AdS/CFT Duality Involving Deformed PP-waves from the Lunin-Maldacena Background

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0.1 Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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6th day of March 2007.

0.2 Abstract

The AdS/CFT conjecture, as stated by Maldacena, postulates an equivalence between a string theory and a field theory. More precisely, it links type IIB string theory on the $AdS_5 \times S^5$ background with $\mathcal{N} = 4$ super Yang-Mills (SYM) field theory in $3 + 1$-dimensional Minkowski space. This thesis aims at performing a specific calculation on both sides of the correspondence, and verifying that we obtain the same result whether we work in the string theory or the field theory. In particular, we will be working with a specific deformation of the $\mathcal{N} = 4$ SYM theory, for which Lunin and Maldacena managed to find the corresponding deformation in the $AdS_5 \times S^5$ background of the matching string theory. By doing this, we will be testing the correspondence for a less supersymmetric field theory, and thus extending the area of duality. More precisely, we will calculate the spectrum of free strings in our deformed background, and find operators dual to supergravity and excited string modes in the field theory.

0.3 Acknowledgements

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0.5 Introduction

The main calculation in this thesis will be split into two sections, based on the two sides of the AdS/CFT correspondence [9]. The first concerns the $\mathcal{N}=4$ SYM field theory side. We will be working with a β-deformation of this theory. The significance of this is that the deformation leaves us with a $\mathcal{N}=1$ supersymmetric field theory (which is a theory with less supersymmetry). We provide a test of this extension of the correspondence to less supersymmetry. Once we have performed the deformation on the field theory side, we will move to the string theory side, and work with the corresponding deformation of the $AdS_5 \times S^5$ background. Furthermore, we will take a pp-wave limit of this deformed Lunin-Maldacena background, which broadly corresponds to a boost followed by a scaling. We perform this limit to probe the "stringy" aspects of the AdS/CFT correspondence. Our aim will be to find the string spectrum in this deformed background, and then try to develop the form of the operators in the deformed SYM dual to the lowest string mode. We will also say something about the operators dual to the excited string modes. In this way, we hope to test an interesting extension to the AdS/CFT correspondence, and add yet another calculation which validates its duality.

The AdS/CFT correspondence is nontrivial to test as it is a strong/weak coupling duality with respect to the 't Hooft coupling. In order to perform a predictive calculation on both sides of the correspondence for comparative purposes, one needs to consider quantities which are not corrected, or at worst receive small corrections. This implies that any results obtained at weak coupling will persist at strong coupling. Berenstein, Maldacena and Nastase [10] showed that near BPS operators dual to excited string modes satisfy this criterion.

The particular β-deformation [11] we will be studying in this thesis is obtained by making the superpotential replacement

$$Tr(\hat{\Phi}^1\hat{\Phi}^2\hat{\Phi}^3 - \hat{\Phi}^1\hat{\Phi}^3\hat{\Phi}^2) \to Tr(e^{i\pi\gamma}\hat{\Phi}^1\hat{\Phi}^2\hat{\Phi}^3 - e^{-i\pi\gamma}\hat{\Phi}^1\hat{\Phi}^3\hat{\Phi}^2).$$

As mentioned, this deformed $\mathcal{N}=4$ super Yang-Mills field theory has $\mathcal{N}=1$ supersymmetry, and is invariant under a $U(1)\times U(1)$ non-$\mathcal{R}$ symmetry. To see how the dual string theory in $AdS_5\times S^5$ is deformed, we note that the $AdS_5\times S^5$ geometry contains a two torus, whose isometries match the $U(1)\times U(1)$ field theory symmetry. If g represents the metric of this two torus, the deformation of the dual gravitational theory corresponds to

$$\tau = B + i\sqrt{g} \to \tau_\gamma = \frac{\tau}{1+\gamma\tau},$$

where B is the NS-NS 2-form. This deformation not only allows us to study the AdS/CFT duality in a less supersymmetric regime, it is also characterized by a continuous adjustable deformation parameter γ which could possibly lead to new scaling limits. The background obtained via the deformation is termed the Lunin-Maldacena (LM) background. This thesis will deal with a particular pp-wave limit of this background, allowing us to probe stringy aspects of the correspondence. More precisely, our pp-wave limit results in the geometry of a homogeneous plane wave, which has been studied [21],[22].

Various aspects of the AdS/CFT duality will need to be elaborated on in order to make the ultimate calculation performed in this thesis lucid. The body of this thesis is divided into two broad parts. The first part deals with the background theory and preliminaries and is itself divided into three chapters. The first of these deals with aspects of the gravitational side of the AdS/CFT duality, while the second considers aspects of the field theory. The third chapter involves the AdS/CFT conjecture itself, and attempts to provide some supporting arguments. The second part deals with the actual core calculation of the thesis, where we match the string spectrum in the pp-wave limit of the LM background to the anomalous dimensions of operators found on the gauge theory side. The calculation is also split into parts corresponding to the two sides of the correspondence.

Part I

Preliminaries

Chapter 1

String theory in $AdS_5 \times S^5$

1.1 String Theory Basics

1.1.1 Some Background

Like all pioneering theories, string theory has a history dotted with ingenius insights and controversy. Born of the long-standing need to find an ultimate unifying theory, and as yet unverified by direct experiment, it is an area of hot debate. To fully appreciate its current state in the physics community, it is useful to examine its roots [8].

As far back as 1921, the Kaluza-Klein theory showed that one can unify electromagnetism and gravity if one includes an extra tiny spatial dimension curled into a circle. This idea was followed much later by the realization that the dual theories which describe the particle spectrum also describe the quantum mechanics of strings, officially beginning the era of string theory. Supersymmetry arose in quick succession, both by introducing fermions into string theory and simply as an aspect of ordinary field theory. A few years later, string theory arose as a possible theory of quantum gravity, when the spin 2 excitation was found to have zero mass. Supergravity followed by introducing supersymmetry into gravity, especially important since gravity appears in the excitation spectrum of string theory. Furthermore, it was realized that adding worldsheet supersymmetry to string theory yields a spectrum with an equal number of bosons and fermions, and hence that string theory can be made spacetime supersymmetric. The resulting theory was named superstring theory.

Pre 1984, a number of anomalies in string theory seemed to undermine its status as a possible unifying theory. But this year marked a true beginning for the theory when these anomalies were found to cancel in specific examples, and it was widely accepted as a candidate ultimate theory. The last decade of the twentieth century lead to further progress, when it was recognized that duality transformations link the five seemingly unrelated types of superstring theory, laying a foundation for a possible nonperturbative model of string theory. The theory also managed to extract the microscopic origin of black hole thermodynamics, and continues to make advances in such areas as black hole quantum mechanics.

In very basic terms, string theory is founded on the idea that a string under tension can give rise to all the elementary particles, which correspond to various excitation modes of the string. Quantum mechanics dictates that even point particles are smeared in space, but this is not the smearing we are talking about when we move to a theory of strings. Indeed, elementary particles aren't pointlike from the point of view of string theory, but the important thing is that the extended nature of strings allows them to be excited. Strings are tiny one dimensional objects, with size the order of the Planck length (10^{-33}cm). The Planck length defines the scale at which general relativity and quantum mechanics become incompatible - the fluctuations arising in quantum mechanics due to the uncertainty principle destroy the smooth spacetime described by general relativity at this scale. To develop a theory of quantum gravity, one needs to be able to probe such exceptionally small distances. We require that the string size is the order of the Planck length to enable us to use the string as such a probe and ensure that string theory is a viable theory of quantum gravity. However, this size requirement makes any tests of the theory currently impossible. It is doubtful whether we will ever be able to build an accelerator powerful enough to see strings directly, but despite this there are many indirect but clever tests for the theory, and new results are emerging regularly. Indeed, string theory at long distances should reproduce quantum field theory describing point particles, since a string looks like a point particle on length scales we can probe. We are very familiar with field theories, and so studying what excitations within string theory participate in the long distance limit should lead to familiar and predictable results.

Another important property of strings is that they are fundamental, meaning that there are no smaller objects which make up the string. This ensures that the string tension remains constant, whether the string is stretched or not. In addition, fundamental strings only exhibit transverse excitations, with longitudinal excitations being prohibited (due to a lack of smaller constituents within the string). Requiring these conditions of the string at the quantum level leads to some interesting consequences: the background in which the strings propagate must have 10 spacetime dimensions, and the nature of the background is determined by generalizing Einstein's field equations in 3+1 dimensions to higher dimensions, with supersymmetry included.

Two types of basic strings exist based on whether they are open-ended or close into a loop, called open and closed strings respectively. String theories are classified broadly by what type of strings and particles they describe. More precisely, string theories can have either open and/or closed strings, and include bosons (force particles) and/or fermions (matter particles). Since supersymmetry relates bosons and fermions, a string theory which has supersymmetry must include fermions. Theoretically, each particle has a supersymmetric partner, but none have been detected experimentally. Any experimental evidence for supersymmetry at high energy is consistent with string theory as the ultimate description of our physical world. Now, a string theory which incorporates supersymmetry is called a superstring theory, while bosonic string theory deals exclusively with bosons. The bosonic theory also deals with both open and closed strings but is flawed in that it also gives rise to tachyons, particles with imaginary mass. In the late eighties we believed that there were five superstring theories, namely the type I, type IIA and IIB, and the heterotic $SO(32)$ and $E_8 \times E_8$ string theories (heterotic implies that right and left moving excitations on the string are considered different). None of these theories involve tachyons, and the type I theory is the only one which includes open and closed strings in its description - each of the other theories deals with only closed strings. The type IIA and IIB theories differ with regard to the chirality (or lack thereof) of massless fermions: in IIA, they spin both ways and are nonchiral, whereas in IIB they spin in only one way and are chiral. The heterotic theories differ in their group symmetry. The heterotic $SO(32)$ theory has an $SO(32)$ symmetry, while the heterotic $E_8 \times E_8$ theory has an $E_8 \times E_8$ group symmetry.

Initially, it was widely believed that only one of these five superstring theories was in fact a Theory of Everything, and that the others were simply mathematical possibilities which didn't manifest themselves in nature. Later it was recognized that these theories are related, and one may transform among them via duality transformations - the theories are simply special cases of one particular underlying theory. In transforming among these theories, we manage to link non-trivially concepts such as strong and weak coupling strengths, and large and small distances. This link obscures our ability to distinguish between these seemingly opposite concepts.

It is the T-duality which relates type IIA and IIB superstring theories and it links large and small distance scales. This is achieved by noting that if one of the nine space dimensions is formed into a circle, a string will not only be able to move around the circle with a quantized momentum, but it will also be able to wrap around the circle with a quantized winding number. In addition, one may interchange the momentum and winding modes if one interchanges the circle's radius R with $\frac{l_s^2}{R}$, where l_s is the string length. This interchange is possible since the energies of the two systems (namely, the string moving around the circle and the string wrapping around the circle) match under such an interchange. Now, a large R leads to a small $\frac{l_s^2}{R}$, so that interchanging the modes leads to an interchange in the size of distance scales. The T-duality tells us that if we compactify the IIA and IIB theories on a circle, interchange momentum and winding modes and thus the distance scales, we transform between the theories. The same is true of the two heterotic string theories.

The S-duality establishes a link between strong and weak coupling strengths. A coupling constant is a measure of the strength of an interaction, and small coupling constants lead to good approximations in perturbation theory. Should a coupling constant be large, it no longer becomes advantageous to use perturbation theory as dropping higher order terms does not lead to a good approximation of the physics. The string theory coupling constant is dependent on the dilaton (which is a particular oscillation mode of the string) and exchanging a dilaton field with minus itself exchanges a large coupling constant with a small one. This dilaton exchange leads to the S-duality, a symmetry which relates type I superstring theory with the heterotic $SO(32)$ theory. This is particulaly useful since a theory with a strong coupling constant cannot be probed via a perturbative series, but its dual theory can.

1.1.2 The Relativistic String

We will discuss the particulars of the classical relativistic string as a basic introduction to string theory. This discussion will allow us to show how we can obtain the action of a relativistic string by considering the proper area of the surface traced out by the string in spacetime, as well as highlight some properites of this action [1].

A string will trace out a surface called a worldsheet as it moves in spacetime. Any particular timeslice of this worldsheet represents the string itself as parametrized by an observer at that particular time. A relativistic string, open or closed, will have an action which must depend on the trajectory of the string, and is proportional to the Lorentz invariant "proper area" of its worldsheet. Such an action is called the Nambu-Goto action. We will begin our discussion by considering spatial surfaces, as these are easier to visualize, and later adapt the arguments to worldsheets. We will focus on 2-dimensional surfaces embedded in 3-dimensional target space, which is simply the surrounding space in which the surface resides.

A 2-dimensional surface is parametrized by two parameters, call them ξ^1 and ξ^2. One can draw a grid on the surface by marking it with lines of constant ξ^1 and ξ^2 respectively. In this way, the parameters act locally as coordinates on the physical surface. We assume that our surface has a 3-dimensional target space described by x^1, x^2 and x^3-coordinates. Thus, the parametrized surface is represented by the set of functions

$$\vec{x}(\xi^1,\xi^2) = (x^1(\xi^1,\xi^2), x^2(\xi^1,\xi^2), x^3(\xi^1,\xi^2)).$$

One more type of space we need to consider is the parameter space, which is determined by the range of the parameters. Specifying the physical surface involves describing the one-to-one map $\vec{x}(\xi^1,\xi^2)$ over all of parameter space.

Ultimately we are interested in calculating the area A of our physical surface. We do so infinitesimally, first considering a tiny area in parameter space with sides of length $d\xi^1$ and $d\xi^2$, and then looking for the image of this object in target space. Such an image will correspond to an infinitesimal piece of the physical surface in target space, its area being dA. The infinitesimal area element in target space is in general a parallelogram with sides denoted by $d\vec{v}_1$ and $d\vec{v}_2$, where

$$d\vec{v}_1 = \frac{\partial \vec{x}}{\partial \xi^1} d\xi^1 \equiv \vec{x}(d\xi^1, 0),$$

$$d\vec{v}_2 = \frac{\partial \vec{x}}{\partial \xi^2} d\xi^2 \equiv \vec{x}(0, d\xi^2).$$

Next, we wish to calculate the area dA of this infinitesimal parallelogram:

$$\begin{aligned} dA &= |d\vec{v}_1||d\vec{v}_2||\sin\theta| \\ &= \sqrt{|d\vec{v}_1|^2|d\vec{v}_2|^2 - |d\vec{v}_1|^2|d\vec{v}_2|^2\cos^2\theta} \\ &= \sqrt{(d\vec{v}_1 \cdot d\vec{v}_1)(d\vec{v}_2 \cdot d\vec{v}_2) - (d\vec{v}_1 \cdot d\vec{v}_2)^2}, \end{aligned}$$

where θ is the angle between $d\vec{v}_1$ and $d\vec{v}_2$. Using the definitions of $d\vec{v}_1$ and $d\vec{v}_2$ in terms of ξ^1 and ξ^2 respectively, we can write

$$dA = d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^1}\right)\left(\frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right) - \left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right)^2}$$

and, furthermore,

$$A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^1}\right)\left(\frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right) - \left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right)^2}.$$

This expression for the area of a parametrized surface is general, and the integral runs over the parameter ranges. Now, the area of the physical surface, as well as the infinitesimal area element, should be independent of the way in which the surface is parametrized, i.e. the area should be reparametrization invariant. We can rewrite this area in such a way as to make the reparametrization invariance manifest. We begin by considering a surface in target space described by $\vec{x}(\xi^1, \xi^2)$ as usual, so that the length ds of a vector $d\vec{x}$ tangent to the surface satisfies

$$ds^2 = d\vec{x}\cdot d\vec{x}.$$

Using

$$d\vec{x} = \frac{\partial\vec{x}}{\partial\xi^1}d\xi^1 + \frac{\partial\vec{x}}{\partial\xi^2}d\xi^2 = \frac{\partial\vec{x}}{\partial\xi^i}d\xi^i$$

we can write

$$ds^2 = \frac{\partial\vec{x}}{\partial\xi^i}\cdot\frac{\partial\vec{x}}{\partial\xi^j}d\xi^i d\xi^j \equiv g_{ij}(\xi)d\xi^i d\xi^j,$$

where

$$g_{ij}(\xi) \equiv \frac{\partial\vec{x}}{\partial\xi^i}\cdot\frac{\partial\vec{x}}{\partial\xi^j} = \begin{pmatrix} \frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^1} & \frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^2} \\ \frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^1} & \frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^2} \end{pmatrix}.$$

The very important quantity $g_{ij}(\xi)$ is the induced metric on the surface, because it uses the metric in the target space to determine distances on the surface. If we set

$$g \equiv \det(g_{ij})$$

we see that

$$A = \int d\xi^1 d\xi^2 \sqrt{g}.$$

In order to test the reparametrization invariance of A, we make the reparametrization

$$\tilde{\xi}^1(\xi^1, \xi^2) \quad and \quad \tilde{\xi}^2(\xi^1, \xi^2)$$

and investigate what happens to g. Now, ds^2 is a geometrical property of $d\vec{x}$ and must be reparametrization invariant. So, we require

$$\begin{aligned} g_{ij}(\xi) d\xi^i d\xi^j &= \tilde{g}_{pq}(\tilde{\xi}) d\tilde{\xi}^p d\tilde{\xi}^q, \\ &= \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j} d\xi^i d\xi^j, \end{aligned}$$

where $\tilde{g}(\tilde{\xi})$ is the metric for the parameters $\tilde{\xi}$. The result above is true for any $d\xi$, so

$$g_{ij}(\xi) = \tilde{g}_{pq}(\tilde{\xi}) \frac{\partial \tilde{\xi}^p}{\partial \xi^i} \frac{\partial \tilde{\xi}^q}{\partial \xi^j}.$$

As a quick aside, note that

$$\begin{aligned} d\xi^1 d\xi^2 &= \left| \det\left(\frac{\partial \xi^i}{\partial \tilde{\xi}^j}\right) \right| d\tilde{\xi}^1 d\tilde{\xi}^2 \equiv |\det M| d\tilde{\xi}^1 d\tilde{\xi}^2, \\ d\tilde{\xi}^1 d\tilde{\xi}^2 &= \left| \det\left(\frac{\partial \tilde{\xi}^i}{\partial \xi^j}\right) \right| d\xi^1 d\xi^2 \equiv |\det \tilde{M}| d\xi^1 d\xi^2, \end{aligned}$$

where

$$M_{ij} = \frac{\partial \xi^i}{\partial \tilde{\xi}^j}, \quad \tilde{M}_{ij} = \frac{\partial \tilde{\xi}^i}{\partial \xi^j}.$$

If we combine these two equations we see that

$$|\det M||\det \tilde{M}| = 1.$$

This observation allows us to write

$$\begin{aligned} g_{ij}(\xi) &= \tilde{g}_{pq}\tilde{M}_{pi}\tilde{M}_{qj} = (\tilde{M}^T)_{ip}\tilde{g}_{pq}\tilde{M}_{qj} \\ \Rightarrow g &= (\det \tilde{M}^T)\tilde{g}(\det \tilde{M}) = \tilde{g}(\det \tilde{M})^2 \\ \Rightarrow \sqrt{g} &= \sqrt{\tilde{g}}|\det \tilde{M}|. \end{aligned}$$

Thus, we see that A is truly reparametrization invariant by noting that

$$\int d\xi^1 d\xi^2 \sqrt{g} = \int d\tilde{\xi}^1 d\tilde{\xi}^2 |\det M| \sqrt{\tilde{g}} |\det \tilde{M}| = \int d\tilde{\xi}^1 d\tilde{\xi}^2 \sqrt{\tilde{g}}.$$

Having considered the precise nature of spatial surfaces, we can now explore spacetime surfaces. Being 2-dimensional, these worldsheets are parametrized by two parameters, τ and σ, which are locally viewed as coordinates on the worldsheet, and represent the parameter space. Broadly, τ corresponds to time on the string and σ corresponds to position along the string. A surface is represented by the mapping

$$X^\mu(\tau,\sigma) = (X^0(\tau,\sigma), X^1(\tau,\sigma), ..., X^d(\tau,\sigma)),$$

where X^0, X^1,..., X^d represent spacetime coordinates particular to the string worldsheet, i.e. string coordinates. To find the area of the spacetime surface we again consider an infinitesimal area element in parameter space (with sides $d\tau$ and $d\sigma$) and evaluate the corresponding area element on the worldsheet. This has sides dv_1^μ and dv_2^μ where

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau, \quad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma,$$

so that the area of the worldsheet is given by

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}\right)\left(\frac{\partial X^\nu}{\partial \sigma}\frac{\partial X_\nu}{\partial \sigma}\right)}$$

$$= \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2}.$$

The action of the relativistic string, which gives us the dynamics of our theory, is proportional to the area of the worldsheet, and all that is left in making this relationship an equality is ensuring our action has the correct dimension. Now

$$[S] = \frac{ML^2}{T} \quad and \quad [A] = L^2,$$

so we need to multiply the area by some suitable constant with units of M/T. The quantity T_0/c has the required units, where T_0 is the string tension. We can now write the string action as

$$S = \frac{-T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2},$$

where $\sigma_1 > 0$ is some constant. This is the so-called Nambu-Goto action. Again, we want to make sure this action is reparametrization invariant, so we need to rewrite it to make the test more transparent. As before, we know that

$$-ds^2 = dX^\mu dX_\mu = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta,$$

where $\xi^1 = \tau$, $\xi^2 = \sigma$ and $\eta_{\mu\nu}$ is the Minkowski metric in target space. We define the induced metric on the worldsheet as

$$\gamma_{\alpha\beta} \equiv \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{bmatrix}_{\alpha\beta}.$$

Thus, the manifestly reparametrization invariant form of the Nambu-Goto action is

$$S = \frac{-T_0}{c} \int d\tau d\sigma \sqrt{-\gamma},$$

where $\gamma \equiv \det(\gamma_{\alpha\beta})$. We can follow the same procedure as before to show that it is reparametrization invariant. However, no new information can be extracted from the details so they are not included.

The Nambu-Goto action involves an unsightly square root. We can get a quadratic action by making use of a worldsheet metric. The resulting equivalent action, termed the Polyakov action [7], is given by

$$S[X,h] = -\frac{T_0}{2}\int d\tau d\sigma \sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}.$$

In the above, the quantity $h_{ab}(\tau,\sigma)$ is an auxiliary rank two symmetric tensor field and can be interpreted naturally as a metric on the string worldsheet. Note that

$$h = \det(h_{ab}), \quad h^{ab} = (h^{-1})_{ab}.$$

The Polyakov action is reparametrization invariant, but it also exhibits an additional local gauge symmetry called Weyl invariance (or conformal invariance). This says that the action is invariant under the transformation

$$h_{ab} \to e^{2\rho(\tau,\sigma)}h_{ab},$$

where $\rho(\tau,\sigma)$ is an arbitrary function on the worldsheet. The two local reparametrization symmetries allow us to choose the conformal gauge, where

$$(h_{ab}) = e^{\phi(\tau,\sigma)}(\eta_{ab}) = e^{\phi(\tau,\sigma)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{ab}.$$

The conformal gauge says that h_{ab} and η_{ab} agree up to a scaling function e^ϕ. In this case, the metric h_{ab} is said to be conformally flat.

We can obtain the relativistic string equations of motion by varying the Nambu-Goto action, which will also give rise to a discussion of the boundary conditions of string endpoints. The Lagrangian density we extract from the Nambu-Goto action is

$$\mathcal{L}(\dot{X}^\mu, X'^\mu) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}.$$

The equations of motion emerge when we set the variation of the action to zero, namely

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\frac{\partial(\delta X^\mu)}{\partial\tau} + \frac{\partial\mathcal{L}}{\partial X'^\mu}\frac{\partial(\delta X^\mu)}{\partial\sigma}\right] = 0,$$

where

$$\delta \dot{X}^\mu = \delta \left(\frac{\partial X^\mu}{\partial \tau} \right) = \frac{\partial (\delta X^\mu)}{\partial \tau},$$

$$\delta X'^\mu = \frac{\partial (\delta X^\mu)}{\partial \sigma}.$$

The Lagrangian density leads us to

$$\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}},$$

$$\frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}.$$

We introduce a useful notation for the above quantities:

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \quad \mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X'^\mu}.$$

We can rewrite the variation in the action in terms of this new notation, which will help us to determine the boundary conditions. The result is

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial}{\partial \tau} (\delta X^\mu \mathcal{P}^\tau_\mu) + \frac{\partial}{\partial \sigma} (\delta X^\mu \mathcal{P}^\sigma_\mu) - \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} \right) \right].$$

In our analysis, we will specify the initial and final states of the string, so that variations in these quantities are zero, i.e.

$$\delta X^\mu(\tau_i, \sigma) = \delta X^\mu(\tau_f, \sigma) = 0.$$

In δS above, the term with the total τ derivative will yield terms proportional to $\delta X^\mu(\tau_i, \sigma)$ and $\delta X^\mu(\tau_f, \sigma)$, which will clearly vanish. The variation then becomes

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau [\delta X^\mu \mathcal{P}^\sigma_\mu]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} \right).$$

For all δX^μ the second term in the variation above has to vanish, so we extract the equation of motion of the relativistic open or closed string:

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

The first term in δS can be written more explicitly as

$$\begin{aligned}\int_{\tau_i}^{\tau_f} d\tau (\delta X^0(\tau,\sigma_1)\mathcal{P}^\sigma_0(\tau,\sigma_1) - \delta X^0(\tau,0)\mathcal{P}^\sigma_0(\tau,0)\\ + \delta X^1(\tau,\sigma_1)\mathcal{P}^\sigma_1(\tau,\sigma_1) - \delta X^1(\tau,0)\mathcal{P}^\sigma_1(\tau,0)\\ +\dots\dots\dots\\ + \delta X^d(\tau,\sigma_1)\mathcal{P}^\sigma_d(\tau,\sigma_1) - \delta X^d(\tau,0)\mathcal{P}^\sigma_d(\tau,0)).\end{aligned} \tag{1.1}$$

This term concerns the string endpoints and will yield $2(d+1)$ boundary conditions, one corresponding to each term in (1.1). To simplify our analysis we will consider only one general term in (1.1). We specify this term by fixing μ and taking σ_* as the σ-coordinate, where $\sigma_* = 0, \sigma_1$. We need to impose boundary conditions on the endpoints which will make each of the terms in (1.1) disappear so that δS will vanish. As a result two possibilities arise for boundary conditions which can be imposed at the endpoints. We can either require that the endpoints of the string remain fixed throughout its motion, or that the endpoints are free to do whatever is necessary for δS to vanish. The former case gives rise to the Dirichlet boundary condition, which is expressed mathematically as

$$\frac{\partial X^\mu}{\partial \tau}(\tau,\sigma_*) = 0, \quad \mu \neq 0.$$

Since τ and time are both timelike we exclude the case where $\mu = 0$. The Dirichlet boundary condition states that the μ-coordinate of the selected string endpoint is fixed in time. An equivalent statement of this boundary condition is $\delta X^\mu(\tau,\sigma_*) = 0$, which obviously ensures that each term in (1.1) vanishes. The second possible boundary condition, called the free endpoint condition or Neumann boundary condition, corresponds to

$$\mathcal{P}^\sigma_\mu(\tau,\sigma_*) = 0$$

and clearly leads to a vanishing of each term in (1.1). This condition doesn't impose any restriction on the string endpoints, hence the nomenclature. One may pick either of the two conditions for each spatial direction and at each endpoint.

The Dirichlet boundary conditions give rise to a widely studied object called a D-brane (where "D" stands for Dirichlet). These particular conditions, as mentioned, require that the endpoints of the strings remain fixed. It is the objects to which these endpoints are attached that are termed D-branes. They can be many-dimensional objects, such that a Dp-brane represents a p-dimensional D-brane. The Dirichlet boundary conditions specify the D-brane. Open string endpoints are allowed to move freely along the directions of the brane. As a visual example of a D-brane, consider a flat D2-brane in 3-dimensional space, specified by the condition $x^3 = 0$ (so that the brane lies in the (x^1, x^2)-plane). The Dirichlet boundary condition is $X^3 = 0$, and the remaining X^1 and X^2 coordinates satisfy free boundary conditions. It is also possible to define a Dp-brane in p-dimensional space. Such a brane is termed a space-filling brane, and implies that all the string spatial coordinates satisfy free endpoint conditions.

D-branes are actual physical objects within string theory, and don't need to be put in by hand. Nor are they necessarily hyperplanes. They are fascinating objects with many interesting properties, but a detailed analysis would veer off the main ideas of this thesis and will thus not be discussed here.

1.1.3 Light-cone Coordinates

Since we will be dealing with the light-cone gauge later on, it is instructive to give a brief overview of light-cone coordinates [1]. The relativistic string can be quantized with relative ease using these coordinates in light-cone quantization.

The two light-cone coordinates are defined as

$$x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^1), \qquad x^- \equiv \frac{1}{\sqrt{2}}(x^0 - x^1).$$

Notably, they consist of two independent linear combinations of the time coordinate x^0 and a spatial coordinate, here chosen as x^1. The remaining coordinates (here we deal with 4-dimensional Minkowski space) are kept as x^2 and x^3. Thus, we have traded our (x^0, x^1) coordinates for (x^+, x^-), and the complete set of light-cone coordinates can be written as

$$(x^+, x^-, x^2, x^3).$$

Now, if we consider a beam of light moving in the positive x^1 direction, we can write

$$x^1 = ct = x^0 \Rightarrow x^- = 0,$$

which corresponds to the x^+ axis. Similarly, for a beam of light moving in the negative x^1 direction, we can write

$$x^1 = -ct = -x^0 \Rightarrow x^+ = 0,$$

which corresponds to the x^- axis. Thus, the nomenclature is obvious as the coordinate axes of the light-cone coordinates correspond to world-lines of beams of light emitted from the origin along the x^1 axis. The light-cone axes are at 45^o to the (x^0, x^1) axes.

