Combinatorial Properties of Lattice Paths

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Declaration

I hereby declare that the work contained in this dissertation is my original work, and that any work done by others or myself previously has been acknowledged and referenced accordingly.

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Abstract

We study a type of lattice path called a skew Dyck path which is a generalization of a Dyck path. Therefore we first introduce Dyck paths and study their enumeration according to various parameters such as number of peaks, valleys, doublerises and return steps. We study characteristics such as bijections with other combinatorial objects, involutions and statistics on skew Dyck paths. We then show enumerations of skew Dyck paths in relation to area, semi-base and semi-length. We finally introduce superdiagonal bargraphs which are associated with skew Dyck paths and enumerate them in relation to perimeter and area.

— Nolwazi M. Dube

Contents

Chapter 1

Dyck Paths

1.0.1 Introduction

In this dissertation, we study Dyck paths and skew Dyck paths and their enumerating properties. We do this by studying research papers titled *Dyck path enumeration* by E. Deutsch, *Skew Dyck paths* and *Skew Dyck paths, area and superdiagonal bargraphs* by E. Deutsch, E. Munarini and S. Rinaldi, see [3], [4], [5]. We mainly chose these papers because they consider the combinatorial properties within these Dyck paths and skew Dyck paths and also consider combinatorial properties in relation to other combinatorial objects.

In Chapter 1, we give an introduction and survey on Dyck paths as we are to use them in later chapters by studying the research paper on Dyck path enumeration. We do this by discussing some preliminaries and terminology used in relation to Dyck paths. We will then give a brief overview of the enumeration of Dyck paths in relation to length and other various parameters, see [3]. In Chapter 2, we study the research paper on skew Dyck paths mainly in relation to their semi-length. We show several combinatorial results that arise from these skew Dyck paths. Some of these combinatorial results include showing a bijection between these skew paths and other combinatorial objects, showing that there exists involutions between skew Dyck paths and there are several statistics on these skew Dyck paths. In the last chapter (Chapter 3), we further study skew Dyck paths according to their area. We then extend the definition of a skew Dyck path to form a superdiagonal bargraph where we enumerate these superdiagonal bargraphs in relation to their area and perimeter.

1.0.2 Preliminaries and terminology on Dyck paths

Definition 1.0.1. *A* ***Dyck path*** *is a path on a square lattice which is in the first quadrant with up steps, $U = (1,1)$ and down steps, $D = (1,-1)$, beginning at $(0,0)$ and ending at $(2m,0)$. The length of a Dyck path is the total number of up and down* *steps, see* [3], [4]*. The semi-length then is equal to half the total number of up and down steps, namely, m. Figure* 1.1 *illustrates a Dyck path.*

Definition 1.0.2. *A **peak** is an up step immediately followed by a down step, i.e., UD.*

Definition 1.0.3. *A **valley** is a down step immediately followed by an up step, i.e., DU.*

Definition 1.0.4. *A **doublerise** is an up step immediately followed by another up step, i.e., UU.*

Definition 1.0.5. *The **height** or **level** of a doublerise, valley or peak is the point of intersection of the two steps involved.*

Definition 1.0.6. *A **return step** is a down step, D, at level* 1 *or at a point where $y = 1$. A **primitive Dyck path** is a path that contains exactly one return step.*

Definition 1.0.7. *Any maximal string of down (up) steps in a Dyck path is called a **descent (ascent)**. The **length** of a Dyck path is its total number of up and down steps.*

Definition 1.0.8. *A **concatenation** of Dyck paths σ and γ is defined as $\sigma\gamma$.*

Definition 1.0.9. *An **elevation** of σ is defined as $\hat{\sigma} = U\sigma U$ where σ and $\hat{\sigma}$ are Dyck paths.*

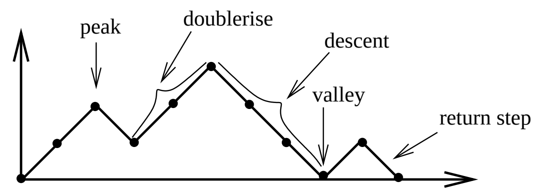


Figure 1.1: A Dyck path with steps $UUDUUDDDUD$, $m = 5$.

We denote an empty path by ρ and graphically by $\bullet$.

Let $\mathbf{C}$ and $\mathbf{D}$ be nonempty sets of Dyck paths. Therefore the concatenation $\mathbf{CD}$ of $\mathbf{C}$ and $\mathbf{D}$ is

$$\mathbf{CD} = \{\sigma\gamma : \sigma \in \mathbf{C},\ \gamma \in \mathbf{D}\}$$

and the elevation $\hat{\mathbf{C}}$ of $\mathbf{C}$ is

$$\hat{\mathbf{C}} = \{\hat{\gamma} : \gamma \in \mathbf{C}\}.$$

Consequently $\mathbf{C} = \mathbf{C}\{\rho\} = \{\rho\}\mathbf{C}$.

Let $\mathbb{K}$ be the class of Dyck paths and $\mathbb{K}_m$ be the class of Dyck paths with semi-length m where $\mathbb{K} = \cup_{m=0}^{\infty}\mathbb{K}_m$. Thus $\mathbb{K}_0 = \{\rho\}$. The decomposition of a Dyck path illustrated in Figure 1.2 below is given by

$$\sigma = U\gamma_1 D\lambda_1 \ \text{ or } \ \rho \tag{1.1}$$

where γ_1 and $\lambda_1 \in \mathbb{K}$. Here $\sigma = U\gamma_1 D\lambda_1$ is known as the *first return decomposition* because the down step, D, is the initial return step of the Dyck path to the x-axis. Figure 1.2 below illustrates this decomposition.

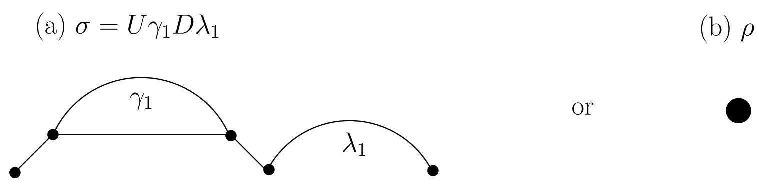


Figure 1.2: The decomposition of a Dyck path.

The decomposition can also be shown as:

$$\sigma = \gamma_2 U\lambda_2 D \ \text{ or } \ \rho \tag{1.2}$$

where γ_2 and $\lambda_2 \in \mathbb{K}$. This is illustrated in Figure 1.3 below.

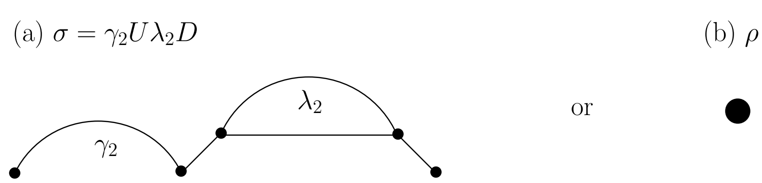


Figure 1.3: The decomposition of a Dyck path.

Now, (1.1) implies that, see [3],

$$\mathbb{K}_m = \hat{\mathbb{K}}_0\mathbb{K}_{m-1} \cup \hat{\mathbb{K}}_1\mathbb{K}_{m-2} \cup ... \cup \hat{\mathbb{K}}_{m-2}\mathbb{K}_1 \cup \hat{\mathbb{K}}_{m-1}\mathbb{K}_0,\ m \geq 1. \tag{1.3}$$

Similarly, (1.2) implies that, see [3],

$$\mathbb{K}_m = \mathbb{K}_0\hat{\mathbb{K}}_{m-1} \cup \mathbb{K}_1\hat{\mathbb{K}}_{m-2} \cup ... \cup \mathbb{K}_{m-2}\hat{\mathbb{K}}_1 \cup \mathbb{K}_{m-1}\hat{\mathbb{K}}_0,\ m \geq 1. \tag{1.4}$$

Since (1.3) and (1.4) are disjoint unions, we hence obtain, see [3],

$$|\mathbb{K}_m| = |\mathbb{K}_0||\mathbb{K}_{m-1}| + |\mathbb{K}_1||\mathbb{K}_{m-2}| + ... + |\mathbb{K}_{m-2}||\mathbb{K}_1| + |\mathbb{K}_{m-1}||\mathbb{K}_0|,\ m \geq 1. \tag{1.5}$$

As $\mathbb{K}_m$ is the class of Dyck paths with semi-length m then the cardinality of Dyck paths with semi-length m, denoted by $|\mathbb{K}_m|$, is the Catalan number, c_m, see [3], [4], for $m \geq 0$ where

$$c_m = \binom{2m}{m}\frac{1}{m+1}. \tag{1.6}$$

The generating function for these c_m is, see [3], [4],

$$\begin{aligned} c(q) &= \sum_{m\geq 0} c_m q^m \\ &= \frac{1-\sqrt{1-4q}}{2q}. \end{aligned} \tag{1.7}$$

1.1 Enumeration of Dyck paths

We now find the generating function for Dyck paths in relation to semi-length and some parameters such as number of peaks, return steps, valleys and doublerises and enumerate some results from these generating functions.

Let b be a fixed parameter of Dyck paths where $b \in \mathbb{N}$. If a parameter b is additive, then $b(\sigma\gamma) = b(\sigma) + b(\gamma)$, for all Dyck paths σ, γ. If T is a finite set of Dyck paths, then $B_T(x)$ is the enumerating polynomial of T corresponding to parameter b:

$$B_T(x) = \sum_{\sigma \in T} x^{b(\sigma)}.$$

If $\mathbf{C}$ and $\mathbf{D}$ are finite sets of Dyck paths and $\mathbf{C} \cap \mathbf{D} = \emptyset$, then, see [3]

$$B_{\mathbf{C}\cup\mathbf{D}}(x) = B_{\mathbf{C}} + B_{\mathbf{D}}. \tag{1.8}$$

Let $B_m(x)$ be the enumerating polynomial of the class of m-Dyck paths, where x represents the parameter in question. By m-Dyck paths, we mean a finite set of Dyck paths with semi-length m. We also define $\hat{B}_m(x)$ to be the enumerating polynomial of the primitive Dyck paths of semi-length $m+1$, see [3]. Let $\Phi(x, y)$ be the generating function for Dyck paths in relation to semi-length, marked by y and parameter, marked by x defined by:

$$\Phi(x, y) = \sum_{m \geq 0} B_m(x) y^m \tag{1.9}$$

and also define

$$\hat{\Phi}(x, y) = \sum_{m \geq 0} \hat{B}_m(x) y^m. \tag{1.10}$$

We let $\Phi(x, y) = \Phi$ and $\hat{\Phi}(x, y) = \hat{\Phi}$ to simplify our calculations. In the enumerating problems we will encounter, we are going to solve them in the following way:

1. We study the impact of concatenation on Dyck paths in relation to parameter then make use of $(1.3), (1.4)$ and (1.8). Finally state B_m in terms of

$$B_{m-1}, B_{m-2}, ..., B_0, \hat{B}_{m-1}, \hat{B}_{m-2}, ..., \hat{B}_0. \tag{1.11}$$

2. We study the impact of elevation on Dyck paths in relation to parameter and then state

$$\hat{B}_m \text{ in terms of } B_m. \tag{1.12}$$

3. Then from (1.9) to (1.12) we eliminate Φ by obtaining two relations between Φ and $\hat{\Phi}$. From that we get an equation satisfied by $\Phi(x, y)$.

Notation: The notation $[y^m]R(y)$ is used in this chapter and later chapters which denotes the coefficient of y^m in the series of $R(y)$.

1.1.1 Enumeration according to the number of peaks

The number of peaks is an additive parameter because if we take any two Dyck paths say σ and γ, then $\sigma\gamma$ consists of the exact number of peaks in σ and in γ. For any additive parameter x, if $\mathbf{J}$ and $\mathbf{K}$ are two finite sets, we obtain $B_{\mathbf{JK}}(x) = B_{\mathbf{J}}(x) B_{\mathbf{K}}(x)$. **JK** is a concatenation of Dyck paths from **J** and **K**. Then (1.11) becomes

$$B_m(x) = B_{m-1}(x)\hat{B}_0(x) + B_{m-2}(x)\hat{B}_1(x) + ... + B_0(x)\hat{B}_{m-1}(x), \; m \geq 1 \tag{1.13}$$

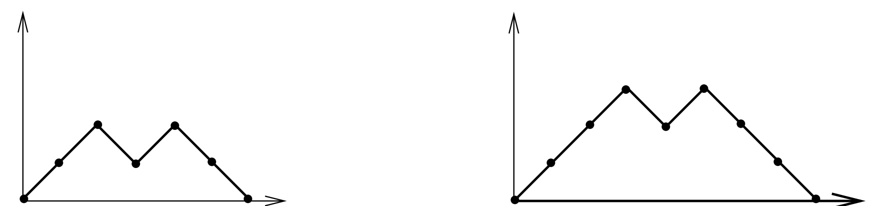
which gives:

$$\Phi - 1 = y\Phi\hat{\Phi}, \tag{1.14}$$

where y denotes the semi-length Here $B_m(x)$ represents the number of peaks in Dyck paths of semi-length m. The elevation on this parameter is given by

$$\hat{B}_m(x) = \begin{cases} x & \text{for } m = 0 \\ B_m(x) & \text{for } m \geq 1. \end{cases} \tag{1.15}$$

This suggests that the number of peaks does not change by elevation except the empty path that becomes UD, see Figure 1.4 below.



(a) Dyck path, σ. (b) Elevation of σ, i.e $\hat{\sigma} = U\sigma D$

Figure 1.4: The elevation of a Dyck path.

From (1.15), we have, see [3]

$$\hat{\Phi} = \Phi + x - 1. \tag{1.16}$$

Solving (1.14) and (1.16) simultaneously, hence eliminating $\hat{\Phi}$, we obtain:

$$\begin{aligned} \Phi - 1 &= y\Phi\hat{\Phi} \\ &= y\Phi(\Phi + x - 1) \\ &= y\Phi^2 + xy\Phi - y\Phi \\ 0 &= y\Phi^2 - \Phi(1 + y - yx) + 1. \end{aligned} \tag{1.17}$$

We then introduce the Narayana function $\varrho = \varrho(x, y)$ which we use to find the generating function for Dyck paths according to the number of peaks. It is defined implicitly by, see [3],

$$(1 + \varrho)(1 + x\varrho)y = \varrho \quad \text{where} \quad \varrho(x, 0) = 0. \tag{1.18}$$

Giving

$$\varrho(x,y) = \frac{1 - y - xy - \sqrt{1 - 2y + y^2 - 2xy - 2xy^2 + x^2y^2}}{2xy}. \tag{1.19}$$

Solving (1.17) and (1.18) simultaneously and hence eliminating y, gives the generating function for Dyck paths in relation to the number of peaks and semi-length, see [3],

$$\Phi = 1 + x\varrho(x,y), \tag{1.20}$$

We now state the Lagrange Inversion Theorem which we use in this chapter and the next two chapters.

Theorem 1.1.1. ***Lagrange Inversion Theorem***, *see* [3],
Let $T(q)$ be a generating function with the assumption that it satisfies the functional equation

$$T(q) = 1 + qF(T(q)) \tag{1.21}$$

where $F(\beta)$ is a polynomial in β. Then (1.21) has a unique solution in $T(q)$ and if $B(\beta)$ is a polynomial in β, then

$$[q^h]B(T(q)) = \frac{1}{h}[\beta^{h-1}]B'(1+\beta)(F(1+\beta))^h \text{ for } h \geq 1. \tag{1.22}$$

Then applying the Lagrange Inversion formula to (1.18) yields

$$[y^m]\Phi = \frac{1}{m}[\beta^{m-1}](1+\beta)^m(1+x\beta)^m$$

Then

$$\begin{aligned} [x^s]\frac{1}{m}[\beta^{m-1}](1+\beta)^m(1+x\beta)^m &= \frac{1}{m}[\beta](1+\beta)^m\beta^s\binom{m}{s} \\ &= \frac{1}{m}\binom{m}{s}[\beta^{m-1-s}](1+\beta)^m \\ &= \frac{1}{m}\binom{m}{s}\binom{m}{m-1-s} \\ &= \frac{1}{m}\binom{m}{s}\binom{m}{s-1}. \end{aligned}$$

Hence

$$[x^s y^m]\Phi = \frac{1}{m}\binom{m}{s}\binom{m}{s-1}, \text{ m} \geq 1, \tag{1.23}$$

which is the Nayarana results, see [3].

1.1.2 Enumeration according to the number of return steps and peaks

Recall that the number of peaks (marked by x) is an additive parameter. A return step is also an additive parameter because any Dyck path that is elevated has precisely one return step. Since we have two parameters we will introduce another marker for the second one, namely, w. Hence we have

$$\hat{B}_m(x,w) = \begin{cases} xw & \text{if } m = 0 \\ wB_m(x,1) & \text{if } m \geq 1. \end{cases} \tag{1.24}$$

From (1.24), the generating function for Dyck paths in relation to number of return steps, number of peaks and semi-length is, see [3],

$$\hat{\Phi}(x,y,w) = xw - w + w\Phi(x,1,y). \tag{1.25}$$

Solving (1.14) and (1.25) simultaneously, hence eliminating $\hat{\Phi}$, we obtain:

$$\begin{aligned} \Phi - 1 &= y\Phi\hat{\Phi} \\ \Phi(x,1,y) &= y\Phi(x,w,y)(xw - w + w\Phi(x,1,y)) \\ &= wy\Phi(x,w,y)(x - 1 + \Phi(x,1,y)). \end{aligned} \tag{1.26}$$

By setting $w = 1$ we have $\Phi(x,1,y) = \Phi(x,y) = \Phi$, then

$$\begin{aligned} \Phi - 1 &= y\Phi(x - 1 + \Phi) \\ \Phi - 1 &= xy\Phi - y\Phi + y\Phi^2 \\ \Phi - 1 &= xy\Phi - y\Phi + y\Phi^2 \\ 0 &= y\Phi^2 - \Phi(1 + y - xy) + 1 \\ 0 &= 1 + x\varrho(x,y), \end{aligned} \tag{1.27}$$

where $\varrho(x,y)$ is the Narayana function and (1.27) is the generating function for Dyck paths in relation to number of peaks and semi-length, see (1.20). Therefore as a result we get, see [3],

$$\Phi(x,w,y) = \frac{1}{1 - xwy(1 + \varrho(x,y))}. \tag{1.28}$$

Now expanding (1.28), we have the following geometric series, see [3]

$$\sum_{f=0}^{\infty}\Big(wxy(1+\varrho(x,y))\Big)^f. \tag{1.29}$$

Then by the use of Nayarana's results, see [3], we obtain

$$[x^s w^r y^m]\Phi(x,w,y) = \begin{cases} \frac{r}{s}\binom{m-1}{s-1}\binom{m-1-r}{s-r} & \text{if } s>0, r<\text{m} \\ 1 & \text{if } s=r=m \\ 0 & \text{otherwise.} \end{cases} \tag{1.30}$$

1.1.3 Enumeration according to the number of return steps

The generating function for the enumeration for Dyck paths in relation to semi-length and number of return steps is found by letting $x=1$ and renaming w by x in (1.28). Using (1.19) in (1.28) we have

$$\begin{aligned}\Phi(1,x,y) &= \frac{1}{1-xy(1+\varrho(1,y))} \\ &= \frac{1}{1-xy\Big(1+\frac{1-2y-\sqrt{1-2y+y^2-2y-2y^2+y^2}}{2y}\Big)} \\ &= \frac{1}{1-xy(1+\frac{1-2y-\sqrt{1-4y}}{2y})} \\ &= \frac{1}{1-xyc(y)},\end{aligned} \tag{1.31}$$

where $c(y)$ is the generating function for Catalan numbers, see [3]. We know from [3] that

$$[y^m]c(y)^h = \frac{h}{2m+h}\binom{2m+h}{m} \quad \text{where} \quad (m,h)\neq(0,0). \tag{1.32}$$

From (1.32) we derive

$$[x^s y^m]\Phi = \frac{s}{2m-s}\binom{2m-s}{m}. \tag{1.33}$$

Now, from (1.31) we obtain,

$$\frac{\partial\Phi}{\partial x}\Big|_{x=1} = yc(y)^3. \tag{1.34}$$

1.1.4 Enumeration according to the number of doublerises

The number of doublerises is an additive parameter because if σ and γ are Dyck paths then the number of double rises for the concatenation, $\sigma\gamma$, consists exactly of the number of doublerises of σ and those of γ. We mentioned earlier that the number of peaks is also additive and therefore the number of doublerises can be enumerated in a similar manner. However since for every m-Dyck path the total number of doublerises and number of peaks is equal to m. By this we mean that each rise will either turn into a doublerise or a peak. Therefore we obtain, see [3],

$$\Phi_{drise}(x, y) = \Phi_{peaks}(x^{-1}, xy). \tag{1.35}$$

From (1.20), we get, see [3]

$$\Phi_{drise}(x, y) = 1 + \varrho(x, y). \tag{1.36}$$

1.1.5 Enumeration according to the number of valleys

The number of valleys can be enumerated in a similar way as the number of peaks. For an m-Dyck path, we find that the number of peaks in one more than the number of valleys, see [3]. Hence from (1.20), we obtain

$$\Phi_{valley}(x, y) = 1 + \varrho(x, y). \tag{1.37}$$

1.2 Summary

In this chapter, we introduced a type of lattice path called a Dyck path and we discussed some definitions and terminology associated to it. We further looked at some enumerations of these Dyck paths according to different parameters namely, number of peaks, return steps, doublerises and valleys. We now move on to a similar type of lattice path called a skew Dyck path.

