On the Geometry of Lightlike Hypersurfaces

A dissertation submitted in fulfillment of a Masters degree at the University of the Witwatersrand, Johannesburg-South Africa.

By

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A dissertation submitted in fulfillment of a masters degree in the school of mathematics at the University of the Witwatersrand, Johannesburg-South Africa.

As the candidate's supervisor, I have approved this thesis for submission.

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Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Jabulile Khoza

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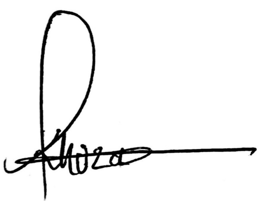

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Dedication

I dedicate this work to **Florah** and **Junus Khoza**

my parents and care givers,

and to my sisters **Suzan Khoza** and **Phindile Khoza**.

for being there during times of tribulation.

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Abstract

The study of lightlike geometry in semi-Riemannian manifolds was introduced by A. Bejancu and K. L. Duggal in their book [3]. The book laid a foundation for the study of lightlike submanifolds by providing some constructions on such subspaces. In this research, we focus on the study of lightlike hypersurfaces of semi-Riemannian manifolds, i.e. submanifolds of codimension one. In this line, we prove necessary and sufficient conditions for a lightlike hypersurface of a semi-Riemannian space form to be totally umbilical, screen conformal and screen almost conformal (see Theorem 2.5.2, 2.5.4 and 2.5.5).

Contents

CHAPTER

INTRODUCTION AND OBJECTIVES

0.1 Introduction

In this research, we study the geometry of lightlike hypersurfaces in a semi-Riemannian manifold. Lightlike hyperspaces have numerous applications in general relativity and electromagnetism. For instance, in general relativity, lightlike hypersurfaces represent different black hole horizons (see details in [3]). Unlike classical theory of nondegenerate hypersurfaces, the normal bundle of a lightlike hypersurface is part of the tangent bundle. This anomaly makes the study of lightlike hypersurfaces extremely difficult. To overcome this challenge, A. Bejancu and K. L. Duggal introduced in [3] the extrinsic approach to the study of lightlike geometry in which their approach depends on a screen distribution of choice. And since the lightlike hypersurface is Hausdorff and paracompact, such a screen distribution always exists. Following their constructions, many scholars have studied lightlike hypersurfaces, see for instance [7][10][11][12][14][15].

In this thesis, we study the geometry of lightlike hypersurfaces, with a specific attention to totally umbilical and screen conformal ones. We prove new and necessary conditions for which a lightlike hypersurface of a semi-Riemannian space form is totally umbilical, screen conformal and almost screen conformal.

The rest of the thesis is arranged as follows: In Chapter 1, we present the basic preliminaries needed in the rest of the document, In Chapter 2 we study the geometry of lightlike hypersurfaces. In particular, we prove Theorem 2.5.2, 2.5.4 and 2.5.5) which are the main contributions to this study.

Chapter One

Preliminaries

1.1 Lightlike Vector spaces

Suppose that V is an m-dimensional vector space with a symmetric bilinear map $g : V \times V \to \mathbb{R}$. Then g is said to be **degenerate** on V if there exists a vector $\xi \in V$ with $\xi \neq 0$, such that $g(\xi, v) = 0,\ \forall v \in V$. We say that V is nondegenerate if for any given $u \in V$ one has $g(u, v) = 0$ for all $v \in V$ implies that $u = 0$. The set of all ξ such that g degenerate is given by $\mathrm{Rad}V = \{\xi \in V : g(\xi, v) = 0, \forall v \in V\}$, and $\mathrm{Rad}V$ is called the **Radical/null space** of V. We call the dimensions of $\mathrm{Rad}V$ the **nullity degree** of g, denoted by $\mathrm{null}V$. Clearly, g is degenerate if and only if $\mathrm{null}V > 0$, and nondegenerate if and only if $\mathrm{null}V = 0$.[3]

Proposition 1.1.1. *[3] g is degenerate if and only if* $\mathrm{null}V > 0$.

Proof. $\Rightarrow$ Suppose that g is degenerate, then $\mathrm{Rad}V \neq \{0\}$ which implies that $\mathrm{null}V \neq 0$. Therefore, $\mathrm{null}V > 0$.

$\Leftarrow$ Suppose that $\mathrm{null}V > 0$, then $\mathrm{Rad}V \neq \{0\}$. This implies that there exists a $\xi \neq 0$ such that $g(\xi, v) = 0$ for all $v \in V$. Therefore, g is degenerate. □

Proposition 1.1.2. *[3] g is nondegenerate if and only if* $\mathrm{null}V = 0$.

Proof. $\Rightarrow$ Suppose that g is nondegenerate, then $\mathrm{Rad}V = \{0\}$ which implies that $\mathrm{null}V = 0$.

$\Leftarrow$ Suppose that $\mathrm{null}V = 0$, then $\mathrm{Rad}V = \{0\}$ which implies that g is nondegenerate. □

In a case where $g(v, v) > 0(g(v, v) < 0)$ for any nonzero $v \in V$, we say that g is **positive(negative) definite** . If $g(v, v) \geq 0(g(v, v) \leq 0)$ for any $v \in V$; and if there exits a nonzero $u \in V$ such that $g(u, v) \neq 0$, we say that g is **positive(negative) semi-definite** for any nonzero $v \in V$.

Proposition 1.1.3. *[3] Any positive or negative semi-definite g is degenerate.*

Proof. Suppose that g positive(negative) semi-definite, then by definition there exists a nonzero $u \in V$ such that $g(u, v) \neq 0$ for all $v \in V$. This implies that $\text{Rad}V \neq \{0\}$, therefore g is degenerate. $\square$

Suppose that W is a subspace of V, then $g_{|W} : W \times W \to \mathbb{R}$ is a restriction of g on $W \times W$ and it is also a symmetric bilinear map on W. If q is the dimension of the largest subspace $W \subset V$ for which $g_{|W}$ is negative definite, then q is called the **index** of g on V.[3]

It is convenient to write g in terms of its quadratic form. Since g is bilinear, then

$$\begin{aligned} g(v+w, v+w) &= g(v, v+w) + g(w, v+w) \\ &= g(v,v) + g(v,w) + g(w,v) + g(w,w) \\ &= g(v,v) + 2g(v,w) + g(w,w), \end{aligned}$$

for any $v, w \in V$. It follows that

$$g(v,w) = \frac{1}{2}[g(v+w, v+w) - g(v,v) - g(w,w).]$$

This equation can be rewritten as

$$g(v,w) = \frac{1}{2}[h(v,w) - h(v) - h(w)],$$

where $h : V \to \mathbb{R}$ is the associated quadratic form of a symmetric bilinear map g with $h(v) = g(v,v)$ for all $v \in V$. Suppose that $E = \{e_1, e_2, \cdots, e_m\}$ is the basis of V, then

$$\begin{aligned} h(v) &= g(v,v) \\ &= \langle v, v\rangle \\ &= \sum_{i=1}^{m} \langle v^i e_i, v^i e_i\rangle \\ &= \sum_{i=1}^{m} (v^i)^2 \langle e_i, e_i\rangle \\ &= \sum_{i=1}^{m} \lambda_i (v^i)^2, \end{aligned}$$

where $\lambda_i \in \mathbb{R}$ and v^i are the components of V with respect to the basis E. h is said to be of type (p, q, r) where $p + q + r = m$, if there exists in p, q, and r coefficients of λ_i which are positive, negative and zero, respectively. In this case q is the index and r is the nullity degree of g in V. Generally, the canonical form h is not unique and the type of h does not depend on the E chosen. [3]

Let h be the associated quadratic form of g of type (p, q, r) on V. Then we have the following propositions:

Proposition 1.1.4. *[3] g is degenerate(nondegenerate) if and only if $r > 0 (r = 0)$.*

Proof. **W**e prove the first case by Proposition 1.1.1 and the second case by Proposition 1.1.2.

Case I: $\Rightarrow$ Suppose that g is degenerate, then $nullV > 0$. It follows that $r = nullV > 0$.

$\Leftarrow$ Suppose that $r > 0$, then $r = nullV > 0$. Thus, by Proposition 1.1.1 g is degenerate.

Case II: $\Rightarrow$ Suppose that g is nondegenerate nondegenerate, then $r = nullV = 0$.

$\Leftarrow$ Suppose that $r = 0$, then $r = nullV = 0$. Thus, by Proposition 1.1.2 g is nondegenerate. □

Proposition 1.1.5. *[3] g is positive(negative) definite if and only if $p = m(q = m)$.*

Proof. **Case I:** $\Rightarrow$ Suppose that g is positive definite, then $g(v, v) > 0$ for all nonzero $v \in V$, and $r = nullV = 0$ by Proposition 1.1.4. Since g is not negative definite, by definition $q = 0$ and $p + q + r = p = m$. $\Leftarrow$ Suppose that $p = m$, then $q = 0$ and $r = 0$. This implies that g is nondegenerate and $g(v, v) > 0$ for all nonzero $v \in V$. Therefore, g is positive definite.

Case II: $\Rightarrow$ Suppose that g is negative definite, then $g(v, v) < 0$ for all nonzero $v \in V$, and $r = nullV = 0$ by Proposition 1.1.4. Since g is negative definite, by definition $q \neq 0$ and $p = 0$. So, $p + q + r = q = m$. $\Leftarrow$ Suppose that $q = m$, then $p = 0$ and $r = 0$. This implies that g is nondegenerate and $g(v, v) < 0$ for all nonzero $v \in V$. Therefore, g is negative definite. □

Proposition 1.1.6. *[3] g is positive(negative) semi-definite if and only if $q = 0, p > 0, r > 0 (p = 0, q > 0, r > 0)$.*

Proof. **Case I :** $\Rightarrow$ Suppose that g is positive semi-definite, then $g(v, v) > 0$ for all nonzero $v \in V$ and there exists a nonzero $u \in V$ such that $g(u, v) = 0$ for all $v \in V$. As $g(v, v) > 0$, $p > 0$ and $q = 0$. Since $g(u, v) = 0$, then $r > 0$.

$\Leftarrow$ Suppose that $q = 0, p > 0, r > 0$. Then $g(v, v) > 0$ and $r = nullV > 0$, which implies g is positive semi-definite by definition.

Case II: $\Rightarrow$ Suppose that g is negative semi-definite, then $g(v, v) < 0$ for all nonzero $v \in V$ and there exists a nonzero $u \in V$ such that $g(u, v) = 0$ for all $v \in V$. As $g(v, v) < 0$, $p = 0$ and $q > 0$. Since $g(u, v) = 0$, then $r > 0$.

$\Leftarrow$ Suppose that $p = 0, q > 0, r > 0$. Then $g(v, v) < 0$ and $r = nullV > 0$, which implies g is negative semi-definite by definition. □

Let $U = \{u_i, \cdots, u_m\}$ be an arbitrary basis of V. Then it is possible to express g by an $m \times m$ matrix $G = [g_{ij}]$ with $g_{ij} = g(u_i, u_j)$ where $1 \leq i, j \leq m$. This matrix is called **the associated matrix** of g with respect to the basis U. Moreover,

$$\begin{aligned} g_{ij} &= g(u_i, u_j) \\ &= \langle u_i, u_j \rangle \\ &= \sum_{i=1}^{m} \langle u_i^i e_i, u_j^i e_i \rangle \\ &= \sum_{i=1}^{m} u_i^i u_j^i \langle e_i, e_i \rangle \\ &= \sum_{i=1}^{m} \lambda_i u_i^i u_j^i, \end{aligned}$$

where $\{e_1, \cdots, e_j\}$ are the eigenvalues of G and λ_i are the corresponding eigenvalues. Clearly, g is nondegenerate(degenerate) on V if and only if $rank\ G = m(rank\ G < m)$.

In the case where g is nondegenerate, g is said to be a **semi-Euclidean metric** and V is called **semi-Euclidean space**. A **proper semi-Euclidean metric** g is such that $p \cdot q \neq 0$ and in this case V is called a **proper semi-Euclidean space**. When g is positive definite, then g is a **Euclidean metric** and V **Euclidean space**. A **Lorentz (Minkowski) space** V is defined by the **Lorentz (Minkowski) metric** g where the the index $q = 1$. In the case where g is degenerate, V is said to be a **lightlike (degenerate) vector space**. If $g(v, v) > 0$ or $g = 0$, we say that g is **spacelike**. If $g(v, v) < 0$, we say that g is **timelike**. If $g(v, v) = 0$ or $v \neq 0$, we say that g is **lightlike**.

Definition 1.1.1. The map $||\cdot|| : V \to \mathbb{R}$ is called the **norm** and it is such that

$$||v|| = |g(v, v)|^{\frac{1}{2}},$$

for all $v \in V$.

Definition 1.1.2. [3] The set of all null vectors, Λ in a vector space V is defined by

$$\Lambda = \{v \in (V - \{0\}), g(v, v) = 0\}.$$

The proposition below provides a prove for the existence the lightlike vector.

Proposition 1.1.7. *There exists a orthonormal basis of a semi-Euclidean space such that* $V \neq \{0\}$.

Proof. Prove provided in page 3 of [3]. □

Example 1.1.1. [3] Suppose that $E = \{e_1 = (1, 0, \cdots, 0), \cdots, e_m = (0, 0, \cdots, 1)\}$ is the canonical basis of $\mathbb{R}^m$ equipped with a proper semi-Euclidean metric such that for any $0 < q < m$ and $x, y \in \mathbb{R}^m$

$$
\begin{aligned}
g(x,y) &= \langle x, y\rangle \\
&= \sum_{i=1}^{m} \langle x^i e_i, y^i e_i\rangle \\
&= \sum_{i=1}^{m} x^i y^i \langle e_i, e_i\rangle \\
&= \sum_{i=1}^{m} x^i y^i g(e_i, e_i) \\
&= -\sum_{i=1}^{q} x^i y^i + \sum_{i=a=q+1}^{m} x^a y^a .
\end{aligned}
$$

The hypersurface Λ_{q-1}^{m-1} of $\mathbb{R}_q^m$ is defined by

$$\Lambda_{q-1}^{m-1} = \big\{ x \in (\mathbb{R}_q^m - \{0\}\big), \sum_{i=1}^{q} (x^i)^2 + \sum_{a=q+1}^{m} (x^i)^2 = 0\}$$

and it is called the **null cone** of V. Finally, $\mathbb{R}^m$ becomes a Euclidean space g given by

$$g(x,y) = \sum_{A=1}^{m} x^A y^A .$$

[3]

The orthonormal basis E of V satisfies the Kronecker delta equation through the relation

$$g(e_i, e_j) = e_i \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

and we can express the vector v of V as

$$v = \sum_{i=1}^{m} \epsilon_i g(v, e_i) e_i .$$

The set of elements in the basis of E are called the **signature** of E and the associated matrix can be rewritten as

$$h(v) = g(v,v) = \sum_{i=1}^{m} g(v, e_i)^2 .$$

Remark 1.1.1. In this research we are only interested in the orthonormal basis of a semi-Euclidean space with a signature of the for $(- \cdots - + \cdots +)$.