We take x^+ as the light-cone time and x^- as a spatial coordinate. Light-cone time is not quite the same as ordinary time, because light-cone time will freeze for certain light rays (as we mentioned, x^- remains constant for a light ray in the positive x^1 direction, while x^+ remains constant for a light ray in the negative x^1 direction).

1.1.4 String Quantization in the Light-cone Gauge

Reparametrization invariance of a string's worldsheet results in the appearance of constraints within the theory, so that quantizing the string amounts to quantizing a constrained system. Notably, two different methods arise naturally when we consider string quantization: covariant quantization via the Neveu-Schwarz-Ramond (NSR) approach, and light-cone quantization via the Green-Schwarz (GS) approach. The former has to contend with non-physical negative norm states because it keeps certain symmetries manifest, such as the Lorentz invariance if it is in flat Minkowski space. The latter is practically the direct opposite of the NSR approach: it makes sure that no unphysical states are allowed thus sacrificing manifest invariance under certain symmetries.

Now, the string worldsheet action is the so-called Polyakov action as discussed previously, and it has a kinetic term of the form

$$g_{\mu\nu}\eta^{ab}\partial_a X^\mu \partial_b X^\nu.$$

The problem arises when you try to canonically quantize the string. The oscillator commutation relations will look like

$$[a^\nu, a^{\dagger\mu}] \propto g^{\mu\nu},$$

where g has one negative eigenvalue and nine positive eigenvalues. Clearly, negative norm states will be created by the oscillator corresponding to the negative eigenvalue. So, the timelike oscillations of the string give rise to the problem, which is cured by the constraints arising from string worldsheet reparametrization invariance.

The beauty of the light-cone gauge is that it makes use of light-cone coordinates, with two light-like and eight spacelike coordinates. Thus, the problematic timelike coordinate doesn't stand on its own - it mixes with one space coordinate. The gauge gets rid of one of the light-like coordinates, and also allows you to solve the constraints to eliminate the second light-like coordinate. This leads to a theory with no constraints and eight quantized space coordinates.

We use the light-cone gauge approach in this thesis, but not directly - we have to take the pp-wave limit of our background to make progress. The original LM background (not in the limit) is complicated by the presence of RR-fields, which are higher tensor fields than the 2-tensor B potential, and couple to D-branes directly. They do not, however, couple to fundamental strings directly. The pp-wave limit allows us to test the AdS/CFT correspondence despite the difficulties within the LM background.

1.1.5 Type IIB Superstring Theory

The AdS/CFT duality involves type IIB string theory, which is a superstring theory, so it is necessary to discuss this theory in some detail. We generalize our previous string analysis to the superstring by generalizing the Polyakov action [4].

We begin by noting that the the string worldsheet has a diffeomorphism and a Weyl symmetry, which leaves us with a large number of gauge choices. For simplicity in evaluating the physical spectrum in type IIB strings we choose the light-cone gauge

$$\tau = X^+ = X^0 + X^9.$$

In this gauge we can eliminate X^-. The beauty of this gauge is that we have managed to remove all unphysical modes (which include oscillations due to negative-norm oscillators in the mode expansion of X^0, as well as longitudinal string excitations which have no meaning). This gauge doesn't give us manifest Lorentz invariance, however, and only the transverse group of rotations, which in this gauge is $SO(8)$ with a covering group of $Spin(8)$, will be manifest. Three particular 8-dimensional representations of this covering group which we will make use of in our analysis are the spinor representation **8s** (with indices a, b, c), the vector representation **8v** (with indices i, j, k) and the conjugate spinor representation **8c** (with indices $\dot{a}, \dot{b}, \dot{c}$). Right-handed spinors are related to left-handed conjugate spinors by parity transformations which act to change the sign of one of the vector components. We are working in the Green-Schwartz formalism, which is actually equivalent to the Neveu-Schwarz-Ramond superstring formalism. In the latter formalism, the **8c** and **8s** comes from the Ramond (R) sector, whereas the **8v** comes from the Neveu-Schwarz (NS) sector.

We now specify the worldsheet action of the type IIB string in the Green-Schwartz formalism. The transverse spatial coordinates of the string are contained within the bosonic fields X^i defined on the worldsheet. The fermionic fields on the worldsheet (their existence guaranteed by supersymmetry) are represented by left-moving S^a and right-moving $\tilde{S}^a$. Their indices tell us that they transform as spinors **8s**. Due to the equivalent spacetime transformation properties of these two spinors, we see that our theory on the worldsheet is non-chiral. It is chiral in spacetime, however, since we only see the right-handed chirality of spacetime fermions and not the parity transform. It is interesting to note the choice which leads us instead to type IIA string theory. Indeed, we could have chosen left-moving $S^{\dot{a}}$ (which transforms as a left-handed conjugate spinor as suggested by its index) and $\tilde{S}^a$ again. The type IIA theory which results has opposite chirality properties, namely it is chiral on the worldsheet but nonchiral in spacetime. Now, the type IIB string has action

$$S = -\frac{1}{2\pi} \int d\sigma d\tau (\partial_+ X^i \partial_- X^i - i S^a \partial_- S^a - i \tilde{S}^a \partial_+ \tilde{S}^a),$$

where τ is time on the worldsheet, σ is the coordinate along the string ($0 \leq \sigma <$

2π), and we have set $\alpha' = \frac{1}{2}$. We note that the bosons X^i will satisfy a periodic boundary condition along σ. Supersymmetry is a spacetime symmetry linking the bosons and the fermions, so the fermions will satisfy the same boundary condition, thus

$$X^i(\sigma + 2\pi) = X^i(\sigma), \quad S^a(\sigma + 2\pi) = S^a(\sigma), \quad etc.$$

Using these boundary conditions we can quantize this free field theory by writing the mode expansions

$$X^i = x^i + \frac{1}{2}p^i\tau + \frac{i}{2}\sum_{n\neq 0}\left[\frac{1}{n}\alpha_n^i e^{-in(\tau+\sigma)} + \frac{1}{n}\tilde{\alpha}_n^i e^{-in(\tau-\sigma)}\right],$$

$$S^a = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)},$$

$$\tilde{S}^a = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty} \tilde{S}_n^a e^{-in(\tau-\sigma)}.$$

Since we are using standard canonical quantization techniques, we need to enforce that the oscillator modes satisfy the standard commutators and anticommutators, given in this case by

$$\begin{aligned} [\alpha_m^i, \alpha_n^j] &= m\delta^{ij}\delta_{m+n}, \quad [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m+n}, \\ \{S_n^a, S_m^b\} &= \delta^{ab}\delta_{m+n}, \quad \{\tilde{S}_n^a, \tilde{S}_m^b\} = \delta^{ab}\delta_{m+n}. \end{aligned} \tag{1.2}$$

We can also discuss the bosonic and fermionic zero modes to determine a labelling of our ground state. Indeed, the zero modes x^i of the bosonic fields X^i allow the ground state to be labelled by the momentum eigenvalue $|p\rangle$ since they satisfy the Heisenberg commutation relations

$$[x^i, p^j] = i\delta^{ij}.$$

The fermionic zero modes S_0^a and $\tilde{S}_0^a$ have an algebra given by

$$\{S_0^a, S_0^b\} = \delta^{ab}, \quad \{\tilde{S}_0^a, \tilde{S}_0^b\} = \delta^{ab}.$$

We expect that the ground state furnishes a representation of this zero mode algebra. We can focus on the left-moving S^a in our analysis (since the treatment of $\tilde{S}^a$ is identical). We can obtain the usual anticommutators from the anticommutators (1.2) by defining four fermionic oscillators

$$\sqrt{2}b_m = (S^{2m-1} + iS^{2m}), \quad m = 1, 2, 3, 4.$$

It is clear that we get the usual anticommutation relations

$$\{b_m, b_n^\dagger\} = \delta_{mn}, \quad \{b_m, b_n\} = 0, \quad \{b_m^\dagger, b_n^\dagger\} = 0.$$

By writing these new fermionic oscillators, we have in effect chosen a specific embedding $SO(8) \supset SU(4) \times U(1)$. Thus, our oscillators $\{b_m\}$ transform in the fundamental representation of $SU(4)$ and have a $\frac{1}{2}$ unit of $U(1)$ charge. Since the $SU(4)$ fundamental representation is labelled by $\mathbf{4}$, our new oscillators are said to transform in $\mathbf{4}(\frac{1}{2})$. The complex conjugate oscillators $\{b_m^\dagger\}$ transform in the complex conjugate representation.

In this particular choice for the embedding, our vector, spinor and conjugate spinor representation will decompose as

$$\begin{aligned}
\mathbf{8s} &= \mathbf{4}\left(\frac{1}{2}\right) + \bar{\mathbf{4}}\left(-\frac{1}{2}\right), \\
\mathbf{8v} &= \mathbf{6}(0) + \mathbf{1}(1) + \mathbf{1}(-1), \\
\mathbf{8c} &= \mathbf{4}\left(-\frac{1}{2}\right) + \bar{\mathbf{4}}\left(\frac{1}{2}\right).
\end{aligned}$$

The ground state should furnish a representation of the zero mode algebra of fermionic zero modes. We wish to work out this representation. We do so by making filled states from the totally "empty" Fock space vacuum $|0\rangle$, which itself is annihilated by all annihilation operators b_m. We act on this vacuum with creation operators to obtain the various filled states. The 16-dimensional representation we obtain is given by

$$\begin{array}{rl}
|0\rangle & \mathbf{1}(1) \\
b_m^\dagger|0\rangle & \bar{\mathbf{4}}\left(\frac{1}{2}\right) \\
b_m^\dagger b_n^\dagger|0\rangle & \mathbf{6}(0)
\end{array}$$

$$b^\dagger_m b^\dagger_n b^\dagger_p |0\rangle \quad \mathbf{4}\left(-\frac{1}{2}\right)$$
$$b^\dagger_m b^\dagger_n b^\dagger_p b^\dagger_q |0\rangle \quad \mathbf{1}(-1).$$

The right-hand column above gives the dimensions of the $SU(4)$ representation followed by the $U(1)$ charge. We note that the 16-dimensional representation of the left-moving ground states can be written as $\mathbf{8v} + \mathbf{8c}$, which is also true for the right-moving ground states.

Now that we have specified our ground state, we can build a physical string state $|\psi\rangle$, which should satisfy the on-shell conditions

$$\alpha' M^2 = -\alpha' p^\mu p_\mu = 4N, \ and \ = 4\tilde{N}$$

where

$$N \equiv \sum_{n=0}^{\infty} (n\alpha^i_{-n}\alpha^i_n + S^a_{-n}S^a_n),$$

$$\tilde{N} \equiv \sum_{n=0}^{\infty} (n\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \tilde{S}^a_{-n}\tilde{S}^a_n).$$

We build our physical string state by acting with creation operators on the 16×16-dimensional ground state which is labelled by spacetime momentum p. The massless states have no oscillator excitations, and we find them by taking the tensor product of left- and right-moving ground states:

$$(|i\rangle \oplus |\dot{a}\rangle) \otimes (|j\rangle \oplus |\dot{b}\rangle).$$

Now, giving what we said previously about the R and NS sectors, the Ramond states turn out to be the spacetime fermions and the Neveu-Schwarz states the spacetime bosons.

When we take the tensor product of left- and right-moving states, we get a number of possibilities. If we consider the indices in the product, we see that we can get states in the NS-R and R-NS sector, the NS-NS sector and the R-R sector.

In the NS-R and R-NS sector we get the spacetime fermions $\psi_{i\dot{a}}$ and $\psi_{j\dot{b}}$ respectively. These correspond to the two gravitini of type IIB string theory. Also, if we recall that i and j are **8v** indices and hence belong to the NS sector, then our states $|i\rangle \otimes |j\rangle$ are obviously NS-NS states. These states give rise to the metric g_{ij} (via the reduction in terms of the symmetric traceless combination), the 2-form B_{ij} (via the reduction in terms of the antisymmetric combination) and the dilaton ϕ (via the reduction in terms of the scalar combination). Lastly, we see that we can reduce the R-R states $|\dot{a}\rangle \otimes |\dot{b}\rangle$ to give a scalar χ, a 2-form B_{ij} and a self dual 4-form D_{ijkl}. All of these quantities form the massless spectrum of type IIB string theory, and their interpretation is not too difficult to elucidate.

In the NS-NS sector, the metric g_{ij} is simple to interpret - it is clearly the metric of spacetime. The dilaton ϕ sets the strength of the string interactions. The field B_{ij} can be interpreted in the same way as the A_μ in electromagnetism. In electromagnetism, the term in the action that we add to account for a point particle's coupling to the electromagnetic field is

$$\int A_\mu dx^\mu,$$

due to its reparametrization invariance - this is a symmetry shared by the action, and allows us to define any parametrization for the particle's worldline without changing the action. Similarly, in the case of the string action, the reparametrization invariant term we can add to represent the B_{ij} field coupling to the fundamental string is given by

$$\int B_{ij} dx^i \wedge dx^j.$$

The fact that B_{ij} is antisymmetric is crucial to obtain a reparametrization invariant action. If we now consider the RR sector, we find that the fields in this sector couple to D-branes. Indeed, it is the number of indices in our field which tell us what type of higher dimensional object we can couple to. In the case of D_{ijkl}, the term we add to the action is of the form

$$\int D_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l. \tag{1.3}$$

The key point in obtaining a reparametrization invariant action is that D_{ijkl} is totally antisymmetric. It is clear that the term in (1.3) couples to a D3-brane, which has a 3+1-dimensional worldvolume. In addition to the electric couplings discussed above, we can also have magnetic couplings to the potential. We can obtain a 3-form field strength H_3 from the NS-NS B_{ij} field by writing

$$H_3 = dB_2.$$

Further, we can get a 7-form dual field strength by contracting H_3 with the 10-dimensional ϵ tensor, so that

$$H_7 =^* H_3,$$

but we can also get it by using a 6-form potential

$$H_7 = dA_6.$$

This 6-form potential will couple to something called a NS5 brane, which has a 5+1-dimensional worldvolume. Fundamental strings in 10 dimensions are related to NS5 branes via electromagnetic duality. A similar argument shows that D1-branes are related to D5-branes by electromagnetic duality and that D3-branes are self dual.

1.2 The $AdS_5 \times S^5$ Background

This background is vital in the duality, as this is the gravitational background in which the strings move on the string theory side of the correspondence. It is thus instructive to discuss this background in more detail, especially the 5-dimensional anti-de Sitter space AdS_5. We will examine the nature of this space and also that of its boundary.

We define the d-dimensional space AdS_d by the hyperboloid

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + ... + X_{d-1}^2 = -\Lambda^2$$

embedded in a $d+1$-dimensional space with metric given by

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + ... + dX_{d-1}^2.$$

It appears as though the embedding space has two time coordinates. We can rewrite the metric of AdS_d in terms of global coordinates μ and t, where $0 \leq \mu \leq \infty$ and $0 \leq t < 2\pi$ (so that time is periodic). We do this by noting that

$$\cosh^2 \mu - \sinh^2 \mu = \frac{(e^{\mu} + e^{-\mu})^2}{4} - \frac{(e^{\mu} - e^{-\mu})^2}{4} = 1.$$

We can now set

$$(X_{-1}, X_0) = \Lambda \cosh \mu \hat{n}_2 \equiv \vec{x},$$

$$(X_1, X_2, ..., X_{d-1}) = \Lambda \sinh \mu \hat{n}_{d-1} \equiv \vec{y},$$

where $\hat{n}_k$ is a k-dimensional unit vector. We use the fact that

$$\hat{n}_2 = (\sin t \cos t), \quad \hat{n}_3 = (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta), \quad etc.$$

to verify that the association above is valid, since

$$\begin{aligned}
&-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + ... + X_{d-1}^2 \\
&= -\Lambda^2 \cosh^2 \mu \hat{n}_2 \cdot \hat{n}_2 + \Lambda^2 \sinh^2 \mu \hat{n}_{d-1} \cdot \hat{n}_{d-1} \\
&= -\Lambda^2 (\cosh^2 \mu - \sinh^2 \mu) = -\Lambda^2.
\end{aligned}$$

Now, using $\hat{n}_2 \cdot \hat{n}_2 = 1$ we can further deduce that

$$d\hat{n}_2 \cdot d\hat{n}_2 = (\cos t, -\sin t) \cdot (\cos t, -\sin t) dt^2 = dt^2,$$

$$\hat{n}_2 \cdot d\hat{n}_2 = (\sin t, \cos t) \cdot (\cos t, -\sin t) dt = 0,$$

which leads us to

$$\begin{aligned}
d\vec{x} \cdot d\vec{x} &= (\Lambda \sinh \mu \hat{n}_2 d\mu + \Lambda \cosh \mu d\hat{n}_2)^2 \\
&= \Lambda^2 \sinh^2 \mu \hat{n}_2 \cdot \hat{n}_2 d\mu^2 + \Lambda^2 \cosh^2 \mu d\hat{n}_2 \cdot d\hat{n}_2 \\
&\quad + 2\Lambda^2 \cosh \mu \sinh \mu \hat{n}_2 \cdot d\hat{n}_2 d\mu. \\
&= \Lambda^2 \cosh^2 \mu dt^2 + \Lambda^2 \sinh^2 \mu d\mu^2.
\end{aligned}$$

Similarly, using

$$\hat{n}_{d-1} \cdot \hat{n}_{d-1} = 1,$$

$$\hat{n}_{d-1} \cdot d\hat{n}_{d-1} = 0,$$

$$d\hat{n}_{d-1} \cdot d\hat{n}_{d-1} = d\Omega_{d-2}^2,$$

where $d\Omega_{d-2}^2$ is the metric of S^{d-2}, we can see that

$$d\vec{y} \cdot d\vec{y} = \Lambda^2 \cosh^2 \mu d\mu^2 + \Lambda^2 \sinh^2 \mu (d\Omega_{d-2})^2.$$

These results yield the metric in global coordinates:

$$\begin{aligned} ds_{AdS_d}^2 &= -d\vec{x} \cdot d\vec{x} + d\vec{y} \cdot d\vec{y} \\ &= -\Lambda^2 \cosh^2 \mu dt^2 + \Lambda^2 d\mu^2 + \Lambda^2 \sinh^2 \mu (d\Omega_{d-2})^2. \end{aligned}$$

We can easily elucidate the topology of AdS_d using the coordinate transformation

$$\sinh \mu = \tan \rho, \qquad 0 \le \rho \le \frac{\pi}{2},$$

from which we can write

$$\cosh \mu = \sqrt{1 + \sinh^2 \mu} = \sqrt{\frac{\cos^2 \rho}{\cos^2 \rho} + \frac{\sin^2 \rho}{\cos^2 \rho}} = \sec \rho,$$

$$d\mu = \frac{d\mu}{d\rho} d\rho = \frac{1}{\cosh \mu} \frac{d\rho}{\cos^2 \rho} = \sec \rho d\rho.$$

Using this change of coordinates the metric of AdS_d becomes

$$\begin{aligned} ds_{AdS_d}^2 &= \Lambda^2 [-\sec^2 \rho dt^2 + \sec^2 \rho d\rho^2 + \tan^2 \rho d\Omega_{d-2}^2] \\ &= -\tilde{R}^2 dt^2 + dR^2 + R^2 d\Omega_{d-2}^2, \end{aligned}$$

where

$$\tilde{R} = \Lambda \sec \rho, \qquad R = \Lambda \tan \rho.$$

Notice that the first term ($\tilde{R}^2 dt^2$) in the metric above corresponds to that of a line, while the second term ($dR^2 + R^2 d\Omega_{d-2}^2$) is the metric of a $d-1$-dimensional ball B_{d-1}. Thus, AdS_d has a topology given by $B_{d-1} \times \Re$. We can also determine the topology of the boundary of AdS_d. We do this by writing the metric as

$$ds^2_{AdS_d} = \frac{\Lambda^2}{\cos^2 \rho}[-dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2_{d-2}],$$

from which we see that the metric blows up at $\rho = \frac{\pi}{2}$. The boundary is thus present at $\rho = \frac{\pi}{2}$. If we perform a conformal transformation, which leaves the topology unchanged, we obtain

$$d\hat{s}^2_{AdS_d} = -dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2_{d-2}$$

so that at the boundary, the metric looks like

$$ds^2 = -dt^2 + d\Omega^2_{d-2}.$$

We have thus shown that the topology of the boundary is $\Re \times S^{d-2}$.

1.3 Lunin-Maldacena Backgrounds

It will be instructive to describe the LM background as the gravitational part of the AdS/CFT correspondence tested in this thesis is just such a background [4].

Lunin-Maldacena backgrounds correspond to the gravitational backgrounds dual to β-deformed $\mathcal{N} = 4$ super Yang-Mills theory. In addition to $U(1)_{\mathcal{R}}$ $\mathcal{R}$-symmetry they exhibit a $U(1) \times U(1)$ isometry and have non-zero NS-NS B fields. Lunin and Maldacena introduced solution generating transformations (those which generate new solutions out of old ones) which yield such backgrounds, hence the nomenclature.

It was previously mentioned that an important insight lead to the understanding that the five different superstring theories are actually all special cases of one underlying theory. The situation being studied dictates which local description is most viable. In fact, it is quite possible that theories existing in a different number of spacetime dimensions can each be a partial description of the same underlying theory. Indeed, some situations are presented most clearly in the little understood 11-dimensional M-theory.

Now, as mentioned before, low energy superstring theories produce supergravity. We get 11-dimensional supergravity by taking a low energy limit of M-theory. Similarly, we get type IIA and IIB supergravity theories by taking the low energy limit of the type IIA and IIB superstring theories respectively. If we compactify 11-dimensional supergravity we are lead to the lower dimensional IIA and IIB supergravities: in fact, conserved charges in IIA and IIB supergravity correspond to momenta along the compactified dimensions of the 11-dimensional theory. These momenta will change under a symmetry of the higher dimensional theory. Due to its link to the 10-dimensional theory, the effect on the momenta will translate into an effect on the conserved charges which label the solution, leading to a new solution. Hence, they are solution generating transformations.

Lunin and Maldacena followed this procedure which allowed them to construct the relevant background. They used the fact that solutions of 11-dimensional supergravity compactified on a three torus (which has an $SL(3,R)$ symmetry) can be related to solutions of 10-dimensional type IIB supergravity compactified on a two torus. The $SL(3,R)$ symmetry then becomes the precise solution generating transformation for IIB supergravity. Now, the S^5 in $AdS_5 \times S^5$ contains a two torus and the solution generating transformation can thus be applied, yielding the LM solution. This solution is given explicitly in Chapter 4.

It turns out that the LM background is set up by a collection of sources, including D3-branes, which act as a source of the five form flux present in the description. The field theory dual to closed strings in such a background should be obtainable via a low energy limit of the open strings attached to these D3-branes. Such a field theory must be conformal due to the fact that AdS_5 space has an $SO(2,4)$ isometry, and in addition has $\mathcal{N}=1$ supersymmetry. Our discussion in section 2.4 will deal with such theories. We also expect the field theory to be non-commutative due to the presence of a B field. As discussed in section 2.5, the usual field theory product is replaced by

$$f(X) * g(X) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial \xi^i}\frac{\partial}{\partial \zeta^j}} f(X+\xi)g(X+\zeta)|_{\xi=\zeta=0} \tag{1.4}$$

when the B fluxes are parallel to the 3-brane, and $\frac{\partial}{\partial \xi^i}$ correspond to momenta. In this case, however, the B fluxes are transverse to the 3-branes, and so we replace momenta in (1.4) above by $U(1) \times U(1)$ charges (Q^1, Q^2). The product is then conjectured to be

$$f * g \equiv e^{i\pi\gamma(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} fg,$$

where fg is the usual product.

Chapter 2

Field Theory Basics

2.1 Nonabelian Gauge Theory

We begin our discussion of nonabelian gauge theory by reviewing global transformations, and then consider the implications arising from a local transformation [2]. C.N. Yang and R. Mills investigated the theory, giving rise to Yang-Mills gauge theory. We label our N-component complex scalar field by $\varphi(x)$, where

$$\varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ . \\ . \\ . \\ \varphi_N(x) \end{bmatrix}.$$

With U an element of $SU(N)$, the scalar field and its adjoint transform as

$$\varphi(x) \to U\varphi(x),$$
$$\varphi^\dagger(x) \to \varphi^\dagger(x)U^\dagger.$$

U is independent of spacetime, making the above a global transformation. We are particularly interested in what happens to the Lagrangian

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi - V(\varphi^\dagger\varphi)$$

under the transformation. Since U is unitary ($U^\dagger U = 1$) we can easily show that $\varphi^\dagger\varphi$ and $\partial\varphi^\dagger\partial\varphi$ are unchanged by the transformation:

$$\varphi^\dagger\varphi \to (\varphi^\dagger U^\dagger)(U\varphi) = \varphi^\dagger(U^\dagger U)\varphi = \varphi^\dagger\varphi,$$

$$\partial\varphi^\dagger\partial\varphi \to \partial(\varphi^\dagger U^\dagger)\partial(U\varphi) = \partial\varphi^\dagger(U^\dagger U)\partial\varphi = \partial\varphi^\dagger\partial\varphi.$$

It is thus clear that the Lagrangian is invariant under $SU(N)$ for any U.

Next, we can consider what happens when the transformation U is allowed to depend on spacetime, making it a local or gauge transformation. If we set $U = U(x)$, $\varphi^\dagger\varphi$ remains invariant under the transformation

$$\varphi \to U(x)\varphi, \quad \varphi^\dagger \to \varphi^\dagger U^\dagger(x), \tag{2.1}$$

since

$$\varphi^\dagger\varphi \to \varphi^\dagger U^\dagger(x)U(x)\varphi = \varphi^\dagger\varphi.$$

However, $\partial\varphi^\dagger\partial\varphi$ is no longer unaffected by the transformation, since $\partial_\mu = \frac{\partial}{\partial x^\mu}$ will act on $U(x)$ in addition to $\varphi(x)$. In fact the transformation (2.1) results in

$$\partial_\mu\varphi \to \partial_\mu(U(x)\varphi) = U\partial_\mu\varphi + (\partial_\mu U)\varphi = U[\partial_\mu\varphi + (U^\dagger\partial_\mu U)\varphi]. \tag{2.2}$$

The first term in (2.2) is precisely how we would like $\partial_\mu\varphi$ to transform to make the Lagrangian invariant, so we need to find some way to remove the term $(U^\dagger\partial_\mu U)\varphi$. In fact, we would like a derivative of φ to transform like φ, or like $\partial_\mu\varphi$ when U does not depend on x, which would ensure our Lagrangian was invariant. A Lagrangian invariant under a gauge transformation is said to be gauge invariant. We can define a covariant derivative D_μ, which acts on φ as follows:

$$D_\mu\varphi(x) = \partial_\mu\varphi(x) - iA_\mu(x)\varphi(x),$$

where $A_\mu(x)$ is a gauge potential, and require that this derivative transforms like φ, namely

$$D_\mu\varphi(x) \to U(x)D_\mu\varphi(x)$$

$$\Rightarrow [D^\mu \varphi(x)]^\dagger D_\mu \varphi(x) \to [D^\mu \varphi(x)]^\dagger D_\mu \varphi(x).$$

Thus, we can use $D_\mu \varphi(x)$ to build an invariant kinetic energy term for the scalar field φ.

Now, requiring that $D_\mu \varphi(x) \to U(x) D_\mu \varphi(x)$ leads us to a transformation law for the nonabelian gauge potential A_μ. This transformation law, termed the nonabelian gauge transformation, is

$$A_\mu \to U A_\mu U^\dagger - i(\partial_\mu U)U^\dagger = U A_\mu U^\dagger + iU\partial_\mu U^\dagger.$$

The A_μ are $N \times N$ matrices which are hermitean. We see this by examining the term $-i(\partial_\mu U)U^\dagger$. If we call T^a the generators of $SU(N)$, we can write

$$U = e^{i\theta^a(x)T^a},$$

so that

$$-i(\partial_\mu U)U^\dagger = \partial_\mu \theta^a T^a U U^\dagger = \partial_\mu \theta^a T^a.$$

This term is clearly in the Lie algebra. A_μ is thus Lie algebra valued, and hence hermitean.

Furthermore, if we write $U = e^{i\theta \cdot T} \simeq 1 + i\theta \cdot T$, we obtain, infinitesimally,

$$A_\mu \to A_\mu + i\theta^a [T^a, A_\mu] + \partial_\mu \theta^a T^a. \tag{2.3}$$

By tracing (2.3), we note that $Tr(A_\mu)$ is invariant under the transformation, so we take A_μ to be traceless. This allows us to decompose the matrix field A_μ into component fields A_μ^a, and write

$$A_\mu = A_\mu^a T^a.$$

The number of these component fields matches the number of generators in the group ($N^2 - 1$ for $SU(N)$).

The Lie algebra of the $SU(N)$ generators is

$$[T^a, T^b] = if^{abc}T^c,$$

where f^{abc} are the structure constants. This leads to

$$A^a_\mu \to A^a_\mu - f^{abc}\theta^b A^c_\mu + \partial_\mu \theta^a,$$

so that A^a_μ transforms as the adjoint representation of the group if θ is independent of x.

As a final comment on the nature of A_μ, we will justify its nomenclature. In the case of the abelian group $U(1)$, with element $U(x) = e^{i\theta(x)}$, A_μ turns out to be the standard abelian gauge potential from electromagnetism. Thus, in the particular case discussed in this section, A_μ is termed the nonabelian gauge potential.

