



Exploring multiplicative reasoning with Grade Four learners through structured problem solving

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Declaration

I, Sameera Hansa, hereby declare that this research report is my own work. It is being submitted for the Degree of Master of Science (Science Education) at the University of the Witwatersrand, Johannesburg. This report has not been submitted for any other degree or examination at any other university.



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9 October 2017

Date

Abstract

South Africa's performance in mathematics education is ranked amongst the world's worst. This performance is not only alarming at an international level, but also nationally. Annual National Assessments (ANA) conducted by the Department of Education have showed that the level of mathematics across the foundation and intermediate phase is poor with a pronounced dip in performance at a Grade 4 level (Department of Basic Education, 2014). Multiplication and division are common challenging areas that contribute to this poor performance. This is concerning as mathematics is globally recognised as a key competence for providing access to higher education and developing a country's society and economy.

My study, aimed at exploring multiplicative reasoning with Grade 4 learners through structured problem solving, is focused on the learning of multiplication and division within the context of an intervention concentrated on developing learners' ability to model multiplicative situations. Shifts in the use of models were investigated following a small-scale intervention in which different modelling approaches (particularly ratio modelling) were introduced and developed. A control group was used to determine the usefulness of the intervention. Questions which I sought to answer were: (a) what kinds of multiplicative reasoning (models) are Grade 4 learners using prior to intervention, (b) what changes, if any, are seen in overall performance, across the intervention and control group, in the post-test, and, (c) what kinds of differences in model use were associated with the shifts in performance?

The main dataset comprised of 61 pre- and post-test scripts across three Grade 4 classes in a former Model C school in a Johannesburg district. A sample of 15 interviews were also conducted across the classes. Document analysis and transcription notes were used to analyse data with a Realistic Mathematics Education (RME) framework informing my analysis. Findings from my study reveal that prior to intervention, Grade 4 learners presented limited multiplicative models which were predominantly confined to traditional algorithms. After the small-scale intervention, learners used a broader range of models with an emerging take up of ratio models. The success rate associated with the models presented by learners also improved. Limited and/or no changes in model use and their respective success rates were seen in the control group suggesting that the intervention program was useful. These findings suggest that, as a future recommendation, it would be worthwhile to investigate the outcomes of running a similar intervention in less privileged settings.

Dedication

To my late father who would have been so proud of this achievement and to my mother and siblings for their unwavering support and constant encouragement during my M.Sc. journey.

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Chapter One: Introduction

1.1 Background to the study

Performances in mathematics and science are often important criteria for a country's standard of education because they are globally recognised as key competences for developing an individual, society and economy (Soudien, 2016).

Studies which measure the educational achievement in countries are becoming increasingly popular and important for conversations around what is working well and/or needs to be improved in an education system. Since 1995, South Africa has participated in TIMSS - Trends in International Mathematics and Science Study. The results of the 2015 TIMSS results ranked South Africa amongst the lowest performing countries in mathematics and science featuring second last out of 48 countries (Mullis, Martin, Foy, & Hooper, 2015). Statistics from the Southern Africa Consortium for Monitoring Educational Quality (SACMEQ) III study also indicated that South Africa's performance in mathematics is below average (Spaull, 2011). These performances are not only alarming at an international level, but also nationally. The Annual National Assessment (ANA) results for mathematics in the intermediate phase (Grades 4-6) for 2014 were low with learners scoring an average of 37% in Grade 4 (Department of Basic Education, 2014) (see Table 1.1). Poor mathematics skills were displayed across the primary grades with average performances declining drastically each year from 68 % in Grade 1 to 43 % in Grade 6 with a pronounced dip in performance at Grade 4 level (DBE, 2014). These results indicate that children from Foundation Phase through Intermediate Phase and consequently Senior Phase are struggling with mathematics and are not performing as well as they are expected to. Studies conducted by both TIMSS and SACMEQ suggest that the poor performance of South African children is due to a lack of mathematical thinking which interferes with mathematical learning.

GRADE	AVERAGE PERCENTAGE MARK (%)
1	68
2	62
3	56
4	37
5	37
6	43

Table 1.1: Summary of ANA results 2014 (Department of Basic Education, 2014)

Multiplication and division are common challenging areas in mathematics education. Learners tend to struggle with these concepts and hence perform poorly (Department of Basic Education, 2011). As a teacher having taught both Grade 4 and 5 mathematics, I have noticed that children struggle with situations involving multiplicative concepts. In the 2014 ANA, only 50% of the 20 Grade 4 learners that I taught correctly answered all questions involving multiplication and division.

Difficulties with multiplication and division are not only unique to South African learners as problems have been alluded to in a wide range of literature (Greer, 1992; Nunes & Bryant, 1996; Anghileri, 2006b; Hansen, 2011). According to the NCTM – National Council of Teachers of Mathematics (2000) – at a Grade 3 to 5 level, learners are expected to: understand various meanings of multiplication and division, the effects of multiplying and dividing whole numbers, as well as identify and use inverse operations to solve problems. These issues, together with a concerning dip in Grade 4 performance, drove my interest to explore and develop multiplicative reasoning using a structured problem-solving approach with a group of Grade 4 South African learners.

In the sections that follow, I outline literature situated around multiplicative reasoning, problem solving and the role of representations. I also explain my decision to use Realistic Mathematics Education (RME) as a broad theoretical framework for my study and outline the research design based on an intervention drawn from Askew (2004)'s work.

1.2 Literature based rationale

1.2.1. Multiplicative Reasoning

Multiplicative reasoning plays an important role in the way children learn and think about mathematics. It is characterized by the ability to work flexibly with concepts, strategies and representations of multiplication and division as they occur in a wide range of contexts (Siemon, Breed, & Virgona, 2005). This kind of thinking is important as it focuses primarily on how children make sense of multiplicative relations between quantities rather than on arithmetic operations. Nunes and Bryant (2007) make three assumptions in relation to multiplicative reasoning:

- For children to understand multiplication and division, they need to understand the inverse relation between the two concepts

- Connections between multiplication and division are conceptually related and
- Despite procedural connections between addition and multiplication, these two forms are different.

