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Domination and distance measures in graphs

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October 31, 2025



Declaration

I, Tshepo Matshisevhe, hereby declare that this dissertation entitled "Domination and distance measures in graphs" is my original work. I certify that:

- This work was completed during my candidature for a master's degree at this University.
- Any previously submitted portions of this dissertation, whether at this University or another institution, have been explicitly identified.
- All references to published works have been appropriately cited.
- Direct quotations from the work of others have been accurately sourced.
- This dissertation is entirely my own work, excluding the cited quotations.
- All significant sources of assistance have been acknowledged.

I affirm the authenticity of this declaration.

Abstract

Let G be a graph. A dominating set of G is a set of vertices such that every vertex in G is either in the set or adjacent to a vertex in the set. The domination number is the size of a smallest dominating set. The minimum degree of graph G is the smallest number of edges incident to any single vertex in the graph. In graph G, the distance between two vertices is the length of a shortest path between them. The eccentricity of a vertex in G is the greatest distance from that vertex to any other vertex in the graph. The radius of G is the smallest eccentricity among all eccentricities of vertices of the graph. The diameter of G is the largest eccentricity amongst all eccentricities of vertices in the graph. The boundary of G consists of all vertices with the highest eccentricity. The average distance of G is mean of all distances between vertex pairs. This dissertation explores the intricate relationships between domination, distance-based parameters, and minimum degree in graphs.

The first part of the dissertation provides bounds on the domination number in terms of radius, order, and minimum degree of the graph. The second part focuses on bounds on domination number involving distances between vertices and the eccentricity of the boundary of the graph. Finally, the third part discusses upper bounds on average distance in terms of the domination number of the graph.

Acknowledgement

I want to thank my supervisors, Professor Simon Mukwembi and Professor Betsie Jonck, for their help and advice. They were patient with me and helped me finish my dissertation on time.

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Chapter 1 Introduction and Preliminaries

1.1 Introduction

Domination, distance, and minimum degree are fundamental concepts in graph theory, with numerous applications in various fields. In recent years, researchers have explored the intricate relationships between these parameters, leading to significant advancements in our understanding of graph structure. In this dissertation, we will establish bounds on the domination number of a graph when its order, minimum degree, radius, diameter and average distance are prescribed.

In this chapter, we lay the groundwork for the dissertation by defining concepts. Section 1.2 introduces graphs and related terms. Section 1.3 focuses on distance measures in graphs, and Section 1.4 discusses domination in graphs.

1.2 Graph theory terminology

We define a *graph* $G = (V, E)$ as a structure consisting of a finite, nonempty set of objects V, called *vertices*, along with a (possibly empty) set E of unordered pairs of distinct vertices of G, called *edges*. The vertex set of G is represented by $V(G)$, and the edge set of G is represented by $E(G)$.

The *order* of a graph G is the number of vertices, $|V(G)|$, in the graph. The *size* of a graph G is the number of edges, $|E(G)|$, in the graph. We typically use n to denote the order and m the size of a graph G.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. We write $v_iv_j \in E(G)$ to indicate the edge $\{v_i, v_j\}$ of G. If $e = v_jv_j \in E(G)$, we say that v_i and v_j are *adjacent* and that e is *incident* to v_i and v_j.

Example 1.2.1

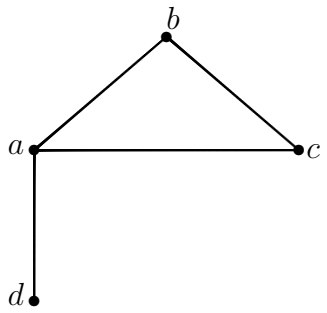


Figure 1.2.1: Graph G.

In the graph G shown in Figure 1.2.1,

$$V(G) = \{a, b, c, d\}$$

and

$$E(G) = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}\}.$$

A graph with only one vertex is called a *trivial graph*, and a graph with two or more vertices is a *nontrivial graph*. A *complete graph*, denoted as K_n, is a graph with n vertices where every pair of vertices is connected by an edge. Note that K_1 represents the trivial graph.

The *open neighbourhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, that is, $N(x) = \{y \in V \mid xy \in E\}$. These vertices are known as

the *neighbours* of x. The *closed neighbourhood* $N[x]$ of x includes x and its open neighbourhood, i.e., $N[x] = \{x\} \cup N(x)$. For a set S of vertices, the *open neighbourhood* $N(S)$ of S is the union of the open neighbourhoods of its vertices, i.e., $N(S) = \bigcup_{x \in S} N(x)$, and the *closed neighbourhood* $N[S]$ is the union of S and its open neighbourhood, i.e., $N[S] = S \cup N(S)$.

The *degree* of a vertex x, denoted $deg_G\ x$, in a graph G is the cardinality of the set $N(x)$. A vertex $x \in V$ is known as an *isolated vertex* if $deg_G\ x = 0$, and as a *leaf* if $deg_G\ x = 1$. In a graph G with n vertices, a vertex x is called a *dominating vertex* if $deg_G\ x = n - 1$. A graph G is termed *k-regular* if each vertex $x \in V$ has $deg_G\ x = k$. The *minimum degree* is the smallest degree in G, denoted as $\delta(G)$. The *maximum degree* of graph G is the largest degree in G, denoted as $\Delta(G)$. The *degree sequence* of G, indicated by $(deg_G\ x_1, deg_G\ x_2, deg_G\ x_3, \ldots, deg_G\ x_n)$, is a list of the degrees of all its vertices, usually arranged in non-increasing order.

In a graph G, a $x - y$ *walk* W between two vertices x and y (which may be the same) is a sequence of vertices that starts at x and ends at y, where each consecutive pair of vertices in W is connected by an edge in G. This walk W can be expressed as:

$$x = x_0, x_1, \ldots, x_k = y,$$

where $x_i x_{i+1} \in E(G)$ for $0 \leq i \leq k - 1$. The walk W can also be denoted by

$$W = x_0 x_1 \cdots x_k.$$

The *length* of W is the count of edges in it. A walk in a graph G that does not repeat any vertex is called a *path*. A *cycle* in a graph is a path whose first and last vertices are identical. A graph G is called a *forest* if it contains no cycles, that is, if it is acyclic.

A graph G is called *connected* if there is a path between every pair of vertices in the graph. A very important class of graphs that is often used when considering distances is a class called *trees*. A connected graph with no cycles is formally known as a *tree*.

We define H as a *subgraph* of G, indicated by $H \leq G$, if $V(H) \subseteq V(G)$ and any edge $xy \in E(H)$ also belongs to $E(G)$. If H also meets the criterion that for any vertices x and y, $xy \in E(H)$ if and only if $xy \in E(G)$, then H is called an *induced subgraph* of G. The induced subgraph H with $S = V(H)$ is referred to as the *subgraph induced by* S and is denoted by $\langle S \rangle$.

In a tree T, a *branch* at a vertex v is a connected subgraph of T that includes v and one of the components that results when v is removed. Equivalently, if u is a neighbour of v, the branch of T at v through u is the subtree consisting of u together with all vertices that are connected to u when v is deleted from T. A subset $P \subseteq V$ is called a *packing* in G if, for any two vertices $x, y \in P$, the condition $N[x] \cap N[y] = \emptyset$ holds. The *packing number* $\rho(G)$ denotes the largest possible size of a packing in G. A subset $P \subseteq V$ is said to be *independent* if no two vertices in P are adjacent in G, and such a set P is termed *maximal* if it is not properly contained in any larger independent set. The *independence number*, $\alpha(G)$, of G denotes the maximum size of an independent set in G. Notably, while $\rho(G)$ measures the maximum number of pairwise non-adjacent vertices that can be "packed" into the graph, $\alpha(G)$ measures the largest set of such vertices that are independent. Essentially, every packing is an independent set, but not every independent set qualifies as a packing, highlighting the connection between these measures.

Example 1.2.2

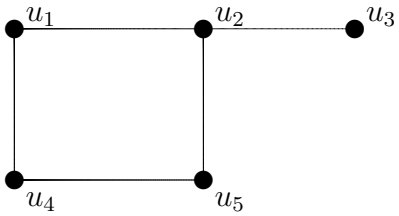


Figure 1.2.2: Graph G.

In the graph shown in Figure 1.2.2, we have $N[u_4] \cap N[u_3] = \emptyset$. Therefore

one packing is $\{u_4, u_3\}$ and $\rho(G) = 2$. We also have the following independent sets

$$\{u_2, u_4\}, \{u_1, u_3\}, \{u_1, u_5\}, \{u_4, u_3\}, \text{ and } \{u_1, u_3, u_5\}.$$

A largest independent set is $\{u_1, u_3, u_5\}$ and therefore $\alpha(G) = 3$.

1.3 Distance concepts

In a graph G, the *distance* $d_G(x, y)$ between vertices x and y is the length of a shortest path connecting x and y. An $x-y$ path of this length is called an $x - y$ *geodesic*.

Example 1.3.1

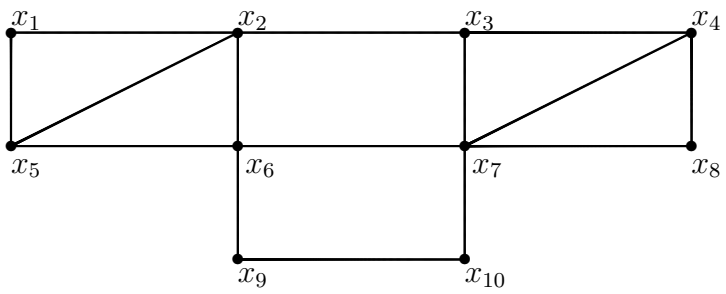


Figure 1.3.3: Graph G.

For the graph G in Figure 1.3.3, we have

$$\begin{aligned} d_G(x_1, x_1) &= 0, \\ d_G(x_1, x_2) &= 1, \\ d_G(x_1, x_6) &= 2, \\ d_G(x_1, x_7) &= 3, \\ d_G(x_1, x_8) &= 4. \end{aligned}$$

For a vertex $x \in V$ in a connected graph G, the *eccentricity* $e(x)$ is defined as $max_{y \in V(G)} d_G(x, y)$. This means $e(x)$ is the maximum distance between x and any other vertex in G.

In a graph G, the *diameter* and *radius* are defined by the *maximum* and *minimum eccentricities* of its vertices, respectively. They are denoted by $diam(G)$ and $rad(G)$.

Example 1.3.2

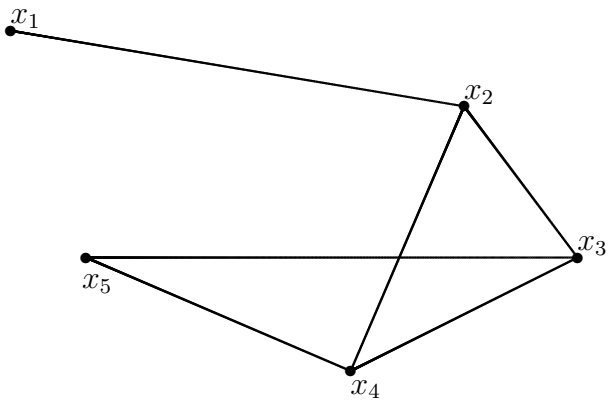


Figure 1.3.4: Graph G.

In the graph G in Figure 1.3.4, we notice that the eccentricities of different vertices are as follows:

$$e(x_1) = 3, e(x_2) = 2, e(x_3) = 2, e(x_4) = 2, e(x_5) = 3.$$

A vertex x is defined as a *central vertex* if $e(x) = rad(G)$. The subgraph induced by the central vertices of G is referred to as the center $Cen(G)$ of G.

In a connected graph $G = (V, E)$ of order n, the *average distance* $\mu(G)$ is the arithmetic mean of shortest-path distances over all unordered pairs of distinct vertices, calculated as

$$\mu(G) = \frac{1}{\binom{n}{2}} \sum_{\{x,y\} \subseteq V} d_G(x, y).$$

The *transmission* or *total distance* $\sigma(v)$ of a vertex v in a connected graph G is given by

$$\sigma(v) = \sum_{y \in V} d_G(v, y).$$

The *total distance* $\sigma(G)$ of G is defined as

$$\sigma(G) = \sum_{y \in V} \sigma(y).$$

Note that

$$\begin{aligned} \mu(G) &= \frac{1}{\binom{n}{2}} \frac{1}{2} \sigma(G) \\ &= \frac{\sigma(G)}{n(n-1)}. \end{aligned}$$

Example 1.3.3

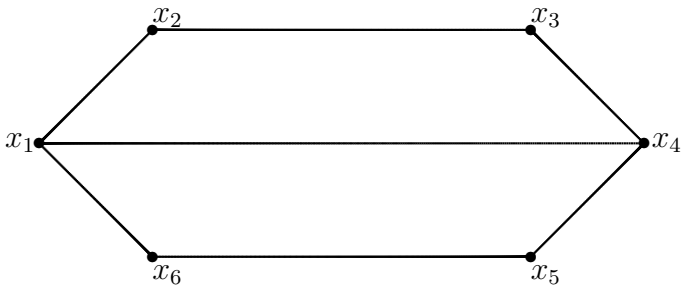


Figure 1.3.5: Graph G.

In the graph G in Figure 1.3.5, the total distances of vertices are calculated

as follows:

$$\begin{aligned}
\sigma(x_1) &= 3 \cdot 1 + 2 \cdot 2 = 7, \\
\sigma(x_2) &= 1 \cdot 2 + 2 \cdot 2 + 3 = 9, \\
\sigma(x_3) &= 2 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 = 9, \\
\sigma(x_4) &= 3 \cdot 1 + 2 \cdot 2 = 7, \\
\sigma(x_5) &= 2 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 = 9, \\
\sigma(x_6) &= 2 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 = 9.
\end{aligned}$$

Then

$$\begin{aligned}
\sigma(G) &= \sum_{x \in V} \sigma(x) = 50, \\
\mu(G) &= \frac{1}{\binom{n}{2}} * \frac{\sigma(G)}{2} \\
&= \frac{\sigma(G)}{n(n-1)} \\
&= \frac{50}{6(6-1)} \\
&= 1.67.
\end{aligned}$$

In a graph G, a geodesic that has a length of $diam(G)$ is known as a *diametral path*. We define the *boundary* of G as the set

$$B(G) = \{x \in V(G) \mid e(x) = diam(G)\},$$

which will be referred to as B subsequently.

We have seen how we define the eccentricity of a vertex. Let us now define the *eccentricity of a set*. Consider a set $S \subseteq V(G)$ and a vertex $x \in V(G)$. The distance $d_G(x, S)$ between vertex x and S is defined as

$$d_G(x, S) = min\{d_G(x, y) \mid y \in S\}.$$

The *eccentricity* of $S \subseteq V(G)$ is given by

$$e(S) = max\{d_G(y, S) \mid y \in V(G)\}.$$

Example 1.3.4

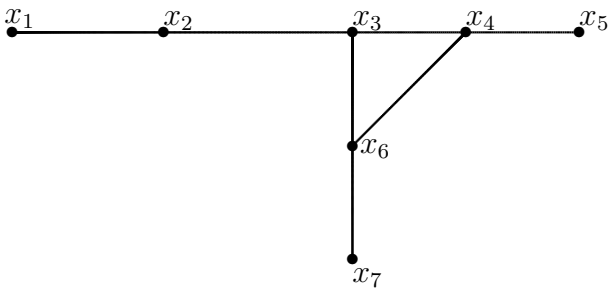


Figure 1.3.6: Graph G.

Consider the graph G in Figure 1.3.6. Given the set $S = \{x_3, x_6\}$, then

$$\begin{aligned}
&d_G(x_1, S) = d_G(x_1, x_3) = 2,\\
&d_G(x_2, S) = d_G(x_2, x_3) = 1,\\
&d_G(x_3, S) = d_G(x_3, x_3) = 0,\\
&d_G(x_4, S) = d_G(x_4, x_3) = d_G(x_4, x_6) = 1,\\
&d_G(x_5, S) = d_G(x_5, x_3) = d_G(x_5, x_6) = 2,\\
&d_G(x_6, S) = d_G(x_6, x_6) = 0,\\
&d_G(x_7, S) = d_G(x_7, x_6) = 1.
\end{aligned}$$

It follows that $e(S) = 2$. Note that $diam(G) = 4$ and so the boundary of G is $B = \{x_1, x_5, x_7\}$.

1.4 Domination concepts

In any graph G, each vertex x is regarded as exerting dominance over itself and all vertices that share an edge with it. This collection of vertices is referred to as the *closed neighbourhood* of x, denoted by $N[x]$. For two subsets $X, Y \subseteq V(G)$, the set X is said to *dominate* Y when every element of Y belongs to the closed neighbourhood of X, that is, $Y \subseteq N[X]$. A subset $S \subseteq V(G)$ is called a *dominating set* if every vertex of G is contained in S or is adjacent to some vertex in S, which can be expressed as $N[S] = V(G)$. The smallest possible size of such a set is known as the *domination number* of G, denoted by $\gamma(G)$. Any dominating set whose cardinality equals $\gamma(G)$ is referred to as a $\gamma(G)$-set.

Example 1.4.1

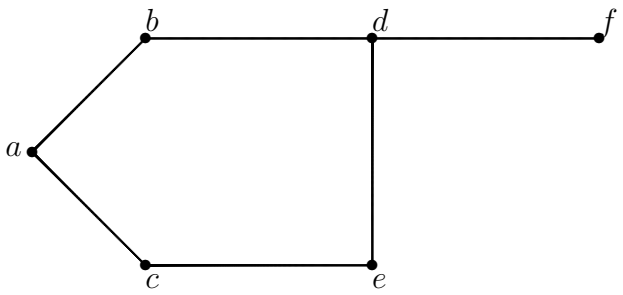


Figure 1.4.7: Graph G.