Chapter 2

Skew Dyck paths

In this chapter we study the characteristics of skew Dyck paths and show that there are bijections between them and other combinatorial objects. We also look at four different types of involutions on skew Dyck paths and finally discuss some statistics on these skew paths.

2.1 Introduction and elementary properties

In this section we give basic definitions,terminology and some properties related to skew Dyck paths.

Definition 2.1.1. *A* ***skew Dyck path*** *is a path in the first quadrant which originates at* $(0,0)$ *and ends at the* x*-axis consisting of up steps,* $U = (1,1)$*, left steps,* $L = (-1,-1)$ *and down steps,* $D = (1,-1)$*, making sure that there is no intersection between up steps and left steps.*

Definition 2.1.2. *The* ***length*** *of a skew Dyck path,* $2h$ *is the total number of up, down and left steps and the semi-length refers to half the number of its steps ie.,* h*, see* [4].

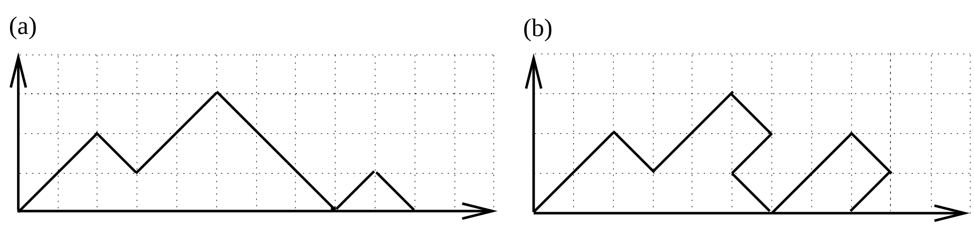


Figure 2.1: (a) A Dyck path with 10 steps ($h = 5$) and (b) a skew Dyck path with 12 steps ($h = 6$).

Figure 2.1 is an example of a Dyck path with steps $UUDUUDDDUD$ and a skew Dyck path with steps $UUDUUDLDUUDL$. We notice that a skew Dyck path of length $2h$ does not necessarily mean that it will end at $(2h, 0)$, see [4]. From Figure 1.2 we note that the difference between (a) and (b) is that (b) includes left steps, L, and (a) does not, which tells us that (b) is a generalization of (a).

Let ρ denote an empty path of a skew Dyck path or represented graphically by $\bullet$. Define $\mathbb{W}$ to be the set of all skew Dyck paths and $\mathbb{W}_h$ to be the set of all skew paths with semi-length h where $\mathbb{W} = \cup_{h=0}^{\infty}\mathbb{W}_h$. The skew Dyck path can either be an empty path or decomposed as $U\gamma D\mu$, where $\gamma, \mu \in \mathbb{W}$ or $U\varphi L$, where $\varphi \in \mathbb{W}$, $\varphi \neq \rho$, see [4]. This decomposition is illustrated in Figure 2.2 below.

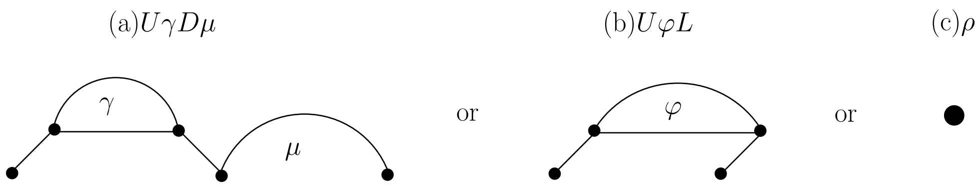


Figure 2.2: A decomposition of a skew Dyck path.

From Figure 2.2 the composition of the generating function, $s(q)$, for the enumeration of skew Dyck paths in terms of semi-length, counted by q, is as follows:

$$s(q) = 1 + qs^2(q) + q(s(q) - 1). \tag{2.1}$$

Where,

1. 1 represents the empty path.,
2. $qs^2(q)$ represents the skew path with decomposition $U\gamma D\mu$. Since with semi-length we consider half the number of steps, we consider up steps only. Then as shown in Figure 2.3 there is only one up step represented by q and there are two different skew Dyck paths γ and μ represented by $s^2(q)$.
3. Here, $q(s(q) - 1)$ represents the skew path with decomposition $U\varphi L$. Since there is only one up step, then we have q. There is only one skew Dyck path represented by φ. However, since $\varphi \neq \rho$ (i.e, φ is not an empty path), we subtract the empty path from $s(q)$ i.e., $(s(q) - 1)$.

Solving for $s(q)$ we get,

$$\begin{aligned}
s(q) &= 1 + qs^2(q) + q(s(q) - 1) \\
&= 1 + qs^2(q) + qs(q) - q \\
s(q)(1-q) &= 1 + qs^2(q) - q \\
0 &= qs^2(q) - (1-q)s(q) + (1-q).
\end{aligned}$$

Hence

$$\begin{aligned}
s(q) &= \frac{1 - q - \sqrt{(1-q)^2 - 4q(1-q)}}{2q} \\
&= \frac{1 - q - \sqrt{1 - 6q + 5q^2}}{2q} && (2.2) \\
&= 1 + q + 3q^2 + 10q^3 + 36q^4 + 137q^5 + ... && (2.3)
\end{aligned}$$

In (2.2) above, we use the negative square root because $\lim_{q\to 0} s(q) = 0$ with the negative root and $\lim_{q\to 0} s(q) = \frac{2}{0} = \infty$ with the positive root. As shown above in (2.3), the first six terms of the sequence $s(q)$ given by the coefficients s_h are $1, 1, 3, 10, 36, 137...$, see [8]. Figure 2.3 shows the ten skew Dyck paths with semi-length 3.

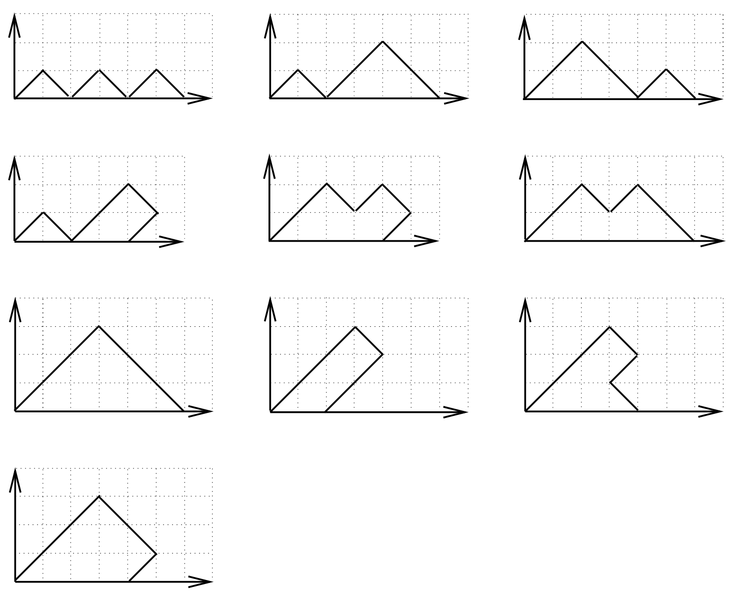


Figure 2.3: Skew Dyck paths with semi-length 3.

If we let $s(q) = 1 + q(s(q)^2 + s(q) - 1)$, then by the Lagrange Inversion theorem, see Theorem 1.1.1, we have

$$s_h = [q^h]s(q) = \frac{1}{h}[\beta^{h-1}](1+3\beta+\beta^2)^h \text{ for every } h \geq 1.$$

We then expand $(1+3\beta+\beta^2)$ into three trinomials namely $(1+\beta)^2+\beta$, $(1-\beta)^2+5\beta$ and $(1+3\beta)+\beta^2$. We then obtain the following lemmas.

Lemma 2.1.3. *see* [4],

$$s_h = \sum_{p=1}^{h} \binom{h-1}{p-1} c_p.$$

Proof.

$$\begin{aligned}
s_h &= [q^h]s(q) \\
&= \frac{1}{h}[\beta^{h-1}](1+3\beta+\beta^2)^h \\
&= \frac{1}{h}[\beta^{h-1}]((1+\beta)^2+\beta)^h \\
&= \frac{1}{h}[\beta^{h-1}]\sum_{p=0}^{h}\binom{h}{p}(1+\beta)^{2p}\beta^{h-p} \\
&= \frac{1}{h}\sum_{p=0}^{h}\binom{h}{p}[\beta^{p-1}](1+\beta)^{2p} \\
&= \frac{1}{h}\sum_{p=0}^{h}\binom{h}{p}\binom{2p}{p-1} \\
&= \frac{1}{h}\sum_{p=0}^{h}\frac{h!}{(h-p)!p!}\times\frac{(2p)!}{(p-1)!(p+1)!} \\
&= \sum_{p=0}^{h}\frac{(h-1)!}{(h-p)!(p-1)!}\times\frac{(2p)!}{p!p!}\times\frac{1}{p+1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{p=1}^{h}\binom{h-1}{p-1}\binom{2p}{p}\frac{1}{p+1} \\
&= \sum_{p=1}^{h}\binom{h-1}{p-1}c_p.
\end{aligned}$$

□

Lemma 2.1.4. *see* [4],

$$s_h = \sum_{p=1}^{h} \binom{h-1}{p-1} (-1)^{p-1} 5^{h-p} c_p.$$

Proof.

$$\begin{aligned}
s_h &= [q^h]s(q) \\
&= \frac{1}{h}[\beta^{h-1}](1+3\beta+\beta^2)^h \\
&= \frac{1}{h}[\beta^{h-1}]((1-\beta)^2+5\beta)^h \\
&= \frac{1}{h}[\beta^{h-1}]\sum_{p=0}^{h}\binom{h}{p}(1-\beta)^{2p}(5\beta)^{h-p} \\
&= \frac{1}{h}\sum_{p=0}^{h}\binom{h}{p}5^{h-p}[\beta^{h-1}](1-\beta)^{2p}\beta^{h-p} \\
&= \frac{1}{h}\sum_{p=0}^{h}\binom{h}{p}5^{h-p}[\beta^{p-1}](1-\beta)^{2p} \\
&= \frac{1}{h}\sum_{p=0}^{h}\binom{h}{p}5^{h-p}\binom{2p}{p-1}(-1)^{p-1} \\
&= \sum_{p=1}^{h}\frac{(h-1)!}{(h-p)!(p-1)!}\times\frac{(2p)!}{(p-1)!(p+1)!}\times\frac{1}{p}\times(-1)^{p-1}5^{h-p} \\
&= \sum_{p=1}^{h}\binom{h-1}{p-1}\binom{2p}{p}\frac{(-1)^{p-1}5^{h-p}}{p+1} \\
&= \sum_{p=1}^{h}\binom{h-1}{p-1}(-1)^{p-1}5^{h-p}c_p.
\end{aligned}$$

□

Lemma 2.1.5. *see* [4],

$$s_h = \sum_{p=1}^{\lfloor\frac{h-1}{2}\rfloor} \binom{h}{p}\binom{h-p}{p+1}3^{h-2p-1}.$$

Proof.

$$\begin{aligned}
s_h &= [q^h]s(q) \\
&= \frac{1}{h}[\beta^{h-1}]((1+3\beta)+\beta^2)^h \\
&= \frac{1}{h}[\beta^{h-1}]\sum_{p=0}^{h}\binom{h}{p}(1+3\beta)^{h-p}\beta^{2p} \\
&= \frac{1}{h}\sum_{p=0}^{h}\binom{h}{p}[\beta^{h-2p-1}](1+3\beta)^{h-p} \\
&= \frac{1}{h}\sum_{p=0}^{\lfloor\frac{h-1}{2}\rfloor}\binom{h}{p}\binom{h-p}{h-2p-1}3^{h-2p-1} \\
&= \frac{1}{h}\sum_{p=0}^{\lfloor\frac{h-1}{2}\rfloor}\binom{h}{p}\frac{(h-p)!}{(p+1)!(h-2p-1)!}3^{h-2p-1} \\
&= \frac{1}{h}\sum_{p=0}^{\lfloor\frac{h-1}{2}\rfloor}\binom{h}{p}\binom{h-p}{p+1}3^{h-2p-1}.
\end{aligned}$$

□

Remark 2.1.6.

Lemma 2.1.4 shows that $(s_h)_{h\geq 1}$ is a binomial transform of $(c_p)_{p\geq 1}$. From this observation with the use of (1.7), we find that, see [4],

$$s(q) = c\Big(\frac{q}{1-q}\Big). \tag{2.4}$$

where $c(q)$ is the generating function for the Catalan numbers.

Proof.

$$\begin{aligned}
c\Big(\frac{q}{1-q}\Big) &= \frac{1-\sqrt{1-4(\frac{q}{1-q})}}{\frac{2q}{1-q}} \\
&= \frac{1-q-\sqrt{(1-q)^2-4q(1-q)}}{2q} \\
&= \frac{1-q-\sqrt{1-6q+5q^2}}{2q} \\
&= s(q).
\end{aligned}$$

□

2.2 Bijections with other combinatorial objects

We shall show that skew Dyck paths and some combinatorically defined objects are equinumerous. Since two of these objects involve trees we will start with some definitions and terminology.

Definition 2.2.1. *A **graph**, P, is a mathematical structure that consists of two sets $V(P)$ and $E(P)$. The elements of $V(P)$ are known as **vertices** and that of $E(P)$ are known as **edges**. That is $V(P) = \{v_1, v_2, v_3, \cdots, v_n\}$ consisting of n vertices and $E(P) = \{e_1, e_2, e_3, \cdots, e_m\}$ consisting of m edges. Each edge has, at most, two vertices associated to it, which are called its endpoints, see* [9].

Figure 2.4 shows an illustration of a graph.

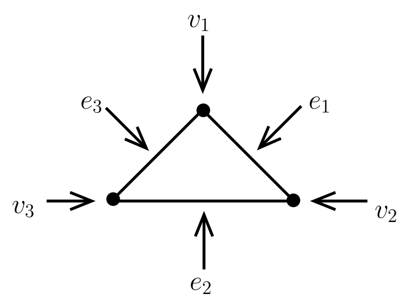


Figure 2.4: A graph called a triangle with 3 with set edges $\{e_1, e_2, e_3\}$ and set vertices $\{v_1, v_2, v_3\}$.

Definition 2.2.2. *A **path** in a graph is a sequence of edges that connects a sequence of vertices. A path is simple if there are no repeated edges and no repeated vertices, except possibly the initial and final vertex. A cycle is a simple path beginning and ending at the same vertex, see* [9].

Definition 2.2.3. *A graph is **simple connected** if every vertex is joined to every other vertex by a path, see* [9].

Definition 2.2.4. *A **tree**, T_h, is a simple connected graph without any cycles consisting of h vertices and $h-1$ edges. Trees may be **rooted** i.e., have a distinguished vertex. The rooted vertex is either at the top or bottom of the tree. In this case, $h-1$ of the vertices are referred to as children except for the root. A subtree of T_h is a tree ${T_m}'$ whose vertices and edges form a subset of T_h i.e., $h \geq m$, see* [9].

Definition 2.2.5. *A **binary tree** is a rooted tree where each vertex has none, one or two vertices connected to it by a path, see* [9].

Definition 2.2.6. *An **ordered tree** is a rooted tree where the arrangement of the subtrees is important, see* [9].

2.2.1 Bijection from Hex trees to skew Dyck paths

Definition 2.2.7. *A Hex tree is a tree which is rooted where each vertex consists of none, one or two vertices connected to it and when only one vertex is available, it is either a right, middle or a left vertex, see* [4]*.*

In this case Hex trees will have their rooted vertices at the bottom. Hex trees and tree-like polyhexes rooted at an edge have a bijection between them. A *polyhex* is an assemble of regular hexagons where two hexagons connect to each other by sharing an edge or rather a common edge. Some polyhexes have points common to three edges called a peri-connexion, see [6]. Other polyhexes form rings of hexagons, (e.g, a central hexagon surrounded by six adjacent hexagons). To form a tree-like polyhex, we need to form a *cata-polyhex*, (a polyhex with no peri-connexion and no rings of hexagons), see [6]. If we take a vertex and position it in the middle of the hexagon and then connect any two vertices by an edge with the condition that the two hexagons share a common edge then we obtain a Hex tree. The term Hex comes from this simple bijection. Figure 2.5 illustrates this bijection.

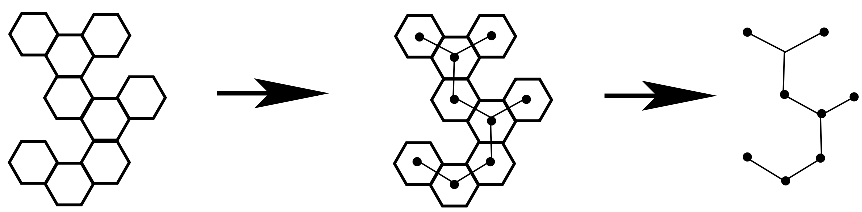


Figure 2.5: Bijection from a tree-like polyhex to a Hex tree.

If we let $\mathbb{X}$ be the class of all Hex trees and $\mathbb{W}$ be the set of all skew Dyck paths. Then the map $(-)' : \mathbb{W}\backslash\{\bullet\} \rightarrow \mathbb{X}$ is defined in Figure 2.6

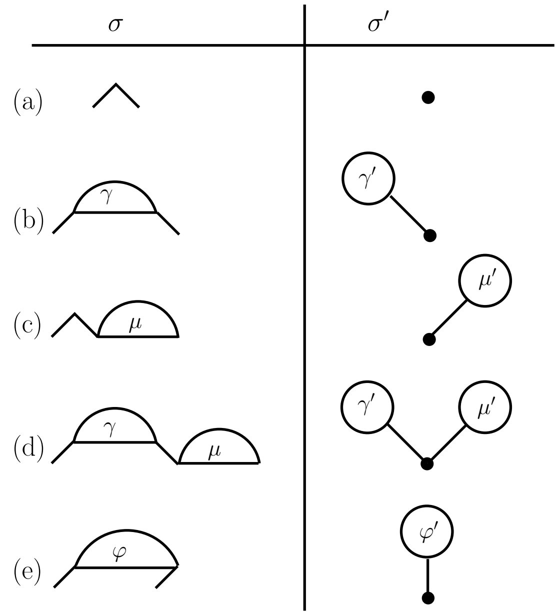
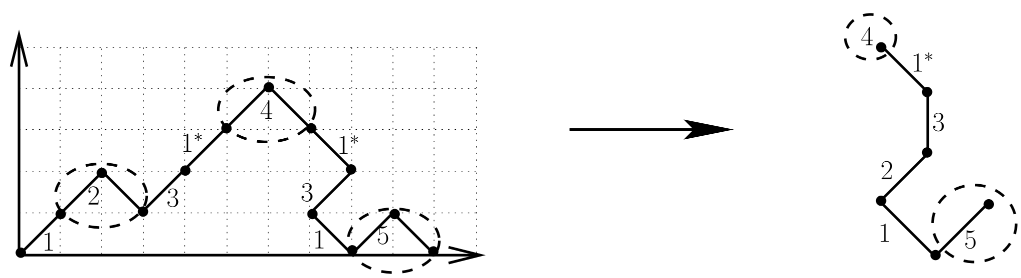


Figure 2.6 A mapping from skew Dyck paths and Hex trees.

where γ, μ and φ are non-empty skew Dyck paths and γ', μ' and φ' are Hex trees. Figure 2.6 shows that by inductive reasoning, this bijection maps skew Dyck paths of semi-length h into Hex trees with $h-1$ edges, see [4]. Figure 2.7(i) is an example of this bijection.



(a) Skew Dyck path with semi-length 6.

(b) Hex tree with 5 edges.

Figure 2.7(i) An example of a bijection from a skew Dyck path to a Hex tree.

In Figure 2.7(i), the points $(1, 1^*, 2, 3, 4, 5)$ in (a) are mapped on to $(1, 1^*, 2, 3, 4, 5)$ in (b). Now 1 and 5 are from Figure 2.6 (d), 2 is from (c), 3 is from (e), 1^* is from (b) and 4 is from (a).

Remark 2.2.8.

If we restrict this bijection to just Dyck paths it then becomes a bijection from binary trees to Dyck paths. By replacing Dyck paths with ordered trees by a standard or glove bijection, see [1]. A glove bijection is described as follows:

1. Traverse the marked tree, with the root at the top, in preorder. Preorder is a process illustrated on the lelft side of the first diagram of Figure 2.7 (ii).
2. When an edge is passed along the left side on the way down, it correlates to an up step, U, in the skew Dyck path.
3. When an edge is passed along the right side on the way up, it correlates with a down step, D.