These assumptions suggest that we cannot see multiplication and division as only repeated addition or subtraction as they are concepts which are much deeper than that. Learners should be given opportunities to explore the connections that these concepts share. Treating multiplication as an extension of addition and division as an extension of subtraction, develops a very narrow view of the meaning of multiplication and division, thus detaching meaning from the multiplicative structure (Barmby, Bilsborough, Harries, & Higgins, 2009).

1.2.2. Problem Solving

One way of assisting learners in developing multiplicative reasoning skills is through problem solving. Problem solving can be defined as “engaging in a task for which the solution method is not known in advance” (NCTM, 2000, p.52). De Corte, Greer, and Verschaffel (2000) argue that when learners solve arithmetical word problems, they tend to apply operations algorithmically with neither realistic considerations nor using common sense. Therefore, there is a lack of making sense of the problems presented. This resonates with my own observations whereby children either relied heavily on concrete strategies to solve word problems or they delved straight into abstract algorithms with little understanding of the algorithm and mathematics involved. Learners are mostly taught to memorize strategies and are habitually drilled with algorithms and so, whilst they are able to calculate and solve problems using algorithms, very few are able to make sense of the context of word problems.

Opportunities for students to grasp the symbolic system of mathematics can also be restricted due to classroom practices (Ensor, Hoadley, Jacklin, Kuhne, Schmitt, Lombard & Van den Heuvel-Panhuizen, 2009). Therefore, the way in which teachers present mathematical ideas to students is fundamentally important as it provides opportunities for students to understand and use those ideas. Teaching using a problem-solving approach is strongly encouraged as it offers learners an opportunity to do mathematics as well as understand mathematical concepts, processes and techniques (NCTM, 1989). Problem solving helps students construct a deep understanding of mathematical ideas and processes engaging them in *doing* mathematics by creating, conjecturing, exploring, testing, and verifying (Lester, Masingila, Mau, Lambdin, dos Santos & Raymond 1994).

1.2.2. *The Role of Representations and Models in Mathematics Education*

The power of mathematics originates from the fact that we can represent real-life or concrete situations in mathematical language, pictures and symbols (Haylock & Cockburn, 2013). Representations are an integral part of mathematical competence as they allow us to think about and communicate mathematical ideas (Hiebert & Carpenter, 1992). Mathematical modelling uses representations to find solutions to problems concretizing formal abstract mathematics making it more accessible to learners (Gravemeijer, 1999; Haylock, 2010). In my study, I define *models* as a representation that learners use to signify problem situations whereas a *strategy* is the process undertaken, perhaps using the model of the situation, to arrive at the answer (correct or incorrect) (Van den Heuvel-Panhuizen, 2003).

In word problems, children make sense of problems through intuitively modelling the action and relations described in them (Carpenter, Fennema, & Franke, 1996). As learner's progress, they should be able to move from concrete to more symbolic representations. Drawing on the work of Gravemeijer (1999), as children move from using informal strategies to more formal ways of reasoning, a transition from *models of* situations to *models for* mathematical thinking occurs. For example; in a multiplication as rate word problem of 10 pots with 7 magic beans, how many beans will there be in 10 pots? A *model of* the situation could be the child drawing 10 pots with 7 beans in each pot. The teacher could provide a model of this situation by drawing a double number line with the number of pots and beans shown on each line to get to the answer of 70. The teacher could also adapt the number line by skip counting in 7's for every two pots. Whilst this may not mirror what the children initially did, Askew (2012b) states that through the joint activity of the children solving the problem on their own and the teacher sharing her model, over time the children will begin to appropriate the teacher's model using it for themselves thus shifting from a *model of* to a *model for* them to use. Over time, with frequent use, these *models for* (formal ways of reasoning), will eventually become *tools* which children think with (Askew, 2012b).

1.2.3. *Context*

My study was focused on the learning of multiplication and division within the context of an intervention aimed at developing learners' ability to model multiplicative situations. Van den Heuvel-Panhuizen (2003) states that models of situations are important as they provide students with opportunities for progress. To effectively comprehend mathematics and reason

multiplicatively, it is important that learners use and produce models of situations. This approach is strongly encouraged as the traditional formal approach to teaching mathematics that has dominated South African classrooms for a number of years has not afforded learners many opportunities to make use of what Gravemeijer (1994) describes as ‘horizontal mathematization’ – which involves informal tools such as diagrams and pictures to solve problems. Rather it has placed learners immediately in the more formal ‘vertical mathematization’ process, a process that involves using formal mathematical language and/or strategies (such as algorithms) (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2000). This has negative consequences such that if learners forget the algorithms taught to them, they will not have had a strategy in place to assist them in solving the problem. Providing learners with the opportunity to explore and develop models for solving problems is important. One way of achieving this is through guidance from the teacher in the form of ‘intervention’ whereby various modelling approaches are explored and developed to ensure adequate conceptual understanding of multiplication and division.

Bearing the above in mind, drawing on Realistic Mathematics Education (RME) was most suitable for this study and was adopted for two main reasons. Firstly, trends around the world show that for learners to grasp mathematics, there has to be a link between the mathematics taught and the world in which they are in (Van den Heuvel-Panhuizen, 2003). Word problems used in my study incorporated contexts that learners could relate to. And second, to encourage, explore and develop multiplicative reasoning, learners were given opportunities to think about and produce their own models for solving problems (Gravemeijer, 1999).

