So we have,

$$\varphi_n(z) - \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)}\varphi_n(d\sigma, z)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\xi)K_{n-1}(d\sigma, \xi, z)\Big[d\sigma(\xi) - \big(\rho(z)\big)^{-1}d\mu(\xi)\Big].$$

Next, divide (5.17) by $\varphi_n(d\sigma, z) \neq 0$ in $|z| = 1$ and [illegible] (1.27) to get the result. ∎

Lemma 5.5

Let $d\mu$ be a finite, positive Borel measure on τ with $\mu' > 0$ a.e. (meas). We define

$$\eta_n = \inf_{\pi_n}\Big\| |\pi_n| - (\mu')^{-1/2}\Big\|_{L^2(d\mu)}.$$

Then the following are true:

(5.18) $$|a_n| \leq \frac{1}{2\pi}\int_\tau \Big| |\varphi_n(z)|^2 \mu'(\theta) - 1\Big| d\theta \leq 2\eta_n;$$

(5.19) $$\eta_n \to 0 \quad \text{as} \quad n \to \infty.$$

Proof:

This is Lemma 1, 2 and 3 of [22]. ∎

Remarks:

(a) We note that 5.18 and 5.19 together yield an alternative proof of (3.6), (3.7) and Theorem 3.9 for $W_n \equiv 1$.

(b) Let $\log \mu' \in L^1(\tau)$ and define

$$\delta_n = \inf_{\pi_n} \|\pi_n - D_{\mu'}\|_{L^2(d\mu)}.$$

δ_n plays a fundamental role in the theory of orthonormal polynomials on τ (see [5] p.21). Also notice by (2.36) $\delta_n = O(\eta_n)$. Thus when $\mu' \notin \log L^1(\tau)$, η_n is its natural analogue.

Lemma 5.6

Let $d\mu(z) = \rho(z)d\sigma(z)$ be a finite Borel measure on τ and $1 - \epsilon \leq \rho(z) \leq 1 + \epsilon$, $z \in \tau$, $\epsilon \in (0, 1/2)$. Then,

$$(1+\epsilon)^{-1} \leq \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)} \leq (1+\epsilon), \; n = 0, 1, 2, \dots . \tag{5.20}$$

There exists C_1 such that

$$\left|\frac{\varphi_n(z)}{\varphi_n(d\sigma, z)} - 1\right| \leq C_1\sqrt{\epsilon}(1 + \mathcal{L}(\rho, z)), \; z \in \tau, \; n \geq N. \tag{5.21}$$

Proof:

By (1.20) if V_1 and V_2 are two finite Borel measures such that $V_1 \leq MV_2$ for some $M > 0$, then

$$\gamma_n(dV_1) \geq M^{-1/2}\gamma_n(dV_2).$$

Thus in our case as

$$d\mu(z) \leq (1+\epsilon)d\sigma(z)$$

and

$$d\sigma(z) \leq (1+\epsilon)^{-1}d\mu(z), \; \epsilon \in \left(0, \frac{1}{2}\right)$$

(5.20) follows by the above considerations. Now write

$$\tau = \left(\tau_{\sqrt{\epsilon}}(z)\right) \bigcup \left(\tau \backslash \tau_{\sqrt{\epsilon}}(z)\right).$$

Then if $\xi \in \left(\tau \backslash \tau_{\sqrt{\epsilon}}(z)\right)$, $|z - \xi| \geq \sqrt{\epsilon}$, thus

$$\left|\frac{\rho(z) - \rho(\xi)}{z - \xi}\right| \leq 3\sqrt{\epsilon}.$$

So that we have,

$$\left|\frac{\rho(z)-\rho(\xi)}{z-\xi}\right| \le \begin{cases} \mathcal{L}(\rho,z), & \xi \in \tau_{\sqrt{\epsilon}}(z), \\ 3\sqrt{\epsilon}, & \xi \in \tau \backslash \tau_{\sqrt{\epsilon}}(z). \end{cases} \tag{5.22}$$

Now use Lemma 5.4, together with (5.22) the Schwarz inequality, 3.14, orthonormality of $d\sigma$ and (5.20) to yield result. ∎

We wish to define the Jackson Sum of a 2π periodic continuous function. For this purpose we need the following two lemmas whose proofs can be found in ([21] pp.79–82).

<u>Lemma 5.7</u>

$$\left(\sin\left(\frac{nt}{2}\right)\left(\sin\left(\frac{t}{2}\right)\right)^{-1}\right)^4$$

$$= L + \sum_{k=1}^{2n-2} \ell_k \cos kt, \quad \text{some} \quad L, \ell_k, \ 1 \le k \le 2n-2; \tag{5.23}$$

$$\int_0^{\pi/2} \left(\sin(nt)(\sin t)^{-1}\right)^4 dt = \frac{\pi n(2n^2+1)}{6}; \tag{5.24}$$

$$\int_0^{\pi/2} t\left((\sin(nt))(\sin t)^{-1}\right)^4 dt < \frac{\pi^2 n^2}{4}. \tag{5.25}$$

<u>Lemma 5.8</u> (The Jackson Sum)

Let f be 2π periodic and continuous with modulus of continuity $w_f(\delta)$. Set

$$u_n(x) = \frac{3}{2\pi n(2n^2+1)} \int_0^{2\pi} f(t) \left[\left(\sin \frac{n(t-x)}{2}\right)\left(\sin\left(\frac{t-x}{2}\right)\right)^{-1}\right]^4 dt$$

then $u_n(x) = J_n(x,f)$ is called the Jackson Sum of f and

$$J_n(t) = \frac{3}{2\pi n(2n^2+1)} \left[\sin n\left(\frac{t-x}{2}\right)\left(\sin\left(\frac{t-x}{2}\right)\right)^{-1}\right]^4$$

the corresponding Jackson Kernel satisfying

$$J_n(x,f) = \int_0^{2\pi} f(t)J_n(t)dt.$$

Using Lemma 5.7 we can then prove:

(a) u_n is a trigonometric polynomial of degree $2n-2$

(5.26) $$\left|u_n(x) - f(x)\right| \le C_2 w_f\left(\frac{1}{n}\right) \le C_3\left(\frac{1}{n}\right)$$

where the last inequality holds if f satisfies a Lipschitz condition of order 1.

Remark:

In our setting let n be an even positive integer and set $m = 1 + \frac{n}{2}$. Then

$$J_n(z) = \frac{3}{2\pi m(2m^2+1)}\left|\frac{z^m - 1}{z-1}\right|^4$$

and

$$J_n(z,\rho) = \int_\tau J_n(\xi)\rho(\xi z)d\theta$$

are the Jackson Sum and corresponding Kernel of a 2π periodic continuous ρ. If $z = e^{i\theta}$, $J_n(e^{i\theta},\rho)$ is a trigonometric polynomial of degree n. Also if $\rho_n(z) = J_n(z,\rho)$ then,

$$|\rho_n(z) - \rho(z)| \le C_2 w_\rho\left(\frac{1}{n}\right) \le C_3\left(\frac{1}{n}\right)$$

with the second inequality holding when ρ satisfies a Lipschitz condition of order 1.

Lemma 5.9

Let ρ be continuous and 2π periodic on τ. Set $\rho_n(z) = J_n(z,\rho)$. Then $\mathcal{L}(\rho_n, z) \le C_5\mathcal{L}(\rho, z)$, $n = 2,4,6,\ldots$, $z \in \tau$.