In the graph G shown in Figure 1.4.7, the sets $\{a, d\}$, $\{a, e, f\}$, $\{b, d, e\}$, and $\{c, d\}$ are all dominating sets of G. Notice that $\{f, a\}$ is not a dominating set because vertex e is not dominated. A $\gamma(G)$-set is $\{a, d\}$, and the domination number is given by $\gamma(G) = 2$.

1.5 Conclusion

This chapter laid the groundwork for the dissertation by introducing key definitions and concepts. Section 1.1 outlined the scope of our research. Section 1.2 presented fundamental graph-theoretic definitions. In Sections

1.3 and 1.4, we focused on specific concepts of graph distance and domination number. The next chapter will delve into the existing literature to contextualize our work.

Chapter 2 Literature Review

2.1 Introduction

Graph theory has experienced significant advancements over the years, particularly with the growth of fields such as computer science, electrical and computer engineering, and operations research. Within this discipline, one of the most rapidly expanding areas is the exploration of domination and related subareas, including independence, covering, and matching. The ongoing evolution of these concepts highlights their importance in both theoretical research and practical implementations, making them integral to advancements in modern computing and engineering.

The foundational study of dominating sets in graphs is attributed to Cockayne and Hedetniemi's 1977 survey paper, which included 20 references and is credited with launching the modern exploration of this topic. By 1990, Hedetniemi and Laskar edited a special issue of *Discrete Mathematics* focused entirely on domination, showcasing an extraordinary growth in the field, with references expanding from 20 to approximately 400 over 13 years. Just seven years later, the body of research on dominating sets and related topics surpassed 1,000 publications, underscoring the field's rapid and ongoing development.

This dissertation explores the bounds on domination in terms of distance measures and minimum degree. Specifically, we study bounds on domination number in terms of three distance measures namely radius, diameter,

and average distance, as considered in [1], [2], and [3]. We investigate bounds on domination number in terms of radius, order, and minimum degree of the graph. Further, we examine bounds on the domination number with regard to the diameter, average distance, and the eccentricity of the boundary. In conclusion, we examine the characterization provided by Dankelmann [4] for connected graphs of a given order and domination number that achieve the maximum possible average distance.

In this chapter, we will begin by discussing the motivation behind studying bounds on the domination number in relation to the three distance measures. Following this, we will review existing background work on the bounds of the domination number.

2.2 Motivation and Background

2.2.1 Motivation

Domination, distance, and minimum degree are key concepts in graph theory, with extensive applications across various disciplines. Understanding the complex relationships between these parameters is essential for analyzing graph structures and properties.

Average distance, which measures the mean distance between pairs of vertices, is a significant graph parameter. It is widely used in fields like network science, chemistry, and social network analysis. For example, in network science, average distance helps evaluate the efficiency of information flow within a network.

In computer science, domination problems play a crucial role in network design, where resources must be allocated efficiently to monitor or control nodes. Distance measures are vital for optimizing communication in routing protocols, while minimum degree considerations are critical for ensuring network reliability, as high-degree nodes enhance robustness.

In social network analysis, domination concepts identify key influencers or groups, while distance measures assess social proximity between individuals. Similarly, in chemistry, distance-based indices like the Wiener index help predict molecular properties.

This research investigates the interplay among domination, distance, and minimum degree, contributing to advancements in graph theory and its applications in fields such as computer science, network science, chemistry, and social network analysis.

2.2.2 Background

The concept of domination in graph theory was first formally introduced by Berge in 1958 and later developed further by Ore in 1962. Since the survey by Cockayne and Hedetniemi [7] in 1977, domination theory has garnered significant attention. The following problem is considered the origin of the study of dominating sets in graphs. On a standard 8×8 chessboard, a queen can move any number of squares horizontally, vertically, or diagonally, as long as no other piece obstructs its path. Chess enthusiasts in Europe, during the 1850s, posed the challenge of determining the minimum number of queens required to ensure that every square on the chessboard is either occupied by a queen or under attack by one. It was correctly determined that five is the minimum number of queens needed to dominate all the squares on an 8×8 chessboard. This problem, known as the Five Queens Problem, involves finding such a dominating set of five queens [6].

Domination theory has attracted substantial interest owing to its diverse real-world applications in areas such as network design, power system monitoring, facility location, social and biological sciences, and other branches of mathematics. The book *Domination in Graphs: Core Concepts* [8] delves into the fundamental concepts of domination in graphs, as suggested by its title: domination, total domination, and independent domination. It highlights key results on these types of domination, accompanied by numerous

proofs that exemplify many proof techniques used in domination theory.

The book [10] provides a broad survey of domination theory, organized into three major parts. It begins with key domination concepts such as broadcast domination, alliances, domatic numbers, dominator colorings, irredundancy, and variations of Roman domination, as well as domination in spectral graph theory. The second part extends these ideas to more specialized structures including hypergraphs, chessboards, digraphs, and tournaments. The final part highlights algorithmic developments and computational complexity, addressing signed, minus, majority, and power domination, along with alliances in graphs, and concludes with an overview of self-stabilizing algorithms.

In contrast, [9] concentrates on major domination types that received little coverage in [8]. Out of the many existing variations, over seventy identified in the literature the book focuses on the most extensively studied ones, including paired, connected, restrained, multiple, distance, fractional, Roman, rainbow, locating, eternal, secure, global, stratified, and power domination.

2.2.3 Strategic Evolution

During the 4th century AD, the Roman Empire faced increasing challenges in maintaining control over its vast territories across Europe, North Africa, and Asia Minor. Under Emperor Constantine (306–337 AD), the army was restructured into mobile field units and local militias to protect the Empire from both internal revolts and external invasions. This defensive organization parallels the concept of domination in graph theory, where each region (vertex) must either be secured directly or be adjacent to a secured region containing a defensive unit, thereby forming a dominating set [8].

2.3 Survey of Important Results

This section offers a brief summary of bounds on various domination parameters, emphasizing those expressed in relation to the graph's order, minimum degree, radius, diameter, and average distance. Furthermore, it includes a set of open problems and conjectures aimed at encouraging further research in this field.

2.3.1 Bounds on domination

Determining the domination number, total domination number, or independent domination number for an arbitrary graph is an NP-complete problem. Owing to the difficulty of computing these exact values, much of the existing research concentrates on deriving precise upper and lower bounds for these parameters. Ore [12] established the following bound for isolate-free graphs of order n, $\gamma(G) \leq \frac{n}{2}$. In 1973, Blank [13] demonstrated that for any connected graph with a minimum degree of at least 2, the domination number bound of $\frac{1}{2}n$ from Ore's Theorem can be refined to $\frac{2}{5}n$, given that the order satisfies $n \geq 8$. In 1996, Reed [37] presented a refined proof demonstrating that for a graph with a minimum degree of at least 3, $\gamma(G) \leq \frac{3}{8}n$.

Since then, several improvements of this bound were proved (see, for instance [7] and references therein). Numerous other bounds on domination number in terms of other graph parameters were presented in [8]. In 2010, DeLaviña, Pepper, and Waller [11] established the following lower bound on the domination number in terms of the radius $\gamma(G) \geq \frac{2}{3}rad(G)$. This inequality is sharp. For instance, when $G = C_{6k}$ with $k \geq 1$, we have $rad(G) = 3k$ and $\gamma(G) = 2k$, which satisfies $\gamma(G) = \frac{2}{3}rad(G)$.

Using the result of [5] on the existence of un-related vertices, Henning and Mukwembi [1] showed that $\gamma(G) \leq n - \frac{2}{3}(r-6)\delta$ provided that G has a minimum degree $\delta \geq 3$ and $r \geq 6$. We will study and understand sharp

bounds on domination number in terms of radius, order and minimum degree of a graph as proved by Henning and Mukwembi [1]. They proved that if G is a connected graph of order n, domination number $\gamma(G)$ and radius r, then $\frac{2}{3}r \leq \gamma(G) \leq n - \frac{4}{3}r + \frac{2}{3}$, where equality is achieved in the upper bound if, and only if, G is a path or a cycle on n vertices with $n \equiv 4 \pmod 6$.

Trees are among the simplest graph families, making it unsurprising that extensive research has been conducted on domination within trees. A large number of problems that are NP-complete in general graphs can be efficiently solved in polynomial time when limited to the class of trees. Interest in trees also stems from their status as minimally connected graphs, meaning that among all connected graphs with n vertices, trees have the fewest possible edges, specifically $n-1$. Because of their hierarchical and branching nature, trees serve as fundamental models in numerous disciplines, including data organization in computer science and molecular representation in chemistry. The study of domination in trees contributes to understanding these and other applications.

In [2] Kang proved, for a graph G with $V(G) = \{x_1, x_2, \ldots, x_n\}$, that

$$\gamma(G) \geq \begin{cases} \frac{1}{r(r-1)} \sum\limits_{1 \leq i < j \leq r} d(x_i, x_j), & \text{if } r \geq 3, \\ \frac{1}{3} \sum\limits_{1 \leq i < j \leq r} d(x_i, x_j), & \text{if } r = 2. \end{cases}$$

for any r vertices $\{x_1, x_2, \ldots, x_r\} \subseteq V(G)$. This result establishes a lower bound on the domination number with regards to the sum of an arbitrary number of vertices in a graph. We will explore bounds on the domination number in relation to distances. Specifically, we will establish a lower bound on the domination number based on the distance between three vertices and subsequently generalize this result to an arbitrary number of vertices.

We will investigate the bounds on domination number in relation to the average distance and as an application of the main result, we will investi-

gate the proof of a conjecture of DeLaviña, Pepper, and Waller [11] that $\gamma(G) \geq \frac{1}{2}(e(B) + 1)$ with $e(B)$ denoting the eccentricity of the boundary of a graph G.

We will further investigate a proof by Dankelmann [4] on an upper bound on average distance in relation to domination number; this proof is given by Rautenbach in [3]. Rautenbach provides a concise and simplified proof that eliminates the complex computations employed in [4].

A lot of research has been done and is continuing to be done on bounds on the domination number with regards to various graph parameters. This includes studies on graph families like trees, which are minimally connected and widely applicable, and results linking the domination number to parameters such as eccentricity, minimum degrees, order, size, radius, diameter, and average distance. In the next section, we will present relevant results that are important for this study.

2.3.2 Domination and order

We begin by stating a basic theorem from book [7]. A dominating set is said to be *minimal* if no proper subset of it is itself a dominating set.

Theorem 2.3.1. *Let S be a minimal dominating set in a graph G. Then each $v \in S$ must satisfy at least one of the following conditions:*

(i) there is a vertex $w \in V(G) \setminus S$ such that $N(w) \cap S = \{v\}$;

(ii) the vertex v is not adjacent to any other vertex in S, making it isolated within the set.

Proof. Assume that S is a minimal dominating set. By definition, removing any vertex v from S results in a set $S \setminus \{v\}$ that is no longer a dominating set. This implies that there must exist a vertex $w \in V(G) \setminus (S \setminus \{v\})$ that is not dominated by $S \setminus \{v\}$, meaning w has no neighbours in $S \setminus \{v\}$. Now, we consider two possibilities for w. If $w = v$, it has no adjacent vertices in S

and condition (ii) holds. If $w \neq v$, then since $w \notin S$ and S is a dominating set, w must have at least one neighbour in S. However, given that w is not adjacent to any vertex in $S \setminus \{v\}$, it follows that $N(w) \cap S = \{v\}$ and condition (i) holds. □

In a graph that contains no isolated vertices, the following Ore's result highlights a property of the complementary set of a minimal dominating set.

Lemma 2.3.1. *[7] Let G be a graph with no isolated vertices, and let S be a minimal dominating set of G. Then, the set of vertices $V(G) \setminus S$ is also a dominating set.*

Proof. We prove the result by contradiction. Assume that S is a minimal dominating set and suppose that $V(G) \setminus S$ is not a dominating set of G. This means that there exists at least one vertex $v \in V(G)$ that is not in $V(G) \setminus S$ and that is not adjacent to any vertex in $V(G) \setminus S$. Since v is not adjacent to any vertex in $V(G) \setminus S$, it must be adjacent to a vertex in S. Since v is not in $V(G) \setminus S$, $v \in S$. Therefore , v would have to be adjacent to at least one vertex in $V(G) \setminus S$ since S is a minimal dominating set. This contradicts our assumption that v is not adjacent to any vertex in $V(G) \setminus S$. □

Ore's Theorem [12], established in 1962, is a cornerstone of domination theory. It offers an upper bound on the domination number $\gamma(G)$ of a graph G with no isolated vertices, based on their order.

Theorem 2.3.2. *[7] In a graph G with order n and no isolated vertices, the inequality*

$$\gamma(G) \leq \frac{n}{2}$$

holds for the domination number.

Proof. Let S be a minimal dominating set of G. Then according to Lemma 2.3.1 $V(G) \setminus S$ also is a dominating set of G. Thus

$$\gamma(G) \leq min\{|S|, |V(G) \setminus S|\} \leq \frac{n}{2}.$$

□

Blank [13] demonstrated in 1973 that if the minimum degree of a connected graph is at least 2, the domination number bound of $\frac{1}{2}n$ from Ore's Theorem can be improved to $\frac{2}{5}n$, for graphs where $n \geq 8$.

Theorem 2.3.3. *[13] For a connected graph G with order n and minimum degree $\delta(G) \geq 2$, the domination number $\gamma(G)$ satisfies*

$$\gamma(G) \leq max\left\{\frac{n+2}{3}, \frac{2}{5}n\right\}.$$

Theorem 2.3.3 directly leads to the following result.

Theorem 2.3.4. *[13] For a connected graph G of order $n \geq 8$ and minimum degree $\delta(G) \geq 2$, the domination number $\gamma(G)$ satisfies*

$$\gamma(G) \leq \frac{2}{5}n.$$

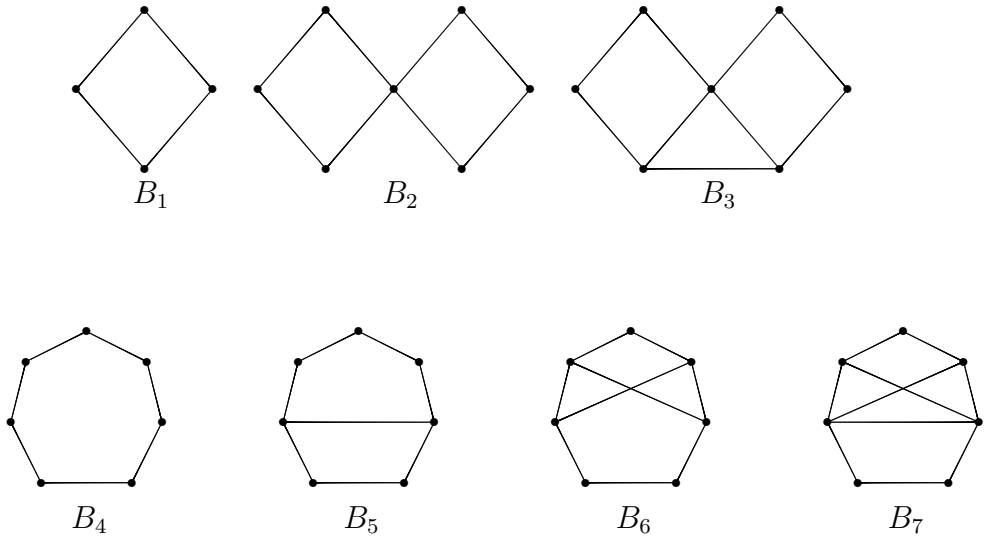


Figure 2.3.1: The family $\mathcal{B}_{dom}$ containing seven graphs G such that $\gamma(G) > \frac{2}{5}n$.

McCuaig and Shepherd [15] presented a novel and insightful proof of Theorem 2.3.4 in 1989. Their proof identified the exceptional graphs of small order $n \leq 7$ that do not satisfy the $\frac{2}{5}n$ bound. Let

$$\mathcal{B}_{dom} = \{B_1, B_2, \ldots, B_7\}$$

denote the set of seven graphs, consisting of one graph of order four and six graphs of order seven. Each graph $G \in \mathcal{B}_{dom}$ with order n satisfies $\delta(G) = 2$ and $\gamma(G) > \frac{2}{5}n$.

In this dissertation we prove the following important result which gives us the domination of a path in terms of the order of the path and will be used to prove other results.

Fact 2.3.1. *For $n \geq 1$, $rad(P_n) = \lfloor \frac{n}{2} \rfloor$. For $n \geq 3$, $rad(C_n) = rad(P_n)$.*

The above result also gives us the following result,

$$\gamma(P_n) \leq \frac{n+2}{3},$$

where equality holds precisely when $n \equiv 1 \pmod 3$.

2.3.3 Domination, order, minimum degree, radius, and diameter

Consider a connected simple graph G with specified parameters: order, diameter, radius, and minimum degree. Utilizing set theory and standard graph-theoretic results, upper bounds on the domination number $\gamma(G)$ are derived based on these parameters through explicit constructions. It is demonstrated that these bounds remain valid for connected graphs.

The exploration of the radius and diameter of a graph in relation to other graph parameters has a rich history, and research in this area continues to evolve. Notable contributions from Erdős et al. [5] and the Danklemann-Entringer [18] method represent some of the effective tools used to establish asymptotically tight bounds on a graph's radius and diameter. Additionally, methods that begin with a diametrical path of graph G have often resulted in asymptotically precise bounds for these parameters.

