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Conserved Vectors and Symmetry Invariant Solutions to Polynomial Wave Models

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Abstract

In the present paper, we consider a special family of equations that are known to incorporate important wave features in oceanography studies. Physically, these equations admit generalized symmetries and non constant separants which implies they are of great relevance to integrability studies. These equations, are highly nonlinear and once reduced, are not solvable by conventional techniques. We construct series to effectively solve these evolution equations wherein recurrence relations occur and the convergence may be tested. Moreover, the conserved vectors of the equations are established.

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1 Introduction

A special class of third-order polynomial equations exist such that they possess nontrivial Lie-Bäcklund symmetries [1]. These evolution equations appear in the study of waves and oceans, as well as several engineering applications. It is known that via differential substitutions, these equations are connected to the KdV, Sawada-Kotera and Kaup equations [2]. Due to these properties, these equations have applications in fluid dynamics, or specifically ion-acoustic waves in plasmas, stratified internal waves and other nonlinear wave phenomena. Such models, among others, have applications in wave propagation studies [3–5]. This family of equations possess Lie algebras that are commutative. They are:

$$u_t = u_x^3 u_{xxx} + \alpha u_x^3, \tag{1}$$

$$u_t = u^3 u_{xxx} + \frac{3}{2} u^2 u_x u_{xx} + \alpha u^2 u_x, \tag{2}$$

$$u_t = u^3 u_{xxx} + \alpha u^3 u_x, \tag{3}$$

where $u(x,t)$ generally represents a displacement from rest (for example, the height of a fluid above or below rest), and $\alpha \neq 0$ is an arbitrary constant. In the literature, the above types of equations

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Table 1 Lie brackets.

[,]	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$-5X_2$
X_3	$-X_1$	$5X_2$	0

and ones related to them, have been analysed from the point of view of bifurcations, phase portraits, periodic solutions etc. [6–9]. Symmetry methods and conservation laws are fundamental components of the analysis of partial differential equations [10–12]. The theoretical aspects relating to symmetries is well known and is best studied from the book [13]. In this paper, symmetry methods are combined with power series to arrive at solutions. The partial differential equations are reduced to ordinary differential equations, but are difficult to solve with common techniques. However, power series are a brilliant tool to find approximate solutions, and the solutions may be exact if a recurrence relation can be found.

The plan of the study follows. In Section 2, we present the symmetry invariant properties possessed by the three equations, with the commutator, adjoint and group details. Section 3, contains the conservation laws and multipliers calculated for the equations. In Section 4, we present reductions and power series solutions and a demonstration of the convergence of a solution. The discussion and conclusion follows in Section 5 and 6, respectively.

2 Invariant properties

Suppose we induce a Lie group of infinitesimal ($\varepsilon \ll 1$) transformations:

$$x \to x+\varepsilon\xi(x,t,u), \quad t \to t+\varepsilon\tau(x,t,u), \quad u \to u+\varepsilon\phi(x,t,u), \tag{4}$$

with symmetry vector field $X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u$. The coefficients $\xi,\ \tau$ and ϕ satisfy the well known Lie symmetry condition, so that the Lie point symmetries of Eq. (1) and Eq. (2) are

$$X_1 = \partial_x,\ X_2 = \partial_t,\ X_3 = x\partial_x - 5t\partial_t + 3u\partial_u. \tag{5}$$

The Lie brackets of (5) are given in Table 1, and the corresponding one-parameter groups K_m $(m = 1,2,3)$ generated by the symmetries X_m, as

$$K_1 : (x,t,u) \mapsto (x+\varepsilon,t,u), \quad K_2 : (x,t,u) \mapsto (x,t+\varepsilon,u), \quad K_3 : (x,t,u) \mapsto (e^{\varepsilon}x, e^{5\varepsilon}t, e^{3\varepsilon}u). \tag{6}$$

Moreover, we require the adjoints to calculate the one-dimensional subalgebras. The approach for the one-dimensional optimal system is provided in [14]. The adjoints are given in Table 2. One takes a general element from the Lie algebra and reduce it to its simplest like form by applying carefully chosen adjoint transformations $\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i,X_j] + \frac{1}{2}\epsilon^2[X_i,[X_i,X_j]] - \cdots$. We consider

$$X = a_1X_1 + a_2X_2 + a_3X_3, \tag{7}$$

where the a_i's are arbitrary constants, and simplify it by adjoint maps. Suppose first that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by $\mathrm{Ad}(\exp(a_1X_1)$ we get $X' = X_3 + a_2X_2$. Now acting on X' by $\mathrm{Ad}(\exp(-a_2/5)X_2)$ results in X_3. If $a_3 = 0, a_2 = 1$, then we act on $X = a_1X_1 + X_2$ by $\mathrm{Ad}(\exp(c_1X_3)$ resulting in $X_2, X_2 \pm X_1$. Next suppose that $a_3 = a_2 = 0, a_1 = 1$ which gives X_1.

Hence, we obtain in this case, the optimal system of one-dimensional subalgebras spanned by $X_1,\ X_2+\epsilon X_1,\ X_3$, where $\epsilon = 0, \pm 1$. On the other hand, the Lie point symmetries of (3) are

$$X_1,\ X_2,\ \bar{X}_3 = -3t\partial_t + u\partial_u,\ X_4 = u\cos\left(\sqrt{\alpha}x\right)\partial_u + \frac{\sin\left(\sqrt{\alpha}x\right)\partial_x}{\sqrt{\alpha}}, X_5 = u\sin\left(\sqrt{\alpha}x\right)\partial_u - \frac{\cos\left(\sqrt{\alpha}x\right)\partial_x}{\sqrt{\alpha}}. \tag{8}$$

Table 2 Adjoint representation of subalgebras.