Figure 2.7(ii) illustrates this bijection

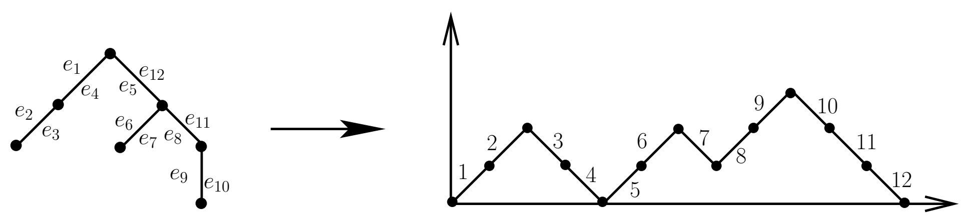


Figure 2.7(ii) Glove bijection where the edge e_t corresponds to the step t where $j = 1, 2, 3, ..., 11, 12$.

If we let binary trees be top rooted, then we get a mapping from ordered trees to binary trees as shown in Figure 2.8, see [4]:

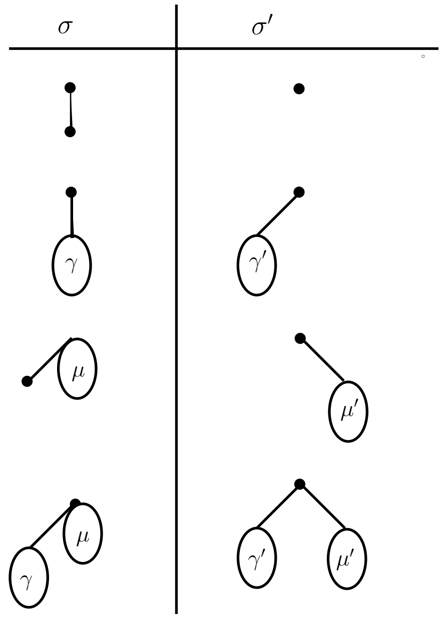


Figure 2.8 A mapping from ordered trees to binary trees.

where γ and μ are non-empty ordered trees. Figure 2.8 exhibits the **first child-next sibling** bijection, see [4]. This bijection has the following properties, see [9]:

1. The vertices of the binary tree, say A, are the vertices of the ordered tree, say D, with the root deleted.

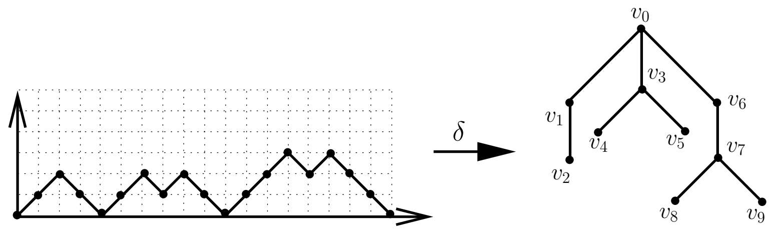
2. The root of A is the first child of the root of D.

3. Vertex v_x is the left child of vertex v_y in A provided that v_x is the left child of v_y in D, where $x \neq y$.

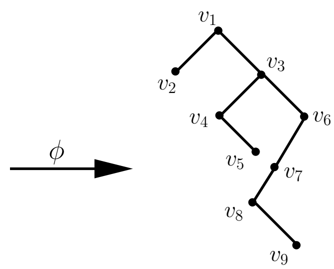
4. Vertex v_x is the right child of vertex v_y in A provided that v_x is the next sibling to the right of v_y in D, where $x \neq y$.

5. In A, we delete the first edge from the root to its only left child.

Figure 2.9 below illustrates these two bijections.



(a) A Dyck path consisting of 18 steps. (b) An ordered tree with 10 vertices.



(c) A binary tree with 9 vertices.

Figure 2.9 The glove bijection, δ, and first child-next sibling bijection, ϕ.

The glove bijection in Figure 2.9 is the same as the one in Figure 2.7(ii), hence the same method can be applied. The first child-next sibling is described is described as follows

- The binary tree in (c) has 9 vertices unlike the ordered tree which has 10 due to the root and edge in (b) being deleted.
- The root v_1 in (c) is the first left child of v_0 which is v_1 in (b).
- Since v_2 is v_1's only child in (b), (we take it as a left child) then it is v_1's left child in (c).
- Since v_3 is v_1's next sibling from the left in (b), then it is v_1's right child in (c).
- Since v_6 is v_3's next sibling in (b) then it is v_3's right child in (c).
- Since v_4 is v_3's left child in (b) then it is v_3's left child in (c).
- Since v_5 is v_4's next sibling in (b) then it is v_4's right child in (c).
- Since v_7 is v_6's only child in (b), (we take it as a left child) then it is v_6's left child in (c).
- Since v_8 is v_7's left child in (b), then it is v_7's left child in (c).
- Since v_9 is v_8's next sibling in (b), then it is v_8's right child in (c).

2.2.2 Bijection from 3-Motzkin paths to skew Dyck paths

Definition 2.2.9. *A* 3-***Motzkin path*** *is a lattice path similar to a Dyck path. The difference is not only does it consist of up and down steps but also horizontal steps denoted by* $H = (1, 0)$*, see Figure 2.10. These horizontal steps can be drawn in* 3 *colours, see* [7]*. Note that the horizontal step can be on the* x*-axis. We say the path is of size* y *because it begins at the origin and ends at* $(y, 0)$*. The length of a Motzkin path refers to the total number of up, down and horizontal steps.*

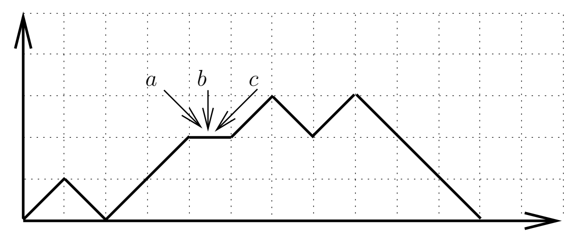


Figure 2.10 A 3-Motzkin path.

Figure 2.10 above shows a 3-Motzkin path of length or size 11 where the horizontal step could be any of colours a, b or c. The 3-Motzkin path is similar to a Dyck path but allows horizontal steps. Counting the 3-Motzkin paths yields Harary-Read Benzene numbers 1,1,3,10,36,137,543..., see [7], [8], which is the same as the sequence given by the coefficients s_h of the series $s(q)$, see (2.3).

There is a bijection from 3-Motzkin paths to Hex trees. We describe the bijection as follows:

1. Traverse the Hex tree starting with the root at the bottom, in preorder. Preorder means that you move through the tree from the root starting from the left side.

2. If we meet an edge for the first time and leads to a right, middle or left vertex is equivalent to a horizontal step of a, b, or c respectively.

3. If we meet an edge for the first time and stems from a branch point, there corresponds a down or up step respectively.

Combining the structure of this bijection and the bijection and that of hex trees to skew Dyck paths gives a bijection from 3-Motzkin paths to skew Dyck paths, is as follows:

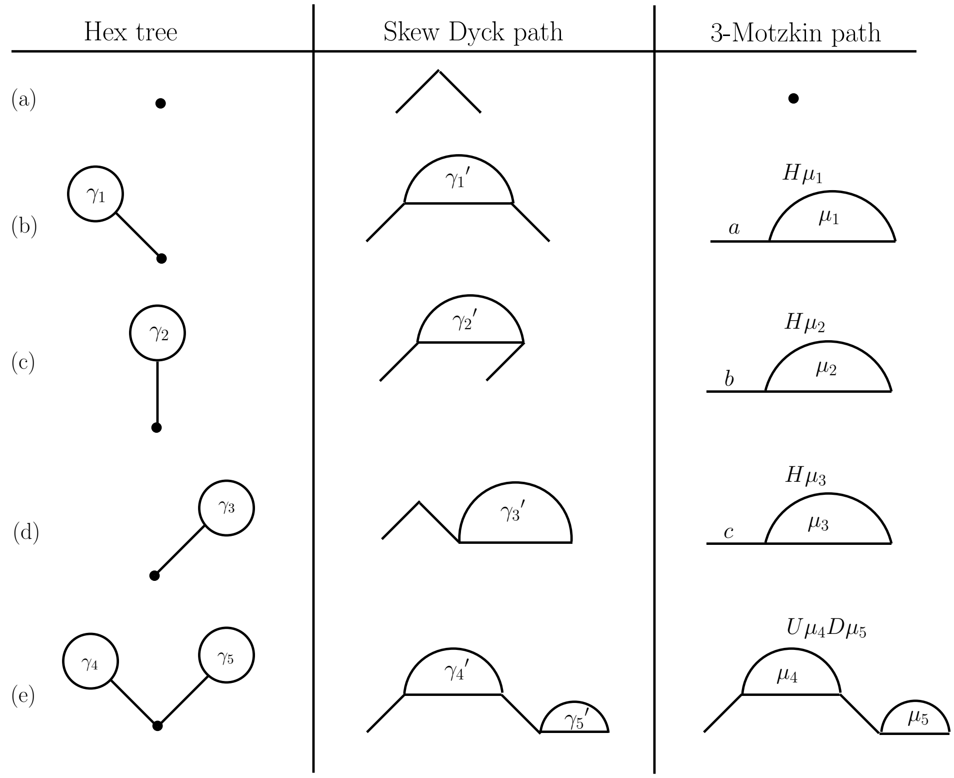
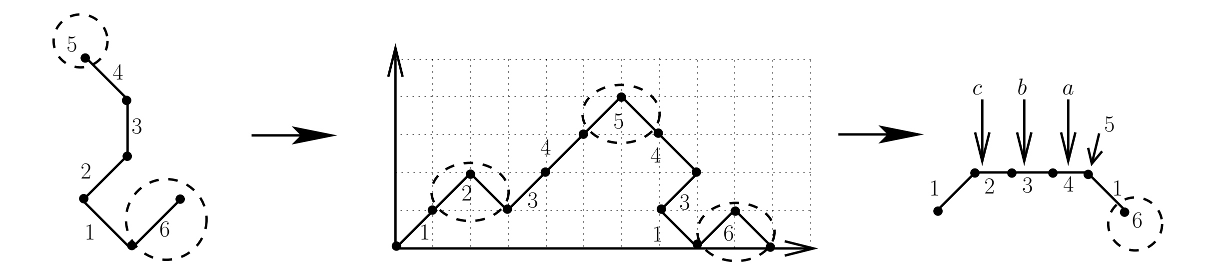


Figure 2.11 A bijection from 3-Motzkin paths to skew Dyck paths.

where γ_k, ${\gamma_k}'$ and μ_k are hex trees, skew Dyck paths and 3-Motzkin paths respectively, where $k = 1, 2, 3, 4, 5$. Figure 2.12 is an example of this bijection.



(a) Hex tree with 5 edges. (b) Skew Dyck path with semi-lenth 6. (c) A 3-Motzkin path of length 5.

Figure 2.12 An example of a bijection from a skew Dyck path to a Motzkin path.

In Figure 2.12, from the root, the left (1) and right (6) edge mean the skew path will be of the form (e) in Figure 2.11. Figure 2.11 also shows the structure of the

Motzkin path (c). The right edge of the root ends in a vertex and from Figure 2.11 (a), this is just a peak (marked by 6). Next is a right edge marked by 2 in (a) and that is mapped onto 2 in (b) and (c), see Figure 2.11 (d). The horizontal step in Figure 2.11 (d) is of colour c. The edge in (a) that follows is the middle edge marked by 3 which is mapped onto 3 in (b) and (c), see Figure 2.11 (c). The horizontal step in Figure 2.11 (c) is b. Then lastly is the left edge marked by 4 and is mapped onto 4 in (b) and (c). The horizontal step in Figure 2.11 (b) is a. Edge 4 ends in vertex 5 which by Figure 2.11 (a) is a peak marked by 4 in Figure 2.12 (b) and a vertex in Figure 2.12 (c).

From Figure 2.12, we can see that this decomposition leads us to conclude that the generating function for the enumeration of 3-Motzkin paths in relation to their length is equal to the generating function of the enumeration of hex trees in relation to the number of edges, see [4]. Hence the common generating function $g(q)$ is,

$$g(q) = 1 + 3qg(q) + q^2 g(q)^2. \tag{2.5}$$

- 1 represents an empty path.,

- $3qg(q)$ represents the three decompositions $H\mu_1$, $H\mu_2$ and $H\mu_3$, and q shows that there is only one horizontal step representing one of the three colours a, b or c in each decomposition. $g(q)$ shows that there is only one 3-Motzkin path, μ_i, for each decomposition, $i = 1, 2, 3$.

- Here, $q^2 g(q)^2$ represents the decomposition $U\mu_4 D\mu_5$, and q^2 represents the up step, U, and down step, D. Also $g(q)^2$ denotes the two 3-Motzkin paths μ_4 and μ_5.

From the generating function $g(q)$ for hex trees and 3-Motzkin paths and the generating function $s(q)$ for skew Dyck paths, we see that their relationship is:

$$s(q) = 1 + qg(q). \tag{2.6}$$

This is due to the bijection that maps skew Dyck paths with semi-length h to hex trees consisting of $h - 1$ edges. Since 3-Motzkin paths with length h are counted by s_{h+1}, we derive combinatorically three equations for s_{h+1}.

1. Select $2m$ steps from h unit steps between the origin, $(0, 0)$, and the end of the path, $(h, 0)$. Then from the $2m$ steps we construct c_m 'disjoint' Dyck paths. Lastly we have the remaining $h - 2m$ horizontal steps each consisting of the three colors. We then get

$$s_{h+1} = \sum_{m=0}^{h} \binom{h}{2m} 3^{h-2m} c_m.$$

2. The difference between 2-Motzkin and 3-Motzkin paths is that the horizontal steps of 2-Motzkin paths are of two colours. We know that 2-Motzkin paths with length h are computed by the Catalan number c_{h+1}, see [7]. From h unit steps between $(0,0)$ and $(h,0)$, we take m steps over which we can perhaps have c_{m+1} 'disjoint' 2-Motzkin paths. Then we have the remaining $h-m$ horizontal steps whereby each step has the last colour. This again gives us

$$s_{h+1} = \sum_{m=0}^{h} \binom{h}{m} c_{m+1}.$$

3. Lastly, we use 1-Motzkin paths to construct 3-Motzkin paths. 1-Motzkin paths are similar to 2-Motzkin paths and 3-Motzkin paths, the only difference is that the horizontal step can be of one colour. Here, from the h unit steps between $(0,0)$ and $(n,0)$, we take m steps to perhaps have M_m 'disjoint' Motzkin paths. M_m is the m-th Motzkin number, see [2], defined by

$$M_m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} c_k.$$

   If we assume that the horizontal steps of the 1-Motzkin paths have colour a, then we give the remaining $h-p$ horizontal steps and for each we allocate one of two colours. We obtain,

$$s_{h+1} = \sum_{m=0}^{h} \binom{h}{m} 2^{h-m} M_m, \tag{2.7}$$

   which is the relation between s_{h+1} and M_m.

2.2.3 Bijection from marked trees to skew Dyck paths

The bijection from marked trees to skew Dyck paths will be shown by deriving a bijection in relation to the glove bijection from ordered trees to Dyck paths we mentioned in section 2.2.1.

Definition 2.2.10. *A* ***marked tree*** *is an ordered tree that has right non-final edges and none, some or all these non-final edges are marked. A* ***right non-final edge*** *is an edge that is rightmost amongst other edges stemming from the exact same vertex and there is more than one vertex that occur at that edge, see Figure 2.13.*

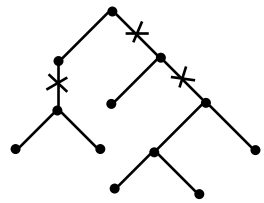


Figure 2.13 A marked tree with all three right non-final edges marked.

We shall find the bivariate generating function for the number of right non-final edges, $T(r, g)$, where g marks the number of edges and r tracks the number of right non-final edges. The decomposition of ordered trees is illustrated in Figure 2.14:

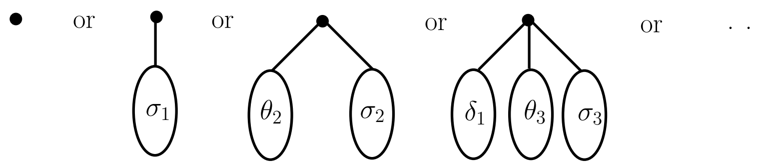


Figure 2.14 The decomposition of ordered trees

where σ_j, δ_j and θ_j are ordered trees, $j \in \mathbb{N}$. From the decomposition, we get:

$$T = 1 + g[r(T-1)+1] + g^2T[r(T-1)+1] + g^3T^2[r(T-1)+1] + ...$$

The 1 denotes the empty path. Then $g[r(T-1)+1]$ represents the ordered tree with the subtree σ_1. Here follows an explaination of the decomposition:

- g: One edge in ordered tree.
- $r(T-1)$: Right non-final edge, r, has to be attached to the subtree σ_1 and σ_1 should not be an empty path hence the empty path is removed i.e., $T-1$.
- $g[r(T-1)+1]$: We add back an empty path, 1, if there is no right final edge, t.

We use similar reasoning for the rest of the generating function. Simplifying $T(r, g)$ gives:

$$gT(r,g)^2 - (1+g-rg)T(r,g) + 1 + g - rg = 0.$$

Applying the Lagrange Inversion Theorem gives the number of ordered trees, v_{hm}, that have h edges and m right non-final edges, see [4]:

$$v_{hm} = [r^m g^h]T(r,g) = \binom{h-1}{m} M_{h-m-1}. \tag{2.8}$$

Combining (2.7) and (2.8) we get a relation between s_h and the number of marked trees. By shifting the index of s_{h+1} to the left and letting $m = j$ from (2.7) we get:

$$s_h = \sum_{j=0}^{h-1} 2^{h-1-j} \binom{h-1}{j} M_j = \sum_{m=0}^{h-1} 2^m \binom{h-1}{m} M_{h-m-1}.$$

Now, we are going to describe the glove bijection between skew Dyck paths and marked trees.

1. Traverse the marked tree, with the root at the top, in preorder. Preorder is a process illustrated on the left side of the first diagram of Figure 2.15.

2. When an edge is passed along the left side on the way down, it correlates to an up step, U, in the skew Dyck path.

3. When an edge is passed along the right side on the way up, it correlates with a down step, D.

4. When an edge is passed along the right side on the way up and it is a marked right non-final edge, then it correlates to a left step, L.

Figure 2.15 is an illustration of this bijection.

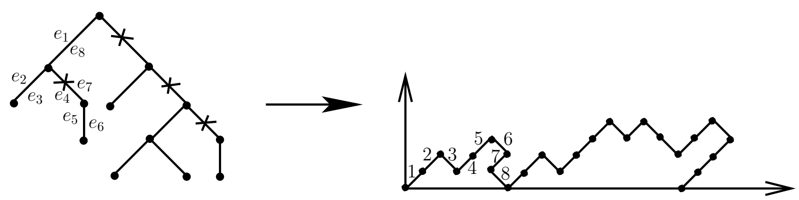


Figure 2.15 Glove bijection where the edge e_t corresponds to the step t where $j = 1, 2, 3, 4, 5, 6, 7, 8$.

2.2.4 Bijection from weighted Dyck paths to skew Dyck paths

Definition 2.2.11. *A* ***weighted Dyck path*** *is a Dyck path where the down steps are assigned a weight, w.*

Earlier in Chapter 1 in (1.7), we introduced $c(q)$ which is the generating function for Dyck paths where q marks the number of down steps. Assign a weight, y ($y \in \mathbb{N}$), to each down step of a Dyck path where q marks the weight-sum of the Dyck paths. Looking at it this way, we would say the down step of the normal Dyck path is of weight 1. Now since each down step in the Dyck path could have a weight of any positive integer, then we will have these possible weights:

$$q + q^2 + q^3 + q^4 + q^5 + ...$$

which gives a geometric sequence and simplifying it gives $\frac{q}{1-q}$. Hence if we substitute q by $\frac{q}{1-q}$ then we get the generating function for Dyck paths where each down-step is associated with a weight y where $y \in \mathbb{Z}$. Then q is the sum-weight of the Dyck path.