In 1869 Jordan [39] gave an important result which we apply in this dissertation, for any tree T, one of the following holds: either $diam(T) = 2rad(T)$, in which case the center $Cen(T)$ consists of a single vertex, or $diam(T) = 2rad(T) - 1$, in which case $Cen(T)$ is formed by two adjacent vertices. Particularly, the next result which will be established later in this dissertation follows from this observation.

Fact 2.3.2. *In any tree T, it holds that $diam(T) \geq 2rad(T) - 1$.*

The exploration of domination in graphs is an ongoing process, as evidenced by recent research in [19], [20], [21], and [22]. It is intriguing to further explore the bounds of these domination parameters.

In [17], Mafuta establishes new bounds on the domination number, building on and strengthening the results of Henning and Mukwembi [1], Mafuta investigates bounds for the domination number by applying set theory in conjunction with Fact 2.3.2, and the Danklemann-Entringer [18] method to enhance the results presented in [1]. Specifically, Mafuta aims to improve

the following results.

Theorem 2.3.5. *[1] Let G be a connected graph with $rad(G) = r \geq 1$. Then, the following inequalities are satisfied:*

$$\gamma(G) \geq \frac{1}{3}(diam(G) + 1)$$

and

$$\frac{2}{3}r \leq \gamma(G) \leq n - \frac{2}{3}(2r - 1).$$

We also give proofs of the bounds in Theorem 2.3.5 in this dissertation. We further explore bounds on domination in terms of order, radius and minimum degree of the graph.

Theorem 2.3.6. *[1] If G is a connected graph with radius $rad(G) = r \geq 6$ and minimum degree $\delta = \delta(G) \geq 3$, then*

$$\gamma(G) \leq n - \frac{2}{3}(r - 6)\delta.$$

The bound is asymptotically tight, as there exist graphs that meet the theorem's conditions such that

$$\gamma(G) = n - \frac{2}{3}(r + \frac{3}{2})\delta.$$

We give a detailed proof of Theorem 2.3.6 in this dissertation.

In [17] Mafuta strengthened Theorem 2.3.6. For all $\delta > 2$, the result also strengthens Theorem 2.3.5.

Theorem 2.3.7. *[17] Let G be a connected graph with $rad(G) = r$ and $\delta \geq 2$. Then,*

$$\gamma(G) \leq n - \frac{1}{3}(diam(G) + 1)\delta$$

and

$$\gamma(G) \leq n - \frac{2}{3}(r - \frac{1}{2})\delta.$$

Furthermore,

$$\gamma(G) \leq n - \frac{2}{3}r\delta$$

for $r \equiv 0 \pmod 3$. *All bounds are sharp.*

It is interesting that the same family C_n for $n \equiv 2 \pmod 6$, used in the proof of the above theorem is also stated in [1] and attains the second bound of Theorem 2.3.7. The ideas employed in the proof of Theorem 2.3.7 yields Theorem 2.3.8 below.

Theorem 2.3.8. *[17] Consider a connected, C_3-free graph G with $\delta \geq 3$ and $rad(G) = r$. Then,*

$$\gamma(G) \leq n - \frac{1}{2}(diam(G) + 1)(\delta - 1)$$

and

$$\gamma(G) \leq n - (r - 1)(\delta - 1).$$

To explore this further, one could aim to improve Theorem 2.3.8 by deriving sharper bounds for each integer $c \in \{0, 1, 2, 3\}$ where $r \equiv c \pmod 4$.

2.3.4 Domination and average distance

In 1997, Dankelmann [4] established an upper bound on the average distance of a graph based on its order and domination number, identifying the extremal graphs that achieve this bound. The *Wiener Index*, which differs from the average distance by a factor and introduced by chemist Wiener in 1947 [27], characterizes various chemical species, such as alkanes, alkenes, and arenes [29]. This index correlates with several physical and chemical properties such as boiling point, refractive index, surface tension, viscosity, melting point, and chromatographic retention time [30]. Numerous bounds for the average distance have been derived in relation to other graph parameters. Plesnik [31] investigated how the diameter, radius, and average distance are interrelated, while Cerf et al. [32] established a lower bound on the average distance based on the Moore bound. Tomescu and Melter [33] identified the extremal values of the average distance for graphs of given order and chromatic number, results that were later refined for 2-connected graphs [34]. Algorithmic approaches to computing average distance have

also been examined in [35] and [36]. The GRAFFITTI program [41] conjectured:

$$\mu(G) \leq \alpha(G)$$

where $\alpha(G)$ is the independence number. Chung [42] proved this conjecture, showing equality only for the complete graph. In [43], sharp bounds on μ, depending on independence number and order, were established. The extremal graphs achieving the upper bound consist of two nearly equal complete graphs or stars connected by a path. Similar structures are likely candidates for solving extremal average distance problems. Indeed, graphs with a given order, matching number, and maximum average distance exhibit this structure [43]. Dankelmann [4] applies similar principles to extremal graphs with a given order, domination number, and maximal average distance. Dankelmann [4] derived upper bounds on vertex transmission based on graph order and domination number, crucial for proving the main theorem. This useful lemma is credited to Walikar and Acharya [44].

Lemma 2.3.2. *[44] Consider a graph G. The conditions $\gamma(G-e) > \gamma(G)$ for each edge $e \in E(G)$, and G being a union of vertex disjoint stars, are equivalent.*

An immediate implication of Lemma 2.3.2, or equivalently from the definition of the domination number, is that every connected graph G possesses a spanning tree T with the same domination number. Consequently, any extremal graph G of fixed order and domination number that attains the maximum average distance must be a tree. If a connected graph G has maximal average distance among all connected graphs of the same order, then G must be a tree. Suppose, for contradiction, that G is not a tree. Then G contains at least one cycle, and we can remove an edge from this cycle without disconnecting G. Let G' be the resulting graph after removing such an edge. Since removing an edge from a cycle does not decrease distances between any pair of vertices, the average distance in G' is at least that of G, and in fact, it strictly increases. This contradicts the assumption that G has maximal average distance. Thus, G must be a tree.

The following lemma, which extends a result of Zelinka [46], asserts that

in a tree, every vertex with maximum transmission is necessarily an end-vertex.

Lemma 2.3.3. *[4] For a connected graph G with a cut vertex $v \in V(G)$, there exists a vertex $w \in N(v)$ such that $\sigma(w) > \sigma(v)$. Additionally, if G is a tree and v is neither an end vertex nor adjacent to an end vertex, then there is a vertex w, adjacent to an end vertex, where $\sigma(w) > \sigma(v)$.*

This lemma is proved by first examining the behavior of G when a cut vertex v is removed, leaving components $G_1, G_2, \ldots, G_k$. Focusing on G_1, the component with the smallest order, and selecting a vertex w from the neighbourhood of v within G_1, they established the inequality $\sigma(w) > \sigma(v)$.

This dissertation explores a streamlined approach to proving Dankelmann's characterization of connected graphs, focusing on those with a specified order and domination number that yield the greatest average distance. The following definition characterizes extremal graphs in which a single vertex attains the maximum transmission among all graphs of a given order and domination number. It suffices to restrict our attention to trees.

Definition 2.3.1. *[4] Let $H_{n,\gamma}$ be defined for positive integers n and γ as follows. When $1 \leq \gamma \leq \frac{n}{3}$, $H_{n,\gamma}$ consists of a path $P_{3\gamma-1} = v_1 v_2 \ldots v_{3\gamma-1}$ and independent vertices $w_1, \ldots, w_{n+1-3\gamma}$, all connected to $v_{3\gamma-1}$ (see Figure 2.3.2). For $\frac{n}{3} < \gamma \leq \frac{n}{2}$, $H_{n,\gamma}$ is formed from a path $P_{2n-3\gamma+1} = v_1 v_2 \ldots v_{2n-3\gamma+1}$ and independent vertices $w_{3n-6\gamma+3}, \ldots, w_{2n-3\gamma+1}$, with vertices v_i joined to w_i for $3n - 6\gamma + 3 \leq i \leq 2n - 3\gamma + 1$ (see Figure 2.3.3).*

Ore [12] showed that any graph with n vertices and no isolated vertices has a domination number of at most $\frac{n}{2}$.

Example 2.3.1

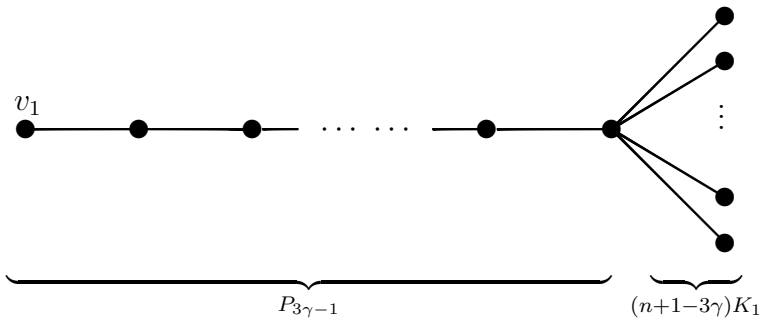


Figure 2.3.2: The extremal graph $H_{n,\gamma}$ from Definition 2.3.1 and Lemma 2.3.4 for $\gamma \leq \frac{n}{3}$.

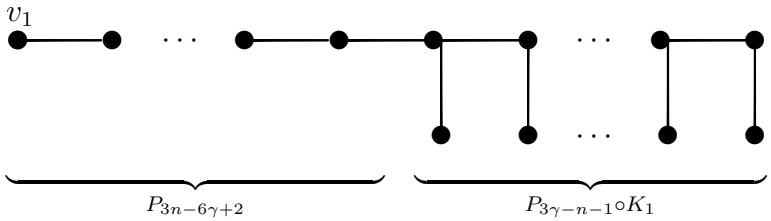


Figure 2.3.3: The extremal graph $H_{n,\gamma}$ from Definition 2.3.1 and Lemma 2.3.4 for $\gamma > \frac{n}{3}$.

The difference in the shapes of the extremal graphs for $\gamma \leq \frac{n}{3}$ and $\gamma > \frac{n}{3}$ arises from the fact that the path P_n, which is the unique graph of order n that maximizes the transmission of a vertex, has a domination number of $\lceil \frac{n}{3} \rceil$.

Lemma 2.3.4 ([4]). *Let G be a tree of order n and domination number $\gamma(G)$. Then, for each vertex $v \in V(G)$,*

$$\sigma(v,G) \leq \begin{cases} (3\gamma-1)(n-\frac{3}{2}\gamma) & \text{if } \gamma \leq \frac{n}{3}, \\ (2n-3\gamma+1)^2 - \frac{1}{2}(3n-6\gamma+3)(3n-6\gamma+2) & \text{if } \gamma > \frac{n}{3}. \end{cases}$$

Equality holds if and only if $G = H_{n,\gamma}$ and $v = v_1$.

The above lemma was proved by first considering the case $\gamma(G) \leq \frac{n}{3}$. Let D be a minimum dominating set of G, and let W represent a shortest

path joining two diametral vertices. It was shown that each vertex in D can dominate at most three vertices of W, which leads to

$$\sigma(v,G) \leq (1+2+\cdots+(3\gamma-2))+(3\gamma-1)(n-3\gamma+1) = (3\gamma-1)\Big(n-\frac{3}{2}\gamma\Big).$$

For $\gamma > \frac{n}{3}$ they first proved the following bound on the diameter of a tree H of order n and domination number at least $\gamma \geq \frac{n+2}{3}$,

$$diam(H) \leq 2n - 3\gamma + 1$$

by induction on n. They then assumed $n \leq 3\gamma - 3$ and let H have a maximum diameter among all trees of order n and domination number at least γ. Let a, z be a diametral pair of vertices. Then a is an end vertex with a unique neighbour b. By their choice of a and z, each neighbour of b except one is an end vertex and this leads to contradiction and together with induction this concludes the proof of the lemma.

To ensure completeness, they verified that the bound in Lemma 2.3.4 applies to all connected graphs and identified the extremal graphs. The previous lemma enabled Dankelmann to prove the following sharp upper bound on the average distance of a graph G in terms of order n and domination number γ. The two cases $\gamma \leq \frac{n}{3}$ and $\gamma > \frac{n}{3}$ were treated separately.

Definition 2.3.2. *For positive integers n, γ with $\gamma \leq \frac{n}{3}$ let $G_{n,\gamma}$ denote the graph obtained from a path $P_{3\gamma-2}$ with end vertices v_1, v_2, which are identical if $\gamma = 1$, and two independent sets of vertices W_1, W_2 of order $\lceil \frac{n-3\gamma+2}{2} \rceil$ and $\lfloor \frac{n-3\gamma+2}{2} \rfloor$, respectively, by joining each vertex of W_i to v_i, $i = 1, 2$.*

Theorem 2.3.9. *[4] Let G be a connected graph of order n and domination number $\gamma \leq \frac{n}{3}$. Then we have*

$$\mu(G) \leq \begin{cases} \frac{n+1}{3} - \frac{(n-3\gamma)(n-3\gamma+2)(2n+3\gamma-7)}{6n(n-1)} & \text{if } n-\gamma \text{ is even,} \\ \frac{n+1}{3} - \frac{(n-3\gamma)(n-3\gamma+2)(2n+3\gamma-7)-9(\gamma-1)}{6n(n-1)} & \text{if } n-\gamma \text{ is odd.} \end{cases}$$

Equality holds if and only if $G = G_{n,\gamma}$.

Proof by induction of Theorem 2.3.9 was done on the order of the graph

n. Suppose that the theorem holds for smaller values of n and then extend the proof to larger values. In the base case, the theorem is verified for small values of n and γ, specifically $n \leq 6$. The proof then has a graph modification where an edge xy was removed from the graph and the resulting components G_x and G_y were examined. Then the relationship between the domination number of the modified graph and the original graph was explored. A contradiction argument was also used to prove the maximality of $\sigma(G)$. Lemma 2.3.2 and Lemma 2.3.3 were also applied.

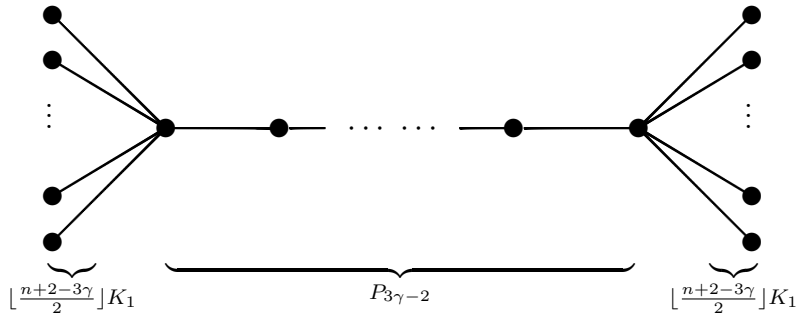


Figure 2.3.4: The extremal graph $G_{n,\gamma}$ from Definition 2.3.2 and Theorem 2.3.9.

In [4], it was demonstrated that $G_{n,\gamma}$ attains the maximum average distance among all graphs with order n and domination number γ. Extremal graphs for $\gamma(G) > \frac{n}{3}$ are now determined.

Definition 2.3.3. *For positive integers n, γ with $\gamma > \frac{n}{3}$ let $G_{n,\gamma}$ denote the graph obtained from a path $P = (v_1, v_2, \ldots, v_{2n-3\gamma})$ and two independent sets of vertices $W_1 = \{w_1, w_2, \ldots, w_{\lceil \frac{3\gamma-n}{2} \rceil}\}$ and $W_2 = \{w'_1, w'_2, \ldots, w'_{\lfloor \frac{3\gamma-n}{2} \rfloor}\}$, respectively, by joining v_j to w_j and $v_{2n-3\gamma+1-j}$ to w'_j for $j = 1, \ldots, \lceil \frac{3\gamma-n}{2} \rceil$.*

Theorem 2.3.10. *[4] Let G be a graph of order n and domination number $\gamma \geq \frac{n}{3}$. Then the average distance satisfies*

$$\mu(G) \leq \begin{cases} \frac{n+1}{3} - \frac{(3\gamma-n)(3\gamma-n-2)(5n-6\gamma-4)}{3n(n-1)}, & \text{if } n-\gamma \text{ is even,} \\ \frac{n+1}{3} - \frac{(3\gamma-n-1)(3\gamma-n-3)(5n-6\gamma-2)+6(2n-3\gamma-1)}{3n(n-1)}, & \text{if } n-\gamma \text{ is odd.} \end{cases}$$

Equality holds if and only if $G = G_{n,\gamma}$.

Dankelmann proved the above Theorem 2.3.10 by induction on n in [4].

In this dissertation we present a shorter and simpler proof. Let n and γ be positive integers. A graph $G(n,\gamma)$ of order n and domination number γ is defined as follows. If $3\gamma \leq n$, then $G(n,\gamma)$ is formed from a path P of order $3\gamma-2$ by adding $\lfloor\frac{n+2-3\gamma}{2}\rfloor$ pendant vertices to one endpoint of P and $\lceil\frac{n+2-3\gamma}{2}\rceil$ pendant vertices to the other endpoint. If $3\gamma \geq n$, then $G(n,\gamma)$ is obtained from a path P of order $2n-3\gamma$ by attaching one pendant vertex to each of the first $\lfloor\frac{3\gamma-n}{2}\rfloor$ vertices and to each of the last $\lceil\frac{3\gamma-n}{2}\rceil$ vertices of P.

We apply the above definition and notation to give a detailed proof of the following lemma and theorem.