1.2 Semi-Riemannian Curvature

Suppose that ∇ is a linear connection on (M, g), then for all $X, Y, Z \in \Gamma(TM)$

$$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \tag{1.1}$$

if and only if g is parallel with respect to g and we call ∇ a **metric (Levi Civita) connection**. A result called **Koszul formula** gives that

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X), \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. We define the **curvature tensor** of a semi-Riemannian manifold by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.2}$$

for all $X, Y, Z \in \Gamma(TM)$. Moreover, the **torsion tensor** is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{1.3}$$

In this case, we say ∇ is torsion free if and only if $T = 0$. If ∇ is torsion free, then ∇ is symmetric by definition. Bianchi developed two identities that described the relationship between curvature tensors. The two **Bianchi identities** are given by

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \tag{1.4}$$

and

$$(\nabla_X R)(Y, Z, W) + (\bar{\nabla}_Y R)(Z, X, W) + (\nabla_Z R)(X, Y, W) = 0. \tag{1.5}$$

One more important definition, is the definition of the type-4 curvature tensor of the semi-Riemannian manifold which is given by

$$R(X, Y, Z, U) = g(R(X, Y)Z, U), \ \ \forall X, Y, Z, U \text{ on } M. \tag{1.6}$$

This definition yields that

$$\begin{aligned} R(X, Y, Z, U) + R(Y, X, Z, U) &= 0, \\ R(X, Y, Z, U) + R(X, Y, U, Z) &= 0, \\ R(X, Y, Z, U) - R(Z, U, X, Y) &= 0. \end{aligned}$$

[6]

Lightlike Hypersurfaces

2.1 Lightlike Transversal Vector Bundle of a Lightlike Hypersurface

By a lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, we mean a co-dimension one submanifold of $\bar{M}$, such that the metric tensor g, which is the restriction $\bar{g}_x|_{T_xM}$, for any $x \in M$, is lightlike (also known as degenerate). In particular, this means that there exists a non-zero section ξ such that $g(\xi, X) = 0$, for any X tangent to M. Now, the radical subspace of T_xM, at each point $x \in M$, denoted by $\operatorname{Rad} T_xM$, is defined as $\operatorname{Rad} T_xM = \{\xi \in T_xM : g_x(\xi, X),\ \forall X \in T_xM\}$. The dimension of $\operatorname{Rad} T_xM$ is called the nullity degree of g. As ξ is also orthogonal to itself, one can easily see that the normal subspace $T_xM^{\perp}$ of M is also lightlike and, moreover, $\operatorname{Rad} T_xM = T_xM \cap T_xM^{\perp}$. Since the dimension of $T_xM^{\perp}$ is one, it follows that $\dim \operatorname{Rad} T_xM = 1$ and that $\operatorname{Rad} T_xM = T_xM^{\perp}$. Then, we call $\operatorname{Rad} TM$ the radical distribution of M.

The complementary and orthogonal vector bundle to $\operatorname{Rad} TM$ in TM is called the screen distribution of M, and often denoted by $S(TM)$. We, therefore, have the decomposition

$$TM = \operatorname{Rad} TM \perp S(TM), \tag{2.1}$$

where "$\perp$" between vector subbundles means an orthogonal sum. It is easy to see that $S(TM)$ is a nondegenerate distribution on M, and its existence is secured by the assumption that M is paracompact. Unfortunately, the screen distribution is not unique, and often a lightlike hypersurface with a specific screen distribution is denoted as $(M, g, S(TM))$. The following result is fundamental to the study of lightlike hypersurface:

Theorem 2.1.1 (Duggal-Bejancu [3]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a unique vector bundle* $\operatorname{tr}(TM)$ *of rank one, over M, such that for any nonzero section ξ of* $\operatorname{Rad} TM$ *on a* *coordinate neighbourhood $\mathcal{U}$ of M, there exists a unique section N of* $\operatorname{tr}(TM)$, *on* $\mathcal{U}$, *such that* $\bar{g}(\xi, N) = 1$, $\bar{g}(N, N) = 0$ *and* $\bar{g}(N, X) = 0$, *for any X tangent to* $S(TM)|_{\mathcal{U}}$.

Consider the decomposition

$$T\bar{M}_{|M} = S(TM) \perp S(TM)^{\perp}.$$

It follows from Duggal-Bejancu theorem that

$$\begin{aligned} T\bar{M}_{|M} &= S(TM) \oplus_{orth} (TM^{\perp} \oplus tr(TM)) \\ &= TM \oplus_{orth} tr(TM). \end{aligned}$$

Proposition 2.1.1. *[3] Any screen distribution $S(TM)$ of a lightlike hypersurface is non-degenerate of constant index $q-1$.*

Proof. Given that $u \neq 0$ and $u \in S(TM)$ with $g(u,v) = 0$ for all $v \in S(TM)$. By definition of $RadTM = TM^{\perp}$, $g(u,\xi) = 0$ for all $\xi \in RadTM$. And $g(u,N) = 0$ for all $N \in tr(TM)$. This implies that $u \in TM^{\perp}$ or $u \in tr(TM)$. This is a contradiction as $S(TM)$ and $S(TM)^{\perp}$ are complimentary subspaces. Thus, $S(TM)$ is non-degenerate. As $S(TM)$ is in M, then the index of $S(TM)$ $t < q$. On the other hand, the restriction of the induced metric on $S(TM)$ is of index of $t \geq q-1$, since $\dim T_uM^{\perp} = 1$. Hence the index of $S(TM)$ is of index $q-1$. □

Proposition 2.1.2. *[3] Any screen distribution on a lightlike hypersurface of a Lorentz manifold is Riemannian.*

Proof. Suppose that (M,g) is a $(n+1)$-dimensional lightlike hypersurface of a $(n+2)$-dimensional Lorentzian manifold $(\bar{M}, \bar{g})$, with $n \geq 2$. Since $S(TM)$ is of index $q-1$, $S(TM)$ is n dimensional. Since $n \in \mathbb{R}^{+}$, $g(v,v) \geq 0$ for all nonzero $v \in S(TM)$. But $g(v,v) \neq 0$ for all $v \in S(TM)$, since $S(TM)$ is nondegenerate. Hence, $g(v,v) > 0$ which implies that $S(TM)$ is positive definite. Since the induced metric on $S(TM)$ is positive definite, then any screen distribution on a lightlike hypersurface of a Lorentz manifold is Riemannian. □

2.2 The Induced Geometrical Objects on a Lightlike Hypersurface

Let (M,g) be a lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. And suppose that ∇ and $\bar{\nabla}$ are linear connections connections on M and $\bar{M}$, respectively. Since $T\bar{M}_{|M} = TM \perp tr(TM)$. Then for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$ we have:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.2}$$

where $\nabla_X Y \in \Gamma(TM)$ and $h(X,Y) \in \Gamma(tr(TM))$. And

$$\bar{\nabla}_X Y = -A_V X + \nabla^t_X V, \tag{2.3}$$

$A_V X \in \Gamma(TM)$ and $\nabla^t_X V \in \Gamma(tr(TM))$. ∇ and ∇^t are called induced linear connection on M and $tr(TM)$, respectively. Moreover, h and A_V are said to be the **second fundamental form** and **the shape operator** of M in $\bar{M}$, respectively.

Proposition 2.2.1. ∇ *is torsion free.[3]*

Proof. Since $\bar{\nabla}$ is symmetric, then by definition $\bar{\nabla}$ is torsion free. Moreover, by the definition of torsion T and equation 2.2 we obtain

$$\begin{aligned} T(X,Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \\ 0 &= \nabla_X Y - h(X,Y) - \nabla_Y X + h(Y,X) - [X,Y] \\ 0 &= h(Y,X) - h(X,Y) + \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned}$$

Equating the components,

$$h(X,Y) = h(Y,X)$$

and

$$[X,Y] \equiv \nabla_X Y - \nabla_Y X = 0.$$

The second equation implies that ∇ is symmetric. Thus, by definition ∇ is torsion free.[9] □

Proposition 2.2.2. *h is a $\Gamma(tr(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$.[3]*

Proof. Suppose that f is a real valued function about p. Since

$$h(X,Y) = \bar{\nabla}_X Y - \nabla_X Y,$$

by the definition of a connection:

1. h is additive in the first argument.

$$\begin{aligned} h(X+Z,Y) &= \bar{\nabla}_{X+Z} Y - \nabla_{X+Z} Y \\ &= (\bar{\nabla}_X Y + \bar{\nabla}_Z Y) - (\nabla_X Y + \nabla_Z Y) \\ &= (\bar{\nabla}_X Y - \nabla_X Y) + (\bar{\nabla}_Z Y - \nabla_Z Y) \\ &= h(X,Y) + h(Z,Y). \end{aligned}$$

2. h is additive in the second argument.

$$\begin{aligned} h(X,Y+W) &= \bar{\nabla}_X (Y+W) - \nabla_X (Y+W) \\ &= \bar{\nabla}_X Y + \bar{\nabla}_X W - \nabla_X Y - \nabla_X W \\ &= h(X,Y) + h(X,W). \end{aligned}$$

3. h is homogeneous in the first argument.

$$\begin{aligned} h(f(p)X,Y) &= \bar{\nabla}_{f(p)X}Y - \nabla_{f(p)X}Y \\ &= f(p)\bar{\nabla}_X Y - f(p)\nabla_X Y \\ &= f(p)[\bar{\nabla}_X Y - \nabla_X Y] \\ &= f(p)h(X,Y). \end{aligned}$$

4. h is homogeneous in the second argument.

$$\begin{aligned} h(X,fY) &= \bar{\nabla}_X fY - \nabla_X fY \\ &= (Xf)Y_p f(p) + f(p)\bar{\nabla}_X Y - [(Xf)Y_p f(p) + f(p)\nabla_X Y] \\ &= f(p)\bar{\nabla}_X Y - f(p)\nabla_X Y \\ &= f(p)[\bar{\nabla}_X Y - \nabla_X Y] \\ &= f(p)h(X,Y). \end{aligned}$$

Since ∇ is a linear connection on $\Gamma(tr(TM))$, it follows that h is a $\Gamma(tr(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$.[9] □

Proposition 2.2.3. *A_V is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$.[3]*

Proof. Suppose that f is a real valued function about p, and let $\nabla^t_X V$ and ∇_X be linear connections on $\Gamma(TM)$. Then

$$\begin{aligned} A_V(X+Z) &= \nabla^t_{X+Z}V - \bar{\nabla}_{X+Z}V \\ &= \nabla^t_X V + \nabla^t_Z V - (\bar{\nabla}_X V + \bar{\nabla}_Z V) \\ &= (\nabla^t_X V - \bar{\nabla}_X V) + (\nabla^t_Z V - \bar{\nabla}_Z V) \\ &= A_V X + A_V Z. \end{aligned}$$

$$\begin{aligned} A_V(f(p))X &= \nabla^t_{f(p)X}V - \bar{\nabla}_{f(p)X}V \\ &= f(p)\nabla^t_X V - [f(p)\bar{\nabla}_X V] \\ &= f(p)[\nabla^t_X V - \bar{\nabla}_X V] \\ &= f(p)A_V X. \end{aligned}$$

Thus, A_V is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$.[9] □

Proposition 2.2.4. *∇^t is a linear connection on the lightlike transversal vector bundle $tr(TM)$.[3]*

Proof. Recall that

$$\nabla^t_X V = -A_V X + \bar{\nabla}_X V.$$

So, by proposition 2.2.3

$$\begin{aligned}
\nabla^t_X(V+W) &= -A_{V+W}X + \bar{\nabla}_X(V+W) \\
&= -A_V X - A_W X + \bar{\nabla}_X V + \bar{\nabla}_X W \\
&= -A_V X + \bar{\nabla}_X V - A_W X + \bar{\nabla}_X W \\
&= \nabla^t_X V + \nabla^t_X W.
\end{aligned}$$

$$\begin{aligned}
\nabla^t_{X+Y} V &= -A_V(X+Y) + \bar{\nabla}_{X+Y} V \\
&= -A_V X - A_V Y + \bar{\nabla}_X V + \bar{\nabla}_Y V \\
&= -A_V X + \bar{\nabla}_X V - A_W X + \bar{\nabla}_X W \\
&= \nabla^t_X V + \nabla^t_Y V.
\end{aligned}$$

$$\begin{aligned}
\nabla^t_{f(p)X} V &= -A_V(f(p)X) + \bar{\nabla}_{f(p)X} V \\
&= -f(p)A_V X + f(p)\bar{\nabla}_X V \\
&= f(p)[-A_V X + \bar{\nabla}_X V] \\
&= f(p)\nabla^t_X V.
\end{aligned}$$

$$\begin{aligned}
\nabla^t_X(fV) &= -A_{fV}X + \bar{\nabla}_X(fV) \\
&= -f(p)A_V X + f(p)\bar{\nabla}_X V \\
&= f(p)[-A_V X + \bar{\nabla}_X V] \\
&= f(p)\nabla^t_X V.
\end{aligned}$$

Thus, ∇^t is a linear connection on the lightlike transversal vector bundle $tr(TM)$.[9] □

Suppose that $\mathcal{U} \subset M$ and the pair $\{\xi, N\}$ is in the neighbourhood of M. We define a $\mathcal{F}(\mathcal{U})$-symmetric bilinear form B and a 1-form τ on $\mathcal{U}$ as follows,

$$B(X,Y) = \bar{g}(h(X,Y),\xi) \tag{2.4}$$

and

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi) \tag{2.5}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$.[3] Then the relationship between the elements of $\Gamma(TM)$ and $\Gamma(tr(TM))$ with respect to the second fundamental form, the shape operator, the $\mathcal{F}(\mathcal{U})$-symmetric bilinear form B and the 1-form τ on $\mathcal{U}$ is given by the Lemma below.