1.3 Research Questions

In order to explore and develop multiplicative reasoning with Grade 4 learners through structured problem solving, the following questions were addressed:

1. *What kinds of multiplicative reasoning (models) are Grade 4 learners using prior to intervention, as seen in pre-test responses?*
2. *What changes, if any, are seen in overall performance, across the intervention and control group, in the post-test?*
3. *What kinds of differences in model use were associated with the shifts in performance?*

1.4 Methodology

To answer my research questions, I worked with 61 learners from three Grade 4 classes in a former Model C school in a Johannesburg district. My specific focus on Grade 4 was motivated by the fact that it is the first year of the intermediate phase and the grade in which a widely documented performance dip is seen in the national landscape. Prior studies have linked this dip to a lack of move beyond concrete counting-based approaches, which become inefficient as the number range increases (Schollar, 2008), as well as difficulties with making sense of word problems. ANA diagnostic reports from 2013 and 2014 also showed that multiplication and division including word problems with these operations were areas of concern. Learners were unable to interpret the word problems correctly nor use the given information appropriately or translate language into mathematical language with the correct number sentences (Department of Basic Education, 2013; 2014).

The participants in my study were classified into two groups; an intervention and control group. My decision to include a control group was primarily to determine if the intervention played a role in any shifts with regards to the way learners reason multiplicatively.

Intervention lessons were structured around the materials, pedagogical approach and lesson structure suggested in Askew (2004)'s Big Book of Word Problems relating to multiplication and division. The Teacher Guides provided with the Big Books advocate for a supportive classroom environment when solving word problems. The idea was to support children in making sense of various problems through an exploratory approach where learners could advance their own models of the problems as well as explore and understand their peers' approaches in the classroom. Intervention lessons took place once a week over a period of eight weeks in a regular pattern of one hour lessons. They comprised of four parts: (1) Solving the problems, (2) Linking up the problems, (3) Follow-up problems and (4) Wrap-up.

A pre- and post-test was designed and incorporated around the intervention lessons and administered with both groups of learners. Tests comprised of both word problems and bald multiplicative calculation questions – questions neither situated in a real world, or imagined context, but rather presented as a mathematical operation to compute.

While my study was primarily focused on multiplicative reasoning word problem situations, I included a small number of bald questions and multiplicative reasoning relationship items to get a broader sense of the impact of the intervention lesson sequence.

The pre-test was administered as an initial assessment to establish the kinds of models learners were using to set up and solve problems prior to any intervention provided. The post-test was then used to determine if any shifts occurred with comparisons made between the two groups so as to establish the effectiveness of the intervention. A sample of learners were also interviewed immediately after the tests to gain a better understanding of how learners worked through the problems in the tests.

Document analysis of the pre- and post-tests were an integral part of my study. To gain insight into the models learners used prior and after the intervention program a set of codes were developed according to literature based on widely used models, as well as models that were introduced in the intervention lessons so that inferences could be made on any gains achieved. A detailed discussion on my methodological approaches can be found in Chapter 4.

1.5 Structure of research report

This chapter is an introduction to the research report, which covers the background, rationale and the guiding research questions that have been used in this study.

Chapter 2 reviews a range of literature regarding multiplicative reasoning, problem solving and the role of representations whereas Chapter 3 discusses RME as a theoretical framework of my study.

Chapter 4 explains the methodology used in my research. The research design and sampling procedure are described as well as data collection techniques and methods of analysis.

Chapter 5 presents and discusses the findings of this study considering the research questions and literature reviewed.

And lastly, Chapter 6 presents the concluding remarks and limitations of my study with some recommendations for future research.

Chapter Two: Literature Review

2.1 Introduction

The way in which children model multiplicative problems was of interest because evidence suggests that these are often challenging areas for primary aged learners. Venkat (2013) reports that poor results are often linked to children's overreliance on unit counting, repeated addition and subtraction strategies which are found well into the intermediate phase and as a result, many children find difficulty in moving beyond these strategies. Difficulties with multiplicative concepts are largely attributed to the kinds of thinking that these concepts require (Barmby, Harries, Higgins, & Suggate, 2009) which are examined later in this chapter.

The purpose of this chapter is to explore a range of literature on multiplicative reasoning, problem solving and modelling approaches. The aim is to connect my own research with existing theory and literature providing a foundation for the intervention lessons.

This chapter comprises four broad areas. Since my study is based on exploring multiplicative reasoning with Grade 4 learners through structured problem solving, I first discuss the nature of multiplication and division. Thereafter, I discuss multiplicative reasoning. Third, I consider the role of problem solving as it was a main component of this study. And last, I explore the role of representations in relation to multiplication and division which form part of the analytical framework used within this study. Problem situations related to multiplication and division are also outlined as well as various multiplicative modelling approaches.

2.2 Multiplication and Division

Anghileri (1989) defines multiplication as a binary operation with two inputs – the first being the size of the set (*multiplicand*) and the second representing the number of replications of that set (*multiplier*). Both of which are distinct elements in multiplication. Division on the other hand can be seen as the inverse of multiplication and involves the act of splitting something into equal parts. Successive splits in division are thus related to the emergence of new relations and number meanings as it provokes a change in the relationship between the

whole and the parts (Nunes & Bryant, 1996). Division consists of a *dividend* – the number which is to be divided and a *divisor*, the number by which it is divided (Haylock, 2010). When considering division as the inverse of multiplication, questions that could be asked are how many groups of the same size are there in a particular set. This type of thinking helps children think about the mathematical structure and connections that are present in this inverse relationship (Anghileri, 2006b; Nunes & Bryant, 1996). Division can also be used to compare quantities thus integrating a ratio structure. When comparing two numbers of some kind, we could consider the ratio of the numbers i.e. how many times is B more than A. This kind of thinking is the inverse of multiplication as what we are trying to find is a factor by which one quantity must be increased in order to match the other (Haylock, 2010).