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3-\varepsilon X_1$
X_2	X_1	X_2	$5\varepsilon X_2+X_3$
X_3	$e^{\varepsilon}X_1$	$e^{-5\varepsilon}X_2$	X_3

Table 3 Lie brackets.

[,]	X_1	X_2	$\bar{X}_3$	X_4	X_5
X_1	0	0	0	$-\sqrt{\alpha}X_5$	$\sqrt{\alpha}X_4$
X_2	0	0	$-3X_2$	0	0
$\bar{X}_3$	0	$3X_2$	0	0	0
X_4	$\sqrt{\alpha}X_5$	0	0	0	$\frac{X_1}{\sqrt{\alpha}}$
X_5	$-\sqrt{\alpha}X_4$	0	0	$-\frac{X_1}{\sqrt{\alpha}}$	0

Table 4 Adjoint representation of subalgebras.

Ad	X_1	X_2	$\bar{X}_3$	X_4	X_5
X_1	X_1	X_2	$\bar{X}_3$	$X_5\sin(\sqrt{\alpha}\varepsilon)$ $+X_4\cos(\sqrt{\alpha}\varepsilon)$	$X_5\cos(\sqrt{\alpha}\varepsilon)$ $-X_4\sin(\sqrt{\alpha}\varepsilon)$
X_2	X_1	X_2	$3\varepsilon X_2+\bar{X}_3$	X_4	X_5
$\bar{X}_3$	X_1	$e^{-3\varepsilon}X_2$	$\bar{X}_3$	X_4	X_5
X_4	$X_5(\frac{1}{2}\sqrt{\alpha}e^{-\varepsilon}-\frac{\sqrt{\alpha}e^{\varepsilon}}{2})$ $+(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2})X_1$	X_2	$\bar{X}_3$	X_4	$X_1(\frac{e^{-\varepsilon}}{2\sqrt{\alpha}}-\frac{e^{\varepsilon}}{2\sqrt{\alpha}})$ $+(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2})X_5$
X_5	$X_4(\frac{\sqrt{\alpha}e^{\varepsilon}}{2}-\frac{1}{2}\sqrt{\alpha}e^{-\varepsilon})$ $+(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2})X_1$	X_2	$\bar{X}_3$	$X_1(\frac{e^{\varepsilon}}{2\sqrt{\alpha}}-\frac{e^{-\varepsilon}}{2\sqrt{\alpha}})$ $+(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2})X_4$	X_5

The Lie brackets of (8) are given in Table 3.

The one-parameter groups for this case are

$$\bar{K}_3 : (x,t,u)\mapsto (x,e^{-3\varepsilon}t,e^{\varepsilon}u), \tag{9}$$

$$K_4 : (x,t,u)\mapsto \left(\frac{2\cot^{-1}(e^{-\varepsilon}\cot(\frac{\sqrt{\alpha}x}{2}))}{\sqrt{\alpha}},t,\frac{e^{\varepsilon}u(\cot^2(\frac{\sqrt{\alpha}x}{2})+1)}{e^{2\varepsilon}+\cot^2(\frac{\sqrt{\alpha}x}{2})}\right), \tag{10}$$

$$K_5 : (x,t,u)\mapsto \left(\frac{\sin^{-1}(\tanh(\varepsilon-\tanh^{-1}(\sin(\sqrt{\alpha}x))))}{-\sqrt{\alpha}},t,\frac{u\,\mathrm{sech}(\varepsilon-\tanh^{-1}(\sin(\sqrt{\alpha}x)))}{\sqrt{1-\sin^2(\sqrt{\alpha}x)}}\right). \tag{11}$$

The adjoints are given in Table 4, and for this equation, the optimal system of one-dimensional subalgebras is found as follows. Here we consider

$$X = a_1X_1+a_2X_2+a_3\bar{X}_3+a_4X_4+a_5X_5, \tag{12}$$

where the a_i's are arbitrary constants, and we simplify this by the applications of adjoint maps. Suppose firstly that $a_3\neq 0$, then we can assume that $a_3=1$. If we act on X by adjoint maps of X_1,X_2,X_4 and X_5 respectively, with appropriate values of a_i's, we get $\bar{X}_3+aX_1$. If $a_4=1,a_3=0$, then acting on X by adjoint maps of X_4,X_5 respectively, with suitable values of a_i's results in X_4,aX_4+X_2. If $a_5=1,a_4=a_3=0$, then further acting on X by an adjoint map of $\bar{X}_3$ with appropriate values of a_i gives rise to $X_5+bX_1,\ X_5+bX_1+\delta X_2$. Now if $a_1=1,a_5=a_4=a_3=0$ and acting on X by adjoint maps of X_3 with appropriate values of a_i gives rise to $X_2+\epsilon X_1$. If $a_2=1,a_1=a_5=a_4=a_3=0$, we get X_2. Therefore, for

this equation, the optimal system of one-dimensional subalgebras is given by ($a(\neq 0), b$ are constants, $\delta = \pm 1$ and $\epsilon = 0, \pm 1$): $X_2,\ X_4,\ \bar{X}_3 + aX_1,\ X_2 + aX_4,\ X_5 + bX_1,\ X_5 + bX_1 + \delta X_2,\ X_2 + \epsilon X_1$.