Lemma 2.2.1. *[3] $h(X, Y) = B(X, Y)N$ and $\nabla^t_X N = \tau(X)N$.*

Proof. By equation 2.4 and 2.5

$$\begin{aligned} \bar{g}(h(X,Y), \xi) &= B(X,Y) \\ &= B(X,Y)\bar{g}(N, \xi) \\ &= \bar{g}(B(X,Y)N, \xi). \end{aligned}$$

And

$$\begin{aligned} \bar{g}(\nabla^t_X N, \xi) &= \tau(X) \\ &= \tau(X)\bar{g}(N, \xi) \\ &= \bar{g}(\tau(X)N, \xi). \end{aligned}$$

Therefore, we have that $h(X, Y) = B(X, Y)N$ and $\nabla^t_X N = \tau(X)N$ hold. □

Now, equation 2.2 and 2.3 becomes

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N \tag{2.6}$$

and

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$, respectively. As B is the only component of h on $\mathcal{U}$ with respect to N, we call B the local second fundamental form of M, and the equations 2.6 , 2.7 the local Gauss and Weingarten formulae, respectively.[3]

Remark 2.2.1. The geometry of lightlike hypersurfaces depends on the choice of the screen distribution. This is why we need to study the relationship between the geometrical objects by two screen distribution that induced them.[3]

Theorem 2.2.1. *[3] Let (g, M) be a lightlike hypersurface with screen distributions $S(TM)$ and $S(TM)'$ and second fundamental forms h and h' with respect to the transversal bundles $tr(M)$ and $tr(M)'$, respectively. Then $B = B'$ on $\mathcal{U}$.*

Proof. By equation 2.2 $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\bar{\nabla}_X Y = \nabla_X Y + h'(X,Y)$. Moreover, by equation 2.4 $B(X,Y) = \bar{g}(h(X,Y),\xi)$ and $B'(X,Y) = \bar{g}(h'(X,Y),\xi)$. So

$$\begin{aligned}
B(X,Y) &= \bar{g}(h(X,Y),\xi) \\
&= \bar{g}(\bar{\nabla}_X Y - \nabla_X Y, \xi) \\
&= \bar{g}(\bar{\nabla}_X Y, \xi) - \bar{g}(\nabla_X Y, \xi) \\
&= \bar{g}(\bar{\nabla}_X Y, \xi).
\end{aligned}$$

Similarly, $B'(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$. Therefore, $B(X,Y) = B'(X,Y)$. □

Remark 2.2.2. The statement $B = B'$ implies that the local fundamental form of M does not depend on the choice of the screen distribution.[3]

We now introduce the following Lemma to prove that the second fundamental form of a lightlike hypersurface is degenerate.

Lemma 2.2.2. *[6] Since $\bar{\nabla}$ is a metric connection on $\bar{M}$, we have that*

$$B(X,\xi) = 0, \qquad \textit{for all } X \in \Gamma(TM|_{\mathcal{U}}). \tag{2.8}$$

Proof. By equation 2.6

$$\bar{\nabla}_X \xi = \nabla_X \xi + B(X,\xi)N.$$

It follows that

$$\begin{aligned}
\bar{g}(\bar{\nabla}_X \xi, \xi) &= \bar{g}(\nabla_X \xi, \xi) + \bar{g}(B(X,\xi)N, \xi) \\
&= \bar{g}(\nabla_X \xi, \xi) + B(X,\xi)\bar{g}(N,\xi) \\
&= B(X,\xi).
\end{aligned}$$

Thus,

$$\bar{g}(\bar{\nabla}_X \xi, \xi) = B(X,\xi). \tag{2.9}$$

Recall that a linear connection $\bar{\nabla}$ on $(\bar{M}, \bar{g})$ is called a metric (Levi-Civita) connection and g is parallel with respect to $\bar{\nabla}$, i.e

$$(\bar{\nabla}_X g)(Y,Z) = X(g(Y,Z)) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = 0$$

for any $X, Y, Z \in \Gamma(TM)$, then

$$X(g(Y,Z)) = g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}_X Z).$$

It follows that,

$$\begin{aligned}
X(\bar{g}(\xi,\xi)) &= \bar{g}(\bar{\nabla}_X \xi, \xi) + \bar{g}(\xi, \bar{\nabla}_X \xi) \\
0 &= \bar{g}(\bar{\nabla}_X \xi, \xi) + \bar{g}(\xi, \bar{\nabla}_X \xi) \\
0 &= 2\bar{g}(\bar{\nabla}_X \xi, \xi) \\
&\Rightarrow \bar{g}(\bar{\nabla}_X \xi, \xi) = 0.
\end{aligned}$$

Therefore,

$$B(X, \xi) = \bar{g}(\bar{\nabla}_X \xi, \xi) = 0.$$

□

Following the procedure from Differential Geometry by N. J. Hicks; We say that two vector fields V, W are **conjugate** if $B(V, W) = 0$. If a vector field V is self-conjugate, i.e. $B(V, V) = 0$. Then we say that V is an **asymptotic vector** field. The following Lemma, follows from this definitions.

Lemma 2.2.3. *Let M be a lightlike hypersurface, then any $\xi \in \Gamma(TM^{\perp}_{|\mathcal{U}})$ is conjugate to any vector field $X \in M$ and ξ is an asymptotic vector field.[3]*

Proof. By Lemma 2.2.2 $B(X, \xi) = 0$, for all $X \in \Gamma(TM|_{\mathcal{U}})$. So ξ is conjugate to any vector field $X \in M$. Moreover, setting $X = \xi$ we obtain $B(\xi, \xi) = 0$. Therefore, ξ is an asymptotic vector field. □

We now define second fundamental form and its corresponding shape operator for the screen distribution. It is defined by projection morphism below.

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition $TM = S(TM) \perp TM^{\perp}$. We obtain

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) \tag{2.10}$$

and

$$\nabla_X U = -A^*_U X + \nabla^{*t}_X U \tag{2.11}$$

for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, where$\nabla^*_X Y$ and $A^*_U X$ are elements of $\Gamma(S(TM))$, ∇ is a linear connection in $\Gamma(S(TM))$, $\nabla^{*t}$is a linear connection in $TM^{\perp}$. h^* is a $\Gamma(TM)^{\perp}$-valued $\mathcal{F}(\mathcal{M})$-bilinear from on $\Gamma(TM) \times \Gamma(S(TM))$ and A^*_U is a $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$. We call h^* the **screen second fundamental form** and equation A^*_U the **screen shape operator** of $S(TM)$.[6]

We now define the **local screen fundamental form** of $S(TM)$ by

$$C(X, PY) = \bar{g}(h^*(X, PY), N) \tag{2.12}$$

and define

$$\varepsilon(X) = \bar{g}(\nabla^*_X \xi, N), \quad \text{for all } X, Y \in \Gamma(TM).[3] \tag{2.13}$$

Lemma 2.2.4. *[3] $\varepsilon(X) = -\tau(X)$ for all $X, Y \in \Gamma(TM)$.*

Proof. By equation 2.13 and 2.7

$$\begin{aligned}
\varepsilon(X) &= \bar{g}(\nabla^{*t}_X \xi, N) \\
&= \bar{g}(\nabla_X \xi + A^*_\xi X, N) \\
&= \bar{g}(\nabla_X \xi, N) + \bar{g}(A^*_\xi X, N) \\
&= \bar{g}(\nabla_X \xi, N) \\
&= \bar{g}(\bar{\nabla}_X \xi, N) \\
&= -\bar{g}(\bar{\nabla}_X N, \xi) \\
&= -\bar{g}(-A_N X + \tau(X) N, \xi) \\
&= \bar{g}(A_N X, \xi) - \tau(X)\bar{g}(N, \xi) \\
&= -\tau(X),
\end{aligned}$$

for all $X, Y \in \Gamma(TM)$. □

by Lemma 2.2.1, equation 2.10 and 2.11 we obtain

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.14}$$

Moreover, by equation 2.15

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \text{ for all } X, Y \in \Gamma(TM). \tag{2.15}$$

Remark 2.2.3. The second fundamental form of the screen distribution is also degenerate. The induced connection ∇ is unique and independent of $S(TM)$ when $h = 0$. In general, the induced connection ∇ is not a metric connection. To express this, we define a local 1-form on $\mathcal{U}$ below.

Define a local 1-form η by

$$\eta(X) = \bar{g}(X, N), \qquad \text{for all } N \in \Gamma(TM)|_{\mathcal{U}}.$$

Theorem 2.2.2. *[3] Let $(g, M, S(TM))$ be a lightlike hypersurface, then:*

1. *A linear connection ∇^* is a metric connection on $S(TM)$ and*

2. *The induced connection ∇ on M satisfies*

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.16}$$

for all $X, Y, Z \in \Gamma(TM|_{\mathcal{U}})$.

Proof. 1. Follows from equation 2.4 and 2.6.

2. By equation 2.4 and Lemma 2.2.1, a metric connection $\bar{\nabla}$ on M and the definition of η, we get

$$
\begin{aligned}
0 &= (\bar{\nabla}_X \bar{g})(Y, Z) \\
&= X(\bar{g}(Y, Z)) - \bar{g}(\bar{\nabla}_X Y, Z) - \bar{g}(Y, \bar{\nabla}_X Z) \\
&= X(g(Y, Z)) - g(\nabla_X Y + h(X, Y), Z) - g(Y, \nabla_X Z + h(X, Z)) \\
&= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(h(X, Y), Z) - g(Y, \nabla_X Z) - g(Y, h(X, Z)) \\
&= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - B(X, Y)\bar{g}(Z, N) - B(X, Z)\bar{g}(Y, N) \\
&= (\nabla_X g)(Y, Z) - B(X, Y)\eta(Z) - B(X, Z)\eta(Y).
\end{aligned}
$$

Thus, the connection ∇ on X is not a metric connection and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y). \tag{2.17}$$

□

Lightlike hypersurfaces, unlike the non-degenerate space do not have a second fundamental form and the shape operator that are related by means of a metric tensor field. By equation 2.4 and Lemma 2.2.1 we can introduce that following two lemmas.

Lemma 2.2.5. *[3]* $B(X, Y) = g(A^*_\xi X, Y)$, $\quad g(A^*_\xi X, N) = 0$. *for all* $X, Y \in \Gamma(TM)$.

Proof. $B(X, Y) = g(A^*_\xi X, Y)$.

By equation 2.4 and Lemma 2.2.1 And

$$
\begin{aligned}
\bar{g}(\bar{\nabla}_X Y, \xi) &= \bar{g}(\nabla_X Y, \xi) + \bar{g}(B(X, Y)N, \xi) \\
&= g(\nabla_X Y, \xi) + B(X, Y)\bar{g}(N, \xi) \\
&= B(X, Y).
\end{aligned}
$$

Therefore,

$$B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi).$$

Moreover, by equation 2.15

$$
\begin{aligned}
B(X, Y) &= \bar{g}(\bar{\nabla}_X Y, \xi) \\
&= -\bar{g}(\bar{\nabla}_X \xi, Y) \\
&= -g(\nabla_X \xi, Y) \\
&= -g(-A^*_\xi X - \tau(X)\xi, Y) \\
&= g(A^*_\xi X, Y) + \bar{g}(\tau(X)\xi, Y) \\
&= g(A^*_\xi X, Y).
\end{aligned}
$$

Note that $\bar{\nabla}_X\xi = \nabla_X\xi$, since $B(X,\xi) = 0$ by Lemma 2.2.2 We now prove that

$$g(A^*_\xi X, N) = 0.$$

By the definition $\nabla_X\xi = -A^*_\xi X - \tau(x)\xi$, for all $X, Y \in \Gamma(TM)$. It follows

$$\begin{aligned}
\bar{g}(\nabla_X\xi, N) &= -\bar{g}(A^*_\xi X, N) - \tau(X)\bar{g}(\xi, N)\\
-\bar{g}(\bar{\nabla}_X\xi, N) &= -\bar{g}(A^*_\xi X, N) - \tau(X)\\
-\bar{g}(\bar{\nabla}_X N, \xi) &= -\bar{g}(A^*_\xi X, N) - \tau(X)\\
-\bar{g}(-A^*_N X + \tau(X)N, \xi) &= -\bar{g}(A^*_\xi X, N) - \tau(X)\\
\bar{g}(A^*_N X, \xi) - \tau(X)\bar{g}(N, \xi) &= -\bar{g}(A^*_\xi X, N) - \tau(X)\\
-\tau(X) &= -\bar{g}(A^*_\xi X, N) - \tau(X)
\end{aligned}$$

Therefore, $g(A^*_\xi X, N) = 0$. □

Lemma 2.2.6. *[3]* $C(X, PY) = g(A_N X, PY), \quad \bar{g}(A^*_N Y, N) = 0$ *for all* $X, Y \in \Gamma(TM)$.

Proof.

$$C(X, PY) = g(A_N X, PY).$$

Note that just like in the case of B,

$$h^*(X, PY) = C(X, PY)\xi.$$

Moreover, $\nabla^*_X Y \in \Gamma(S(TM)$ and $N \in tr(TM))$ gives

$$\begin{aligned}
C(X, PY) &= \bar{g}(h^*(X, PY), N)\\
&= \bar{g}(\nabla_X PY - \nabla^*_X PY, N)\\
&= \bar{g}(\nabla_X PY, N) - \bar{g}(\nabla^*_X PY, N)\\
&= \bar{g}(\nabla_X PY, N)\\
&= -\bar{g}(\bar{\nabla}_X PY, N)\\
&= -\bar{g}(\bar{\nabla}_X N, PY)\\
&= -\bar{g}(-A_N X + \tau(X)N, PY)\\
&= \bar{g}(A_N X, PY) - \tau(X)\bar{g}(N, PY)\\
&= g(A_N X, PY).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\bar{g}(A_N Y, N) &= \bar{g}(\tau(Y)N - \bar{\nabla}_Y N, N)\\
&= \bar{g}(\tau(Y)N, N) - \bar{g}(\bar{\nabla}_Y N, N)\\
&= 0.
\end{aligned}$$

□

2.3 Gauss-Codazzi equations

This section is dedicated to the derivation of Gauss-Codazzi equations and other related properties. Let ∇ be the connection on M and $\bar{\nabla}$ be the connection on $\bar{M}$ with curvature tensors R and $\bar{R}$, respectively. And suppose that $(M, g, S(TM))$ is a lightlike hypersurface of of a semi-Riemannian manifold $(\bar{M}, \bar{g})$.