According to our arguments above, the gauge invariant Lagrangian is

$$\mathcal{L} = (D^\mu \varphi)^\dagger (D_\mu \varphi) - V(\varphi^\dagger \varphi).$$

By introducing the covariant derivative, we have introduced an additional field $A_\mu(x)$ into the theory, and we need to include its dynamics into the Lagrangian. We do so by requiring that the quantity we introduce is gauge and Lorentz invariant and contains quadratic terms for A_μ. Thus, we need to find a field strength which depends on A_μ, such that these conditions are met. This is done succinctly using differential forms. First, we define for simplicity

$$A_\mu \equiv A^M_\mu \equiv -iA^P_\mu,$$

where M refers to mathematician notation, and P to physicist notation. This notation allows us to write the covariant derivative as

$$D_\mu = \partial_\mu + A_\mu.$$

We also introduce the matrix 1-form $A = A_\mu dx^\mu$, which is a form and a matrix in the defining representation of the Lie algebra. We can write

$$A^2 = A_\mu A_\nu dx^\mu dx^\nu = \frac{1}{2}[A_\mu, A_\nu]dx^\mu dx^\nu \neq 0$$

for a nonabelian gauge potential. The transformation of the gauge potential becomes

$$A \to UAU^\dagger + UdU^\dagger.$$

It turns out that the object we are looking for to describe the dymanics of A_μ is a 2-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$, and we want to build it out of the 1-form A. Two possibilities arise for building a 2-form out of A: we could either have the 2-form dA or the 2-form A^2, and we expect that F will be a linear combination of the two. Now, U is a 0-form, so we can write

$$dU^\dagger = \partial_\mu U^\dagger dx^\mu.$$

With the action of d, the gauge potential transformation law becomes

$$dA \to UdAU^\dagger + dUAU^\dagger - UAdU^\dagger + dUdU^\dagger,$$

while A^2 transforms like

$$\begin{aligned} A^2 &\to UA^2U^\dagger + UAdU^\dagger + UdU^\dagger UAU^\dagger + UdU^\dagger UdU^\dagger \\ &= UA^2U^\dagger + UAdU^\dagger - dUAU^\dagger - dUdU^\dagger. \end{aligned} \tag{2.4}$$

In the second line in (2.4) we used $d(UU^\dagger) = d(1) \Rightarrow UdU^\dagger = -dUU^\dagger$. So, if we now add dA and A^2, we see that the combination transforms as

$$dA + A^2 \to U(dA + A^2)U^\dagger.$$

The quantity $dA + A^2$ obviously transforms homogeneously, and we can thus define the field strength 2-form

$$F = dA + A^2. \tag{2.5}$$

Now that we have the quantity we were looking for, we can return to our more explicit notation, and write (2.5) as

$$F = (\partial_\mu A_\nu + A_\mu A_\nu)dx^\mu dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])dx^\mu dx^\nu.$$

If we further define

$$F \equiv \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu,$$

we can write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Recall that we used $A_\mu \equiv A_\mu^M = -iA_\mu^P$, and similarly we can write $F_{\mu\nu} \equiv F_{\mu\nu}^M = -iF_{\mu\nu}^P$. Thus, in physicist notation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

If we further extend our results to include both group and Lorentz indices, using

$$A_\mu = A_\mu^a T^a, \quad F_{\mu\nu} = F_{\mu\nu}^a T^a,$$

we get

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c.$$

It turns out that the quantity we need to include in the Lagrangian to account for the dynamics of the field $A_\mu(x)$ is proportional to $TrF_{\mu\nu}F^{\mu\nu}$. More precisely, the Yang-Mills (or nonabelian) Lagrangian is

$$\mathcal{L} = -\frac{1}{2g^2}TrF_{\mu\nu}F^{\mu\nu}.$$

Furthermore, since we normalize T^a by $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}.$$

This Lagrangian contains a quadratic term $(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$, which corresponds to the propagation of the Yang-Mills boson (or the nonabelian gauge boson), which is a massless vector boson carrying an internal index a. In addition the Lagrangian has a cubic term $f^{abc} A^{b\mu} A^{c\nu} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)$ and a quartic term $(f^{abc} A^b_\mu A^c_\nu)^2$, which describe the self-interaction of the nonabelian gauge boson. This self-interaction arises because the Yang-Mills bosons couple to all fields which transform nontrivially under the gauge group, but as we have shown the Yang-Mills bosons have a nontrivial transformation, and so couple to themselves. Recall that in Maxwell theory a field's charge reveals how the field transforms under the $U(1)$ gauge group, so that the analogue of the charge of a nonabelian gauge field is the representation of the field. Furthermore, in this simpler theory, the photon which is itself uncharged will couple to charged fields. The presence of this self-interaction in pure Yang-Mills theory makes it a highly nontrivial theory. The self-interactions and their relative strengths are determined by symmetry, since group theory completely fixes the structure constants.

Yang-Mills theory has an elegant formulation; this elegant formulation makes it a non-linear theory which is, in turn, hard. As a quantum field theory, we have no reliable approximation techniques to study its low energy dynamics.

2.2 Supersymmetry Basics

Supersymmetry (SUSY) is intriguing because it is a basic transformation linking bosons and fermions while all other experimentally verifiable symmetries link the same kind of particles to themselves. It is also tempting to think that we might be able to solve the cosmological constant problem by getting the boson and fermion contributions to cancel each other since the fermions have a negative contribution to the vacuum energy. Supersymmetry would guide us in doing this. For all the good it would do, it is unfortunately still experimentally unverified, and no supersymmetric partners in the current theories with supersymmetry have been detected.

Supersymmetry requires the same number of bosonic and fermionic degrees of freedom. The simplest fermionic field one could use to build a field theory with this symmetry is the two-component Weyl spinor ψ, which introduces a complex degree of freedom, so that we have to add in a complex scalar field φ. Instead of proceeding by trial and error to build a supersymmetric theory, we will follow the superspace and superfield formalism. Firstly, however, we will discuss the supersymmetry algebra [3], and then move to a more detailed explanation of superspace [2]. Note that we now use m and n as spacetime indices, where μ and ν were used previously.

We begin our analysis by considering the S-matrix and noting its symmetries. A symmetry of the S-matrix corresponds to a transformation which serves to move between the single and multiparticle asymptotic states affected by the matrix. There are three broad groups to which the symmetries of the S-matrix belong. The S-matrix has:

(i) Poincaré invariance, which involves translations, rotations and boosts and has generators P_m and M_{mn};

(ii) Internal global symmetries (which are space-independent), where the conserved quantities are electic charge and isospin. The generators, which are unchanged by Lorentz transformations making them Lorentz scalars, satisfy a Lie algebra

$$[B_l, B_k] = iC^i_{lk}B_i,$$

where the structure constants are represented by C^i_{lk};

(iii) Discrete symmetries, namely charge conjugation (C), parity transformation (P) and time translation (T).

Some decades ago, the Coleman and Mandula theorem proved rigorously that there are no further symmetries of the S-matrix. However, weakening a condition in the theorem allows the possiblity of supersymmetry to arise. The condition imposed in the assumptions of the theorem was that only commutators were present in the S-matrix symmetry algebra. Weakening this assumption to allow the generators to satisfy anticommutation relations in addition to the standard commutation relations gives rise to this additional symmetry. A number of years after the Coleman and Mandula theorem, it was further shown that no other symmetries are possible by adding anticommutators into the mix. In more precise terms, supersymmetry results via the introduction of anticommuting symmetry generators which transform in the spinor representation (that is, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations) of the Lorentz group. Since internal symmetries are Lorentz scalars, it is clear that supersymmetry is not an internal symmetry. It is an extension of the Poincaré spacetime symmetries to include the anticommuting spinor generators, and can be further extended to more or less than four spacetime dimensions.

We will denote our supersymmetry generators using Weyl spinor notation, so that they are given by Q^{α}_{A}, where $A = 1, ..., \mathcal{N}$. $\mathcal{N}$ tells us the number of generators and α, which can be dotted or undotted, represents the spinor nature of the generators. The logical next step is to write down the supersymmetry algebra of these generators. We will focus on the 4-dimensional SUSY algebra. This algebra is given by

$$\begin{aligned}
\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} &= 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B, \\
\{Q^A_\alpha, Q^C_\beta\} &= \epsilon_{\alpha\beta} a^{lAC} B_l, \\
\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}C}\} &= -\epsilon_{\dot{\alpha}\dot{\beta}} a^*_{lAC} B^l, \\
[Q^A_\alpha, P_m] &= [\bar{Q}^A_{\dot{\alpha}}, P_m] = 0, \\
[Q^A_\alpha, M_{mn}] &= \sigma_{mn\alpha}{}^{\beta} Q^A_\beta, \\
[\bar{Q}^{\dot{\alpha}}_A, M_{mn}] &= \bar{\sigma}^{\dot{\alpha}}_{mn\dot{\beta}} \bar{Q}^{\dot{\beta}}_A, \\
[Q^A_\alpha, B_l] &= S_l^A{}_C Q^C_\alpha, \\
[\bar{Q}_{\dot{\alpha}A}, B_l] &= -S^*_{lA}{}^C \bar{Q}_{\dot{\alpha}C}, \\
[B_l, B_k] &= iC_{lk}{}^j B_j \\
[P_m, B_l] &= [M_{mn}, B_l] = 0,
\end{aligned} \tag{2.6}$$

where $\bar{Q}$ is the adjoint of Q, together with the usual Poincaré algebra. This particular form of the algebra follows almost uniquely by requiring consistency with Lorentz transformation properties of the generators. For example, consider the anticommutator between the supercharges. If we were to start from scratch and try to elucidate the form of this anticommutator, we would note that the left-hand side transforms as $(\frac{1}{2}, \frac{1}{2})$, and look for an object made up of the other possible generators (that is, P_m, M_{mn} and B_l) which transforms in the same way, to make up the right-hand side. The most general such object turns out to be

$$\sigma^m_{\alpha\dot{\beta}} P_m C^A_B,$$

where C^A_B are complex Lorentz scalar coefficients. Thus, we are lead to a possibility for the algebra, given by

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = \sigma^m_{\alpha\dot{\beta}} P_m C^A_B. \tag{2.7}$$

We can make progress if we complex conjugate the above expression. Using the fact that

$$(\sigma^m_{\alpha\dot{\beta}})^\dagger = \sigma^m_{\beta\dot{\alpha}},$$

$$(Q^A_\alpha)^\dagger = \bar{Q}^A_{\dot{\alpha}},$$

we can conclude that C^A_B is a hermitean matrix. In addition, C^A_B turns out to be positive definite, since $\{Q, \bar{Q}\}$ is positive definite. Since the coefficients are c-numbers we can pick a basis for the generators such that C^A_B is proportional to δ^A_B, since δ^A_B is itself a positive definite hermitean matrix. Adding a factor of two to our algebra in (2.7) doesn't change its transformation properties, and thus we are lead by the series of arguments above to the anticommutation relation given in (2.6).

We follow our statement of the SUSY algebra by a basic theorem involving any physical state in a supersymmetric theory. If we contract the algebra with $\bar{\sigma}^{n\dot{\beta}\alpha}$ we obtain

$$4P^n = \bar{\sigma}^{n\dot{\beta}\alpha}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}.$$

If we consider only the time component of this equation we get

$$4H = \sum_\alpha \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = \sum_\alpha \{Q_\alpha, Q^\dagger_\alpha\} = \sum_\alpha (Q_\alpha Q^\dagger_\alpha + Q^\dagger_\alpha Q_\alpha).$$

If $|S\rangle$ is any physical state in a supersymmetric theory, then the above leads us to

$$\langle S|H|S\rangle = \frac{1}{2}\sum_\alpha \sum_{S'} |\langle S'|Q_\alpha|S\rangle|^2 \geq 0.$$

This leads us to conclude that any physical state in a supersymmetric field theory must have nonnegative energy.

We can also examine more closely the relationship between the SUSY generators and the other generators, most notably the translation generators P_m. Again, constrained by the need for the left hand side (corresponding to the commutator) to transform correctly under the Lorentz group (and noting that there are no $(\frac{1}{2}, 1)$ and $(1, \frac{1}{2})$ symmetry generators), we can write

$$\begin{aligned} [Q^A_\alpha, P_m] &= Z^A_B \sigma_{\alpha\dot{\beta}m} \bar{Q}^{\dot{\beta}B}, \\ [\bar{Q}^{\dot{\alpha}A}, P_m] &= (Z^A_B)^* Q^B_\beta \bar{\sigma}^{\dot{\alpha}\beta}_m, \end{aligned} \tag{2.8}$$

where Z^A_B are complex Lorentz scalar coefficients. To extract the nature of these coefficients, we insert our commutator (2.8) above into the Jacobi identity

$$[[Q^A_\alpha, P_m], P_n] + [[P_m, P_n], Q^A_\alpha] + [[P_n, Q^A_\alpha], P_m] = 0,$$

and use

$$\sigma^{mn\beta}_\alpha = \frac{i}{4}[\sigma^m_{\alpha\dot{\gamma}} \bar{\sigma}^{n\dot{\gamma}\beta} - \sigma^n_{\alpha\dot{\gamma}} \bar{\sigma}^{m\dot{\gamma}\beta}].$$

This leads to

$$-4i(ZZ^*)^A_B \sigma^\beta_{mn\alpha} Q^B_\beta = 0 \Rightarrow ZZ^* = 0.$$

This helps, but we need more information to find the form of Z^A_B itself. We get this information by considering the general anticommutator of two SUSY generators, using the fact that it must transform as $(0,0) \oplus (1,0)$ under the Lorentz group. The relation is

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} X^{AB} + \epsilon_{\beta\gamma} \sigma^{mn\gamma}_\alpha M_{mn} Y^{AB}.$$

Again, we rely on the Jacobi identity to get extra information, but this time we contract it with $\epsilon^{\alpha\beta}$:

$$[\{Q^A_\alpha, Q^B_\beta\}, P_m] + \{[P_m, Q^A_\alpha], Q^B_\beta\} - \{[Q^B_\beta, P_m], Q^A_\alpha\} = 0.$$

This identity, along with

$$[X^{AB}, P_m] = 0,$$

$$\sigma^m_{\alpha\dot{\beta}}\bar{\sigma}^{\dot{\gamma}\delta}_m = 2\delta^\delta_\alpha\delta^{\dot{\gamma}}_{\dot{\beta}},$$

$$\bar{\sigma}^{m\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta}\sigma^m_{\delta\dot{\gamma}}; \quad \sigma^m_{\alpha\dot{\beta}} = \epsilon_{\dot{\delta}\dot{\beta}}\epsilon_{\gamma\alpha}\bar{\sigma}^{m\dot{\delta}\gamma},$$

leads to

$$-4(Z^{AB} - Z^{BA})P_m = 0,$$

and hence that Z^A_B is symmetric. Using this in addition to $ZZ^* = 0$ that we got before, we can conclude that $ZZ^\dagger = 0$. Thus, $Z^A_B = 0$, and we can now write

$$[Q^A_\alpha, P_m] = [\bar{Q}^A_{\dot{\alpha}}, P_m] = 0.$$

In addition, our analysis leads us to a form for the anticommutator of two SUSY generators. The Jacobi identity leads us to conclude that

$$[M_{mn}Y^{AB}, P_m] = 0 \Rightarrow Y^{AB} = 0,$$

so that

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}X^{AB}.$$

We can further note that the complex objects X^{AB} commute with all the SUSY generators Q^A_α and their adjoints $\bar{Q}_{\dot{\alpha}A}$, making them central charges. They also generate an Abelian invariant subalgebra of the compact Lie algebra generated by B_l, so that

$$X^{AB} = a^{lAB}B_l,$$

where a^{lAB} are complex and obey the relation $S_l{}^A{}_C a^{kCB} = -a^{kAC}S^*_{lC}{}^B$. Here, we used the commutators

$$[Q^A_\alpha, B_l] = S_l{}^A{}_B Q^B_\alpha,$$

$$[\bar{Q}_{\dot{\alpha}A}, B_l] = -S^{*}_{lA}{}^{B}\bar{Q}_{\dot{\alpha}B}.$$

The simplest case we can deal with is supersymmetry with only one generator, corresponding to $\mathcal{N} = 1$ SUSY. Since this thesis deals with this type of non-maximal supersymmetry, we will state its properties. The $\mathcal{N} = 1$ supersymmetry algebra is a special case of the general algebra (2.6). For example, the anticommutator between the supercharges is

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}}P_m.$$

The cental charges X^{AB} disappear when $\mathcal{N} = 1$, so we obtain

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.$$

We also note that the coefficients S_l are real for this particular type of SUSY, and the Jacobi identity for Q and two B's tells us that the constants C^k_{ml} vanish, so the internal symmetry algebra is Abelian. Now, we know that

$$[Q_\alpha, B_l] = S_l Q_\alpha,$$

$$[\bar{Q}_{\dot{\alpha}}, B_l] = -S_l \bar{Q}_{\dot{\alpha}},$$

and if we rescale B_l we obtain

$$[Q_\alpha, B_l] = Q_\alpha,$$

$$[\bar{Q}_{\dot{\alpha}}, B_l] = -\bar{Q}_{\dot{\alpha}}.$$

We denote by $\mathcal{R}$ the single independent combination of the B_l whose commutator with Q_α and $\bar{Q}_{\dot{\alpha}}$ does not vanish. This $\mathcal{R}$ is a $U(1)$ generator and satisfies

$$[Q_\alpha, \mathcal{R}] = Q_\alpha,$$

$$[\bar{Q}_{\dot{\alpha}}, \mathcal{R}] = -\bar{Q}_{\dot{\alpha}},$$

and we call the internal global $U(1)$ symmetry of $\mathcal{N} = 1$ SUSY provided by this generator $\mathcal{R}$ symmetry. By noting that, in general, the relations are

$$[Q_\alpha, \mathcal{R}] = \lambda_\mathcal{R} Q_\alpha, \quad [\bar{Q}_{\dot{\alpha}}, \mathcal{R}] = -\lambda_\mathcal{R} \bar{Q}_{\dot{\alpha}},$$

we see that the SUSY generators have an $\mathcal{R}$-charge of $\lambda_\mathcal{R}$ =+1 and -1 respectively.

Basically, what we have shown is that a single hermitean $U(1)$ generator $\mathcal{R}$ does not commute with the supercharges due to associativity of the super-Poincaré algebra. We label the corresponding symmetry the $U(1)_\mathcal{R}$ symmetry. Due to the nature of $\mathcal{R}$, not all component fields in a multiplet transform in the same way under this $U(1)_\mathcal{R}$ symmetry.

To complete our analysis of $\mathcal{N} = 1$ SUSY, we will list the remaining commutators of the generators in this particular realm:

$$[Q_\alpha, P_m] = [\bar{Q}_{\dot{\alpha}}, P_m] = 0,$$

$$\begin{aligned}
[Q_\alpha, M_{mn}] &= \sigma_{mn\alpha}{}^{\beta} Q_\beta, \\
[\bar{Q}^{\dot{\alpha}}, M_{mn}] &= \bar{\sigma}^{\dot{\alpha}}_{mn\dot{\beta}} \bar{Q}^{\dot{\beta}}, \\
[P_m, P_n] &= 0, \\
[M_{mn}, P_p] &= i(\eta_{np} P_m - \eta_{mp} P_n), \\
[M_{mn}, M_{pq}] &= -i(\eta_{mp} M_{nq} - \eta_{mq} M_{np} - \eta_{np} M_{mq} + \eta_{nq} M_{mp}), \\
[P_m, \mathcal{R}] &= [M_{mn}, \mathcal{R}] = 0.
\end{aligned}$$

Next, we need to find the Casimir operators, which will allow us to write down the irreducible representations of supersymmetry on asymptotic single particle states. We will do this for $\mathcal{N} = 1$ SUSY. We begin by considering the Casimirs of the Poincaré algebra. These are $P^2 = P_m P^m$ and $W^2 = W_m W^m$, where P^2 is the mass operator with eigenvalues m^2 and W_m is the Pauli-Ljubanski vector

$$W_m = \frac{1}{2} \epsilon_{mnpq} P^n M^{pq},$$

such that the eigenvalues of W^2 for massive states are $-m^2 s(s+1)$, with $s = 0, \frac{1}{2}, 1, ...$ For massless states $W_m = \lambda P_m$ where λ is the helicity. Helicity is the projection of spin in the direction of motion (i.e. $\lambda = \hat{p} \cdot \vec{S}$). It takes on discrete values, and is positive for right-handed particles, and negative for left-handed particles. To find the Casimirs of $\mathcal{N}=1$ SUSY we need to find the objects which commute with Q and $\bar{Q}$. P commutes with both but M does not, hence P^2 is a Casimir of $\mathcal{N}=1$ SUSY while W^2 is not. However, we can define a quantity which contains W_m, namely C^2, where

$$B_m = W_m - \frac{1}{4}\bar{Q}_{\dot{\alpha}}\bar{\sigma}_m^{\dot{\alpha}\beta}Q_\beta,$$

$$C_{mn} = B_m P_n - B_n P_m,$$
$$C^2 = C_{mn}C^{mn}.$$

Using this definition, we can easily evaluate

$$[W_m, Q_\alpha] = -i\sigma_{mn\alpha}^{\beta} Q_\beta P^n,$$

$$[\bar{Q}_{\dot{\beta}}\bar{\sigma}_m^{\dot{\beta}\gamma}Q_\gamma, Q_\alpha] = -2P_m Q_\alpha + 4i\sigma_{nm\alpha}^{\beta} P^n Q_\beta,$$

which lead to

$$[C_{mn}, Q_\alpha] = [B_m, Q_\alpha]P_n - [B_n, Q_\alpha]P_m = 0.$$

Hence, C^2 is a Casimir of $\mathcal{N}=1$ SUSY. Now that we have established what the Casimirs are, we can construct all possible irreducible representations of $\mathcal{N}=1$ SUSY on asymptotic states. We will split our analysis into cases of massive and massless states.

2.2.1 Massive States in $\mathcal{N}=1$ SUSY

We will confine our analysis to the rest frame of massive states, where $P_m = (m, \vec{0})$. This is the simplest example we can use to give us the general form of our states. We can immediately write down the corresponding algebra:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\sigma^0_{\alpha\dot{\beta}} = 2m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.9}$$

Now, in the case of a rest frame we see that we can write our Casimir C^2 in terms of the spin operator S_i, for $i = 1,2,3$:

$$C^2 = 2m^4 J_i J^i = 2m^4 J^2, \tag{2.10}$$

where

$$J_i \equiv S_i - \frac{1}{4m}\bar{Q}\bar{\sigma}_i Q.$$

We can write the algebra of J_i by realizing that both S_i and $\bar{\sigma}_i^{\dot{\alpha}\beta}$ satisfy the algebra of $SU(2)$:

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

The eigenvalues of J^2 are $j(j+1)$, where $j = \mathcal{Z}$ or $\frac{1}{2}\mathcal{Z}$. In order to verify that (2.10) is indeed a Casimir, we need to show that J^2 is a Casimir. It turns out that

$$[Q_\alpha, J_i] \propto \vec{P}, \quad [\bar{Q}_{\dot{\alpha}}, J_i] \propto \vec{P},$$

but $\vec{P} = 0$ is the rest frame, so J_i commutes with Q and $\bar{Q}$. If we study the $\mathcal{N} = 1$ SUSY algebra in the rest frame (2.9) more closely, we see that $\bar{Q}_{\dot{\alpha}}$ and Q_α actually correspond to two pairs of (unnormalized) creation/annihilation operators. They fill out the $\mathcal{N} = 1$ massive SUSY irreducible representation (irrep) of fixed m and j. Thus, we can use these generators to define a new state given a definite state $|m, j\rangle$. This new state is

$$|\Omega\rangle = Q_1 Q_2 |m, j\rangle,$$

which satisfies $Q_1|\Omega\rangle = Q_2|\Omega\rangle = 0$, so that $|\Omega\rangle$ is a "vacuum state" (a state annihilated by the "annihilation operators" $\propto Q_\alpha$). The values of j_3 range from $-j, ..., j$, so the degeneracy of this state is $2j+1$. In addition, $|\Omega\rangle$ is an eigenstate of spin, since

$$J_i|\Omega\rangle = S_i|\Omega\rangle.$$

Hence we can write

$$|\Omega\rangle = |m, s, s_3\rangle,$$

which means that we can use mass and spin to label all the states in the SUSY irreducible representation. Instead of using $\bar{Q}$ and Q as the creation/annihilation operators directly, we normalize them, and use the result as our conventional creation/annihilation operators:

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2}, \quad a_{1,2}^\dagger = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}}.$$

Using this notation we can write down the full massive $\mathcal{N} = 1$ SUSY irrep, which contains $4(2j+1)$ states:

$$\begin{gathered}
|\Omega\rangle \\
a_1^\dagger|\Omega\rangle \\
a_2^\dagger|\Omega\rangle \\
\frac{1}{\sqrt{2}} a_1^\dagger a_2^\dagger|\Omega\rangle = -\frac{1}{\sqrt{2}} a_2^\dagger a_1^\dagger|\Omega\rangle.
\end{gathered}$$

Furthermore, we can use

$$[S_3, \begin{pmatrix} a_2^\dagger \\ a_1^\dagger \end{pmatrix}] = \frac{1}{2} \begin{pmatrix} a_2^\dagger \\ -a_1^\dagger \end{pmatrix}$$

to show that $|\Omega\rangle = |m, j, j_3\rangle$ has possible spins given by $s_3 = j_3, j_3 - \frac{1}{2}, j_3 + \frac{1}{2}, j_3$.

2.2.2 Massless States in $\mathcal{N} = 1$ SUSY

In this case, we will see that there is only one pair of creation/annihilation operators, which means that there will only be 2×2 states (the spin degeneracy of the massless particle is 2) in each massive $\mathcal{N} = 1$ SUSY irreducible representation, half the number of states available in the massive case. We will work in the light-like reference frame, such that $P_m = (E, 0, 0, E)$. The SUSY algebra in this case is

$$\{Q_{\dot{1}}, \bar{Q}_{\dot{1}}\} = 4E, \quad \{Q_{\dot{2}}, \bar{Q}_{\dot{2}}\} = 0. \tag{2.11}$$

If we define our vacuum state $|\Omega\rangle$ in this case as we did in the massive case, we notice that (2.11) has an interesting consequence:

$$\langle\Omega|Q_2 \bar{Q}_{\dot{2}}|\Omega\rangle = (\bar{Q}_{\dot{2}}|\Omega\rangle)^\dagger (\bar{Q}_{\dot{2}}|\Omega\rangle) = 0.$$

Evidently, the (unnormalized) creation operator $\bar{Q}_{\dot{2}}$ creates zero-norm states, so it equals the zero operator. Hence, the only non-trivial (unnormalized) creation/annihilation operator pair is $\bar{Q}_{\dot{1}}$ and Q_1. Again, we normalize these operators and write

$$a = \frac{1}{2\sqrt{E}}Q_1, \quad a^\dagger = \frac{1}{2\sqrt{E}}\bar{Q}_{\dot{1}}.$$

The operator $a^\dagger$ transforms as $(0, \frac{1}{2})$ under the Lorentz group, so that it acts by increasing the helicity of a state by $\frac{1}{2}$. Since $|\Omega\rangle$ has a definite helicity and is non-degenerate, we can write the two possible states in each massless $\mathcal{N} = 1$ SUSY irrep as

$$\begin{array}{ll} |\Omega\rangle & \textit{helicity} \;\; \lambda \\ a^\dagger|\Omega\rangle & \textit{helicity} \;\; \lambda + \dfrac{1}{2}. \end{array}$$

However, we still have four states with helicities $\lambda, \lambda + \frac{1}{2}, -\lambda - \frac{1}{2}$ and $-\lambda$, since the above is generally not a CPT state, so that we end up having to pair two massless SUSY irreps.

We can use the results in the $\mathcal{N} = 1$ SUSY cases above to extrapolate to cases with more supersymmetry.

2.2.3 Massless States with No Central Charges in $\mathcal{N} > 1$ SUSY

In this general case, which we will study very briefly, we deal with $\mathcal{N}$ SUSY generators, and hence $\mathcal{N}$ creation operators $a_A^\dagger$. There are $2^{\mathcal{N}}$ states in the irrep, and they look like

$$\frac{1}{\sqrt{n!}}a_{A_1}^\dagger ... a_{A_n}^\dagger|\Omega\rangle.$$

The degeneracy of the states is given by $\binom{\mathcal{N}}{n} \times 2$ (again the 2 corresponds to the spin degeneracy), and the possible helicities are $\lambda, \lambda + \frac{1}{2}, ..., \lambda + \frac{\mathcal{N}}{2}$.

2.2.4 Massive States With No Central Charges in $\mathcal{N} > 1$ SUSY

There are $2\mathcal{N}$ creation operators $(a_\alpha^A)^\dagger$ in this case, giving rise to $2^{2\mathcal{N}}(2j+1)$ states in the SUSY irrep. It is rather instructive to consider a particular example to illustrate the properties of this case. We will thus focus on the case where

$\mathcal{N} = 2$, and study the massive irreducible representation, labelled Ω_0, obtained by taking $|\Omega\rangle$ to be a spin 0 state. There are a total of 16 states in this irrep, five of spin 0, eight of spin $\frac{1}{2}$ and three of spin 1. There are five spin 0 irreps, four spin $\frac{1}{2}$ irreps, and one spin 1 irrep. A detailed breakdown of the possible states is given:

1 spin 0 state: $|\Omega\rangle$

4 spin $\frac{1}{2}$ states: $(a^A_\alpha)^\dagger|\Omega\rangle$

3 spin 1 and 3 spin 0 states: $(a^{A_1}_{\alpha_1})^\dagger(a^{A_2}_{\alpha_2})^\dagger|\Omega\rangle$

4 spin $\frac{1}{2}$ states: $(a^{A_1}_{\alpha_1})^\dagger(a^{A_2}_{\alpha_2})^\dagger(a^{A_3}_{\alpha_3})^\dagger|\Omega\rangle$

1 spin 0 state: $(a^{A_1}_{\alpha_1})^\dagger(a^{A_2}_{\alpha_2})^\dagger(a^{A_3}_{\alpha_3})^\dagger(a^{A_4}_{\alpha_4})^\dagger|\Omega\rangle$.