Proof:

If $z_0 \in \tau$ is fixed and $\mathcal{L}(\rho, z_0) = \infty$ then we are done. So suppose without loss of generality that $\mathcal{L} = \mathcal{L}(\rho; z_0) < \infty$. Let $u(z) = |z - z_0|$, $z \in \tau$, continuous and 2π periodic. Further using $\big||a| - |b|\big| \le |a - b|$,

$$|u(z) - u(\xi)| = \Big||z - z_0| - |\xi - z_0|\Big| \le |z - \xi|. \tag{5.27}$$

So u satisfies a Lipschitz condition of order 1, thus if $u_n(z) = J_n(z, u)$ we have by Lemma 5.8 that,

$$|u_n(z) - u(z)| \le C_2 w_u\Big(\frac{1}{n}\Big) \le C_3\Big(\frac{1}{n}\Big). \tag{5.28}$$

Then

$$\begin{aligned}
&|\rho_n(z) - \rho_n(z_0)| \\
&\le \int_0^{2\pi} J_n(\xi)\big|\rho(\xi z) - \rho(\xi z_0)\big| d\theta \\
&\le \int_0^{2\pi} J_n(\xi)\big|\rho(\xi z) - \rho(z_0)\big| d\theta + \int_0^{2\pi} J_n(\xi)\big|\rho(\xi z_0) - \rho(z_0)\big| d\theta \\
&\le \mathcal{L}\Big(\int_0^{2\pi} J_n(\xi)\big|\xi z - z_0\big| d\theta\Big) + \mathcal{L}\Big(\int_0^{2\pi} J_n(\xi)|\xi z_0 - z_0| d\theta\Big) \\
&\le \mathcal{L}\big(u_n(z) + u_n(z_0)\big) = \mathcal{L}\big(u_n(z) - u(z) + u_n(z_0) - u(z_0) + u(z) + u(z_0)\big) \\
&= \mathcal{L}\big(u_n(z) - u(z) + u_n(z_0) - u(z_0) + u(z) - u(z_0)\big) \quad (\text{as } u(z_0) = 0) \\
&\le \mathcal{L}\Big(|z - z_0| + \frac{2C_3}{n}\Big)
\end{aligned}$$

by (5.27) and (5.28). So we have shown that,

$$|\rho_n(z) - \rho_n(z_0)| \le \mathcal{L}\Big(|z - z_0| + \frac{2C_3}{n}\Big). \tag{5.29}$$

Now let

$$t_{2n}(z) = \frac{\big(\rho_n(z) - \rho_n(z_0)\big)^2}{|z - z_0|^2}, \quad z \in \tau.$$

Then using (5.29),

$$t_{2n}(z) \leq \mathcal{L}^2\Big(1 + \frac{2C_3}{n|z-z_0|}\Big)^2, \quad z \in \tau. \tag{5.30}$$

So (5.30) implies that

$$0 \leq t_{2n}(z) \leq C_4^2\mathcal{L}^2, \; z \in \tau\backslash\tau_{1/4n}(z_0), \; C_4 = 1 + 9C_3. \tag{5.31}$$

Now suppose for some $n = 2, 4, 6, \ldots$ we have

$$M_n = \max_{z\in\tau} t_{2n}(z) > \mathcal{L}^2C_3^2.$$

Then we may choose $z_1 \in \tau_{1/4n}(z_0)$ such that $M_n = t_{2n}(z_1)$ and consequently a point $\xi \in \tau_{1/4n}(z_0)$ such that

$$\big|t'_{2n}(\xi)\big| \geq 4n(M_n - C_3^2\mathcal{L}^2). \tag{5.32}$$

Next, $t_{2n}(e^{i\theta})$ is a positive trigonometric polynomial of degree $< 2n$, so by Bernstein's inequality (see [2] p.91) $|t'_{2n}(z)| \leq 2nM_n$ for all $z \in \tau$. Thus with (5.32), we must have $M_n \leq 2\mathcal{L}^2C_3^2$ and so we have shown that

$$\mathcal{L}(\rho_n, z) \leq C_4\mathcal{L}(\rho, z), \; z \in \tau_{1/4n}(z_0). \tag{5.33}$$

So (5.33) with (5.31) can yield the result. ∎

<u>Lemma 5.10</u>

Let $\rho > 0$ be continuous and 2π periodic on τ such that $\mathcal{L}(\rho, z) \leq C_5$, $z \in \gamma \subseteq \tau$ where γ is an arbitrary subset of τ. Let $\rho_n(z) = J_n(z, \rho)$. Then,

$$\lim_{n\to\infty} D(\rho_n, z, ex) = D(\rho, z, ex) \tag{5.34}$$

uniformly for $z \in \gamma$.

Proof:

Using (5.26) we can deduce that

$$a_n(z) = \log\left[\rho_n(z)\big(\rho(z)\big)^{-1}\right] \to 0 \text{ as } n \to \infty. \tag{5.35}$$

Use the definition D, Lemma 2.8, Lemma 2.9, and $|e^z| = \exp\big(Re(z)\big)$ to deduce that $\left|D(\rho_n, z, ex)\right| \to \left|D(\rho, z, ex)\right|$ uniformly for $z \in \tau$ and that to show convergence of the arguments merely requires that we show (using Lemma 2.9 again that)

$$\widetilde{a_n}(z) = \frac{1}{\pi}\int_0^{2\pi} \frac{a_n(\xi) - a_n(z)}{|\xi - z|^2} Im(\bar{\xi}z)d\theta \to 0 \text{ as } n \to \infty,\ z \in \tau. \tag{5.36}$$

Now by Lemma 5.9 as $\mathcal{L}(\rho, z) \le C_5$ we have

$$\mathcal{L}(a_n, z) \le C_6,\ z \in \tau,\ \frac{n}{2} \in \mathbb{N}. \tag{5.37}$$

Moreover $\left|Im(\bar{\xi}z)\right| = \left|Im(1 - \bar{\xi}z)\right| \le |\xi - z|$, $\xi, z \in \tau$. So if we choose $\delta \in (0, \pi)$ then,

$$\begin{aligned}
\left|\widetilde{a_n}(z)\right| &\le \frac{1}{\pi}\int_\tau \frac{|a_n(\xi) - a_n(z)|}{|\xi - z|}d\theta =: \frac{1}{\pi}\int_\tau I_1 d\theta \\
&= \frac{1}{\pi}\int_{\tau_\delta(z)} I_1 d\theta + \frac{1}{\pi}\int_{\tau\setminus\tau_\delta(z)} I_1 d\theta \\
&\le \frac{1}{\pi}\int_{\tau_\delta(z)} \mathcal{L}(a_n, z)d\theta + (\delta)^{-1}\int_{\tau\setminus\tau_\delta(z)} \left|a_n(\xi) - a_n(z)\right| d\theta \\
&\le \frac{2\delta}{\pi}\mathcal{L}(a_n, z) \text{ as } n \to \infty \quad \text{(by (5.35))} \\
&\le C_7\delta \quad \text{(by (5.37))}.
\end{aligned}$$

Let $\delta \to 0$ and the result is proved. ∎

We can now give the proof of Theorem 5.11.

Proof:

Let $\rho_k(z) = J_k(z,\rho)$, $d\mu_k = \rho_k d\sigma$, k a positive integer. Then we have for each k,

$$\frac{\varphi_n(z)}{\varphi_n(d\sigma,z)} - D(\rho,z,ex)$$
$$= \Big(\frac{\varphi_n(z)}{\varphi_n(d\mu_k,z)} - 1\Big)\Big(\frac{\varphi_n(d\mu_k,z)}{\varphi_n(d\sigma,z)}\Big) +$$
$$+ \Big(\frac{\varphi_n(d\mu_k,z)}{\varphi_n(d\sigma,z)} - D(\rho_k,z,ex)\Big) + (D_{\rho_k} - D_\rho). \tag{5.38}$$

Let $\epsilon \in (0,1/2)$ be arbitrary. Then by Lemma 5.10 we may fix k such that

$$1-\epsilon \le \rho_k(z)\big(\rho(z)\big)^{-1} \le 1+\epsilon,\ z \in \tau, \tag{5.39}$$

$$\Big|D(\rho_k,z) - D(\rho,z)\Big| \le \epsilon,\ z \in \tau. \tag{5.40}$$

Using Lemma 5.9, $\mathcal{L}\Big(\frac{\rho_k}{\rho},z\Big) \le C_8$ and $\mathcal{L}(\rho,z) \le C_9$, $z \in \gamma$. So by Lemma 5.6,

$$\lim_{n\to\infty}\sup\Big|\frac{\varphi_n(z)}{\varphi_n(d\mu_k,z)} - 1\Big| \le C_{10}\sqrt{\epsilon} \tag{5.41}$$

uniformly for $z \in \gamma$. Also by Lemma 5.3

$$\lim_{n\to\infty}\frac{\varphi_n(d\mu_k,z)}{\varphi_n(d\sigma,z)} = D(\rho_k,z) \tag{5.42}$$

uniformly for $z \in \gamma$.