Both multiplication and division have different properties. Multiplication has three distinct properties: *commutative*, *associative* and *distributive* whereas division only has a *distributive* property. The commutative property for multiplication states that the order in which numbers are multiplied does not change the result, for example; $a \times b = b \times a$ (Haylock & Cockburn, 2013). The associative property applies when two or more numbers (a , b and c) are being multiplied. These numbers may be grouped in an order and still give the same result - $(a \times b) \times c = a \times (b \times c)$. Division does not have an associative or commutative property as $(a \div b) \div c$ does not equal to $a \div (b \div c)$ (Haylock & Cockburn, 2013). The distributive property of multiplication on the other hand states that for all real numbers, $a \times (b + c) = a \times b + a \times c$ therefore a can be distributed across both b and c . The distributive property applies to division too where $(a + b) \div c = (a \div c) + (b \div c)$ and $(a - b) \div c = (a \div c) - (b \div c)$ provided that c does not equal to 0 (Haylock & Cockburn, 2013).

It is common practice that multiplication and division are concepts which are taught as separate arithmetic operations following from addition and subtraction (Nunes & Bryant, 1996). While multiplication can be worked out using repeated addition and division can be worked out using repeated subtraction, to treat these concepts just as another form of addition and subtraction has been argued to be a limited way of looking at these operations (Nunes & Bryant, 1996). Instead, both multiplication and division require a significant qualitative change in children's thinking which goes beyond purely computing and involves building up complex networks of connections (Haylock & Cockburn, 2013; Nunes & Bryant, 1996).

2.3 Multiplicative Reasoning

Brown, Hodgen, & Küchemann (2014) define multiplicative reasoning as different ways of thinking about and representing situations which involve multiplication, division, scale factor and rates as well as ratio and proportion. These situations could involve whole numbers, integers, rational or real numbers. Unlike addition and subtraction which involve an act of joining and separating, multiplicative situations take different forms that cannot only be solved by joining and separating actions (Nunes & Bryant, 1996).

Multiplicative reasoning is characterised by the capacity to work flexibly with an extended range of numbers, recognise and solve a range of problems involving multiplication and division including direct and indirect proportion and the means to communicate this effectively in a variety of ways (words, diagrams, symbolic expressions and written algorithms) (Siemon et al., 2005). This kind of thinking is important as it focuses primarily on problem structures and how children make sense of multiplicative relations between quantities rather than on arithmetic operations.

Nunes & Bryant (2007, p.8) make three assumptions in relation to multiplicative reasoning:

1. To fully understand multiplication and division, children need to understand the inverse relationship between the two concepts
2. Connections between multiplication and division are conceptual and relate to connections within the domains of reasoning
3. Despite procedural connections between addition and multiplication, these two forms are sufficiently different.

These assumptions reiterate the need to steer away from teaching multiplication and division as separate operations and give learners a chance to see the relationships that these concepts share. Children need to be able to think about these concepts and the calculations which emerge thereof as interrelated (Anghileri, 2006b). Situations which give rise to understanding these connections can be categorized into various categories: (1) equivalent groups – groups of, (2) partitioning, (3) rectangular arrays, (4) multiplicative comparison – scale factor and (5) Cartesian products (Greer, 1992; Siemon et al., 2005).

The *groups of* idea progresses from an additive model of multiplication when children start counting larger numbers of objects (Siemon et al., 2005). Children should be able to see that the one-to-one counting is a tedious process and thus should try using more efficient strategies either by skip counting in twos, fives and tens or through recalled multiplication facts. The progress from ‘one-to-one’ to ‘one-to-many’ counting can be difficult because children tend to lose sight of what they are doing; counting a count (Siemon et al., 2005). *Partitioning* supports multiplicative thinking as it focuses learner’s attention on a known number of shares. Siemon et al. (2005) state that children should be encouraged to work with tasks that allow them to explore how many ways a number of objects can be shared equally so as to lead to the realization that a collection can be partitioned in a number of ways. Additionally, the *array or area* idea more neutrally represents all multiplicative quantities in a relationship i.e. the number of groups, the equal number in each group and the product. The notion of area to represent multiplicative structures generalizes more overtly to the multiplicative scale factor (Greer, 1992) or factor-factor-product idea (Siemon et al., 2005) which supports fractions, multiple factor situations, exponentiation and algebraic factorization (Siemon et al., 2005). And lastly, Cartesian products correspond to the formal definition of $m \times n$ in terms of the number of distinct ordered pairs that can be formed when the first member of each pair belongs to a set with m elements and the second to a set with n elements (Greer, 1992). In this situation, the ‘for each’ idea which is often seen in rate and proportion is emphasized. Each of these situations can be associated with particular ways of asking a question with each represented in ways that shows *repeated sets*, *many-to-one* correspondence, *arrays* of rows and columns and *many-to-many* correspondences (Anghileri, 2006b) (see figure 2.1).