3 Conservation laws and components

In this section, we derive the conservation laws associated with Eqs. (1)-(3). A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies $D_i T^i = 0$, along the solutions of a given differential system, say $G_\alpha\left(x, u^{(k)}\right) = 0$. Multipliers that produce conservation laws, are denoted by Λ^α [15]. They satisfy the relation

$$D_i T^i = \Lambda^\alpha G^\alpha, \tag{13}$$

for all functions u^α and the overdetermined equations for Λ^α are

$$\frac{\delta}{\delta u^\alpha}[\Lambda^\alpha G^\alpha] = 0, \tag{14}$$

where $\frac{\delta}{\delta u^\alpha}$ is the Euler operator. Eq. (1) admits two conservation laws, corresponding to the multipliers: $\Lambda^1(x,t,u,u_x,u_{xx}) = -\frac{1}{2}\frac{-2u_{xx}x+u_x}{u_x^3}$, with

$$T_t^1 = -\frac{1}{2}\frac{x}{u_x}, \quad T_x^1 = \frac{1}{2}\frac{-2\alpha u_x^3 x - u_x^2 u_{xx}^2 x + 3\alpha u u_x^2 + u_x^3 u_{xx} - u_t x}{u_x^2}, \tag{15}$$

and $\Lambda^2(x,t,u,u_x,u_{xx}) = \frac{u_{xx}}{u_x^3}$,

$$T_t^2 = -\frac{1}{2u_x}, \quad T_x^2 = -\frac{1}{2}\frac{2\alpha u_x^3 + u_x^2 u_{xx}^2 + u_t}{u_x^2}. \tag{16}$$

Eq. (2) admits four conservation laws, viz. $\Lambda^3(x,t,u,u_x,u_{xx}) = \frac{1}{u^{\frac{3}{2}}}$, admits

$$T_t^3 = -\frac{2}{\sqrt{u}}, \quad T_x^3 = -u^{3/2}u_{xx} - \frac{2}{3}\alpha u^{3/2}. \tag{17}$$

$\Lambda^4(x,t,u,u_x,u_{xx}) = \frac{-2}{u^2}$, admits

$$T_t^4 = \frac{2}{u}, \quad T_x^4 = 2uu_{xx} + \frac{1}{2}u_x^2 - +2\alpha u. \tag{18}$$

$\Lambda^5(x,t,u,u_x,u_{xx}) = \frac{2}{3}\alpha + u_{xx}$, admits

$$T_t^5 = \frac{2}{3}\alpha u - \frac{1}{2}{u_x}^2, \quad T_x^5 = u_t u_x - \frac{2}{3}\alpha u^3 u_{xx} - \frac{2}{9}\alpha^2 u^3 - \frac{1}{2}u^3 {u_{xx}}^2. \tag{19}$$

$\Lambda^6(x,t,u,u_x,u_{xx}) = \frac{2\alpha t u^2 + 3tu^2 u_{xx} + 2x}{u^2}$, admits

$$\begin{aligned} T_t^6 &= \frac{1}{2}\frac{4\alpha t u^2 - 3tu{u_x}^2 - 4x}{u}, \\ T_x^6 &= -\frac{2}{3}\alpha^2 t u^3 - \frac{3}{2}tu^3{u_{xx}}^2 - 2\alpha ux - 2uu_{xx}x - \frac{1}{2}x{u_x}^2 + 2uu_x + 3tu_t u_x - 2\alpha t u^3 u_{xx}. \end{aligned} \tag{20}$$

Eq. (3) admits five conservation laws, viz. $\Lambda^7(x,t,u,u_x,u_{xx}) = \frac{1}{u^2}$, provides the conserved quantities

$$T_t^7 = -\frac{1}{u}, \quad T_x^7 = \frac{1}{2}u_x^2 + \frac{1}{2}\alpha u^2 - uu_{xx}. \tag{21}$$

$\Lambda^8(x,t,u,u_x,u_{xx}) = \frac{1}{u^3}$, provides the conserved quantities

$$T_t^8 = -\frac{1}{2u^2}, \quad T_x^8 = -\alpha u - u_{xx}. \tag{22}$$

$\Lambda^9(x,t,u,u_x,u_{xx}) = \frac{\sin(\sqrt{\alpha}x)}{u^3}$, gives

$$T_t^9 = -\frac{1}{2}\frac{\sin(\sqrt{\alpha}x)}{u^2}, \quad T_x^9 = \cos(\sqrt{\alpha}x)u_x\sqrt{\alpha} - \sin(\sqrt{\alpha}x)u_{xx}, \tag{23}$$

$\Lambda^{10}(x,t,u,u_x,u_{xx}) = \frac{\cos(\sqrt{\alpha}x)}{u^3}$, admits the components

$$T_t^{10} = -\frac{1}{2}\frac{\cos(\sqrt{\alpha}x)}{u^2}, \quad T_x^{10} = -\sin(\sqrt{\alpha}x)u_x\sqrt{\alpha} - \cos(\sqrt{\alpha}x)u_{xx}. \tag{24}$$

$\Lambda^{11}(x,t,u,u_x,u_{xx}) = \frac{1}{2}\frac{\alpha u^2 + 2uu_{xx} - {u_x}^2}{u^2}$, admits the components

$$T_t^{11} = \frac{1}{2}\frac{\alpha u^2 - {u_x}^2}{u}, \quad T_x^{11} = -\frac{1}{8}\frac{\alpha^2u^5 + 4\alpha u^4u_{xx} - 2\alpha u^3{u_x}^2 + 4u^3{u_{xx}}^2 - 4u^2{u_x}^2u_{xx} + u{u_x}^4 - 8u_t u_x}{u}. \tag{25}$$

The above conserved quantities may be expressed using the notation in [16], as follows. Let $\bar{\epsilon}$ denote the space of solutions of a given model, and consider the spatial domain $\Omega(-\infty,\infty)$, then the integration of a non-trivial conservation law over Ω produces a conserved integral $C = \int_\Omega T_t\, dx\,|_{\bar{\epsilon}}$, satisfying $\frac{dC}{dt} = -T_x|_{\partial\Omega}\,\big|_{\bar{\epsilon}}$. That is, we may denote the conserved integrals as $C^k = \int_{-\infty}^{\infty} T_t^k\, dx$, where $k = 1,\ldots,11$. We remark that the physical interpretation of these conservation laws would be more elegant under a Hamiltonian formulation, and while these equations may not all possess an apparent Hamiltonian function, we shall present an approach to solve this problem elsewhere in the near future.