Lemma 2.3.5 ([3])**.** *Let l be an integer such that $l \geq 2$, and let T be a tree. Suppose $x_0, x_1, \ldots, x_{l+2} \in V(T)$ satisfy $x_{i-1}x_i \in E(T)$ for $1 \leq i \leq l+1$, $x_l x_{l+2} \in E(T)$, $\deg_T(x_0) = 1$, and $\deg_T(x_j) = 2$ for $2 \leq j \leq l-1$. If we define $T_1 = T - x_l x_{l+1} + x_1 x_{l+1}$ and $T_2 = T - x_l x_{l+2} + x_1 x_{l+2}$, then it follows that $\gamma(T) \in \{\gamma(T_1), \gamma(T_2)\}$.*

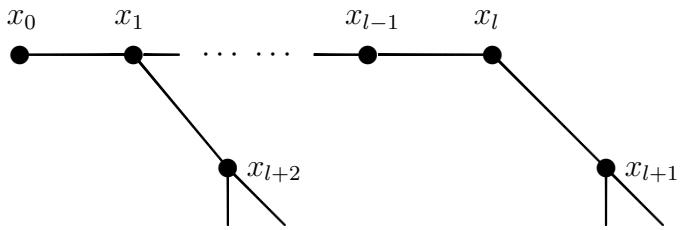


Figure 2.3.5: Tree T.

We will use this lemma to prove the following theorem.

Theorem 2.3.11. *[3] Let G be a connected graph of order n and domination number γ. Then*

$$\mu(G) \leq \mu(G(n,\gamma)).$$

The challenge of establishing a precise lower bound on the average distance in terms of the order and domination number remains unresolved. The graphs illustrated below demonstrate that this bound approaches a value close to 1. Sanchis [26] established that these graphs are the only connected graphs of a given order and domination number that attain the maximum possible size. For $4 \leq \gamma \leq \frac{n}{2}$, let $G^{n,\gamma}$ denote the graph formed by combining an $(n-\gamma)$-clique with an independent set of size γ, where every vertex of the clique is connected to exactly one vertex of the independent set, ensuring that $G^{n,\gamma}$ has no isolated vertices. In this case, the average distance is given by

$$\mu(G^{n,\gamma}) = \frac{n-3+2\gamma}{n-1}.$$

We remark that the problem restricted to trees is much simpler. Let $T^{n,\gamma}$ be the tree obtained from a star with end vertices $v_1, \ldots, v_{n-\gamma}$ by adding vertices $w_1, \ldots, w_{\gamma-1}$ and joining each w_i with v_i. It is easy to prove by induction that for every tree T of order n and domination number γ

$$\mu(T) \geq \mu(T^{n,\gamma}),$$

with equality if and only if $T = T^{n,\gamma}$.

2.4 Conclusion

In this chapter, we introduced the relevance of graph theory to various fields of study, highlighting the central role of distance in many graph-theoretic concepts. We then motivated the study of graph parameters, particularly focusing on domination and its relationship to other graph parameters.

To provide context, we delved into the historical development of domination theory, acknowledging the contributions of prominent researchers in the field. We also presented a comprehensive survey of existing results closely related to the research presented in this dissertation.

Chapter 3 Bounds on domination number in terms of radius

3.1 Introduction

In this chapter we will study the relationship between graph's domination number and its radius. More specifically, we consider a graph G that possesses a $\gamma(G)-$set S, we will prove the existence of a spanning tree T with S as a $\gamma(T)-$set. Using this result, we will establish a lower bound on the domination number in terms of radius. Furthermore, in this chapter, we will also prove upper bounds on the domination number in terms of radius. Our proofs will follow methods presented in [1].

3.2 Existence of a spanning tree with the same domination number

Spanning trees have been instrumental in many areas. We begin by presenting an important result which, for any $\gamma(G)-$set S, justifies the existence of a spanning tree T with S as a $\gamma(T)-$set. The result, which is presented in [1], will help us in establishing the interdependence of the domination number and the radius in graphs while order and minimum degree are pre-

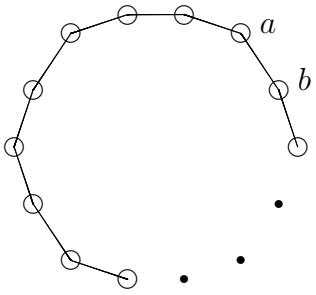
scribed.

The result has acted as a foundation to many other results. We present it below.

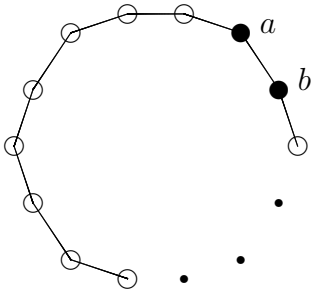
Lemma 3.2.1. *[1] Given a connected graph G with a $\gamma(G)$-set S, one can always obtain a spanning tree T of G for which S is also a $\gamma(T)$-set.*

Proof. If the graph G itself is a tree, then it trivially satisfies the condition with $T = G$, and the statement holds immediately. Suppose instead that G is not a tree; in that case, let C denote a cycle contained within G. We are going to consider three cases and one of them must occur.

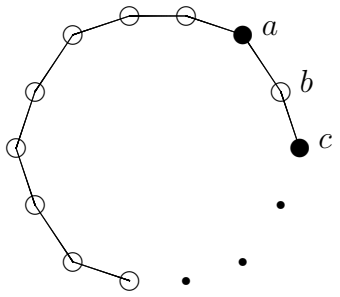
Case 1: Within the cycle C, there exist two adjacent vertices, say a and b, such that neither a nor b belongs to S.



Case 2: The cycle C contains a pair of neighbouring vertices, denoted a and b, such that both a and b belong to the set S.



Case 3: The cycle C contains two adjacent vertices, say a and b, such that a belongs to S whereas b does not.



In each of the three cases, let $e = ab$.

Claim 1: $G - e$ is dominated by S.

Proof of Claim 1:

Case 1:

Since a and b are not in the dominating set S, they must be dominated by other vertices in S. Removing edge e from the graph does not disconnect a or b from any vertices in S. Therefore, $G - e$ is dominated by S.

Case 2:

Vertices a and b are in the dominating set S; hence in $G-e$, $a(b)$ dominates itself.

Case 3:

Let $c \neq a$ be a vertex on C that is adjacent to b. If $c \notin S$, then, let $e = bc$ and proceed as in Case 1. If $c \in S$ that means it dominates vertex b in $G - e$ and since a is also in the dominating set S, a dominates itself in the graph $G - e$ and so $G - e$ is dominated by S.

By repeatedly removing edges from the cycles of G until no cycles remain, we obtain a spanning tree T of G. From the construction and the preceding claim, it follows that S is a dominating set of T, implying that $\gamma(T) \leq |S|$. Since deleting edges from G does not reduce its domination number, we have $|S| = \gamma(G) \leq \gamma(T)$. Therefore, $\gamma(T) = |S|$, and S serves as a $\gamma(T)-$set. □

Example 3.2.1

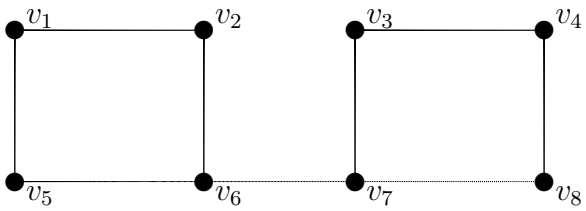


Figure 3.2.1: Graph G.

Consider Figure 3.2.1 above. Let S be a $\gamma(G)-$set where $S = \{v_1, v_4, v_6\}$. Consider the cycle $C_1 = v_1v_2v_6v_5v_1$. As in Case 3, let $a = v_1$, $b = v_5$, and $c = v_6$. Removing edge $e = ab = v_1v_5$ removes the cycle C_1 from the graph G which results in $G - e$ as shown in Figure 3.2.2 below.

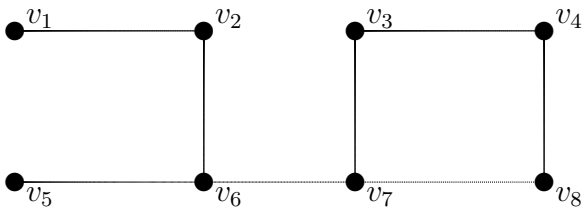


Figure 3.2.2: Graph $G - e$.

Now consider the cycle $C_2 = v_3v_4v_8v_7v_3$. As in Case 1, let $a = v_3$, $b = v_7$. Removing edge $e = ab = v_3v_7$ removes the cycle C_2 from the graph G which results in $G - e$ as shown in Figure 3.2.3 below. Now we are left with a connected graph with no cycles and thus a spanning tree T.

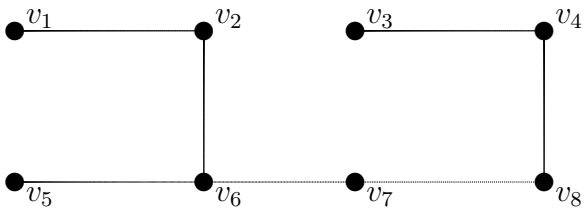


Figure 3.2.3: Spanning tree T.

Note that $S = \{v_1, v_4, v_6\}$ is a $\gamma(T)-$set.

3.3 Lower bound on domination in terms of radius

Before establishing a lower bound on domination in relation to radius, the following facts are needed.

Fact 3.3.1. *For every connected graph G, $\gamma(G) \geq \frac{1}{3}(diam(G) + 1)$.*

Proof. Let G be a connected graph and $P = v_0v_1v_2 \dots v_d$ be a diametral path of G. Consider the set $S = \{v_i \in V(P) | i \equiv 0 \pmod 3\}$. Since $d(x, y) \geq 3$ for all $x, y \in S$, no two vertices in S can be dominated by one vertex. Therefore,

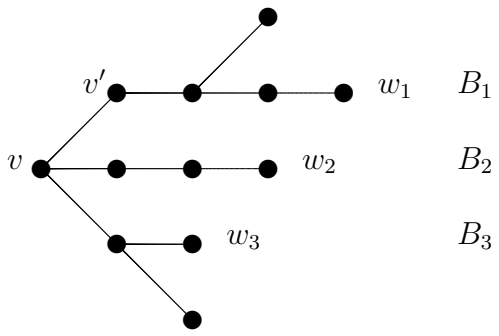
$$\begin{aligned} \gamma(G) \geq |S| &= \left\lfloor \frac{d}{3} \right\rfloor + 1 \\ &\geq \frac{d+1}{3} \\ &= \frac{1}{3}(diam(G) + 1), \end{aligned}$$

as desired. □

The following fact will also be needed.

Fact 3.3.2. *For every tree T, $diam(T) \geq 2rad(T) - 1$.*

Proof. Let v be a centre vertex, $B_1, B_2, \ldots, B_k$ its branches. Let w_i be a vertex in B_i furthest from v. Let $r_i := d(v, w_i)$. Assume, without loss of generality, that $r_1 \geq r_2 \geq \cdots \geq r_k$.



Clearly, $r_1 = rad(T)$.

Claim:

$$r_2 \geq rad(T) - 1.$$

Suppose to the contrary that $r_2, \ldots, r_k \leq rad(T) - 2$. Let v' be a neighbour of v in B_1. We show that $e(v') \leq rad(T) - 1$. Let x be a vertex of T. Then $d(v', x) \leq r_1 - 1$ if x is in B_1. For x in B_i, $i = 2, 3, \ldots, k$, we have

$$\begin{aligned} d(v', x) &= d(v', v) + d(v, x) \\ &\leq 1 + r_2 \\ &\leq 1 + rad(T) - 2 \\ &= rad(T) - 1. \end{aligned}$$

It then follows that $e(v') \leq rad(T) - 1$, a contradiction.
Now from the Claim,

$$\begin{aligned} diam(T) &\geq d(w_1, w_2) \\ &= r_1 + r_2 \\ &\geq rad(T) + rad(T) - 1 \\ &= 2rad(T) - 1, \end{aligned}$$

and the proof is complete. □

We now use Fact 3.3.1, Fact 3.3.2 and Lemma 3.2.1 to establish a lower bound on domination in relation to the radius.

Theorem 3.3.1. *[1] Let G be a connected graph. Then the inequality $\gamma(G) \geq \frac{2}{3}rad(G)$ holds. This lower bound is best possible, even for graphs with a fixed minimum degree that can be chosen to be arbitrarily large.*

Proof. Consider S to be a $\gamma(G)$-set of a connected graph G. There exists a spanning tree T of G for which S is also a $\gamma(T)$-set. Consequently, $\gamma(T) = \gamma(G)$.

By Fact 3.3.1,

$$\gamma(T) \geq \frac{1}{3}(diam(T) + 1)$$

and from Fact 3.3.2

$$diam(T) \geq 2rad(T) - 1.$$

Thus,

$$\gamma(T) \geq \frac{1}{3}(diam(T) + 1) \geq \frac{1}{3}((2rad(T) - 1) + 1) = \frac{2}{3}rad(T).$$

Since adding edges does not enlarge the radius of a graph, it follows that $rad(T) \geq rad(G)$. As a result, $\gamma(G) = \gamma(T) \geq \frac{2}{3}rad(T) \geq \frac{2}{3}rad(G)$, as desired.

We will now illustrate that the bound is sharp.

Consider an integer $k \geq 1$ and let $H = P_{6k}$ denote a path on $6k$ vertices. Define $V(H) = \{v_1, v_2, \ldots, v_{6k}\}$, and let $S = \{v_i \mid i \equiv 2 \pmod 3\}$. It is observed that S forms the unique $\gamma(H)$-set. Denote by $\mathcal{G}_1$ the collection of all graphs obtained from H by adjoining pendant vertices to some or all vertices of S.

Then for any graph $G \in \mathcal{G}_1$, the minimum degree satisfies $\delta(G) = 1$,

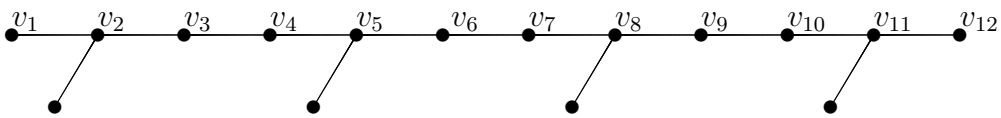


Figure 3.3.4: Graph G in $\mathcal{G}_1$.

the radius is $rad(G) = 3k$, and the domination number is $\gamma(G) = |S| = 2k$. Therefore, $\gamma(G) = \frac{2}{3} rad(G)$. □

3.4 Upper bounds on domination in terms of radius

Before establishing an upper bound on domination in terms of radius, the following facts are needed.

Fact 3.4.1. *For $n \geq 1$, $\gamma(P_n) \leq \frac{1}{3}$ diam$(P_n) + 1$ with equality if and only if $n \equiv 1 \pmod 3$.*

Proof. Let $P_n = v_0 v_1 v_2 v_3 \dots v_{n-1}$. Then $diam(P_n) = d(v_0, v_{n-1}) = n - 1$. To establish the upper bound, it suffices to prove that

$$\gamma(P_n) \leq \frac{1}{3}(n-1) + 1 = \frac{n+2}{3}.$$

Let

$$S = \begin{cases} \{v_i \in V(P_n) | i \equiv 0 \pmod 3\} & \text{if } n \equiv 1 \pmod 3, \\ \{v_i \in V(P_n) | i \equiv 1 \pmod 3\} & \text{if } n \equiv 0, 2 \pmod 3. \end{cases}$$

Then S dominates P_n and so

$$\gamma(P_n) \leq |S| = \begin{cases} \frac{n}{3} \text{ if } n \equiv 0 \pmod 3, \\ \frac{n+2}{3} \text{ if } n \equiv 1 \pmod 3, \\ \frac{n+1}{3} \text{ if } n \equiv 2 \pmod 3. \end{cases}$$

Note that $\gamma(P_n) = \frac{1}{3}diam(P_n) + 1$

iff

$$\gamma(P_n) = \frac{1}{3}(n-1) + 1$$

iff

$$\gamma(P_n) = \frac{n+2}{3}$$

iff

$$n \equiv 1 \pmod 3.$$

□

Fact 3.4.2. *For $n \geq 1$, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. For $n \geq 3$, $\gamma(C_n) = \gamma(P_n)$. Furthermore, if $n \equiv 1 \pmod 3$ and v is any specified vertex of P_n, then there exists a $\gamma(P_n)-$set that contains v.*

Proof. Since any three consecutive vertices on P_n are dominated by at least one vertex, we have

$$\gamma(P_n) \geq \left\lceil \frac{n}{3} \right\rceil.$$

From Fact 3.4.1, we know that

$$\gamma(P_n) \leq |S| = \begin{cases} \frac{n}{3} \text{ if } n \equiv 0 \pmod 3, \\ \frac{n+2}{3} \text{ if } n \equiv 1 \pmod 3, \\ \frac{n+1}{3} \text{ if } n \equiv 2 \pmod 3. \end{cases}$$

We conclude, from the two inequalities, that $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$, as desired. Let $C_n = v_0 v_1 v_2 \ldots v_{n-1} v_0$, $n \geq 3$. Then the set

$$S = \begin{cases} \{v_i \in V(C_n) | i \equiv 0 \pmod 3\} \text{ if } n \equiv 1 \pmod 3, \\ \{v_i \in V(C_n) | i \equiv 1 \pmod 3\} \text{ if } n \equiv 0, 2 \pmod 3, \end{cases}$$

is a $\gamma(C_n)-$set with $\left\lceil \frac{n}{3} \right\rceil$ vertices and so

$$\gamma(C_n) = \gamma(P_n).$$

Now assume that $n \equiv 1 \pmod 3$ and let $P_n = v_1v_2v_3 \dots v_n$. Assume that $v = v_i$ for some $i = 1, 2, \dots, n$. We show that a $\gamma(P_n)$-set S can be chosen with $v \in S$.