Proposition 2.3.1. *[3] The curvature tensor of $\bar{R}$ is given by*

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. We begin by the applying the definition of curvature tensor

$$\begin{aligned}
\bar{R}(X,Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\
&= \bar{\nabla}_X[\nabla_Y Z + h(Y,Z)] - \bar{\nabla}_Y[\nabla_X Z + h(X,Z)] - [\nabla_{[X,Y]}Z + h([X,Y],Z)] \\
&= \bar{\nabla}_X\nabla_Y Z + \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y\nabla_X Z - \bar{\nabla}_Y h(X,Z) - \nabla_{[X,Y]}Z - h([X,Y],Z) \\
&= \nabla_Y[\nabla_X Z + h(X,Z)] - \nabla_X[\nabla_Y Z + h(Y,Z)] - \nabla_{[X,Y]}Z - h([X,Y],Z) \\
&+ \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y h(X,Z).
\end{aligned}$$

Rearranging the equation we get

$$\begin{aligned}
\nabla_X\nabla_Y Z &+ \nabla_Y h(X,Z) - \nabla_X\nabla_Y Z - \nabla_X h(Y,Z) - \nabla_{[X,Y]}Z - h([X,Y],Z) \\
&+ \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y h(X,Z) \\
&= \nabla_X\nabla_Y Z - \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z + \nabla_Y h(X,Z) - \nabla_X h(Y,Z) - h([X,Y],Z) \\
&+ \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y h(X,Z) \\
&= R(X,Y)Z + \nabla_Y h(X,Z) - \nabla_X h(Y,Z) + \bar{\nabla}_X h(Y,Z) - \bar{\nabla}_Y h(X,Z) - h([X,Y],Z).
\end{aligned}$$

Since $\bar{\nabla}_X V = -A_V X + \nabla^t_X V$ then $\bar{\nabla}_X h(Y,Z) = -A_{h(Y,Z)}X + \nabla^t_X h(Y,Z)$. It follows that

$$\begin{aligned}
R(X,Y)Z &+ \nabla_Y h(X,Z) - \nabla_X h(Y,Z) + [-A_{h(Y,Z)}X + \nabla^t_X h(Y,Z)] \\
&- [-A_{h(X,Z)}Y + \nabla^t_Y h(X,Z)] - h([X,Y],Z) \\
&= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X - \nabla^t_Y h(X,Z) + \nabla^t_X h(Y,Z) + \nabla_Y h(X,Z) \\
&- \nabla_X h(Y,Z) - h([X,Y],Z) \\
&= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X - \nabla^t_Y h(X,Z) + \nabla^t_X h(Y,Z) + \nabla_Y h(X,Z) \\
&- \nabla_X h(Y,Z) - h([\nabla_Y X - \nabla_X Y],Z) \\
&= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X - \nabla^t_Y h(X,Z) + \nabla^t_X h(Y,Z) + \nabla_Y h(X,Z) \\
&- \nabla_X h(Y,Z) - h(\nabla_Y X,Z) + h(\nabla_X Y,Z).
\end{aligned}$$

Set

$$(\nabla_X h)(Y,Z) = \nabla^t_X(h(Y,Z)) - h(\nabla_X y, Z) - h(Y, \nabla_X Z). \tag{2.18}$$

Then

$$\begin{aligned}
\bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + [\nabla^t_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)] \\
& - [\nabla^t_Y h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)] \\
= {} & R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z).
\end{aligned}$$

□

Proposition 2.3.2. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(\bar{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW) - \bar{g}(h(X,Z), h^*(Y,PW)) - \bar{g}(h(Y,Z), h^*(X,PW)).$$

Proof. By Proposition 2.3.1

$$\begin{aligned}
\bar{g}(\bar{R}(X,Y)Z, PW) = {} & \bar{g}(R(X,Y)Z, PW) + \bar{g}(A_{h(X,Z)}Y, PW) \\
& - \bar{g}(A_{h(Y,Z)}X, PW) + \bar{g}((\nabla_X h)(Y,Z), PW) - \bar{g}((\nabla_Y h)(X,Z), PW).
\end{aligned}$$

Since $A_{h(X,Z)}Y = (\nabla^t_Y h(X,Z) - \bar{\nabla}_Y h(X,Z))$ we have

$$\begin{aligned}
& \bar{g}(R(X,Y)Z, PW) + \bar{g}((\nabla^t_Y h(X,Z) - \bar{\nabla}_Y h(X,Z)), PW) - \bar{g}((\nabla^t_X h(Y,Z) - \bar{\nabla}_X h(Y,Z)), PW) \\
& \quad + \bar{g}((\nabla_X h)(Y,Z), PW) - \bar{g}((\nabla_Y h)(X,Z), PW) \\
& \quad = \bar{g}(R(X,Y)Z, PW) + \bar{g}(\nabla^t_Y h(X,Z), PW) - \bar{g}(\bar{\nabla}_Y h(X,Z), PW) \\
& \quad - \bar{g}(\nabla^t_X h(Y,Z), PW) + \bar{g}(\bar{\nabla}_X h(Y,Z), PW) \\
& \quad + \bar{g}((\nabla_X h)(Y,Z), PW) - \bar{g}((\nabla_Y h)(X,Z), PW) \\
& \quad = \bar{g}(R(X,Y)Z, PW) + \bar{g}(\nabla^t_Y h(X,Z), PW) - \bar{g}(\bar{\nabla}_Y h(X,Z), PW) \\
& \quad - \bar{g}(\nabla^t_X h(Y,Z), PW) + \bar{g}(\bar{\nabla}_X h(Y,Z), PW) \\
& \quad + \bar{g}([\nabla^t_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)], PW) \\
& \quad - \bar{g}([\nabla^t_Y h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)], PW).
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \bar{g}(R(X,Y)Z, PW) + \bar{g}(\nabla^t_Y h(X,Z), PW) - \bar{g}(\bar{\nabla}_Y h(X,Z), PW) - \bar{g}(\nabla^t_X h(Y,Z), PW) \\
& \quad + \bar{g}(\bar{\nabla}_X h(Y,Z), PW) + \bar{g}(\nabla^t_X h(Y,Z), PW) - \bar{g}(h(\nabla_X Y, Z), PW) - \bar{g}(h(Y, \nabla_X Z), PW) \\
& \quad - \bar{g}(\nabla^t_Y h(X,Z), PW) + \bar{g}(h(\nabla_Y X, Z), PW) + \bar{g}(h(X, \nabla_Y Z), PW) \\
& \quad = \bar{g}(R(X,Y)Z, PW) - \bar{g}(h(Y, \nabla_X Z), PW) + \bar{g}(h(X, \nabla_Y Z), PW) \\
& \quad = \bar{g}(R(X,Y)Z, PW) - \bar{g}(h(Y,Z), \nabla_X PW) + \bar{g}(h(X,Z), \nabla_Y PW).
\end{aligned}$$

By definition $\nabla_X PW = \nabla^*_X PW + h^*(X, PW)$, so

$$
\begin{aligned}
&\bar{g}(R(X,Y)Z,PW) - \bar{g}(h(Y,Z), \nabla^*_X PW + h^*(X,PW)) + \bar{g}(h(X,Z), \nabla^*_Y PW + h^*(Y,PW)) \\
&\qquad = \bar{g}(R(X,Y)Z,PW) - \bar{g}(h(Y,Z),\nabla^*_X PW) - \bar{g}(h(Y,Z),h^*(X,PW)) \\
&\qquad + \bar{g}(h(X,Z),\nabla^*_Y PW) + \bar{g}(h(X,Z),h^*(Y,PW)) \\
&\qquad = \bar{g}(R(X,Y)Z,PW) + \bar{g}(h(X,Z),h^*(Y,PW)) - \bar{g}(h(Y,Z),h^*(X,PW)) \\
&\qquad - \bar{g}(h(Y,Z),\nabla^*_X PW) + \bar{g}(h(X,Z),\nabla^*_Y PW).
\end{aligned}
$$

Since $h(X,Y) = B(X,Y)N$ we obtain

$$
\begin{aligned}
&\bar{g}(R(X,Y)Z,PW) + \bar{g}(h(X,Z),h^*(Y,PW)) - \bar{g}(h(Y,Z),h^*(X,PW)) \\
&\qquad - \bar{g}(B(Y,Z)N,\nabla^*_X PW) + \bar{g}(B(X,Z)N,\nabla^*_Y PW) \\
&\qquad = \bar{g}(R(X,Y)Z,PW) + \bar{g}(h(X,Z),h^*(Y,PW)) - \bar{g}(h(Y,Z),h^*(X,PW)) \\
&\qquad - B(Y,Z)\bar{g}(N,\nabla^*_X PW) + B(X,Z)\bar{g}(N,\nabla^*_Y PW) \\
&\qquad = g(R(X,Y)Z,PW) + \bar{g}(h(X,Z),h^*(Y,PW)) - \bar{g}(h(Y,Z),h^*(X,PW)),
\end{aligned}
$$

where $\nabla^*_X W$, $\nabla^*_Y W \in \Gamma(S(TM))$ and $N \in tr(TM))$. □

Proposition 2.3.3. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(\bar{R}(X,Y)Z,U) = \bar{g}((\nabla_X h)(Y,Z) - \bar{g}(\nabla_Y h)(X,Z),U)$$

for any $X, Y, Z, W \in \Gamma(TM)$ and $U \in \Gamma(TM^\perp)$.

Proof. By Proposition 2.3.1, Proposition 2.3.2 and since $(X, Y, Z \in \Gamma(TM)$ and $U \in \Gamma(Rad\ (TM))$

$$
\begin{aligned}
&\bar{g}(\bar{R}(X,Y)Z,U) \\
&\qquad = \bar{g}(R(X,Y)Z,U) + \bar{g}(A_{h(X,Z)}Y,U) - \bar{g}(A_{h(Y,Z)}X,U) \\
&\qquad + \bar{g}((\nabla_X h)(Y,Z),U) - \bar{g}((\nabla_Y h)(X,Z),U) \\
&\qquad = R(X,Y)\bar{g}(Z,U) + A_{h(X,Z)}\bar{g}(Y,U) - A_{h(Y,Z)}\bar{g}(X,U) \\
&\qquad + \bar{g}((\nabla_X h)(Y,Z),U) - \bar{g}((\nabla_Y h)(X,Z),U) \\
&\qquad = \bar{g}((\nabla_X h)(Y,Z),U) - \bar{g}((\nabla_Y h)(X,Z),U) \\
&\qquad = \bar{g}((\nabla_X h)(Y,Z) - \bar{g}(\nabla_Y h)(X,Z),U).
\end{aligned}
$$

□

Proposition 2.3.4. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(\bar{R}(X,Y)Z,V) = \bar{g}(R(X,Y)Z,V)$$

and

$$R(X,Y)U = \bar{R}(X,Y)U$$

for all $X, Y, Z, W \in \Gamma(TM)$, $U \in \Gamma(TM^\perp)$ and $V \in \Gamma(tr(TM))$.

Proof. By Proposition 2.3.1, Proposition 2.3.2

$$
\begin{aligned}
&\bar{g}(\bar{R}(X,Y)Z,V)\\
&\quad = \bar{g}(R(X,Y)Z,V) + \bar{g}(A_{h(X,Z)}Y,V) - \bar{g}(A_{h(Y,Z)}X,V)\\
&\quad + \bar{g}((\nabla_X h)(Y,Z),V) - \bar{g}((\nabla_Y h)(X,Z),V)\\
&\quad = \bar{g}(R(X,Y)Z,V) + \bar{g}(\nabla^t_Y h(X,Z),V) - \bar{g}(\bar{\nabla}_Y h(X,Z),V)\\
&\quad - \bar{g}(\nabla^t_X h(Y,Z),V) + \bar{g}(\bar{\nabla}_X h(Y,Z),V)\\
&\quad + \bar{g}([\nabla^t_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)],V)\\
&\quad - \bar{g}([\nabla^t_Y h(X,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)],V)\\
&\quad = \bar{g}(R(X,Y)Z,V) + B(X,Z)\bar{g}(\nabla^t_Y N,V) - B(X,Z)\bar{g}(\bar{\nabla}_Y N,V) - B(Y,Z)\bar{g}(\nabla^t_X N,V)\\
&\quad + B(Y,Z)\bar{g}(\bar{\nabla}_X N,V) + \bar{g}[(\nabla^t_X B(Y,Z) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z)]N,V)\\
&\quad - \bar{g}([\nabla^t_Y B(X,Z) - B(\nabla_Y X,Z) - B(X,\nabla_Y Z)]N,V)\\
&\quad = \bar{g}(R(X,Y)Z,V),
\end{aligned}
$$

where $N \in tr(TM)$ and $V \in \Gamma(tr(TM))$. If we replace Z by U in the above equation we obtain

$$R(X,Y)U = \bar{R}(X,Y)U,$$

for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$. □

Note that the relations we proved in Proposition 2.3.2, Proposition 2.3.3 and Proposition 2.3.4 are called the **Gauss-Codazzi equations** of the lightlike hypersurface $(M, g, S(TM))$. However, replacing Z by U in the mentioned propositions yields $\bar{R}(X,Y)U = R(X,Y)U$ for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$. Thus, the restriction of the lightlike hypersurface $(M, g, S(TM))$ of curvature ∇ on $TM^{\perp}$ does not depend on the screen distribution $S(TM)$. If we consider $\{\xi, N\}$ on a neighbourhood $\mathcal{U} \subset M$ then we have the following relations stated as propositions below.