Thus from (5.38), (5.40), (5.41), (5.42) we get

$$\lim_{n\to\infty}\sup\Big|\frac{\varphi_n(z)}{\varphi_n(d\sigma,z)} - D(\rho,z)\Big| \le C_{11}\sqrt{\epsilon} \tag{5.43}$$

uniformly for $z \in \tau$. As $\epsilon > 0$ is arbitrary in (5.43) we have the result uniformly for $z \in \gamma \subseteq \tau$. ∎

We now proceed to prove the last theorem in this chapter, Theorem 5.13 which is the comparative mean analogue of (2.58), (2.59) and (3.11). We need a lemma first whose proof is not difficult but long so we give the reference ([15], Lemma 1.1).

Lemma 5.12

Assume $\mu' > 0$ a.e. (meas) in $[0,2\pi]$, $g \geq 0 \in L^1(d\mu)(\tau)$ such that $\hat{Q}g^{\pm 1} \in L^\infty(d\mu)(\tau)$ for a suitable polynomial Q. Let f be 2π periodic such that $f \in L^\infty(d\mu)(\tau)$. Then, for $u = e^{it}$,

$$\lim_{n\to\infty} \int_0^{2\pi} f(t)\varphi_n(gd\mu, u)\overline{\varphi_{n+\ell}(u)}\sqrt{g(t)}\mu'(t)dt$$

$$= \int_0^{2\pi} f(t)u^{-\ell}D(g,u)\left(\sqrt{g(t)}\right)^{-1}dt \tag{5.44}$$

for all integers ℓ.

Proof of Theorem 5.13

Proof:

$$\int_0^{2\pi}\left|\varphi_n(gd\mu,u)\overline{D(g,u)} - \varphi_n(u)\right|^2\mu'(t)dt$$

$$= \int_0^{2\pi}|\varphi_n(gd\mu,u)|^2|D(g,u)|^2\mu'(t)dt + \int_0^{2\pi}|\varphi_n(u)|^2\mu'(t)dt -$$

$$- 2Re\int_0^{2\pi}\frac{\overline{D(g,u)}}{(g(t))^{1/2}}\varphi_n(gd\mu,u)\overline{\varphi_n(u)}(g(t))^{1/2}\mu'(t)dt$$

$$\leq 2\pi + 2\pi - 2Re\int_0^{2\pi}\frac{\overline{D(g,u)}}{(g(t))^{1/2}}\varphi_n(gd\mu,u)\overline{\varphi_n(u)}(g(t))^{1/2}\mu'(t)dt$$

by orthonormality and (2.36). Also by (2.36)

$$\overline{D(g)}(g(t))^{-1/2} \in L^\infty(d\mu)(\tau).$$

So by Lemma 5.12 as $n \to \infty$ with $\ell = 0$ the last integral becomes

$$\int_0^{2\pi}\frac{\overline{D(g,u)}D(g,u)}{\left((g(t))^{1/2}\right)^2}dt = 2\pi.$$

So the result follows. ■

We conclude this chapter with some open questions:

(a) Recall that [15] gives a comparative analogue to Theorem 2.19 (b) which is stronger than (5.2) and (5.3) in the sense that weaker conditions are imposed on g. It is worth asking whether an analogue of this type exists for Theorem 5.11 in this setting? Further, we saw that for the result in [15] uniform boundedness was required of φ_n in a neighbourhood of θ which was not required in Theorem 2.19 (b) of Freud. Can this boundedness be weakened or removed here?

(b) What form would Theorem 2.19 (c) take in the setting of non Szegö weights, that is, are there mean, ratio or comparative analogues? What form would the strong analogue take if any?

(c) Also for (c) of the Remarks after Theorem 5.13, does there exist an analogue for Theorem 5.11?

(d) In Theorem 5.11 ρ was (a) continuous and positive on τ and (b) $\mathcal{L}(\rho, z) \leq C_1$. Can (a) be weakend?

(e) Let μ and σ be finite Borel measures on τ. Let $\gamma \subseteq \tau$ and suppose $\mu|_\gamma = \sigma|_\gamma$. Then is it true that

$$\lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi_n(d\sigma, z)} = 1$$

for every closed $\gamma' \subseteq \gamma$?

We note that (d) and (e) are raised in [22].

Chapter 6

Uniform asymptotics on the unit circle

Notation

Throughout μ will denote a positive, Borel measure on τ, and $w = \mu' > 0$ *a.e.* $(meas)$. $\varphi_n(d\mu, z) = \varphi_n(z)$ will be the corresponding orthonormal polynomials with respect to $d\mu$, $supp(d\mu)$ is infinite. Let $\gamma \subseteq \tau$ be an arc on τ and let $W(f,\delta)(\gamma)$ denote the modulus of continuity of a function f on γ. Define $DL(\gamma) = \{f : f \text{ continuous such that } \frac{W(f,\delta)(\gamma)}{t} \in L^1(\tau)\}$ that is, $DL(\gamma)$ is the set of functions that satisfy a Dini–Lipschitz condition on $\gamma \subseteq \tau$. $\tilde{w}$ will as usual denote the conjugate function of w and $C_1, C_2, \ldots$ will denote constants independent of n, z, ξ, w.

Statement 6.1

We can now state the main results to be proved in this chapter.

Theorem 6.6

Let $w \in DL(\tau)$, also on the open arc $\gamma \subseteq \tau$, let w' exist and satisfy $w' \in DL(\gamma)$. Then

$$\lim_{n\to\infty} |\varphi_n(z)|^{-1} = (w(z))^{1/2}$$

uniformly on each closed $\gamma' \subseteq \gamma$. Note here that we write

$$d\mu(\theta) = w(\theta)d\theta.$$

Theorem 6.8

Let $w \in DL(\tau)$. Then

$$\lim_{n\to\infty} |\varphi_n(z)| = (w(z))^{-1/2}$$

uniformly for all arbitrary subsets $\gamma \subseteq \tau$ for which

$$m = \inf_{z \in \gamma} w(z) > 0.$$

The main tool used in the proofs above is the concept of polynomials of the Second Kind, hence we begin with the following definition.

Definition 6.2 (Polynomials of the Second Kind)

Let μ be a finite Borel measure on τ and write $d\mu(e^{i\theta}) = w(\theta)d\theta$. Now we can define

(a) $$F_\mu(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\mu(e^{i\theta}),\ |z|<1,\ z = re^{i\alpha}$$

as in Lemma 2.9.

Let $\mu_0 = \frac{1}{2\pi}\int_0^{2\pi} d\mu(\theta) = F_\mu(0) < \infty$ and define a new measure μ^* on τ as follows:

(b) $\dfrac{d\mu^*(z)}{d\mu(z)} = \dfrac{\mu_0^2}{|F_\mu(z)|^2}$ satisfying $F_{\mu^*}(z)\cdot F_\mu(z) = \mu_0^2,\ |z|<1.$

(c) We can also define orthonormal polynomials $\psi_n(d\mu^*, z)$ with respect to $d\mu^*$ as follows:

$$\psi_n(d\mu^*, z) = \frac{1}{2\pi\mu_0}\int_0^{2\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\Big(\varphi_n(e^{i\theta}) - \varphi_n(z)\Big)d\mu(\theta),\ |z|<1.$$

Lemma 6.3

Let $w \in DL(\tau)$. Then

(a) μ^* is absolutely continuous on $|z|<1$, further if w and $\tilde{w}$ have no zeros in common then μ^* is absolutely continuous with respect to $d\mu$ on $|z|=1$, that is, $d\mu^*(z) = \dfrac{\mu_0^2}{|F_\mu(z)|^2}d\mu(z),\ |z|=1$ where $F_\mu(z) = \lim_{r\to 1-} F_\mu(rz)$ exists a.e. $(d\mu)$, $(r<1)$.