Now, we look at 3 cases separately.

Case A: $i \equiv 0 \pmod 3$

Let $S = \{v_0, v_3, v_6, \dots, v_{n-1}\}$. Then S contains v and S is a dominating set of P_n. Further,

$$|S| = \frac{n-1}{3} + 1 = \left\lceil \frac{n}{3} \right\rceil,$$

and so S is a $\gamma(P_n)-$set.

Case B: $i \equiv 1 \pmod 3$

Let $S = \{v_1, v_4, v_7, \ \dots, v_n\}$. Then S contains v and S is a dominating set of P_n. Moreover,

$$S = \frac{n-1}{3} + 1 = \left\lceil \frac{n}{3} \right\rceil,$$

and so S is a $\gamma(P_n)-$set.

Case C: $i \equiv 2 \pmod 3$

Let $S = \{v_2, v_5, v_8, \dots, v_{n-3}, v_n\}$. Then S contains v and S is a dominating set of P_n. Moreover,

$$|S| = \frac{n-1}{3} + 1 = \left\lceil \frac{n}{3} \right\rceil,$$

and so S is a $\gamma(P_n)-$set.

Therefore, S is a $\gamma(P_n)-$set that contains v, as desired. □

Fact 3.4.3. *For $n \geq 1$, rad $(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$. For $n \geq 3$, rad (C_n) =rad(P_n).*

Proof. Let $P_n = v_1v_2 \dots v_n$ and $C_n = v_1v_2 \dots v_nv_1$. Then for n even,

$$e(v_{\frac{n}{2}}) = \frac{n}{2}$$

and n odd,

$$e(v_{\frac{n+1}{2}}) = \frac{n-1}{2}$$

are the smallest eccentricities. It follows that

$$\begin{aligned} rad(P_n) &= \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases} \\ &= \left\lfloor \frac{n}{2} \right\rfloor \\ &= rad(C_n). \end{aligned}$$

□

Theorem 3.4.1. *[1] For a connected graph G of order n,*

$$\gamma(G) \leq n - \frac{4}{3}rad(G) + \frac{2}{3}.$$

With equality holding if and only if $G \in \{P_n, C_n\}$ and $n \equiv 4 \pmod 6$.

Proof. Denote the maximum degree of G by $\Delta(G)$ and let v be a vertex of G with $deg(v) = \Delta(G)$. For every $i = 1, \dots, e(v)$, let $N_i = \{x \in V(G) : d_G(x, v) = i\}$. Given any $x \in N_i$, pick a vertex $x' \in N_{i-1}$ such that $xx' \in E(G)$, where $1 \leq i \leq e(v)$. The collection of edges $\{xx' : x \in V(G) - \{v\}\}$ defines a spanning tree $T \subseteq G$ with the property that $deg_T(v) = deg_G(v)$.

Claim: T is distance-preserving from v.
Proof of Claim:
Assume that $x \in N_i$. By definition of N_i, $d_G(v, x) = i$. We show that $d_G(x, v) = d_T(x, v)$, i.e., $d_T(x, v) = i$. By construction of T, letting $x = x_i$, there exists a vertex $x_{i-1} \in N_{i-1}$ such that $x_ix_{i-1} \in E(T)$. Inductively there exists vertices $x_{i-2} \in N_{i-2}$, $x_{i-3} \in N_{i-3}$, $\dots, x_1 \in N_1$, such that $x_{i-1}x_{i-2}, x_{i-2}x_{i-3}, \dots, x_2x_1, xv \in E(T)$. So it follows that $vx_1x_2 \dots x_{i-2}x_{i-1}x$

is a $v - x$ path in T of length i and so $d_T(x, v) = i = d_G(x, v)$.

Now, to prove the bound of the theorem we consider two cases on the tree T. A diametral path of T is denoted by P.

Case 1: $T \neq P$
Assume that $T \neq P$. Then there exists a vertex y on P of degree at least 3. Let w be a neighbour of y on T, but not on the path P. The forest T_p is obtained by removing all vertices of the path P from T, that is, $T_p = T - V(P)$. Because the addition of edges to a graph does not increase its domination number, we have

$$\gamma(G) \leq \gamma(T).$$

Moreover, note that

$$\gamma(T) \leq \gamma(P \cup w) + \gamma(T_p - w). \tag{3.1}$$

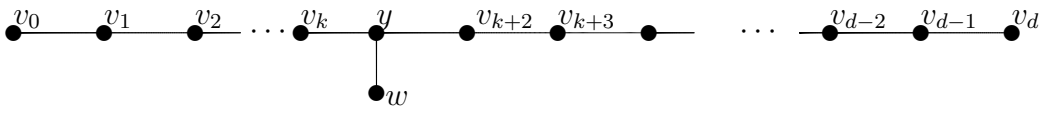


Figure 3.4.5: $P \cup w$.

Let the diameter of T be denoted by $d = \text{diam}(T)$. Hence, the path P contains $d + 1$ vertices.
Subcase 1: Assume that $d + 1 \equiv 1 \pmod 3$. By Fact 3.4.2, let S_y be a $\gamma(P)-$set containing y such that $|S_y| = \left\lceil \frac{d+1}{3} \right\rceil$. Since y dominates w, we get

$$\gamma(P \cup w) = |S_y| = \left\lceil \frac{d+1}{3} \right\rceil \leq \frac{d+1}{3} + 1 = \frac{d+4}{3}.$$

Since $\gamma(T_p - w)$ cannot exceed the order of $T_p - w$, using (3.1), we get

$$\begin{aligned}\gamma(T) &\leq \gamma(P \cup w) + \gamma(T_p - w) \\ &= \frac{d+4}{3} + n - d - 1 - 1 \\ &= n - \frac{2d}{3} - \frac{2}{3},\end{aligned}$$

and so,

$$\gamma(G) \leq \gamma(T) \leq n - \frac{2}{3}d - \frac{2}{3}.$$

By Fact 3.3.2, we know that $d = \text{diam}(T) \geq 2rad(T) - 1$. Hence,

$$\begin{aligned}\gamma(G) &\leq n - \frac{2}{3}(2rad(T) - 1) - \frac{2}{3} \\ &= n - \frac{4}{3}rad(T) + \frac{2}{3} - \frac{2}{3} \\ &= n - \frac{4}{3}rad(T).\end{aligned}$$

Since $\text{rad}(T) \geq rad(G)$, then

$$\gamma(G) \leq n - \frac{4}{3}rad(G) < n - \frac{4}{3}rad(G) + \frac{2}{3}$$

for graph G, proving the upper bound is strict.

Subcase 2: Assume that $d + 1 \equiv 0, 2 \pmod 3$. Then by Fact 3.4.1, and since $d + 1 \not\equiv 1 \pmod 3$, it follows that

$$\gamma(P) < \frac{1}{3}\text{diam}(P) + 1 = \frac{1}{3}d + 1.$$

It follows that

$$\gamma(P) + \gamma(T[w]) = \gamma(P \cup w) < \frac{1}{3}d + 1 + 1,$$

and so

$$\begin{aligned}\gamma(T) &< \gamma(P \cup w) + \gamma(T_p - w) \\ &< \frac{1}{3}d + 1 + 1 + n - d - 1 - 1 \\ &= n - \frac{2}{3}d.\end{aligned}$$

Therefore,

$$\gamma(G) \leq \gamma(T) < n - \frac{2}{3}d.$$

By Fact 3.3.2, we know that $d = \text{diam}(T) \geq 2rad(T) - 1$. Hence,

$$\begin{aligned}\gamma(G) &< n - \frac{2}{3}(2rad(T) - 1) \\ &= n - \frac{4}{3}rad(T) + \frac{2}{3}.\end{aligned}$$

Thus since $rad(T) \geq rad(G)$, we have that

$$\gamma(G) < n - \frac{4}{3}rad(G) + \frac{2}{3}.$$

for graph G, proving the upper bound is strict.

Case 2: $T = P$

In this case, the maximum degree of G is at most 2, which implies that $G \in \{P_n, C_n\}$. By Fact 3.4.2 and Fact 3.4.3 we have that

$$\begin{aligned}\gamma(G) &= \left\lceil \frac{n}{3} \right\rceil \\ &\leq \frac{n+2}{3} \\ &= n - \frac{4n}{6} + \frac{2}{3} \\ &= n - \frac{4}{3} \cdot \frac{n}{2} + \frac{2}{3} \\ &\leq n - \frac{4}{3}\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} \\ &= n - \frac{4}{3}rad(G) + \frac{2}{3}, \end{aligned} \tag{3.2}$$

and the bound holds as desired. We now show that equality holds iff

$$G \in \{P_n, C_n\} \text{ and } n \equiv 4 \pmod 6. \tag{3.3}$$

Firstly, we assume that equality holds. Then from Case 2 above, $G \in \{P_n, C_n\}$. We show that $n \equiv 4 \pmod 6$. By Fact 3.4.2, and by Fact 3.4.3,

$$\gamma(G) = \left\lceil \frac{n}{3} \right\rceil$$

and

$$rad(G) = \left\lfloor \frac{n}{2} \right\rfloor,$$

respectively. Since we have equality, it follows that

$$\left\lceil \frac{n}{3} \right\rceil = n - \frac{4}{3} rad(G) + \frac{2}{3} = n - \frac{4}{3} \left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3}. \tag{3.4}$$

By the division algorithm there exists integers q, r such that $n = 6q + r$ for some $r \in \{0, 1, ..., 5\}$. We first show that $r = 4$. For each of the following cases we will consider the RHS and the LHS OF (3.4),

Subcase 1: $r = 0, n = 6q$.

RHS:

$$\begin{aligned} n - \frac{4}{3} \left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} &= 6q - \frac{4}{3} \left\lfloor \frac{6q}{2} \right\rfloor + \frac{2}{3} \\ &= 6q - \frac{4}{3}(3q) + \frac{2}{3} \\ &= 2q + \frac{2}{3}. \end{aligned}$$

LHS:

$$\left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{6q}{3} \right\rceil = 2q,$$

therefore LHS $\neq$ RHS.

Subcase 2: $r = 1, n = 6q + 1$.

RHS:

$$\begin{aligned} n - \frac{4}{3}\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} &= 6q + 1 - \frac{4}{3}\left\lfloor \frac{6q+1}{2} \right\rfloor + \frac{2}{3} \\ &= 6q + 1 - \frac{4}{3}(3q) + \frac{2}{3} \\ &= 2q + \frac{5}{3}. \end{aligned}$$

LHS:

$$\left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{6q+1}{3} \right\rceil = 2q + 1,$$

therefore LHS $\neq$ RHS.

Subcase 3: $r = 2, n = 6q + 2$.

RHS:

$$\begin{aligned} n - \frac{4}{3}\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} &= 6q + 2 - \frac{4}{3}\left\lfloor \frac{6q+2}{2} \right\rfloor + \frac{2}{3} \\ &= 6q + 1 - \frac{4}{3}(3q+1) + \frac{2}{3} \\ &= 2q + \frac{4}{3}. \end{aligned}$$

LHS:

$$\left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{6q+2}{3} \right\rceil = 2q + 1,$$

therefore LHS $\neq$ RHS.

Subcase 4: $r = 3, n = 6q + 3$.

RHS:

$$\begin{aligned} n - \frac{4}{3}\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} &= 6q + 3 - \frac{4}{3}\left\lfloor \frac{6q+3}{2} \right\rfloor + \frac{2}{3} \\ &= 6q + 3 - \frac{4}{3}(3q+1) + \frac{2}{3} \\ &= 2q + \frac{7}{3}. \end{aligned}$$

LHS:

$$\left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{6q+3}{3} \right\rceil = 2q+1,$$

therefore LHS $\neq$ RHS.

Subcase 5: $r = 5, n = 6q+5$.

RHS:

$$\begin{aligned} n - \frac{4}{3}\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} &= 6q+5 - \frac{4}{3}\left\lfloor \frac{6q+5}{2} \right\rfloor + \frac{2}{3} \\ &= 6q+5 - \frac{4}{3}(3q+2) + \frac{2}{3} \\ &= 2q+3. \end{aligned}$$

LHS:

$$\left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{6q+5}{3} \right\rceil = 2q+2,$$

therefore LHS $\neq$ RHS.

Since $r = 0, 1, 2, 3,$ and 5 does not satisfy (3.4), we conclude that $r = 4$. Hence $n = 6q+4$ and so $n \equiv 4 \pmod 6$.

We now show the converse in (3.3). Assume that $G \in \{P_n, C_n\}$ and $n \equiv 4 \pmod 6$ with $n = 6q+4$, for some integer q. We prove that $\gamma(G) = n - \frac{4}{3}rad(G) + \frac{2}{3}$. Since $G \in \{P_n, C_n\}$, from Fact 3.4.2 and Fact 3.4.3,

$$\begin{aligned} \gamma(G) &= \left\lceil \frac{n}{3} \right\rceil \\ &= \left\lceil \frac{6q+4}{3} \right\rceil \\ &= \left\lceil 2q + \frac{4}{3} \right\rceil \\ &= 2q+2 \\ &= 2(q+1), \end{aligned}$$

and the LHS

$$\begin{aligned} n - \frac{4}{3}\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} &= 6q + 4 - \frac{4}{3}\left\lfloor \frac{6q+4}{2} \right\rfloor + \frac{2}{3} \\ &= 6q + 4 - \frac{4}{3}(3q+2) + \frac{2}{3} \\ &= 2q + 2 \\ &= 2(q+1), \end{aligned}$$

and so equality holds, as desired. □

Using Fact 3.3.2, Fact 3.4.1, Fact 3.4.2, and Fact 3.4.3, [1] improved the bound of Theorem 3.4.1 if minimum degree is prescribed. To prove the upper bound on domination number in terms of minimum degree and radius, the following definition and fact, which were presented in [1], are needed.

Let G be a connected graph. From now onwards z will be a fixed central vertex, and $\text{rad}(G) = r = e(z)$. Consider $N_i = \{v \in V(G) : d_G(v,z) = i\}$ for every $i = 0, 1, \ldots, \text{e}(z)$. We define $N_{\leq j}$ and $N_{\geq j}$ as

$$N_{\leq j} = \bigcup_{0 \leq i \leq j} N_i \quad \text{and} \quad N_{\geq j} = \bigcup_{j \leq i \leq r} N_i,$$

respectively. Let T be a spanning tree of G, that is distance-preserving from z then,

$$d_T(v,z) = d_G(v,z) \quad \text{for all } v \in V.$$

For a vertex $v \in N_i$, let $T(z,v)$ denote the set of vertices on the unique z–v path in T. The construction of this spanning tree was presented in Theorem 3.4.1.

Definition 3.4.1. *Let G be a connected graph with $\text{rad}(G) = r \geq 6$. Let z be a fixed central vertex of G and $z_r \in N_r$. We say that a vertex $y \in N_i$ is related to the vertex z_r if there exists vertices u, v, where $u \in T(z, z_r) \bigcap N_{\geq 5}$ and $v \in T(z,y) \bigcap N_{\geq 5}$ such that $d_G(u,v) \leq 2$.*

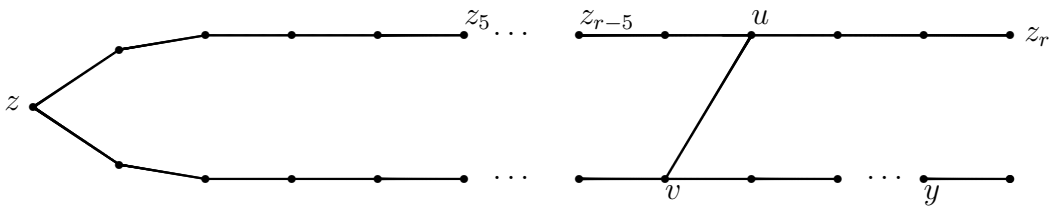


Figure 3.4.6: Vertex y related to vertex z_r.

We can now state the result due to Erdős et al [5].

Fact 3.4.4. *Let G be a connected graph with $rad(G) = r \geq 6$ and let z be a central vertex of G. For each vertex $z_r \in N_r$, there exists a vertex in $N_{\geq r-5}$ which is not related to z_r.*

Proof. We prove Fact 3.4.4 by contradiction. Suppose $z_r \in N_r$ is a vertex such that every vertex in $N_{\geq r-5}$ is related to z_r. We will then derive a contradiction, showing that this assumption cannot hold.

Let z_5 be the unique vertex in N_5 on $T(z, z_r)$. We show that $e(z_5) < r$. Let x be a vertex in G. We will consider two cases in which we will show that the distance between the vertex z_5 and vertex x is at most $r - 1$.

Case 1: $x \in N_{\leq r-6}$
Then

$$d_G(z_5, x) \leq d_T(z_5, z) + d_T(z, x) \leq 5 + r - 6 = r - 1.$$

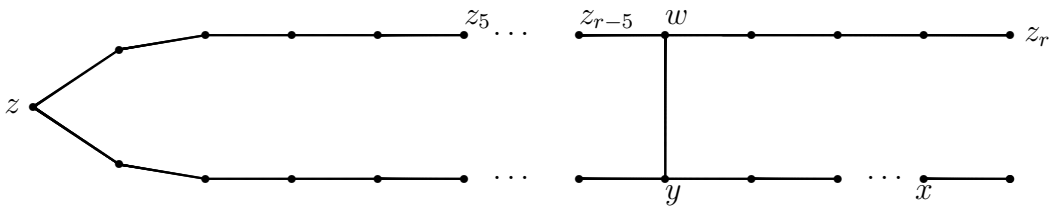


Figure 3.4.7: $T(z, z_r)$ and $T(z, x)$ with z_r related to x.

