40}{37}+\frac{203}{37}i, \frac{-40}{37}+\frac{204}{37}i, \frac{-19}{17}+\frac{94}{17}i, \frac{-11}{10}+\frac{11}{2}i\right\} \\
\mathbb{O}_6 &= \tau_{-2+5i}\sigma\tau_i\sigma\tau_2\sigma(\mathbb{O}_0)
\end{aligned}$$

$$= \left\{ \frac{-11}{10} + \frac{11}{2}i, \frac{-19}{17} + \frac{93}{17}i, \frac{-19}{17} + \frac{94}{17}i, \frac{-40}{37} + \frac{204}{37}i, \frac{-40}{37} + \frac{203}{37}i, -1 + \frac{11}{2}i \right\}$$

$$\mathbb{O}_7 = \quad \tau_{-2+5i}\sigma\tau_i\sigma\tau_2\sigma\tau_{-2i}(\mathbb{O}_0)$$

$$= \left\{ \frac{-11}{10} + \frac{11}{2}i, \frac{-307}{277} + \frac{1523}{277}i, \frac{-307}{277} + \frac{1524}{277}i, \frac{-108}{97} + \frac{534}{97}i, \frac{-108}{97} + \frac{533}{97}i, \frac{-10}{9} + \frac{11}{2}i \right\}$$

$$\mathbb{O}_8 = \quad \tau_{-2+5i}\sigma\tau_i\sigma\tau_2\sigma\tau_{-2i}\sigma(\mathbb{O}_0)$$

$$= \left\{ \frac{-10}{9} + \frac{11}{2}i, \frac{-307}{277} + \frac{1523}{277}i, \frac{-108}{97} + \frac{533}{97}i, \frac{-108}{97} + \frac{534}{97}i, \frac{-307}{277} + \frac{1524}{277}i, \frac{-11}{10} + \frac{11}{2}i \right\}$$

We note that Ford [16] establishes the result relating the peaks of the pentahedra and the convergents to ζ_0 as follows.

Theorem 6.5.5. *Let a moving point trace a continuous curve from a point above the plane $t = 1$ to the point ζ_0 in $\mathbb{C}$. Let this curve lie entirely in $\mathbb{H}^3$ except at the point of termination, and let it intersect no base of a pentahedron more than a finite number of times. Then the z-coördinates of the successive peaks of the pentahedra through which the point passes are the convergents, in order, of a regular continued fraction whose value is ζ_0.*

Conversely any regular continued fraction converging to the value ζ_0 can be generated this way.

Thus we see that the successive peaks of the pentahedra, corresponding to the convergents, also correspond to the Ford spheres at the points on $\mathbb{C}$. This sequence of pentahedra is "contained" in a sequence of ideal octahedra. The images of ∞ as explored in Example 6.5.4 will correspond to a subsequence of this chain of peaks. The chain of octahedra gives rise to a chain of Ford spheres that converge to ζ_0 [1].

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