2.2.5 $\mathcal{N} > 1$ SUSY With Non-Zero Central Charges

The last two cases we considered both had no central charges. This section will deal with $\mathcal{N} > 1$ SUSY with central charges present. In this case we can write

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} X^{AB}, \quad \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = -\epsilon_{\dot{\alpha}\dot{\beta}} X^*_{AB},$$

as we obtained before. The central charge X^{AB} is antisymmetric, and we will adopt the convention $X^{AB} = -X_{AB}$. When central charges are nonzero, we need to rediagonalize the basis in order to interpret $\bar{Q}_{\dot{\alpha}A}$ and Q^A_α as creation/annihilation operators. Any basis can be chosen when describing the central charges as they commute with all the generators. We begin by performing a standard similarity transformation: we can write

$$X^{AB} = U^A_C \tilde{X}^{CD} (U^T)^B_D,$$

where $\tilde{X}^{CD}$ looks like

$$\begin{pmatrix} (Z_1 i\sigma^2) & 0 & \ldots & 0 \\ 0 & (Z_2 i\sigma^2) & \ldots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & \ldots & (Z_{\frac{\mathcal{N}}{2}} i\sigma^2) \end{pmatrix}$$

for $\mathcal{N}$ even, and

$$\begin{pmatrix} (Z_1 i\sigma^2) & 0 & \ldots & 0 & 0 \\ 0 & (Z_2 i\sigma^2) & \ldots & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & \ldots & (Z_{\frac{\mathcal{N}}{2}} i\sigma^2) & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

for $\mathcal{N}$ odd. The eigenvalues $Z_1, Z_2, ..., Z_{\frac{\mathcal{N}}{2}}$ are real and can be chosen to be nonnegative. We will apply the rest of this analysis to the massive states in the rest frame. In our newly defined basis, we can write

$$\{Q^{aL}_\alpha, \bar{Q}_{\dot{\beta}bM}\} = 2m\sigma^0_{\alpha\dot{\beta}}\delta^a_b\delta^L_M,$$

$$\{Q^{aL}_\alpha, Q^{bM}_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{LM}Z_M,$$
$$\{\bar{Q}_{\dot{\alpha}aL}, \bar{Q}_{\dot{\beta}bM}\} = -\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab}\delta_{LM}Z_M,$$

where $a, b = 1, 2$ and $L, M = 1, 2, ..., \frac{\mathcal{N}}{2}$ (note that the repeated M index is not summed). The following $2\mathcal{N}$ pairs of operators turn out to be the creation/annihilation operators:

$$a^L_\alpha = \frac{1}{\sqrt{2}}[Q^{1L}_\alpha + \epsilon_{\alpha\beta}\bar{Q}_{\dot{\gamma}2L}\bar{\sigma}^{0\dot{\gamma}\beta}],$$

$$(a^L_\alpha)^\dagger = \frac{1}{\sqrt{2}}[\bar{Q}_{\dot{\alpha}1L} + \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{0\dot{\beta}\gamma}Q^{2L}_\gamma],$$
$$b^L_\alpha = \frac{1}{\sqrt{2}}[Q^{1L}_\alpha - \epsilon_{\alpha\beta}\bar{Q}_{\dot{\gamma}2L}\bar{\sigma}^{0\dot{\gamma}\beta}],$$
$$(b^L_\alpha)^\dagger = \frac{1}{\sqrt{2}}[\bar{Q}_{\dot{\alpha}1L} - \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{0\dot{\beta}\gamma}Q^{2L}_\gamma]. \tag{2.12}$$

The creation operators $(a^L_\alpha)^\dagger$ and $(b^L_\alpha)^\dagger$ create states of definite spin since $\bar{Q}_{\dot{\alpha}}$ and Q_α transform equivalently under spatial rotations. Using the relations (2.12) and

$$\epsilon_{\alpha\delta}\bar{\sigma}^{0\dot{\gamma}\delta}\epsilon_{\dot{\gamma}\dot{\beta}} = -\sigma^0_{\alpha\dot{\beta}},$$

$$\epsilon_{\dot{\beta}\dot{\delta}}\bar{\sigma}^{0\dot{\delta}\gamma}\epsilon_{\alpha\gamma} = \sigma^0_{\alpha\dot{\beta}},$$

we can calculate the anticommutators for these operators, which turn out to be

$$\begin{aligned} \{a^L_\alpha, (a^M_\beta)^\dagger\} &= (2m + Z_M)\sigma^0_{\alpha\dot{\beta}}\delta^L_M, \\ \{b^L_\alpha, (b^M_\beta)^\dagger\} &= (2m - Z_M)\sigma^0_{\alpha\dot{\beta}}\delta^L_M. \end{aligned} \tag{2.13}$$

2.2.6 BPS-saturated States

In this section, we will consider the possibilities which arise when we study the anticommutators in (2.13). Firstly, we note that $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ are positive definite quantities, and the Z_M are nonnegative, which leads us to conclude that $Z_M \leq 2m$ for all Z_M in a supersymmetry irreducible representation. Furthermore, we can consider what happens when $Z_M < 2m$ and $Z_M = 2m$ separately. When $Z_M < 2m$, we find that the massive irrep multiplicities match those for the case of no central charges. The case when we saturate the bound for some or all Z_M, namely when $Z_M = 2m$, is rather interesting, particularly if all the Z_M saturate the bound. In this case, all the $(b^L_\alpha)^\dagger$ operators project onto zero norm states, since

$$\{b^L_\alpha, (b^M_\beta)^\dagger\} = 0.$$

As we saw before, this results in a loss of half of the creation operators, so that the massive SUSY irrep has only $2^{\mathcal{N}}(2j+1)$ states, and not the full $2^{2\mathcal{N}}(2j+1)$. Such massive multiplets, with reduced multiplicity, are called short multiplets, and the states are called BPS-saturated states. The BPS monopoles in supersymmetric gauge theories are an example.

2.2.7 Superspace and Superfields

Having discussed the SUSY algebra and its various properties and consequences in some detail, the next step involves building actual supersymmetric field theories. In basic terms, this involves studying how the SUSY algebra will affect fields, and precisely how they will transform under this symmetry [2]. We again turn to the basic statement for $\mathcal{N} = 1$ supercharges

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}}P_m.$$

This relation says that acting with Q and then $\bar{Q}$ generates a translation P_m, but $P_m \equiv i\frac{\partial}{\partial x^m}$ and hence generates translations in x^m. Now, Q_α is Grassmann, so this argument might suggest that Q_α could generate translations in some abstract Grassmann coordinate θ^α. In the same way, the adjoint $\bar{Q}_{\dot{\beta}}$ would generate translations in $\bar{\theta}^{\dot{\beta}}$. Using this, one can define superspace as a space with bosonic and fermionic coordinates given by $\{x^m, \theta^\alpha, \bar{\theta}^{\dot{\beta}}\}$. Translations in this space would be directly representative of the supersymmetry algebra. We might expect that

$$Q_\alpha \sim \frac{\partial}{\partial \theta^\alpha}, \quad \bar{Q}_{\dot{\beta}} \sim \frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}},$$

since these supercharges should correspond to some sort of translations. This doesn't satisfy the SUSY algebra, and with some more effort we find the following quantities which do:

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m,$$

$$\bar{Q}_{\dot{\beta}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} + i\theta^\beta\sigma^m_{\beta\dot{\beta}}\partial_m.$$

The above relations actually represent a translation in the fermionic direction accompanied by a slight translation in the bosonic direction respectively.

Having defined a superspace, the next logical step is to define a field which lives in this space. Such a field is termed a superfield and is given by $\Phi(x^m, \theta^\alpha, \bar{\theta}^{\dot{\beta}})$. These fields transform as follows under an infinitesimal SUSY transformation:

$$\Phi \to \Phi' = (1 + i\zeta^\alpha Q_\alpha + i\bar{\zeta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})\Phi,$$

where ζ and $\bar{\zeta}$ are Grassmann parameters. We can define a special type of superfield by imposing a specific condition on it. This condition requires us to introduce two new quantities which anticommute with Q_α and $\bar{Q}_{\dot{\beta}}$, namely

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m,$$

$$\bar{D}_{\dot{\beta}} = -\left[\frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} + i\theta^\beta\sigma^m_{\beta\dot{\beta}}\partial_m\right].$$

The condition we impose on Φ is

$$\bar{D}_{\dot{\beta}}\Phi = 0,$$

and the significance of this arises when we note that the transform Φ' of Φ will obey the same condition, i.e. $\bar{D}_{\dot{\beta}}\Phi' = 0$. A superfield which satisfies this condition is called a chiral superfield. What implications does such a condition have for Φ? It turns out that if we define

$$y^m \equiv (x^m + i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}),$$

then a chiral superfield is a superfield $\Phi(y, \theta)$ which depends only on y and θ, since

$$\bar{D}_{\dot{\beta}} y^m = -\left[\frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} + i\theta^\beta \sigma^n_{\beta\dot{\beta}} \partial_n\right] y^m = -[-i\theta^\alpha \sigma^m_{\alpha\dot{\beta}} + i\theta^\beta \sigma^m_{\beta\dot{\beta}}] = 0.$$

We can further study the chiral superfield by expanding it in powers of θ (where θ has two components (θ^1, θ^2)) keeping y fixed. Due to the anticommuting nature of θ, the highest power of θ that can be present in the expansion is two, namely $\theta\theta$. As a result, our power series in Grassmann variables will not extend forever, but will terminate:

$$\Phi(y, \theta) = \varphi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y).$$

The objects $\varphi(y)$, $\psi(y)$ and $F(y)$, which will be elaborated on later, can be regarded as the standard series coefficients for now. Since the definition of y involves x, we can go one step further by Taylor expanding the existing expansion about x, which gives

$$\begin{aligned} \Phi(y, \theta) = {} & \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^m\bar{\theta}\partial_m\varphi(x) \\ & - \frac{1}{2}\theta\sigma^m\bar{\theta}\theta\sigma^n\bar{\theta}\partial_m\partial_n\varphi(x) + \sqrt{2}\theta i\theta\sigma^m\bar{\theta}\partial_m\psi(x). \end{aligned} \tag{2.14}$$

We see that ψ corresponds to a Weyl fermion field, while φ and F are two complex scalar fields. θ has an $\mathcal{R}$ charge of -1. Since Φ has a definite $\mathcal{R}$ charge, another benefit of superfields is that they efficiently allow one to determine the $\mathcal{R}$ charges of component fields.

We can impose symmetry arguments and dimensional analysis to find how the objects in (2.14) transform under the supersymmetry transformation. We find that if our superfield changes by

$$\delta\Phi = i(\zeta Q + \bar{\zeta}\bar{Q})\Phi$$

under an infinitesimal SUSY transformation, then the change in the component fields F, ψ and φ is

$$\delta F \sim \partial_m \psi^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}},$$

$$\delta\psi \sim \zeta F + \partial_m \varphi \sigma^m \bar{\zeta},$$

$$\delta\varphi \sim \zeta\psi.$$

We have omitted the overall constants, as they contain little useful information. What is important, however, is that δF looks like a total divergence, which means that $\int d^4x F$ is an invariant of supersymmetry. We can make a more general statement for the expansion of any superfield Φ, if we denote the coefficient of $\theta\theta$ in the expansion by $[\Phi]_F$. Our result tells us that $\delta([\Phi]_F)$ is a total divergence under a supersymmetry transformation, making $\int d^4x [\Phi]_F$ invariant under supersymmetry.

We have defined our superspace and the fields which live in it, and now we would like to find the supersymmetry action. To do this, we make another observation: if Φ is a chiral superfield, then so is Φ^2, Φ^3, etc. Thus, according to our discussion, the following quantity is invariant under supersymmetry:

$$\int d^4x \left[\frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3 + ...\right]_F. \tag{2.15}$$

We can find expressions for $[\Phi^2]_F$, $[\Phi^3]_F$, etc. by squaring/cubing/... our expansion (2.14) and looking for the coefficients of $\theta\theta$. We get

$$[\Phi^2]_F = (2F\varphi - \psi\psi),$$

$$[\Phi^3]_F = 3(F\varphi^2 - \varphi\psi\psi), \quad etc.$$

These expressions suggest that we have a mass term for the Weyl fermion ψ, and also for the coupling of this fermion to the scalar field φ, but a true action should contain the dymanics of these fields. Currently, the quantity (2.15) doesn't have any kinetic energy terms, which we expect to come from the form $\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{m\alpha\dot{\alpha}}\partial_m\psi_\alpha$. However, $\bar{\psi}_{\dot{\alpha}}$ does not appear in Φ, so we have to be more inventive to introduce kinetic energy terms to our action. Since this is a conjugate field of ψ, we need to involve $\Phi^\dagger$. The simplest possibility to consider is $\Phi^\dagger\Phi$, which is actually a vector superfield. A vector superfield is any superfield $V(x,\theta,\bar{\theta})$ which satisfies $V = V^\dagger$. If we now consider the expansion for a general vector superfield V in powers of θ and $\bar{\theta}$, we discover (using the properties of Grassmann variables) that the highest power is $\bar{\theta}\bar{\theta}\theta\theta$. As we did before, we denote the coefficient of $\bar{\theta}\bar{\theta}\theta\theta$ in the expansion of V by $[V]_D$. We are lead, by dimensional analysis, to conclude that under an infinitesimal supersymmetry transformation $\delta V = i(\zeta Q + \bar{\zeta}\bar{Q})V$, the change in $[V]_D$ is a total divergence. Since $\Phi^\dagger\Phi$ is an acceptable vector superfield, we have in fact shown that the action $\int d^4x[\Phi^\dagger\Phi]_D$ is invariant under supersymmetry.

If we look at our expansion (2.14) for Φ, we see that $\int d^4x[\Phi^\dagger\Phi]_D$ contains the kinetic energy terms we have been looking for:

$$\int d^4x\varphi^\dagger\partial^2\varphi,$$

$$\int d^4x\partial\varphi^\dagger\partial\varphi,$$

$$\int d^4x\bar{\psi}\bar{\sigma}^m\partial_m\psi,$$

$$\int d^4xF^\dagger F.$$

We now have enough information to write the supersymmetric action, given a superfield Φ:

$$S = \int d^4x\{[\Phi^\dagger\Phi]_D + ([f(\Phi)]_F + h.c.)\},$$

where $f(\Phi)$ is any polynomial in Φ and we can show that $[f(\Phi)]_F = F[df(\varphi)/d\varphi]$+terms not involving F. If we choose

$$f(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3,$$

we can write an explicit example of the action, which is

$$S = \int d^4x\{\partial\varphi^\dagger\partial\varphi + i\bar{\psi}\bar{\sigma}^m\partial_m\psi + F^\dagger F$$

$$- (mF\varphi - \frac{1}{2}m\psi\psi + gF\varphi^2 - g\varphi\psi\psi + h.c.)\}.$$

This particular choice for the superpotential f corresponds to the Wess-Zumino model.

We can expand our analysis of the field F by noting that it does not propagate like a dynamical field, but is instead an auxiliary field and can be integrated out in the path integral $\int DF^\dagger DFe^{iS}$. We see this by collecting all the terms in S that depend on F. We get

$$F^\dagger F - F(m\varphi + g\varphi^2) - F^\dagger(m\varphi^\dagger + g\varphi^{\dagger 2}) = |F - (m\varphi + g\varphi^2)^\dagger|^2 - |m\varphi + g\varphi^2|^2.$$

If we now integrate over F and $F^\dagger$, we get

$$S = \int d^4x \left[\partial\varphi^\dagger\partial\varphi + i\bar{\psi}\bar{\sigma}^m\partial_m\psi - |m\varphi + g\varphi^2|^2 + \left(\frac{1}{2}m\psi\psi - g\varphi\psi\psi + h.c.\right)\right].$$

At the end of section 2.4 we talk about a superpotential, so it will be instructive to expand slightly on the nature of this object [4]. At low energy, the terms with the fewest derivatives will dominate the action. The superpotential, which is the leading term, can be written as an integral over one half of superspace. A contribution to the superpotential could be

$$\int d^4xd^2\theta_L V(\Phi), \tag{2.16}$$

if we restrict our attention to left chiral superfields. Note that Φ depends on θ_L and Φ^* on θ_R. The superpotential is said to be holomorphic because it depends on Φ but not on its conjugate Φ^*. It is crucial for the superpotential to be holomorphic: should V depend on Φ^*, integrating it over the left half of superspace will lead to terms which cannot be present in the action (i.e. we will have terms with θ_R dependence, but our action should not depend on Grassmann coordinates at all). If V depends only on Φ it is analytic, which means that it is completely determined by its magnitude (overall scale) and its singularities.

The superpotential allows us to generate potential terms in the Lagrangian. The Kahler potential, given by

$$\int d^4x d^2\theta_L d^2\theta_R K(\Phi, \Phi^*),$$

is the most important correction to the superpotential. It generates the usual field theory kinetic terms and is clearly not holomorphic.

2.3 $\mathcal{N} = 4$ Super Yang-Mills Theory

Using the framework of the Yang-Mills theory, and applying supersymmetry, leads to the $\mathcal{N} = 4$ super Yang-Mills theory [6]. In Euclidean space, its Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{YM}[\mathcal{D}_\mu, \Psi_{\alpha a}, \dot{\Psi}^a_{\dot{\alpha}}, \Phi_m] = \frac{1}{4}Tr\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+\frac{1}{2}Tr\mathcal{D}^\mu\Phi^n\mathcal{D}_\mu\Phi_n-\frac{1}{4}g^2Tr[\Phi^m, \Phi^n][\Phi_m, \Phi_n] \\ + Tr\dot{\Psi}^a_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu\mathcal{D}^\mu\Psi_{\beta\alpha} - \frac{1}{2}igTr\Psi_{\alpha a}\sigma^{ab}_m\epsilon^{\alpha\beta}[\Phi^m, \Psi_{\beta b}] \\ - \frac{1}{2}igTr\dot{\Psi}^a_{\dot{\alpha}}\sigma^m_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}[\Phi_m, \dot{\Psi}^b_{\dot{\beta}}],\end{aligned}$$

where Greek letters refer to the spacetime $SO(4) = SU(2) \times SU(2)$ symmetry, so that μ, ν are spacetime vector indices ranging over four values, and $\alpha, \beta, \dot{\alpha}, \dot{\beta}$ are spinor indices which take values $1, 2$. Latin indices belong to the internal $SO(6) = SU(4)$ symmetry so that internal vector indices m, n take six values and spinor $\mathcal{R}$-symmetry indices a, b take values $1, 2, 3, 4$. There are four spinors Ψ, six scalars Φ, a gauge field $\mathcal{A}$ and a covariant derivative $\mathcal{D}$. More precisely,

$$\mathcal{D}_\mu = \partial_\mu - ig\mathcal{A}_\mu,$$

where g is a dimensionless coupling constant, and

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu].$$

This supersymmetric field theory is rich and broad and a full analysis would exceed the needs of this thesis.

2.4 Marginal Deformations of $\mathcal{N} = 4$ SYM Theory

A driving force behind the AdS/CFT duality is its potential to unravel QCD. This particular field theory has no conformal symmetry or supersymmetry, so we cannot apply the duality directly to QCD yet. We can make some strides in our understanding, though, by studying theories which are somewhat close to QCD. There are Yang-Mills theories that, unlike QCD, are conformally invariant and are supersymmetric. Since $\mathcal{N} = 4$ super Yang-Mills present in the duality is maximally supersymmetric, to get closer to QCD we could try to alter this theory by some deformation to yield a less supersymmetric theory. To still obtain a simple example, we could further require that the less supersymmetric theory is still conformally invariant. Marginal deformations do just that [4]. It is interesting to obtain the corresponding gravity duals of marginal deformations of Yang-Mills theory. This thesis does precisely this and confirms the AdS/CFT duality in this less supersymmetric regime with a spectrum calculation.

To expand on the concept of a marginal deformation, we first comment on conformal field theories, which have exact scale invariance. Such field theories are special, as this condition is non-trivial to maintain in a quantum field theory which comes with a "cutoff". Such cutoffs are dimensionful quantities necessary in quantum field theory to overcome the difficulties introduced by its many divergences. Should we have any dependence on such dimensionful quantities within the theory, it would no longer be conformally invariant.

We use non-renormalization theorems to show that a theory has exact conformal invariance, broadly by assigning charges within the problem which ultimately leads us to a number of exact results. These quantities that we can compute exactly are called effective actions. They can be used to compute low energy correlation functions in quantum field theory. We can build operators $\mathcal{O}_i$ out of fields $\{\phi_i\}$ in the field theory, and in the path integral formalism the correlators are given by

$$\langle \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n \rangle = \int \mathcal{D}\phi_i \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n e^{iS}.$$

We use a momentum cutoff κ to define our field

$$\phi(x) = \int_{p^2 < \kappa^2} (a(p) e^{-ip\cdot x} + a^*(p) e^{ip\cdot x}) dp$$

using the standard mode expansions. Note that we can separate this field into low- and high-momentum pieces, respectively ϕ_l and ϕ_h such that

$$\phi(x) = \phi_l(x) + \phi_h(x),$$
$$\phi_l(x) = \int_{p^2<\mu^2} (a(p)e^{-ip\cdot x} + a^*(p)e^{ip\cdot x})dp,$$
$$\phi_h(x) = \int_{\mu^2\leq p^2<\kappa^2} (a(p)e^{-ip\cdot x} + a^*(p)e^{ip\cdot x})dp.$$

We can eliminate the high-momentum piece from our fields if we consider only correlators of operators $\mathcal{O}_i$ made up solely of ϕ_l, so that

$$\begin{aligned}\langle \mathcal{O}_1\mathcal{O}_2...\mathcal{O}_n\rangle &= \int \mathcal{D}\phi_{i,l}\mathcal{D}\phi_{i,h}\mathcal{O}_1\mathcal{O}_2...\mathcal{O}_n e^{iS}\\ &= \int \mathcal{D}\phi_{i,l}\mathcal{O}_1\mathcal{O}_2...\mathcal{O}_n e^{iS_{eff}}.\end{aligned}$$

We can see that the effective action S_{eff} governs the low energy dynamics of the theory. Indeed, it is a special type of effective action, the so-called infrared effective action, which can sometimes be determined exactly. These are low energy effective actions obtained by taking the limit $\mu \to 0$, which amounts to retaining only the leading terms in the low energy fields. Indeed, a low energy effective action is confined to describing the degrees of freedom of a theory, below some given energy scale μ. Although the degrees of freedom at low energy and the original degrees of freedom generally differ, we simplify our analysis by only considering the cases where these degrees of freedom match. The process of moving to a low energy description and "integrating out the high energy degrees of freedom" is characterized by a change in the coefficients of terms in the Lagrangian, the so-called couplings. The change in couplings g_i as a result of a flow to lower energy is captured in the β functions, where

$$\mu\frac{\partial g_i}{\partial \mu} = \beta_i(g_k, \mu).$$

Using quantum field theory, one can compute these β functions in perturbation theory, and in a conformally invariant theory which is unaffected by changes in scale, we expect all these β functions to disappear. This is not difficult to see: in a scale invariant theory, we expect our couplings to be independent of scale, so that flowing from high to low energy should not affect g_i, making $\frac{\partial g_i}{\partial \mu} = 0$. A marginal deformation of a theory corresponds to adding a term whose coefficient has a vanishing β function to its Lagrangian. We could use the Feynman rules derived from the action S or the Feynman rules derived from the action

$$S + \int d^4 x \lambda \mathcal{O}_{marginal}.$$

Clearly, these two sets of Feynman rules differ. If $\beta_\lambda = 0$ (where β_λ is the β function corresponding to the coefficient λ) with the first set of Feynman rules, we say $\mathcal{O}$ is marginal. If $\beta_\lambda = 0$ also with the second set of Feynman rules we say $\mathcal{O}$ is exactly marginal or integrable. Along with marginal deformations, we can also have relevant and irrelevant deformations, which correspond to situations where flows to lower energy lead to a coupling which grows or becomes smaller respectively.

This project is centered around the β-deformed $\mathcal{N} = 4$ super Yang-Mills theory. The deformation makes the theory non-maximally supersymmetric - it has $\mathcal{N} = 1$ supersymmetry, and it has the superpotential

$$f = e^{i\pi\gamma} Tr(\Phi^1 \Phi^2 \Phi^3) + e^{-i\pi\gamma} Tr(\Phi^1 \Phi^3 \Phi^2).$$

The fact that the theory does not have maximal supersymmetry makes it particularly interesting. It allows us to test the AdS/CFT correspondence in this realm also, and not just in the standard $\mathcal{N} = 4$ SYM case.

2.5 Non-commutative Field Theories and NS-NS B-Fields

Now that we have explained what we mean by a β deformation of a field theory, we note that these β deformations match nicely with the deformations we use to move from standard to non-commutative field theories. These non-commutative field theories can also be obtained by considering low energy string theory with a background NS-NS B field. This section aims to show that the presence of this magnetic field results in a change of the way in which we define a product between two functions in field theory [4]. The new product is that of functions on a non-commutative space. These results will aid us in our search of the gravitational background dual to the β-deformed $\mathcal{N} = 4$ super Yang-Mills theory.

We will consider open strings interacting with a constant NS-NS B field. In order to make our analysis precise, we need to specify the dynamics of our open strings by means of an action. The strings are under the influence of a magnetic field, namely a constant NS-NS B_{ij} field, and move in a flat spacetime with metric g_{ij}. We label the string's worldsheet by σ and assume that it has Eucildean signature. We also put an extra condition on the nature of our B field: we require that B_{ij} is a matrix with an even rank r, with $r \leq (p+1)$. The sigma model is thus given by

$$
\begin{aligned}
S &= \frac{1}{4\pi\alpha'} \int_{\Sigma} (g_{ij}\partial_a X^i \partial^a X^j - 2\pi i \alpha' B_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j) \\
&= \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{ij}\partial_a X^i \partial^a X^j - \frac{i}{2}\int_{\partial\Sigma} B_{ij} X^i \partial_t X^j.
\end{aligned}
$$

The expression in the second line above requires some clarification: $\partial\Sigma$ represents the boundary of the worldsheet, and ∂_t is a tangential derivative along this boundary. To specify the motion of our strings fully, we need to specify how their endpoints move (since they are open strings and have endpoints). More precisely, we need to specify the boundary conditions for the open string endpoints attached to a Dp-brane (which we do here for i along the Dp-branes):

$$
g_{ij}\partial_n X^j + 2\pi i \alpha' B_{ij}\partial_t X^j|_{\partial\Sigma} = 0.
$$

The boundary conditions are clearly dependent on B, and are either Neumann (if $B = 0$) or Dirichlet boundary conditions (B has rank $r = p$ and $B \to \infty$). The above are boundary conditions for a general Σ. In this case, we will consider Σ a disc, so that it can be conformally mapped to the upper half plane. If we write

$$
\partial \equiv \frac{\partial}{\partial z}, \quad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}},
$$

and take $Imz \geq 0$, the boundary conditions for this case become

$$
g_{ij}(\partial - \bar{\partial})X^j + 2\pi\alpha' B_{ij}(\partial + \bar{\partial})X^j|_{z=\bar{z}} = 0.
$$

Furthermore, with the quantities

$$
\begin{aligned}
G^{ij} &= \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}_S = \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B}\right)^{ij}, \\
G_{ij} &= g_{ij} - (2\pi\alpha')^2 (Bg^{-1}B)_{ij},
\end{aligned}
$$

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}_A = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B}\right)^{ij},$$

we can write the propagator as

$$\langle X^i(z) X^j(z') \rangle = -\alpha' [g^{ij} \log|z - z'| - g^{ij} \log|z - \bar{z}'| + G^{ij} \log|z - \bar{z}'|^2 + \frac{1}{2\pi\alpha'} \theta^{ij} \log \frac{z - \bar{z}'}{\bar{z} - z'} + D^{ij}].$$

Let's make a few comments with regard to this propagator. The D^{ij} are B-dependent constants with no z or z' dependence, and can be set to any desired value. The fourth term, like the first three in the propagator, is single-valued but only if the branch cut due to the logarithm is in the lower half plane. The coefficient θ^{ij} has a rather nice intuitive interpretation, which will become clear in a bit.