Case 2: $x \in N_{\geq r-5}$

By assumption x is related to z_r. Let w and y be vertices in $V(G)$ where $w \in T(z, z_r) \cap N_{\geq 5}$ and $y \in T(z, x) \cap N_{\geq 5}$ such that $d_G(w, y) \leq 2$. Then,

$$d_G(z_5, w) = d_G(z, w) - 5, \tag{3.5}$$

and

$$d_G(y, x) = d_G(z, x) - d_G(z, y) \leq r - d_G(z, y). \tag{3.6}$$

Therefore, by the triangle inequality, (3.5) and (3.6), we get

$$\begin{aligned} d_G(z_5, x) &\leq d_G(z_5, w) + d_G(w, y) + d_G(y, x) \\ &\leq (d_G(z, w) - 5) + 2 + (r - d_G(z, y)) \\ &= d_G(z, w) - d_G(z, y) + r - 3. \end{aligned} \tag{3.7}$$

By the triangle inequality, we get

$$d_G(z, w) \leq d_G(z, y) + d_G(w, y),$$

and so,

$$d_G(z, w) - d_G(z, y) \leq d_G(w, y) \leq 2. \tag{3.8}$$

Hence from (3.7) and (3.8), we get

$$\begin{aligned} d_G(z_5, x) &\leq d_G(z, w) - d_G(z, y) + r - 3 \\ &\leq 2 + r - 3 \\ &= r - 1. \end{aligned}$$

Thus, from the two cases, we have $d_G(z_5, x) \leq r - 1$ for every $x \in V(G)$. Hence $e(z_5) \leq r - 1$, a contradiction. □

We are now ready to present the proof of the upper bound on domination number in terms of order, radius and minimum degree.

Theorem 3.4.2. *For a connected graph G of order n with minimum degree $\delta \geq 3$ and radius $r \geq 6$, the domination number satisfies*

$$\gamma(G) \leq n - \frac{2}{3}(r-6)\delta.$$

This bound is asymptotically sharp, as there exist graphs meeting these criteria for which

$$\gamma(G) = n - \frac{2}{3}\left(r + \frac{3}{2}\right)\delta.$$

Proof. Let us fix a central vertex z in graph G, which means $e(z) = rad(G)$. Using the notation introduced in Definition 3.4.1 and by Fact 3.4.4, we find a fixed vertex $x \in N_{r-5}$ which is not related to the vertex $z_r \in N_r$. Recall that $T(z, z_r)$ is the set of vertices on a z to z_r path on T. Let the set of vertices in $T(z, z_r)$ be $z = y_0, y_1, y_2, ..., y_{r-1}, y_r = z_r$.

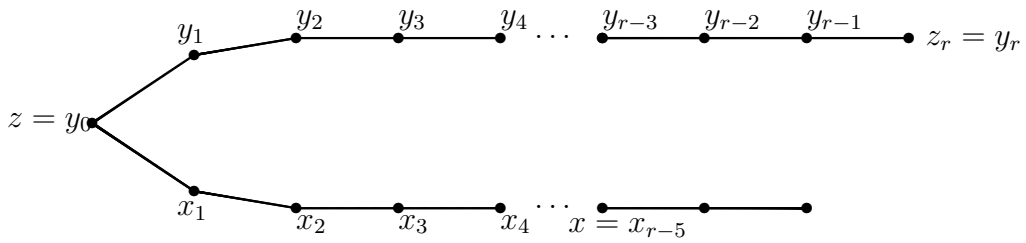


Figure 3.4.8: $T(z, z_r)$ and $T(z, x)$.

Thus $y_i \in N_i$ for $i = 0, 1, \ldots, r$. We then define the set P to contain every third vertex, starting from the second vertex. Specifically, let

$$P = \{y_{3i+1} \mid i = 0, 1, ..., q-1\},$$

where $q = \left\lfloor \frac{r+1}{3} \right\rfloor$. For any two vertices u and v in set P, their neighbourhoods do not overlap, i.e.,

$$N[u] \cap N[v] = \emptyset \ \forall u, v \in P. \tag{3.9}$$

This is because $d(u, v) \geq 3 \ \forall \ u, v \in P$.

Next, we recall that $T(z, x)$ is the set of vertices on a z to x path on T. Assume that $T(z, x)$ consists of vertices $x_0, x_1, x_2, ..., x_{r-6}, x_{r-5} = x$. We define the set Q to contain every third vertex, starting from the second vertex, in the set $T(z, x)$. Specifically, let

$$Q = \{x_{3i+1} \mid i = 2, 3, ..., t + 1\},$$

where $t = \left\lfloor \frac{r-10}{3} \right\rfloor$. As above, for any two vertices u and v in set Q, their neighbourhoods do not overlap, i.e.,

$$N[u] \cap N[v] = \emptyset \ \forall u, v \in Q. \tag{3.10}$$

We then define the set A as the union of sets P and Q, i.e., $A = P \cup Q$, and we define the set B as the set of vertices that are not in $N[A]$, i.e., $B = V - N[A]$. Since x and z_r are unrelated, (3.9) and (3.10) imply that for any two vertices u and v in A, their neighbourhoods do not overlap, i.e.,

$$N[u] \cap N[v] = \emptyset \ \forall u, v \in A. \tag{3.11}$$

Using the minimum degree condition ($\delta \geq 3$), (3.11) implies that

$$n = |N[A]| + |B| \geq |A|(\delta + 1) + |B|,$$

and so

$$|B| \leq n - |A|(\delta + 1). \tag{3.12}$$

Since $A \cup B$ is a dominating set of G, we can, by (3.12), express the size of

a minimum dominating set $\gamma(G)$ as follows

$$\begin{aligned}\gamma(G) &\leq |A| + |B| \\ &\leq |A| + n - |A|(\delta + 1) \\ &= n - \delta|A| \\ &= n - \delta\left(\left\lfloor \frac{r+1}{3} \right\rfloor + \left\lfloor \frac{r-10}{3} \right\rfloor\right) \\ &\leq n - \frac{2}{3}(r-6)\delta.\end{aligned}$$

This concludes the proof of the upper bound of the theorem.

To demonstrate the asymptotic optimality of the upper bound, we construct a family of graphs $\mathcal{G}_\delta$ with minimum degree $\delta \geq 3$ as follows. For $\delta \geq 1$, define $\mathcal{G}_\delta$ as the family of graphs having minimum degree δ. Let $k \geq 1$ be an integer and consider the path $H = P_{6k}$ of order $6k$. Let $V(H) = \{v_1, v_2, \ldots, v_{6k}\}$, and define

$$S = \{v_i \mid i \equiv 2 \pmod 3\}.$$

Observe that S is the unique $\gamma(H)$-set.

For $\delta \geq 2$, the family $\mathcal{G}_\delta$ consists of all graphs G obtained from H by performing the following operations: - Replace the end-vertices v_1 and v_{6k} with a complete graph K_δ; - Replace each vertex v_i satisfying $i \equiv 0 \pmod 3$ and $i \neq 6k$ with a complete graph $K_{\lfloor \frac{\delta}{2} \rfloor}$; - Replace each vertex v_i satisfying $i \equiv 1 \pmod 3$ and $i \neq 1$ with a complete graph $K_{\lceil \frac{\delta}{2} \rceil}$; - Add all possible edges between vertices of two cliques, that correspond to adjacent vertices in H. If $G \in \mathcal{G}_\delta$, we have $\delta(G) = \delta$,

$$n = |V(G)| = 2(\delta + 1)k + \delta, \tag{3.13}$$

and

$$rad(G) = 3k, \tag{3.14}$$

and

$$\gamma(G) = 2k. \tag{3.15}$$

We now show that these graphs have a radius $r \geq 6$ and satisfy

$$\gamma(G_\delta) = n - \frac{2}{3}(r + \frac{3}{2})\delta. \tag{3.16}$$

By (3.13) - (3.15)

$$\begin{aligned} n - \frac{2}{3}\left(r + \frac{3}{2}\right)\delta &= 2(\delta+1)k + \delta - \frac{2}{3}\left(3k + \frac{3}{2}\right)\delta \\ &= 2k\delta + 2k + \delta - 2k\delta - \delta \\ &= 2k. \end{aligned}$$

and

$$\gamma(G) = 2k,$$

therefore equality in (3.16) holds. □

Example 3.4.1

For Theorem 3.4.2, we will consider $\delta = 3$ and so $K_\delta = K_3$, $K_{\lfloor \frac{\delta}{2} \rfloor} = K_1$, and $K_{\lceil \frac{\delta}{2} \rceil} = K_2$. Let $k = 3$. Then we have $H_{6k} = H_{18}$.

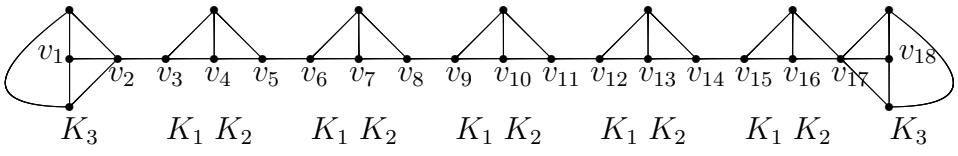


Figure 3.4.9: Graph G in $\mathcal{G}_3$.

3.5 Conclusion

This chapter was divided into five sections. In the first section we gave a brief introduction of what this chapter covered. In the second section we proved the existence of a spanning tree with the same domination number as the graph G, while in the third section we started by proving results that were later used to prove a lower bound on domination number with regards to the radius of the graph. In section four we focused on proving upper bounds on domination in terms of radius, order, and the minimum degree of the graph. In the next chapter we study bounds on domination number in relation to distances in the graph.

Chapter 4 Bounds on domination number in terms of distance

4.1 Introduction

Recall that in the previous chapter we proved that if x, y is a diametral pair in a graph G, then

$$\gamma(G) \geq \frac{1}{3}\Big(diam(G) + 1\Big) = \frac{1}{3}\Big(d_G(x, y) + 1\Big).$$

It follows that for any two vertices $x_1, x_2 \in V(G)$,

$$\gamma(G) \geq \frac{1}{3}\Big(d_G(x_1, x_2)\Big). \tag{4.1}$$

In this chapter, we begin by generalising (4.1) for the total distance of any r vertices, $x_1, x_2, \ldots, x_r \in V(G)$, where $r \geq 3$. In addition, we will explore the relationship between domination number, diameter, and eccentricity of the boundary of the graph as presented in [2].

4.2 Lower bound on domination in relation to the distance between vertices

We begin by proving a theorem that gives us a lower bound on domination number in relation to the distance between any three vertices $x_1, x_2,$ and x_3. To achieve this, we will use a lower bound on domination number in relation to the diameter and several other results.

Theorem 4.2.1. *Given any three vertices x_1, x_2, x_3 of a connected graph G we have*

$$\gamma(G) \geq \frac{1}{6}\Big(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)\Big). \tag{4.2}$$

Proof. By Lemma 3.2.1, there exists a spanning tree T of G with $\gamma(T) = \gamma(G)$. Since $d_T(u, v) \geq d_G(u, v)$ for any two vertices $u, v \in V(T) = V(G)$, it suffices to prove inequality (4.2) on T. We will consider two cases separately.

Case 1: Let $u, v \in \{x_1, x_2, x_3\}$ and assume that x_1, x_2, x_3 lie on the same $u - v$ path P^1 in T.

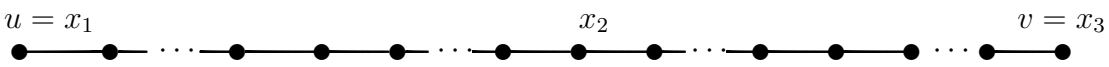


Figure 4.2.1: $u - v$ path P^1 in T.

By Fact 3.3.1,

$$\gamma(T) \geq \frac{1}{3}\Big(\text{diam}(T) + 1\Big). \tag{4.3}$$

Since $d_T(x_1, x_3) = d_T(x_1, x_2) + d_T(x_2, x_3)$,

$$\begin{aligned}\frac{1}{6}\Big(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3)\Big) &= \frac{1}{6}\Big(2d_T(x_1, x_3)\Big) \\ &= \frac{1}{3}\Big(d_T(x_1, x_3\Big). \end{aligned} \tag{4.4}$$

We already know that the diameter is greater than or equal to the distance between any two vertices and so (4.3) in conjuction with (4.4), gives

$$\begin{aligned}\gamma(T) &\geq \frac{1}{3}\Big(diam(T) + 1\Big) \\ &\geq \frac{1}{3}\Big(d_T(x_1, x_3) + 1\Big) \\ &= \frac{1}{6}\Big(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3)\Big) + \frac{1}{3};\end{aligned}$$

hence

$$\begin{aligned}\gamma(G) = \gamma(T) &> \frac{1}{6}\Big(d_T(x_1, x_2) + d_T(x_2, x_3) + d_T(x_1, x_3)\Big) \\ &\geq \frac{1}{6}\Big(d_G(x_1, x_2) + d_G(x_2, x_3) + d_G(x_1, x_3)\Big),\end{aligned}$$

as desired.

Case 2: Now assume that the three vertices do not lie on the same path. Let P^1 be the $x_1 - x_3$ path in T. Then there exists a vertex y on P^1 that is closest to x_2 such that $d_T(x_1, y) = a$, $d_T(x_2, y) = b$, and $d_T(x_3, y) = c$ with $0 \notin \{a, b, c\}$ as shown in the figure below.

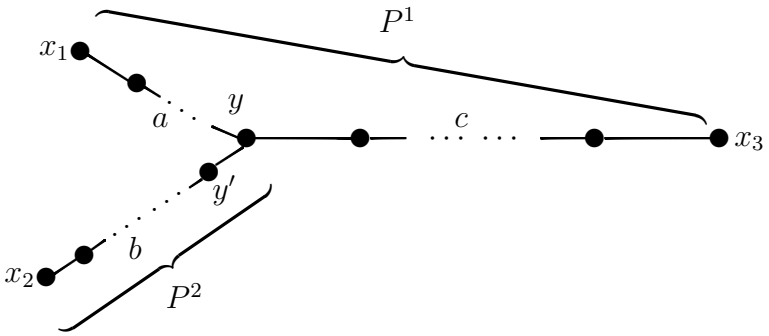


Figure 4.2.2: Case when x_1, x_2, x_3 do not lie on the same path.

Then, inequality (4.2) becomes,

$$\gamma(T) \geq \frac{1}{6}\Big((a+b)+(a+c)+(b+c)\Big) = \frac{1}{3}\Big(a+b+c\Big). \tag{4.5}$$

Let y' be the vertex lying on the $x_2 - y$ path and adjacent to y. Let P^2 denote the $x_2 - y'$ path, so that $diam(P^2) = b - 1 \geq 0$.

Let M be a $\gamma(T)-$set. Assume on one hand that $y \notin M$.

Subcase 1:

Assume that $y' \notin M$. We proceed by applying the inequality (4.3) on P^1 and P^2. This gives,

$$\begin{aligned}\gamma(P^2) \geq \frac{1}{3}\Big(diam(P^2)+1\Big) &\geq \frac{1}{3}\Big(b-1+1\Big) \\ &= \frac{1}{3}b,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\gamma(P^1) \geq \frac{1}{3}\Big(diam(P^1)+1\Big) &> \frac{1}{3}\Big(d_T(x_1,y)+d_T(x_3,y)\Big) \\ &= \frac{1}{3}\Big(a+c\Big).\end{aligned} \tag{4.7}$$

Therefore,

$$\gamma(T) \geq \gamma(P^1 \cup P^2) > \frac{1}{3}\Big(a+b+c\Big), \tag{4.8}$$

and so (4.5) is proven.

Subcase 2:

Assume that y' is in M. Then y' dominates y and so

$$\gamma(P^1) \geq \frac{1}{3}\Big((a-1)+1+(c-1)+1\Big) = \frac{1}{3}\Big(a+c\Big), \tag{4.9}$$

and

$$\gamma(P^2) \geq \frac{1}{3}\Big((b-1)+1\Big). \tag{4.10}$$

It follows that

$$\gamma(T) \geq \gamma(P^1 \cup P^2) \geq \frac{1}{3}(a+c)+\frac{1}{3}b,$$

and (4.5) is proven.

On the other hand, assume that $y \in M$. Then, since y dominates all its neighbours,

$$\begin{aligned}\gamma(T) \geq \gamma(P^1 \cup P^2) &\geq \frac{1}{3}\Big(a-1\Big)+\frac{1}{3}\Big(b-1\Big)+\frac{1}{3}\Big(c-1\Big)+|\{y\}| \\ &= \frac{1}{3}\Big(a+b+c-3\Big)+1 \\ &= \frac{1}{3}\Big(a+b+c\Big),\end{aligned} \tag{4.11}$$

and so (4.5) is proven. □

We now find a lower bound on the domination number in relation to the distance between a set of vertices $\{x_1, x_2, \ldots, x_r\}$ where $r \geq 3$.

Theorem 4.2.2. *[2] Let G be a connected graph and $S = \{x_1, x_2, \ldots, x_r\} \subseteq V(G)$, where $r \geq 3$. Then*

$$\gamma(G) \geq \frac{1}{r^2-r}\sum_{\{u,v\}\subseteq S} d_G(u,v).$$

Moreover the bound is sharp.