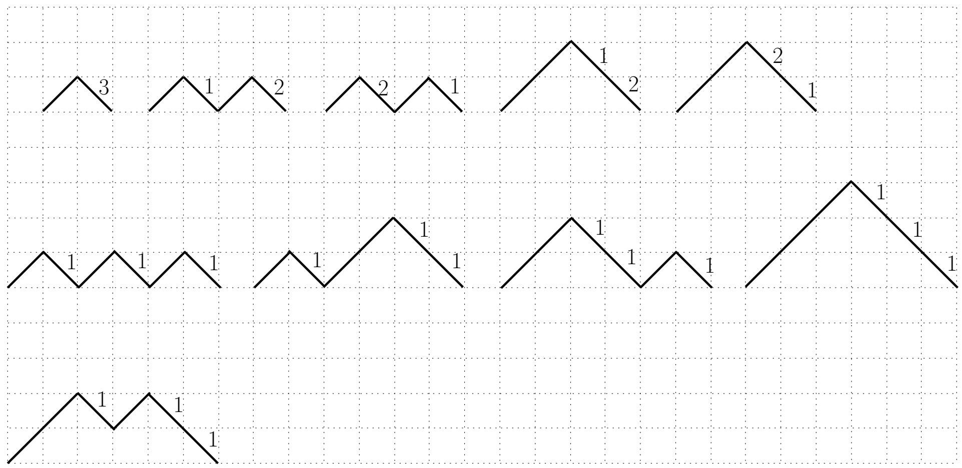


Figure 2.16 An example of the 10 weighted Dyck paths whose sum-weight is 3

We took the 10 weighted Dyck paths from the relation we mentioned earlier in (2.4), namely $s(q) = c\left(\frac{q}{1-q}\right)$. A Dyck path of semi-length h can have a weight sum k, so the number of weighted Dyck paths can be counted in $\binom{h-1}{k-1}$ ways, [4]. This leads us to,

$$s_h = \sum_{k=1}^{h} \binom{h-1}{k-1} c_k.$$

Now, we introduce the bijection from weighted Dyck paths to skew Dyck paths. Recall that $\mathbb{W}_h$ is the set of skew Dyck paths of semi-length h in section 2.1. Define $\mathbb{P}_h$ to be the class of weighted Dyck paths. They have semi-length h and weight-sum h respectively. We then define the mapping $(-)' : \mathbb{P}_h \to \mathbb{W}_h$ by,

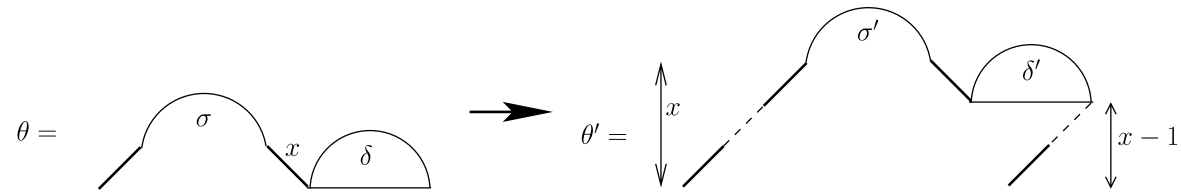


Figure 2.17 A bijection from a weighted Dyck path to a skew Dyck path.

where x is the weight of the first return step (to the x-axis) of the Dyck path, i.e., their last weight. Also, an empty path of a Dyck path is mapped onto an empty path of the skew Dyck path. We are now going to illustrate this bijection through an example.

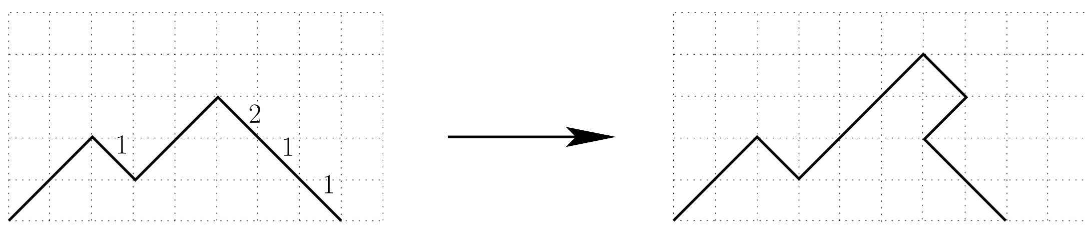


Figure 2.18 A bijection from a weighted Dyck path to a skew Dyck path.

Figure 2.18 shows an example of a Dyck path of sum-weight 5 and its mapping to the skew Dyck path of semi-length 5. It is also easy to notice that for this bijection the number of peaks and down steps do not change. Hence the number of skew Dyck paths with semi-length h consisting of y down steps is $\binom{h-1}{y-1}$.

2.3 Involutions on skew Dyck paths

Definition 2.3.1. *An* ***involution*** *is a bijection from a set which when applied twice gives the identity i.e it is its inverse. In our case the set is* $\mathbb{W}_h$*, the set of skew Dyck paths of semi-length h.*

Definition 2.3.2. *A* ***fixed or invariant point*** *is an element in the domain of a function that is mapped to itself.*

2.3.1 Involution I

Let $(-)^* : \mathbb{W}_h \to \mathbb{W}_h$ be an involution on $\mathbb{W}_h$ defined recursively by:

(a) $\rho^* = \rho$ where ρ is an empty path.

(b) $(UD)^* = UD$.

(c) $(U\gamma L)^* = UD\gamma^*$.

(d) $(U\gamma D\mu)^* = U\gamma^* D\mu^*$.

(e) $(UD\gamma)^* = U\gamma^* L$.

where $\gamma, \mu \in \mathbb{W}$ and $\gamma \neq \rho$. Graphically, we show the involution in Figure 2.19:

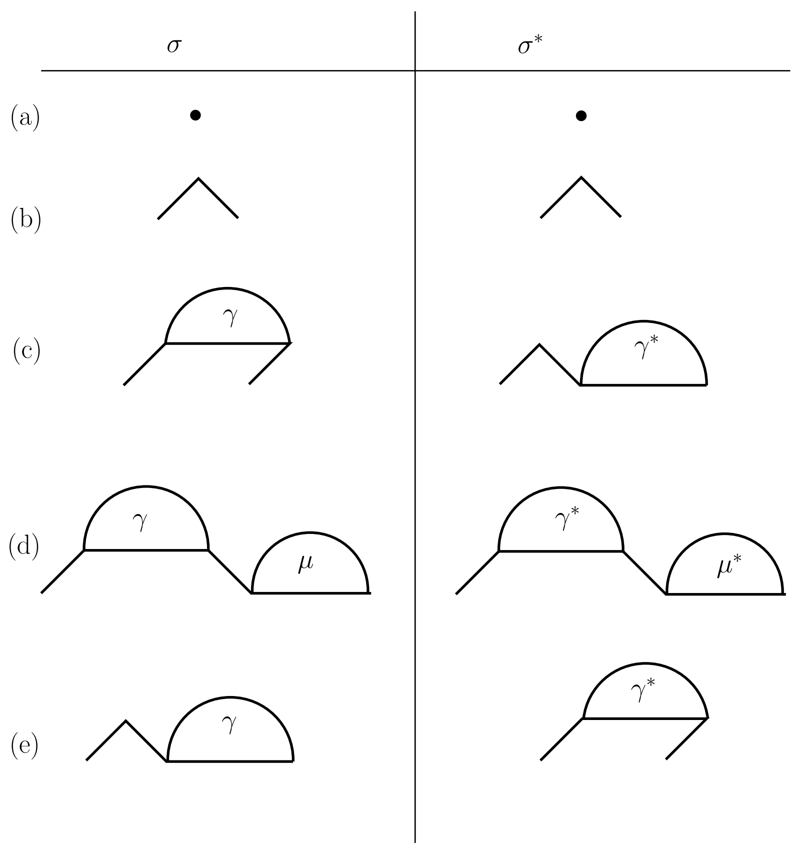


Figure 2.19 Involution I

where $\gamma \neq \rho$. We see that from the definition of this involution that given a skew Dyck path γ we have that the number of left steps, L, is equal to UDU steps in γ^*. Figure 2.20 below illustrates an example of this involution.

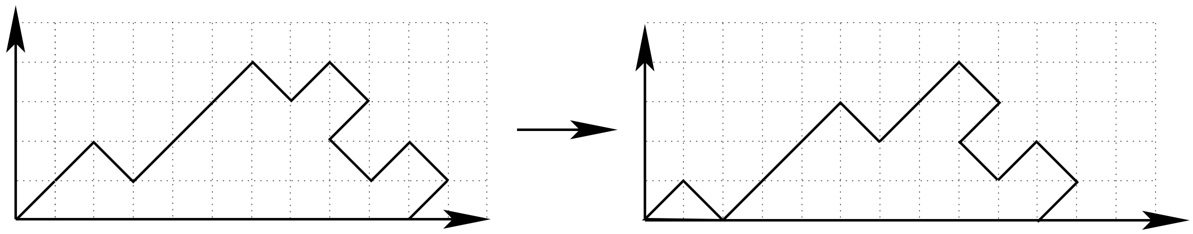


Figure 2.20 An example of involution I.

We also see that three cases form from this involution, namely:

(a) The skew Dyck paths $UD\gamma$ and $U\gamma L$ are not fixed points. A fixed point in this case means that the skew Dyck path does not change after the bijection.

(b) ρ and skew path UD are fixed points.

(c) The skew path $U\gamma D\mu$, $(\gamma \neq \rho)$, is a fixed point with the condition that: $\gamma^* = \gamma$ and $\mu^* = \mu$.

Therefore, let $\psi(q)$ be the generating function for the number of fixed points, based on semi-length. We see that $\psi(q)$ is defined by:

$$\psi(q) = 1 + q + q(\psi(q) - 1)\psi(q),$$

where,

- 1 represents the empty path.,
- q represents the UD path.
- $q(\psi(q) - 1)\psi(q)$ represents the $U\gamma D\mu$ path. Since $\gamma \neq \rho$ we remove the empty path i.e., $\psi(q) - 1$. $\psi(q)$ represents the skew path μ.

Simplifying $\psi(q)$ gives:

$$\begin{aligned}
\psi(q) &= 1 + q + q\psi^2(q) - q\psi(q) \\
0 &= q\psi^2(q) - (1+q)\psi(q) + (1+q) \\
\psi(q) &= \frac{1 + q - \sqrt{(1+q)^2 - 4q(1+q)}}{2q} && (2.9) \\
&= \frac{1 + q - \sqrt{1 - 2q - 3q^2}}{2q}. && (2.10)
\end{aligned}$$

We take the negative root of (2.9) because $\lim_{q\to 0} \psi(q) = 0$ with the negative root and $\lim_{q\to 0} \psi(q) = \frac{2}{0} = \infty$ with the positive root. The Motzkin number generating function $M(q)$ is defined by, see [2]:

$$M(q) = \frac{1 - q - \sqrt{1 - 2q - 3q^2}}{2q^2}.$$

Hence the number of skew Dyck paths with semi-length h that are fixed points of involution I is M_{h-1}, see [4]. However, we can construct this involution through an involution on hex trees. For skew paths which are not fixed points, there is an interchanging of the edges which lead to the right child and the edges that leads to the middle child. For skew paths that are fixed points, the hex tree has to have at most two edges attached to each vertex. We know that the Motzkin number, M_h, counts such hex trees, see [4]. Now we can say that skew Dyck paths of semi-length h are counted by the Motzkin number, M_{h-1}, since there is a bijection from hex trees with $h - 1$ edges to skew Dyck paths of semi-length h.

2.3.2 Involution II

Let $(-)^a : \mathbb{W}_h \to \mathbb{W}_h$ be an involution on $\mathbb{W}_h$ and defined recursively by:

(a) $\rho^a = \rho$ where ρ is an empty path.

(b) $(UD)^a = UD$.

(c) $(U\gamma D)^a = U\gamma^a L$.

(d) $(U\gamma L)^a = U\gamma^a D$.

(e) $(U\gamma D\mu)^a = U\gamma^a D\mu^a$, $\mu \neq \rho$.

Pictorically, this involution is shown in Figure 2.21.

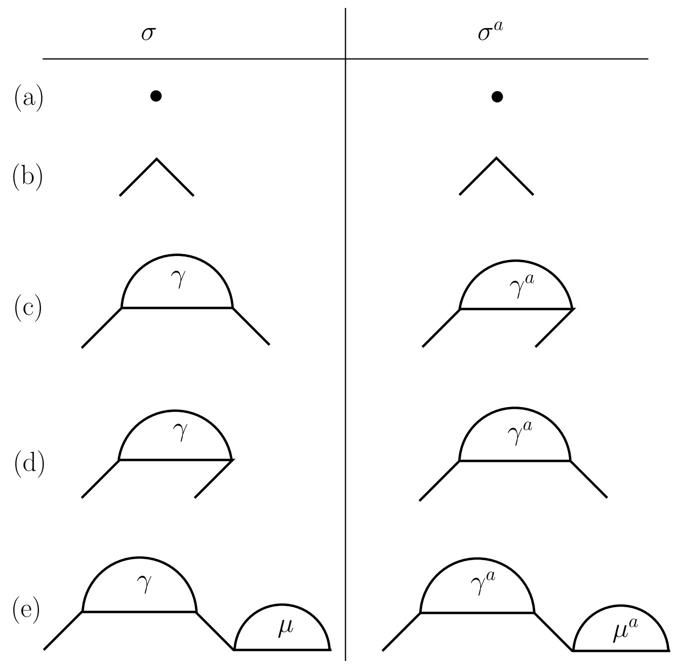


Figure 2.21 Involution II.

Figure 2.22 is an example of this involution.

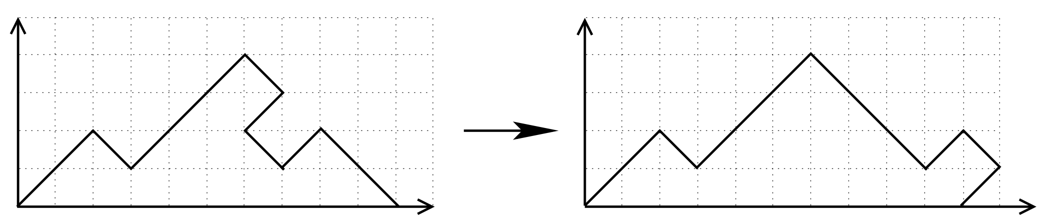


Figure 2.22 An example of involution II.

Figure 2.22 shows the involution that is derived from $(U\gamma\mu D)^a = U\gamma^a\mu^a L$. From the definition of involution II and Figure 2.22, we observe that the following are preserved by this involution:

1. The number of peaks.
2. The number of valleys.
3. The number of doublerises at any given point.

4. The maximal number of consecutive up steps, U.

This involution is constructed again via the involution on hex trees where the edge leading to the middle and left children interchange. Using the logic we used in the first involution, we have that the Motzkin number, M_{h-1}, counts the cardinality of skew Dyck paths with semi-length h which are fixed points.

2.3.3 Involution III

Let $(-)^b : \mathbb{W}_h \to \mathbb{W}_h$ be an involution on $\mathbb{W}_h$ and defined recursively by:

(a) $\rho^b = \rho$ where ρ is an empty path.

(b) $(UD)^b = UD$.

(c) $(U\gamma L)^b = U\gamma^b L$.

(d) $(U\gamma D)^b = UD\gamma^b$.

(e) $(UD\gamma)^b = U\gamma^b D$.

(f) $(U\gamma D\mu)^b = U\gamma^b D\mu^b$, $\mu \neq \rho$.

where $\gamma, \mu \neq \rho$.

Pictorically, this involution is shown in Figure 2.23.

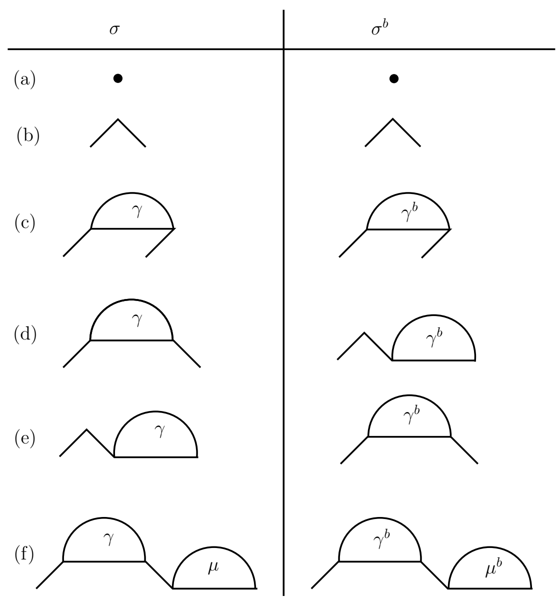


Figure 2.23 Involution III.

Figure 2.24 below illustrates this involution.

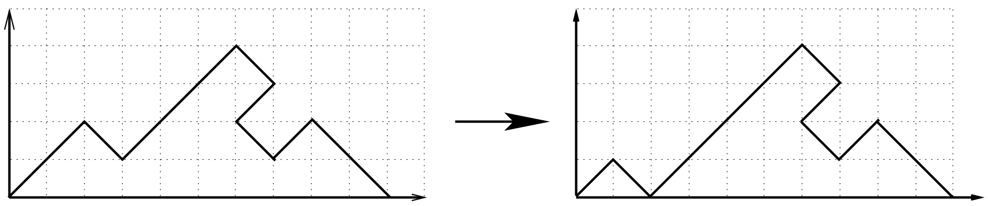


Figure 2.24 An example of involution III.

From the definition and Figure 2.24, we can observe that the number of left steps, L, is preserved and if we restrict Involution III to Dyck paths, then it becomes an involution on Dyck paths. Involution III is also constructed by an involution on Hex trees. The edge leading to the right and left children interchange. Just like the previous cases, we observe that the number of skew Dyck paths is counted by the Motzkin number, M_{h-1}.

2.3.4 Involution IV

Let $(-)^c : \mathbb{W}_h \to \mathbb{W}_h$ be an involution on $\mathbb{W}_h$ and defined recursively by:

(a) $\rho^c = \rho$ where ρ is an empty path.

(b) $(U\gamma L)^c = U\gamma^c L$, $\gamma \neq \rho$.

(c) $(U\gamma D\mu)^c = U\mu^c D\gamma^c$.

Pictorically, this involution is shown in Figure 2.25.

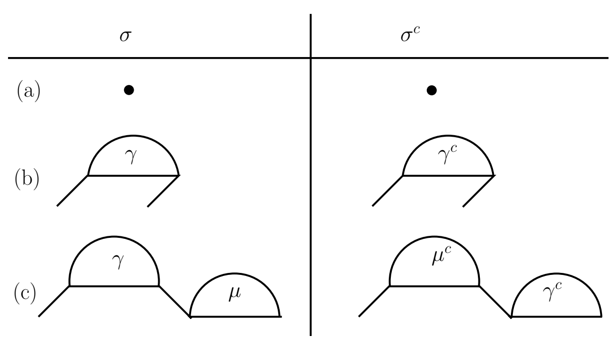


Figure 2.25 Involution IV.

Figure 2.26 below illustrates this involution.

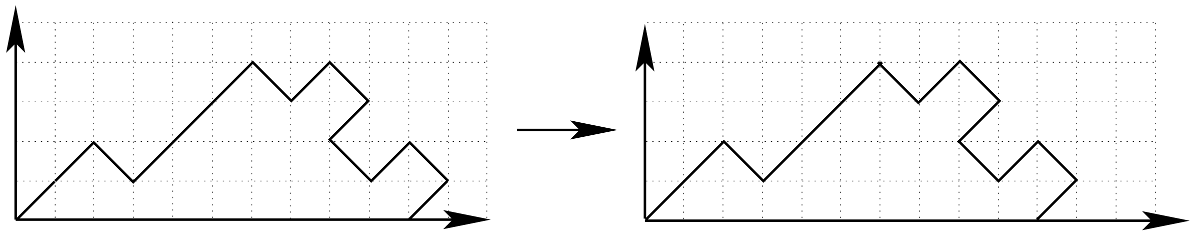


Figure 2.26 An example of involution IV.

Figure 2.26 illustrates the involution $(U\gamma L)^c = U\gamma^c L$. Involution IV is derived from the involution that has a reflection on Hex trees. We take a reflection of a Hex tree where the mirror line is the vertical axis that passes through the root of the Hex tree [4]. From the definition, we have the following circumstances:

- $U\gamma L$ ($\gamma \in \mathbb{W}$) is a fixed point of the involution provided that γ is also a fixed point e.g. Figure 2.26

- $U\gamma D\mu$ ($\gamma, \mu \in \mathbb{W}$) is a fixed point provided that it has an odd size and $\mu = \gamma^c$, take γ to be arbitrary. Now if we let f_h be the number of fixed points in $\mathbb{W}$, then we get

$$f_{2h+1} = s_h + f_{2h}$$

where $U\gamma D\mu$ ($\gamma, \mu \in \mathbb{W}$) with the odd size is represented by f_{2h+1}. The path $U\gamma L$ ($\gamma \in \mathbb{W}$) is represented by f_h and $\mu = \gamma^c$, for arbitrary γ, is represented by s_h. We also have that $f_{2h} = f_{2h-1}$ since we are only considering the odd sizes of the skew paths. We therefore derive

$$f_{2h} = f_{2h-1} = s_0 + s_1 + s_2 + ... + s_{h-1}. \tag{2.11}$$

Here, f_h is equivalent to the cardinality of Hex trees which are symmetric with $h-1$ edges. Now define $v(q)$ to be the generating function of the Hex trees that are symmetric where z represents the edge of a Hex tree. The decomposition of $v(q)$, graphically, is as follows :

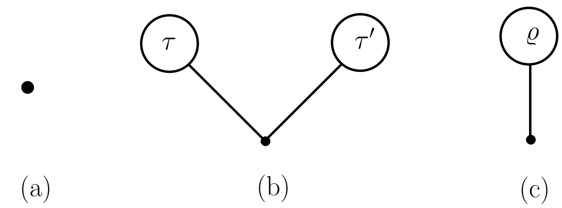


Figure 2.27 A decomposition of a symmetric hex tree.

where (a) is an empty path. Also (b) and (c) are non empty symmetric Hex trees where ϱ is a symmetric Hex tree and τ and τ' is a random Hex tree and the image of τ after reflection respectively. From the decomposition we derive $v(q)$ to be:

$$v(q) = 1 + qv(q) + q^2 g(q^2),$$

where $g(q)$ is the generating function for the number of Hex trees based on the number of edges defined in (2.7).

- 1 is the empty path (a).
- $qv(q)$ is the symmetric Hex tree (c).
- $q^2 f(q^2)$ is the symmetric Hex tree (b).