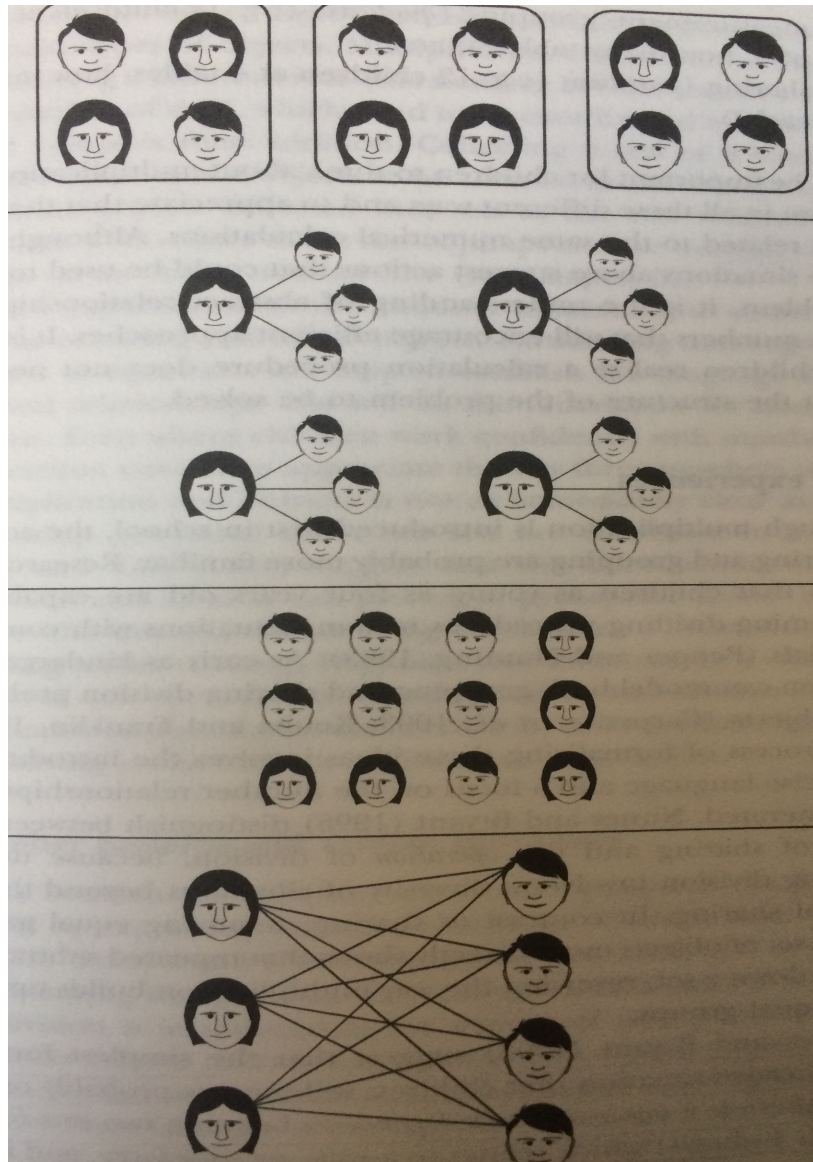


Figure 2.1: Repeated sets, many-to-one correspondence, an array of rows and columns, and a many-to-many correspondence (Anghileri, 2006b, p.85)

In the different situations shown in figure 2.1, we can see that all the situations involve three numbers: the number of objects in each set, the number of sets and the total. These numbers can be seen from both a multiplication and division perspective. In multiplication, we try to find the total number whereas when we try to find the other two numbers (number of objects in each set or the number of sets), we use division (Anghileri, 2006). In working with division situations, two distinct types emerge; (1) measurement/grouping (quotitive) situations and (2) sharing (partitive) situations. In *partitive problems*, a dividend is shared equally among a certain number of recipients and the size of the portion (the quotient) depends on the number of recipients (the divisor) whereas in *quotitive situations*, the

dividend is divided into fixed portions (divisor) and the number of recipients (quotient) depends on the size of the portion (Squire & Bryant, 2002).

The ideas discussed above provide a basis upon which an intervention developing multiplicative thinking can be conducted. It can also be seen that developing multiplicative thinking is not an easy process and often the shift from additive reasoning to multiplicative reasoning can be difficult. Watson (n.d) as cited in Hurst (2015) mentions that not everyone is able to make the shift from additive thinking to multiplicative thinking as the enforcement of multiplication as repeated addition tends to linger as a dominant image for many learners. There is a common consensus in the literature which suggests that multiplicative reasoning does not develop naturally, instead it requires deliberate instruction (Lamon, 2007; Moss & Case, 1999; Siemon et al., 2005). Previous studies by Siemon et al. (2005) have found that children need to be given some opportunity to think through various problem situations, what they mean, how they can be represented, how we can use what we know, and which strategies are better and why. Therefore, in this study, I drew largely on a variety of multiplication and division problem types so that I could explore how learners dealt with these concepts prior to intervention and, if at all, any shifts occurred after working rigorously with them in developing multiplicative reasoning. Prior to reviewing the various problem types used in this study, I discuss the importance of problem solving in multiplicative reasoning.

2.4 Problem Solving

One way of assisting learners in developing multiplicative reasoning skills is through problem solving. A problem in its most literal form is when the solution is not evident and requires a goal that needs to be reached. Therefore problem solving can be defined as “engaging in a task for which the solution method is not known in advance” (NCTM, 2000, p.52).

Teaching using a problem-solving approach is highly encouraged and favoured with educational reforms (Lampert, 2001). It is believed that this approach offers learners an opportunity to do mathematics as well as understand mathematical concepts, processes and techniques (NCTM, 1989). Problem solving should not be taught as a separate topic rather it should be integrated into mathematics teaching so as to support learning with understanding.

Haylock & Cockburn (2013) talk about ways in which problem solving should be approached. They discuss the ‘gap’ between what is given and the end goal– the *three G’s* of problem solving (*Givens, Goal and Gap*). The *gap* is the route one takes to getting to the goal which is done through clarifying what is given and what is required. Children should be encouraged to ask themselves questions around what they know about the problem, what are they trying to find and what needs to be done to be able to find a solution.

Thinking about questions like these, could also allow learners better ways of dealing with unnecessary information in the text (superfluous information). Superfluous information can sometimes lead children to selecting all the numbers from the problems – ‘number grabbing’ which results in poor problem solving as they are unable to distinguish information that is relevant from irrelevant information (Cook, 2006; Littlefield & Rieser, 1993; Hegarty, Mayer, & Monk, 1995). In studies conducted by Cook (2006), it was found that when learners poorly discriminated information in word problems, they often used a simple comparison strategy in which they looked back and forth between relevant and irrelevant information without closely looking at the question. Contrarily, the study found that learners who were able to accurately discriminate information in the word problems, used either a question-guided strategy prior to using any computation whereby they referred to the question a number of times or they used a strategy of discriminating information in the initial reading of the problem which notices and uses higher order relations in the problem text (Cook, 2006). Through being able to discriminate information in the problem text and think about questions around what the problems are like, children are able to make better sense of the problems which in turn makes them better problem solvers.