Proof. We first establish the bound. Consider $r \geq 3$. We notice that the base case with $r = 3$ is proven in Theorem 4.2.1 above. Thus, let $x_1, x_2, \ldots, x_r$ be any arbitrary $r \geq 4$ vertices of G. Since

$$\gamma(G) \geq \frac{1}{6}\Big(d_G(x_i,x_j)+d_G(x_i,x_k)+d_G(x_j,x_k)\Big)$$

holds for any triplets $\{x_i, x_j, x_k\} \subseteq \{x_1, x_2, \ldots, x_r\}$, we have

$$\begin{aligned}\binom{r}{3}\gamma(G) &\geq \frac{r-2}{6} \sum_{1\leq i<j<k\leq r} \Big(d_G(x_i, x_j) + d_G(x_i, x_j) + d_G(x_j, x_k)\Big) \\ &= \frac{r-2}{6} \sum_{1\leq i<j\leq r} d_G(x_i, x_j);\end{aligned}$$

note that the last equality comes from the fact that there are $r-2$ triplets containing any given pair of vertices. Thus

$$\gamma(G) \geq \frac{1}{\binom{r}{3}} \frac{r-2}{6} \sum_{\{u,v\}\subseteq S} d_G(u, v) = \frac{1}{r^2 - r} \sum_{\{u,v\}\subseteq S} d_G(u, v),$$

and the bound in the theorem is proven.

We now prove that the bound is sharp. To see that the bound is sharp, let $G = K_{1,r}$ be a star with r leaves labeled $x_1, \ldots, x_r$. Then $\gamma(G) = 1$ because the central vertex dominates all leaves and since $d(x_i, x_j) = 2$ for any leaves, we have

$$\sum_{1\leq i<j\leq r} d(x_i, x_j) = \binom{r}{2} \cdot 2 = \frac{r(r-1)}{2} \cdot 2 = r(r-1),$$

hence there is equality in the bound. □

We now turn our focus to considering a lower bound on domination number in relation to the eccentricity of the boundary of the graph, that is, we will focus on vertices with a maximum eccentricity in G.

4.3 Lower bound on domination with regards to the eccentricity of the boundary

Say B denotes the set of vertices of maximum eccentricity in G. Recall that $e_G(B)$ denotes the eccentricity of B in G, i.e., the distance between B and a vertex furthest away from B. The following theorem, which is

deduced from Theorem 4.2.1, provides a proof to a conjecture of DeLaViña et al. [2] on a lower bound on the domination number in terms of $e_G(B)$. The theorem was presented in [2].

Theorem 4.3.1. *Let G be a connected graph. Then*

$$\gamma(G) \geq \frac{1}{2}\Big(e_G(B) + 1\Big).$$

Proof. **Case 1:** Assume that $B = V(G)$.
If the boundary B includes all vertices of the graph, then

$$e_G(B) = 0,$$

and thus

$$\gamma(G) \geq \frac{1}{2}(0 + 1) = \frac{1}{2}.$$

In this case, the inequality holds trivially since $\gamma(G) \geq 1$.

Case 2: Assume $B \neq V(G)$.
Then $|B| \geq 2$ and so $|V(G)| \geq 3$. We choose two vertices x and y from B such that

$$d(x, y) = diam(G).$$

Let $e_G(B) = r$. We select a vertex $z \in V(G) \setminus B$ such that $d(z, B) = r$. From this setup, we can derive the following inequalities

$$d(x, z) \geq r \text{ and } d(y, z) \geq r.$$

Since x, y is a diametral pair, we have

$$d(x, y) = diam(G) \geq r + 1.$$

To see this suppose to the contrary that $d(x, y) = r$. Then from $d(z, B) = r$, we get that $e(z) = r$ and so $z \in B$, a contradiction. Hence, we have

$$d(x, y) + d(x, z) + d(y, z) \geq 3r + 1. \tag{4.12}$$

If equality holds in the above inequality (4.12), i.e., if

$$d(x,y)+d(x,z)+d(y,z)=3r+1$$

then

$$r=d(x,z)=d(y,z)=d(x,y)-1.$$

However, we cannot have both $d(x,z)$ and $d(x,y)$ being congruent to 2 (mod 3). Using Theorem 4.2.1, we find

$$\gamma(G)>\frac{1}{6}\Big(d(x,y)+d(x,z)+d(y,z)\Big)=\frac{1}{6}\Big(3r+1\Big)=\frac{1}{2}r+\frac{1}{6}.$$

Therefore,

$$\gamma(G)\geq\frac{1}{2}\Big(e_G(B)+1\Big),$$

and the bound in the theorem is proven. On the other hand, if

$$d(x,y)+d(x,z)+d(y,z)>3r+1,$$

then Theorem 4.2.1 implies that

$$\gamma(G)\geq\frac{1}{2}r+\frac{1}{2},$$

and so

$$\gamma(G)\geq\frac{1}{2}\Big(e_G(B)+1\Big),$$

which proves the bound in the theorem. □

4.4 Conclusion

We saw in Chapter 3 that

$$\gamma(G)\geq\frac{1}{3}\Big(d_G(x,y)+1\Big)$$

for any two vertices $x,y\in V(G)$. In this chapter we studied lower bounds on domination number in terms of sum of distances between any three

vertices x_1, x_2, x_3. In particular, we showed that

$$\gamma(G) \geq \frac{1}{6}\Big(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)\Big).$$

We then generalised this result for the total distance of any r vertices, $x_1, x_2, \ldots, x_r \in V(G)$, where $r \geq 3$, and showed that

$$\gamma(G) \geq \frac{1}{r^2 - r} \sum_{\{u,v\} \subseteq S} d_G(u, v),$$

for any set $S \subseteq V(G)$ with r vertices. We then concluded this chapter by presenting a lower bound on domination number in terms of another distance measure which is the eccentricity of the boundary,

$$\gamma(G) \geq \frac{1}{2}\Big(e_G(B) + 1\Big).$$

In the following chapter we focus on bounds on domination number in relation to the average distance of the graph.

Chapter 5 Bounds on domination number in terms of average distance

5.1 Introduction

The domination number and the average distance are two fundamental graph parameters that have been extensively investigated in graph theory. The first bound on average distance in terms of domination number was given by Dankelmann [4]. In this chapter we will study the relationship between domination number and the average distance of the graph. We will look at the relationship between the domination number and average distance as presented in [3]. There, a much shorter proof of Dankelmann's theorem was presented.

Fact 5.1.1. *Let D be a dominating set of a graph G, u an endvertex of G, and v a neighbour of u. Then $D \cup \{v\} \setminus \{u\}$ dominates G.*

Proof. Assume the hypothesis of the fact. Since u is an endvertex, it can only be dominated by itself or by v. Therefore, $D \cup \{v\} \setminus \{u\}$ dominates u. □

5.2 Domination in trees

We begin the study by first presenting a tree transformation that does not change the domination number. The transformation is through moving a branch to a well chosen position in the graph. This result, which is stated and proved below, will be used to establish an upper bound on average distance in terms of domination number in the next section.

Lemma 5.2.1. *Let l be an integer such that $l \geq 2$, and let T be a tree. Suppose $x_0, x_1, \ldots, x_{l+2} \in V(T)$ satisfy $x_{i-1}x_i \in E(T)$ for $1 \leq i \leq l+1$, $x_l x_{l+2} \in E(T)$, $\deg_T(x_0) = 1$, and $\deg_T(x_j) = 2$ for $2 \leq j \leq l-1$. If we define $T_1 = T - x_l x_{l+1} + x_1 x_{l+1}$ and $T_2 = T - x_l x_{l+2} + x_1 x_{l+2}$, then it follows that $\gamma(T) \in \{\gamma(T_1), \gamma(T_2)\}$.*

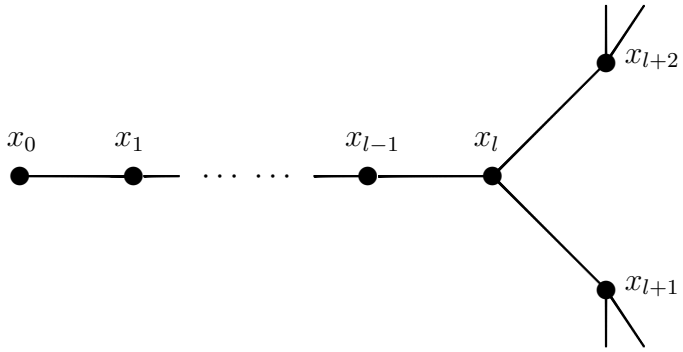


Figure 5.2.1: Tree T.

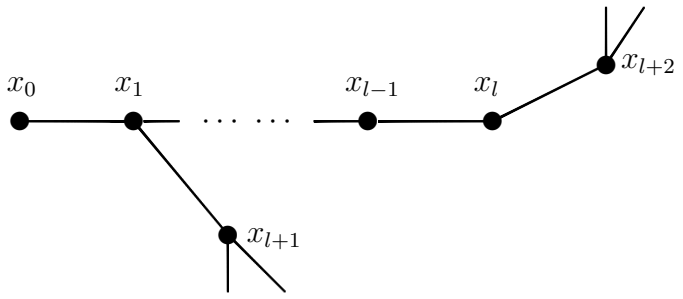


Figure 5.2.2: Tree T_1.

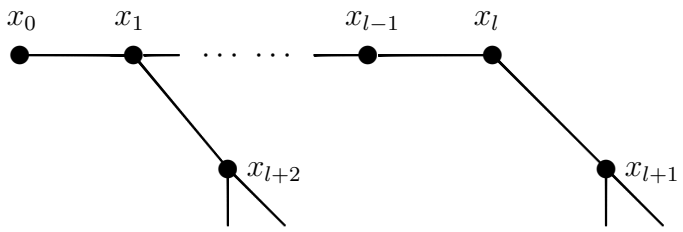


Figure 5.2.3: Tree T_2.

Proof. Let $\gamma = \gamma(T)$ and let D be a $\gamma(T)-$set. For $i \in \{1, 2\}$, let $\gamma_i = \gamma(T_i)$ and D_i be a $\gamma(T_i)-$set. Note that x_1 has a neighbour, x_0, of degree 1 in T, T_1, and T_2. Therefore, by Fact 5.1.1, we can assume that $x_1 \in D \cap D_1 \cap D_2$.
Claim:

(i) $min\{\gamma_1, \gamma_2\} \leq \gamma$.

(ii) $|\gamma_1 - \gamma_2| \leq 1$.

(iii) $\gamma - 1 \leq min\{\gamma_1, \gamma_2\}$.

Proof of Claim:
We first prove (i) by considering two cases.
Case 1: Assume that $x_l \in D$.
Then since $x_1 \in D \cap D_1 \cap D_2$, x_1 dominates x_{l+1} in T_1 and x_1 dominates x_{l+2} in T_2. Therefore, D is a dominating set of T_1 and of T_2 so that

$$\gamma_1, \gamma_2 \leq \gamma$$

and (i) is true.

Case 2:
Assume that $x_l \notin D$.
If x_l is dominated by x_{l-1}, then D dominates T_1 and T_2 and so

$$\gamma_1, \gamma_2 \leq \gamma.$$

If $x_{l-1} \notin D$, then one of x_{l+1} or x_{l+2} is in D and dominates x_l. It follows

that D dominates T_1 or T_2, i.e., $\gamma_1 \leq \gamma$ or $\gamma_2 \leq \gamma$. We conclude that

$$min\{\gamma_1, \gamma_2\} \leq \gamma$$

and (ii) is proven.

We now prove (ii). Since $D_1 \cup \{x_{l+1}\}$ dominates both T and T_2, we get

$$\gamma \leq |D_1 \cup \{x_{l+1}\}| \leq \gamma_1 + 1,$$

and

$$\gamma_2 \leq |D_1 \cup \{x_{l+1}\}| \leq \gamma_1 + 1.$$

Hence,

$$\gamma_1 \geq \gamma - 1 \tag{5.1}$$

and

$$\gamma_1 \geq \gamma_2 - 1. \tag{5.2}$$

Similarly since $D_2 \cup \{x_{l+2}\}$ dominates both T and T_1, we get

$$\gamma_2 \geq \gamma - 1 \tag{5.3}$$

and

$$\gamma_2 \geq \gamma_1 - 1. \tag{5.4}$$

From (5.2) and (5.4), we get

$$\gamma_2 - 1 \leq \gamma_1 \leq \gamma_2 + 1$$

and

$$\gamma_1 - 1 \leq \gamma_2 \leq \gamma_1 + 1.$$

Therefore,

$$|\gamma_1 - \gamma_2| \leq 1$$

and (ii) holds. From (5.1) and (5.3), we get that

$$min\{\gamma_1, \gamma_2\} \geq \gamma - 1.$$

and (iii) is proven. This completes the proof of the Claim.

We now proceed with the proof by way of contradiction. Suppose to the contrary that $\gamma \neq \gamma_1, \gamma_2$. From the Claim (i) and (ii), we get $\gamma - 1 \leq min\{\gamma_1, \gamma_2\} \leq \gamma$. This, in conjuction with Claim 1 (ii) and $\gamma \neq \gamma_1, \gamma_2$, yields $\gamma_1 = \gamma_2 = \gamma - 1$. Since $\gamma_1 = \gamma_2 = \gamma - 1$, neither D_1 nor D_2 dominates T. Hence $x_l \notin D_1 \cup D_2$.

We now construct a dominating set of T with cardinality less than γ. For $i \in \{1, 2\}$, let V_i be the set of vertices of the component of $T - x_l x_{l+i}$ that contains x_{l+i}.

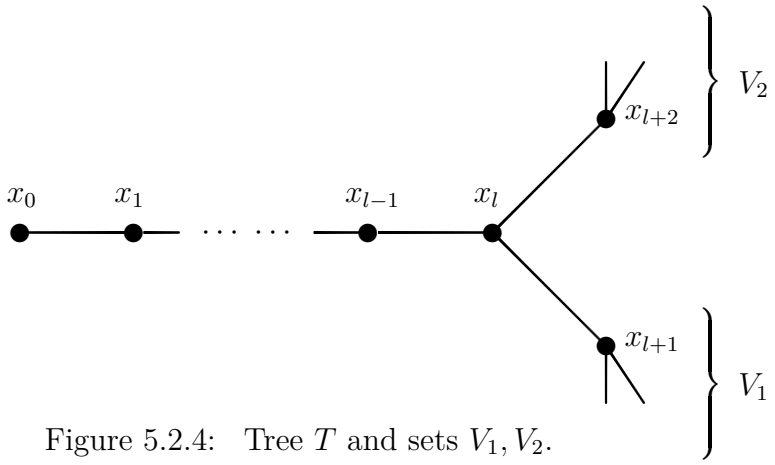


Figure 5.2.4: Tree T and sets V_1, V_2.

Let

$$A = V_1 \cap D,$$

and

$$B = V_2 \cap D.$$

Let

$$C = D \setminus (V_1 \cup V_2).$$

Then

$$D = A \cup B \cup C.$$

For $i \in \{1, 2\}$, let

$$A_i = V_1 \cap D_i,$$

$$B_i = V_2 \cap D_i,$$

and

$$C_i = D_i \setminus (V_1 \cup V_2),$$

and so

$$A_i \cup B_i \cup C_i = D_i. \tag{5.5}$$

Note that

$$A_i \cup \{x_l\} \cup B \cup C$$

is a dominating set of T. If

$$|A_i| \leq |A| - 2,$$

then

$$A_i \cup \{x_l\} \cup B \cup C$$

is a dominating set of T of order

$$|A_i \cup \{x_l\} \cup B \cup C| \leq |A| - 2 + 1 + |B| + |C| = |D| - 1 < \gamma,$$

which is a contradiction. This implies,

$$|A_i| \geq |A| - 1$$

and similarly,

$$|B_i| \geq |B| - 1$$

and

$$|C_i| \geq |C| - 1.$$

If

$$|A_i| \geq |A| + 1,$$

then by (5.5),

$$A \cup \{x_l\} \cup B_i \cup C_i$$

is a dominating set of T of order at most

$$\begin{aligned} |A \cup \{x_l\} \cup B_i \cup C_i| &= |A| + 1 + |B_i| + |C_i| \\ &\leq |A_i| - 1 + 1 + |B_i| + |C_i| \\ &= \gamma_i \\ &< \gamma, \end{aligned}$$

which is a contradiction. This implies,

$$|A_i| \leq |A|$$

and similarly,

$$|B_i| \leq |B|$$

and

$$|C_i| \leq |C|.$$

Since $x_l \notin D_1 \cup D_2$, the two sets

$$A_2 \cup B_1 \cup C_1$$

and

$$A_2 \cup B_1 \cup C_2$$

are dominating sets of T. This implies

$$|A_1| + 1 = |A_2| = |A|,$$

$$|B_1| = |B_2| + 1 = |B|,$$

and

$$|C_1| = |C_2| = |C|.$$

Now

$$D' = \{x_l\} \cup A_1 \cup B_2 \cup C_1$$

is a dominating set of T of order

$$\begin{aligned} |\{x_l\}| + |A_1| + |B_2| + |C_1| &= 1 + |A_1| + |B_1| - 1 + |B_2| \\ &= \gamma_1 \\ &< \gamma, \end{aligned}$$

which is a contradiction. □

5.3 Graph maximizing average distance

We first construct a particular graph $G(n, \gamma)$ of order n and domination number γ. The next section demonstrates that among all connected graphs with order n and domination number γ, the graph $G(n, \gamma)$ achieves the greatest average distance.