(b) $F_\mu(z) \cdot F_\mu(z) = \mu_0^2$, $|z| = 1$ if w, $\tilde{w}$ have no zeros in common.

(c) $D(\rho, z, ex) = (\mu_0)^{-1}\overline{F_\mu}((z)^{-1})$, $|z| \geq 1$ where $\rho(z) = \dfrac{\mu_0^2}{|F_\mu(z)|^2}$.

<u>Proof:</u>

We note first that the existence of the boundary values a.e. $(d\mu)$ follows from a similar argument used in Lemma 2.15 when defining D for $|z| = 1$. Now if w, $\tilde{w}$ have no zeros in common by (2.21) we have, $|F_\mu(z)|^2 \geq C > 0$, $z \in r$. Thus $\rho(z)$ is continuous and positive on $|z| \leq 1$ and so by the theorem of F and M Riesz ([5] p.220) μ^* is absolutely continuous with respect to $d\mu$ on $|z| = 1$ and $d\mu^* = \rho(z)d\mu(z)$. Next

$$\begin{aligned} Re\Big((F_\mu(z))^{-1}\Big) &= Re\Big(\overline{F_\mu(z)}(|F_\mu(z)|)^{-2}\Big) \\ &= Re(F_\mu(z))(|F_\mu(z)|)^{-2} > 0 \end{aligned}$$

in $|z| < 1$. Thus $Re\Big((F_\mu(z))^{-1}\Big)$ is a positive harmonic function in $|z| < 1$. So letting $r < 1$, we may write by the famous Poisson representation theorem (see [1] p.167)

$$\mu_0^2|F_\mu(rz)|^{-2} = \mu_0^2\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta)|F_\mu(z)|^{-2}d\theta, \ |z| = 1, \ r < 1$$

which implies that

$$\text{(6.1)} \qquad |F_{\mu^*}(rz)|^2 = \frac{\mu_0^2}{2\pi}\int_0^{2\pi} P(r,\theta)|F_\mu(z)|^{-2}d\theta, \ |z| = 1, \ r < 1.$$

Also we have,

$$\text{(6.2)} \qquad |F_{\mu^*}(rz)|^2 = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta)|F_{\mu^*}(z)|^2 d\theta, \ |z| = 1, \ r < 1.$$

So by (6.1) and (6.2) $\mu_0^2|F_\mu(z)|^{-2} = |F_{\mu^*}(z)|^2$, $|z| = 1$ where this is understood to mean the points where $F_\mu(z)$ exists on $|z| = 1$ that is a.e. $(d\mu)$. Finally by definition of $D(\rho, z, ex)$ and (2.45) we note that as

$\left|(\mu_0)^{-1}F_\mu((z)^{-1})\right|^2 = \rho(z), \quad z = e^{i\theta}, \quad (\mu_0)^{-1}\overline{F_\mu}((\infty)^{-1}) = 1$ and $(\mu_0)^{-1}\overline{F_\mu}((z)^{-1})$ is analytic in $|z| > 1$. We can choose an exterior Szegō function $D(\rho, z, ex) = (\mu_0)^{-1}\overline{F_\mu}((z)^{-1})$, $|z| > 1$ and extend the definition to $|z| \geq 1$.

The following results also hold and the proofs can be found in ([5] pp.10–11).

Lemma 6.4

(6.3) $$\mu_0 Re\Big(\frac{\psi_n^*(d\mu^*, z)}{\varphi_n^*(z)}\Big) = \left|\varphi_n^*(z)\right|^2, \ |z| = 1,$$

(6.4) $$\psi_n(d\mu, z) = \varphi_n(d\mu^*, z), \ \varphi_n(z) = \psi_n(d\mu^*, z), \ |z| \leq 1.$$

Lemma 6.5

Let $f \in DL(\tau)$, $f > 0$ a.e. (meas) on τ. Then there exists $g \in C'(\tau) = \{h : h : \tau \to \mathbf{C}\,,\ h' \text{ continuous}\}$, positive and a $\delta > 0$ such that if $f_0 = fg$ then f_0 satisfies $(f_0^2 + \tilde{f}_0^2) \geq \delta^2$, $z \in \tau$ that is f_0, $\tilde{f}_0$ have no common zeros.

Proof:

Since $f(z) > 0$ a.e. (meas) on τ, there exists $z \in \tau$ such that $f(z) > 0$ and $f(-z) > 0$. Suppose without loss of generality that $z = 1$. Since f is continuous there exists $\delta > 0$ such that $f(z) \geq \delta$, $z \in \tau_\delta(1)$ and $z \in \tau_\delta(-1)$. Now choose a function $h(z) \in C'(\tau)$ such that $h(z) > 0$ for $z \in \tau_\delta(1)$ and $h(z) = 0$ for $z \in \tau \backslash \tau_\delta(1)$.

Also let
$$\tilde{f}(z) = \frac{1}{\pi}\int_0^{2\pi} \frac{f(\xi)Im(\bar{\xi}z)}{|\xi - z|^2}d\theta, \ \xi = e^{i\theta}$$
be the conjugate function of f (see Lemma 2.9). Now $Im(\bar{\xi}z)$ changes sign at $\xi = z$ and $\xi = -z$. Hence the zeros of $Im(\bar{\xi}z)$ cancel with one of the zeros

of $|\xi - z|^2$ so that we obtain

$$(6.5) \qquad |\widetilde{fh}(z)| \geq m > 0, \; z \in \tau \backslash (\tau_\delta(1) \cup \tau_\delta(-1)).$$

Also $\tilde{f}$ is continuous and $M = \max_\tau |\tilde{f}| < \infty$. So we define $\lambda = (M + \delta)(m)^{-1}$, $f_0 = f(1 + \lambda h)$ and then by the linearity of $\tilde{f}$ and by (6.5), we have for $z \in \tau \backslash (\tau_\delta(1) \cup \tau_\delta(-1))$,

$$|\tilde{f}_0| = |\tilde{f} + \lambda(\widetilde{fh})| \geq \lambda |\widetilde{fh}| - |\tilde{f}| \geq \lambda m - M = \delta.$$

Also we have, $|f_0| \geq \delta$ on $\tau_\delta(1) \cup \tau_\delta(-1)$ so $f_0^2 + \tilde{f}_0^2 \geq \delta^2$ on τ and so letting $g = 1 + \lambda h$ we are done. ■

We can now give the proof of Theorem 6.6.

Proof:

We first prove the result assuming w, $\tilde{w}$ have no zeros in common. Then we have, $|F_\mu(z)|^2 > 0$, $z \in \tau$, $\rho(z) = \mu_0^2 |F_\mu(z)|^{-2}$ is continuous and positive on τ (see Lemma 6.3), and has a continuous derivative on $\gamma \subseteq \tau$ so that on every closed $\gamma' \subseteq \gamma$ we have $\mathcal{L}(\rho, z) \leq C_2$, $z \in \gamma' \subseteq \gamma$. Also by absolute continuity of μ^* we may write $d\mu^*(z) = \rho(z) d\mu(z)$, $|z| \leq 1$. We can now apply Theorem 5.11 and (6.4) to deduce

$$(6.6) \qquad \lim_{n\to\infty} \frac{\psi_n(d\mu, z)}{\varphi_n(z)} = \lim_{n\to\infty} \frac{\varphi_n(d\mu^*, z)}{\varphi_n(z)} = D(\rho, z, ez)$$

uniformly for $z \in \gamma'$. Also by (6.3) we have that

$$(6.7) \qquad \mu_0 Re\left\{\frac{\psi_n(d\mu^*, z)}{\varphi_n(z)}\right\} = \mu_0 Re\left\{\frac{\psi_n^*(d\mu^*, z)}{\varphi_n^*(z)}\right\} = |\varphi_n(z)|^{-2}, \; z \in \tau.$$