Problem solving should therefore support learners in constructing a deeper understanding of mathematical ideas and processes (Lester et al., 1994). Using word problems which incorporate familiar contexts can also lead to learners seeing mathematics as worthwhile and making more sense. Hence, teaching through problem solving draws on and weaves together the strands of mathematical proficiency i.e. conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (Kilpatrick, Swafford, & Findell, 2001). In the section that follows, different classes of word problems are discussed.

2.5 Classes of multiplicative problem situations

In this section, I discuss literature on the classes of multiplication and division word problems that I worked with from Askew (2004)'s Big Book of Word Problems. These classes were:

- Multiplication as Repeated Addition
- Multiplication as Rate
- Multiplication as Scale
- Multiplication as Array
- Division as Sharing
- Division as Repeated Subtraction
- Division as Grouping (*with remainder*)

2.5.1. *Multiplication as Repeated Addition*

Multiplication as repeated addition situations are situations in which there are several groups – all the same size that need to be added together (Askew, 2004). For example: *Sam loves collecting bottle tops. She collected 11 bottle tops on Tuesday, 11 bottle tops on Wednesday, and 11 bottle tops on Thursday. How many bottle tops will Sam have collected?*

The repeated addition model is often seen as a primitive model associated with multiplication and tacitly affects the meaning and use of multiplication (Fischbein, Deri, Nello, & Marino, 1985). In the above problem, Sam collects the same number of bottle tops which can be recorded mathematically as $11 + 11 + 11 = 33$ or $11 \times 3 = 33$. A salient feature in this type of problem is that there are three sets of eleven. Haylock & Cockburn (2013) mention that central to this type of structure is the idea of 'so many sets of so many' which is simply an extension of the aggregation structure of addition (Haylock, 2010). Wallace & Gurganus (2005) iterate that in these types of problems, one factor describes the number of items in each group and the other describes the number of groups with the product being the number of items in total.

11. Sam loves collecting bottle tops. She collected 11 bottle tops on Tuesday, 11 bottle tops on Wednesday, and 11 bottle tops on Thursday. How many bottle tops will Sam have collected?

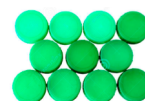
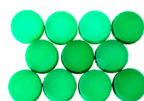
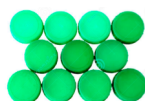


Figure 2.2: Multiplication as repeated as repeated addition ($11 + 11 + 11$ or $11 \times 3 = 33$)

2.5.2. *Multiplication as Rate*

Multiplication as rate situations are situations where there is an implicit ratio, where implicitly or explicitly, there is a 'per' in the context (Askew, 2004). Key words associated with this type of problems are 'each', 'per' and 'every'. In rate problems, we often work with comparing two quantities or a constant relation of *one-to-many* correspondence between two sets (Nunes & Bryant, 1996). This idea relates to that of repeated addition as there are equal groups however they cannot be considered the same. In one-to-many correspondence situations, the constant one-to-many correspondence is the invariant which is usually not seen in additive reasoning. For example; if there are 4 chairs per table, each time we add an extra table we must add 4 chairs to the set of chairs. Nunes & Bryant (1996) highlight that this contrasts with additive situations because in order to keep the difference between two sets constant, we must add the same number of objects to each set. Therefore, when maintaining the ratio invariant in such situations, the notion of replicating emerges. If one table has four chairs, then the unit to be considered in the set of tables is one whereas the unit in the set of chairs is a composite unit of four chairs (Nunes & Bryant, 1996). Thus, ratio situations express a pair of numbers that remain invariant even if the set varies.

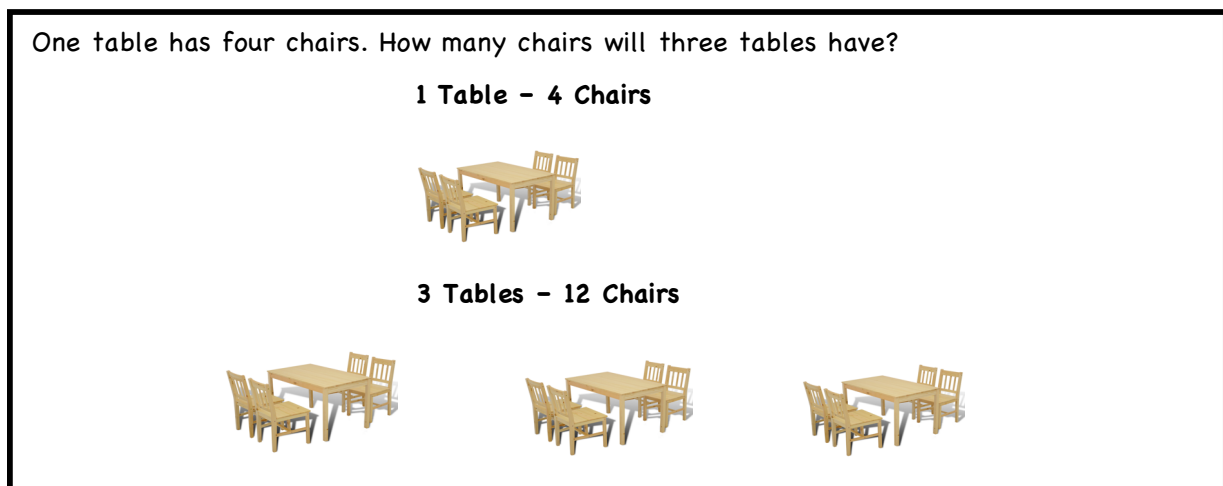


Figure 2.3: Multiplication as rate ($3 \times 4 = 12$)

2.5.3. *Multiplication as Scale*

Multiplication as scale situations are ones where a continuous quantity is increased in size by a scaling factor (Askew, 2004). The idea of scaling is slightly more difficult because a scale factor is involved. A scalar factor refers to the number of replications relating two set sizes of the same type (Nunes & Bryant, 1996).