Case 1: $\gamma \leq \frac{n}{3}$.
The graph $G(n, \gamma)$ is formed by taking a path P of order $3\gamma - 2$ and extending it by attaching $\left\lfloor \frac{n+2-3\gamma}{2} \right\rfloor$ vertices to one end and $\left\lceil \frac{n+2-3\gamma}{2} \right\rceil$ vertices to the other.

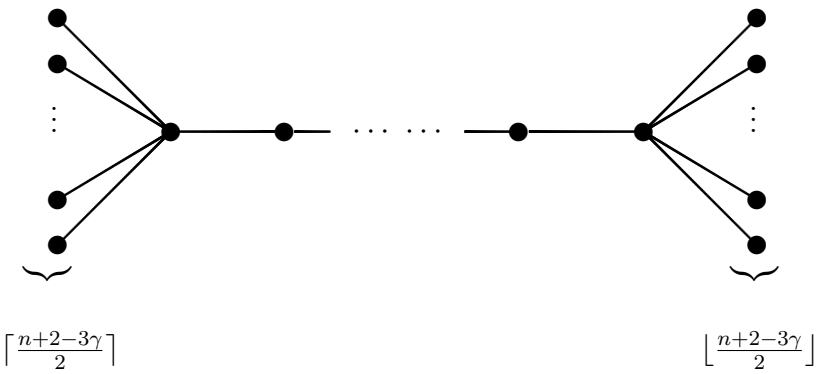


Figure 5.3.5: $G(n, \gamma)$ for the case $\gamma \leq \frac{n}{3}$.

Case 2: $\gamma \geq \frac{n}{3}$.

We take a path P of order $2n - 3\gamma$ and attach one vertex to each of the first $\left\lfloor \frac{3\gamma - n}{2} \right\rfloor$ vertices and one vertex to each of the last $\left\lceil \frac{3\gamma - n}{2} \right\rceil$ vertices of P.



Figure 5.3.6: $G(n, \gamma)$ for the case $\gamma \geq \frac{n}{3}$.

This section analyzes the average distance of $G(n, \gamma)$ based on different values of γ. Note that when $3\gamma = n$, both Case 1 and Case 2 result in a path of order n, which is significant because it is the only connected graph of order n that achieves the maximum possible average distance. In the case where $\gamma = 1$, the construction, in Case 1, produces the star graph $K_{1,n-1}$. The star is the only tree T of order n and domination number 1, and it also has the minimum possible average distance among all trees of order n. The behavior of the average distance is further explained by structural modifications to the graph.

Observation 5.3.1. *Let T be a tree, u be a vertex of T adjacent to end-vertices $u_1, u_2, \ldots, u_r$. Let T' be the tree obtained from T by removing the edges $uu_2, \ldots, uu_r$ and adding the edges $u_1u_2, \ldots, u_1u_r$.*

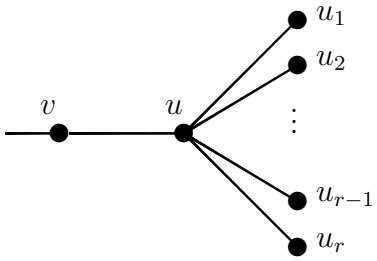


Figure 5.3.7: Tree T.

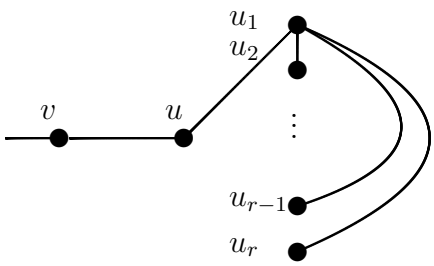


Figure 5.3.8: Tree T'.

Note that the total distance of vertices in $S = \{v, u, u_1, \ldots, u_r\}$ in T is $\frac{2r^2+4r+2}{2}$ and in T' is $\frac{2r^2+6r-6}{2}$. Hence, by considering

$$\sigma(T') - \sigma(T),$$

we get

$$\mu(T) < \mu(T').$$

Observation 5.3.2. *Suppose that $2 \leq r \leq \frac{n}{3}$. Then the graph $G(n, \gamma)$ can be constructed from $G(n, \gamma - 1)$ by performing the operation described in Observation 5.3.1. Consequently,*

$$\mu(G(n, \gamma - 1)) \leq \mu(G(n, \gamma)).$$

Observation 5.3.3. *Let T be a tree that contains a path $abcd$ such that $deg_T\ b = deg_T\ c = 2$. Let T' be obtained from T by replacing edge ab with ac. Then*

$$\mu(T') < \mu(T).$$

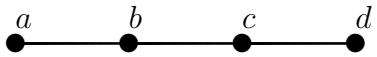


Figure 5.3.9: Tree T.

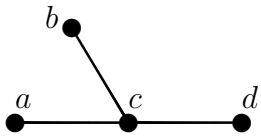


Figure 5.3.10: Tree T'.

This transformation is applied three times when constructing $G(n, \gamma+1)$ from $G(n, \gamma)$ in the case where $\gamma \geq \frac{n}{3}$ and $\gamma \leq \frac{n}{2} - 1$. It follows that

$$\mu(G(n, \gamma + 1)) < \mu(G(n, \gamma)).$$

Hence, for a fixed n, the average distance of $G(n, \gamma)$ increases strictly for $\gamma \leq \frac{n}{3}$, attains its maximum at $\gamma = \frac{n}{3}$, and then decreases strictly for $\gamma \geq \frac{n}{3}$.

5.4 An upper bound on average distance in terms of domination number

To establish an upper bound on the average distance in terms of the domination number, we consider a graph G that maximizes the average distance among all connected graphs of order n with domination number γ. We begin by leveraging structural properties of spanning trees and domination sets, then use distance-based transformations to demonstrate that any deviation from the proposed structure results in a decrease in the average distance.

Theorem 5.4.1. *Let G be a connected graph of order n and domination number γ. Then*

$$\mu(G) \leq \mu(G(n, \gamma)).$$

Proof. Assume that G is a connected graph of order n and domination number γ that maximizes the average distance.
Claim 1: G is a tree.
Proof of Claim 1:

By Lemma 3.2.1, let T be a spanning tree of G that has the same domination number γ as G. We show that $G = T$. Since T is a subgraph of G, $V(T) = V(G)$ and $E(T) \subseteq E(G)$. It remains to show that $E(G) \subseteq E(T)$. Suppose to the contrary that $e = uv \in E(G)$ but $e \notin E(T)$. Since $d_G(x, y) \leq d_T(x, y)$, for all $x, y \in V$, we have $\mu(G) \leq \mu(T)$. Since G maximizes the average distance amongst all graphs of order n and domination number γ and T has order n and domination number γ, we can assume that

$$\mu(G) = \mu(T).$$

However,

$$\begin{aligned}
\mu(G) &= \frac{1}{\binom{n}{2}} \sum_{\{x,y\} \subseteq V} d_G(x, y) \\
&= \frac{1}{\binom{n}{2}} \left[\sum_{\{x,y\} \subseteq V \setminus \{u,v\}} d_G(x, y) + \sum_{(x,y) \in (V \setminus \{u,v\}) \times \{u,v\}} d_G(x, y) + d_G(u, v) \right] \\
&< \frac{1}{\binom{n}{2}} \left[\sum_{\{x,y\} \subseteq V \setminus \{u,v\}} d_T(x, y) + \sum_{(x,y) \in (V \setminus \{u,v\}) \times \{u,v\}} d_T(x, y) + d_T(u, v) \right] \\
&= \mu(T),
\end{aligned}$$

a contradiction and so $e \in E(T)$, i.e., $E(T) = E(G)$.Let P^0 be the graph obtained from G by deleting all of its endvertices.

Claim 2: P^0 is a path

Proof of Claim 2: For contradiction, we assume that $u \in V(P^0)$ such that $\deg_{P^0}(u) \geq 3$. Let $uv \in E(P^0)$ be such that the component of $G - u$ that contains v has minimum order and is a path. We define $P^1 : x_1 x_2 \dots x_l$ with $x_{l-1} = v$ and $x_l = u$. Assume that x_0 is an end vertex adjacent to x_1. Note that $\deg_G(x_j) = 2$ for $2 \leq j \leq l - 1$. Let $x_{l+1}, x_{l+2} \in N[x_l]$.

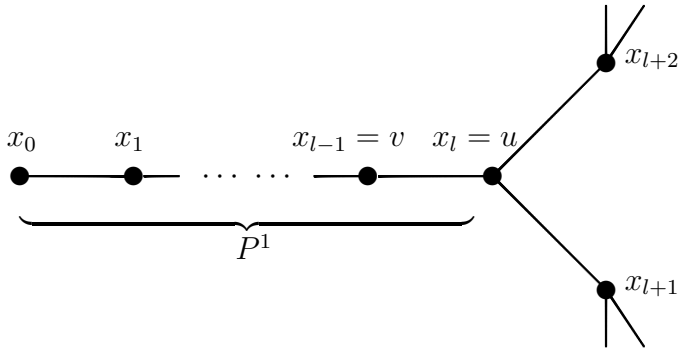


Figure 5.4.11: Path P^1.

Define two trees T_1 and T_2 as done in Lemma 5.2.1. Let a be the order of the component of $G-u$ containing x_{l+1}.Let b denote the sum of the orders of all components of $G-u$ that exclude both x_{l-1} and x_{l+1}, and let c be the sum of the orders of all components of $G-x_1$ that do not contain u. Since uv has been chosen such that P^1 has minimum order, $b > c$. Since

$$\sigma(T_1) - \sigma(G) = a \cdot (b-c) \cdot (l-1)$$

and

$$\sigma(T_2) - \sigma(G) = b \cdot (a-c) \cdot (l-1),$$

we get

$$\mu(T_1) > \mu(G)$$

and

$$\mu(T_2) > \mu(G.)$$

Either T_1 or T_2 has order n and domination number γ according to Lemma 5.2.1, which is a contradiction to the fact that G maximizes the average distance among all graphs of order n and domination number γ.

We now consider two cases and for these cases, let $T_0 : u_1u_2 \dots u_k$.
Case 1: $\gamma \leq \frac{n}{3}$.
We assume that G has domination number at most γ and not exactly γ since the domination of $G(n,\gamma)$ is strictly increasing for $\gamma \leq \frac{n}{3}$ according to the observation above. If $deg_G(u_i) = 2$ for $2 \leq i \leq k-1$, then the fact that G has maximum average distance implies that the difference in the number of endvertices adjacent to u_1 and u_k is at most 1, which would then imply

that $G = G(n, \gamma)$.

We now consider proof by contradiction. Assume that i is the smallest index with $2 \leq i \leq k-1$ for which $\deg_T(u_i) \geq 3$. Let v_i be an endvertex adjacent to u_i. Denote by n_1 the number of endvertices adjacent to u_1, and by n_k the order of the component of $T - u_i$ that contains u_k. We define the two trees $T' = G - u_i v_i + u_1 v_i$ and $T'' = G - u_i v_i + u_k v_i$ from Lemma 5.2.1. Then,

$$\sigma(T') - \sigma(G) = (n_k - n_1) \cdot (i - 1)$$

and

$$\sigma(T'') - \sigma(G) \geq (n_1 + i - n_k) \cdot (k - i).$$

We can therefore conclude that one of these trees has an average distance larger than that of G. There exists some dominating set of G which contains u_1 and u_k but does not contain v_i, the two trees have domination number at most $\gamma(G) \leq \gamma$, which is a contradiction.

Case 2: $3\gamma \geq n$.
We assume that G has a domination number at least γ instead of exactly γ since domination of $G(n, \gamma)$ is strictly decreasing for $3\gamma \geq n$. Given that $n \geq 4$, suppose there exists some $1 \leq i \leq k$ where u_i is adjacent to two endvertices v_i and v_i' in G. Then, modifying G by replacing $u_i v_i$ with $v_i v_i'$ results in a tree $T' = G - u_{i_2} v_i' + v_i v_i'$ that has a greater average distance than G while still maintaining $\gamma(T') \geq \gamma(G) \geq \gamma$, leading to a contradiction. Thus, each vertex in P^0 connects to at most one endvertex. Suppose the degree sequence $(deg_G(u_1), deg_G(u_2), \ldots, deg_G(u_k))$ consists of a single 2, followed by n_1 vertices of degree 3, then n_2 vertices of degree 2, then n_3 vertices of degree 3, and ending with another vertex of degree 2. Then,

$$\gamma(G) = n_1 + \left\lceil \frac{n_2 - 2}{3} \right\rceil + n_3.$$

Additionally, the maximum average distance of G implies that $|n_1 - n_3| \leq 1$, which confirms that $G = G(n, \gamma)$. For contradiction, assume indices i_1, i_2, i_3

with $1 \leq i_1 < i_2 < i_3 \leq k$ are selected such that $u_{i_1}, u_{i_2}, u_{i_3}$ are all adjacent to endvertices. Moreover, suppose that there are at least n_1 vertices of degree 2 between u_{i_1} and u_{i_2} in P^0, and at least n_2 such vertices between u_{i_2} and u_{i_3}. Let v_{i_2} be the endvertex adjacent to u_{i_2}. Since

$$\left\lceil \frac{n_1 - 2}{3} \right\rceil + \left\lceil \frac{n_2 - 2}{3} \right\rceil \leq \left\lceil \frac{(n_1 + n_2 + 1) - 2}{3} \right\rceil,$$

consider the trees

$$T' = G - u_{i_2}v_{i_2} + u_{i_1}v_{i_2}$$

and

$$T'' = G - u_{i_2}v_{i_2} + u_{i_3}v_{i_2}.$$

Both trees must have a domination number at least γ. Let n_1' be the order of the component of $G - u_{i_2}$ containing u_{i_1}, and n_2' the order of the component containing u_{i_3}. Then

$$\sigma(T') - \sigma(G) = (n_2' - n_1' + n_1 + 1) \cdot (n_1 + 1).$$

and

$$\sigma(T'') - \sigma(G) = (n_1' - n_2' + n_2 + 1) \cdot (n_2 + 1).$$

Hence, one of the two trees has larger average distance than G, since this contradicts our initial assumption, the proof is complete. □

5.5 Conclusion

In this chapter we proved results which discuss the structure of graphs that maximize the average distance among all connected graphs of a given order n and domination number γ. In the next chapter we conclude this dissertation.

Chapter 6 Conclusion

In this dissertation, we explored upper and lower bounds on the domination number in terms of various graph parameters, including order, minimum degree, radius, diameter, distance between vertices, eccentricity of the boundary, and average distance. Chapter 1 served as an introductory chapter, where we defined the basic terminology used throughout the dissertation.

In Chapter 2, we provided a historical background on the study of domination in graphs, motivating the importance of the results presented in this work. Additionally, we surveyed significant contributions from other researchers, highlighting bounds on domination number in terms of parameters such as order, minimum degree, radius, independence number, transmission, and average distance.

In Chapter 3, we explored bounds on the domination number of a graph in terms of its radius. We began by proving the existence of a spanning tree T with the same domination number as the graph G. This result allowed us to establish both lower and upper bounds on the domination number in relation to the graph's radius. Before deriving these bounds, we proved several preliminary results. First, we showed that the domination number is bounded below by the diameter of G,

$$\gamma(G) \geq \frac{1}{3}(diam(G) + 1).$$

We also established a relationship between the diameter and radius of a tree T,

$$diam(T) \geq 2rad(T) - 1.$$

The first major theorem in this chapter provided a lower bound on the domination number in relation to the graph's radius,

$$\frac{2}{3}rad(G) \leq \gamma(G).$$

To proceed, we proved a result bounding the domination number of a path P_n of order n in relation of its diameter,

$$\gamma(P_n) \leq diam(P_n) + 1.$$

We also derived expressions for the domination number and radius of P_n in terms of its order,

$$\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil, \; rad(P_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Using these results, we established an upper bound on the domination number of G in relation to its order n and radius,

$$\gamma(G) \leq n - \frac{4}{3}rad(G) + \frac{2}{3}.$$

To conclude the chapter, we introduced the concept of unrelated vertices and used it to prove a final bound on the domination number in relation to the graph's order n, minimum degree $\delta \geq 3$, and radius $r \geq 6$:

$$\gamma(G) \leq n - \frac{2}{3}(r-6)\delta.$$

In Chapter 4, we studied the relationship between the domination number and the diameter of a graph. The first theorem we proved gave us a lower bound on the domination number with regards to the distances between any three vertices $x_1, x_2,$ and x_3,

$$\gamma(G) \geq d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3).$$

We then proved a theorem that generalizes this result to provide a lower bound on the domination number in terms of the distances between any arbitrary number of vertices,

$$\gamma(G) \geq \frac{1}{r^2 - r} \left(\sum_{1 \leq i < j \leq r} d_G(x_i, x_j) \right).$$

Finally, we concluded this chapter by establishing a lower bound on the domination number in relation to the eccentricity of the boundary,

$$\gamma(G) \geq \frac{1}{2}(e(B) + 1).$$

In Chapter 5, we studied the relationship between the domination number and the average distance of the graph G. We began by proving an important lemma, which served as the foundation for proving Dankelmann's theorem. This theorem discusses the structure of graphs that maximize the average distance among all connected graphs of a given order n and domination number γ.

To further improve bounds on the domination number, future research could focus on deriving bounds in terms of other graph parameters, such as the independence number, chromatic number, or edge density. Another interesting direction could involve exploring domination number bounds in relation to spectral graph theory, such as eigenvalues of the adjacency or laplacian matrix. Additionally, it would be worthwhile to apply our findings to specific graph classes, such as bipartite graphs, planar graphs, or regular graphs. Investigating these specialized cases may yield deeper insights into the interplay between domination number and structural properties of graphs.

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