When we consider the interaction of two strings, we see that we are confined to treating the boundary of the worldsheet Σ as the boundary represents the end-point history and open strings interact with eachother via the joining or splitting at the ends. In turn, we concentrate on the propagator at the boundary $\partial\Sigma$, where z and z' (denoted τ and τ' respectively):

$$\langle X^i(\tau) X^j(\tau') \rangle = -\alpha' G^{ij} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{ij} \epsilon(\tau - \tau'),$$

where $\epsilon(\tau)$ is 1 for $\tau > 0$ and -1 for $\tau < 0$. We can say something about the nature of θ^{ij} if we interpret τ as time. Indeed, conformal field theory allows us to use the short distance behaviour of operator products to compute commutators of operators, if we equate time ordering and operator ordering. In this case, with τ as time

$$[X^i(\tau), X^j(\tau)] = T(X^i(\tau) X^j(\tau^-) - X^i(\tau) X^j(\tau^+)) = i\theta^{ij},$$

where T is the time-ordering operator. According to the relation above, we see that the coordinates do not commute. Thus, the X^i are coordinates on a non-commutative space and are governed by the noncommutativity parameter θ. We mentioned at the beginning of this section that we want to consider the open string theory at low energy, and we do so now by taking $\alpha' \to 0$. This limit leads us to the propagator

$$\langle X^i(\tau)X^j(\tau')\rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau').$$

This propagator allows us to write the following for normal ordered operators:

$$: e^{ip_iX^i(\tau)} : : e^{iq_iX^i(0)} := e^{-\frac{1}{2}\theta^{ij}p_iq_j\epsilon(\tau)} : e^{ipX(\tau)+iqX(0)} : .$$

This can be extended to more general functions

$$: f(X(\tau)) : : g(X(0)) :=: e^{\frac{i}{2}\epsilon(\tau)\theta^{ij}\frac{\partial}{\partial X^i(\tau)}\frac{\partial}{\partial X^j(0)}} f(X(\tau))g(X(0)),$$

and we can also write

$$lim_{\tau\to 0^+} : f(X(\tau)) : : g(X(0)) :=: f(X(0)) * g(X(0)) : .$$

In the above, the product $*$ of functions on a non-commutative space is given by

$$f(X) * g(X) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^i}} f(X+\xi)g(X+\zeta)|_{\xi=\zeta=0},$$

which is the product which replaces the normal product between functions due to the magnetic field.

2.6 The Holographic Principle

A vastly explored problem in physics involves the unification of quantum mechanics and general relativity into a theory of quantum gravity. The holographic principle provides one approach in an attempt to resolve this problem of unification [4]. In broad terms, the holographic principle states that in a quantum theory of gravity, the number of degrees of freedom available in the system scales like the surface area of the system, and not its volume. This is reminiscent of the way a hologram works: 2-dimensional information yields a 3-dimensional result. The physics of black holes provides evidence for the principle. Indeed, the entropy of a black hole is found to be proportional to its horizon area, and not its volume. However, the principle requires that this type of scaling be a general property of gravity, and not just of black holes. It is thus important to note that horizons are not restricted to the scenario of black holes. For example, a uniformly accelerated observer in Minkowski space will see the Rindler spacetime, which has a horizon. The holographic principle will associate an entropy with this horizon.

Chapter 3

AdS/CFT Correspondence

The AdS/CFT correspondence is a statement of the equivalence between a maximally supersymmetric $SU(N)$ Yang-Mills theory in 4-dimensional Minkowski spacetime and a type IIB closed superstring theory in 10-dimensional $AdS_5 \times S^5$ space. The $\mathcal{N} = 4$ Yang-Mills theory is "maximally supersymmetric" because it has the most supersymmetry possible for a field theory. It also lacks dimensionful parameters, making it a conformal field theory (CFT). The S^5 part of the 10-dimensional $AdS_5 \times S^5$ background refers to a 5-sphere, and AdS_5 represents a non-compact anti-de Sitter space. Even more than a correspondence, this relationship between a gauge theory and a string theory can be viewed as a duality, where two different systems are used to describe the same physics. AdS/CFT also provides a concrete realization of the holographic principle, since it claims an equivalence between a theory of quantum gravity (type IIB string theory) and a non-gravitational theory living on its boundary (the boundary of $AdS_5 \times S^5$ space is $3 + 1$-dimensional Minkowski space) [1], [4].

Theoretical physics has been trying to describe strongly interacting particles, and string theory was in fact first investigated as a theory of strongly interacting hadrons since the energy dependence of angular momentum of hadronic excitations ($J \propto E^2$) emerged by studying a rotating open string. Furthermore, quark confinement can be easily "visualized" in the string picture by considering a meson (quark-antiquark pair) as the ends of an open string, which itself comprises a tube of colour flux lines. The quarks can never be pulled apart because the tension in the string joining the quarks is constant - no matter the distance between them the tension remains. QCD, an $SU(3)$ Yang-Mills theory of quarks and gluons, eventually replaced string theory as a more consistent description of strongly interacting particles. We are still hoping for a string theory description of QCD (which has no supersymmetry) to emerge. Despite the fact that $SU(N)$ gauge theory and QCD are quite different, the existence of a duality between a maximally supersymmetric gauge theory and string theory brings us one step closer to achieving a similar QCD description.

As yet unproved, the AdS/CFT correspondence has been widely and successfully tested. Two major arguments arise for the existence of the correspondence, based on symmetry and on a low energy limit. The first argument is based on counting symmetries on both sides of the correspondence, and matching the algebras that their generators satisfy. Fifteen generators give rise to the set of conformal symmetries (which are field transformations) present on the field theory side. Ten of these generators fall under spacetime translations and Lorentz transformations, four generate the special conformal transformations and one generates the scale transformation. These generators form the 4-dimensional conformal Lie algebra. We need to find the matching symmetries on the string theory side of the correspondence. We locate them by considering the AdS_5 part of the string background. It turns out that fifteen operators generate the isometries of this space, and these generators obey the same algebra as that of the conformal symmetry generators on the field theory side. In addition, the isometries of S^5 are matched in the field theory by the $\mathcal{R}$ symmetries which rotate among elements of the scalar fields and fermions. So, we see that the isometries of the $AdS_5 \times S^5$ space on the string theory side are matched to symmetries on the field theory side.

There is another rather elegant argument for the existence of the AdS/CFT correspondence which involves the low energy limit. It is based on considering a configuration of N D3-branes and noting that two descriptions of such a system arise. In the low energy limit, each description gives rise to two decoupled subsystems. One subsystem matches in both cases (specifically, this subsystem is supergravity in flat spacetime), and we expect the other two subsystems to match as well. Let's consider the arguments in more detail.

Superstring theory is not only confined to the study of strings, but includes membranes of higher internal dimensions. These Dirichlet branes (or D-branes) can be described in different ways. One such description involves viewing these branes as hyperplanes embedded in a spacetime, so that strings are allowed to end on them. It encompasses even closed string theories - strings closed in the bulk of spacetime will "unfurl" when in contact with a brane and become open strings whose endpoints are confined to the brane. More precisely, we can view these Dirichlet p-branes (Dp-branes) as $p+1$-dimensional hyperplanes in $9+1$-dimensional spacetime. For the string endpoints, there are $9-p$ string coordinates transverse to the brane, and as a result of the endpoints being confined to move along the brane these coordinates have to satisfy fixed Dirichlet boundary conditions. The remaining $p+1$ longitudinal coordinates will satisfy the free Neumann boundary conditions, since the endpoints are free to move anywhere along the brane. A Dp-brane with the abovementioned dynamics turns out to be a BPS saturated object preserving half of the bulk supersymmetries. In addition, it carries an elementary unit of charge with respect to the $p+1$ form gauge potential from the Ramond-Ramond sector of type IIB superstring theory.

Now, there are two different ways to describe a stack of N Dp-branes. The first involves considering the branes in terms of the $U(N)$ supersymmetric gauge theory on the world volume of the stack. Alternatively, we can study the p-brane background of the type IIB closed superstring theory, with its classical Ramond-Ramond charge. In the first case we have used the fact that the massless spectrum of a maximally supersymmetric $U(1)$ gauge theory in $p+1$ dimensions matches that of the massless spectrum of open strings living on a Dp-brane, making a worldvolume gauge theory realization on the Dp-branes obvious. If we now consider N parallel Dp-branes (stacked on top of eachother with zero separation, as we will consider the case where all scalar expectation values vanish), we see that N^2 different open string species are possible (their endpoints can lie on any of the N Dp-branes). But the adjoint representation of $U(N)$ has dimension N^2, leading us to the maximally supersymmetric $U(N)$ gauge theory.

In the second case, should N be very large, the stack of branes will constitute a heavy object in a closed string theory with gravity causing the space to curve. In this case, it could well be described by a metric and some background fields (like the Ramond-Ramond $p+1$ form potential). Considering the low energy limit of both of these descriptions will ultimately lead to the AdS/CFT conjecture.

When we speak of a low energy limit, we mean energies that are smaller than the string energy scale $\frac{1}{l_s}$, namely $E \ll \frac{1}{\sqrt{\alpha'}}$. Basically, it involves considering the string coupling constant g fixed and keeping all energies bounded, letting $\alpha' \to 0$. In this limit the massless $U(N)$ Yang-Mills fields will dominate on the N D3-branes under consideration, as the massive states of the open strings on the branes decouple. Also, the 10-dimensional Newton constant (which is α'-dependent) associated with the closed string fields propagating over the entire spacetime, will disappear in the limit. In addition, the interactions between the $U(N)$ fields and spacetime fields on the branes vanish. In this description we are thus left with two decoupled subsystems: decoupled closed strings on the 10-dimensional Minkowski spacetime, and the supersymmetric $U(N)$ Yang-Mills theory, from which one field decouples leaving a fully interacting $SU(N)$ Yang-Mills gauge theory.

We can now consider the N coincident D3-branes in a different way, namely in the IIB closed string theory. More precisely, we study 3-branes whose metric is

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2),$$

where r is a radial coordinate for six spatial dimensions. The D3-branes have Ramond-Ramond charge and energy, and are thus described by a nontrivial solution of the field equations for the massless fields of the theory, which includes a horizon at the end of an infinite throat. More precisely, the D3-branes stretch along x^1, x^2 and x^3 but appear as a point along the 6 transverse spatial dimensions, where the branes are surrounded by five-dimensional spheres whose volume tends to a constant as we approach the horizon in the transverse space. Now, an observer at infinity will see excitations of extremely small energy when observing finite energy excitations near the horizon of the D3-branes, because the excitations are red-shifted. So if an observer perceives excitations of very low energy, they could either be finite energy near-horizon excitations, or ones of low energy far away from the brane. These two possibilities of excitations are decoupled: the near-horizon excitations can never reach infinity while the far away excitations are almost never captured by the branes. In fact the three-branes are tiny compared to the long wavelength of the low energy supergravity modes, and hence are not detected by them, which means the two sets of modes don't interact. Once again, this description has split into two subsystems: the near-horizon region and the far away region represented by IIB supergravity on flat space.

As mentioned before, in the low energy limit we expect the two subsystems of both descriptions to match. One subsystem in each case was IIB supergravity in flat spacetime. We expect the other subsystems in each description to match as well. These are the $SU(N)$ Yang-Mills theory and the near-horizon region of the IIB background for a system of N D3-branes, which turns out to be $AdS_5 \times S^5$. More precisely, the near horizon geometry of the 3-branes corresponds to taking the limit when $r \ll L$, which leads to

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) + L^2 d\Omega_5^2,$$

where $z = \frac{L^2}{r} \gg L$. This is the metric of $AdS_5 \times S^5$. So, our low energy limit has lead us to long wavelength supergravity modes propagating in flat 10-dimensional Minkowski space, as well as all the modes of IIB string theory in the $AdS_5 \times S^5$ geometry. Thus, we conjecture that IIB string theory in the $AdS_5 \times S^5$ background is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ which is precisely the AdS/CFT conjecture.

It is somewhat surprising that such a conjecture should exist. On the one hand, the $\mathcal{N} = 4$ super-Yang Mills theory lives in $3+1$-dimensional Minkowski space, but the IIB string theory it is supposed to be equivalent to has a 10-dimensional $AdS_5 \times S^5$ background. This seems irreconcilable until one considers that the boundary of $AdS_5 \times S^5$ is $3 + 1$-dimensional Minkowski space, leading us to a connection with the holographic principle. In addition, we can consider the parameters of the theories on each side of the correspondence and study how they are related. There is a puzzle which emerges immediately. Both the gauge theory and string theory have a coupling constant controlling the size of quantum corrections. The string theory, however, has an additional parameter due to the extended nature of strings. This is the string tension, and controls these "stringy" corrections. The number of parameters in each theory doesn't seem to match. Due to a rather ingenius insight by 't Hooft however, this difficulty was overcome by realizing that there are two parameters one can use to write the Feynman diagram expansion in the field theory, namely $\frac{1}{N}$ and $\lambda = g_{YM}^2 N$, appropriately called the 't Hooft coupling.

Thus, we now see that both theories have two parameters: the $SU(N)$ Yang Mills theory has the dimensionless parameters N and g_{YM} (the coupling constant), while the IIB string theory is governed by the string coupling g_s and the radius of S^5 in units of string length R/l_s. The theories are equivalent when

$$g_s = g_{YM}^2, \qquad \frac{R^2}{l_s^2} = \sqrt{g_{YM}^2 N}.$$

At first glance, it might seem possible to test the correspondence directly by working on various quantities on both sides at weak coupling and comparing the results. Indeed the first parameter relation above equates weak coupling on both sides of the correspondence. The second relation, however, leads to a problem. Since g_{YM}^2 controls the size of quantum corrections in the field theory, we can only compute things in the field theory when g_{YM}^2 is small. Alternatively, in order to make calculations simpler on the string theory side, we require R to be large (and hence the curvature of both the AdS_5 and the S^5 small) so that we do not have large string tension corrections. Intuitively, if R is big in string units the string is much smaller than the space it moves in - it will see a smooth small-curvature space and can be treated as a particle, not an extended object. The presence of stringy corrections governed by l_s would lead to large modifications of our result, so that we can only compute things in the string theory if l_s is small, hence $\frac{R^2}{l_s^2}$ is large. The second relation above tells us that small g_{YM}^2 leads to small R, so that a calculation which is possible in field theory is intractable in string theory, and vice versa. We know that in order to make some sort of predictive calculation of the correspondence, we need small g_{YM}^2 and large R at finite N, but our relation prohibits this possibility. Even at large N, where it might seem like we can have small g_{YM} and large λ, it turns out that λ plays the role of the effective coupling constant, so again we are restricted.

In order to proceed, we need to consider quantities not directly governed by the second relation, namely quantities in the field theory which are protected (or nearly protected) from corrections in g_{YM}^2. Since such quantities are λ-independent, they can be calculated at zero coupling on the field theory side and the result compared with a corresponding string theory calculation. Even if we compute such quantities at weak coupling we would be sure to get a matching result at strong coupling, making our conjecture testable in this realm. The beauty of the AdS/CFT duality is that it allows us to evaluate intractable large-λ effects in the field theory by rather considering the 10-dimensional supergravity theory in which calculations are viable.

Due to its nontrivial nature the AdS/CFT conjecture has not yet been proved, but numerous tests can be devised to provide strong supporting evidence. One such test would be a matching of the conserved charges in the theories on either side of the correspondence. Noether's theorem requires a symmetry to identify each conserved charge. The field theory exhibits invariance under an $SU(4)$ $\mathcal{R}$ symmetry and an $SO(2,4)$ conformal symmetry, and states in the theory are labelled by the $SO(2) \times SO(4)$ and the $\mathcal{R}$ symmetry group quantum numbers, where $SO(2) \times SO(4)$ is the maximal subgroup of the conformal group. We can use the isometries of the background to classify supergravity states, but the field theory symmetries and the isometries of $AdS_5 \times S^5$ match exactly since $SU(4)$ and $SO(6)$ are locally equivalent. This gives us a clear map between the field theory and supergravity states.

The main part of this thesis is centered on the $SO(2)$ symmetry, whose quantum number on the field theory side is the conformal dimension Δ of the operator. Now, we know that $\mathcal{N} = 4$ SYM is conformally invariant, which leads to a very specific form for the two point correlator of an operator

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \propto \frac{1}{|x-y|^{\Delta}}.$$

The AdS/CFT conjecture proposes a full link between the dynamics of the participating theories, and in this case it equates the spectrum of conformal dimensions to the spectrum of string energies. This is precisely the link tested in this thesis.

Part II

Deformed PP-Waves from the Lunin-Maldacena Background

Chapter 4

Analysis on the Gravity Side of the Correspondence

4.1 PP-wave Limit of the Lunin-Maldacena Geometry

The first part of our analysis [5] involves taking the pp-wave limit of the Lunin-Maldacena background, whose metric is given by [11]

$$ds^2 = R^2(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \sum_i d\mu_i^2$$

$$+ G\sum_{i=1}^{3}\mu_i^2 d\phi_i^2 + \gamma^2\mu_1^2\mu_2^2\mu_3^2 G(\sum_i d\phi_i)^2), \tag{4.1}$$

where

$$\mu_1 = \cos\alpha, \quad \mu_2 = \sin\alpha\cos\theta, \quad \mu_3 = \sin\alpha\sin\theta,$$

and

$$G = \frac{1}{1+\gamma^2(\mu_1^2\mu_2^2 + \mu_1^2\mu_3^2 + \mu_3^2\mu_2^2)}.$$

The B field of the relevant string sigma model which corresponds to this background is

$$B_{\varphi_1\varphi_2}\mathcal{D}\varphi_1 \wedge \mathcal{D}\varphi_2 = G\gamma g R^2 \mathcal{D}\varphi_1 \wedge \mathcal{D}\varphi_2,$$

where

$$\mathcal{D}\varphi_1 = d\varphi_1 - d\psi + \frac{3\mu_1^2\mu_2^2}{g}d\psi, \qquad \mathcal{D}\varphi_2 = d\varphi_2 - d\psi + \frac{3\mu_3^2\mu_2^2}{g}d\psi.$$

Here we have used the angles ψ, φ_1 and φ_2; they serve to make our computations look tidier, and are defined by

$$\phi_1 = \psi - \varphi_2, \quad \phi_2 = \psi + \varphi_1 + \varphi_2, \quad \phi_3 = \psi - \varphi_1.$$

Since we ultimately want to obtain the spectrum of free strings in this background, we need to consider only the metric and the B field, as this is sufficient when writing down the relevant string sigma model. As a result we will not work with the RR-fluxes C_2 and C_4, despite the fact that they are also non-zero in this background. We consider the pp-wave limit of the metric, and then focus on the B field. We will begin by describing how we obtained the LM metric defined above, and how the pp-wave limit is taken, and then move on to a detailed analysis of how we performed the limit.

4.2 The Lunin-Maldacena Background

We start our analysis by stating the equation for an S^5 of radius 1:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = 1.$$

This equation can be rewritten by noting that

$$\begin{aligned}(x^1)^2 + (x^4)^2 &= (\mu_1)^2,\\ (x^2)^2 + (x^5)^2 &= (\mu_2)^2,\\ (x^3)^2 + (x^6)^2 &= (\mu_3)^2,\\ (\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2 &= 1.\end{aligned}$$

From our equation for S^5 we can write down the metric, using the fact that the coordinates (x^1, x^4) are described by radius μ_1 and angle ϕ_1, (x^2, x^5) are described by radius μ_2 and angle ϕ_2 and (x^3, x^6) are described by radius μ_3 and angle ϕ_3. This metric becomes

$$\begin{aligned} ds^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2 + (dx^6)^2 \\ &= (d\mu_1)^2 + (\mu_1)^2(d\phi_1)^2 + (d\mu_2)^2 + (\mu_2)^2(d\phi_2)^2 + (d\mu_3)^2 + (\mu_3)^2(d\phi_3)^2. \end{aligned}$$

Since the μ_i's square to 1 and the angles ϕ_i take values from 0 to 2π and are independent, we may parametrize the μ_i's as

$$\mu_1 = \cos\alpha, \quad \mu_2 = \sin\alpha\cos\theta, \quad \mu_3 = \sin\alpha\sin\theta,$$

and use this to write, for $0 \leq \alpha \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}$,

$$(d\mu_1)^2 + (d\mu_2)^2 + (d\mu_3)^2 = d\alpha^2 + \sin^2\alpha d\theta^2.$$

Putting all of this together, we obtain the following for our metric:

$$ds^2 = d\alpha^2 + \sin^2\alpha d\theta^2 + (\cos\alpha)^2(d\phi_1)^2 + (\sin\alpha\cos\theta)^2(d\phi_2)^2 + (\sin\alpha\sin\theta)^2(d\phi_3)^2.$$

Using the AdS/CFT correspondence we may identify the scalar components of the chiral superfields Φ^1, Φ^2 and Φ^3 with the spacetime coordinates $(x^1, x^2, x^3, x^4, x^5, x^6)$, using the map

$$\Phi^1 = x^1 + ix^4, \quad \Phi^2 = x^2 + ix^5, \quad \Phi^3 = x^3 + ix^6.$$

This identification between scalar fields and spacetime coordinates is arbitrary - the only thing that is fixed is the need to have one real scalar Higgs field in the adjoint of the gauge group for each spacelike direction in the string theory transverse to the brane. In addition, invariance under the following transformation leads to $U(1)$ symmetries in our metric:

$$\Phi^i \to e^{i\phi_i}\Phi^i.$$

We need to find the parameters of the $U(1) \times U(1)$ symmetry which will be used in the deformation of the field theory, and the ϕ's are not the correct parameters. As mentioned at the beginning of this section, it turns out that to get the correct parameters we transform

$$\phi_1 = \psi - \varphi_2, \quad \phi_2 = \psi + \varphi_1 + \varphi_2, \quad \phi_3 = \psi - \varphi_1.$$

This change of variables leads to

$$\begin{aligned} ds^2 = d\alpha^2 + \sin^2\alpha d\theta^2 + (\cos\alpha)^2(d\psi - d\varphi_2)^2 + (\sin\alpha\sin\theta)^2(d\psi - d\varphi_1)^2 \\ + (\sin\alpha\cos\theta)^2(d\psi + d\varphi_1 + d\varphi_2)^2. \end{aligned}$$

The next step is the crucial one as it involves the deformation. To do this deformation, we need to find the volume of the torus, i.e. τ. This torus has coordinates φ_1,φ_2, and we compute the volume at fixed ψ, α and θ. This means that any part of the metric which contains $d\psi$, $d\alpha$ or $d\theta$ is irrelevant to our calculation of τ, and we only need to consider

$$ds^2 = (\cos\alpha)^2(d\varphi_2)^2 + (\sin\alpha\cos\theta)^2(d\varphi_1 + d\varphi_2)^2 + (\sin\alpha\sin\theta)^2(d\varphi_2)^2.$$

Using the fact that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$

we can read off

$$\begin{aligned} g_{\varphi_1\varphi_1} &= \sin^2\alpha = \mu_2^2 + \mu_3^2, \\ g_{\varphi_2\varphi_2} &= \cos^2\alpha + \sin^2\alpha\cos^2\theta = \mu_1^2 + \mu_2^2, \\ g_{\varphi_1\varphi_2} &= \sin^2\alpha\cos^2\theta = \mu_2^2. \end{aligned}$$

We can now evaluate

$$\det g_{ij} = g = \sin^2\alpha(\cos^2\alpha + \sin^2\alpha\cos^2\theta\sin^2\theta) = \mu_1^2\mu_2^2 + \mu_3^2\mu_2^2 + \mu_1^2\mu_3^2,$$

and use this to calculate the volume density of the two torus

$$\sqrt{g} = \sqrt{\sin^2\alpha(\cos^2\alpha + \sin^2\alpha\cos^2\theta\sin^2\theta)}$$

$$= \sqrt{\mu_1^2\mu_2^2 + \mu_3^2\mu_2^2 + \mu_1^2\mu_3^2}.$$

So, we will scale the volume of the torus by

$$\begin{aligned}
G &= \frac{1}{1 + \gamma^2 \sin^2\alpha(\cos^2\alpha + \sin^2\alpha\cos^2\theta\sin^2\theta)} \\
&= \frac{1}{1 + \gamma^2(\mu_1^2\mu_2^2 + \mu_3^2\mu_2^2 + \mu_1^2\mu_3^2)} \\
&= \frac{1}{1 + \gamma^2 g}.
\end{aligned}$$

We need to scale the terms (which belong to the torus)

$$(\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta)\left[d\varphi_2 + \frac{-\cos^2\alpha + 2\sin^2\alpha\sin^2\theta\cos^2\theta}{\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta}d\psi\right]^2$$
$$+\sin^2\alpha[d\varphi_1 + \cos^2\theta d\varphi_2 + \cos(2\theta)d\psi]^2$$

by G, and not the terms (which do not belong to the torus)

$$d\alpha^2 + \sin^2\alpha d\theta^2 + \frac{9\cos^2\alpha\sin^2\alpha\sin^2\theta\cos^2\theta}{\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta}d\psi^2.$$

We will rather scale the following terms by G

$$(\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta)\left[d\varphi_2 + \frac{-\cos^2\alpha + 2\sin^2\alpha\sin^2\theta\cos^2\theta}{\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta}d\psi\right]^2$$
$$+\frac{9\cos^2\alpha\sin^2\alpha\sin^2\theta\cos^2\theta}{\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta}d\psi^2 + \sin^2\alpha[d\varphi_1 + \cos^2\theta d\varphi_2 + \cos(2\theta)d\psi]^2$$
$$= \sum_{i=1}^{3}\mu_i^2 d\phi_i^2,$$

and add

$$\begin{aligned}
\frac{9\cos^2\alpha\sin^2\alpha\sin^2\theta\cos^2\theta}{\cos^2\alpha + \sin^2\alpha\sin^2\theta\cos^2\theta}d\psi^2(1-G) &= \frac{\mu_1^2\mu_2^2\mu_3^2}{g}(\sum_i d\phi_i)^2(1-G) \\
&= \frac{\mu_1^2\mu_2^2\mu_3^2}{g}(\sum_i d\phi_i)^2\frac{\gamma^2 g}{1+\gamma^2 g}
\end{aligned}$$

$$= \gamma^2 \mu_1^2 \mu_2^2 \mu_3^2 G(\sum_i d\phi_i)^2.$$

Thus, due to our arguments above, the relevant part of the metric becomes

$$ds^2 = \sum_i d\mu_i^2 + G \sum_{i=1}^{3} \mu_i^2 d\phi_i^2 + \gamma^2 \mu_1^2 \mu_2^2 \mu_3^2 G(\sum_i d\phi_i)^2,$$

and the full metric is given by

$$ds^2 = R^2(-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \sum_i d\mu_i^2$$

$$+ G \sum_{i=1}^{3} \mu_i^2 d\phi_i^2 + \gamma^2 \mu_1^2 \mu_2^2 \mu_3^2 G(\sum_i d\phi_i)^2).$$

as shown at the beginning of this section.

4.3 PP-Wave Analysis

We would like to be able to test Lunin and Maldacena's conjecture that the β-deformed $\mathcal{N} = 4$ SYM is dual to string theory on the Lunin-Maldacena background, and a possibility which arises is to check that the spectrum of anomalous dimensions of operators in the dual field theory matches the mass spectrum of strings in the Lunin-Maldacena background. A problem arises because the quantization of a free string on the LM background has not been developed, so we have to settle on studying a pp-wave limit of the LM background: we know how to quantize the string in that case. Although not directly the comparison we were looking to make, it is quite strong evidence of the proposed duality.

In this section we will begin by describing how to take the pp-wave limit, and then we will show the details of such an analysis applied to the metric and the B field.

If we let u and v be lightlike coordinates and x^i transverse spacelike coordinates, then the pp-wave limit involves rescaling

$$u \to u, \quad v \to \Omega^2 v, \quad x^i \to \Omega x^i, \quad g_{\mu\nu} \to \Omega^{-2} g_{\mu\nu},$$

and taking $\Omega \to 0$. There is a particularly physical and useful way to think of the pp-wave limit. Notice that it corresponds to making a boost

$$u \to \Omega^{-1} u, \quad v \to \Omega v, \quad x^i \to x^i,$$

and following this by a change in length units

$$u \to \Omega u, \quad v \to \Omega v, \quad x^i \to \Omega x^i.$$

There are two reasons why thinking of the limit in this way is instructive:

(i) Since $[S] = kg.m^2.s^{-1}$, the scaling takes

$$S \to \Omega^2 S,$$

which means that this is not a symmetry of the action. It is, however, a symmetry of the equations of motion, and is called a solution generating transformation.

(ii) The sugra action doesn't change under boosts, which are global transformations mapping solutions into each other only if the asymptotic space is invariant under boosts; if not, these are gauge transformations and the state is left invariant.