Thus taking the real part of (6.6), using Lemma 6.3 (c), (6.7) and Lemma 2.9 we have,

$$(6.8) \qquad \lim_{n\to\infty} |\varphi_n(z)|^{-2} = \mu_0 Re\{D(\rho, z, ez)\} = Re\overline{\{F_\mu(z)\}} = w(z)$$

uniformly for $z \in \gamma'$. Now suppose the general case. Define $f_0 = wg$ as in Lemma 6.5 such that f_0 and $\widetilde{f}_0$ have no common zeros. Write $d\sigma = f_0 d\theta$ so that by (6.8) we have that

$$(6.9) \qquad \lim_{n\to\infty} |\varphi_n(d\sigma, z)|^{-2} = f_0(z) \text{ uniformly for } z \in \gamma'$$

but $d\sigma = f_0 d\theta = wg d\theta = g d\mu$ and g is continuous. So by Theorem 5.11

$$(6.10) \qquad \lim_{n\to\infty} \left|\frac{\varphi_n(d\sigma, z)}{\varphi_n(d\mu, z)}\right|^2 = |D(g, z, ex)|^2 = (g(z))^{-1}, \; z \in \gamma.$$

So we have by (6.9) and (6.10)

$$(6.11) \qquad \lim_{n\to\infty} |\varphi_n(d\mu, z)|^{-2} = (f_0(g)^{-1})(z) = f(z)$$

uniformly for $z \in \gamma'$. ■

We can now turn to the proof of Theorem 6.8. However we need one lemma.

Lemma 6.7

Let $f \in DL(\tau)$. Define $\gamma = \overline{\tau_{r_0}(1)} = \{e^{i\theta}: |\theta| \le r_0\}$, $r_0 \in (0, \pi)$. Now since f is continuous choose $r_5 \in (r_0, \pi)$ such that $f(z) \ge m/2 > 0$, $z \in \tau_5 = \tau_{r_5}(1) = \{e^{i\theta}: |\theta| < r_5\}$ where $m = \inf_{z\in\gamma} f(z) > 0$ (as in the Statement of Theorem 6.8).

Set $r_k = r_0 + \dfrac{k(r_5 - r_0)}{5}$, $k = 0, \ldots, 5$ such that $\gamma \subseteq \tau_0 \subseteq \tau_1 \ldots \subseteq \tau_5$. Then there exist two functions f_1, f_2 on τ such that

$$(6.12) \qquad f_1 \in C'(\tau_5 \backslash \overline{\tau_2});$$

$$(6.13) \qquad f_1 = f \text{ on } \tau_1;$$

$$(6.14) \qquad 1 - \varepsilon \le f \cdot (f_1)^{-1} \le 1 + \varepsilon, \; z \in \tau;$$

(6.15) $$f_2 \in C'(\tau);$$

(6.16) $$1-\epsilon \le f_1(f_2)^{-1} \le 1+\epsilon, \ z \in \tau;$$

(6.17) $$w(f_2,\delta) \le C_4 w(f_1,\delta) \le C_3 w(f,\delta) \ \text{ some } \ \delta \in (0,\pi).$$

<u>Proof:</u>

Choose $\delta \in (0,\pi)$ and let Ω denote the Steklov smoother function of f, that is,

$$\Omega(e^{ix}) = (\delta)^{-1} \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f(e^{it})dt.$$

Then $\Omega \in C'(\tau)$. Also

$$\Omega'(e^{ix}) = \left[f\left(\exp i\left(x+\frac{\delta}{2}\right)\right) - f\left(\exp i\left(x-\frac{\delta}{2}\right)\right)\right](\delta)^{-1}$$

which implies $\|\Omega'\|_{L^\infty} \le w(f,\delta)(\delta)^{-1}$ so that

(6.18) $$w(\Omega,\delta) \le \|\Omega'\|_{L^\infty}\delta \le w(f,\delta).$$

Now write $\tau_5 = \tau_1 \cup \tau_2 \cup (\tau_5 \backslash (\tau_1 \cup \tau_2))$. Let $A \cup B = \tau_2 \backslash \tau_1$. Let A_1, B_1 be the endpoints of τ_2 and $\tau \backslash \tau_2$ and A_2, B_2 the endpoints of τ_1 and $A \cup (\tau_5 \backslash \tau_2)$ and $B \cup (\tau_5 \backslash \tau_2)$ where $\tau_2 = \tau_1 \cup A \cup B$. Choose a polynomial π_3 of degree 3 such that π_3 agrees with Ω and Ω' at A_1, B_1 and with f at A_2 and B_2, and non zero in $A \cup B \backslash (A_1 \cup B_1 \cup A_2 \cup B_2)$. Define

$$\begin{aligned} f_1 &= f \quad z \in \tau_1 \\ &= \Omega \quad z \in \tau \backslash \tau_2 \\ &= \pi_3 \quad z \in A \cup B \end{aligned}$$

Then $f_1 \in C'(\tau_5 \backslash \overline{\tau_2})$. Also there exists C_3 such that $w(\pi_3,\delta) \le C_3 w(f,\delta)$. Thus by (6.18) we have $w(f_1,\delta) \le C_3 w(f,\delta)$. To prove (6.14) note if

$f_1(z) = 0$. Then by definition of f_1 and π_3 (6.14) holds. If $f_1(z) \neq 0$ $\frac{f}{f_1} - 1 = \frac{f - f_1}{f_1}$ and $|f - f_1| \leq w_f(\delta) \to 0$ as $\delta \to 0$ by uniform continuity of f on τ. So (6.18) holds. Next define

$$f_2 = \frac{1}{\delta}\int_{z-\frac{\delta}{2}}^{z+\frac{\delta}{2}} f_1(e^{it})dt$$

and proceed as before. ■

The proof of Theorem 6.8:

<u>Proof:</u>

We note first that it is sufficient to prove the result for every closed arc $\gamma \subseteq \tau$. To see this we assume that the theorem fails for a general set $\gamma \subseteq \tau$. Then there exists $\epsilon > 0$, a sequence of positive integers n and a sequence of points $z_n \subseteq \gamma$, $n \in \Omega$ such that $\left||\varphi_n(z_n)| - w(z_n)^{-1/2}\right| \geq \epsilon$, $n \in \Omega$. By compactness we can find $\Omega' \subseteq \Omega$ such that $z_n \to z_0 \in \tau$, $n \in \Omega'$. Now since $z_n \in \gamma$, $|w(z_n)| \geq m$. Thus by continuity of f, there exists a $\delta > 0$ such that $|w(z_n)| > \frac{m}{2}$, $z \in \overline{\tau_\delta(z_0)}$. Thus for large enough $n \in \Omega' \subseteq \Omega$, $z_n \in \overline{\tau_\delta(z_0)}$ and so the theorem fails for this closed arc as well. Thus we may suppose without loss of generality that this arc is γ with centre 1. Write $\gamma = \tau_{r_0}(1) = \tau_C = \{e^{i\theta} \colon |\theta| \leq r_0\}$, $r_0 \in (0,\pi)$. Now as $m > 0$, by continuity of f there exists $r_5 \in (r_0, \pi)$ such that $w(z) \geq \frac{m}{2} > 0$ for $z \in \tau_5$. Define τ_k, $0 \leq k \leq 5$ as in Lemma 6.7. Let $\epsilon_0 = \min\left\{\frac{1}{5}, \frac{(r_1 - r_0)^2}{25}\right\}$ and choose $\epsilon \in (0, \xi_0)$ arbitrary. Let w_1, w_2 be chosen as in Lemma 6.7 with $\delta \in (0,\pi)$. Also let $J(f,\delta) = \int_0^\delta \frac{w(f,t)}{t}dt$. Then as $w \in DL(\tau)$, there exists C_5 such that $J(f,\delta) \leq C_5$ and $\lim_{\delta\to 0^+} J(f,\delta) = 0$. Let $\varphi_n(d\mu_i, z)$ be the orthonormal polynomials corresponding to w_i, $i = 1,2$. Now note that by the assumptions on w_1, w_2 we have by Theorem 6.6

$$\lim_{n\to\infty} |\varphi_n(d\mu_1, z)| = (w_1(z))^{-1/2} \tag{6.19}$$

uniformly for $z \in \overline{\tau_4} \backslash \tau_3$

$$\lim_{n\to\infty} \left|\varphi_n(d\mu_2, z)\right| = \left(w_2(z)\right)^{-1/2} \tag{6.20}$$

uniformly for $z \in \tau_4$. Hence we can assume that

$$\left|\varphi_n(d\mu_1, z)\right| \leq C_6, \; z \in \overline{\tau_4} \backslash \tau_3, \; n \in \mathbb{N} \tag{6.21}$$

and that

$$\left|\varphi_n(d\mu_2, z)\right| \leq C_7, \; z \in \overline{\tau_4}, \; n \in \mathbb{N}. \tag{6.22}$$