4 Power law solutions

In this section, we draw on the optimal system of one-dimensional subalgebras to reduce Eqs. (1)-(3) to ordinary differential equations. Some of the subsequent ordinary differential equations are highly nonlinear and may be solved using the power series method [17, 18]. Reducing Eqs. (1)-(3) by the generator X_1 gives $y'(t) = 0$, where $u(x,t) = y(t)$. Solving gives the obvious solution $y(t) = c_0$, where c_0 is an arbitrary constant. Hence, each of the three equations possess the constant solution $u(x,t) = c_0$.

4.1 Series solutions for Eq. (1)

Reducing the equation by $X_2 + \epsilon X_1$ gives

$$\epsilon^4 - 3p(s)^2(\alpha\epsilon^3 + p'''(s)) = 0, \tag{26}$$

where $s = \frac{-x+t\epsilon}{\epsilon}$ and $u(x,t) = p(s)$. In search of series solutions, a substitution of the power series

$$p(s) = \sum_{r_1=0}^{\infty} a_{r_1}s^{r_1}, \tag{27}$$

is done for the reduced equation. Hence, we find the series (we set $\epsilon = 1$ for simplicity)

$$p(s) = a_0 + a_1s + a_2s^2 + \frac{s^3(3\alpha a_0^2+1)}{18a_0^2} + \frac{s^4(\alpha a_1 - \frac{a_1(3\alpha a_0^2+1)}{3a_0^2})}{12a_0}$$

$$+\frac{s^5(\alpha a_1^2-\frac{a_1^2(3\alpha a_0^2+1)}{3a_0^2}-4a_1(\alpha a_1-\frac{a_1(3\alpha a_0^2+1)}{3a_0^2})+2\alpha a_0 a_2-\frac{2a_2(3\alpha a_0^2+1)}{3a_0})}{60a_0^2}+\ldots. \tag{28}$$

Reducing Eq. (1) by X_3 gives the ODE

$$\begin{aligned}&9y(z)^3(\alpha+525z^2y''(z)+125z^3y'''(z)+340zy'(z))\\&+15zy(z)^2y'(z)(\alpha+525z^2y''(z)+125z^3y'''(z)+330zy'(z))-y'(z)+54y(z)^4=0,\end{aligned} \tag{29}$$

where $z=tx^5$ and $u(x,t)=x^3y(z)$. A substitution of the power series

$$y(z)=\sum_{r_1=0}^{\infty}a_{r_1}z^{r_1}, \tag{30}$$

into (29), provides the power series solution

$$\begin{aligned}y(z)=a_0+9z(\alpha a_0^3+6a_0^4)+&+21z^2(702a_0^3(\alpha a_0^3+6a_0^4)+9\alpha a_0^2(\alpha a_0^3+6a_0^4))\\&+z^3(110502a_0^3(702a_0^3(\alpha a_0^3+6a_0^4)+9\alpha a_0^2(\alpha a_0^3+6a_0^4))+390258a_0^2(\alpha a_0^3+6a_0^4)^2\\&+399\alpha a_0^2(702a_0^3(\alpha a_0^3+6a_0^4)+9\alpha a_0^2(\alpha a_0^3+6a_0^4))+1539\alpha a_0(\alpha a_0^3+6a_0^4)^2)+\ldots.\end{aligned} \tag{31}$$

4.2 Series solutions for Eq. (2)

Reducing the equation by $X_2+\epsilon X_1$ gives

$$p'(s)(2\epsilon^3+p(s)^2(2\alpha\epsilon^2+3p''(s)))+2p(s)^3p'''(s)=0, \tag{32}$$

where $s=\frac{-x+t\epsilon}{\epsilon}$ and $u(x,t)=p(s)$. Similar to above, the series solution for (32) is

$$\begin{aligned}p(s)=a_0+a_1s+a_2s^2+&\frac{s^3(-\alpha a_1a_0^2-3a_1a_2a_0^2-a_1)}{6a_0^3}\\&\frac{s^4(-2\alpha a_2a_0^2-2\alpha a_1^2a_0-\frac{9a_1(-\alpha a_1a_0^2-3a_1a_2a_0^2-a_1)}{2a_0}-6a_2^2a_0^2-6a_1^2a_2a_0-2a_2)}{24a_0^3}+\ldots.\end{aligned} \tag{33}$$

Reducing the equation by X_3 gives

$$\begin{aligned}&(12y(z)^4-2y'(z)+3zy(z)^2y'(z)(2\alpha+54zy'(z)+27z^2y''(z))+y(z)^3\\&(4\alpha+246zy'(z)+270z^2y''(z)+54z^3y'''(z)))=0,\end{aligned} \tag{34}$$

where $z=tx^3$ and $u(x,t)=x^2y(z)$. In this case, we have the series solution

$$\begin{aligned}y(z)=a_0+2z(\alpha a_0^3+3a_0^4)+&\frac{3}{2}z^2(98a_0^3(\alpha a_0^3+3a_0^4)+6\alpha a_0^2(\alpha a_0^3+3a_0^4))\\&+2z^3(135a_0^3(98a_0^3(\alpha a_0^3+3a_0^4)+6\alpha a_0^2(\alpha a_0^3+3a_0^4))+324a_0^2(\alpha a_0^3+3a_0^4)^2\\&+3\alpha a_0^2(98a_0^3(\alpha a_0^3+3a_0^4)+6\alpha a_0^2(\alpha a_0^3+3a_0^4))+8\alpha a_0(\alpha a_0^3+3a_0^4)^2)+\ldots.\end{aligned} \tag{35}$$