In what we have defined above, γ is the deformation parameter. One can choose any null geodesic to perform the pp-wave limit; we will do it using the null geodesic $\tau = \psi$, with $\alpha_0 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_0 = \frac{\pi}{4}$. Furthermore, in order to perform the limit we set

$$\theta = \frac{\pi}{4} + \sqrt{\frac{2}{3}}\frac{x^1}{R}, \quad \alpha = \alpha_0 - \frac{x^2}{R}, \quad \rho = \frac{r}{R},$$

$$\varphi_1 = \frac{x'^3}{R}, \quad \varphi_2 = \frac{x'^4}{R}, \quad t = x^+ + \frac{x^-}{R^2}, \quad \psi = \frac{x^-}{R^2} - x^+,$$

$$x^3 = \sqrt{\frac{2}{3+\gamma^2}}\left(x'^3 + \frac{1}{2}x'^4\right), \quad x^4 = \sqrt{\frac{3}{2(3+\gamma^2)}}x'^4,$$

and then take $R \to \infty$. We obtain the following pp-wave metric and B field:

$$
\begin{aligned}
ds^2 &= -4dx^+dx^- - \left[r^2 + \frac{4\gamma^2}{3+\gamma^2}((x^1)^2 + (x^2)^2)\right](dx^+)^2 + dr^2 + r^2 d\Omega_3^2 \\
&\quad + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + \frac{4\sqrt{3}}{\sqrt{3+\gamma^2}}(x^1 dx^3 + x^2 dx^4)dx^+, \qquad (4.2) \\
B &= \frac{\gamma}{\sqrt{3}} dx^3 \wedge dx^4 + \frac{2\gamma}{\sqrt{3+\gamma^2}}(x^2 dx^3 \wedge dx^+ + x^1 dx^+ \wedge dx^4).
\end{aligned}
$$

We will describe the calculations that lead to these results in some detail, to make the analysis transparent, beginning with the metric:

We want to work in the vicinity of the null geodesic, so we expand

$$
\theta = \frac{\pi}{4} + \sqrt{\frac{3}{2}}\frac{x^1}{R}, \qquad \alpha = \alpha_0 - \frac{x^2}{R}.
$$

Note that x^1 and x^2 only parametrize the vicinity of the null geodesic, and we divide these by R because we want them to be lengths. So, we will use the approximations

$$
\begin{aligned}
\cos\left(\theta_0 + \sqrt{\frac{3}{2}}\frac{x^1}{R}\right) &\approx \frac{1}{\sqrt{2}}\left[1 - \sqrt{\frac{3}{2}}\frac{x^1}{R} - \frac{3}{4}\frac{(x^1)^2}{R^2}\right], \\
\sin\left(\theta_0 + \sqrt{\frac{3}{2}}\frac{x^1}{R}\right) &\approx \frac{1}{\sqrt{2}}\left[1 + \sqrt{\frac{3}{2}}\frac{x^1}{R} - \frac{3}{4}\frac{(x^1)^2}{R^2}\right], \\
\cos\left(\alpha_0 - \frac{x^2}{R}\right) &\approx \frac{1}{\sqrt{3}}\left[1 + \frac{\sqrt{2}x^2}{R} - \frac{(x^2)^2}{2R^2}\right], \\
\sin\left(\alpha_0 - \frac{x^2}{R}\right) &\approx \frac{1}{\sqrt{3}}\left[\sqrt{2} - \frac{x^2}{R} - \frac{(x^2)^2}{\sqrt{2}R^2}\right].
\end{aligned}
$$

We will also choose to expand ρ about zero

$$
\rho = \frac{r}{R},
$$

and approximate

$$\cosh(\rho) \approx 1 + \frac{r^2}{2R^2}, \quad \sinh(\rho) \approx \frac{r}{R}.$$

Finally we set

$$\varphi_1 = \frac{x'^3}{R}, \quad \varphi_2 = \frac{x'^4}{R}.$$

We will take the pp-wave limit by holding $r, u, v, x^1, x^2, x'^3, x'^4, \Omega_3$ and γ fixed and taking $R \to \infty$, so that we are indeed in the vicinity of the null geodesic and that $G = \mathcal{O}(1)$ with R^{-2} corrections. We can determine the light-cone coordinates by looking at the $\mathcal{O}(R^2)$ piece of the metric

$$R^2(-dt^2 + d\psi^2) = -4dx^+dx^-.$$

They are

$$x^+ = \frac{1}{2}(t - \psi), \quad x^- = \frac{R^2}{2}(t - \psi),$$

so that we can write

$$t = x^+ + \frac{x^-}{R^2}, \quad \psi = \frac{x^-}{R^2} - x^+.$$

Our approximations lead to

$$\mu_1 = \frac{1}{\sqrt{3}}\left[1 + \frac{\sqrt{2}x^2}{R} - \frac{(x^2)^2}{2R^2}\right],$$

$$\mu_2 = \frac{1}{\sqrt{3}}\left[1 - \left(\sqrt{\frac{3}{2}}x^1 + \frac{x^2}{\sqrt{2}}\right)\frac{1}{R} - \left(\frac{3}{2}\frac{(x^1)^2}{2} - \sqrt{\frac{3}{2}}\frac{x^1x^2}{\sqrt{2}} + \frac{(x^2)^2}{2}\right)\frac{1}{R^2}\right],$$

$$\mu_3 = \frac{1}{\sqrt{3}}\left[1 + \left(\sqrt{\frac{3}{2}}x^1 - \frac{x^2}{\sqrt{2}}\right)\frac{1}{R} - \left(\frac{3}{2}\frac{(x^1)^2}{2} + \sqrt{\frac{3}{2}}\frac{x^1x^2}{\sqrt{2}} + \frac{(x^2)^2}{2}\right)\frac{1}{R^2}\right],$$

$$g = \frac{1}{3} - \frac{6(x^1)^2 + 6(x^2)^2}{9R^2},$$

$$G = \frac{3}{3+\gamma^2} + \frac{\gamma^2}{(3+\gamma^2)^2}\frac{6(x^1)^2 + 6(x^2)^2}{R^2},$$

$$\begin{aligned}\sum_{i=1}^{3}\mu_i^2 d\phi_i^2 &= d\psi^2 - \sqrt{\frac{3}{2}}\frac{8x^1}{3R^2}dx'^3 d\psi + \left(-2\sqrt{2}x^2 - \sqrt{\frac{3}{2}}\frac{4}{3}x^1\right)\frac{dx'^4 d\psi}{R^2} \\ &+ \frac{2}{3R^2}(dx'^3)^2 + \frac{2}{3R^2}(dx'^4)^2 + \frac{2}{3R^2}dx'^3 dx'^4,\end{aligned}$$

$$\begin{aligned}G\sum_{i=1}^{3}\mu_i^2 d\phi_i^2 &= \frac{3}{3+\gamma^2}d\psi^2 + \frac{3}{3+\gamma^2}[-\sqrt{\frac{3}{2}}\frac{8x^1}{3R^2}dx'^3 d\psi + \left(-2\sqrt{2}x^2 - \frac{4}{3}\sqrt{\frac{3}{2}}x^1\right)\frac{dx'^4 d\psi}{R^2} \\ &+ \frac{2}{3R^2}(dx'^3)^2 + \frac{2}{3R^2}(dx'^4)^2 + \frac{2}{3R^2}dx'^3 dx'^4] \\ &+ \frac{\gamma^2}{(3+\gamma^2)^2}\frac{6(x^1)^2 + 6(x^2)^2}{R^2}d\psi^2,\end{aligned}$$

$$\mu_1^2\mu_2^2\mu_3^2(\sum_i d\phi_i)^2 = \left[\frac{1}{3} - \frac{2(x^1)^2}{R^2} - \frac{2(x^2)^2}{R^2}\right]d\psi^2,$$

$$G\mu_1^2\mu_2^2\mu_3^2(\sum_i d\phi_i)^2 = \frac{3}{3+\gamma^2}\left[\frac{1}{3} - \frac{2(x^1)^2}{R^2} - \frac{2(x^2)^2}{R^2}\right]d\psi^2 + \frac{d\psi^2}{3}\frac{\gamma^2}{(3+\gamma^2)^2}\frac{6(x^1)^2 + 6(x^2)^2}{R^2},$$

$$\begin{aligned}&R^2\sin^2\alpha d\theta^2 = (dx^1)^2, \\ &R^2 d\alpha^2 = (dx^2)^2, \\ &R^2 d\rho^2 = dr^2, \\ &R^2 dt^2\cosh^2\rho = R^2 dt^2 + r^2 dt^2, \\ &R^2\sinh^2\rho d\Omega_3^2 = r^2 d\Omega_3^2.\end{aligned}$$

Our last step is to take $R \to \infty$; we obtain

$$ds^2 = -4dx^+dx^- - \left[r^2 + \frac{4\gamma^2}{3(3+\gamma^2)}(3(x^1)^2 + 3(x^2)^2)\right](dx^+)^2 + dr^2 + r^2 d\Omega_3^2$$
$$+ (dx^1)^2 + (dx^2)^2 + \frac{2}{3+\gamma^2}((dx'^4)^2 + dx'^3 dx'^4 + (dx'^3)^2)$$
$$+ \frac{6}{3+\gamma^2}\left(\sqrt{\frac{3}{2}}\frac{4x^1}{3}dx'^3 + \left(\sqrt{2}x^2 + \frac{2}{3}\sqrt{\frac{3}{2}}x^1\right)dx'^4\right)dx^+.$$

We can further simplify the metric using the following

$$(dx'^4)^2 + dx'^3 dx'^4 + (dx'^3)^2 = \left(dx'^3 + \frac{1}{2}dx'^4\right)^2 + \frac{3}{4}(dx'^4)^2 \equiv d\tilde{x}_3^2 + \frac{3}{4}(dx'^4)^2,$$

$$\begin{aligned}\frac{4}{3}\sqrt{\frac{3}{2}}x^1 dx'^3 + \left(\sqrt{2}x^2 + \frac{2}{3}\sqrt{\frac{3}{2}}x^1\right)dx'^4 &= \frac{4}{3}\sqrt{\frac{3}{2}}x^1 dx'^3 + \frac{2}{3}\sqrt{\frac{3}{2}}x^1 dx'^4 + \sqrt{2}x^2 dx'^4 \\ &= \frac{4}{3}\sqrt{\frac{3}{2}}x^1\left(dx'^3 + \frac{1}{2}dx'^4\right) + \sqrt{2}x^2 dx'^4 \\ &= \frac{4}{3}\sqrt{\frac{3}{2}}x^1 d\tilde{x}_3 + \sqrt{2}x^2 dx'^4,\end{aligned}$$

so that

$$ds^2 = -4dx^+dx^- - \left[r^2 + \frac{4\gamma^2}{3(3+\gamma^2)}(3(x^1)^2 + 3(x^2)^2)\right](dx^+)^2$$
$$+ dr^2 + r^2 d\Omega_3^2 + (dx^1)^2 + (dx^2)^2 + \frac{2}{3+\gamma^2}\left(d\tilde{x}_3^2 + \frac{3}{4}(dx'^4)^2\right)$$
$$+ \frac{6}{3+\gamma^2}\left(\frac{4}{3}\sqrt{\frac{3}{2}}x^1 d\tilde{x}_3 + \sqrt{2}x^2 dx'^4\right)dx^+.$$

As a final step, we set

$$x^3 = \sqrt{\frac{2}{3+\gamma^2}}\tilde{x}_3, \quad x^4 = \sqrt{\frac{3}{2(3+\gamma^2)}}x'^4,$$

so that we produce the metric quoted previously, namely

$$ds^2 = -4dx^+dx^- - \left[r^2 + \frac{4\gamma^2}{3+\gamma^2}((x^1)^2 + (x^2)^2)\right](dx^+)^2 + dr^2 + r^2 d\Omega_3^2$$

$$+ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + \frac{4\sqrt{3}}{\sqrt{3+\gamma^2}}(x^1 dx^3 + x^2 dx^4)dx^+. \quad (4.3)$$

We can perform a rudimentary check to verify that this is indeed the correct pp-wave limit of the metric. We do so by ensuring that we can reproduce the correct pp-wave limit of $AdS_5 \times S^5$ when we set $\gamma = 0$. Applying this condition to the above result (4.3) leads us to the metric

$$ds^2 = -4dx^+dx^- - r^2(dx^+)^2 + dr^2 + r^2 d\Omega_3^2 + (dx^1)^2 + (dx^2)^2$$

$$+ (dx^3)^2 + (dx^4)^2 + 4(x^1 dx^3 + x^2 dx^4)dx^+. \quad (4.4)$$

To guarantee a successful check, we need to show that the metric (4.4) is equivalent to the expected metric (i.e. that of the pp-wave limit of $AdS_5 \times S^5$), which is

$$ds^2 = -4dx^+dx^- - (r^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2)(dx^+)^2 + dr^2$$

$$+ r^2 d\Omega_3^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \quad (4.5)$$

To show that this is the case we change

$$x^- \to x^- + \frac{\sqrt{3}}{2\sqrt{3+\gamma^2}}(x^1x^3 + x^2x^4)$$

and also perform the following change of variables in (4.3)

$$y^1 = \cos\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^1 - \sin\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^3,$$

$$y^3 = \cos\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^3 + \sin\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^1,$$

$$y^2 = \cos\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^2 - \sin\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^4,$$

$$y^4 = \cos\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^4 + \sin\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)x^2.$$

We find that the metric (4.3) becomes

$$\begin{aligned}
ds^2 &= -4dx^+dx^- + \sum_{i=5}^{8}(dx^i)^2 + \sum_{i=1}^{4}(dy^i)^2 \\
&- \left((x^5)^2 + (x^6)^2 + (x^7)^2 + (x^8)^2 + \frac{(y^1)^2 + (y^2)^2 + (y^3) + (y^4)^2}{1+\frac{\gamma^2}{3}}\right)(dx^+)^2 \\
&+ \frac{4\gamma^2}{3+\gamma^2}[(\cos\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)y^1 + \sin\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)y^3)^2 \\
&+ (\cos\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)y^2 + \sin\left(\frac{\sqrt{3}x^+}{\sqrt{3+\gamma^2}}\right)y^4)^2](dx^+)^2,
\end{aligned}$$

which is equivalent to (4.5) when we set $\gamma = 0$.

Next, we will show the details of the calculation to obtain the pp-wave limit of the B field. We are interested in

$$B_{\varphi_1\varphi_2}\mathcal{D}\varphi_1 \wedge \mathcal{D}\varphi_2 = G\gamma g R^2 \mathcal{D}\varphi_1 \wedge \mathcal{D}\varphi_2. \tag{4.6}$$

We require

$$\mathcal{D}\varphi_1 \wedge \mathcal{D}\varphi_2 = \mathcal{O}(R^{-2})$$

to ensure that the pp-wave limit of the above quantity (4.6) doesn't blow up. Since we cannot be sure that this is necessarily the case, we will compute $\mathcal{D}\varphi_1$ and $\mathcal{D}\varphi_2$ explicitly. We begin by stating that

$$d\Omega^2 = \mu_1^2 d\phi_1^2 + \mu_2^2 d\phi_2^2 + \mu_3^2 d\phi_3^2,$$

which can be written as

$$\begin{aligned}
d\Omega^2 &= d\psi^2 + d\varphi_1^2(\mu_2^2 + \mu_3^2) + d\varphi_2^2(\mu_2^2 + \mu_1^2) + 2d\varphi_1 d\varphi_2 \mu_2^2 + 2d\psi d\varphi_2(\mu_2^2 - \mu_1^2) \\
&+ 2d\psi d\varphi_1(\mu_2^2 - \mu_3^2)
\end{aligned}$$

$$= (\mu_2^2+\mu_3^2)\left(d\varphi_1 + \frac{\mu_2^2}{\mu_2^2+\mu_3^2}d\varphi_2 + \frac{\mu_2^2-\mu_3^2}{\mu_2^2+\mu_3^2}d\psi\right)^2 - \frac{\mu_2^4}{\mu_2^2+\mu_3^2}d\varphi_2^2 + d\psi^2$$
$$- \frac{(\mu_2^2-\mu_3^2)^2}{\mu_2^2+\mu_3^2}d\psi^2 - \frac{2\mu_2^2(\mu_2^2-\mu_3^2)}{\mu_2^2+\mu_3^2}d\psi d\varphi_2 + d\varphi_2^2(\mu_2^2+\mu_1^2) + 2d\psi d\varphi_2(\mu_2^2-\mu_1^2).$$

We simplify this further by noting that the coefficient for $d\varphi_2^2$ (outside the squared bracket) can be written

$$\mu_2^2+\mu_1^2 - \frac{\mu_2^4}{\mu_2^2+\mu_3^2} = \frac{g}{\mu_2^2+\mu_3^2},$$

so that we have

$$d\Omega^2 = (\mu_2^2+\mu_3^2)\left(d\varphi_1 + \frac{\mu_2^2}{\mu_2^2+\mu_3^2}d\varphi_2 + \frac{\mu_2^2-\mu_3^2}{\mu_2^2+\mu_3^2}d\psi\right)^2 + \frac{g}{\mu_2^2+\mu_3^2}d\varphi_2^2$$
$$+ \left(1 - \frac{(\mu_2^2-\mu_3^2)^2}{\mu_2^2+\mu_3^2}\right)d\psi^2 + 2\left[\mu_2^2-\mu_1^2 - \frac{\mu_2^2(\mu_2^2-\mu_3^2)}{\mu_2^2+\mu_3^2}\right]d\psi d\varphi_2.$$

Furthermore, we can rewrite the coefficient of $d\varphi_2 d\psi$ as

$$\mu_2^2-\mu_1^2 - \frac{\mu_2^2(\mu_2^2-\mu_3^2)}{\mu_2^2+\mu_3^2} = \frac{3\mu_2^2\mu_3^2-g}{\mu_2^2+\mu_3^2},$$

which allows us to write

$$d\Omega^2 = (\mu_2^2+\mu_3^2)\left(d\varphi_1 + \frac{\mu_2^2}{\mu_2^2+\mu_3^2}d\varphi_2 + \frac{\mu_2^2-\mu_3^2}{\mu_2^2+\mu_3^2}d\psi\right)^2$$
$$+ \frac{g}{\mu_2^2+\mu_3^2}\left[d\varphi_2 - d\psi + \frac{3\mu_2^2\mu_3^2}{g}d\psi\right]^2 + \left(1 - \frac{(\mu_2^2-\mu_3^2)^2}{\mu_2^2+\mu_3^2} - \frac{9\mu_2^4\mu_3^4}{g(\mu_2^2+\mu_3^2)}\right)d\psi^2.$$

From this we can read off

$$\mathcal{D}\varphi_2 = d\varphi_2 - d\psi + \frac{3\mu_3^2\mu_2^2}{g}d\psi$$

and also note from the $d\varphi_2$ dependence that

$$d\varphi_1 + \frac{\mu_2^2}{\mu_2^2+\mu_3^2}d\varphi_2 + \frac{\mu_2^2-\mu_3^2}{\mu_2^2+\mu_3^2}d\psi = \mathcal{D}\varphi_1 + \frac{\mu_2^2}{\mu_2^2+\mu_3^2}\mathcal{D}\varphi_2$$

$$\Rightarrow \mathcal{D}\varphi_1 = d\varphi_1 - d\psi + \frac{3\mu_1^2\mu_2^2}{g}d\psi.$$

We conclude that

$$\mathcal{D}\varphi_1 = d\varphi_1 - d\psi + \frac{3\mu_1^2\mu_2^2}{g}d\psi = d\varphi_1 - \frac{1}{3gR}\left(2\sqrt{\frac{3}{2}}x^1 - \sqrt{2}x^2\right)d\psi$$

$$= \frac{dx'^3}{R} + \frac{1}{3gR}\left(2\sqrt{\frac{3}{2}}x^1 - \sqrt{2}x^2\right)dx^+,$$

$$\mathcal{D}\varphi_2 = d\varphi_2 - d\psi + \frac{3\mu_3^2\mu_2^2}{g}d\psi = d\varphi_2 - \frac{1}{3gR}2\sqrt{2}x^2 d\psi = \frac{dx'^4}{R} + \frac{1}{3gR}2\sqrt{2}x^2 dx^+,$$

which leads us to the B field

$$B_{\varphi_1\varphi_2}\mathcal{D}\varphi_1\mathcal{D}\varphi_2 = G\gamma g\left(dx'^3 \wedge dx'^4 + 2\sqrt{2}x^2 dx'^3 \wedge dx^+ + \left(2\sqrt{\frac{3}{2}}x^1 - \sqrt{2}x^2\right)dx^+ \wedge dx'^4\right)$$

$$= \frac{\gamma}{3+\gamma^2}\left(d\tilde{x}_3 \wedge dx'^4 + 2\sqrt{2}x^2 d\tilde{x}_3 \wedge dx^+ + 2\sqrt{\frac{3}{2}}x^1 dx^+ \wedge dx'^4\right)$$

$$= \frac{\gamma}{\sqrt{3}}dx^3 \wedge dx^4 + \frac{2\gamma}{\sqrt{3+\gamma^2}}(x^2 dx^3 \wedge dx^+ + x^1 dx^+ \wedge dx^4) \qquad (4.7)$$

as shown before. In addition, we obtain the following field strengths

$$H_{23+} = \frac{2\gamma}{\sqrt{3+\gamma^2}}, \quad H_{14+} = \frac{-2\gamma}{\sqrt{3+\gamma^2}}.$$

Clearly the field strength is null as is required in the pp-wave limit.

4.4 String Sigma Model in the PP-wave Limit of the LM Geometry

As mentioned, we wish to obtain the string spectrum in the sigma model corresponding to the pp-wave limit of the LM geometry. We do this by showing that the background obtained in the previous section corresponds to a homogeneous pp-wave [21], and then using an existing result [22] to extract the spectrum. We will be working in lightcone gauge.

For simplicity we drop the fermions from our analysis, and are left with the following string worldsheet action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma[\sqrt{\eta}\eta^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \epsilon^{ab}B^{NS}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \alpha'\sqrt{\eta}\phi(x)R],$$

where R is the scalar curvature on the worldsheet, η^{ab} is the worldsheet metric and $\eta = |\det \eta_{ab}|$. We choose $\sqrt{\eta}\eta^{ab}$ diagonal with $\sqrt{\eta}\eta^{00} = -1$ and $\sqrt{\eta}\eta^{11} = 1$. We shift the x^- coordinate

$$x^- \to x^- + \frac{\sqrt{3}}{2\sqrt{3+\gamma^2}}(x^1x^3 + x^2x^4),$$

so that our metric becomes

$$ds^2 = -4dx^+dx^- - \left[\sum_{i=5}^{8}(x^i)^2 + \frac{4\gamma^2}{3+\gamma^2}((x^1)^2 + (x^2)^2)\right](dx^+)^2$$
$$+\sum_{i=1}^{8}(dx^i)^2 + \frac{2\sqrt{3}}{\sqrt{3+\gamma^2}}(x^1dx^3 - x^3dx^1 + x^2dx^4 - x^4dx^2)dx^+.$$

According to [21] this metric is that of a homogeneous pp-wave, which allows us to use the results of [22] when investigating the sigma model for this background. We continue this section by reviewing these results.

We use the gauge $x^+ = \tau$ and evaluate the Lagrangian density from the action above, allowing σ's value to vary from 0 to π and setting $\alpha' = \frac{1}{2\pi}$. Let's consider the first term in the action, which can be evaluated to give

$$-\frac{1}{2}\sqrt{\eta}\eta^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu = -2\frac{\partial x^-}{\partial\tau} - \frac{1}{2}\left[\sum_{i=5}^{8}(x^i)^2 + \frac{4\gamma^2}{3+\gamma^2}((x^1)^2 + (x^2)^2)\right]$$
$$-\frac{1}{2}\sum_{i=1}^{8}\partial_a x^i\partial^a x^i + \frac{\sqrt{3}}{\sqrt{3+\gamma^2}}\left(x^1\frac{\partial x^3}{\partial\tau} - x^3\frac{\partial x^1}{\partial\tau} + x^2\frac{\partial x^4}{\partial\tau} - x^4\frac{\partial x^2}{\partial\tau}\right),$$

where the signs of the first and last terms above are completely arbitrary. Using equation (4.7) for the pp-wave limit of B, namely

$$B = \frac{\gamma}{\sqrt{3}}dx^3\wedge dx^4 + \frac{2\gamma}{\sqrt{3+\gamma^2}}(x^2dx^3\wedge dx^+ + x^1dx^+\wedge dx^4),$$

we can evaluate the second term in the action. We do so by noting that the field strength is unaffected if we add or subtract a constant to B, which allows us to use

$$B = \frac{2\gamma}{\sqrt{3+\gamma^2}}(x^2 dx^3 \wedge dx^+ + x^1 dx^+ \wedge dx^4).$$

From this, we obtain

$$-\frac{1}{2}\epsilon^{ab} B^{NS}_{\mu\nu} \partial_a x^\mu \partial_b x^\nu = \frac{2\gamma}{\sqrt{3+\gamma^2}} \left(x^2 \frac{\partial x^3}{\partial \sigma} - x^1 \frac{\partial x^4}{\partial \sigma} \right).$$

We can thus write down the Lagrangian density, which turns out to be

$$\mathcal{L} = -2\frac{\partial x^-}{\partial \tau} - \frac{1}{2}\left[\sum_{i=5}^{8} (x^i)^2 + \frac{4\gamma^2}{3+\gamma^2}((x^1)^2 + (x^2)^2) \right] + \frac{2\gamma}{\sqrt{3+\gamma^2}} \left(x^2 \frac{\partial x^3}{\partial \sigma} - x^1 \frac{\partial x^4}{\partial \sigma} \right)$$

$$+ \frac{\sqrt{3}}{\sqrt{3+\gamma^2}} \left(x^1 \frac{\partial x^3}{\partial \tau} - x^3 \frac{\partial x^1}{\partial \tau} + x^2 \frac{\partial x^4}{\partial \tau} - x^4 \frac{\partial x^2}{\partial \tau} \right) - \frac{1}{2} \sum_{i=1}^{8} \partial_a x^i \partial^a x^i.$$

Note that using light cone gauge ensures that only the transverse coordinates remain, and we don't have to deal with timelike modes or any residual constraints.

Since all the difficulty we might have encountered due to the gauge invariance has been eliminated by moving to light cone gauge, we can quantize the string theory as usual. Hence, we use canonical quantization in what follows, and begin by computing the canonical momenta:

$$p^1(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^1} = \dot{x}^1(\tau,\sigma) - \sqrt{\frac{3}{3+\gamma^2}} x^3, \quad p^2(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^2} = \dot{x}^2(\tau,\sigma) - \sqrt{\frac{3}{3+\gamma^2}} x^4,$$

$$p^3(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^3} = \dot{x}^3(\tau,\sigma) + \sqrt{\frac{3}{3+\gamma^2}} x^1, \quad p^4(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^4} = \dot{x}^4(\tau,\sigma) + \sqrt{\frac{3}{3+\gamma^2}} x^2,$$

$$p^k(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \dot{x}^k(\tau,\sigma) \quad k = 5,6,7,8.$$

Next we impose the equal time commutation relations

$$[p^k(\tau,\sigma), x^j(\tau,\sigma')] = -i\delta^{jk}\delta(\sigma - \sigma').$$

Using a standard Lagrange transform we can write down the Hamiltonian

$$H = \frac{1}{2}\int_0^\pi d\sigma[\sum_{k=1}^{8}\left(p^k p^k + \frac{\partial x^k}{\partial \sigma}\frac{\partial x^k}{\partial \sigma}\right)+\frac{3}{3+\gamma^2}\sum_{k=1}^{4}(x^k)^2+\sum_{k=5}^{8}(x^k)^2+\frac{4\gamma^2}{3+\gamma^2}\sum_{k=1}^{2}(x^k)^2$$
$$-\frac{4\gamma}{\sqrt{3+\gamma^2}}\left(x^2\frac{\partial x^3}{\partial \sigma} - x^1\frac{\partial x^4}{\partial \sigma}\right)+\frac{2\sqrt{3}}{\sqrt{3+\gamma^2}}(p^1x^3+p^2x^4-p^3x^1-p^4x^2)]. \quad (4.8)$$

The coordinates $x^5,\ x^6,\ x^7,\ x^8$ come from the AdS_5 part of the space and so we expect that they are unaffected by the deformation, since this part of the space does not participate in the deformation. This is obvious in the Hamiltonian we have obtained, as the masses corresponding to the modes of these coordinates have no γ dependance. As a result, our analysis only deals with $x^1,\ x^2,\ x^3,\ x^4$. The quantization we perform here is virtually identical to the quantization of the single free scalar field, in which we write the most general classical solution in terms of plane waves, each of which has a specific energy. To ensure that we quantize each of these modes, we enforce the oscillator commutation relations on the coefficients of these plane waves. Basically, this amounts to requiring that the standard equal time commutation relations hold. In our case, the classical solutions are no longer plane waves, but will still yield the exact quantum field and spectrum, since we are dealing with a Lagrangian which is quadratic in the fields.