We show that there exists C_{14} such that

$$\left|\varphi_n(d\mu_1, z)\right| \leq C_{14}, \; z \in \overline{\tau_4}, \; n \in \mathbb{N}. \tag{6.23}$$

So let

$$m_r = \max_{z \in \overline{\tau_4}} \left|\varphi_n(d\mu_1, z)\right| = \left|\varphi_n(d\mu_1, z_n)\right|$$

say. Let $\mathbb{N} = \Omega_1 + \Omega_2$ where $n \in \Omega_1$ iff $z_n \in \overline{\tau_4} \backslash \tau_3$ and $n \in \Omega_2$ iff $z_n \in \tau_3 \subseteq \overline{\tau_4}$. Well if $n \in \Omega_1$ choose $C_{14} = C_6$ by (6.21). Now let $n \in \Omega_2$ and define $\delta(z) = \left(w_1(z)\right)\left(w_2(z)^{-1}\right)$, $z \in \tau_3 \subseteq \overline{\tau_4}$. Now by Lemma 5.4 we have,

$$\begin{aligned}
&\left|\frac{\varphi_n(d\mu_1, z)}{\varphi_n(d\mu_2, z)} - \frac{\gamma_n(d\mu_1)}{\gamma_n(d\mu_2)}\right| \\
&\quad \leq \left(\pi\delta(z)\right)^{-1}\left[\int_{\tau_{\sqrt{\epsilon}}} + \int_{\tau \backslash \tau_{\sqrt{\epsilon}}(\xi)} \left|\frac{\rho(\xi) - \rho(z)}{\xi - z}\right| \times \right. \\
&\qquad \left. \times \left|\varphi_n(d\mu_1, \xi)\right| \left|\varphi_n(d\mu_2, \xi)\right| \times f_2(\xi)\, d\theta, \; \xi = e^{i\theta}\right] \\
&\quad = I_1 + I_2.
\end{aligned}$$

I_1: notice that $r_4^2 > (r_4 - r_0)^2 = \frac{16(r_5 - r_0)^2}{25} > \frac{(r_5 - r_0)^2}{25} > \frac{(r_1 - r_0)^2}{25} > \epsilon_0$. Thus $r_4 > \sqrt{\epsilon_0}$, and therefore

$$\tau_{\sqrt{\epsilon}} \subseteq \tau_4 \subseteq \overline{\tau_4}. \tag{6.25}$$

Also by (6.14) and (6.17) we have, $w(\rho,\delta) \le C_8$. So

$$J(\rho,\delta) \le C_9,\ J(\rho,\delta). \tag{6.26}$$

So by (6.24), (6.25), (6.26) we have that $I_1 \le C_{10} M_n J(f,\sqrt{\epsilon})$. Now using (5.20) and the Schwarz inequality we have $I_2 \le C_{11}\sqrt{\epsilon}$, so putting $z = z_n$, $n \in \Omega_2$ and using (6.22) with (5.22) we get $M_n \le C_{12} M_n J(\rho,\sqrt{\epsilon}) + C_{13}$. Also as $J(f,\sqrt{\epsilon}) \to 0$ as $\epsilon \to 0$, $J(\rho,\sqrt{\epsilon}) \le \frac{1}{2} C_{12}$, $n \in \Omega_2$ which says that $M_n \le 2C_{13}$, $n \in \Omega_2$. So we have $|\varphi_n(d\mu_1, z)| \le C_{14}$, $n \in \Omega_2$. Thus using (6.23), (6.24) and (5.20) we have that,

$$\limsup_{n\to\infty}\left|\frac{\varphi_n(d\mu_1,z)}{\varphi_n(d\mu_2,z)} - 1\right| \le C_{15} J(f,\sqrt{\epsilon}) \tag{6.27}$$

uniformly for $z \in \gamma$. Also by (5.21), (6.13), (6.14) we have that

$$\limsup_{n\to\infty}\left|\frac{\varphi_n(d\mu,z)}{\varphi_n(d\mu_1,z)} - 1\right| \le C_{16}\sqrt{\epsilon} \tag{6.28}$$

uniformly for $z \in \gamma$. Now from (6.14) and (6.16) we get

$$\left|w(z)^{-1/2} - w_2(z)^{-1/2}\right| \le C_{17}\epsilon,\ z \in \tau. \tag{6.29}$$

So by (6.29), (6.28), (6.27), (6.19) and (6.20) that we get

$$\limsup_{n\to\infty}\Big||\varphi_n(d\mu,z)| - (w(z))^{-1/2}\Big| \le C_{18} J(f,\sqrt{\epsilon}) \tag{6.30}$$

uniformly for $z \in \gamma$. As $\epsilon \in (0,\epsilon_0)$ is arbitrary the result holds. ∎

We end this chapter with some open questions raised in ([20] and [22]).

(1) Let $w > 0$ a.e. (meas) be a weight on τ and suppose $w(z) = 1$ in some open $\gamma \subseteq \tau$. Then it is true that $\lim_{n\to\infty}|\varphi_n(z)| = 1$ uniformly for every closed $\gamma' \subseteq \gamma$? We note that the above is true when $\log w \in L^1(\tau)$ (see [5] pp.201–203).

(2) Is it possible to extend Szegő's theory to measures supported on a finite number of Jordan curves in the complex plane? If so how?

(3) Is it possible to extend Szegő's theory to the case where the derivative of the measure might vanish on sets of positive measure? If so how?

Chapter 7

Transition to $[-1,1]$

In this dissertation we have dealt up to now with the unit circle, however there exists naturally an equally vast and rich theory for the real line. We shall not delve too deeply into this area except for one particular weight which arises in a very natural way. It is well known that the map $x \to \cos x$ maps the interval $[-1,1]$ onto τ. So given a positive Borel measure on τ we consider its equivalent form on $[-1,1]$ and it will be these measures that we will deal with in this last chapter.

Section 7A : Introduction and Notation

Definition 7.1

Let $d\alpha$ be a finite positive Borel measure on $[-1,1]$ with $supp(d\alpha) \subseteq [-1,1]$ infinite. Let $p_n(d\alpha, x) = \beta_n(d\alpha)x^n + \cdots$, $n = 0,1,2,\ldots$, $\beta_n > 0$ be the corresponding system of orthonormal polynomials uniquely determined by

$$\int_{-1}^{1} p_n(d\alpha, x)p_m(d\alpha, x) = \delta_{mn},\ \beta_n > 0.$$

We now define the associated measure on τ as follows:

$$\mu(\theta) = \begin{cases} \alpha(1) - \alpha(\cos\theta), & 0 \leq \theta \leq \pi, \\ \alpha(\cos\theta) - \alpha(1), & -\pi \leq \theta \leq 0. \end{cases}$$

We often write $d\mu(\theta) = |d\alpha(\cos\theta)|$. Also note if $\alpha'(x) = w(x)$ then $d\mu$ is also absolutely continuous with weight $f(\theta) = w(\cos\theta)|\sin\theta|$. The following lemma is true and the proof can be found in ([4] p.6, 7, 189, 190).