4.3 Series solutions for Eq. (3)

A reduction of the equation by the symmetry combination $X_2+\epsilon X_1$ gives

$$\epsilon^2(\epsilon+\alpha p(s)^3)p'(s)+p(s)^3p'''(s)=0, \tag{36}$$

where $s = \frac{-x+t\epsilon}{\epsilon}$ and $u(x,t) = p(s)$. Thus, the series solution is

$$p(s) = a_0 + a_1 s + a_2 s^2 + \frac{s^3(-\alpha a_1 a_0^3 - a_1)}{6a_0^3} + \frac{s^4(-2\alpha a_2 a_0^3 - 3\alpha a_1^2 a_0^2 - \frac{3a_1(-\alpha a_1 a_0^3 - a_1)}{a_0} - 2a_2)}{24a_0^3} + \dots \tag{37}$$

A reduction by X_4 yields $y'(t) = 0$, where $u(x,t) = \sin(\sqrt{\alpha}x)y(t)$. Hence a solution is $u(x,t) = \sin(\sqrt{\alpha}x)c_1$. A reduction by $\bar{X}_3 + aX_1$ yields

$$-3zy(z)^3((\alpha a^2 + 21)y'(z) + 9z(4y''(z) + zy^{(3)}(z))) + a^3 y'(z) - (\alpha a^2 + 1)y(z)^4 = 0, \tag{38}$$

where $u(x,t) = e^{x/a}y(z)$ and $z = te^{\frac{3x}{a}}$, having power series

$$\begin{aligned} y(z) = a_0 &+ \frac{a_0^4 z(\alpha a^2 + 1)}{a^3} + \frac{z^2(\frac{7\alpha a_0^7(\alpha a^2+1)}{a} + \frac{67a_0^7(\alpha a^2+1)}{a^3})}{2a^3} \\ &+ \frac{z^3}{3a^3}(\frac{15\alpha a_0^{10}(\alpha a^2+1)^2}{a^4} + \frac{195a_0^{10}(\alpha a^2+1)^2}{a^6} + \frac{5\alpha a_0^3(\frac{7\alpha a_0^7(\alpha a^2+1)}{a} + \frac{67a_0^7(\alpha a^2+1)}{a^3})}{a} \\ &+ \frac{173a_0^3(\frac{7\alpha a_0^7(\alpha a^2+1)}{a} + \frac{67a_0^7(\alpha a^2+1)}{a^3})}{a^3}) + \dots. \end{aligned} \tag{39}$$

A reduction by $X_2 + aX_4$ yields

$$a^2 y'(z)(a - \alpha^{3/2} y(z)^3) + \alpha^{3/2} y(z)^3 y^{(3)}(z) = 0, \tag{40}$$

where $z = \frac{at - \log(\sin(\frac{\sqrt{\alpha}x}{2})) + \log(\cos(\frac{\sqrt{\alpha}x}{2}))}{a}$ and $u(x,t) = \sin(\sqrt{\alpha}x)y(z)$. The corresponding power series is

$$\begin{aligned} y(z) = a_0 &+ a_1 z + a_2 z^2 + \frac{z^3(a^2\alpha^{3/2}a_0^3 a_1 - a^3 a_1)}{6\alpha^{3/2}a_0^3} \\ &+ \frac{z^4(3a^2 a_0^2 a_1^2 \alpha^{3/2} + 2a^2 a_0^3 a_2 \alpha^{3/2} - \frac{3a_1(a^2\alpha^{3/2}a_0^3 a_1 - a^3 a_1)}{a_0} - 2a^3 a_2)}{24\alpha^{3/2}a_0^3} + \dots. \end{aligned} \tag{41}$$

For this case, X_2 produces a reduced equation $y^{(3)}(x) + \alpha y'(x) = 0$, where $u(x,t) = y(x)$ and we solve to find $y(x) = -\frac{C_2\cos(\sqrt{\alpha}x)}{\sqrt{\alpha}} + \frac{C_1\sin(\sqrt{\alpha}x)}{\sqrt{\alpha}} + C_3$, with C_1, C_2, C_3 as arbitrary. A reduction by $X_5 + bX_1$ gives us $u(x,t) = (\sqrt{\alpha}b - \cos(\sqrt{\alpha}x))y(t)$, where $y'(t) = 0$. Hence the invariant solution is $u(x,t) = (\sqrt{\alpha}b - \cos(\sqrt{\alpha}x))c_0$, where c_0 is an arbitrary constant as before. Finally, a reduction by $X_5 + bX_1 + \delta X_2$ yields the equation

$$\alpha^{3/2}\delta(\alpha b^2 - 1)y(z)^3 + 1)y'(z) + \alpha^{3/2}\delta^3 y(z)^3 y^{(3)}(z) = 0,$$

where $u(x,t) = (\sqrt{\alpha}b - \cos(\sqrt{\alpha}x))y(z)$ and $z = \frac{2\delta\tanh^{-1}(\frac{(\sqrt{\alpha}b+1)\tan(\frac{\sqrt{\alpha}x}{2})}{\sqrt{1-\alpha b^2}})}{\sqrt{1-\alpha b^2}} + t$. A solution to the reduced equation via power series is