We also know that $s(q) = 1 + g(q)$ from (1.8). Using this relation with the $v(q)$, we find that $v(q) = \frac{s(q^2)}{1-q}$. It is clear that $v(q)$ is a geometric series consisting of partial sums of the sequence $s_0, 0, s_1, 0, s_2, 0...$, since we only require even powers of q.

2.4 Statistics on skew Dyck paths

We look at some basic statistics on skew Dyck paths which help us discover some combinatorial characteristics of the sequence $(s_h)_{h \geq 0}$.

2.4.1 Number of skew Dyck paths that end in D and in L

A skew Dyck path that ends with a down step, D, is in one of the forms: $U\beta_1 D$ or $U\beta_1 DU\beta_2 D$ or $U\beta_1 DU\beta_2 DU\beta_3 D$ and so on, where $\beta_i \in \mathbb{W}$, $i \in \mathbb{N}$. From this form, we then get their generating function to be:

$$\begin{aligned} D(q) &= qs(q) + q^2 s^2(q) + q^3 s^3(q) + ... \\ &= \frac{qs(q)}{1 - qs(q)}. \end{aligned}$$

Recall that $s(q) = \frac{1-q-\sqrt{1-6q+5q^2}}{2q}$ and substituting this to the equation above, we find that:

$$D(q) = \frac{2}{1+q+\sqrt{1-6q+5q^2}}.$$

Also recall that a skew Dyck path either ends with a down step, D, or left step, L. If we let $L(z)$ be the generating function for skew Dyck paths that end with a left step, L, then it follows that:

$$\begin{aligned}
L(q) &= s(q) - 1 - D(q) \\
&= s(q) - 1 - \frac{qs(q)}{1-qs(q)} \\
&= \frac{q(s(q)-1)}{1-qs(q)} \\
&= \frac{q\left(\frac{1-q-\sqrt{1-6q+5q^2}}{2q}\right) - q}{1-q\left(\frac{1-q-\sqrt{1-6q+5q^2}}{2q}\right)} \\
&= \frac{1-q-\sqrt{1-6q+5q^2-2q}-2q+1-1}{1+q+\sqrt{1-6q+5q^2}} \\
&= \frac{2(1-q)-1-q-\sqrt{1-6q+5q^2}}{1+q+\sqrt{1-6q+5q^2}} \\
&= \frac{2(1-q)}{1+q+\sqrt{1-6q+5q^2}} - 1.
\end{aligned}$$

Let w_h and t_h be the number of skew Dyck paths of semi-length h that end with a down step, D, and a left step, L, respectively. Below is a table showing these statistics. $w_h = A033321(h)$ and $t_h = A128714(h)$, see [4].

h	1	2	3	4	5	6
w_h	1	2	6	21	79	311
t_h	0	1	4	15	58	232

Table 2.1 The number of skew Dyck paths that end with a down step, w_h, and left step, t_h.

Figure 2.28 and 2.29 are graphical representations of t_3 and w_3 respectively.

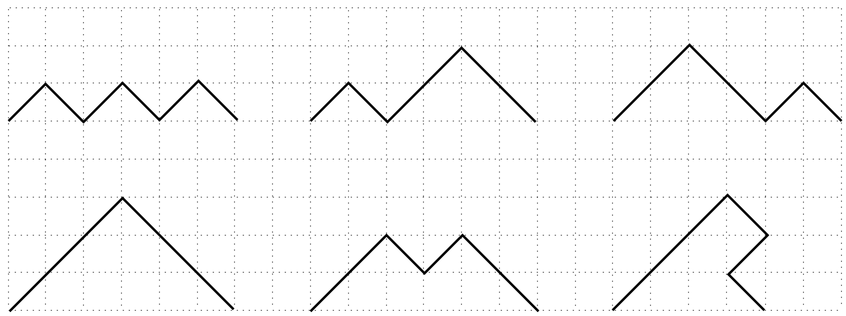


Figure 2.28 The six skew Dyck paths that end with a down step i.e., w_3.

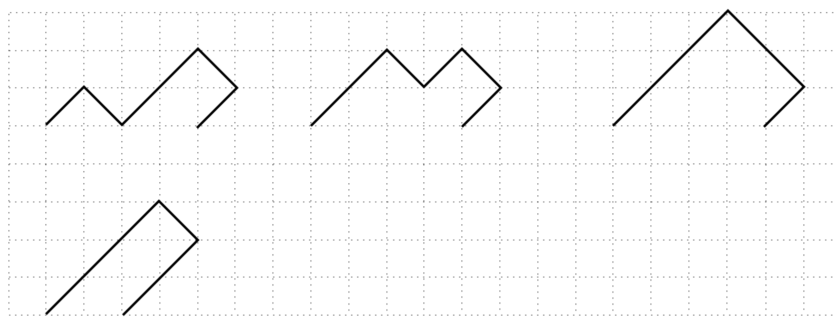


Figure 2.29 The four skew Dyck paths that end with a left step., t_3

Since $D(q) = \frac{qs(q)}{1-qs(q)}$, then rewriting it as $D(q) = qs(q) + qs(q)D(q)$ gives us the following:

$$w_h = s_{h-1} + \sum_{k=1}^{h-1} w_k s_{h-1-k}, \quad w_0 = 0. \tag{2.12}$$

The sum in (2.12) is due to the convolution between $s(q)$ and $D(q)$.

Definition 2.4.1. *A* ***convolution*** *of $R(q)$ and $G(q)$ is defined by*

$$R(q)G(q) = \sum_{k\geq 0}\Big(\sum_{0\leq m\leq k} x_k y_{k-m}\Big)q^k,$$

where $R(q) = \sum_{k=0}^{\infty} x_k q^k$ and $G(q) = \sum_{k=0}^{\infty} y_k q^k$ are generating functions.

Similarly, rewriting $L(q) = \frac{q(s(q)-1)}{1-qs(q)}$ as $L(q) = qs(q) + qs(q)L(q) - q$ gives

$$t_h = s_{h-1} + \sum_{k=2}^{h-1} t_k s_{h-1-k}, \tag{2.13}$$

where the initial conditions are $t_0 = t_1 = 0$.

The sum in (2.13) is due to the convolution between $s(q)$ and $L(q)$. From (2.12) and (2.13) we find the relation $w_h - w_{h-1} = t_h$ and we prove this bijectively. As w_h represents the number of skew Dyck paths that end in a down step, we know that these will either end in an UD step or ηD where η represents a down or left step. Now, the number of skew Dyck paths in $\mathbb{W}$ that end with UD is w_{h-1}, see [4]. The rest of the skew paths that end in ηD are similar to the skew paths that end with a down step, we just replace the down step, D, with a left step, L. Hence $w_h - w_{h-1} = t_h$ is true.

From the above relation we get $s_h = w_h + t_h$ and we know that $\lim\limits_{h\to\infty} \frac{s_h}{s_{h-1}} = 5$, see [4]. From this we get

$$\begin{aligned}
\lim_{h\to\infty} \frac{w_h}{s_h} &= \lim_{h\to\infty} \frac{s_h - t_h}{s_h}. \\
&= \lim_{h\to\infty} \left(1 - \frac{t_h}{s_h}\right). \\
&= 1 - \lim_{h\to\infty} \frac{w_h - w_{h-1}}{s_h}. \\
2\lim_{h\to\infty} \frac{w_h}{s_h} &= 1 + \lim_{h\to\infty} \left(\frac{w_{h-1}}{s_h} \times \frac{s_{h-1}}{s_{h-1}}\right). \\
2\lim_{h\to\infty} \frac{w_h}{s_h} &= 1 + \lim_{h\to\infty} \left(\frac{w_{h-1}}{s_{h-1}} \times \frac{1}{5}\right).
\end{aligned}$$

Because $\lim\limits_{h\to\infty} \frac{w_h}{s_h} = \lim\limits_{h\to\infty} \frac{w_{h-1}}{s_{h-1}}$, we then have,

$$\begin{aligned}
\left(2 - \frac{1}{5}\right) \lim_{h\to\infty} \frac{w_h}{s_h} &= 1 \\
\lim_{h\to\infty} \frac{w_h}{s_h} &= \frac{5}{9},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to\infty}\frac{t_h}{s_h} &= \lim_{h\to\infty}\frac{s_h - w_h}{s_h}\\
&= \lim_{h\to\infty}\left(1-\frac{w_h}{s_h}\right)\\
&= 1-\lim_{h\to\infty}\frac{w_h}{s_h}\\
&= 1-\frac{5}{9}\\
&= \frac{4}{9}.
\end{aligned}$$

2.4.2 Number of left steps

In this section we determine the number of left steps in any skew Dyck path with semi-length h. Define $R(q,p)$ to be the number of skew Dyck paths with semi-length h marked by q and the number of left steps marked by p. Let b_{hj} be the number of skew Dyck paths with semi-length h and containing j left steps. From the decomposition of a skew Dyck path we met earlier in Figure 2.2, we derive the generating function:

$$R(q,p) = 1 + qR(q,p)^2 + pq(R(q,p)-1). \tag{2.14}$$

Simplifying this equation, we get:

$$\begin{aligned}
R(q,p) &= 1 + qR(q,p)^2 + pq(R(q,p)-1)\\
&= qR(q,p)^2 + pqR(q,p) - pq + 1\\
0 &= qR(q,p)^2 + (pq-1)R(q,p) + 1 - pq\\
R(q,p) &= \frac{1-pq-\sqrt{(pq-1)^2+4q(pq-1)}}{2q}.
\end{aligned}$$

In the above we use the negative square root because $\lim_{q,p\to 0} R(q,p) = 0$ with the negative root and $\lim_{q,p\to 0} s(q) = \frac{2}{0} = \infty$ with the positive root.

From here, we get,

Lemma 2.4.2.

$$R(q,p) = c\Big(\frac{q}{1-pq}\Big). \tag{2.15}$$

Proof.

$$
\begin{aligned}
c\Big(\frac{q}{1-pq}\Big) &= \frac{1-\sqrt{1-4(\frac{q}{1-pq})}}{2(\frac{q}{1-pq})} \\
&= \frac{1-\sqrt{1-(\frac{4q}{1-pq})}}{2q}\times(1-pq) \\
&= \frac{1-pq-\sqrt{(1-pq)^2-4q(1-pq)}}{2q} \\
&= R(q,p).
\end{aligned}
$$

□

By the Lagrange Inversion Theorem, see [5], (2.4) and (2.5), we get

$$b_{h,j} = [p^j q^h]R(q,p) = \binom{h-1}{j} c_{h-k}.$$

The first values of b_{hj} are shown in the Table 2.2, (sequence A126181 in Sloane), see [8].

$h\backslash j$	0	1	2	3	4	5
0	1					
1	1					
2	2	1				
3	5	4	1			
4	14	15	6	1		
5	42	56	30	**8**	1	
6	132	210	140	50	10	1

Table 2.2 The number of skew Dyck paths with semi-length h consisting of j left steps i.e., b_{hj} where $h = 1, 2, 3, 4, 5, 6$ and $j = 0, 1, 2, 3, 4, 5$.

Figure 2.30 shows the eight skew Dyck paths where $j = 3$ and $h = 5$.

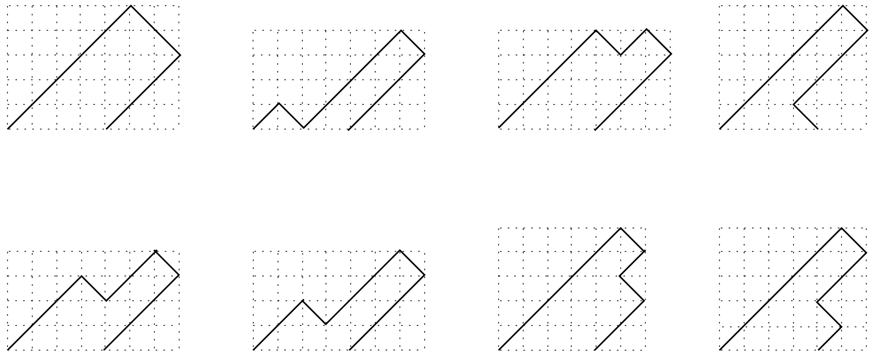


Figure 2.30 Eight skew Dyck paths with semi-length of 5 and 3 left steps i.e., b_{53}.

Now, we define φ_h to be the number of left steps in all skew Dyck paths of semi-length h. Now, $\varphi_h = \sum_{j\geq 0} jb_{hj} = [z^h]\frac{\partial R}{\partial p}\Big|_{p=1}$.

From (2.14), we get that

$$\begin{aligned}
\frac{\partial R}{\partial p} &= 2qR(q,p)\frac{\partial R}{\partial p} + pq\frac{\partial R}{\partial p} + qR(p,q) - q \\
(1 - 2qR(p,q) - pq)\frac{\partial R}{\partial p} &= q(R(p,q) - 1) \\
\frac{\partial R}{\partial p} &= \frac{q(R(p,q-1))}{1 - 2qR(p,q) - pq} \\
\frac{\partial R}{\partial p}\Big|_{p=1} &= \frac{q(s(q)-1)}{1 - 2qs(q) - q}.
\end{aligned}$$

Also we find that,

Lemma 2.4.3.

$$\frac{\partial R}{\partial p}\Big|_{p=1} = q^2 s'(q),$$

where $s(q)$ is defined in (2.1).

Proof.

$$\begin{aligned}
s'(q) &= 2qs(q)s'(q) + s(q)^2 + qs'(q) + s(q) - 1 \\
(1 - 2qs(q) - q)s'(q) &= s(q) - 1 \\
s'(q) &= \frac{s(q) - 1}{1 - 2qs(q) - q}.
\end{aligned}$$

Hence,

$$\left.\frac{\partial R}{\partial p}\right|_{p=1} = qs'(q).$$

□

Now, from this we get,

$$\begin{aligned} \varphi_h &= [q^h]qs'(q) \\ &= [q^{h-1}]s'(q) \\ &= (h-1)s_{h-1}. \end{aligned} \tag{2.16}$$

From $\lim\limits_{h\to\infty} \frac{s_h}{s_{h-1}} = 5$, see [5], we find that the expected number of left steps, $\frac{\varphi_h}{s_h}$, in a skew Dyck path of semi-length h is

$$\frac{\varphi_h}{s_h} = \frac{(h-1)s_{h-1}}{s_h} \sim \frac{1}{5}h.$$

2.4.3 Summary

In this chapter, we introduced a generalized type of Dyck path called the skew Dyck path. We discussed several bijections with a graphical example in each bijection, from skew Dyck paths to other combinatorially explained objects. We further covered four different involutions on these skew paths and included some graphical examples. We finally discussed some statistics on skew Dyck paths and illustrated some of the statistics graphically where possible.

Chapter 3

Skew Dyck paths and superdiagonal bargraphs

In this chapter we study skew Dyck paths in relation to superdiagonal bargraphs. We enumerate the superdiagonal bargraphs in terms of area and perimeter. We begin by defining terminology we will use in the chapter.

Definition 3.0.4. *The **base** of a skew Dyck path is* $2k$ *if the skew Dyck path starts at the origin and ends at* $(2k, 0)$.

Remark 3.0.5.

Therefore the semi-base is k. From this we find that the length of a skew Dyck path is greater or equal to its base. Recall that the length refers to the total number of steps in a skew Dyck path.

Definition 3.0.6. *The **area** of a skew Dyck path is the number of unit right angle triangles that can be formed between a skew Dyck path and the* x*-axis, see Figure 3.1.*

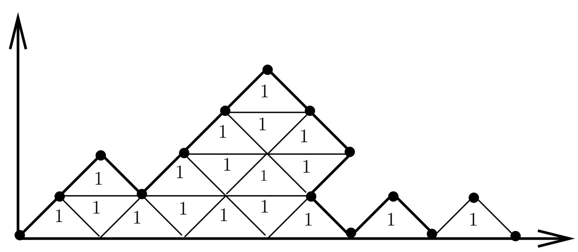


Figure 3.1 A skew Dyck path with area 18 and of base 12.

3.1 Enumeration of skew Dyck paths

We enumerate skew Dyck paths in relation to area and semi-base. We then enumerate these skew paths according to area only. Let $\mathbb{W}_h$ be the class of skew Dyck paths with semi-length h and $\mathbb{K}_h$ the class of all skew Dyck paths of semi-base h. We note that $|\mathbb{W}_h| = j$ where $j \in \mathbb{N}$ and $|\mathbb{K}_h| = \infty$.

Definition 3.1.1. *The y-coordinate of a point in the skew Dyck path is called an* ***ordinate****. If a point on a skew Dyck path has ordinate m, then it is at level m.*

3.1.1 Enumeration of skew Dyck paths coded by semi-base and area

Now, we find the number of skew Dyck paths, d_{hk} coded by semi-base(k) and area(h). Lastly we find the number of these skew paths, d_h according to area. We do this by decomposing skew Dyck paths from the leftmost peak by these two classifications:

1. All skew Dyck paths which are non empty that begin with an up and down step, (UD), intersecting at level 1. By removing the first peak formed by the UD step, the skew path left is arbitrary with the area and semi-base decreased by one. See Figure 3.2 below

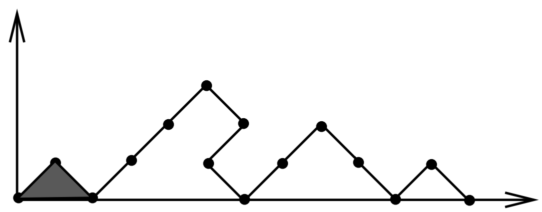


Figure 3.2

2. All skew Dyck paths which are non empty with the leftmost step at level at least two. If the leftmost square is removed in the leftmost peak then the skew Dyck path formed is arbitrary whose area decreases by two but the semi-base stays unchanged. See Figure 3.3 below.

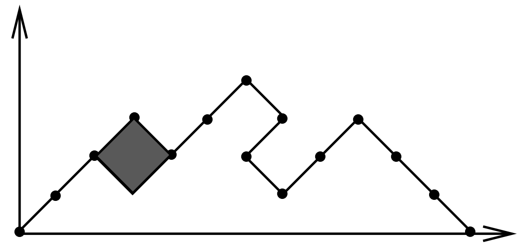


Figure 3.3

If we let $d(a,b)$ be the generating function for skew Dyck paths in relation to semi-base coded by b and area coded by a, then from the decomposition we have,

$$d(a,b) = 1 + abd(a,b) + a^2(d(a,b) - 1).$$

where

- 1 represents an empty path.

- Here $abd(a,b)$ represents all skew Dyck paths on which decomposition 1 applies, where the area and semi-base decrease by one.

- Then $a^2(d(a,b) - 1)$ represents all skew Dyck paths on which decomposition 2 applies, where we exclude the empty path and the area decreases by two.

Solving for $d(a,b)$, we get,

$$\begin{aligned} d(a,b) &= 1 + abd(a,b) + a^2(d(a,b) - 1) \\ d(a,b)(1 - ab - a^2) &= 1 - a^2 \\ d(a,b) &= \frac{1 - a^2}{1 - ab - a^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(a,b) &= \sum_{hk \geq 0} d_{hk} a^h b^k \\ &= \frac{1 - a^2}{1 - ab - a^2}. \end{aligned} \tag{3.1}$$

If we let $d_{k+1} = \sum_{h \geq 0} d_{hk} a^h$, then

$$d_{k+1}(a) = ad_k(a) + a^2 d_{k+1}(a), \tag{3.2}$$

where $ad_k(a)$ represents decomposition 1 and $a^2 d_{k+1}(a)$ represents decomposition 2 in terms of the semi-base k. In decomposition 1, the semi-base decreases by one and in decomposition 2, the semi-base does not change. Thus,

$$d_{k+1}(a) = \frac{a}{1-a^2} d_k(a). \tag{3.3}$$

Since $d_0 = 1$, then

$$\begin{aligned}
d_1(a) &= \frac{a}{1-a^2} d_0 = \frac{a}{1-a^2} \\
d_2(a) &= \frac{a}{1-a^2} d_1 = \frac{a^2}{(1-a^2)^2} \\
d_3(a) &= \frac{a}{1-a^2} d_2 = \frac{a^3}{(1-a^2)^3} \\
d_4(a) &= \frac{a}{1-a^2} d_3 = \frac{a^4}{(1-a^2)^4} \\
&\cdot \\
&\cdot \\
&\cdot \\
d_k(a) &= \frac{a^k}{(1-a^2)^k}, \qquad (3.4)
\end{aligned}$$

for every $k \in \mathbb{N}$. To obtain the numbers d_{hk} we expand this series as follows:

$$\begin{aligned}
[a^h]d_k(a) &= [a^{h-k}] \frac{1}{(1-a^2)^k} \\
&= [a^{h-k}](1-a^2)^{-k} \\
&= [(a^2)^{(h-k)/2}](1-a^2)^{-k} \\
&= \binom{(h-k)/2 + k - 1}{(h-k)/2} \\
&= \frac{((h-k)/2 + k - 1)!}{(k-1)!((h-k)/2)!} \\
&= \binom{(h-k)/2 + k - 1}{k-1} \text{ if } h \equiv k (mod\, 2). \qquad (3.5)
\end{aligned}$$

Theorem 3.1.2. *The number of all skew Dyck paths with semi-base b and area h is*

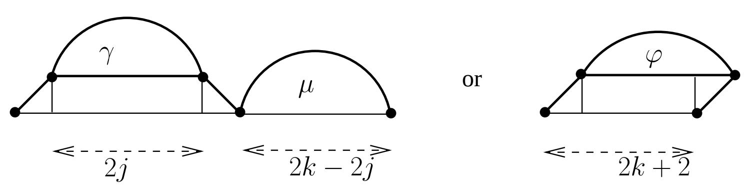
$$d_{hk} = \begin{cases} \binom{(h-k)/2+k-1}{k-1} & \text{if } h \equiv k (mod\, 2) \\ 0 & \text{elsewhere} \end{cases}.$$

The number d_h of skew Dyck paths with area a is equal to the h^{th} Fibonacci number F_h. So,

Definition 3.1.3. ***Fibonacci numbers*** *are a series of numbers where each number in the series is the sum of the two numbers that came prior to it e.g.* $1, 1, 2, 3, 5...$.