For example, in the problem: *Leti and Ayesha have string. Leti's string is 14 meters long and Ayesha's string is 7 times as long. How long is Ayesha's string?* there are two sets of string – Leti's string and Ayesha's string. And whilst the calculation for this problem is 14×7 , it would be incorrect to confuse the problem with repeated addition as there aren't 14 sets of 7 strings but rather two sets of string. One set (Leti's string) has a measurement of 14 meters and the second set is '7 times as long'. Haylock & Cockburn (2013) state that 'times as many' is used to express a relationship between the two sets involved in the story. Children often misunderstand the 'times as many' idea and tend to default to additive operations instead. This is because they fail to understand the relationship between the two sets of numbers in the problem (Hart, 1981). Learners need to be made aware that an object is being enlarged by a factor of x and not how much longer one piece of string is than the other (Askew, 2004). It is therefore important to use and emphasize the language of scaling as well as understand multiplication as "a process of enlargement, of scaling up by a factor; a process of making a quantity 'so many times bigger'" (Haylock & Cockburn, 2013, p.103).

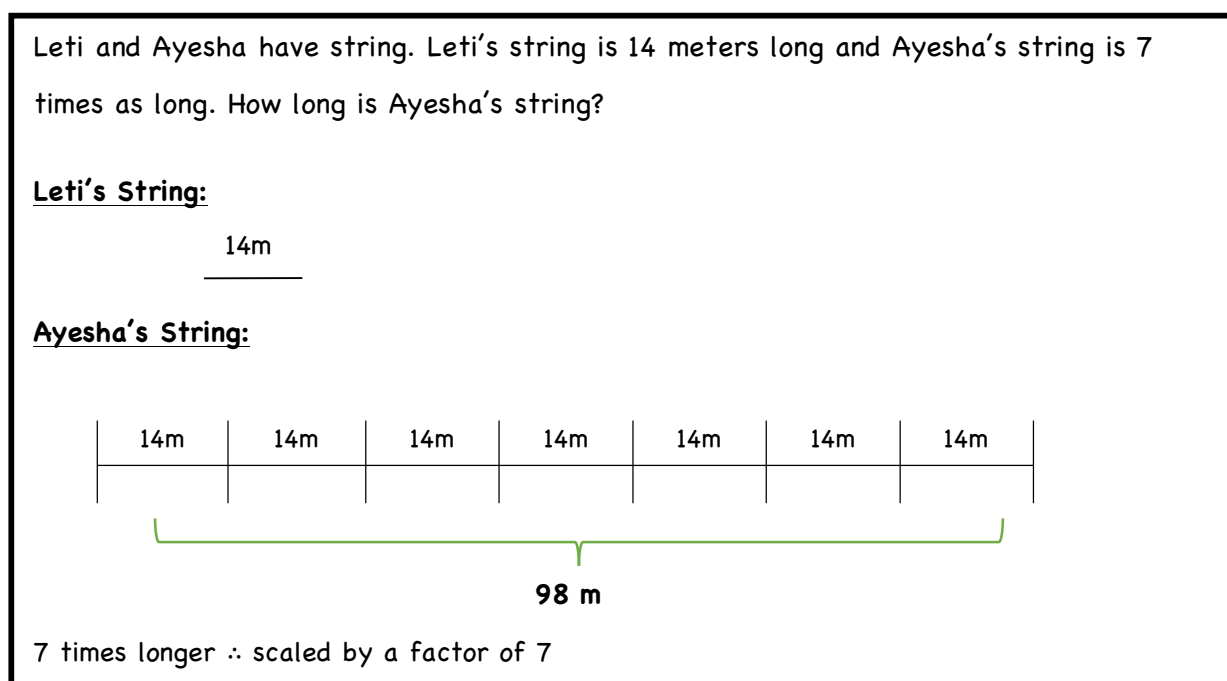


Figure 2.4: Multiplication as scale (14×7)

2.5.4. Multiplication as Array

In order for learners to establish commutativity in multiplication, attention needs to be given to connecting one particular picture of multiplication into a network of experiences associated with this concept (Anghileri, 2006b; Haylock & Cockburn, 2013). Arrays are considered powerful ways in which to represent multiplication. They refer to rectangular

arrays in which the multiplier and the multiplicand are exchangeable making the commutative property transparent (Hurst & Hurrell, 2014).

In figure 2.5, the array (6×3) could be seen as either, three rows of six pies or six columns of three pies which links to the notion that six threes and three sixes are the same. Arrays also have the potential to allow students to visualise commutativity, associativity and distributivity and should be employed alongside other representations to allow students to develop a deeper and more flexible understanding of multiplication and/or division (Young-Loveridge, 2005).