Proposition 2.3.5. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(\bar{R}(X,Y)Z,PW) = g(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW).$$

Proof. By Proposition 2.3.1 to Proposition 2.3.4

$$
\begin{aligned}
&\bar{g}(\bar{R}(X,Y)Z,PW)\\
&= g(R(X,Y)Z,PW) + \bar{g}(h(X,Z),h^*(Y,PW)) - \bar{g}(h(Y,Z),h^*(X,PW))\\
&= g(R(X,Y)Z,PW) + \bar{g}(B(X,Z)N,C(Y,PW)\xi) - \bar{g}(B(Y,Z)N,C(X,PW)\xi)\\
&= g(R(X,Y)Z,PW) + B(X,Z)C(Y,PW)\bar{g}(N,\xi) - B(Y,Z)C(X,PW)\bar{g}(N,\xi)\\
&= g(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW).
\end{aligned}
$$

□

Proposition 2.3.6. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y).$$

Proof. By Proposition 2.3.1 to Proposition 2.3.4

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z,\xi) &= \bar{g}(R(X,Y)Z,\xi) + \bar{g}(A_{h(X,Z)}Y,\xi) - \bar{g}(A_{h(Y,Z)}X,\xi) + \bar{g}((\nabla_X h)(Y,Z),\xi)\\ &\quad - \bar{g}((\nabla_Y h)(X,Z),\xi)\\ &= R(X,Y)\bar{g}(Z,\xi) + \bar{g}(A_{h(X,Z)}Y,\xi) - \bar{g}(A_{h(Y,Z)}X,\xi) + \bar{g}([\nabla^t_X h(Y,Z)\\ &\quad - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)],\xi) - \bar{g}([\nabla^t_Y h(X,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)],\xi).\end{aligned}$$

Set $(\nabla_X B)(Y,Z) = X(B(Y,Z)) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z)$ then

$$\begin{aligned}&\bar{g}(\bar{R}(X,Y)Z,\xi)\\ &\quad= \bar{g}(A_{h(X,Z)}Y,\xi) - \bar{g}(A_{h(Y,Z)}X,\xi) + \bar{g}([\nabla^t_X B(Y,Z) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z)]N,\xi)\\ &\quad- \bar{g}([\nabla^t_Y B(X,Z) - B(\nabla_Y X,Z) - B(X,\nabla_Y Z)]N,\xi)\\ &\quad= \bar{g}(A_{h(X,Z)}Y,\xi) - \bar{g}(A_{h(Y,Z)}X,\xi) + [\nabla^t_X B(Y,Z) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z)]\bar{g}(N,\xi)\\ &\quad- [\nabla^t_Y B(X,Z) - B(\nabla_Y X,Z) - B(X,\nabla_Y Z)]\bar{g}(N,\xi)\\ &\quad= \bar{g}(A_{h(X,Z)}Y,\xi) - \bar{g}(A_{h(Y,Z)}X,\xi) + (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\quad= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \bar{g}(\nabla^t_Y h(X,Z),\xi) - \bar{g}(\bar{\nabla}_Y h(X,Z),\xi)\\ &\quad- \bar{g}(\nabla^t_X h(Y,Z),\xi) - \bar{g}(\bar{\nabla}_X h(Y,Z),\xi).\end{aligned}$$

Since $\bar{\nabla}_X N, \xi = -A_N Y + \tau(Y)N$, it follows that

$$\begin{aligned}&\bar{g}(\bar{R}(X,Y)Z,\xi)\\ &\quad= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(X,Z)\bar{g}(\nabla^t_Y N,\xi) - B(X,Z)\bar{g}(\bar{\nabla}_Y N,\xi)\\ &\quad- (Y,Z)\bar{g}(\nabla^t_X N,\xi) - B(Y,Z)\bar{g}(\bar{\nabla}_X N,\xi)\\ &\quad= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(X,Z)\bar{g}(\nabla^t_Y N,\xi) - B(X,Z)\bar{g}(-A_N Y + \tau(Y)N,\xi)\\ &\quad- (Y,Z)\bar{g}(\nabla^t_X N,\xi) - B(Y,Z)\bar{g}(-A_N X + \tau(X)N,\xi)\\ &\quad= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(X,Z)\bar{g}\tau(X) - B(X,Z)\bar{g}(-A_N Y,\xi) - B(X,Z)\bar{g}(\tau(Y)N,\xi\\ &\quad- B(Y,Z)\tau(Y) - B(Y,Z)\bar{g}(-A_N X,\xi) - B(Y,Z)\bar{g}(\tau(X)N,\xi).\end{aligned}$$

By the definition of ξ

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) - B(Y,Z)\tau(X) + B(X,Z)\tau(Y).$$

□

Proposition 2.3.7. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(\bar{R}(X,Y)Z,N) = \bar{g}(R(X,Y)Z,N).$$

Proof. By Proposition 2.3.1 to Proposition 2.3.4 $\bar{g}(\bar{R}(X,Y)Z,N) = \bar{g}(R(X,Y)Z,N)$ for all $N \in tr(TM)$). □

Remark 2.3.1. Note that the above proposition gives that these lightlike hyperspaces have symmetric shape operators for any given screen distribution $S(TM)$ with respect to the second fundamental form B of M. So the relations below hold.

Proposition 2.3.8. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$g(R(X,Y)PZ,PW) = g(R^*(X,Y)PZ,PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW).$$

Proof. We have shown that $B(X,Y) = \bar{g}(\nabla_X \bar{Y}, \xi)$. Rearranging the equation we obtain

$$\begin{aligned}
&g((\nabla_X h^*)(Y,PZ),PW) - g((\nabla_Y h^*)(X,PZ),PW)\\
&\quad= g([\nabla^t_X h^*(Y,PZ) - h^*(\nabla_X Y,PZ) - h^*(Y,\nabla_X PZ)],PW)\\
&\quad\quad - g([\nabla^t_Y h^*(X,PZ) - h^*(\nabla_Y X,PZ) - h^*(X,\nabla_Y PZ)],PW)\\
&\quad\quad = g(\nabla^t_X h^*(Y,PZ),PW) - g(h^*(\nabla_X Y,PZ),PW) - g(h^*(Y,\nabla_X PZ),PW)\\
&\quad\quad - g(\nabla^t_Y h^*(X,PZ),PW) + g(h^*(\nabla_Y X,PZ),PW) + g(h^*(X,\nabla_Y PZ),PW).
\end{aligned}$$

By definition of C, $C(X,PY)\xi = h^*(X,PY)$, so

$$\begin{aligned}
&g(\nabla^t_X h^*(Y,PZ),PW) - g(h^*(\nabla_X Y,PZ),PW) - g(h^*(Y,\nabla_X PZ),PW)\\
&\quad - g(\nabla^t_Y h^*(X,PZ),PW) + g(h^*(\nabla_Y X,PZ),PW) + g(h^*(X,\nabla_Y PZ),PW)\\
&\quad = C(Y,PZ)g(\nabla^t_X \xi,PW) - C(Y,PZ)g(\nabla_X \xi,PZ),PW) - C(Y,PZ)g(\nabla_X \xi,PW)\\
&\quad - C(X,PZ)g(\nabla^t_Y \xi,PW) + C(X,PZ)g(\nabla_Y \xi,PW) + C(X,PZ)g(\nabla_Y \xi,PW)\\
&\quad = -2C(Y,PZ)B(X,PW) + 2C(X,PZ)B(Y,PW).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&g(A_{h^*(X,PZ)}Y,PW) - g(A_{h^*(Y,PZ)}X,PW)\\
&\quad = g(\nabla^t_Y h^*(X,PZ),PW) - g(\nabla_Y h^*(X,PZ),PW) - g(\nabla^t_X h^*(Y,PZ),PW)\\
&\quad + g(\nabla_X h^*(Y,PZ),PW)\\
&\quad = C(X,PZ)g(\nabla^t_Y \xi,PW) - C(X,PZ)g(\nabla_Y \xi,PW) - C(Y,PZ)g(\nabla^t_X \xi,PW)\\
&\quad + C(Y,PZ)g(\nabla_X \xi,PW)\\
&\quad = C(Y,PZ)B(X,PZ) - C(X,PZ)B(Y,PW)).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
g(&R(X,Y)PZ, PW)\\
&= g(R(X,Y)PZ, PW) + g(A_{h(X,PZ)}Y, PW) - g(A_{h(Y,PZ)}X, PW)\\
&+ g((\nabla_X h)(Y, PZ), PW) - g((\nabla_Y h)(X, PZ), PW)\\
&= g(R^*(X,Y)PZ, PW) + g(A_{h^*(X,PZ)}Y, PW) - g(A_{h^*(Y,PZ)}X, PW)\\
&+ g((\nabla_X h^*)(Y, PZ), PW) - g((\nabla_Y h^*)(X, PZ), PW)\\
&= g(R^*(X,Y)PZ, PW) + C(Y, PZ)B(X, PZ) - C(X, PZ)B(Y, PW)\\
&- 2C(Y, PZ)B(X, PW) + 2C(X, PZ)B(Y, PW)\\
&= g(R^*(X,Y)PZ, PW) + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW).
\end{aligned}
$$

□

Proposition 2.3.9. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(R(X,Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ).$$

Proof.

$$
\begin{aligned}
\bar{g}(A_{h(X,PZ)}Y, N) &= C(X, PZ)\bar{g}(\nabla^t_Y \xi, N) - C(X, PZ)\bar{g}(\bar{\nabla}_Y \xi, N)\\
&= C(X, PZ)\bar{g}(\nabla^t_Y N, \xi) - C(X, PZ)\bar{\nabla}_Y \bar{g}(\xi, N)\\
&= C(X, PZ)\tau(Y).
\end{aligned}
$$

Similarly,

$$\bar{g}(A_{h(Y,PZ)}X, N) = C(Y, PZ)\tau(X).$$

Moreover,

$$
\begin{aligned}
\bar{g}((\nabla_X h)(Y, PZ) &= [\nabla^t_X C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla_X PZ)]\bar{g}(\xi, N)\\
&= (\nabla_X C)(Y, PZ).
\end{aligned}
$$

Since $\bar{g}((\nabla_Y h)(X, PZ) = (\nabla_Y C)(X, PZ)$, thus

$$
\begin{aligned}
\bar{g}(R(X,Y)PZ, N) &= g(R(X,Y)PZ, N) + g(A_{h(X,PZ)}Y, N) - g(A_{h(Y,PZ)}X, N)\\
&+ g((\nabla_X h)(Y, PZ), N) - g((\nabla_Y h)(X, PZ), N)\\
&= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ).
\end{aligned}
$$

□

Proposition 2.3.10. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$ then*

$$\bar{g}(R(X,Y)\xi, N) = C(Y, A^*\xi X) - C(X, A^*\xi Y) - 2d\tau(X, Y).$$

Proof. By Proposition 2.3.9

$$
\begin{aligned}
\bar{g}(R(X,Y)\xi,N) &= -\bar{g}(h(X,\nabla_X\xi),N)+\bar{g}(h(Y,\nabla_Y\xi),N)\\
&= \bar{g}(h(Y,A^*_\xi X,N)+\bar{g}(h(Y,\tau(X)\xi),N)-\bar{g}(h(X,A^*_\xi Y,N)-\bar{g}(h(X,\tau(Y)\xi),N)\\
&= C(Y,A^*\xi X)+\bar{g}(h(Y,\tau(X)\xi),N)-C(X,A^*\xi Y)-\bar{g}(h(X,\tau(Y)\xi),N)\\
&= C(Y,A^*\xi X)-C(X,A^*\xi Y)-2d\tau(X,Y).
\end{aligned}
$$

□

We now have sufficient information to define total curvature for lightlike hypersurfaces of the form $(M,g,S(TM))$. We begin by defining a null plane H of $T_x\bar{M}$.

Definition 2.3.1. [3] Let U be the null vector of $T_x\bar{M}$ with $x\in\bar{M}$ such that for any $W\in H$ there exists $W_o\in H$ with $\bar{g}_x(W_o,W_o)\neq 0$. Then we say a plane H of $T_x\bar{M}$ is a **null plane** directed by U if $U\in H$.

Definition 2.3.2. [3] The **null sectional curvature** of H with respect to U and ∇ is given by

$$\overline{K}_U(H)=\frac{\bar{g}_x(\bar{R}(W,U)U,W)}{\bar{g}_x(W,W)},$$

where W is an arbitrary nondegenerate vector in H.

So, $\overline{K}_U(H)$ depends on a quadratic fashion on U but it is independent of W.

2.4 Umbilical hypersurfaces

Recall that for a $(m+2)$-dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ with a lightlike hypersurface $(M,g,S(TM))$, we say that $u\in M$ is umbilical if

$$B_u(X_u,Y_u)=kg_u(X_u,Y_u),\qquad \text{for all } X_u,Y_u\in T_uM, \tag{2.19}$$

where B is the local second fundamental form of M about u and $k\in\mathbb{R}$. Note that B is independent of $\mathcal{U}$ which is the neighbourhood around u. Take $\mathcal{U}^*$ about u, then we see that $B^*=\alpha B$ on $\mathcal{U}\cap\mathcal{U}*$ where $\alpha\neq 0$ and α is a C^∞ function. So

$$B^*=\alpha B=\alpha B_u(X_u,Y_u)=\alpha k g_u(X_u,Y_u),\qquad \text{for all } X_u,Y_u\in T_uM.$$

Furthermore, M is said to be **totally umbilical** if it is umbilical for each $u\in M$.[3]

Proposition 2.4.1. *[3] M is totally umbilical if and only if locally, on each $\mathcal{U}$ there exists a C^∞ function ρ such that*

$$B(X,Y)=\rho g(X,Y)\qquad \textit{for all } X,Y\in\Gamma(TM|_{\mathcal{U}}).$$

Proof. $\Rightarrow$ Suppose that M is totally umbilical, then by definition, each $u \in M$ is totally umbilical. If we take $\mathcal{U}^*$ about u, then we see that $B^* = \alpha B$ on $\mathcal{U} \cap \mathcal{U}*$ where $\alpha \neq 0$ and α is a C^∞ function. So

$$B^* = \alpha B = \alpha B_u(X_u, Y_u) = \alpha k g_u(X_u, Y_u), \qquad \text{for all } X_u, Y_u \in T_uM.$$

If we set $\rho = \alpha k$ we get that

$$B(X,Y) = \rho g(X,Y) \qquad \text{for all } X, Y \in \Gamma(TM|_{\mathcal{U}}),$$

and ρ is a C^∞ function on $\mathcal{U}$. Since ρ was arbitrary, this statement holds for all $\rho \in \mathcal{U}$.