Proof. In addition to (3.4) and (3.5) we have the identity $d_{h+3} = d_{h+2} + d_{h+1}$, where d_{h+2} represents skew Dyck paths from decomposition 1 and d_{h+1} represents skew Dyck paths from decomposition 2 in relation to area, h. There is only one skew Dyck path with area 0, namely the empty path. There is also one skew Dyck path with area 1, namely the skew Dyck path with steps UD. There is also one skew Dyck path with area 2, namely the skew Dyck path with steps $UDUD$. We therefore have the initial conditions $d_0 = d_1 = d_2 = 1$, it then follows that $d_h = F_h$, for every $h \geq 1$, where F_h are the h^{th} Fibonacci numbers. □

Another way we can prove this thereom is by using the main decomposition in Figure 1.3 in chapter 1. We do this by considering skew Dyck paths with semi-base $k+1$. The decomposition has the following two cases:



where γ,μ and φ are skew Dyck paths and $\varphi \neq \rho$. We claim $d_k(a)$ satisfies the recurrence:

$$d_{k+1}(a) = \sum_{j=0}^{k} a^{2j+1} d_j(a) d_{k-j}(a) + a^{2k+2} d_{k+1}(a). \tag{3.6}$$

To prove (3.6), we use (3.4) to simplify the right hand side as follows:

$$\begin{aligned}
\text{Right hand side of (3.6)} &= \sum_{j=0}^{k} a^{2j+1} d_j(a) d_{k-j} + a^{2k+2} d_{k+1}(a) \\
&= \sum_{j=0}^{k} a^{2j+1} \frac{a^j}{(1-a^2)^j} \times \frac{a^{k-j}}{(1-a^2)^{k-j}} + a^{2k+2} \times \frac{a^{k+1}}{(1-a^2)^{k+1}} \\
&= \sum_{j=0}^{k} a^{2j+1} \frac{a^k}{(1-a^2)^k} + a^{2k+2} \times \frac{a^{k+1}}{(1-a^2)^{k+1}} \\
&= \frac{a^k}{(1-a^2)^k} \times a \Big(\sum_{j=0}^{k} a^{2j} + \frac{a}{1-a^2} \times a^{2k+1} \Big) \\
&= \frac{a^k}{(1-a^2)^k} \times a \Big(\frac{1-a^{2(k+1)}}{1-a^2} + \frac{a}{1-a^2} \times a^{2k+1} \Big) \\
&= \frac{a^{k+1}}{(1-a^2)^{k+1}} \\
&= d_{k+1}. \qquad (3.7)
\end{aligned}$$

Theorem 3.1.4. *There is a bijection between skew Dyck paths with semi-base k and area h and the compositions of h with k odd parts.*

Proof. We will show this bijection between skew Dyck paths and a specific composition using Figure 3.4. We first decompose the area of the skew Dyck path, γ into triangles and squares guided by the skew path and the x-axis as illustrated by Figure 3.4. We then represent each triangle by 1 and each square as 2. Then we finally sum diagonally and we get a set, $\sigma(\gamma)$ odd positive integers.

Now, we know that a composition h, of positive integers of length k is an ordered set of k-elements, $\sigma = (p_1, p_2, ..., p_k)$ (k-tuple) where $p_1 + ... + p_k = h$. Therefore connecting this composition with the decomposition we have just described above produced the bijection because if the skew Dyck path γ has area h semi-base k, then $\sigma(\gamma)$ is a composition of h with k odd components.

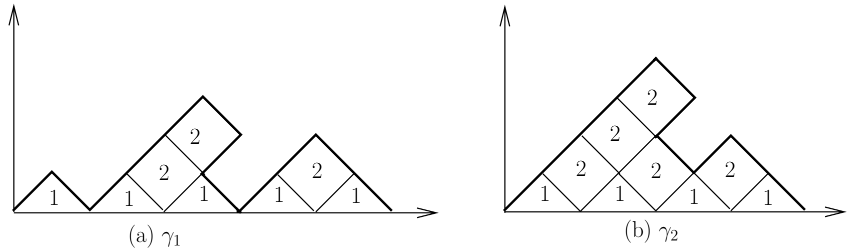


Figure 3.4 Two skew Dyck paths with compositions: (a) $\sigma(\gamma_1) = (1, 5, 1, 3, 1)$ and (b) $\sigma(\gamma_2) = (7, 3, 3, 1)$.

□

Theorem 1 provides a combinatorial interpretation of Fibonacci numbers. Since, $d_h = \sum_{k=1}^{h} d_{hk}$, we have the following lemmas,

Lemma 3.1.5. $F_{2h} = \sum_{k=1}^{h} \binom{h+k-1}{2k-1}$.

Proof.

$$\begin{aligned} F_{2h} &= d_{2hk} \\ &= \sum_{k=1}^{h} \binom{(2h-2k)/2 + 2k - 1}{2k-1} \\ &= \sum_{k=1}^{h} \binom{h+k-1}{2k-1}. \end{aligned} \tag{3.8}$$

□

Lemma 3.1.6. $F_{2h+1} = \sum_{k=0}^{h} \binom{h+k}{2h}$.

Proof.

$$\begin{aligned} F_{2h+1} &= \sum_{k=0}^{h} d_{2h+1} \\ &= \sum_{k=0}^{h} \binom{((2h+1)-(2k+1))/2 + (2h+1) - 1}{2h+1-1} \\ &= \sum_{k=0}^{h} \binom{h+k}{2h}. \end{aligned} \tag{3.9}$$

□

3.1.2 Enumeration of skew Dyck paths coded by area and semi-length

Let $y_h(a)$ be the generating function for skew Dyck paths with semi-length h and where a tracks the area. Since h tracks the semi-length, $y_h(a)$ is a polynomial of degree h^2 and is divisible by h^2 (the area has to be at least h). For example, Figure 3.5 shows skew Dyck paths of semi-length 2. We can see that the areas are 2 (from (a)), 3 (from (b)) and 4 (from (c)).

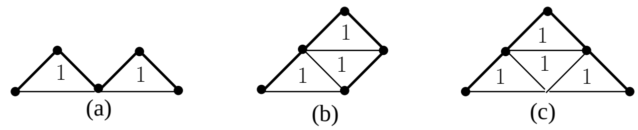


Figure 3.5 Skew Dyck paths of semi-length 2.

Let $Y_r(a;q)$ be the generating function for all skew Dyck paths where q tracks the semi-length. Here, r denotes the ordinate of the leftmost peak. Let,

$$y(a;q) := \sum_{r\geq 0} Y_r(a;q) = \sum_{h\geq 0} y_h(a)q^h.$$

As per 1 and 2 below, the recurrence of $Y_r(a;q)$ is,

$$Y_{r+2}(a;q) = a^{2r+3}\sum_{i=0}^{r} q^{k+2-i}Y_i(a;q) + a^{2r+3}q\Big[y(a;q) - \sum_{i=0}^{r} Y_i(a;q)\Big], \qquad (3.10)$$

where $Y_0(a;q) = 1$ (initial value) which corresponds to the empty path. Since every skew Dyck path that has ordinate one at the leftmost peak can decompose as $UD\gamma$ where $\gamma \in \mathbb{W}$ (see Figure 3.6 (a)). We also have $Y_1(a;q) = aqy(a;q)$.

The recurrence of $Y_r(a;q)$ is due to these two cases following from skew Dyck paths where the leftmost peak has ordinate $r+2$:

1. $\gamma = U^{r+2}DL^{r-i+1}\varphi$, where φ starts with a down step, assuming that it is non-empty, and $i \leq r$ (see Figure 3.6 (b)). We notice that there is a bijection between γ and a class of skew Dyck paths γ', where $\gamma' = U^i\varphi$ and has the left-most peak with ordinate i. Here, the semi-length and area decreased by $r+2-i$ and $2r+3$ respectively.

2. $\gamma = U^{r+2}D\varphi$, where φ starts with an up or down step (U or D), see Figure 3.6 (c). Here γ has a bijection with a class of skew Dyck paths γ', where $\gamma' = U^{r+1}\varphi$. The leftmost peak has ordinate at least $r+1$ where the semi-length and area has decreased by one and $2r+3$ respectively.

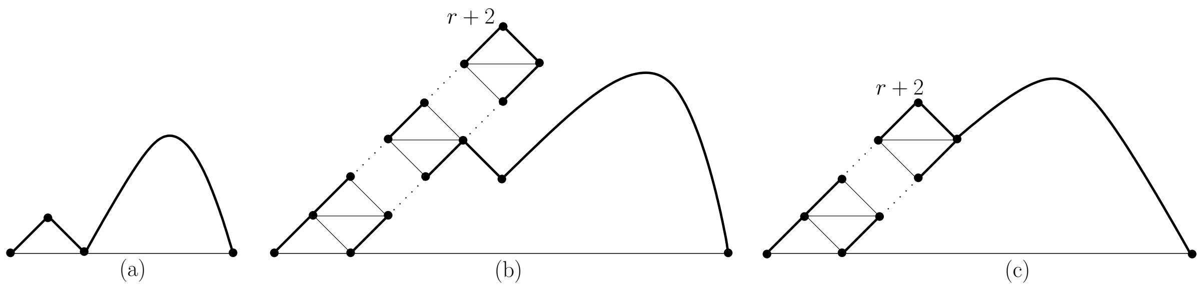


Figure 3.6 A decomposition of non-empty skew Dyck paths based on left most peak.

We now define the series:

$$Y(a;q,s) := \sum_{r\geq 0} Y_r(a;q)s^r.$$

Then the recurrence (3.10) becomes, (see [4])

$$Y(a;q,s) = 1 + \frac{aqs}{1-a^2s}y(a;q) - \frac{a^3(q-q^2)s^2}{(1-a^2qs)(1-a^2s)}Y(a;q,a^2s). \tag{3.11}$$

From this recurrence we find the following identity (see [4]):

$$\begin{aligned} Y(a;q,s) &= \sum_{r\geq 0} \frac{(-1)^r a^{r(2r+1)}(q-q^2)^r s^{2r}}{(1-a^2qs)...(1-a^{2r}qs)(1-a^2s)...(1-a^{2r}s)} \\ &+ \Big[\sum_{r\geq 1} \frac{(-1)^{r-1}a^{r(2r-1)}q^r(1-q)^{r-1}s^{2r-1}}{(1-a^2qs)...(1-a^{2(r-1)}qs)(1-a^2s)...(1-a^{2r}s)}\Big] y(a;q). \end{aligned}$$

Since $Y(a;q,1) = y(a;q)$, then we obtain, (see [4]):

$$y(a;q) = \frac{\sum_{r\geq 0} \frac{(-1)^r a^{r(2r+1)}}{(1-a^2)...(1-a^{2r})} \frac{(q-q^2)^r}{(1-a^2q)(1-a^4q)...(1-a^{2r}q)}}{\sum_{r\geq 0} \frac{(-1)^r a^{r(2r-1)}}{(1-a^2)...(1-a^{2r})} \frac{(q-q^2)^r}{(1-q)(1-a^2q)...(1-a^{2r-2}q)}}. \tag{3.12}$$

We then expand (3.12) to get the following polynomials, see [4]:

$$
\begin{aligned}
y_0(a) &= 1, \\
y_1(a) &= a, \\
y_2(a) &= a^2(1+a+a^2), \\
y_3(a) &= a^3(1+a+3a^2+2a^3+a^4+a^5+a^6), \\
y_4(a) &= a^4(1+a+4a^2+5a^3+5a^4+3a^5+5a^6+4a^7+3a^8+2a^9+a^{10}+a^{11}+a^{12}), \\
y_5(a) &= a^5(1+a+5a^2+7a^3+12a^4+10a^5+11a^6+12a^7+14a^8+12a^9+10a^{10}+8a^{11} \\
&\quad +10a^{12}+7a^{13}+5a^{14}+4a^{15}+3a^{16}+2a^{17}+a^{18}+a^{19}+a^{20}), \\
y_6(a) &= a^6(1+a+6a^2+9a^3+19a^4+23a^5+28a^6+26a^7+36a^8+38a^9+38a^{10}+32a^{11} \\
&\quad +36a^{12}+36a^{13}+34a^{14}+29a^{15}+27a^{16}+25a^{17}+21a^{18}+15a^{19}+16a^{20}+13a^{21} \\
&\quad +10a^{22}+7a^{23}+5a^{24}+4a^{25}+3a^{26}+2a^{27}+a^{28}+a^{29}+a^{30}).
\end{aligned}
$$

The coefficients of the polynomials are found in Sloane as sequence $A129172$, see [5], [8].

Remark 3.1.7.

There is another alternative for deriving $y_h(a)$. Let $w_h(a,b)$ be the generating function for skew Dyck paths with semi-length h in relation to area coded by a and semi-base coded by b. Then $y_h(a) = w_h(a,1)$. Now, let $R(a,b,q)$ be the generating function for all skew Dyck paths where q denotes the semi-length. From the skew Dyck path decomposition from Figure (2.2), we obtain the identity

$$R(a,b,q) = 1 + abqR(a,a^2b,q)R(a,b,q) + q(R(a,a^2b,q) - 1). \tag{3.13}$$

1. 1 represents the empty path, ρ in Figure 2.2.

2. Here $abqR(a,a^2b,q)R(a,b,q)$ represents the skew Dyck path with decomposition $U\gamma D\mu$ in Figure 2.2. $R(a,a^2b,q)$ represents the generating function for the skew Dyck path γ and $R(a,b,q)$ represents the generating function for the skew Dyck path μ. The expression abq shows that the area, semi-base and semi-length is one. The area is formed by the up and down steps.

3. Lastly, $q(R(a,a^2b,q) - 1)$ represents the skew Dyck path with decomposition $U\varphi L$ in Figure 2.2. Here the semi-length is *one* represented by z and $R(a,a^2b,q)$ is the generating function for the skew Dyck path φ. Since φ is nonempty, we then remove the empty path, i.e., $R(a,a^2b,q) - 1$.

Now, we let

$$R(a,b,q) = \sum_{h\geq 0} y_h(a;b)q^h. \tag{3.14}$$

We then obtain the coefficients of q^h from (3.14) with the help of (3.13) and find that $y_0 = 1$, $y_1 = ab$ and

$$y_h(a;b) = ab\sum_{j=0}^{h-1} y_j(a;b)y_{h-1-j}(a,a^2b) + y_{h-1}(a;a^2b). \tag{3.15}$$

Then we find,

$$\begin{aligned} y_2(a;b) &= a^2b^2 + a^3b + a^4b^2, \\ y_3(a;b) &= a^3b^3 + a^4b^2 + 2a^5b^3 + 2a^6b^2 + a^7b^3 + a^8b^2 + a^9b^3, \end{aligned} \tag{3.16}$$

etc.

We now illustrate $2a^6b^2$, one of the terms in $y_3(a;b)$ graphically below in Figure 3.7. The coefficient 2 in $2a^6b^2$ means that there are two skew Dyck paths with area six and semi-base two.

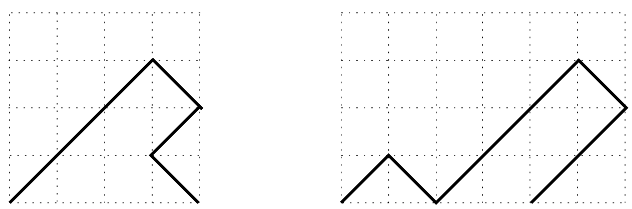


Figure 3.7 Skew Dyck paths with area six and semi-base two.

3.1.3 Total area of skew Dyck paths of given semi-length

Let $A(q) = \sum_{h\geq 0} a_h q^h$ be the generating function for the numbers a_h which give the sum of areas of all skew Dyck paths with semi-length h. We now take the partial derivative of (3.13) with respect to a and b respectively and we denote them as R_a and R_b respectively. From this we obtain the system:

1.

$$\begin{aligned} R_a &= aqR'R + abq(R'_aR + R_aR') + qR'_a \\ &= aqR'R + abqR'_aR + abqR'R_a + 2abR'_a \\ &+ \; 2abR'_a + qR'_a. \end{aligned} \tag{3.17}$$

2.

$$\begin{aligned} R_b &= aqR'R + abq(a^2R'_bR + R_bR') + qa^2R'_b \\ &= aqR'R + a^3bqR'_bR + abqR_bR' + qa^2R'_b. \end{aligned} \tag{3.18}$$

Here the prime means that b is substituted by a^2b i.e., $R' = R(a, a^2b, q)$. For $b = a = 1$ and hence $R(1,1,q) = R'(1,1,q)$. We also find that

$$\begin{aligned} R(1,1,q) &= 1 + qR^2(1,1,q) + qR(1,1,q) - q \\ 0 &= qR^2 + (q-1)R + (1+q). \end{aligned}$$

Solving the quadratic equation, we get

$$\begin{aligned} R &= \frac{1-q-\sqrt{(q-1)^2-4q(1-q)}}{2q} \\ &= \frac{1-q-\sqrt{5q^2-6q+1}}{2q} \\ &= s(q), \end{aligned} \tag{3.19}$$

where $s(q)$ is defined in (2.1).

Now, if we let $A(q) = R_a(1,1,q)$ and $B(q) = R_b(1,1,z)$ where $B(q)$ is the generating function for the numbers b_h which give the sum of semi-bases over skew Dyck paths with semi-length h. Hence the following lemmas:

Lemma 3.1.8.

$$(1 - q - 2qs(q))B(q) = qs(q)^2. \tag{3.20}$$

Proof. From (3.18) we get:

$$\begin{aligned} B(q) &= qs(q)^2 + qB(q)s(q) + qB(q)s(q) + qB(q) \\ qs(q)^2 &= B(q) - 2qB(q)s(q) - qB(q) \\ &= B(q)(1 - 2qs(q) - q). \end{aligned} \tag{3.21}$$

□

Lemma 3.1.9.

$$(1 - q - 2qs(q))A(q) - 2q(1 + s(q))B(q) = qs(q)^2. \tag{3.22}$$

Proof. From (3.17) we get:

$$\begin{aligned} A(q) &= qs(q)^2 + qA(q)s(q) + qA(q)s(q) + 2A(q) + 2A(q) + qA(q) \\ qs(q)^2 &= A(q)(1 - 2qs(q) - q) - 4A(q). \end{aligned}$$

Using (3.20), see [5], we get:

$$(1 - q - 2qs(q))A(q) - 2q(1 + s(q))B(q) = qs(q)^2. \tag{3.23}$$

□

From Lemma 3.1.8 we get:

$$\begin{aligned} B(q) &= \frac{qs(q)^2}{1 - q - 2qs(q)} \\ &= \frac{q(\frac{1-q-\sqrt{5q^2-6q+1}}{2q})^2}{1 - q - 2q\frac{1-q-\sqrt{1-6q+5q^2}}{2q}} \\ &= \frac{(1-q)(1-3q-\sqrt{5q^2-6q+1})}{2q\sqrt{5q^2-6q+1}}, \end{aligned} \tag{3.24}$$

and, see [5],

$$\begin{aligned} A(q) &= \frac{qs(q)^2 + 2q(1+s(q))B(q)}{1 - q - 2qs(q)} \\ &= \frac{qs(q)^2 + 2q(1+s(q))\frac{qs(q)^2}{1-q-2qs(q)}}{1 - q - 2qs(q)} \\ &= \frac{qs(q)^2(1 - q - 2qs(q)) + 2q(1+s(q))qs(q)^2}{(1 - q - 2qs(q))^2} \\ &= \frac{qs(q)^2(1 - q - 2qs(q) + 2q + 2qs(q))}{(1 - q - 2qs(q))^2} \\ &= \frac{(q+q^2)s(q)^2}{(1 - q - 2qs(q))^2}. \end{aligned} \tag{3.25}$$

Now we are going to break down $A(q)$ into two parts to simplify it. We are going to simplify the numerator and then the denominator.