$$\begin{aligned} y(z) = a_0 &+ a_1 z + a_2 z^2 + \frac{z^3(a_0^3 a_1 \alpha^{3/2}\delta - a_0^3 a_1 \alpha^{5/2} b^2 \delta - a_1)}{6\alpha^{3/2}a_0^3\delta^3} \\ &\frac{z^4(3a_0^2 a_1^2 \alpha^{3/2}\delta + 2a_0^3 a_2 \alpha^{3/2}\delta - 3a_0^2 a_1^2 \alpha^{5/2} b^2 \delta - 2a_0^3 a_2 \alpha^{5/2} b^2 \delta - \frac{3a_1(a_0^3 a_1 \alpha^{3/2}\delta - a_0^3 a_1 \alpha^{5/2} b^2 \delta - a_1)}{a_0} - 2a_2)}{24\alpha^{3/2}a_0^3\delta^3} + \dots \end{aligned}$$

4.4 Testing for convergence

Each of the series solutions given above may be expressed as a recurrence relation. We illustrate this below for Eq. (26) with the series (27). Eq. (26) can be expressed as

$$1+3\alpha\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}a_{r_1-r_2}a_{r_2}s^{r_1}$$
$$-3\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}s^{r_1}=0. \tag{42}$$

From (42) comparing coefficients of the power of s, we find that a_0, a_1, a_2 are arbitrary,

$$a_3=\frac{\left(3\alpha a_0^2+1\right)}{18a_0^2}, \tag{43}$$

and in general for $r_1\geq 1$, we have

$$a_{r_1+3}=\frac{-\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{(r_1+3)(r_1+2)(r_1+1)a_0^2}$$
$$+\frac{\alpha\sum_{r_2=0}^{r_1}a_{r_1-r_2}a_{r_2}}{(r_1+3)(r_1+2)(r_1+1)a_0^2}. \tag{44}$$

Therefore, from (44), we can obtain the constants in the power series solution. Next we prove the convergence of the power series solution (27). From (44) we obtain

$$|a_{r_1+3}|\leq M(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}|a_{r_1-r_2+3}||a_{r_2-r_3}||a_{r_3}|+\sum_{r_2=0}^{r_1}|a_{r_1-r_2}||a_{r_2}|), \tag{45}$$

where $r_1=1,2,3...$, and $M=\max\{\frac{1}{a_0^2},\frac{|\alpha|}{a_0^2}\}$. Suppose we have the power series $\mu=R(s)=\Sigma_{r_1=0}^{\infty}p_{r_1}s^{r_1}$ where

$$p_k=|a_k|\ k=0,1,\ldots,3 \text{ and} \tag{46}$$

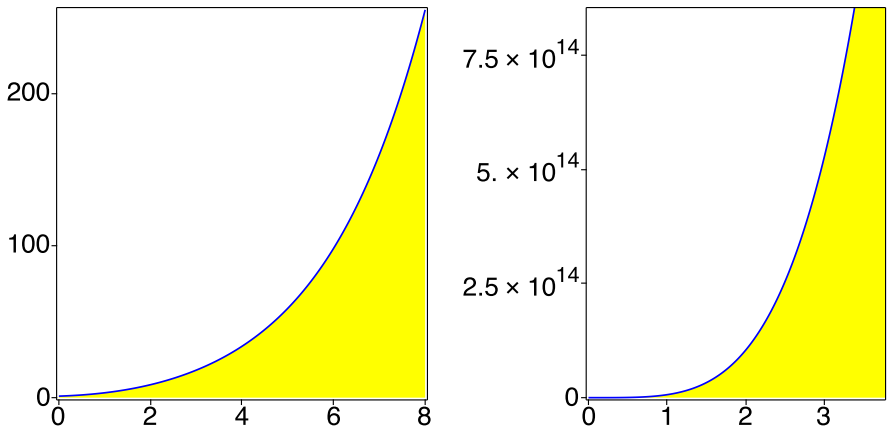
$$p_{r_1+3}=M(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3}+\sum_{r_2=0}^{r_1}p_{r_1-r_2}p_{r_2}). \tag{47}$$

Hence $|a_{r_1}|\leq p_{r_1},\ r_1=0,1,2...$ Next we prove that μ is convergent in a neighbourhood of a point, and μ is a majorant series of Eq. (27), hence:

$$\begin{aligned}R(s)&=p_0+p_1s+p_2s^2+p_3s^3+\sum_{r_1=1}^{\infty}p_{r_1+3}s^{r_1+3}=p_0+p_1s+p_2s^2+p_3s^3\\&\quad+M\sum_{r_1=1}^{\infty}(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3}+\sum_{r_2=0}^{r_1}p_{r_1-r_2}p_{r_2})s^{r_1+3}\\&=p_0+p_1s+p_2s^2+p_3s^3\\&\quad+M(\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3}s^{r_1+3}+\sum_{r_1=1}^{\infty}\sum_{r_2=0}^{r_1}p_{r_1-r_2}p_{r_2}s^{r_1+3})\\&=p_0+p_1s+p_2s^2+p_3s^3+M[R^3(s)+R^2(s^3-p_1s-p_2s^2)+\nu(s)],\end{aligned} \tag{48}$$

where $\nu(s)$ is a polynomial with respect to s, that is $\nu(s)=-2p_0p_1p_2s^3-3p_0^2p_3s^3-p_0p_1^2s^2-2p_0^2p_2s^2-p_0^2s^3-2p_0^2p_1s-p_0^3$. Let

$$F(s,\mu)=\mu-p_0-p_1s-p_2s^2-p_3s^3-M[R^3(s)+R^2(s^3-p_1s-p_2s^2)+\nu(s)], \tag{49}$$