The Heisenberg equations of motion that we obtain from the Hamiltonian (4.8) are given by

$$\frac{\partial p^i}{\partial t} = i[H, p^i],$$

where we use t instead of τ to make its meaning explicit. In full, we get

$$\frac{\partial p^1}{\partial t} = \frac{\partial^2 x^1}{\partial \sigma^2} - \frac{3}{3+\gamma^2}x^1 - \frac{4\gamma^2}{3+\gamma^2}x^1 - \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^4}{\partial \sigma} + \sqrt{\frac{3}{3+\gamma^2}}p^3,$$
$$\frac{\partial p^2}{\partial t} = \frac{\partial^2 x^2}{\partial \sigma^2} - \frac{3}{3+\gamma^2}x^2 - \frac{4\gamma^2}{3+\gamma^2}x^2 + \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^3}{\partial \sigma} + \sqrt{\frac{3}{3+\gamma^2}}p^4,$$
$$\frac{\partial p^3}{\partial t} = \frac{\partial^2 x^3}{\partial \sigma^2} - \frac{3}{3+\gamma^2}x^3 - \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^2}{\partial \sigma} - \sqrt{\frac{3}{3+\gamma^2}}p^1,$$
$$\frac{\partial p^4}{\partial t} = \frac{\partial^2 x^4}{\partial \sigma^2} - \frac{3}{3+\gamma^2}x^4 + \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^1}{\partial \sigma} - \sqrt{\frac{3}{3+\gamma^2}}p^2,$$

$$\frac{\partial p^k}{\partial t} = \frac{\partial^2 x^k}{\partial \sigma^2} - x^k, \quad k = 5, 6, 7, 8.$$

from which we can easily write down the following equations of motion

$$\frac{\partial^2 x^1}{\partial t^2} - \frac{\partial^2 x^1}{\partial \sigma^2} - 2\sqrt{\frac{3}{3+\gamma^2}}\frac{\partial x^3}{\partial t} + \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^4}{\partial \sigma} + \frac{4\gamma^2}{3+\gamma^2}x^1 = 0,$$

$$\frac{\partial^2 x^2}{\partial t^2} - \frac{\partial^2 x^2}{\partial \sigma^2} - 2\sqrt{\frac{3}{3+\gamma^2}}\frac{\partial x^4}{\partial t} - \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^3}{\partial \sigma} + \frac{4\gamma^2}{3+\gamma^2}x^2 = 0,$$

$$\frac{\partial^2 x^3}{\partial t^2} - \frac{\partial^2 x^3}{\partial \sigma^2} + 2\sqrt{\frac{3}{3+\gamma^2}}\frac{\partial x^1}{\partial t} + \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^2}{\partial \sigma} = 0,$$

$$\frac{\partial^2 x^4}{\partial t^2} - \frac{\partial^2 x^4}{\partial \sigma^2} + 2\sqrt{\frac{3}{3+\gamma^2}}\frac{\partial x^2}{\partial t} - \frac{2\gamma}{\sqrt{3+\gamma^2}}\frac{\partial x^1}{\partial \sigma} = 0,$$

$$\frac{\partial^2 x^k}{\partial t^2} - \frac{\partial^2 x^k}{\partial \sigma^2} + x^k = 0 \quad k = 5, 6, 7, 8.$$

The last four equations, being trivial to solve, will not be considered for now. Instead we will focus on the first four equations, which are given by

$$\frac{\partial^2 x^i}{\partial t^2} - \frac{\partial^2 x^i}{\partial \sigma^2} + f^{ij}\frac{\partial x^j}{\partial t} + h^{ij}\frac{\partial x^j}{\partial \sigma} + k_i x^i = 0,$$

where

$$f^{ij} = \begin{bmatrix} 0 & 0 & -2\sqrt{\frac{3}{3+\gamma^2}} & 0 \\ 0 & 0 & 0 & -2\sqrt{\frac{3}{3+\gamma^2}} \\ 2\sqrt{\frac{3}{3+\gamma^2}} & 0 & 0 & 0 \\ 0 & 2\sqrt{\frac{3}{3+\gamma^2}} & 0 & 0 \end{bmatrix}$$

and

$$h^{ij} = 2\begin{bmatrix} 0 & 0 & 0 & \frac{\gamma}{\sqrt{3+\gamma^2}} \\ 0 & 0 & -\frac{\gamma}{\sqrt{3+\gamma^2}} & 0 \\ 0 & \frac{\gamma}{\sqrt{3+\gamma^2}} & 0 & 0 \\ -\frac{\gamma}{\sqrt{3+\gamma^2}} & 0 & 0 & 0 \end{bmatrix}$$

$$k_1 = k_2 = \frac{4\gamma^2}{3+\gamma^2}, \quad k_3 = k_4 = 0.$$

We introduce the mode expansions

$$x^i(t,\sigma) = \sum_{n=-\infty}^{\infty} x^i_n(t) e^{2in\sigma}$$

where, due to the fact that the sting field is real, we have

$$x^i_n = (x^i_{-n})^*.$$

So, instead of the equations of motion we are left with the mode equations

$$\frac{\partial^2 x^i_n}{\partial t^2} + 4n^2 x^i_n + f^{ij}\frac{\partial x^j_n}{\partial t} + 2inh^{ij}x^j_n + k_i x^i_n = 0.$$

The mode expansions are transparent: they ensure that the σ coordinate runs from 0 to π and that the string fields are periodic in σ. We would like to extract a spectrum from our analysis, thus using the perscription of [22] we make the ansatz

$$x^i_n(t) = a^{(n)}_j A^{(n)}_{ij} e^{i\omega^{(n)}_j t},$$

where $a^{(n)}_j$ represents an annihilation operator, $A^{(n)}_{ij}$ is a unitary transformation diagonalizing the equation of motion and $\omega^{(n)}_j$ is the spectrum we are after. After plugging this ansatz into the equations for the modes we are left with

$$(-(\omega^{(n)}_k)^2\delta^{ij} + 4n^2\delta^{ij} + if^{ij}\omega^{(n)}_k + 2inh^{ij} + k_i\delta^{ij})a^{(n)}_k A^{(n)}_{jk} e^{i\omega^{(n)}_k t} = 0.$$

A nontrivial solution arises when

$$\det(-(\omega^{(n)}_k)^2\delta^{ij} + 4n^2\delta^{ij} + if^{ij}\omega^{(n)}_k + 2inh^{ij} + k_i\delta^{ij}) = 0,$$

from which we are able to extract the quartic equation

$$\omega^4 - (4+8n^2)\omega^2 + 16n^4 = 0.$$

This equation is solved by

$$\omega = 1 \pm \sqrt{1 + 4n^2},$$

which agrees exactly with [23]. It is worth noting that this spectrum is independent of the deformation parameter γ, which is unexpected - evidently, any change in the background of the type suggested by LM will yield the same spectrum in the pp-wave limit. It seems that the relative γ dependences of the metric and B field somehow compensate for eachother.

Chapter 5

Analysis on the Field Theory Side of the Correspondence

In this section we deal with the dual field theory analysis linked via the AdS/CFT correspondence with strings in the deformed LM background [5]. We show the field theory deformation which corresponds to the geometrical deformation discussed previously. In the field theory, this deformation involves the superpotential, which is made up of traces of chiral superfields $\hat{\Phi}$:

$$Tr(\hat{\Phi}^1\hat{\Phi}^2\hat{\Phi}^3 - \hat{\Phi}^1\hat{\Phi}^3\hat{\Phi}^2) \to Tr(e^{i\pi\gamma}\hat{\Phi}^1\hat{\Phi}^2\hat{\Phi}^3 - e^{-i\pi\gamma}\hat{\Phi}^1\hat{\Phi}^3\hat{\Phi}^2).$$

In our analysis we will only be considering Higgs fields, which are the bosonic lowest component of $\hat{\Phi}$ denoted by Φ. In the previous section we found the string modes, and our goal in this section will be to find the Higgs field operators dual to these modes. The deformation does not affect the kinetic terms and D terms for the Higgs fields, but it does have an effect on the F terms, which become

$$V = Tr(|[\Phi^2, \Phi^3]_\gamma|^2 + |[\Phi^3, \Phi^1]_\gamma|^2 + |[\Phi^1, \Phi^2]_\gamma|^2) \qquad (5.1)$$

where, for arbitrary A and B the γ-commutators are defined by

$$[A, B]_\gamma \equiv e^{i\pi\gamma}AB - e^{-i\pi\gamma}BA.$$

The above F terms can be obtained by first considering the superfield expansion

$$\hat{\Phi}^i = \Phi^i(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F^i(y) \quad (y^m = x^m + i\theta\sigma^m\bar{\theta})$$

$$= \Phi^i(x) + \theta^\alpha\theta_\alpha F^i(x) + i\theta\sigma^m\bar\theta\partial_m\Phi^i(x) - \frac{1}{4}\theta^\alpha\theta_\alpha\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\partial^m\Phi^i(x),$$

where we have only kept the bosonic fields in the second line, since these are the only fields we are interested in. Using this expansion we find

$$\begin{aligned} V_F &\equiv \int d^2\theta Tr(e^{i\pi\gamma}\hat\Phi^1\hat\Phi^2\hat\Phi^3 - e^{-i\pi\gamma}\hat\Phi^1\hat\Phi^3\hat\Phi^2) \\ &= e^{i\pi\gamma}Tr(F^1\Phi^2\Phi^3 + \Phi^1F^2\Phi^3 + \Phi^1\Phi^2F^3) \\ &- e^{-i\pi\gamma}Tr(F^1\Phi^3\Phi^2 + \Phi^1\Phi^3F^2 + \Phi^1F^3\Phi^2), \end{aligned}$$

which can also be written

$$V_F = Tr(F^1[\Phi^2,\Phi^3]_\gamma + F^2[\Phi^3,\Phi^1]_\gamma + F^3[\Phi^1,\Phi^2]_\gamma).$$

The anti-chiral piece of the superpotential yields a hermitean conjugate term. We don't need to concern ourselves with the fields F^i, or their complex conjugates. They can be eliminated using their equations of motion, which is obvious as they appear quadratically in the action. More precisely, we can find these equations of motion if we look at the kinetic terms of the action before coupling to the vector superfield (keeping only bosonic terms):

$$\int d^4x \int d^4\theta \hat\Phi^{i\dagger}\hat\Phi^i = \int d^4x\left(-\frac{1}{4}\bar\Phi^i\partial_m\partial^m\Phi^i - \frac{1}{4}\partial_m\partial^m\bar\Phi^i\Phi^i + F^i(F^i)^* + \frac{1}{2}\partial^m\bar\Phi^i\partial_m\Phi^i\right).$$

We use this and $V_F + V_F^*$ to elucidate the equations of motion, and after elimination of the auxiliary fields we obtain the F terms in (5.1), so named because they came from an auxiliary field called F.

We will be dealing with correlators of traces which involve only Φ^1, Φ^2, and Φ^3 or $\bar\Phi^1$, $\bar\Phi^2$, and $\bar\Phi^3$. According to [24], one does not need to include D term contributions, self energy corrections or gluon exchange at order g_{YM}^2 in Yang-Mills perturbation theory when computing such correlators in the undeformed theory. First, we will argue that this follows through for the deformed theory at leading order in N, and then use this insight to build our operators. We follow this by showing how to build the operators dual to the vacuum of the sigma model in the deformed theory. The arguments for building the operators dual to excited string states prove to be more subtle, so we study operators dual to excited states in the undeformed theory, and lastly provide an analysis in the deformed theory.

5.1 Only F-terms Contribute

There is another type of auxiliary field , the D field, which can be obtained by coupling to the vector superfield. It leads to new contributions to the potential which are called D terms, and are of the form

$$\sim [[\Phi^1, \bar{\Phi}^1] + [\Phi^2, \bar{\Phi}^2] + [\Phi^3, \bar{\Phi}^3]]^2.$$

It is quite clear that these terms are unchanged under the LM deformation due to the absence of γ in the above. Coupling to the vector superfield is not the only source of interaction possible, as the gluon can couple to the Higgs fields also. This coupling comes from the terms

$$D_\mu \Phi^1 D^\mu \bar{\Phi}^1 + D_\mu \Phi^2 D^\mu \bar{\Phi}^2 + D_\mu \Phi^3 D^\mu \bar{\Phi}^3,$$

where D_μ is the gauge covariant derivative.

We are interested in correlators of the form

$$\langle \mathcal{O}\bar{\mathcal{O}} \rangle,$$

where

$$\mathcal{O} = f_{i_1 i_2 \ldots i_k} Tr(\Phi^{i_1} \Phi^{i_2} \ldots \Phi^{i_k}), \qquad i_1, \ldots, i_k \in \{1, 2, 3\}$$

and the coefficient $f_{i_1 i_2 \ldots i_k}$ is unspecified. In the undeformed theory [24] has argued that the D terms, gluon exchange and self energy corrections are all flavour blind at one loop, so if we work to one loop order and replace $\mathcal{O} \to Tr((\Phi^1)^k)$, the corrections to our correlator are identical to the corrections to

$$\langle \mathcal{O}\bar{\mathcal{O}} \rangle = \langle Tr((\Phi^1)^k) Tr((\bar{\Phi}^1)^k) \rangle.$$

Since $Tr(\Phi^{1k})$ is half BPS, it will receive no radiative corrections at $O(g_{YM}^2)$ [26]. Thus, we see that only F terms contribute in the undeformed case. In the deformed theory, the D term and gluon exchange (coupling of the Higgs fields to the gluons) contributions are unaffected by the deformation. In fact, these two sources of interaction are the same in the deformed and undeformed theories.

The same is not true for the F term contributions to the self energy and this warrants a more thorough discussion of these terms. More precisely, we would like to argue (to leading order in N) that the one loop self energy contributions are also the same in the deformed and undeformed theories, because this would mean that the contributions from the D terms plus one loop self energy insertion plus gluon exchange cancel. This is the case because non-renormalization theorems for the correlator

$$\langle Tr((\Phi^1)^k Tr((\bar{\Phi})^1)^k\rangle$$

imply that these quantities cancel in the undeformed theory. Our discussion makes use of the free field theory propagators

$$\langle \bar{\Phi}^j_{ab}(x)\Phi^k_{cd}(y)\rangle = \frac{\tilde{a}}{|x-y|^2}\delta^{jk}\delta_{ad}\delta_{bc},$$

where $j, k = 1, 2, 3;\ a, b, c, d = 1, ..., N$ are colour labels, and $\tilde{a} = \frac{1}{4\pi^2}$.

It is possible to split the F terms into two parts, one being affected by the deformation (which we will call V_{def}) and one unaffected (or V_{inv}), so that it will be the same as in the undeformed theory. Thus we see that we can write

$$V_F = V_{inv} + V_{def},$$

where

$$V_{inv} = 2Tr(\Phi^1\Phi^2\bar{\Phi}^2\bar{\Phi}^1 + \Phi^2\Phi^1\bar{\Phi}^1\bar{\Phi}^2),$$

$$V_{def} = -2Tr(e^{-2\pi i\gamma}\Phi^2\Phi^1\bar{\Phi}^2\bar{\Phi}^1 + e^{2\pi i\gamma}\Phi^1\Phi^2\bar{\Phi}^1\bar{\Phi}^2).$$

In the above we are only focusing on the Φ^1 and Φ^2 dependent terms, as the argument follows through for the other terms. Notice the γ dependence in V_{def}. The Feynman diagrams corresponding to self energy contributions coming from these two vertices are shown in Figure 1, with the V_{inv} contribution in (A) and the V_{def} contribution in (B). It is clear that (B) is a non-planar diagram, and so can be dropped at large N. So, to leading order in N, we see that the only part of V_F which survives is V_{inv}, and thus, the only contribution to the self energy coming from the F terms is invariant under the deformation. Using these arguments we see that one does not need to consider D term contributions, self energy corrections or gluon exchange at order g_{YM}^2 and at leading order in N when evaluating the correlators in question.

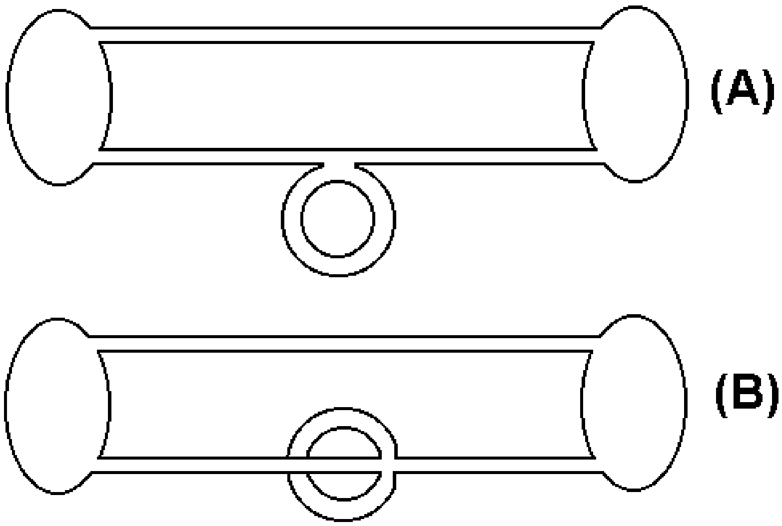

Figure 1 This plot shows the Feynman diagrams corresponding to self energy contributions coming from the F terms. (A) shows the contribution from V_{inv} and is $O(g_{YM}^2 N^3)$. (B) shows the contribution from V_{def} and is $O(g_{YM}^2 N)$.

5.2 Operators Dual to the Vacuum

The vacuum state is not the state with no strings, as one might naively expect. It is the state with one unexcited string: it is the string's excitation which is the important factor in our analysis, not the number of strings, i.e. we deal with the first quantized and not the second quantized approach. Furthermore, this particular mode of the string, i.e. the lowest mode, is important because it does not receive any string tension corrections due to the fact that it is a supergravity mode. This fact is particularly useful when we consider building the operator dual to the vacuum of the string sigma model, which is what we will do in this section. Now, the string tension and the Yang-Mills coupling constant in the dual field theory are related by

$$g_{YM}^2 N = R^4 T^2, \qquad T = \frac{1}{l_s^2},$$

so the fact that there are no string tension corrections to the vacuum gives us a condition that our perspective dual operator needs to satisfy: namely, if we use perturbation theory to compute corrections to the two point function of the operator dual to the vacuum, these corrections should disappear.

Since the vacuum dual operator is BPS, the $U(1)_{\mathcal{R}}$ charge of the operator should be equal to its conformal dimension. In addition, because we boosted along ψ in taking the pp-wave limit so that there is no momentum in the φ_1 and φ_2 directions, we expect that the operator is uncharged under the $U(1) \times U(1)$ symmetry of the field theory. Previously we mentioned that our operators will be built out of Higgs fields, and the arguments above allow us to write down the following charges and dimension for the three fields:

$$\begin{array}{ccccc} & U(1) & U(1) & U(1)_{\mathcal{R}} \equiv J & \Delta \\ \Phi^1 & 0 & -1 & 1 & 1 \\ \Phi^2 & 1 & 1 & 1 & 1 \\ \Phi^3 & -1 & 0 & 1 & 1 \end{array}.$$

We start this analysis by showing how to build the operator dual to the vacuum for small values of J, namely $J = 3$ and $J = 6$. The results obtained highlight an obvious rule which can be used to build the operator for all J.

For $J = 3$, we might consider the field

$$M = \frac{1}{3}(\Phi^1\Phi^2\Phi^3 + \Phi^2\Phi^3\Phi^1 + \Phi^3\Phi^1\Phi^2 + \Phi^1\Phi^3\Phi^2 + \Phi^2\Phi^1\Phi^3 + \Phi^3\Phi^2\Phi^1),$$

which has dimension $\Delta = 3$ and is neutral under $U(1) \times U(1)$. From this field we deduce that the operator

$$Tr(M) = Tr(\Phi^1\Phi^2\Phi^3) + Tr(\Phi^1\Phi^3\Phi^2)$$

has dimension $\Delta = 3$, $U(1)_{\mathcal{R}}$ charge $J = 3$ and is neutral under $U(1) \times U(1)$. This operator is a possibility because it is a symmetric traceless combination of Higgs fields, and hence protected in the undeformed theory. However, it can't be dual to the vacuum of the deformed theory because the F terms correct the two point function of this operator at $\mathcal{O}(g_{YM}^2)$. So we need to do a bit more work to find the operator.

For $J = 3$ we notice that there are two independent loops we could use to build the operator dual to the vacuum, namely

$$\mathcal{O}_1 = Tr(\Phi^1\Phi^2\Phi^3), \quad \mathcal{O}_2 = Tr(\Phi^1\Phi^3\Phi^2).$$

We wish to write down some combination of these two operators to give us an operator dual to the vacuum. We are ultimately interested in calculating correlators involving such operators, and so we need to make use of the two point function of the Higgs fields as defined before:

$$\langle \bar{\Phi}^j_{ab}(x)\Phi^k_{cd}(0)\rangle = \delta^{jk}\delta_{ad}\delta_{bc}\frac{1}{4\pi^2|x|^2} \equiv \delta^{jk}\delta_{ad}\delta_{bc}\frac{\tilde{a}}{|x|^2},$$

where $a, b, c, d = 1, ..., N$ are colour indices, $j, k = 1, 2, 3$ and $\tilde{a} = \frac{1}{4\pi^2}$. To determine what linear combination of $\mathcal{O}_1$ and $\mathcal{O}_2$ we need, we compute the planar contribution to $\langle\bar{\mathcal{O}}_i(x_1)V_F(y)\mathcal{O}_j(x_2)\rangle$. We obtain

$$\langle\bar{\mathcal{O}}_i(x_1)V_F(y)\mathcal{O}_j(x_2)\rangle = \mathcal{M}_{ij}\frac{\tilde{a}^5}{|x_1-y|^4|x_2-y|^4|x_1-x_2|^2}N^4, \tag{5.2}$$

where

$$\mathcal{M}^T = \begin{bmatrix} 6 & -6e^{-2\pi i\gamma} \\ -6e^{2\pi i\gamma} & 6 \end{bmatrix}.$$

The two point function of a BPS operator is not corrected, and so we expect the quantity $\langle\bar{\mathcal{O}}_i(x_1)V_F(y)\mathcal{O}_j(x_2)\rangle$ to be zero. Indeed we find that the matrix $\mathcal{M}$ has a single zero eigenvalue and note that the operator dual to the vacuum is obtained via a linear combination which corresponds to this zero eigenvalue, using the relevant null vector. In addition, this linear combination is unique due to the fact that we only have a single zero eigenvalue. The operator we obtain is

$$\mathcal{O}_\gamma = Tr(\Phi^1\Phi^2\Phi^3) + e^{-2\pi i\gamma}Tr(\Phi^1\Phi^3\Phi^2),$$

and as expected it has dimension $\Delta = 3$, $U(1)_{\mathcal{R}}$ charge $J = 3$ and is unchanged under a $U(1) \times U(1)$ symmetry. Also, $\mathcal{O}_\gamma$ is invariant under the Z_3 symmetry of the deformed theory, which involves cyclic permutations of the three Higgs fields. As a further check, notice that setting $\gamma = 0$ recovers the expected operator for the deformed case.

For $J = 6$, we can write down a basis for the operators involving $2\Phi^1$'s, $2\Phi^2$'s and $2\Phi^3$'s. There are 16 independent loops we could use to build our operator:

$$\mathcal{O}_1 = Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3),\ \mathcal{O}_2 = Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^2\Phi^3),\ \mathcal{O}_3 = Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^3\Phi^2),$$
$$\mathcal{O}_4 = Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^2\Phi^3),\ \mathcal{O}_5 = Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^3\Phi^2),\ \mathcal{O}_6 = Tr(\Phi^1\Phi^1\Phi^3\Phi^3\Phi^2\Phi^2),$$
$$\mathcal{O}_7 = Tr(\Phi^1\Phi^2\Phi^1\Phi^2\Phi^3\Phi^3),\ \mathcal{O}_8 = Tr(\Phi^1\Phi^2\Phi^1\Phi^3\Phi^2\Phi^3),\ \mathcal{O}_9 = Tr(\Phi^1\Phi^2\Phi^1\Phi^3\Phi^3\Phi^2),$$
$$\mathcal{O}_{10} = Tr(\Phi^1\Phi^2\Phi^2\Phi^1\Phi^3\Phi^3),\ \mathcal{O}_{11} = Tr(\Phi^1\Phi^2\Phi^2\Phi^3\Phi^1\Phi^3),\ \mathcal{O}_{12} = Tr(\Phi^1\Phi^2\Phi^3\Phi^1\Phi^2\Phi^3),$$
$$\mathcal{O}_{13} = Tr(\Phi^1\Phi^2\Phi^3\Phi^1\Phi^3\Phi^2),\ \mathcal{O}_{14} = Tr(\Phi^1\Phi^2\Phi^3\Phi^2\Phi^1\Phi^3),\ \mathcal{O}_{15} = Tr(\Phi^1\Phi^3\Phi^1\Phi^3\Phi^2\Phi^2),$$
$$\mathcal{O}_{16} = Tr(\Phi^1\Phi^3\Phi^2\Phi^1\Phi^3\Phi^2).$$

We found these operators by requiring that they have $\Delta = J = 6$, and no $U(1) \times U(1)$ charge. We wish to identify the linear combination of the above operators which is BPS, i.e. the linear combination giving rise to a two point function which is not corrected at $O(g_{YM}^2)$ and at leading order in N. Again, the obvious linear combination of the basis operators, namely

$$\begin{aligned}
\mathcal{O} = {} & Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3) + Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^2\Phi^3) \\
& + Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^3\Phi^2) + Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^2\Phi^3) \\
& + Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^3\Phi^2) + Tr(\Phi^1\Phi^1\Phi^3\Phi^3\Phi^2\Phi^2) \\
& + Tr(\Phi^1\Phi^2\Phi^1\Phi^2\Phi^3\Phi^3) + Tr(\Phi^1\Phi^2\Phi^1\Phi^3\Phi^2\Phi^3) \\
& + Tr(\Phi^1\Phi^2\Phi^1\Phi^3\Phi^3\Phi^2) + Tr(\Phi^1\Phi^2\Phi^2\Phi^1\Phi^3\Phi^3) \\
& + Tr(\Phi^1\Phi^2\Phi^2\Phi^3\Phi^1\Phi^3) + Tr(\Phi^1\Phi^2\Phi^3\Phi^1\Phi^2\Phi^3) \\
& + Tr(\Phi^1\Phi^2\Phi^3\Phi^1\Phi^3\Phi^2) + Tr(\Phi^1\Phi^2\Phi^3\Phi^2\Phi^1\Phi^3) \\
& + Tr(\Phi^1\Phi^3\Phi^1\Phi^3\Phi^2\Phi^2) + Tr(\Phi^1\Phi^3\Phi^2\Phi^1\Phi^3\Phi^2),
\end{aligned}$$

does not receive $\mathcal{O}(g_{YM}^2)$ corrections at leading order in N, in the undeformed theory.

Thus, we again make use of the quantity $\langle \bar{\mathcal{O}}_i(x_1) V_F(y) \mathcal{O}_j(x_2) \rangle$, and evaluate the matrix $\mathcal{M}$. By extracting the null vectors of $\mathcal{M}$ we will have succeeded in writing down our desired BPS operators. We obtain, at leading order in N

$$\mathcal{M}^T = \begin{bmatrix}
3 & b & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
b^* & 5 & b & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & b^* & 4 & 0 & b & 0 & b^* & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b^* & 0 & 4 & b & 0 & 0 & 0 & 0 & 0 & b^* & 0 & 0 & 0 & b & 0 \\
0 & 0 & b^* & b^* & 5 & b & 0 & b^* & 0 & 0 & 0 & 0 & 0 & b^* & 0 & 0 \\
0 & 0 & 0 & 0 & b^* & 3 & 0 & 0 & b^* & 0 & 0 & 0 & 0 & 0 & b^* & 0 \\
b^* & 0 & b & 0 & 0 & 0 & 5 & b & 0 & b & 0 & 0 & b & 0 & 0 & 0 \\
0 & b^* & 0 & 0 & b & 0 & b^* & 6 & b & 0 & 0 & b^* & 0 & 0 & 0 & b \\
0 & 0 & b^* & 0 & 0 & b & 0 & b^* & 5 & b^* & 0 & 0 & b^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b^* & 0 & b & 4 & b^* & 0 & 0 & 0 & b & 0 \\
b^* & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & b & 5 & 0 & b & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 6 & 2b & 2b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b^* & 0 & b & 0 & b^* & b^* & 6 & 0 & b & b \\
0 & b^* & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & b^* & b^* & 0 & 6 & b & b \\
0 & 0 & 0 & b^* & 0 & b & 0 & 0 & 0 & b^* & 0 & 0 & b^* & b^* & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2b^* & 0 & 0 & 0 & 0 & 2b^* & 2b^* & 0 & 6
\end{bmatrix},$$

where $b = -e^{-2\pi i\gamma}$ and $b^* = -e^{2\pi i\gamma}$. $\mathcal{M}$ has a single zero eigenvalue so we are once again able to find a unique operator which does not get corrected at $O(g_{YM}^2)$ (this operator is quoted later, after some discussion). We can repeat this analysis for $J = 9$ and again we obtain a matrix with a single zero eigenvalue, which allows us to write down a unique operator made up of a distinct linear combination of the 188 basis loops. The results for $J = 9$ are too long to write here explicitly, and doing so would not add to the discussion, but using the results for the operators obtained for $J = 3, 6, 9$ we can write down a rule which will allow us to extract the operator for *any* J. We know that the operator will be a symmetric traceless tensor when $\gamma = 0$, contracted with an equal number of Φ^1's, Φ^2's and Φ^3's. In order to elucidate this general operator, we first classify the type of *exchanges* available to us. The following will be called *even* exchanges

$$\Phi^1\Phi^2 \to \Phi^2\Phi^1, \quad or \quad \Phi^2\Phi^3 \to \Phi^3\Phi^2, \quad or \quad \Phi^3\Phi^1 \to \Phi^1\Phi^3,$$

and

$$\Phi^2\Phi^1 \to \Phi^1\Phi^2, \quad or \quad \Phi^3\Phi^2 \to \Phi^2\Phi^3, \quad or \quad \Phi^1\Phi^3 \to \Phi^3\Phi^1$$

odd exchanges. We identify the *even* exchanges with a factor $\alpha = e^{-2\pi i\gamma}$, and the *odd* exchanges with a factor $\alpha^* = e^{2\pi i\gamma}$. We can use these rules to generate the coefficient of each operator in the linear combination. For example, for $J = 6$ we use the operator $\mathcal{O}_1 = Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3)$ as a template, and determine what combination of exchanges will lead to each of the other operators $\mathcal{O}_i$ where $i = 1, ..., 16$. For each exchange we perform on the way to our final operator, we append a multiplicative factor of α or α^*, depending on the nature of the exchange. As an example, let us determine what coefficient the term $Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^3\Phi^2)$ should have in the linear combination. We obtain this term by performing two even exchanges on $Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3)$

$$Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3) \to Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^2\Phi^3) \to Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^3\Phi^2),$$

so the overall coefficient is $\alpha^2 = (e^{-2\pi i\gamma})^2$. If we repeat this procedure for each of the 16 operators, we obtain

$$\begin{aligned}
\mathcal{O}_\gamma = {} & Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3) + \alpha Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^2\Phi^3) \\
& + \alpha^2 Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^3\Phi^2) + \alpha^2 Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^2\Phi^3) \\
& + \alpha^3 Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^3\Phi^2) + \alpha^4 Tr(\Phi^1\Phi^1\Phi^3\Phi^3\Phi^2\Phi^2) \\
& + \alpha Tr(\Phi^1\Phi^2\Phi^1\Phi^2\Phi^3\Phi^3) + \alpha^2 Tr(\Phi^1\Phi^2\Phi^1\Phi^3\Phi^2\Phi^3) \\
& + \alpha^3 Tr(\Phi^1\Phi^2\Phi^1\Phi^3\Phi^3\Phi^2) + \alpha^2 Tr(\Phi^1\Phi^2\Phi^2\Phi^1\Phi^3\Phi^3) \\
& + \alpha Tr(\Phi^1\Phi^2\Phi^2\Phi^3\Phi^1\Phi^3) + \alpha Tr(\Phi^1\Phi^2\Phi^3\Phi^1\Phi^2\Phi^3) \\
& + \alpha^2 Tr(\Phi^1\Phi^2\Phi^3\Phi^1\Phi^3\Phi^2) + \alpha^2 Tr(\Phi^1\Phi^2\Phi^3\Phi^2\Phi^1\Phi^3) \\
& + \alpha^3 Tr(\Phi^1\Phi^3\Phi^1\Phi^3\Phi^2\Phi^2) + \alpha^3 Tr(\Phi^1\Phi^3\Phi^2\Phi^1\Phi^3\Phi^2),
\end{aligned}$$

which matches the result obtained via the null vectors exactly. Again we reproduce the BPS operator of the underformed theory by setting $\gamma = 0$ in the result above. In addition, we have checked that the order in which the exchanges are taken to yield a particular operator does not affect the coefficient obtained for that particular operator, so our analysis is unambiguous. We will show this check for a particular example, using $Tr((\Phi^1)^2\Phi^3\Phi^2\Phi^3\Phi^2)$. We could arrive at this combination of Φ^i's starting from $Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3)$ by performing two different sets of exchanges, namely

$$\begin{aligned}
& Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3) \to Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^2\Phi^3) \\
& \to Tr(\Phi^1\Phi^1\Phi^2\Phi^3\Phi^3\Phi^2) \to Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^3\Phi^2)
\end{aligned}$$

or

$$Tr(\Phi^1\Phi^1\Phi^2\Phi^2\Phi^3\Phi^3) \to Tr(\Phi^3\Phi^1\Phi^2\Phi^2\Phi^3\Phi^1)$$
$$\to Tr(\Phi^1\Phi^3\Phi^2\Phi^2\Phi^3\Phi^1) \to Tr(\Phi^1\Phi^3\Phi^2\Phi^3\Phi^2\Phi^1).$$

Notice that both paths involve performing three even permutations, so both paths would assign a coefficient of $(e^{-2\pi i\gamma})^3 = \alpha^3$ to $Tr(\Phi^1\Phi^1\Phi^3\Phi^2\Phi^3\Phi^2)$.