Looking at the numerator denoted by N, we have:

$$\begin{aligned}
N &= (q+q^2)s(q)^2 \\
&= q(1+q)\frac{(1-q-\sqrt{1-6q+5q^2})^2}{4q^2} \\
&= \frac{1+q}{4q}\times((1-q)^2 - 2(1-q)\sqrt{1-6q+5q^2} + 1 - 6q + 5q^2) \\
&= \frac{1+q}{4q}\times(1-2q+q^2 - 2(1-q)\sqrt{1-6q+5q^2} + 1 - 6q + 5q^2) \\
&= \frac{1+q}{4q}\times(2-8q+6q^2 - 2(1-q)\sqrt{1-6q+5q^2}) \\
&= \frac{1+q}{4q}\times(2(1-q) - 6q(1-q) - 2(1-q)\sqrt{1-6q+5q^2}) \\
&= \frac{1+q}{4q}(1-q)(2-6q-2\sqrt{1-6q+5q^2}) \\
&= \frac{2(1+q)(1-q)}{4q}(1-3q-\sqrt{1-6q+5q^2}) \qquad (3.26) \\
&= (1+q)(1-q)\frac{1-2q-q-\sqrt{1-6q+5q^2}}{2q} \\
&= (1+q)(1-q)(s(q)-1).
\end{aligned}$$

Looking at the denominator, denoted by D, we have:

$$\begin{aligned}
D &= (1-q-2qs(q))^2 \\
&= \Big(1-q-2q(\frac{1-q-\sqrt{1-6q+5q^2}}{2q})\Big)^2 \\
&= 1-6q+5q^2 \\
&= (1-5q)(1-q). \qquad (3.27)
\end{aligned}$$

Hence,

$$\begin{aligned}
A(q) &= \frac{N}{D} \\
&= \frac{(1+q)(1-q)(s(q)-1)}{(1-5q)(1-q)} \\
&= \frac{(1+q)(s(q)-1)}{(1-5q)}. \qquad (3.28)
\end{aligned}$$

Now substituting (2.2) into (3.28), we have

$$A(q) = \frac{(1+q)(1-3q-\sqrt{1-6q+5q^2})}{2q(1-5q)}. \tag{3.29}$$

From [5] we have $a_h \sim 6.5^{h-1}$ and $a_{h+1} \sim 5a_h$. We can express a_h in terms of the $s_h{}'s$. From (3.28), by dividing by q on both sides of the equation, we have

$$\frac{A(q)}{q} = \frac{1+q}{1-5q} \times \frac{s(q)-1}{q}.$$

Then since,

$$\frac{1+q}{1-5q} = \frac{1-5q+6q}{1-5q} = 1+6\Big(\frac{q}{1-5q}\Big) = 1+6\sum_{h\geq 1} 5^{h-1}q^h, \tag{3.30}$$

we have,

$$\frac{A(q)}{q} = 1+6\Big(\frac{q}{1-5q}\Big) = 1+6\sum_{h\geq 1} 5^{h-1}q^h \times \frac{s(q)-1}{q}. \tag{3.31}$$

Now, we find the coefficient of q^h on both sides of (3.31)

$$\begin{aligned}
[q^h]\frac{A(q)}{q} &= [q^h]\Big(1+6\sum_{h\geq 1} 5^{h-1}q^h \times \frac{s(q)-1}{q}\Big) \\
[q^{h+1}]A(q) &= [q^{h+1}]s(q) + [q^h]6\sum_{h\geq 1} 5^{h-1}q^h \times (s(q)-1)q^{-1},
\end{aligned}$$

which gives

$$a_{h+1} = s_{h+1} + 6\sum_{m=0}^{h-1} 5^m s_{h-m}. \tag{3.32}$$

Equality (3.32) gives the following congruences

$$a_{h+1} \equiv s_{h+1} \ (mod\, 2,3,6) \quad \text{and} \quad a_{h+1} \equiv s_{h+1} + s_h \ (mod\, 5).$$

From (3.28) we also obtain the equation

$$\begin{aligned}
(1-5q)A(q) &= (1+q)(s(q)-1) \\
&= (1+q)s(q) - 1 - q.
\end{aligned} \tag{3.33}$$

To find the coefficients a_h from (3.33), we have

$$[q^h](A(q) - 5qA(q)) \quad = \quad [q^h]((1+q)(s(q) - 1 - q)), \tag{3.34}$$

which yields

$$a_{h+2} - 5a_{h+1} \quad = \quad s_{h+2} + s_{h+1}. \tag{3.35}$$

Since we ignore the two terms $(-1-q)$ in (3.34) because $[q^h]1 = 0$ and $[q^h]q = 0$, we start at t_2. Expanding (3.29), see [5], for $h \geq 2$ we get

$$a_h = 6 \times 5^{h-1} - \sum_{m=0}^{h-1} \binom{h-1}{m} \binom{2m+1}{m} \frac{2h-m}{h-m} - \binom{2h+1}{h}. \tag{3.36}$$

D-elevated skew Dyck paths

In Chapter 2 we introduced φ_h which denotes the number of left steps in all skew Dyck paths with semi-length h. We also found that $\varphi_h = (h-1)s_{h-1}$, see [4]. Now, $\varphi_h + b_h = hs_h$ (b_h gives the sum of semi-bases over skew paths of semi-length h and s_h gives the number of all skew paths of semi-length h) since for a skew Dyck path with a semi-length h, the semi-base plus the number of left steps of a skew Dyck path is equal to h. Therefore

$$\begin{aligned} b_h &= hs_h - \varphi_h \\ &= hs_h - (h-1)s_{h-1}. \end{aligned} \tag{3.37}$$

Definition 3.1.10. ***D-elevated*** *skew Dyck paths are skew paths that end with a down step, D and are of the form $U\sigma D$ where $\sigma \in \mathbb{W}$ ($\mathbb{W}$ is a set of all skew Dyck paths).*

Let $N(q) = \sum_{h \geq 0} n_h q^h$ denote the generating function for the numbers n_h which gives the number of all skew Dyck paths which are D-elevated, with semi-length h. To find $N(q)$ we use the generating function $J(a, b, q) = abqR(a, a^2, q)$ where $R(a, a^2, q)$ is the generating function for all skew Dyck paths where a codes the area, b codes the semi-base and q codes the semi-length. We now take the partial derivative of J with respect to a.

$$\begin{aligned} J_a &= bqR'(a, a^2b, q) + abq(R_a)' + abq(2ab)(R_b)' \\ &= bqR'(a, a^2b, q) + abq(R_b)' + 2a^2qb^2(R_a)'. \end{aligned} \tag{3.38}$$

Here the prime means (as before) that b is replaced by $a2b$, i.e., $R' = R(a, a^2b, q)$. Since $R'(1,1,q) = s(q)$, $A(q) = (R_a)(1,1,q) = (R_a)'(1,1,q)$ and $B(q) = (R_b)(1,1,q) = (R_b)'(1,1,q)$, from (3.24) and (3.29), we have

$$
\begin{aligned}
N(q) &= J_a(1,1,q) \\
&= qs(q) + qA(q) + 2qB(q) \\
&= \frac{1-q-\sqrt{1-6q+5q^2}}{2} + \frac{(1+q)(1-3q-\sqrt{1-6q+5q^2})}{2(1-5q)} \\
&\quad + \frac{(1-q)(1-3q-\sqrt{1-6q+5q^2})}{\sqrt{1-6q+5q^2}} \\
&= \frac{(1-5q)(1-q-\sqrt{1-6q+5q^2}) + (1+q)(1-3q-\sqrt{1-6q+5q^2})}{2(1-5q)} \\
&\quad + \frac{2\sqrt{1-6q+5q^2}(1-3q-\sqrt{1-6q+5q^2})}{2(1-5q)} \\
&= \frac{1-6q+5q^2-\sqrt{1-6q+5q^2}+1-2q-3q^2-\sqrt{1-6q+5q^2}}{2(1-5q)} \\
&\quad - \frac{q\sqrt{1-6q+5q^2}+2\sqrt{1-6q+5q^2}-6q\sqrt{1-6q+5q^2}-2(1-6q+5q^2)}{2(1-5q)} \\
&= \frac{4q-8q^2-2q\sqrt{1-6q+5q^2}}{2(1-5q)} \\
&= \frac{q(2-4q-\sqrt{1-6q+5q^2})}{1-5q}.
\end{aligned} \tag{3.39}
$$

L-elevated skew Dyck paths

Definition 3.1.11. ***L-elevated*** *skew Dyck paths are skew paths that end with a left step, L.*

Let $V(q) = \sum_{h \geq 0} v_h q^h$ be the generating function for v_h which are numbers that give the sum of areas for skew Dyck paths that are L-elevated, with semi-length h. To obtain $V(q)$ we use the generating function $M(a,b,q) = (qR(a,a^2,q) - 1)$. Therefore,

$$
\begin{aligned}
V(q) &= M_a(1,1,q) \\
&= q(R_a)'(1,1,q) + 2qab(R_b)'(1,1,q) \\
&= qA(q) + 2qB(q),
\end{aligned} \tag{3.40}
$$

giving

$$
\begin{aligned}
V(q) &= qA(q) + 2qB(q) \\
&= \frac{(1+q)(1-3q-\sqrt{1-6q+5q^2})}{2(1-5q)} + \frac{(1-q)(1-3q-\sqrt{1-6q+5q^2})}{\sqrt{1-6q+5q^2}} \\
&= \frac{-(1+q)(1-3q-\sqrt{1-6q+5q^2})}{-2(1-5q)} \\
&\quad - \frac{2\sqrt{1-6q+5q^2}(1-3q-\sqrt{1-6q+5q^2})}{-2(1-5q)} \\
&= \frac{-1+2q+3q^2+\sqrt{1-6q+5q^2}+q\sqrt{1-6q+5q^2}}{-2(1-5q)} \\
&\quad - \frac{2\sqrt{1-6q+5q^2}+6q\sqrt{1-6q+5q^2}+2(1-6q+5q^2)}{-2(1-5q)} \\
&= \frac{1-10q+13q^2+7q\sqrt{1-6q+5q^2}-\sqrt{1-6q+5q^2}}{-2(1-5q)} \\
&= \frac{1-10q+13q^2+(7q-1)\sqrt{1-6q+5q^2}}{-2(1-5q)}. \qquad (3.41)
\end{aligned}
$$

We now let $P(q)$ denote the generating function for p_h which give the number of all elevated (either L-elevated or D-elevated) skew Dyck paths. Since skew Dyck paths end only with a down step or a left step, it then follows that,

$$
\begin{aligned}
P(q) &= Q(q) + V(q) \\
&= \frac{q(2-4q-\sqrt{1-6q+5q^2})}{1-5q} + \frac{1-10q+13q^2+(7q-1)\sqrt{1-6q+5q^2}}{-2(1-5q)} \\
&= \frac{-2(2q-4q^2-q\sqrt{1-6q+5q^2})}{-2(1-5q)} \\
&\quad + \frac{(1-10q+13q^2+(7q-1)\sqrt{1-6q+5q^2})}{-2(1-5q)} \\
&= \frac{1-14q+21q^2+(9q-1)\sqrt{1-6q+5q^2}}{-2(1-5q)}. \qquad (3.42)
\end{aligned}
$$

3.2 Superdiagonal bargraphs

In this subsection, we look at a special type of bargraph whose structure can be traced from a skew Dyck path called a **superdiagonal bargraph**. A **bargraph** is a class of column-convex polyominos, see [5]. Let $G(\sigma)$ denote a superdiagonal bargraph obtained from a skew Dyck path. $G(\sigma)$ is structured in such a way that the skew Dyck path, σ, is rotated anticlockwise by $\frac{\pi}{4}$ from the origin, $(0,0)$ to $(0,2h)$ and has boundaries at $x=h$ and $y=0$, see Figure 3.8. Superdiagonal bargraphs are recognized by the feature that the apex of each column is located above the diagonal $y=x$, see [5] and Figure 3.8. We are going to study these superdiagonal bargraphs in terms of their perimeter and area.

Definition 3.2.1. *The* ***semi-perimeter*** *of a superdiagonal bargraph is half the length of its boundary, see* [5].

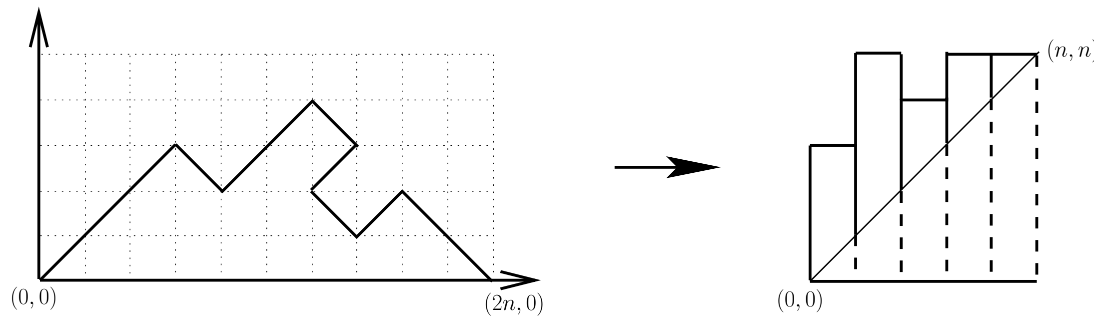


Figure 3.8: Superdiagonal bargraph and the skew Dyck path it is structured from.

3.2.1 Enumeration of superdiagonal bargraphs in relation to perimeter

The enumeration of these bargraphs will be in relation to semi-perimeter. The length of a skew Dyck path, σ, extended by $2h$ is the the perimeter of a superdiagonal bargraph. For example, in Figure 3.8, the perimeter of the superdiagonal bargraph is 22 which is the length of the skew Dyck path, 12, added with the base of the skew Dyck path, 10. If we let $q(\sigma)$, $g(\sigma)$ and $r(\sigma)$ be the number of left, down and up steps, then $g(\sigma)=h$ and $g(\sigma)+q(\sigma)=r(\sigma)$. This is illustrated Figure 3.8 where $g(\sigma)=5$, $q(\sigma)=1$ and $r(\sigma)=6$. Therefore the perimeter of the superdiagonal bargraph is as follows

$$\begin{aligned} G(\sigma) &= q(\sigma)+g(\sigma)+r(\sigma)+2h \\ &= (r(\sigma)-g(\sigma))+g(\sigma)+r(\sigma)+2h \\ &= 2(r(\sigma)+2h) \\ &= 2(r(\sigma)+g(\sigma)). \end{aligned} \tag{3.43}$$

Let $P(u,d)$ denote the generating function for skew Dyck paths where d and u denote the number of down and up steps respectively. Let $g(u)$ be the generating function $g(u)$ for the superdiagonal bargraphs in relation to semi-perimeter. Then $g(u)$ can be derived from $P(u,d)$, in particular, $g(u) = P(u,u)$. From the main decomposition, (2.1), we have

$$P(u,d) = 1 + udP(u,d)^2 + u(P(u,d) - 1), \tag{3.44}$$

where,

1. From Figure 2.2, 1 represents the empty path.

2. Then $udP(u,d)^2$ represents the skew path with decomposition $U\gamma D\mu$ in Figure 2.2. Then as shown in Figure 2.2 there is one up step represented by u and one down step represented by d. There are two different skew Dyck paths γ and μ represented by $P(u,d)^2$.

3. From Figure 2.2 $u(P(u,d) - 1)$ represents the skew path with decomposition $U\varphi L$. There is only one up step represented by u. There is only one skew Dyck path represented by φ. However, since $\varphi \neq \rho$, we subtract the empty path from $P(u,d)$ i.e., $(P(u,d) - 1)$.

Then

$$P(u,d) = \frac{1 - u - \sqrt{1 - 2u + u^2 - 4ud + 4u^2d}}{2ud}. \tag{3.45}$$

Now to obtain $g(u)$, we set $u = d$ in (3.44) and (3.45). We then get

$$\begin{aligned} P(u,u) &= 1 + u^2P(u,u)^2 + u(P(u,u) - 1) \\ g(u) &= 1 + u^2g(u)^2 + ug(u) - u \\ 0 &= u^2g(u)^2 + 1 - u - (1-u)g(u). \end{aligned} \tag{3.46}$$

Therefore

$$g(u) = \frac{1 - u - \sqrt{1 - 2u - 3u^2 + 4u^3}}{2u^2}. \tag{3.47}$$

Lemma 3.2.2. $g(u) = c\left(\frac{u^2}{1-u}\right)$, *where* $c(u)$ *is defined in section* 1.0.1.

Proof.

$$\begin{aligned}
c(u) &= \frac{1-\sqrt{1-4u}}{2u} \\
c\Big(\frac{u^2}{1-u}\Big) &= \frac{1-\sqrt{1-4\Big(\frac{u^2}{1-u}\Big)}}{2\Big(\frac{u^2}{1-u}\Big)} \\
&= \frac{1-u-\sqrt{(1-u)^2-4u^2(1-u)}}{2u^2} \\
&= \frac{1-2u-3u^2+4u^3}{2u^2} \\
&= g(u). \qquad (3.48)
\end{aligned}$$

□

Then g_h becomes, see [5], for $h \geq 1$

$$g_h = \sum_{m=1}^{h-1} \binom{h-m-1}{m-1} c_m. \qquad (3.49)$$

The coefficients g_h appear in Sloane as sequence $A090345$ $(1, 0, 1, 1, 3, 5, 12, 24$,55, $119, 272, ...)$, see [5], [8]. If we have Motzkin paths of length t whose horizontal steps are not at even levels, then the coefficients g_t count the number of these Motzkin paths. A Motzkin path with length t (length defined in section 2.2.2) is a lattice path which is in the first quadrant that originates from $(0, 0)$ and ends at $(h, 0)$, does not fall below the line $y = 0$ and consists of up steps, down steps, and horizontal steps, where each step is denoted by $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$ respectively.

Bijection between Motzkin paths and superdiagonal bargraphs

The relation between Motzkin paths and the coefficients of g_t means that Motzkin paths of length t whose horizontal steps are at odd levels are equinumerous with superdiagonal bargraphs with a semi-perimeter of t. Let $\mathbb{G}_t$ be the set of all superdiagonal bargraphs with a semi-perimeter t. Let $\mathbb{Y}$ be the set of Motzkin paths and $\mathbb{Y}_t$ be the set of Motzkin paths with a length of t whose horizontal steps are at odd levels, see Figure 3.1.7 (a) below. Likewise, let $\mathbb{E}$ be the class of Motzkin paths and $\mathbb{E}_t$ be the class of Motzkin paths with length t whose horizontal steps are at even levels, see Figure 3.9 (b) below.

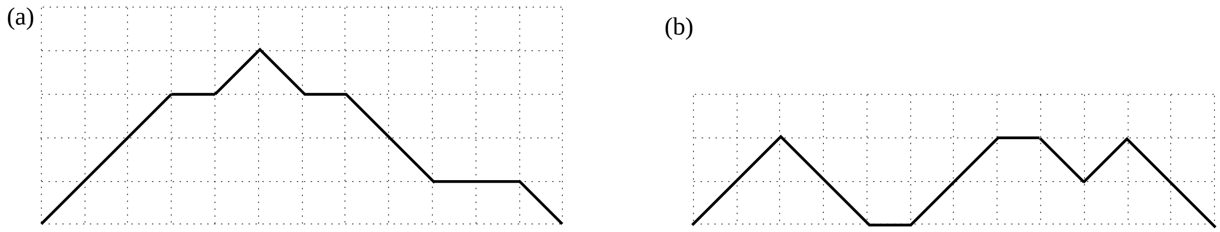


Figure 3.9: (a) A Motzkin path of length 12 whose horizontal steps are at odd levels, (b) A Motzkin path of length 12 whose horizontal steps are at even levels.

We define the bijection $\zeta : \mathbb{Y}_t \rightarrow \mathbb{G}_t$ as follows:

1. Select a superdiagonal bargraph $G(\sigma)$ in $\mathbb{G}_t$ where σ is the skew Dyck path that it is structured from.
2. The equation $r(\sigma) + g(\sigma) = t$ must be satisfied, see [5].

We manipulate (3.46) to obtain (3.50) below

$$\begin{aligned} 0 &= u^2 g(u)^2 + 1 - u - (1-u)g(u) \\ (1-u)g(u) &= u^2 g(u)^2 + 1 - u \\ g(u) &= 1 + \frac{u^2}{1-u} g(u)^2, \end{aligned} \tag{3.50}$$

where u marks the semi-perimeter for superdiagonal bargraphs. Since $\frac{u^2}{1-u} = u^2 + u^3 + u^4 + ...$ then

$$g(u) = 1 + u^2 g(u)^2 + u^3 g(u)^2 + u^4 g(u)^2 + ... + u^{m+2} g(u)^2 + ..., \tag{3.51}$$

where

- The 1 represents the empty path.
- Here $u^{m+2}g(u)^2$ is the superdiagonal bargraph which takes up the form $U^{m+1}\sigma' D \sigma'' L^m$. This is shown in Figure 3.10 below.

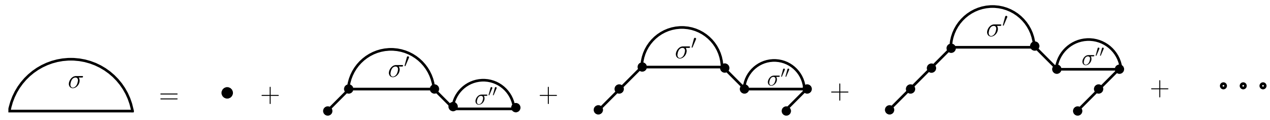


Figure 3.10 Generating function, $g(u)$, for the superdiagonal bargraphs in relation to semi-perimeter.