(a) (b)

Fig. 1 We select the parameter values $\alpha = a_k = 1$: Plots of (a) Eq. (28); (b) Eq. (31).

be the implicit function equation, where we obtain that $F(0,p_0) = 0$ and $F_\mu(0,r_0) = 1 - 3Mp_0^2 \neq 0$. By the implicit function theorem [19], $\mu = R(s)$ is analytic and convergent in a neighbourhood of the point $(0,p_0)$ in the plane and with a positive radius, then the power series solution (27) is convergent in the neighbourhood of a point $(0,p_0)$. Therefore (27) can be written as

$$\begin{aligned} p(s) &= a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \Sigma_{r_1=1}^{\infty} a_{r_1+3} s^{r_1+3} = a_0 + a_1 s + a_2 s^2 + a_3 s^3 \\ &+ \Sigma_{r_1=1}^{\infty} \Big(\frac{-\sum_{r_2=1}^{r_1} \sum_{r_3=0}^{r_2} (r_1 - r_2 + 3)(r_1 - r_2 + 2)(r_1 - r_2 + 1) a_{r_1 - r_2 + 3} a_{r_2 - r_3} a_{r_3}}{(r_1+3)(r_1+2)(r_1+1)a_0^2} \\ &+ \frac{\alpha \sum_{r_2=0}^{r_1} a_{r_1 - r_2} a_{r_2}}{(r_1+3)(r_1+2)(r_1+1)a_0^2} \Big) s^{r_1+3}. \end{aligned} \tag{50}$$

Thus the exact power series solution of Eq. (1) can be written as

$$\begin{aligned} u(x,t) &= a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^3 + \Sigma_{r_1=1}^{\infty} a_{r_1+3}(t-x)^{r_1+3} \\ &= a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^4 \\ &+ \Sigma_{r_1=1}^{\infty} \Big(\frac{-\sum_{r_2=1}^{r_1} \sum_{r_3=0}^{r_2} (r_1 - r_2 + 3)(r_1 - r_2 + 2)(r_1 - r_2 + 1) a_{r_1 - r_2 + 3} a_{r_2 - r_3} a_{r_3}}{(r_1+3)(r_1+2)(r_1+1)a_0^2} \\ &+ \frac{\alpha \sum_{r_2=0}^{r_1} a_{r_1 - r_2} a_{r_2}}{(r_1+3)(r_1+2)(r_1+1)a_0^2} \Big) (t-x)^{r_1+3}, \end{aligned} \tag{51}$$

where $a_0 \neq 0, a_1, a_2$, are arbitrary constants, a_3 is given in (43) and the rest of the constants may be determined by (44). This produces the solution (28).

5 Graphical results and discussions

In this section, we plot truncated solutions for some of the results found above. In Figs.1-3, we have graphical solutions for the ODEs from reductions of Eq. (1) and (3), and the resulting graphical solutions for Eq. (1) and (3). The solution (28) corresponds to a travelling wave reduction and we observe that the solution (31) grows much larger than (28). The graphical solutions for Eq. (2) are quite similar to those of Eq. (1). The truncated solution of (37) is positive at small values of the independent variable