5.3 Operators Dual to Excited String Modes in the Undeformed Theory

Since this section deals with the undeformed theory, we set $\gamma = 0$, and reconsider the way in which we take the pp-wave limit [10]. Let us assume that we take the pp-wave limit by boosting along the Φ^2 direction, and not the original ψ direction. We can define the BMN operators

$$\tilde{O}_{(n)} = Tr(\Phi^1(\Phi^2)^n\Phi^3(\Phi^2)^{J-n}), \tag{5.3}$$

to which correspond the two point funcions

$$\langle \tilde{O}_{(n)}(x_1)\bar{\tilde{O}}_{(m)}(x_2)\rangle = \delta_{mn}\frac{N^{J+2}\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}}.$$

We can also write

$$\langle \tilde{O}_{(i)}(x_1)V_F(y)\bar{\tilde{O}}_{(j)}(x_2)\rangle = Q_{ij}\frac{N^{J+3}\tilde{a}^{J+4}}{|x_1-x_2|^{2J}|x_1-y|^4|x_2-y|^4}$$

where, at leading order in N we find

$$Q = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 & ... & 0 & 0 & -1 \\ -2 & 4 & -2 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & ... & -2 & 4 & -2 \\ -1 & 0 & 0 & 0 & 0 & ... & 0 & -2 & 3 \end{bmatrix}.$$

The anomalous dimensions of $\tilde{O}_{(n)}$ are determined by the eigenvalues of Q, which needs to be diagonalized in order to extract these eigenvalues. The eigenvectors of Q determine the operators that are dual to excited string modes. We are ultimately interested in the anomalous dimensions of the loops obtained in the original pp-wave limit (with a boost along the ψ direction). Since the undeformed theory has an $SO(6)$ rotational invariance, the anomalous dimensions of the loops defined above should match. We will show that we can reproduce these dimensions using loops relevant to the case that we boost along the ψ direction.

If we consider $\tilde{O}_{(n)} = Tr(\Phi^1(\Phi^2)^n\Phi^3(\Phi^2)^{J-n})$ we can imagine that the fields Φ^1 define a lattice, and that the fields Φ^1 and Φ^2 "hopping on this lattice" represent the Yang-Mills interaction. In the pp-wave limit we are interested in, there isn't a field which is the obvious candidate for the lattice, but we can define an analogous operator to that in (5.3)

$$\mathcal{O}_{(n)} \equiv C_{i_1 i_2 i_3 \ldots i_J} Tr(\Phi^1\Phi^{i_1}\ldots\Phi^{i_n}\Phi^3\Phi^{i_{n+1}}\ldots\Phi^{i_J}).$$

Next, we define the quantity

$$C^2 \equiv C_{i_1 i_2 \ldots i_J} C_{i_1 i_2 \ldots i_J},$$

and note that

$$C_{1 i_2 \ldots i_J} C_{1 i_2 \ldots i_J} = C_{2 i_2 \ldots i_J} C_{2 i_2 \ldots i_J} = C_{3 i_2 \ldots i_J} C_{3 i_2 \ldots i_J} = \frac{1}{3} C^2,$$

$$C_{12\ldots i_J} C_{12\ldots i_J} = C_{23\ldots i_J} C_{23\ldots i_J} = C_{31\ldots i_J} C_{31\ldots i_J} = AC^2,$$

$$C_{11 i_3 \ldots i_J} C_{11 i_3 \ldots i_J} = C_{22 i_3 \ldots i_J} C_{22 i_3 \ldots i_J} = C_{33 i_3 \ldots i_J} C_{33 i_3 \ldots i_J} = BC^2,$$

where $6A+3B = 1$. We obtain the above relations by realizing that the operator dual to the sigma model vacuum relevant for our pp-wave limit has an equal number of Φ^1's, Φ^2's and Φ^3's. Also, we used the fact that C is a symmetric tensor. Let us expand on these facts. In the undeformed case, we know that the protected operators are of the form

$$C_{i_1 i_2 i_3 \ldots i_J} Tr(\Phi^{i_1}\Phi^{i_2}\Phi^{i_3}\ldots\Phi^{i_J}),$$

where $C_{i_1 i_2 i_3 ... i_J}$ is any symmetric traceless tensor. We are dealing with complex fields, so any tensor built out of only Φ^i's and not $\bar{\Phi}^i$'s is traceless, so we are only left with the condition that it needs to be symmetric. The super Yang-Mills operator dual to the string theory vacuum state is

$$C_{i_1 i_2 i_3 ... i_J} Tr(\Phi^{i_1} \Phi^{i_2} \Phi^{i_3} ... \Phi^{i_J}).$$

We would like to add in two "impurities" Φ^1 and Φ^3, so we introduce the operators

$$\mathcal{O}_{(n)} \equiv C_{i_1 i_2 i_3 ... i_J} Tr(\Phi^1 \Phi^{i_1} ... \Phi^{i_n} \Phi^3 \Phi^{i_{n+1}} ... \Phi^{i_J}).$$

We can prove that the operator

$$\mathcal{O} = \sum_{n=0}^{J} \mathcal{O}_{(n)}$$

is protected and hence receives no perturbative corrections, by showing that

$$\mathcal{O} = \sum_{n=0}^{J} \mathcal{O}_{(n)} = C_{i_1 i_2 i_3 ... i_{J+2}} Tr(\Phi^{i_1} \Phi^{i_2} \Phi^{i_3} ... \Phi^{i_{J+2}}),$$

with $C_{i_1 i_2 i_3 ... i_{J+2}}$ a completely symmetric traceless tensor. We do not need to show it is traceless - this is obviously true. We will show the symmetric nature explicitly for $J = 3$, where

$$\begin{aligned}
\mathcal{O}_{(0)} &= C_{i_1 i_2 i_3} Tr(\Phi^1 \Phi^3 \Phi^{i_1} \Phi^{i_2} \Phi^{i_3}), \\
\mathcal{O}_{(1)} &= C_{i_1 i_2 i_3} Tr(\Phi^1 \Phi^{i_1} \Phi^3 \Phi^{i_2} \Phi^{i_3}), \\
\mathcal{O}_{(2)} &= C_{i_1 i_2 i_3} Tr(\Phi^1 \Phi^{i_1} \Phi^{i_2} \Phi^3 \Phi^{i_3}), \\
\mathcal{O}_{(3)} &= C_{i_1 i_2 i_3} Tr(\Phi^1 \Phi^{i_1} \Phi^{i_2} \Phi^{i_3} \Phi^3).
\end{aligned}$$

We thus find

$$\mathcal{O}_{(0)} + \mathcal{O}_{(1)} + \mathcal{O}_{(2)} + \mathcal{O}_{(3)} = C_{i_1 i_2 i_3 i_4 i_5} Tr(\Phi^{i_1} \Phi^{i_2} \Phi^{i_3} \Phi^{i_4} \Phi^{i_5}),$$

where

$$
\begin{aligned}
5C_{i_1i_2i_3i_4i_5} = {} & \delta_{i_11}\delta_{i_23}C_{i_3i_4i_5} + \delta_{i_11}\delta_{i_33}C_{i_2i_4i_5} + \delta_{i_11}\delta_{i_43}C_{i_2i_3i_5} + \delta_{i_11}\delta_{i_53}C_{i_2i_3i_4} \\
& + \delta_{i_21}\delta_{i_13}C_{i_3i_4i_5} + \delta_{i_21}\delta_{i_33}C_{i_1i_4i_5} + \delta_{i_21}\delta_{i_43}C_{i_1i_3i_5} + \delta_{i_21}\delta_{i_53}C_{i_1i_3i_4} \\
& + \delta_{i_31}\delta_{i_13}C_{i_2i_4i_5} + \delta_{i_31}\delta_{i_23}C_{i_1i_4i_5} + \delta_{i_31}\delta_{i_43}C_{i_1i_2i_5} + \delta_{i_31}\delta_{i_53}C_{i_1i_2i_4} \\
& + \delta_{i_41}\delta_{i_13}C_{i_2i_3i_5} + \delta_{i_41}\delta_{i_23}C_{i_1i_3i_5} + \delta_{i_41}\delta_{i_33}C_{i_1i_2i_5} + \delta_{i_41}\delta_{i_53}C_{i_1i_2i_3} \\
& + \delta_{i_51}\delta_{i_13}C_{i_2i_3i_4} + \delta_{i_51}\delta_{i_23}C_{i_1i_3i_4} + \delta_{i_51}\delta_{i_33}C_{i_1i_2i_4} + \delta_{i_51}\delta_{i_43}C_{i_1i_2i_3}.
\end{aligned}
$$

It is easy to verify that this is a completely symmetric tensor. Although we performed the analysis for three fields with two impurities, the same holds for an arbitrary number of fields. In our attempt to build the operator which is arbitrarily close to supersymmetric we consider

$$\sum_{n=0}^{J} c_n \mathcal{O}_{(n)},$$

in the limit that the c_n are arbitrarily close to 1.

At this point we make a rather non-trivial assumption: that the operators dual to the excited string states can be built out of the $\mathcal{O}_{(n)}$. This is indeed not obvious. There are a very large number of basis operators for all of loop space, but only $J+1$ operators $\mathcal{O}_{(n)}$, so why should we only use these when building our general operator? For the $J = 9$ case with two impurities added, there are over 1000 basis operators, but we choose to use only 10 of these, which is an obviously massive simplification which gets even more pronounced as J gets larger. Another reason is that of hindsight, where we use our newly acquired knowledge of the operator dual to the vacuum to study the excited string case. We expect that in general the conformal dimensions of operators will get extremely large corrections unless we get some special cancellations (we are taking $N \to \infty$ holding g_{YM}^2 fixed but small). As luck would have it, this does happen for the operators dual to supergravity modes - this precise linear combination of operators lead to a cancellation of corrections. "Nearly protected" operators are formed so that corrections almost cancel. Thus, the operators used to build the vacuum state and the operator dual to the excited string mode coincide. Furthermore, we can check this assumption for the $\gamma = 0$ case, as it has been studied extensively.

The first quantity we are interested in calculating is $\langle \mathcal{O}_{(l)}\bar{\mathcal{O}}_{(k)}\rangle$. Using our results above we can evaluate this overlap:

$$\langle \mathcal{O}_{(l)}(x_1)\bar{\mathcal{O}}_{(k)}(x_2)\rangle = \left[(J-2)AC^2 + \frac{2}{3}C^2 + \delta_{l,J-k}AC^2\right] N^{J+2}\frac{\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}}, \quad k \neq l$$

$$\langle \mathcal{O}_{(l)}(x_1)\bar{\mathcal{O}}_{(l)}(x_2)\rangle = [(J-1)AC^2 + C^2]N^{J+2}\frac{\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}},$$

where in the second equation above l is not summed. At large J

$$\langle \mathcal{O}_{(l)}(x_1)\bar{\mathcal{O}}_{(k)}(x_2)\rangle = \left[(J-2)AC^2 + \frac{2}{3}C^2\right] N^{J+2}\frac{\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}}.$$

Putting these results together we can write (for k and l unrestricted)

$$\begin{aligned}\langle \mathcal{O}_{(l)}(x_1)\bar{\mathcal{O}}_{(k)}(x_2)\rangle &\equiv M_{lk}\frac{N^{J+2}\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}} \\ &= \left[\left((J-2)AC^2 + \frac{2C^2}{3}\right)L + \left(A+\frac{1}{3}\right)C^2 I\right]_{lk}\frac{N^{J+2}\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}},\end{aligned}$$

where L is a matrix with a 1 in every single entry and I is the identity matrix. We require the eigenvalues and eigenvectors of M_{kl}. J of the eigenvectors have eigenvalue $(A+\frac{1}{3})C^2$ and look like

$$|n\rangle = \frac{1}{\sqrt{n^2+n}}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ -n \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the first n entries are 1's and $n = 1, 2, ..., J$, while a single eigenvalue has the form

$$|J+1\rangle = \frac{1}{\sqrt{J+1}}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix},$$

with eigenvalue $(J+1)((J-2)AC^2 + \frac{2C^2}{3}) + (A+\frac{1}{3})C^2$. We can use these eigenvalues and eigenvectors to define new operators $\mathcal{K}_{(n)}$ that have a diagonal two point function, namely

$$\mathcal{K}_{(n)} = \frac{\langle n|_l \mathcal{O}_l}{\sqrt{\lambda_n}}, \quad \bar{\mathcal{K}}_{(n)} = \frac{\bar{\mathcal{O}}_l |n\rangle_l}{\sqrt{\lambda_n}}.$$

Their two point function looks like

$$\langle \bar{\mathcal{K}}_{(n)}(x_1)\mathcal{K}_{(m)}(x_2)\rangle = \delta_{mn} \frac{N^{J+2}\tilde{a}^{J+2}}{|x_1 - x_2|^{2J+4}}.$$

Further, we wish to evaluate the anomalous dimensions for this set of operators at $O(g_{YM}^2)$. We begin by computing

$$\langle \mathcal{O}_{(i)}(x_1) V_F(y) \bar{\mathcal{O}}_{(j)}(x_2)\rangle = H_{ik} M_{kj} \frac{N^{J+3}\tilde{a}^{J+4}}{|x_1 - x_2|^{2J}|x_1 - y|^4 |x_2 - y|^4}$$

where, at large N, we find

$$H = 2 \begin{bmatrix} 3 & -2 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 \\ -2 & 4 & -2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & -2 & 4 & -2 \\ -1 & 0 & 0 & 0 & 0 & \ldots & 0 & -2 & 3 \end{bmatrix}.$$

We can now use this result to evaluate the corresponding overlap containing $\mathcal{K}$ operators. We obtain

$$\langle \mathcal{K}_{(i)}(x_1) V_F(y) \bar{\mathcal{K}}_{(j)}(x_2)\rangle = K_{ij} \frac{N^{J+3}\tilde{a}^{J+4}}{|x_1 - x_2|^{2J}|x_1 - y|^4 |x_2 - y|^4},$$

where

$$K_{nm} = \sqrt{\frac{\lambda_m}{\lambda_n}} \langle n|H|m\rangle.$$

The eigenvalues of K will yield the anomalous dimensions, and its eigenvectors will give the corresponding operators dual to excited string modes. Now, $\sqrt{\frac{\lambda_m}{\lambda_n}} = 1$ for all m and n except if $m = J+1$ or if $n = J+1$. Using this, and the fact that

$$\langle J+1|H = 0 = H|J+1\rangle,$$

we see that we can write

$$K_{nm} = \langle n|H|m\rangle,$$

which tells us that K and H are related by a unitary transformation, and thus have equivalent eigenvalue problems, so we can obtain our result by solving the eigenvalue problem for H. As we mentioned before, we expect, due to the rotational invariance of the background, that the spectrum of our pp-wave limit agrees with the spectrum of the pp-wave limit taken in [10]. Since Q and H are equal to eachother, we see that this is indeed the case, and the agreement between the two computations is a positive sign that we have identified the operators dual to excited string states.

It is instructive at this point to add a few remarks about our analysis. We have focused on $J+1$ operators, but there are many more operators with specific $U(1)_{\mathcal{R}}$ charge $J+2$ and $U(1) \times U(1)$ charge (1,1). In particular, for $J = 6$ we have only kept 7 out of a possible 70 operators, and for $J = 9$ we have only kept 10 of a possible 1050 operators with the correct quantum numbers. For $J = 3, 6$ we have checked explicitly, using the full set of loops, that the $J+1$ BMN operators we have obtained by keeping only this subset of $J+1$ operators do indeed provide operators with a definite anomalous dimension at $O(g_{YM}^2)$. In addition, we checked that the anomalous dimensions obtained using this restricted set of operators is the same as that obtained using the full set of operators. Understanding this decoupling of a small set of nearly protected states is an important existing problem in field theory [10],[27], and this section lends some insight into the decoupling.

5.4 Operators Dual to Excited String Modes in the Deformed Theory

This section serves to build operators dual to excited string modes, but here we consider the deformed theory. We will study these operators for both large and small J, so that we may verify the γ independence of the large J spectrum, and illuminate a γ dependance for finite J.

We will first consider the large J limit, and construct the "background" on which the impurities move using an even number of Φ^1, Φ^2 and Φ^3 fields. We begin by selecting one of the Higgs fields which will make up the background, and then placing a second field to the left of this first one. We then allow this second field to hop to the right, over the first, assigning phases for even and odd exchanges as defined previously. This generates two terms. We follow this by placing a third Higgs field to the left of each of these two terms, and again allowing it to hop all the way to the right, so that we are left with a total of 6 terms with their corresponding phases. We continue in this way until all background Higgs fields have been selected. To illustrate this, let us consider building the background out of one Φ^1, one Φ^2 and one Φ^3 field. The steps which we take are as follows:

$$\Phi^1 \rightarrow \Phi^2\Phi^1 + e^{2\pi i\gamma}\Phi^1\Phi^2$$
$$\rightarrow \Phi^3\Phi^2\Phi^1 + e^{2\pi i\gamma}\Phi^2\Phi^3\Phi^1 + \Phi^2\Phi^1\Phi^3 + e^{2\pi i\gamma}\Phi^3\Phi^1\Phi^2 + \Phi^1\Phi^3\Phi^2 + e^{2\pi i\gamma}\Phi^1\Phi^2\Phi^3.$$

We could have selected the background fields in a different order, changing the overall phase of the above operator, but since this is arbitrary, there is no ambiguity. Recall that we are trying to find a BPS state, which is achieved by building the operators in the way described above. We see this because each exchange term we add by hand will correspond to a matching exchange performed by the potential, but with an opposite sign, giving us a BPS state. However, tracing the above operator will not in general reduce to the BPS state we identified in section 5.2, due to our neglect of the exchange swapping the first and last Higgs field. In the leading order of a large J expansion we expect that neglecting this exchange is justified.

Next we will describe how to build operators corresponding to excited string states with two impurities, which we will take to be Φ^1 and Φ^3. Let Φ^3 hop into the nth position using the same rules for hopping described before, and denote the operator obtained in this way by $\mathcal{O}^\gamma_n$. For each $\mathcal{O}^\gamma_n$ let Φ^1 hop into the mth position and call the resulting operator $\mathcal{O}^\gamma_{n,m}$. Now define

$$\mathcal{O}^\gamma_{(p)} = \sum_{n,m} Tr(\mathcal{O}^\gamma_{n,m})\delta_{m-n,p},$$

where $p = 0, 1, ..., J$, and the delta function sets $m - n = p \text{mod} J$. Using these results we can show that

$$\langle \mathcal{O}^\gamma_{(i)}(x_1)\bar{\mathcal{O}}^\gamma_{(j)}(x_2)\rangle = M^\gamma_{ij}\frac{N^{J+2}\tilde{a}^{J+2}}{|x_1-x_2|^{2J+4}}$$

and

$$\langle \mathcal{O}^\gamma_{(i)}(x_1)V_F(y)\bar{\mathcal{O}}^\gamma_{(j)}(x_2)\rangle = H^\gamma_{ik}M^\gamma_{kj}\frac{N^{J+3}\tilde{a}^{J+4}}{|x_1-x_2|^{2J}|x_1-y|^4|x_2-y|^4},$$

with

$$H^\gamma_{ik} = 8\delta_{ik} - 4\delta_{i+1k} - 4\delta_{ik+1}.$$

When computing these correlators, we sum over all contractions except the ones involving the fields that were at the endpoints of $\mathcal{O}^\gamma_{n,m}$, which should give the correct answer in the large J limit. The H above is equivalent to that in section 5.3 except that it does not have the -1 elements in $H_{0,J}$ and $H_{J,0}$, but we expect that this is not important in the large J limit, which turns out to be the case numerically. Thus, in this limit, H^γ has the same spectrum as H in section 5.3. Our proposal for the BMN operators is then to build them using the eigenvectors of H^γ, so that the spectrum of anomalous dimensions coincides with the spectrum of anomalous dimensions of the undeformed theory, just as the string theory predicts.

Now, we have assumed that the anomalous dimensions of our operators are determined by the eigenvalues of H^γ. We studied the eigenvalues and eigenvectors of M_{ij} in the undeformed case and argued the validity of this statement. Should we wish to prove the statement in the deformed case, we would need to provide a parallel study of M^γ_{ij}, which is an open problem.

Lastly, we will consider the small J limit, in which we can work with all of the possible loops $\mathcal{O}_i$ which exhibit $U(1)_\mathcal{R}$ charge $J+2$ and $U(1)\times U(1)$ charge $(1,1)$. We organize our operators so that they have the two point function

$$\langle \mathcal{O}_i\bar{\mathcal{O}}_j\rangle \propto \delta_{ij}$$

at large N and $O(g^0_{YM})$. We repeat our previous analysis, and compute correlators of the form

$$\langle \mathcal{O}_i(x_1) V_F(y) \bar{\mathcal{O}}_j(x_2) \rangle = T_{ij} \frac{N^{J+3} \tilde{a}^{J+4}}{|x_1 - x_2|^{2J} |x_1 - y|^4 |x_2 - y|^4}.$$

Again, the matrix T will determine the operators which have a definite anomalous dimension, and will yield the dimension itself to $O(g_{YM}^2)$. We find that for $J = 3$ and $\gamma = 0.1$ the smallest eigenvalue of T is 0.07843... .For $J = 6$, the smallest eigenvalue takes the value 0.04124... At infinite J, our string theory prediction in section 4.4 tells us that the smallest eigenvalue should be zero. This discrepancy in the smallest eigenvalue for finite and infinite J makes it impossible to compare our finite J field theory results with the string theory results. We also managed to devise an expansion for T in terms of γ, from which it is possible to develop a perturbative expansion for the anomalous dimension, by treating γ as a small number and using the results of Appendix B. We find that the $O(\gamma)$ term is zero. Although we did not extend our analysis to higher orders in the perturbation series, this would be possible in principle.

Chapter 6

Conclusion

Our aim in this work has been to provide further support for the AdS/CFT conjecture in a less supersymmetric setting, which we have achieved via a study of the Lunin-Maldacena background. We have shown that the geometry obtained by taking a particular pp-wave limit of this deformed background results in a homogeneous plane wave geometry, and further that the string spectrum is γ-independent and is thus unaffected by the deformation. The statement of the conjecture then led us to consider the dual $\mathcal{N} = 1$ supersymmetric gauge theory. In this setting, we argued that at $O(g_{YM}^2)$ and leading order in N, it is only the F-terms that contribute to the anomalous dimension for the class of operators under consideration. Using this insight we managed to find the operators in the deformed field theory dual to the vacuum of the string sigma model. Further, in anticipation of a parallel calculation in the deformed theory, we first identified the set of operators dual to excited string modes in the undeformed theory, and noticed that much less than the total number of possible operators (those with adequate quantum numbers) actually participate. This allowed us some insight into the decoupling of a small set of nearly protected states. In the deformed theory, we wrote down a large J proposal for the operators dual to excited string states. Our confidence in the proposal is further cemented by the fact that the anomalous dimensions of these operators are γ-independent, a result which matches that of the string spectrum. A further consideration at finite J and $O(g_{YM}^2)$ yields a γ-dependence in the anomalous dimensions.

Appendix A

SUSY Notation and Conventions

The convention adopted by [3] is that of the "West Coast" metric, namely

$$\eta^{mn} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which most notably affects the Pauli matrices σ^0 and $\bar{\sigma}^0$, which are 2×2 identity matrices as a result.

In terms of the spinors, the irreducible representations of $SL(2, C) \approx SO(1, 3)$ are $(\frac{1}{2}, 0)$ (which is the left-handed 2 component Weyl spinor), and $(0, \frac{1}{2})$ (which is the right-handed 2 component Weyl spinor). Undotted indices correspond to the $(\frac{1}{2}, 0)$ representation, and dotted indices to $(0, \frac{1}{2})$, so that

$$\begin{aligned} \left(\frac{1}{2}, 0\right) \quad &: \quad \psi_\alpha, \\ \left(0, \frac{1}{2}\right) \quad &: \quad \bar{\psi}^{\dot{\alpha}} \equiv (\psi_\alpha)^*. \end{aligned}$$

We can also write

$$\bar{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^\dagger, \qquad \psi^\alpha \equiv (\bar{\psi}_{\dot{\alpha}})^*.$$

The two dimensional Levi-Civita symbols are used to raise and lower these spinor indices. They are defined by

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2,$$

so that we get

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta,$$

$$\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \qquad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}.$$

In addition, we define

$$\sigma^m = (I, \vec{\sigma}) = \bar{\sigma}_m,$$

$$\bar{\sigma}^m = (I, -\vec{\sigma}) = \sigma_m,$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I denotes the 2×2 identity matrix. The bar notation does not mean complex conjugation, so that σ^m has undotted-dotted indices ($\sigma^m_{\alpha\dot{\beta}}$), and $\bar{\sigma}^m$ has dotted-undotted indices ($\bar{\sigma}^{m\dot{\alpha}\beta}$). The Levi-Civita symbols relate σ^m and $\bar{\sigma}^m$ as follows:

$$\bar{\sigma}^{m\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta}\sigma^m_{\delta\dot{\gamma}}, \quad \sigma^m_{\alpha\dot{\beta}} = \epsilon_{\dot{\delta}\dot{\beta}}\epsilon_{\gamma\alpha}\bar{\sigma}^{m\dot{\delta}\gamma}.$$

Finally, we define the $SL(2,C)$ generators as

$$\sigma^{mn\beta}_{\alpha} = \frac{i}{4}[\sigma^m_{\alpha\dot{\gamma}}\bar{\sigma}^{n\dot{\gamma}\beta} - \sigma^n_{\alpha\dot{\gamma}}\bar{\sigma}^{m\dot{\gamma}\beta}],$$

$$\bar{\sigma}^{mn\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}[\bar{\sigma}^{m\dot{\alpha}\gamma}\sigma^n_{\gamma\dot{\beta}} - \bar{\sigma}^{n\dot{\alpha}\gamma}\sigma^m_{\gamma\dot{\beta}}].$$

Appendix B

Eigenvalue Problem

Section 5.3 deals with the operator H, and this appendix solves the eigenvalue problem of this operator [5].

We write the eigenvalue problem as usual

$$H|i\rangle = \lambda_i|i\rangle,$$

and denote the components of the eigenvectors by

$$|i\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{J-1} \\ v_J \end{bmatrix}.$$

This gives us the expression

$$-4v_{n-1} + 8v_n - 4v_{n+1} = \lambda v_n \tag{B.1}$$

for $1 \leq n \leq J-1$ and

$$3v_0 - 2v_1 - v_J = \lambda v_0, \qquad 3v_J - 2v_{J-1} - v_0 = \lambda v_J. \tag{B.2}$$

We solve for the eigenvector components by making the ansatz

$$v_n = Ae^{ikn} + Be^{-ikn},$$

which leads to

$$\lambda = 8 - 8\cos(k)$$

from (B.1), and (B.2) leads us to the equation solved by the allowed values for k:

$$Imag[(\lambda - 3 + 2e^{-ik} + e^{-iJk})(3e^{ikJ} - 2e^{i(J-1)k} - 1 - \lambda e^{iJk})] = 0.$$

As a final step, we can use (B.2) to find A in terms of B, and then use the normalization of the eigenvector to find B.

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