$\Leftarrow$ Suppose that locally, on each $\mathcal{U}$ there exists a C^∞ function ρ such that

$$B(X,Y) = \rho g(X,Y) \quad \text{for all } X, Y \in \Gamma(TM|_{\mathcal{U}}).$$

This implies that all the points ρ of M are umbilical. Then by definition M is totally umbilical. $\square$

As we have discussed in the previous chapters B is independent of the screen distribution of M.

Theorem 2.4.1. *[3] Let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold of constant curvature $(\bar{M}(c), \bar{g})$; then ρ from $B(X,Y) = \rho g(X,Y)$ satisfies*

$$\xi\rho + \rho\tau(\xi) - \rho^2 = 0,$$

and the curvature tensor of M is given by

$$R(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y] + \rho[g(Y,Z)A_NX - g(X,Z)A_NY]$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$. Furthermore, if $m > 1$ then ρ satisfies the partial differential equation

$$PX(\rho) + \rho\tau(PX) = 0,$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$.

Proof. By Proposition 2.3.3 we have that

$$\bar{g}(\bar{R}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y).$$

Consider the relation

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y).$$

By the definition of totally umbilical lightlike surfaces and product rule we obtain

$$\begin{aligned}(\nabla_X B)(Y,Z) &= XB(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)\\ &= X[\rho g(Y,Z)] - \rho g(\nabla_X Y, Z) - \rho g(Y, \nabla_X Z)\\ &= (X\rho)g(Y,Z) + \rho X g(Y,Z) - \rho g(\nabla_X Y, Z) - \rho g(Y, \nabla_X Z)\\ &= (X\rho)g(Y,Z) + \rho[Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)]\\ &= (X\rho)g(Y,Z) + \rho[(\nabla_X g)(Y,Z)]\\ &= (X\rho)g(Y,Z) + \rho[B(X,Y)\eta(Z) + B(X,Z)\eta(Y)]\\ &= (X\rho)g(Y,Z) + \rho^2[g(X,Y)\eta(Z) + g(X,Z)\eta(Y)].\end{aligned}$$

Similarly for the second term we obtain

$$(\nabla_Y B)(X,Z) = (Y\rho)g(X,Z) + \rho^2[g(Y,X)\eta(Z) + g(Y,Z)\eta(X)].$$

The third term and the fourth term are given by

$$B(Y,Z)\tau(X) = \rho g(Y,Z)\tau(X)$$

and

$$B(X,Z)\tau(Y) = \rho g(X,Z)\tau(Y),$$

respectively. Since $\bar{g}(\bar{R}(X,Y)Z,\xi) = 0$, it follows that

$$(\nabla_X B)(Y,Z) + B(Y,Z)\tau(X) = (\nabla_Y B)(X,Z) + B(X,Z)\tau(Y).$$

By substitution

$$\begin{aligned}&(X\rho)g(Y,Z) + \rho^2[g(X,Y)\eta(Z) + g(X,Z)\eta(Y)] + \rho g(Y,Z)\tau(X)\\ &= (Y\rho)g(X,Z) + \rho^2[g(Y,X)\eta(Z) + g(Y,Z)\eta(X)] + \rho g(X,Z)\tau(Y).\end{aligned}$$

Eliminating the like terms

$$\begin{aligned}&(X\rho)g(Y,Z) + \rho^2[g(X,Z)\eta(Y)] + \rho g(Y,Z)\tau(X)\\ &= (Y\rho)g(X,Z) + \rho^2[g(Y,Z)\eta(X)] + \rho g(X,Z)\tau(Y).\end{aligned}$$

Rearranging the equation and by definition $\eta(X) = \bar{g}(N,X) = 1$, so

$$[X\rho + \rho\tau(X) - \rho^2\eta(X)]g(Y,Z) = [Y\rho + \rho\tau(Y) - \rho^2\eta(Y)]g(X,Z). \tag{2.20}$$

Setting $X = \xi$, we obtain

$$[\xi\rho + \rho\tau(\xi) - \rho^2]g(Y,Z) = [Y\rho + \rho\tau(Y) - \rho^2\eta(Y)]g(\xi,Z)$$

By the definition of ξ, the right hand side is equal to zero. So

$$[\xi\rho + \rho\tau(\xi) - \rho^2)]g(Y, Z) = 0.$$

Since $Y, Z \in \Gamma(TM|\mathcal{U})$ are in $S(TM)$ then $g(Y, Z) \neq 0$. Thus

$$\xi\rho + \rho\tau(\xi) - \rho^2 = 0,$$

as required.

By definition $\bar{R}(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y]$, then by Proposition 2.3.1

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z).$$

It follows that

$$\begin{aligned} A_{h(X,Z)}Y &= \nabla^t_Y h(X, Z) - \bar{\nabla}_Y h(X, Z) \\ &= \nabla^t_Y B(X, Z)N - \bar{\nabla}_Y B(X, Z)N \\ &= \nabla^t_Y N\rho g(X, Z) - \bar{\nabla}_Y N\rho g(X, Z) \\ &= \rho g(X, Z)\tau(Y)N - \rho g(X, Z)[-A_N Y + \tau(Y)N] \\ &= \rho g(X, Z)A_N Y. \end{aligned}$$

Similarly,

$$A_{h(Y,Z)}X = \rho g(Y, Z)A_N X.$$

So

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ c[g(Y, Z)X - g(X, Z)Y] &= R(X, Y)Z + \rho g(X, Z)A_N Y - \rho g(Y, Z)A_N X + (\nabla_X h)(Y, Z) \\ &\quad - (\nabla_Y h)(X, Z). \end{aligned}$$

Since $(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z)$ is not tangential, when we equate the components we obtain

$$c[g(Y, Z)X - g(X, Z)Y] = R(X, Y)Z + \rho g(X, Z)A_N Y - \rho g(Y, Z)A_N X.$$

Therefore,

$$R(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y] + \rho[g(Y, Z)A_N X - g(X, Z)A_N Y].$$

Since

$$[X\rho + \rho\tau(X) - \rho^2]g(Y, Z) = [Y\rho + \rho\tau(Y) - \rho^2\eta(Y)]g(X, Z).$$

Let $X = PX, Y = PY$ and $Z = PZ$, when we take into account that $S(TM)$ is nondegenerate we obtain

$$[PX(\rho) + \rho\tau(PX) - \rho^2\eta(PX)]g(PY, PZ) = [PY(\rho) + \rho\tau(PY) - \rho^2\eta(PY)]g(PX, PZ) \tag{2.21}$$

$$[PX(\rho) + \rho\tau(PX)]PY = [PY(\rho) + \rho\tau(PY)]PX. \tag{2.22}$$

Suppose that there exist $X_0 \in \Gamma(TM|_{\mathcal{U}})$ with $PX_0(\rho) + \rho\tau(PX_0) \neq 0$ at a point $u \in M$. This implies that equation 2.21 is such that all the fibres of $S(TM)_u$ are colinear with $(PX_0)_u$. This is a contradiction since *dim* $(S(TM)_u) = m > 1$. Therefore,

$$PX(\rho) + \rho\tau(PX) = 0,$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$ as required. □

Theorem 2.4.2. *[3] Let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold of constant curvature $(\bar{M}(c), \bar{g})$, then ρ from $B(X, Y) = \rho g(X, Y)$. Then*

(i) If $\lambda = 0$ on some $u \in M$, then $c = 0$. If $\lambda = 0$ for all $\mathcal{U} \subset M$, then the lightlike immersion of M in $\mathbb{R}^{m+2}_q$ is affinely equivalent to the graph immersion of a function $F : M \to \mathbb{R}$,

(ii) If $\lambda \neq 0$ for all $\mathcal{U} \subset M$ then M is totally umbilical immersed in $\bar{M}(c)$ if and only if λ is such that

$$\xi(\lambda) + \lambda\tau(\xi) + c = 0.$$

Proof. By Proposition 2.3.9

$$\bar{g}(\bar{R}(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ).$$

We start by solving the left hand side by the use of Gauss equations and we put into consideration that the curvature tensor is given by

$$\bar{R}(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y].$$

It follows that

$$\begin{aligned}
\bar{g}(\bar{R}(X, Y)PZ, N) &= \bar{g}(R(X, Y)PZ, N) \\
&= \bar{g}(c[g(Y, PZ)PX - g(X, PZ)Y], N) \\
&= \bar{g}(cg(Y, PZ)PX, N) - \bar{g}(cg(X, PZ)Y, N) \\
&= cg(Y, PZ)\bar{g}(X, N) - cg(X, PZ)\bar{g}(Y, N) \\
&= cg(Y, PZ)\eta(X) - cg(X, PZ)\eta(Y).
\end{aligned}$$

We now tackle the right hand side and we solve for each term separately. Since the screen distribution $S(TM)$ is totally umbilical we have smooth functions λ such that $C(X, PY) = \lambda g(X, PY)$. It follows that

$$\begin{aligned}
(\nabla_X C)(Y, Z) &= XC(Y, Z) - C(\nabla_X Y, Z) - C(Y, \nabla_X Z) \\
&= X[\lambda g(Y, Z)] - \lambda g(\nabla_X Y, Z) - \lambda g(Y, \nabla_X Z) \\
&= X(\lambda) g(Y, Z) + \lambda X g(Y, Z) - \lambda g(\nabla_X Y, Z) - \lambda g(Y, \nabla_X Z) \\
&= X(\lambda) g(Y, Z) + \lambda[(\nabla_X g(Y, Z)] \\
&= X(\lambda) g(Y, Z) + \lambda[B(X, Y)\eta(Z) + B(X, Z)\eta(Y)].
\end{aligned}$$

Similarly,

$$(\nabla_Y C)(X, Z) = Y(\lambda) g(X, Z) + \lambda[B(Y, X)\eta(Z) + B(Y, Z)\eta(X)].$$

Moreover, $\tau(Y)C(X, Z) = \lambda g(X, Z)\tau(Y)$ and $\tau(X)C(Y, Z) = \lambda g(Y, Z)\tau(X)$. Let $X = PX$ and $Y = PY$ then substituting back to the original equation, we obtain

$$\begin{aligned}
&cg(PY, PZ)\eta(X) - cg(PX, PZ)\eta(Y) \\
&\quad = X(\lambda) g(PY, PZ) + \lambda B(PX, PY)\eta(Z) + \lambda B(PX, PZ)\eta(Y) \\
&\quad - [Y(\lambda) g(PX, PZ) + \lambda B(PY, PX)\eta(Z) + \lambda B(PY, PZ)\eta(X)] \\
&\quad + \lambda g(PX, PZ)\tau(Y) - \lambda g(PY, PZ)\tau(X).
\end{aligned}$$

Grouping the like terms we get

$$\begin{aligned}
[X(\lambda) - \lambda\tau(X) - c\eta(X)] g(PY, PZ) &= [Y(\lambda) - \lambda\tau(Y) - c\eta(Y)] g(PX, PZ) \\
&\quad + \lambda[B(PY, PZ)\eta(X) - B(PX, PZ)\eta(Y)],
\end{aligned}$$

for all $X, Y, Z \in \Gamma(TM|_{\mathcal{U}})$. Setting $X = \xi$ and since $\eta(X) = \bar{g}(X, N) = 1$ we obtain

$$\begin{aligned}
[\xi(\lambda) - \lambda\tau(\xi) - c] g(PY, PZ) &= [Y(\lambda) - \lambda\tau(Y) - c\eta(Y)] g(P\xi, PZ) \\
&\quad + \lambda[B(PY, PZ) - B(P\xi, PZ)\eta(Y)].
\end{aligned}$$

By definition of ξ and since c is arbitrary, it follows that

$$[\xi(\lambda) - \lambda\tau(\xi) + c] g(PY, PZ) = \lambda B(PY, PZ). \tag{2.23}$$

In the case where $\lambda = 0$ and in view of equation 2.23

$$c = 0.$$

Since $S(TM)$ is nondegenerate, $\lambda = 0$ gives that $C(X, PY) = \lambda g(X, PY) = 0$, which implies that $S(TM)$ is parallel with respect to ∇. So assertion (i) has been proven and (ii) follows from equation 2.20. □

Proposition 2.4.2. *[3] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Suppose $S(TM)$ is integrable and any leaf of $S(TM)$ is totally umbilical, immersed in M as non-degenerate submanifold. Then M is totally umbilical.*

Proof. Let M^* be a leaf of $S(TM)$. Recall that

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N$$

and

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi.$$

Rewriting the second equation in terms of Y only, instead of PY and summing it up with the first equation we obtain

$$\bar{\nabla}_X Y + \nabla_X Y = \nabla_X Y + B(X,Y)N + \nabla^*_X Y + C(X,Y)\xi.$$

It follows that

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y - \nabla_X Y + \nabla^*_X Y + C(X,Y)\xi + B(X,Y)N \\ &= \nabla^*_X Y + C(X,Y)\xi + B(X,Y)N,\end{aligned}$$

for all $X, Y \in \Gamma(TM^*)$. Now let H^* be the mean curvature vector of M^*. Since $(tr(TM) \oplus TM^{\perp})$ is the normal bundle to M, we can find C^∞ functions α and ρ such that $H^* = \alpha\xi + \rho N$. Moreover, M^* is totally umbilical and it is immersed in $\bar{M}$. It follows from the identities in section 2.3 that

$$\begin{aligned}C(X,Y)\xi + B(X,Y)N &= g(\alpha X, Y)\xi + g(\rho X, Y)N \\ &= \alpha g(X,Y)\xi + \rho g(X,Y)N \\ &= g(X,Y)(\alpha\xi + \rho N).\end{aligned}$$

□

Thus, by definition M is totally umbilical immersed in $\bar{M}$. Note that totally umbilical lightlike surfaces are a large class of lightlike hypersurfaces. Let us take a look at two more results.