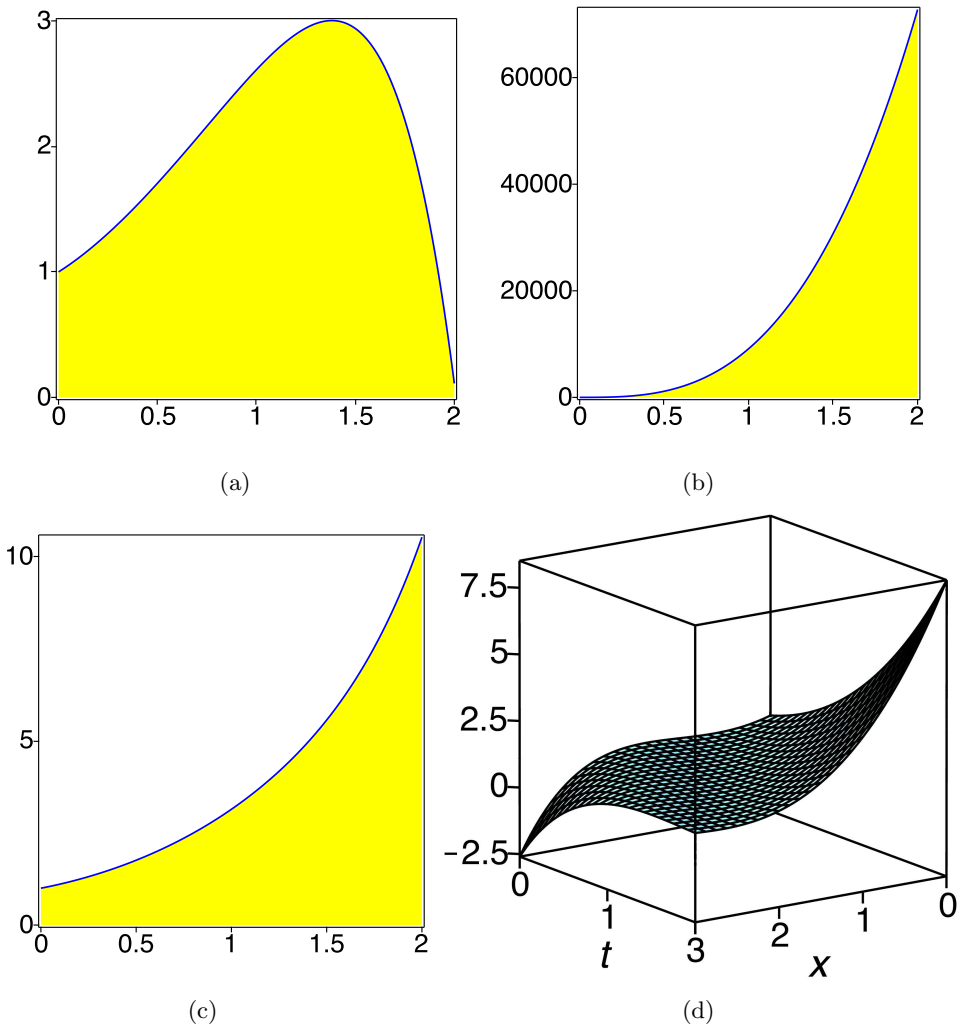


Fig. 2 We select the parameter values $\alpha = a_k = 1$: Plots of (a) Eq. (37); (b) Eq. (39); (c) Eq. (41); (d) Eq. (28).

but becomes negative at higher values while the solution for (39) grows the fastest. We have shown that (28) converges but, similarly, and very tediously, we can test for convergence to all reduced equations in this paper. The solutions can then be transformed back into original variables, given the invertible transformations stated for each reduction.

6 Conclusion

In this study, we have compiled the reversible transformations that render a class of polynomial equations invariant. The reduced equations are difficult to solve exactly using most techniques, but with power series, solutions are obtainable. In fact, we observe that a power series is effective in solving ordinary differential equations admitted by the Lie symmetry reduction approach. The convergence of such series may be tested algorithmically in a manner shown above. The optimal system of subalgebras and conserved quantities were also derived to gain further insights into the family of equations.

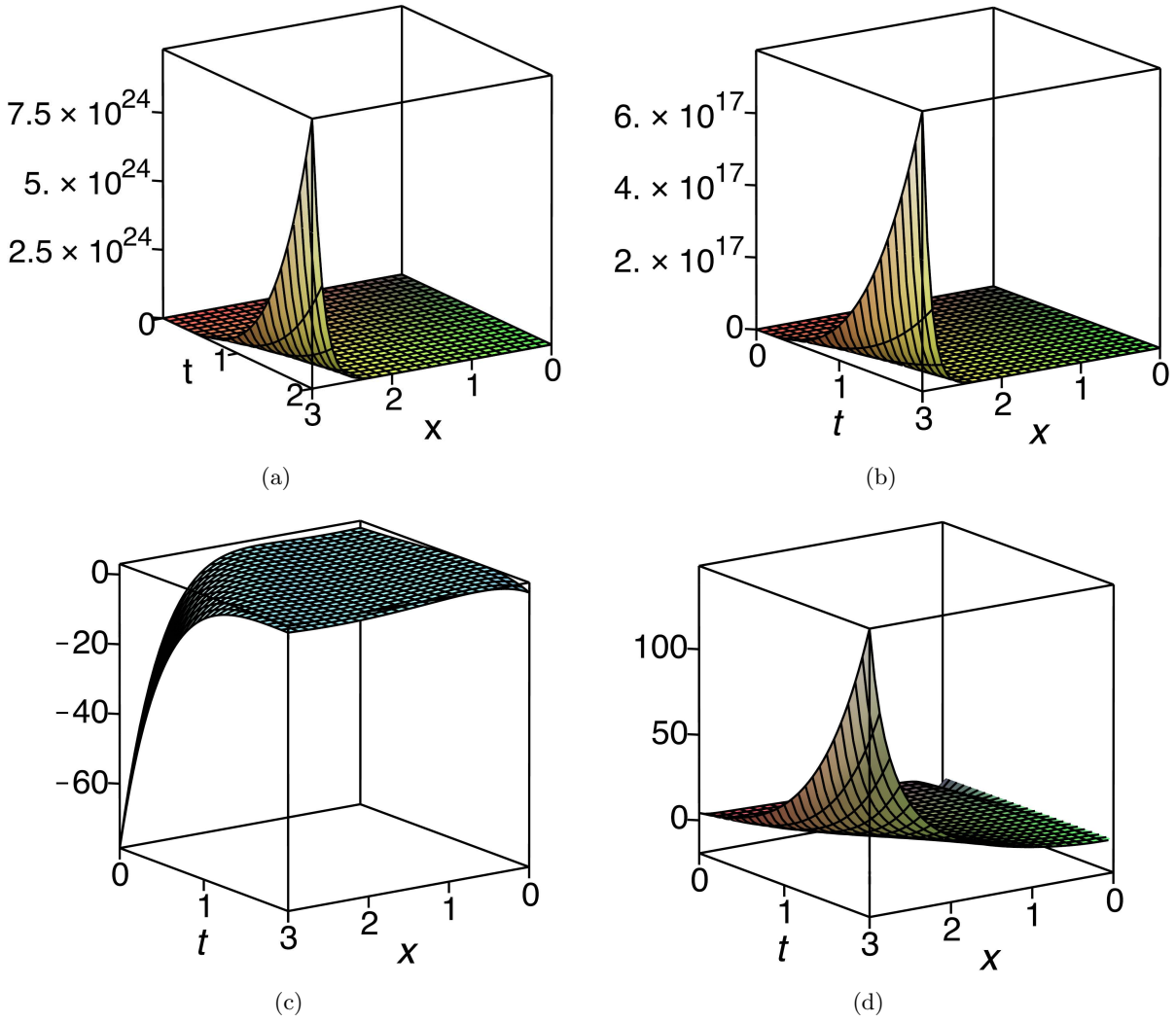


Fig. 3 Parameter values $\alpha = a_k = 1$: 3D plots of (a) Eq. (31); (b) Eq. (39); (c) Eq. (37); (d) Eq. (41).

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