Lemma 7.2

(a) The coefficients of $\varphi_n(d\mu, z)$ are real and

$$(7.1) \qquad -1 < \Phi_k(0) < 1, \; k \geq 1.$$

(b) Let $x = \cos\theta$, $z = e^{i\theta}$ then we have

$$p_n(d\alpha, x)$$
$$(7.2) \qquad = \frac{1}{2\pi}\left[1 + \frac{\varphi_{2n}(d\mu, 0)}{\varphi_{2n}(d\mu)}\right]^{-1/2}\left[z^n\varphi_{2n}(d\mu, z) + z^n\varphi_{2n}(d\mu, z^{-1})\right];$$

$$(7.3) \qquad xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \; n = 0, 1, \ldots$$

$p_0 = b_0$, $p_{-1} = 0$, $a_n = a_n(d\alpha) = \dfrac{\beta_{n-1}}{\beta_n}$ and $b_n = b_n(d\alpha) = \int_{-1}^{1} xp_n^2(x)d\alpha(x)$.

Statement 7.3

We now state the main theorems in this section.

Theorem 7.4

Let $d\alpha$ be such that $supp(d\alpha) \subset [-1, 1]$ infinite and let $d\mu$ be its associated measure on T as in Definition 7.1. Then we have,

$$(7.4) \qquad \lim_{n\to\infty} a_n(d\alpha) = 1/2 \quad \text{iff} \quad \lim_{n\to\infty} \Phi_n(d\mu, 0) = 0;$$

$$(7.5) \qquad \lim_{n\to\infty} a_n(d\alpha) = 1/2 \text{ implies } \lim_{n\to\infty} b_n(d\alpha) = 0.$$

If $\alpha' > 0$ a.e. (meas) in $[-1, 1]$, then we have,

$$(7.6) \qquad \lim_{n\to\infty} a_n(d\alpha) = 1/2, \; \lim_{n\to\infty} b_n(d\alpha) = 0.$$

Theorem 7.5

Let $supp(d\alpha) \subseteq [-1,1]$ and let g be a non negative $d\mu$ integrable function. Assume there exists a polynomial R such that either

(1) $\lim_{n\to\infty} a_n(d\alpha) = 1/2$ and $Rg^{\pm 1}$ is Riemann integrable on $[-1,1]$, or

(2) $\alpha' > 0$ a.e. (meas) on $[-1,1]$ and $Rg^{\pm 1} \in L^\infty(d\mu)$. Then

$$(7.7) \qquad \lim_{n\to\infty} a_n(gd\alpha) = 1/2, \ \lim_{n\to\infty} b_n(gd\alpha) = 0.$$

Write $g(\theta) = g(\cos\theta)$ then,

$$(7.8) \qquad \lim_{n\to\infty} \frac{\beta_n(gd\alpha)}{\beta_n(d\alpha)} = \left(D(g,0)\right)^{-1}.$$

If $0 < r < 1$ then

$$(7.9) \qquad \lim_{n\to\infty} \frac{p_n(gd\alpha,x)}{p_n(d\alpha,x)} = D^{-1}(g,z), \ x = 1/2\left(z+(z)^{-1}\right)$$

uniformly for $0 \le |z| \le r$.

Remarks:

(1) We note that these results are partial analogues to the Ratio Asymptotic $\lim \Phi_n(0) = 0$ in Chapter 3 and the comparative Asymptotics in Chapter 4 and 5.

(2) From 2.56 we deduce the following mean result.

Let $d\mu(\theta)$ be a non negative Borel measure on $[0,2\pi]$ satisfying Szegös condition. Then,

$$\lim_{n\to\infty} \int_0^{2\pi} \left|\varphi_n(d\mu,z) - \frac{z^n}{\left|D(\mu',z)\right|^2}\right|^2 d\mu(\theta) = 0, \ (z = e^{i\theta}).$$

We can now give the corresponding result for $[-1,1]$ (see [5] p.177).

Define Sign $(z) = \frac{z}{|z|}$, $z \in \mathbb{C} \setminus \{0\}$. Then if $d\alpha(x)$ is a mass distribution on $[-1,1]$ satisfying Szegö's condition on $[-1,1]$

$$\int_{-1}^{1} \log \alpha'(x)(1-x^2)^{-1/2} dx > -\infty.$$

Let

$$\mu'(\theta) = \alpha'(\cos\theta)|\sin\theta|$$

$\theta \in [0, 2\pi]$, then for $z = e^{i\theta}$,

$$\lim_{n\to\infty} \int_0^{2\pi} \Bigg| p_n(d\alpha, \cos\theta)\mu'(\theta)^{1/2} - \left(\frac{2}{\pi}\right)^{1/2} Re\Big\{z^n Sign\big(D(\mu', z)\big)\Big\}\Bigg|^2 d\theta = 0.$$

Section 7B : Introduction and Notation

We generalize Section 7.1 (A) as we did in Chapter 3.

Definition 7.6

Let $\{w_{2n}\}_{n\in\mathbb{N}}$ be a sequence of polynomials with real coefficients such that for each $n \in \mathbb{N}$, $\deg w_{2n} = i_n$, $0 \le i_n \le 2_n$ and $w_{2n} > 0$ on $[-1, 1]$. If $i_n > 0$ then $(x_{2n,i})$, $2n - i_n + 1 \le i \le 2n$ denote the zeros of w_{2n}. For $i_n = 2n$, $(x_{2n,i})$, $1 \le i \le 2n$ is the set of zeros of w_{2n}. Also let all the zeros of $w_{2n} \in \mathbb{C}\setminus[-1, 1]$. Now let $d\alpha_{2n} = (w_{2n})^{-1}d\alpha$. Then for each $n \in \mathbb{N}$, if $\|d\alpha_n\| < \infty$, construct $p_{n,m}(x) = \beta_{n,m}(x)^m + \ldots$, $\beta_{n,m} > 0$ to be the mth orthonormal polynomial with respect to $d\alpha_{2n}$, that is

$$\int_{-1}^{1} p_{n,k}(x)p_{n,m}(x)d\alpha_n(x) = \delta_{k,m}.$$

Also let $P_{n,m} = \frac{1}{\beta_{n,m}}p_{n,m}$ be the corresponding mth monic orthogonal polynomial with respect to $d\alpha_{2n}$.

Remark:

Notice that $w_{2n}(\cos\theta) \ge 0$, $\theta \in R$, so by Lemma 2.3 there exists an algebraic polynomial $\hat{W}_{2n}(w)$ of degree i_n with zeros in $|w| \le 1$ such that $w_{2n}(\cos\theta) = |\hat{W}_{2n}(e^{i\theta})|^2$, $\theta \in [0, 2\pi]$. Now consider the conformal map $J : \mathbb{C}\setminus[-1, 1] \to$