Proposition 2.4.3. *[3] Let $\bar{M}$ be a 3-dimensional Lorentz manifold. Then all the lightlike surfaces $M \in \bar{M}$ are either totally umbilical or geodesic.*

Proof. Let W be the span of $S(TM)$ on the coordinate neighbourhood $\mathcal{U} \subset M$. We know that the screen distribution $S(TM)$ is non-degenerate. Let

$$\rho = \frac{B(W,W)}{g(W,W)}$$

then

$$\rho g(W,W) = B(W,W).$$

Thus, $B = \rho g$ on $\mathcal{U}$. □

Proposition 2.4.4. *[3] The lightlike cone Λ^{m+1}_{q-1} of $\mathbb{R}^{m+2}_q$ is a totally umbilical lightlike hypersurface.*

Proof. Consider a linear connection ∇ defined on $\mathbb{R}^m_q$ by

$$\nabla_X Y = \sum_{i=1}^{m} X(Y^i)\frac{\partial}{\partial x^i}$$

we have that

$$\nabla_X \xi = \sum_{i=1}^{m} X(\xi^i)\frac{\partial}{\partial x^i} = X\sum_{i=1}^{m}\nabla_{x^i}\xi^i = X$$

We know that the curvature tensor R is of the form

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

for all $X, Y, Z \in \Gamma(TM)$. Moreover,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N$$

and

$$B(X,\xi) = 0$$

gives that

$$\bar{\nabla}_X \xi = \nabla_X \xi + B(X,\xi)N = \nabla_X \xi.$$

So

$$\bar{\nabla}_X \xi = \nabla_X \xi = X.$$

Furthermore, identity

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi$$

yields the relation

$$A^*_\xi X + \tau(X)\xi + X = 0.$$

Since A^*_ξ is $\Gamma(S(TM))$-valued, if we set $X = PX$ we obtain

$$A^*_\xi PX + \tau(PX)\xi + PX = 0 \Rightarrow A^*_\xi PX + 0 + PX = 0 \Rightarrow A^*_\xi PX = -PX.$$

Then by definition this lightlike cone is totally umbilical with $\rho = -1$. □

2.5 Screen conformal hypersurfaces

Unlike in the case of Riemannian manifolds, lightlike hypersurfaces have interrelations between the second fundamental form and the screen distribution with respect to their shape operators. Here we want to find out if there is a class of lightlike hypersurfaces which have the same geometry as the screen distribution of choice.

Definition 2.5.1. [6] **(screen locally conformal)** Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold, we say that $(M, g, S(TM))$ is screen locally conformal if for a non-vanishing smooth function ϕ on a neighbourhood $\mathcal{U}$ in M, we have

$$A_N = \phi A^*_\xi, \tag{2.24}$$

where A_N and A^*_ξ are shape operators of M and $S(TM)$ respectively. Equivalently, we have

$$C(X, PY) = \phi B(X, PY), \tag{2.25}$$

for any X and Y tangent to M. In case where ϕ is a non-zero constant then M is said to be **screen homothetic**. Furthermore, suppose that $\mathcal{U}$ is connected and maximal such that there does not exist a domain larger than $\mathcal{U} \subset \mathcal{U}'$ on which equation (2.24) holds. Then if $\mathcal{U} = M$ we say that M is **globally screen conformal**.

Example 2.5.1. [6] Consider a lightlike cone Λ_0^{n+1} of $\mathbb{R}_1^{n+1}$. And suppose that $\mathbb{R}_1^{n+1}$ is the space $\mathbb{R}^{n+2}$ endowed with the semi-Euclidean metric

$$\bar{g}(x, y) = -x^0 y^0 + \sum_{a=1}^{n+1} x^a y^a \qquad \big(x = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A}\big).$$

The light cone Λ_0^{n+1} is expressed as $-(x^0)^2 + \sum_{a=1}^{n+1}(x^a)^2 = 0, x \neq 0$. We know that Λ_0^{n+1} is a lightlike hypersurface of $\mathbb{R}_1^{n+1}$ and the radical distribution is spanned by a global vector field

$$\xi = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A} \tag{2.26}$$

on Λ_0^{n+1} .The unique section N satisfying (4.42) is given by

$$N = \frac{1}{2(x^0)^2}\Big\{ - x^0 \frac{\partial}{\partial x^0} + \sum_{a=0}^{n+1} x^a \frac{\partial}{\partial x^a}\Big\} \tag{2.27}$$

and is also globally defined. As ξ is the position vector field we get

$$\bar{\nabla}_X \xi = \nabla_X X = X, \qquad \text{for all } X \in \Gamma(TM).$$

It follows that $A^*_\xi X + \tau(X)\xi + X = 0$. Since A^*_ξ is $\Gamma(S(TM))$-values we have that

$$A^*_\xi X = -PX, \qquad \text{for all } M \in \Gamma(TM). \tag{2.28}$$

Moreover, we express any $X \in \Gamma(S(T\Lambda_0^{n+1}))$ by $X = \sum_{a=1}^{n+1} X^a \frac{\partial}{\partial x^a}$, where $(X^1, \cdots, X^{n+1})$ satisfy

$$\sum_{a=1}^{n+1} x^a X^a = 0. \tag{2.29}$$

It follows that

$$\nabla_\xi X = \bar{\nabla}_\xi X = \sum_{A=1}^{n+1}\sum_{a=1}^{n+1} x^A \frac{\partial X^a}{\partial x^a}\frac{\partial}{\partial x^a}, \tag{2.30}$$

$$\bar{g}(\nabla_\xi X, \xi) = \sum_{A=1}^{n+1}\sum_{a=1}^{n+1} x^a x^A \frac{\partial X^a}{\partial x^a} = -A = 1\sum_{a=1}^{n+1} x^a X^a = 0 \tag{2.31}$$

Since $\nabla_\xi X \in \Gamma(S(T\Lambda_0^{n+1}))$, it follows that $A_N\xi = 0$. Now, we solve $A_N X$ where $M \in \Gamma(S(T\Lambda_0^{n+1}))$. Suppose that $X, Y \in \Gamma(S(T\Lambda_0^{n+1}))$, then

$$C(X,Y) = g(\nabla_X Y, X) = \bar{g}(\bar{\nabla}_X Y, N) = -\frac{1}{2(x^0)^2} g(X,Y),$$

It follows that

$$g(A_N X, Y) = -\frac{1}{2(x^0)^2} g(X,Y) \qquad X, Y \in \Gamma(S(T\Lambda_0^{n+1})).$$

Thus,

$$A_N X = \frac{1}{2(x^0)^2} A^*_\xi X, \qquad X \in \Gamma(S(T\Lambda_0^{n+1})). \tag{2.32}$$

This implies that Λ_0^{n+1} is a screen globally conformal lightlike hypersurface of $\mathbb{R}_1^{n+1}$ with positive conformal function $\psi = \frac{1}{2(x^0)^2}$ globally defined on Λ_0^{n+1}.

Example 2.5.2. [6] Let $\mathcal{U}$ be an open set of $\mathbb{R}_q^{n+2}$ and $F : \mathcal{U} \to \mathbb{R}$ be a smooth function. A lightlike Monge hypersurfaces is given by

$$X = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}_1^{n+2} : m_0 = F(x_1, \cdots, x_{n+1})\}.$$

We know that M is a lightlike hypersurface if and only if F is a solution of the partial differential equation

$$1 + \sum_{i=1}^{q-1} (F'_{x^t})^2 = \sum_{a=q}^{n+1} (F'_{x^a})^2.$$

The radical distribution is spanned by a global vector field

$$\xi = \frac{\partial}{\partial x^0} - \sum_{i=1}^{q-1} F'_{x^i}\frac{\partial}{\partial x^i} + \sum_{a=q}^{n+1} F'_{x^a}\frac{\partial}{\partial x^a}$$

Let us consider a constant timelike section $V^* \in \mathbb{R}_q^{n+2}$ along M, such that

$$\frac{\partial}{\partial x^0}.$$

So

$$H^* = span\{V^*, \xi\}$$

which is a nondegenerate the vector bundle on M. The screen vector bundle $S^*(TM)$ is complementary to H^* in $T\mathbb{R}_q^{n+1}$ and it is am integral distribution on M. $S^*(TM)$ is said to be the natural screen distribution on M. Moreover, $tr^*(TM)$ is the transversal vector bundle spanned by $N = -V^* + \frac{1}{2}\xi$ and $\tau(X)$ for all $M \in \Gamma(TM)$. So

$$\begin{aligned}\tau(X) &= \bar{g}(\bar{\nabla}_X N, \xi)\\ &= \bar{g}(\bar{\nabla}_X(-V^* + \frac{1}{2}\xi), \xi)\\ &= \frac{1}{2}\bar{g}(\bar{\nabla}_X \xi, \xi).\end{aligned}$$

Thus, the Weingarten equations takes the two form $\bar{\nabla}_X Y = -A_N Y$ and $\nabla\xi = -A^*_\xi X$. So we ca deduce that

$$A_N X = \frac{1}{2}A^*_\xi X \qquad \text{for all } X \in \Gamma(TM).$$

This implies that the lightlike Monge hypersurfaces in $\mathbb{R}_q^{n+1}$ are screen globally conformal with the constant positive conformal function $\phi(x) = \frac{1}{2}$.

Proposition 2.5.1. *[6] Suppose that $(M, g, S(TM))$ is a lightlike hypersurface of $(\bar{M}, \bar{g})$ with the condition that $S(TM)$ is integrable and any leaf of $S(TM)$ is totally umbilical immersed in $\bar{M}$ as a codimension 2 non-degenerate submanifold with nowhere vanishing spacelike mean curvature vector field. If the screen distribution is parallel along integral curves of the radical distribution, then M is screen locally conformal.*

Proof. The proof is provided in page (53-54) of [6]. □

Theorem 2.5.1. *[6] Given that $(M, g, S(TM))$ is a lightlike hypersurface of a Lorentzian manifold $(\bar{M}, \bar{g})$, then the following statements hold and are equivalent:*

- ***i.*** *$(M, g, S(TM))$ is screen locally conformal.*
- ***ii.*** *there exists a (maximal) domain $\mathcal{U} \in M$ whereby M and the shape operators of M and its screen distribution are commutative. Furthermore, we have that their corresponding principal curvatures are the same up to a nowhere vanishing smooth function ϕ on $\mathcal{U} \subset M$.*

Proof. The proof is provided in page (54) of [6]. □

Example 2.5.3. [6] Consider a 4-dimensional semi-Euclidean space $(R^4_2, \bar{g})$ with signature $(-,-,+,+)$ of index 2,with respect to a canonical basis $(\partial_0, \partial_1, \partial_2, \partial_3)$. Suppose that we are given a hypersurface M expressed as

$$x_o = x_1 + sin(x_1 + x_2), \quad x_1 + x_2 \neq n\pi, \;\; n \in \mathbb{Z}.$$

Now we see that M is a lightlike hypersurface with a radical distribution $Rad\,TM$ which is spanned by

$$\xi = \partial_0 + \cos(x_1 + x_2)\partial_1 - \cos(x_1 + x_2)\partial_2 + \partial_3.$$

Now let

$$V = \partial_0 + \cos(x_1 + x_2)\partial_1,$$

, then

$$g(V, V) = g(\xi, V) = -(1 + \cos^2(x_1 + x_2)).$$

It follows that the lightlike transversal vector bundle is expressed as $tr(TM) = Span\{N\}$, where

$$N = \frac{-1}{2(1 + \cos^2(x_1 + x_2))}\{\partial_0 + \cos(x_1 + x_2)\partial_1 - \cos(x_1 + x_2)\partial_2 + \partial_3\}.$$

Moreover, the tangent bundle $T\mathbb{R}^4_2$ is spanned by

$$\{\frac{\partial}{\partial u_0} = \partial_0 + \partial_2, \frac{\partial}{\partial u_1} = \partial_1 + \cos(x_1 + x_2)\partial_3, \frac{\partial}{\partial u_2} = \partial_2 + \cos(x_1 + x_2)\partial_3\}.$$

Hence the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \cos^2(x_1 + x_2)\partial_0 - \partial_1, W_2 = \partial_2 + \cos^2(x_1 + x_2)\partial_3\}.$$

Note that in this case $[W_1, W_2] - \bar{\nabla}_{W_1} W_2 - \bar{\nabla}_{W_2} W_1 = \sin(x_1 + x_2)\{\partial_0 + \partial_3\}$. Therefore,

$$\bar{g}([W_1, W_2], Y) = \frac{\sin(x_1 + x_2)}{1 + \cos^2(x_1 + x_2)},$$

this means that $S(TM)$ is not integrable.

Lemma 2.5.1. *On every screen conformal lightlike hypersurface of a semi-Riemanian manifold, the following holds.*

$$(\nabla_X C)(Y, PZ) = (X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ),$$

for any X, Y and Z tangent to M.

Proof. By a direct calculation, while considering the fact that $C(X, PY) = \phi B(X, PY)$, we have

$$\begin{aligned}(\nabla_X C)(Y, PZ) &= X \cdot C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ)\\ &= X\{\phi B(Y, PZ)\} - \phi B(\nabla_X Y, PZ) - \phi B(Y, \nabla^*_X PZ)\\ &= (X\phi)B(Y, PZ) + \phi\{X \cdot B(Y, PZ) - B(\nabla_X Y, PZ) - B(Y, \nabla_X PZ)\}\\ &= (X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ),\end{aligned}$$

for any X, Y and Z tangent to M, which completes the proof. □

With the help of Lemma 2.5.1, we have the following result.

Proposition 2.5.2. *Let (M, g) be a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$. Then, the following relation holds*

$$\{\xi\phi - 2\phi\tau(\xi)\}B(X, PY) = cg(X, PY)$$

for any X and Y tangent to M.

Proof.