We notice that Motzkin paths whose horizontal steps are at odd levels have a similar decomposition and this is where our bijection comes from. We notice that every Motzkin path, $\gamma \in \mathbb{Y}$, (with γ nonempty) has a decomposition in the form

$$U\gamma' D\gamma'' \tag{3.52}$$

where $\gamma' \in \mathbb{E}$ and $\gamma'' \in \mathbb{Y}$. In the same way, every Motzkin path, $\gamma \in \mathbb{E}$, (with γ nonempty) has a decomposition in the form

$$H\gamma' \text{ (with } \gamma' \in \mathbb{Y}) \text{ or } U\gamma' D\gamma'' \text{ (with } \gamma' \in \mathbb{Y} \text{ and } \gamma'' \in \mathbb{E}) \tag{3.53}$$

From the two decompositions, we derive system of equations that involve the generating functions for $\mathbb{Y}$ and $\mathbb{E}$ and they are as follows:

$$Y(u) = 1 + u^2 E(u) Y(u).$$

So,

$$Y(u) = \frac{1}{1 - u^2 E(u)}, \tag{3.54}$$

$$\begin{aligned} E(u) &= 1 + uE(u) + u^2 E(u) Y(u) \\ (1-u)E(u) &= 1 + u^2 Y(u) E(u). \end{aligned} \tag{3.55}$$

Solving (3.54) and (3.55) simultaneously gives

$$\begin{aligned} (1-u)E(u) &= 1 + u^2 E(u) \times \frac{1}{1 - u^2 E(u)} \\ (1 - u^2 E(u))(1-u)E(u) &= 1 - u^2 E(u) + u^2 E(u) \\ 0 &= (u^2 - u^3) E^2(u) + (u-1)E(u) + 1. \end{aligned} \tag{3.56}$$

Therefore

$$
\begin{aligned}
E(u) &= \frac{1-u-\sqrt{u^2-2u+1-4(u^2-u^3)}}{2(u^2+u^3)} \\
&= \frac{1-u-\sqrt{1-2u-3u^2+4u^3}}{2(1-u)u^2}. \qquad (3.57)
\end{aligned}
$$

Substituting $E(u)$ into $Y(u)$, we get

$$
\begin{aligned}
Y(u) &= \frac{2(1-u)u^2}{2(1-u)u^2-u^2(1-u-\sqrt{1-2u-3u^2+4u^3})} \\
&= \frac{2(1-u)}{1-u+\sqrt{1-2u-3u^2+4u^3}} \times \frac{1-u-\sqrt{1-2u-3u^2+4u^3}}{1-u-\sqrt{1-2u-3u^2+4u^3}} \\
&= \frac{2(1-u)(1-u-\sqrt{1-2u-3u^2+4u^3})}{(1-u)^2-1+2u+3u^2-4u^3} \\
&= \frac{2(1-u)(1-u-\sqrt{1-2u-3u^2+4u^3})}{4u^2(1-u)} \\
&= \frac{1-u-\sqrt{1-2u-3u^2+4u^3}}{2u^2}. \qquad (3.58)
\end{aligned}
$$

From (3.54) and (3.55) we also get the identity $E(u)=\frac{Y(u)}{1-u}$ as follows:

We can restructure (3.54) to be

$$
E(u)=\frac{Y(u)-1}{Y(u)u^2}. \qquad (3.59)
$$

Then substituting (3.59) into (3.55) gives

$$
\begin{aligned}
(1-u)E(u) &= 1+u^2Y(u)\times\frac{Y(u)-1}{u^2Y(u)} \\
E(u) &= \frac{Y(u)}{1-u} \\
&= Y(u)+uY(u)+u^2Y(u)+\ldots+u^jY(u)+\ldots \quad . \qquad (3.60)
\end{aligned}
$$

Equation (3.60) can be explained combinatorially as follows:

1. Each Motzkin path $\gamma \in \mathbb{E}$ can be distinctively decomposed as:

- $Y(u)$ corresponds to $\gamma = \gamma_0$
- $uY(u)$ corresponds to $\gamma = H\gamma_1$
- $u^2Y(u)$ corresponds to $\gamma = HH\gamma_2$
- $u^jY(u)$ corresponds to $\gamma = H^j\gamma_j$

2. Now each γ_j can be distinctively decomposed as $\gamma_j = U\gamma_j' D\gamma_j''$, where $\gamma_j' \in \mathbb{Y}$ and $\gamma_j'' \in \mathbb{E}$.

3. Decompositions described in 1 and 2 above are illustrated in Figure 3.11 below

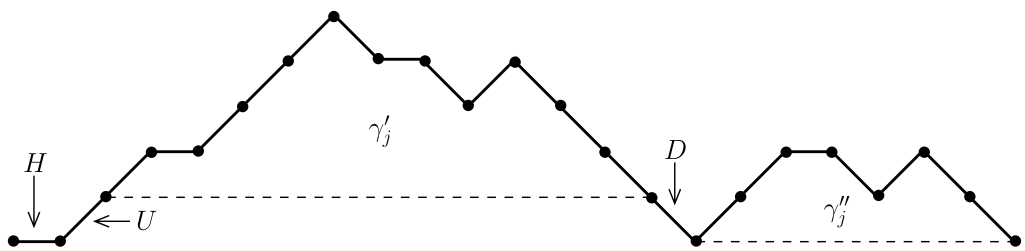


Figure 3.11 An example of the decomposition of the generating function for the set of Motzkin paths whose horizontal steps are at even levels.

Now the path γ_j can be distinctively mapped into a Motzkin path in $\mathbb{Y}$ by simply swapping the places of γ_j' and γ_j'' in the decomposition $U\gamma_j' D\gamma_j''$ becoming

$$\gamma_j{}^* = U\gamma_j'' D\gamma_j'. \tag{3.61}$$

Note that the mapping in (3.61) has a reversible counterpart illustrated in Figure 3.12. Now using this mapping, all Motzkin paths $\gamma \in \mathbb{E}$ can be distinctively represented as $H^j(\gamma_j)^*$, for $j \geq 0$ where $(\gamma)^* \in \mathbb{Y}$. This describes (3.58). We will use γ_j' and γ_j'' from Figure 3.11 to illustrate an example for $H^j(\gamma_j)^*$ in Figure 3.12 below.

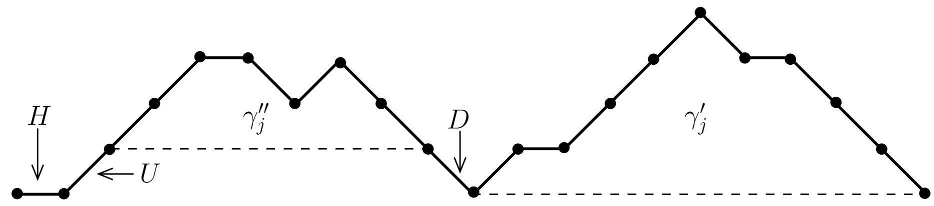


Figure 3.12 An example of the decomposition of the generating function for the set of Motzkin paths whose horizontal steps are at odd levels.

Now that we have collected all the information we need, we can define the bijection $\zeta : \mathbb{Y}_t \to \mathbb{G}_t$. Firstly, the empty Motzkin path will be mapped into the empty superdiagonal bargraph. Now from the distinct decomposition of $\gamma \in \mathbb{E}$ and (3.52) we find that $\gamma \in \mathbb{Y}_h$ can be distinctly be decomposed $\gamma = UH^j\gamma' D\gamma''$, where $j \geq 0$, $\gamma' \in \mathbb{E}$ and $\gamma'' \in \mathbb{Y}$. We can then let

$$\zeta(\gamma) = U^j U\zeta((\gamma')^*)D\zeta(\gamma'')L^j. \tag{3.62}$$

Figure 3.12 below shows the bijection ζ

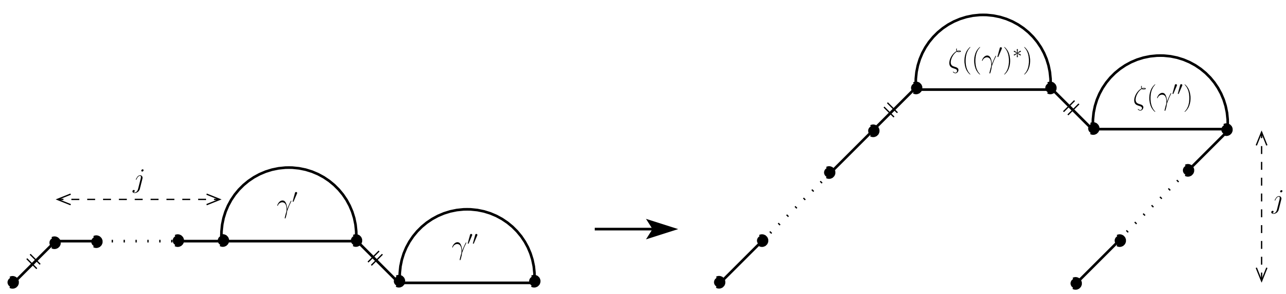


Figure 3.12 The bijection between Motzkin paths and skew Dyck paths i.e., $\zeta(\gamma) = U^j U\zeta((\gamma')^*)D\zeta(\gamma'')L^j$.

Figure 3.13 below shows an example of this bijection

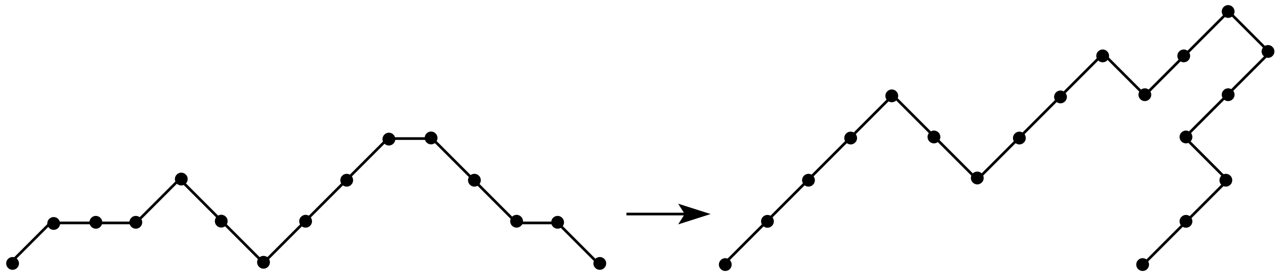


Figure 3.13 An example of the bijection between a Motzkin path and a skew Dyck path.

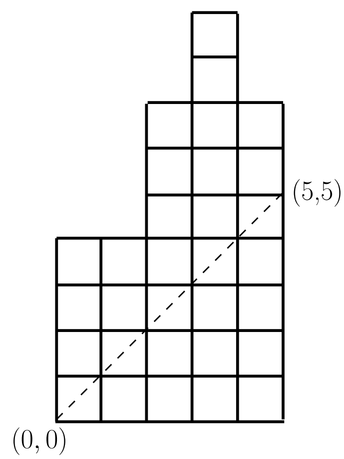


Figure 3.14 A superdiagonal bargraph with a perimeter of 28, formed from the skew Dyck path in Figure 3.13

3.2.2 Enumeration of superdiagonal bargraphs in relation to area

In this subsection we find the number T_{xl} of superdiagonal bargraphs with an area of x and which also consist of l columns. We will finally obtain the number T_x of all superdiagonal bargraphs with an area of x. Earlier in Chapter 3, in Theorem 3.1.4, we introduced a type of composition of the form $\sigma = (p_1, ..., p_l)$ with a length of l. We find that there is a relation between this composition and a bargraph with a base of l and an area of x where each j^{th} column consists of p_j cells. We then also find that there is a correspondence between this composition and superdiagonal bargraphs.

Theorem 3.2.3. *There is a bijection between superdiagonal bargraphs with an area x and a base l and the compositions of $x - \binom{l}{2}$ with a length l, see* [5].

Proof. Let Υ define the bijection as follows

$$\Upsilon : (p_1, p_2, p_3, ..., p_l) \rightarrow (p_1, p_2, p_3, ..., p_l) - (0, 1, 2, ..., l-1) \tag{3.63}$$

From (3.63) we observe the following:

1. The number of columns does not change.

2. The area decreases by $1 + 2 + ... + (l-1)$

Figure 3.15 illustrates an example of this bijection

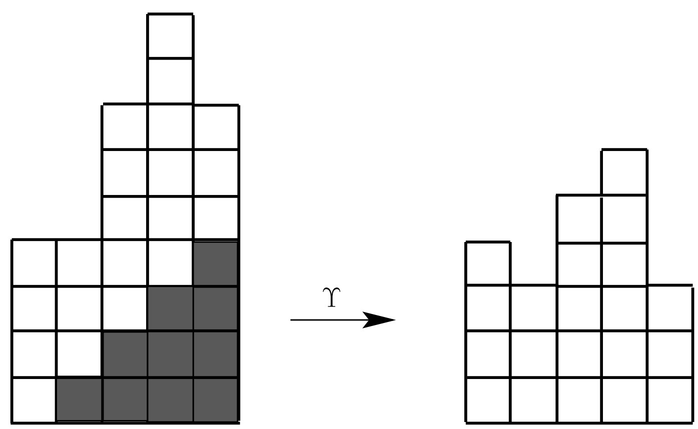


Figure 3.15 $\Upsilon : (4, 4, 7, 9, 7) \rightarrow (4, 3, 5, 6, 3)$

The shaded cells represent the area that is removed after the bijection.

□

As a result, we have

Theorem 3.2.4. *The number of all superdiagonal bargraphs with area x and with l columns is, see* [5],

$$T_{xl} = \binom{x - \binom{l}{2} - 1}{l-1}, \tag{3.64}$$

and the number of all superdiagonal bargraphs with area x is, see [5],

$$
\begin{aligned}
T_x &= \sum_{l=1}^{w} T_{xl} \\
&= \sum_{l \geq 1}^{x} \binom{x - \binom{l}{2} - 1}{l - 1}, \qquad (3.65)
\end{aligned}
$$

where $w = \frac{\sqrt{1+8x}-1}{2}$, *see* [5].

Proof. Equation (3.64) follows from Theorem 3.2.3 and it also follows from the fact that the number of all compositions of x with l parts is equal to the binomial coefficient $\binom{x-1}{l-1}$, where $x, k \geq 1$, see [5]. □

Remark 3.2.5.

Theorem 3.2.4 can also be explained in the following way. Define V to be a set of all superdiagonal bargraphs and let V_j be the set of all superdiagonal bargraphs that consist of j columns. Now V_{j+1} can be divided in such a way that there will be two subdivisions namely ${V'}_{j+1}$ and ${V''}_{j+1}$ such that $V_{j+1} = {V'}_{j+1} \bigcup {V''}_{j+1}$. We define these two subdivisions to be:

1. ${V'}_{j+1}$ is the subdivision of these bargraphs where the first column consists of precisely one cell.

2. ${V''}_{j+1}$ is the subdivision of these bargraphs where the first column consists of two cells or more.

From these definitions we observe the following:

1. A superdiagonal bargraph,${V'}_{j+1}$, can be uniquely formed by adding one row at the bottom of V_j. Figure 3.16 (a) illustrates this.

2. A superdiagonal bargraph, ${V''}_{j+1}$ can be uniquely formed by incorporating an additional cell at the apex of the first column of V_j. Figure 3.16 (b) illustrates this.

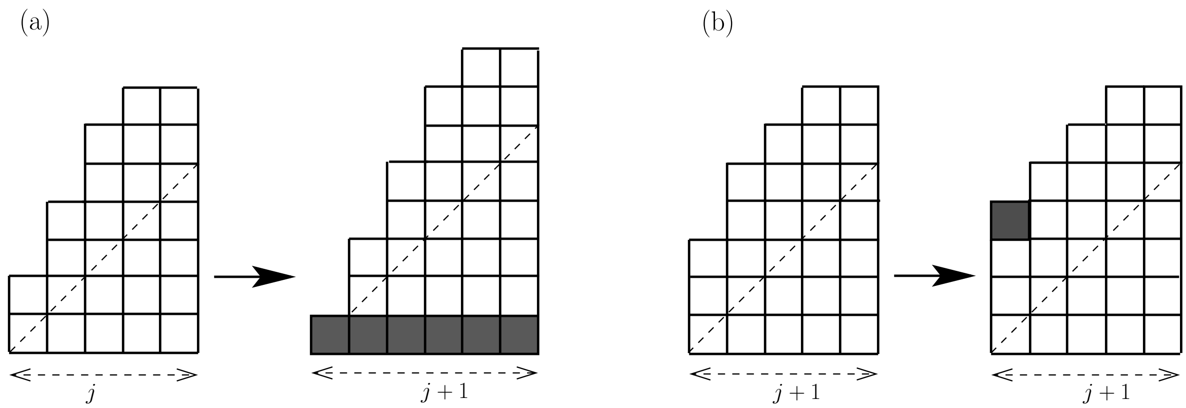


Figure 3.16 Superdiagonal bargraphs (a) V'_{j+1} formed from V_j, and (b) V''_{j+1} formed from V_j.

Now we define $T_j(q)$ to be the generating function of superdiagonal bargraphs that have a base j where q represents the area. Then from the information we just collected above we have

$$T_{j+1}(q) = q^{j+1}T_j(q) + qT_{j+1}(q), \tag{3.66}$$

where

- $q^{j+1}T_j(q)$ represents the case where the additional row is incorporated, as shown in Figure 3.16 (a).

- $qT_{j+1}(q)$ represents the case where the additional cell is incorporated, as shown in Figure 3.16 (b).

Now since $T_0(q) = 1$, it then follows from (3.66) that

$$T_j(q) = \frac{q^j}{1-q} \times T_{j-1}(q). \tag{3.67}$$

We then derive $T_j(q)$ in terms of q and j.

$$
\begin{aligned}
T_1(q) &= \frac{q}{1-q}\times T_0(q) = \frac{q}{1-q} \\
T_2(q) &= \frac{q^2}{1-q}\times T_1(q) = \frac{q^3}{(1-q)^2} \\
T_3(q) &= \frac{q^3}{1-q}\times T_2(q) = \frac{q^6}{(1-q)^3} \\
T_4(q) &= \frac{q^4}{1-q}\times T_3(q) = \frac{q^{10}}{(1-q)^4} \\
&. \\
&. \\
&. \\
T_j(q) &= \frac{q^{j(j+1)/2}}{(1-q)^j}\times T_0(q) \\
&= \frac{q^{\binom{j+1}{2}}}{(1-q)^j}. \qquad (3.68)
\end{aligned}
$$

Letting

$$
T(q) := \sum_{x\geq 0} T_x q^n, \qquad (3.69)
$$

where T_x is the total number of superdiagonal bargraphs with area x. Then

$$
\begin{aligned}
T(q) &= \sum_{j\geq 0} T_j(q) \\
&= \sum_{j\geq 0} \frac{q^{\binom{j+1}{2}}}{(1-q)^j}. \qquad (3.70)
\end{aligned}
$$

From direct inspection, we find that $\frac{T_x}{T_{x-1}} \to 1$, as $x \to \infty$, see [5]. This conjecture was proved by Louchard, see [5]. The table below shows the first ten numbers, V_x, see [5].

x	0	1	2	3	4	5	6	7	8	9
T_x	1	1	1	2	3	4	6	9	13	18

Figure 3.17 shows T_5, the four superdiagonal bargraphs each with area 5.

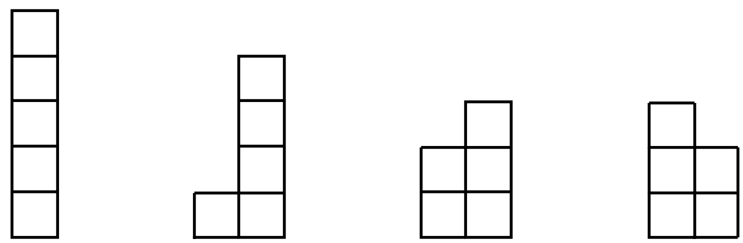


Figure 3.17 The four superdiagonal bargraphs each with area 5 ie., T_5.

3.2.3 Summary

In this chapter, where we close our study for skew Dyck paths, we continued to build on skew Dyck paths and we based the study according to area. We mainly focused on the enumeration in relation to area, semi-base and semi-length for these skew Dyck paths. We then established a connection between skew Dyck paths and a type of bargraphs called superdiagonal bargraphs. We finally went on to enumerate the area of these superdiagonal bargraphs in relation to their perimeter and area.

3.3 Conclusion

This dissertation was mainly about the study of skew Dyck paths. However, we first introduced Dyck paths, in Chapter 1, and a brief enumeration of them with other parameters such as number of peaks, valleys, doublerises and return steps. The concept of skew Dyck paths was then introduced in Chapter 2, as these are a generalization of Dyck paths. We showed that there is a bijection from skew Dyck paths to other combinatorial objects namely Hex trees, Motzkin paths, marked trees and weighted Dyck paths. We then studied four different types of involutions on skew Dyck paths, illustrating graphical examples in each. We ended Chapter 2 with some statistics on skew Dyck paths. The last chapter, Chapter 3, was still an extension on skew Dyck paths but focusing on their area. We showed an enumeration of these skew Dyck paths in relation to area and semi-length. We finally closed the chapter by introducing superdiagonal bargraphs which we showed are constructed from skew Dyck paths. We showed their enumeration in relation to area and perimeter.

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