$|w| > 1$, given by $J(z) = \left(z + \sqrt{z^2 - 1}\right)$. Then the zeros of $\hat{W}_{2n-1}$ are precisely the points $\left[J(x_{2n,i})\right]^{-1}$, $2n - i_n + 1 \leq i \leq 2n$ such that $J(\infty) = \infty$ and $J'(\infty) > 0$. Let $W_{2n} = (w)^{2n-i_n}\hat{W}_{2n}$. Then the degree of $W_{2n} = 2n$ and $w_{2n}((\cos\theta)) = \left|W_{2n}(e^{i\theta})\right|^2$. Now define

$$d\mu_{2n}(\theta) = d\alpha_n(\cos\theta) = \left|W_{2n}(z)\right|^{-2} d\alpha(\cos\theta), \; (z = e^{i\theta});$$

that is, $d\mu_{2n}(E) = d\alpha_n([\cos\theta; \theta \in E])$ whenever $E \subseteq [0,\pi]$ or $E \subseteq [\pi, 2\pi]$. Thus, writing $\mu_{2n}(\theta) = d\mu_{2n}[0 \leq t < \theta]$, we have

$$\mu_{2n}(\theta) = \begin{cases} G_n(\cos\theta) & 0 < \theta \leq \pi \\ -G_n(\cos\theta) & \pi \leq \theta \leq 2\pi, \end{cases}$$

where $G_n(x) = \int_{-1}^{x} d\alpha_n(t)$, $x \in [-1,1]$, at every point θ, where μ_{2n} is continuous and so a.e. (meas) in $[0, 2\pi]$; furthermore we have

$$\begin{aligned} \mu_{2n}'(\theta) &= |\sin\theta| G_n(\cos\theta) = |\sin\theta| \left|W_{2n}(e^{i\theta})\right|^{-2} \alpha'(\cos\theta) \\ &= |\sin\theta| \alpha_n'(\cos\theta), \end{aligned}$$

whenever either side exists, that is a.e. (meas). Also $\mu' > 0$ a.e. (meas) if $\alpha' > 0$ a.e. (meas) where,

$$\mu(\theta) = \begin{cases} \alpha(\cos\theta) & 0 \leq \theta \leq \pi \\ -\alpha(\cos\theta) & \pi \leq \theta \leq 2\pi. \end{cases}$$

$\varphi_n(d\mu, z) = \varphi_n(z)$, $\varphi_{n,m}(d\mu_n, z) = \varphi_{n,m}(z)$, $p_n(d\alpha, x) = p_n(x)$, $p_{n,m}(d\alpha_n, x) = p_{n,m}(x)$.

$C_1, C_2, \ldots$ will denote constants independent of n, ξ, z, x, w.

Lemma 7.7

(a) The coefficients of $p_{n,m}(d\alpha_n, x)$ are real.

(b) The following relations hold:

(7.10)

$$p_{n,m}(x) = (2\pi)^{1/2}\left(1 + \Phi_{2n,2m}(0)\right)^{-1/2} w^{-m}\left[\varphi_{2n,2m}(w) + \varphi_{2n,2m}^*(w)\right]$$

where $\Phi_{2n,2m} = \left(\gamma_{2n,2m}(d\mu_n)\right)^{-1}\varphi_{2n,2m}$ and $x = \frac{1}{2}\left(w + \frac{1}{w}\right)$.

(7.11)

$$P_{n,m}(x) = 2^{-m}\left[1 + \Phi_{2n,2m}(0)\right]^{-1}(w)^{-m}\left[\Phi_{2n,2m}(w) + \Phi^*_{2n,2m}(w)\right];$$

(7.12)

$$xp_{n,n+k}(x) = a_{n,k,1}p_{n,n+k+1}(x) + a_{n,k,0}p_{n,n+1}(x) + a_{n,k-1}p_{n,n+k-1}(x).$$

Definition 7.8

Let k be a fixed integer. Then $(\alpha, (w_{2n}), k)$ will be admissible on $[-1, 1]$ if $(\mu, (W_{2n}), k)$ is admissible on $[0, 2\pi]$. That is

$$\alpha' > 0 \ a.e.\ (meas); \tag{7.13}$$

$$\|\alpha_n\| = \int_{-1}^{1} d\alpha_n(x) < \infty,\ n \in \mathbb{N}; \tag{7.14}$$

$$\int \prod_{i=1}^{-k} |1 - x(x_{2n,i})^{-1}|^{-1} d\alpha(x) \leq m < \infty,\ k = -1, -2, \dots, \tag{7.15}$$

where $x(x_{2n,i})^{-1} = 0$ if $x_{2n,i} = \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{2n}\left(1 - |J(x_{2n,i})|^{-1}\right) = \infty. \tag{7.16}$$

Statement 7.9

We now state the main theorems to be proved in this section. The first result is a ratio analogue of Szegö's classical strong asymptotic formula for non Szegö weights. We also give other forms of this result as proved by Nevai et al using recurrence coefficients and Lubinsky et al using exponential weights. We also give generalizations of Theorem 7.4 in this setting and some strong and weak convergence analogues to Chapter 3.

Theorem 7.10

Let $(\alpha, (w_{2n}), 2k)$ be admissible on $[0, 2\pi]$. Then

$$\lim_{n\to\infty} \frac{p_{n,n+k+1}}{p_{n,n+k}}(x) = \frac{1}{2} \lim_{n\to\infty} \frac{\overline{p_{n,n+k+1}}}{\overline{p_{n,n+k}}}(x) = \frac{1}{2}J(x), \quad x \in \mathbb{C} \setminus [-1,1]$$

where we have uniform convergence on each compact subset of $\mathbb{C} \setminus [-1,1]$ and $\lim_{n\to\infty} \dfrac{\beta_{n,n+k+1}}{\beta_{n,n+k}} = 2.$

Theorem 7.12

Let $(\alpha, (w_{2n}), 2k-2)$ be admissible on $[-1,1]$. Then,

$$\lim_{n\to\infty} a_{n,k,1} = \lim_{n\to\infty} a_{n,k,-1} = \frac{1}{2}$$

and $\lim_{n\to\infty} a_{n,k,0} = 0$.

Remark

Recall that T_n the nth Chebyshev polynomial is given by $T_n(\cos\theta) = \cos n\theta$. Also the Chebyshev polynomials become an orthonormal system with weight function $(1-x)^{-1/2}$, $x \in (-1,1)$ if we define

$$p_0(x) = (\pi)^{-1/2} T_0(x), \; p_i(x) = (2\pi)^{-1/2} T_i(x), \; 1 \le i \le n$$

(see [4] p.34).

Theorem 7.13

Let $(\mu, w_{2n}, 2k)$ be admissible on $[-1,1]$ and let T_n be the nth Chebyshev polynomial as in the remark after Theorem 7.12. Then for each $m \in \mathbb{N}$ and for all bounded Borel measurable functions f on $[-1,1]$,

(7.17)
$$\lim_{n\to\infty} \int_{-1}^{1} \frac{f(x)p_{n,n+k}(x)p_{n,n+k+m}(x)\alpha'(x)}{w_{2n}(x)}dx = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)T_m(x)dx}{(1-x^2)^{1/2}};$$

(7.18)

$$\lim_{n\to\infty}\int_{-1}^{1}\frac{f(x)p_{n,n+k}(x)p_{n,n+k+m}(x)d\alpha(x)}{w_{2n}(x)}dx=\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)T_m(x)dx}{(1-x^2)^{1/2}}.$$

Theorem 7.14

Let $(\alpha,(w_{2n}),2k)$ be admissible on $[-1,1]$. Define

$$g_{n,k}(w)=\int_{-1}^{1}p_{n,k}(x)(w-x)^{-1}d\alpha_n(x).$$

Then,

$$\lim_{n\to\infty}p_{n,n+k}(w)g_{n,n_k}(w)=(w^2-1)^{1/2}; \tag{7.19}$$

$$\lim_{n\to\infty}\frac{g_{n,n+k+1}(w)}{g_{n,n+k}(w)}=\left(J(w)\right)^{-1}, \tag{7.20}$$

where in both (7.19) and (7.20) the convergence is uniform in each compact subset of $\mathbb{C}\setminus[-1,1]$.

Remarks

(a) Notice that Theorem 7.12 is the analogue of (3.6). Theorem 7.13 is the analogue of Theorem 3.10. Finally Theorem 7.14 is the analogue of Theorem 3.13.

(b) Various stronger and weaker versions of Theorem 7.10 have been investigated and proved both for weights on $[-1,1]$ and $\mathbb{R}$. We highlight some of these results.

(1) (Szegõ): Let $w(x)$ be a Szegõ weight function on $[-1,1]$ such that $w(\cos\theta)|\sin\theta|=f(\theta)$. Then for $x=\frac{1}{2}\left(z+\frac{1}{z}\right)$, $|z|\geq R>1$,

$$\left|p_n(x)-\frac{1}{\sqrt{2\pi}}z^n\cdot\frac{1}{D(z^{-1})}\right|\to 0\ \text{ as }\ n\to\infty.$$