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ)\\ &= [(X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ)] - [(Y\phi)B(X, PZ) + \phi(\nabla_Y B)(X, PZ)]\\ &+ \tau(Y)\phi B(X, PZ) - \tau(X)\phi B(Y, PZ)(Y)B(X, PZ) + \tau(X)B(Y, PZ)].\end{aligned}$$

By Proposition 2.3.6

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + B(Y, Z)\tau(X) - B(X, Z)\tau(Y).$$

So,

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, N) &= (X\phi)B(Y, PZ) - (Y\phi)B(X, PZ) + \tau(Y)\phi B(X, PZ) - \tau(X)\phi B(Y, PZ)\\ &+ \phi[\bar{g}(\bar{R}(X, Y)PZ, \xi) - B(Y, PZ)\tau(X) + B(X, PZ)\tau(Y)]\\ &= [X\phi - 2\phi\tau(X)]B(Y, PZ) - [Y\phi - 2\phi\tau(Y)]B(X, PZ) + \phi\bar{g}(\bar{R}(X, Y)PZ, \xi).\end{aligned}$$

Setting $Y = \xi$, by the definition of curvature tensor $\bar{R}(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y]$. Then the left hand side becomes

$$\begin{aligned}\bar{g}(\bar{R}(X, \xi)PZ, N) &= \bar{g}(c[g(\xi, PZ)X - g(X, PZ)\xi], N)\\ &= \bar{g}(cg(\xi, PZ)X, N) - \bar{g}(cg(X, PZ)\xi, N)\\ &= -cg(X, PZ)\bar{g}(\xi, N)\\ &= -cg(X, PZ).\end{aligned}$$

And the right hand side becomes

$$\begin{aligned}
&[X\phi - 2\phi\tau(X)]B(\xi, PZ) - [\xi\phi - 2\phi\tau(\xi)]B(X, PZ) + \phi\bar{g}(\bar{R}(X,\xi)PZ, \xi)\\
&\quad = [X\phi - 2\phi\tau(X)]\rho g(\xi, PZ) - [\xi\phi - 2\phi\tau(\xi)]B(X, PZ) + \phi\bar{g}(\bar{R}(X,\xi)PZ, \xi)\\
&\quad = -[\xi\phi - 2\phi\tau(\xi)]B(X, PZ) + \phi\bar{g}(\bar{R}(X,\xi)PZ, \xi)\\
&\quad = -[\xi\phi - 2\phi\tau(\xi)]B(X, PZ) + \phi\bar{g}(c[g(\xi, PZ)X - g(X, PZ)\xi], \xi)\\
&\quad = -[\xi\phi - 2\phi\tau(\xi)]B(X, PZ).
\end{aligned}$$

Thus, $[\xi\phi - 2\phi\tau(\xi)]B(X, PY) = cg(X, PY)$. □

In view of Proposition 2.5.2, we have the following result.

Theorem 2.5.2. *Let (M, g) be a screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ such that $\xi\phi - 2\phi\tau(\xi) \neq 0$. Then, the following holds*

1. *M is totally umbilical.*
2. *$S(TM)$ is totally umbilical in M.*

Proof. 1. By Proposition 2.3.9

$[\xi\phi - 2\phi\tau(\xi)]B(X, PY) = cg(X, PY)$, so

$$\begin{aligned}
B(X, PY) &= \frac{c}{\xi\phi - 2\phi\tau(\xi)} g(X, PY)\\
&= \rho g(X, PY).
\end{aligned}$$

Thus, M is totally umbilical.

2. As $A_N = \phi A^*_\xi$, then

$$\begin{aligned}
g(A_N, PY) &= g(\phi A^*_\xi, PY)\\
&= \phi g(A^*_\xi, PY).
\end{aligned}$$

It follows

$$C(X, PY) = \phi B(X, PY) = \phi\rho g(X, PY).$$

Thus, $S(TM)$ is totally umbilical in M.

□

We also have the following result.

Theorem 2.5.3. *Every screen conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c = 0), \bar{g})$ is either totally geodesic or ϕ satisfy the partial differential equation $\xi\phi - 2\phi\tau(\xi) = 0$.*

Proof. By Proposition 2.5.2, we know that

$$[\xi\phi - 2\phi\tau(\xi)]B(X, PY) = cg(X, PY),$$

for any X and Y tangent to M. Since $c = 0$, then either $B(X, PY) = 0$ or $\xi\phi - 2\phi\tau(\xi) = 0$. Thus, (M, g) is either totally geodesic or ϕ satisfies $\xi\phi - 2\phi\tau(\xi) = 0$, as required. □

Definition 2.5.2. [6] A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is locally screen quasi-conformal if the shape operators A_N and A^*_ξ of M and $S(TM)$ satisfy

$$A_N = \phi A^*_\xi + \varphi P, \tag{2.33}$$

for some functions ϕ and φ. In case the above property holds everywhere on M, we say that $(M, g, S(TM))$ is globally screen quasi-conformal, and (ϕ, φ) is called the quasi-conformal pair associated to $(M, g, S(TM))$.

Remark 2.5.1. It is clear from the above definition that every screen quasi-conformal lightlike hypersurface is screen conformal whenever $\varphi = 0$.

Lemma 2.5.2. *On every screen conformal lightlike hypersurface of a semi-Riemannian manifold, the following holds.*

$$\begin{aligned}(\nabla_X C)(Y, PZ) &= (X\phi)B(Y, PZ)\\ &\quad + \phi(\nabla_X B)(Y, PZ) + (X\varphi)g(Y, PZ) + \varphi B(X, PZ)\eta(Y),\end{aligned} \tag{2.34}$$

for any X, Y and Z tangent to M.

Proof. As $A_N = \phi A^*_\xi + \varphi P$, it follows

$$\begin{aligned}g(A_N Y, PZ) &= g(\phi A^*_\xi Y, PZ) + g(\varphi PY, PZ)\\ &= \phi g(A^*_\xi Y, PZ) + \varphi g(PY, PZ).\end{aligned}$$

So,

$$C(Y, PZ) = \phi B(Y, PZ) + \varphi g(Y, PZ).$$

It follows

$$\begin{aligned}(\nabla_X C)(Y, PZ) &= X \cdot C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ)\\ &= X \cdot [\phi B(Y, PZ) + \varphi g(Y, PZ)] - [\phi B(\nabla_X Y, PZ) + \varphi g(\nabla_X Y, PZ)]\\ &\quad - [\phi B(Y, \nabla^*_X PZ) + \varphi g(Y, \nabla^*_X PZ)]\\ &= [X \cdot \varphi g(Y, PZ) - \varphi g(\nabla_X Y, PZ) - \varphi g(Y, \nabla^*_X PZ)]\\ &\quad + (X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ)\\ &= (X\varphi)g(Y, PZ)) + [\varphi X \cdot g(Y, PZ) - \varphi g(\nabla_X Y, PZ) - \varphi g(Y, \nabla^*_X PZ)]\\ &\quad + (X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ).\end{aligned}$$

Since $PZ \in S(TM)$, it follows that

$$\begin{aligned}\varphi X \cdot g(Y, PZ) - \varphi g(\nabla_X Y, PZ) - \varphi g(Y, \nabla^*_X PZ) &= \varphi(\nabla_X g)(Y, PZ) \\ &= \varphi B(X, Y)\eta(PZ) + \varphi B(X, PZ)\eta(Y) \\ &= \varphi B(X, Y) g(PZ, N) + \varphi B(X, PZ)\eta(Y) \\ &= \varphi B(X, PZ)\eta(Y).\end{aligned}$$

Therefore,

$$(\nabla_X C)(Y, PZ) = (X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ) + (X\varphi)g(Y, PZ) + \varphi B(X, PZ)\eta(Y).$$

□

Proposition 2.5.3. *Let (M, g) be a screen quasi-conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$. Then, the following relation holds*

$$\{\xi\phi - 2\phi\tau(\xi) - \varphi\}B(X, PY) = \{-\xi\varphi + \varphi\tau(\xi) + c\}g(X, PY)$$

for any X and Y tangent to M.

Proof. Following the results from Lemma 2.5.2 Proposition 2.5.2

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ) \\ &= [(X\phi)B(Y, PZ) + \phi(\nabla_X B)(Y, PZ) + (X\varphi)g(Y, PZ) + \varphi B(X, PZ)\eta(Y)] \\ &- [(Y\phi)B(X, PZ) + \phi(\nabla_Y B)(X, PZ) + (Y\varphi)g(X, PZ) + \varphi B(Y, PZ)\eta(X)] \\ &+ \tau(Y)[\phi B(X, PZ) + \varphi g(X, PZ)] - \tau(X)[\phi B(Y, PZ) + \varphi g(Y, PZ)] \\ &= [X\phi - 2\phi\tau(X)]B(Y, PZ) - [Y\phi - 2\phi\tau(Y)]B(X, PZ) + \phi\bar{g}(\bar{R}(X, Y)PZ, \xi) \\ &+ (X\varphi)g(Y, PZ) + \varphi B(X, PZ)\eta(Y) - (Y\varphi)g(X, PZ) - \varphi B(Y, PZ)\eta(X) \\ &+ \varphi g(X, PZ)\tau(Y) - \varphi g(Y, PZ)\tau(X).\end{aligned}$$

Setting $Y = \xi$,

$$\begin{aligned}-cg(X, PZ) &= -[\xi\phi - 2\phi\tau(\xi)]B(X, PZ) + \varphi B(X, PZ)\eta(\xi) - (\xi\varphi)g(X, PZ) + \varphi g(X, PZ)\tau(\xi) \\ &= -[\xi\phi - 2\phi\tau(\xi)]B(X, PZ) + \varphi B(X, PZ)g(\xi, N) - (\xi\varphi)g(X, PZ) + \varphi g(X, PZ)\tau(\xi) \\ &= -[\xi\phi - 2\phi\tau(\xi) - \varphi]B(X, PZ) - (\xi\varphi)g(X, PZ) + \varphi g(X, PZ)\tau(\xi).\end{aligned}$$

Therefore,

$$[\xi\phi - 2\phi\tau(\xi) - \varphi]B(X, PZ) = [-\xi\varphi + \varphi\tau(\xi) + c]g(X, PZ).$$

□

Next, in view of Proposition 2.5.3, we have the following result.

Theorem 2.5.4. *Let (M, g) be a screen quasi-conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ such that $\xi\phi - 2\phi\tau(\xi) - \varphi \neq 0$. Then, the following holds*

1. *M is totally umbilical.*
2. *$S(TM)$ is totally umbilical in M.*

Proof. 1. If $\xi\phi - 2\phi\tau(\xi) - \varphi \neq 0$ then by Proposition 3.5.2 we obtain

$$\begin{aligned} B(X, PY) &= \frac{-\xi\varphi + \varphi\tau(\xi) + c}{\xi\phi - 2\phi\tau(\xi) - \varphi} g(X, PY) \\ &= \rho g(X, PY). \end{aligned} \tag{2.35}$$

Thus, M is totally umbilical.

2. As $A_N = \phi A^*_\xi + \varphi P$, it follows

$$\begin{aligned} g(A_N X, PY) &= g(\phi A^*_\xi X, PY) + g(\varphi PX, PY) \\ &= \phi g(A^*_\xi X, PY) + \varphi g(PX, PY). \end{aligned}$$

So,

$$C(X, PY) = \phi B(X, PY) + \varphi g(PX, PY).$$

Thus, $S(TM)$ is totally umbilical in M.

□

Finally, we have the following result.

Theorem 2.5.5. *Every screen quasi-conformal lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c = \xi\varphi - \varphi\tau(\xi)), \bar{g})$ is either totally geodesic or ϕ satisfy the partial differential equation $\xi\phi - 2\phi\tau(\xi) - \varphi = 0$.*

Proof. If $c = \xi\varphi - \varphi\tau(\xi)$, then

$$\begin{aligned} \{\xi\phi - 2\phi\tau(\xi) - \varphi\}B(X, PY) &= \{-\xi\varphi + \varphi\tau(\xi) + c\}g(X, PY) \\ &= \{-\xi\varphi + \varphi\tau(\xi) + \xi\varphi - \varphi\tau(\xi)\}g(X, PY) \qquad (2.36) \\ &= 0. \qquad (2.37) \end{aligned}$$

By Theorem 3.5.2 $(\bar{M}(c = \xi\varphi - \varphi\tau(\xi)), \bar{g})$ is either totally geodesic when $B = 0$ or ϕ satisfy the partial differential equation $\xi\phi - 2\phi\tau(\xi) - \varphi = 0$. □

2.6 Conclusions and Perspectives

In this research we have proven new and necessary conditions for which a lightlike hypersurface of a semi-Riemannian space form is totally umbilical, screen conformal and almost screen conformal.

In section 2.1 we constructed the screen distribution and its corresponding lightlike transversal distribution on a lightlike hypersurface M. This screen distribution aided us in defining the induced geometric machinery necessary for obtaining the Gauss-Codazzi equations leading to the theorems that provided the conditions for which a lightlike hypersurface is totally umbilical, screen conformal and screen almost conformal.

In section 2.2 we introduced a local fundamental form. We used this local fundamental form to prove that lightlike hypersurfaces do not depend on the choice of screen distribution. We proved that the second fundamental form of a lightlike hyperspace is degenerate by showing that the local fundamental form is degenerate. Through the projection morphism of the tangent bundle to the screen distribution, we defined the second fundamental form, the shape operator, the Gauss and the Weingarten formulas of the screen distribution.

section 2.3 was dedicated for constructing the Gauss-Codazzi equations for a lightlike hypersurface. We used the Weingarten formula and the Gauss equations of lightlike hypersurfaces to provide an equation for curvature tensor, which was necessary for formulating the Gauss-Codazzi equations for lightlike hypersurfaces. These equations played a role in linking the induced metric and the second fundamental form of M. We observed that these equations depend on the screen distribution. And through these equations we were able to show that the shape operator of any screen distribution of M is symmetric.

In section 2.4 we defined what it means for M to be totally umbilical. This definition aided us in proving the conditions for which M is totally umbilical.

In section 2.5 we defined what it means for M to be screen conformal and screen almost conformal. We then stated and proved the new conditions for which a screen conformal or almost screen conformal hypersurface is totally umbilical.

In future, we hope to expand this work and provide the conditions for which lightlike hypersurfaces are totally umbilical, screen conformal and screen almost conformal for co-dimensions that are greater than one.

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