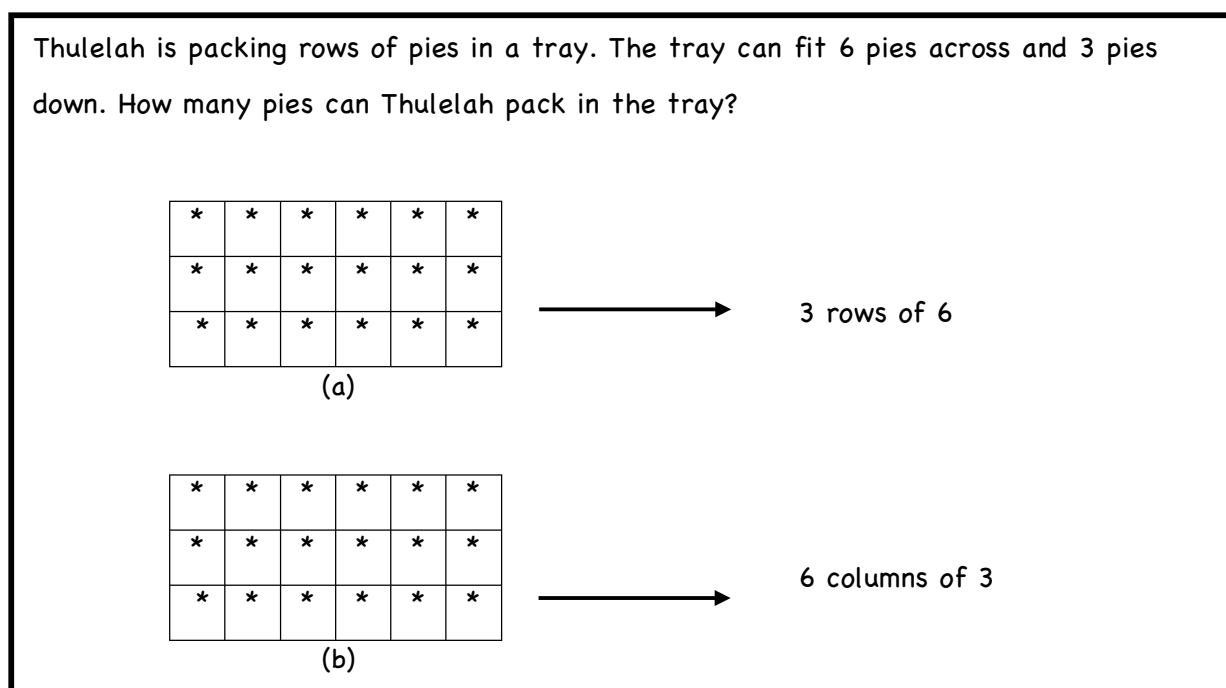


Figure 2.5: Rectangular arrays for 6×3

2.5.5. Division as sharing

Situations where an amount has to be put into equal sized groups and the number of the groups or portion is given are known as ‘division as sharing’ (Nunes & Bryant, 1996; Askew, 2004; Haylock & Cockburn, 2013). For example: *9 friends were given a bag of 72 sweets. They shared them out equally. How many sweets did they each get?* In this problem, 72 sweets need to be shared equally between 9 friends. The notion of sharing is not new to children as they are accustomed to practically working with this idea in their everyday contexts. From an early age, children tend to apply the sharing concept in a one-for-me one-for-you fashion. Sharing differs from addition and subtraction because it involves establishing a multiplicative relationship between two or more sets (Anghileri, 2006b). Three

elements define sharing situations – the size of the whole, the number of parts, and the size of the parts which has to be the same for all the parts. Whilst sharing problems can be quite similar to one-to-many correspondence situations, they differ in that there is an inverse relation between the number of children and the number of sweets-per-child. Anghileri (2006b) states that another difference between one-to-many correspondence situations and division as sharing is that division may naturally result in fractions whereas one-to-one correspondence situations are more commonly connected to whole numbers. Figure 2.6 illustrates the sharing structure where 72 sweets have been shared between 9 children.

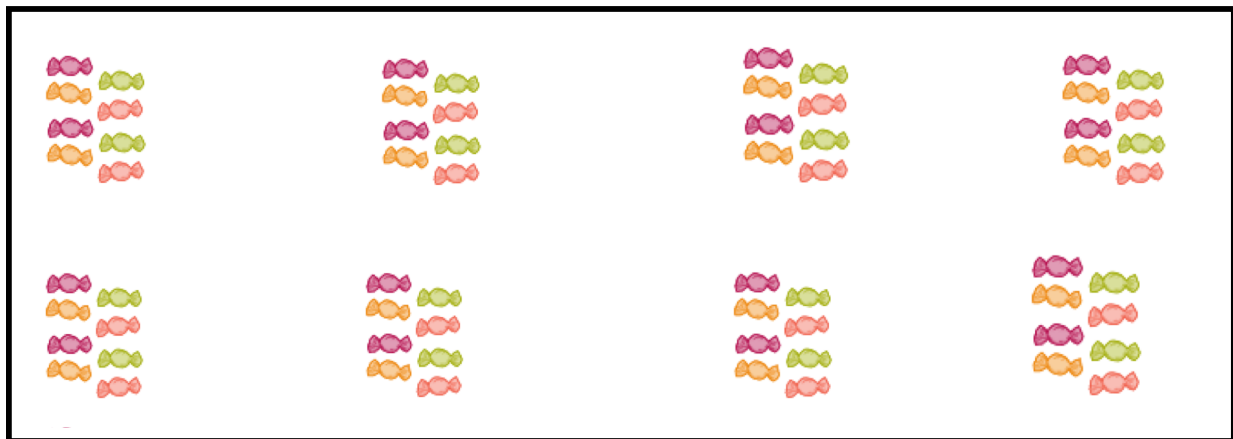


Figure 2.6: Division as Sharing ($72 \div 9$)

2.5.6. *Division as Repeated Subtraction*

Situations where an amount has to be put into equal sized groups or portions and the size of the group is given is known as repeated subtraction (Askew, 2004). Interpreting a calculation such as $24 \div 6$ might be thought of as how many times can I take six away from twenty-four until there is nothing left over? Haylock & Cockburn (2013) state that this question is very similar to how many sixes are there in twenty-four? So even though this is given a different structure, it is essentially just another way of thinking about grouping.

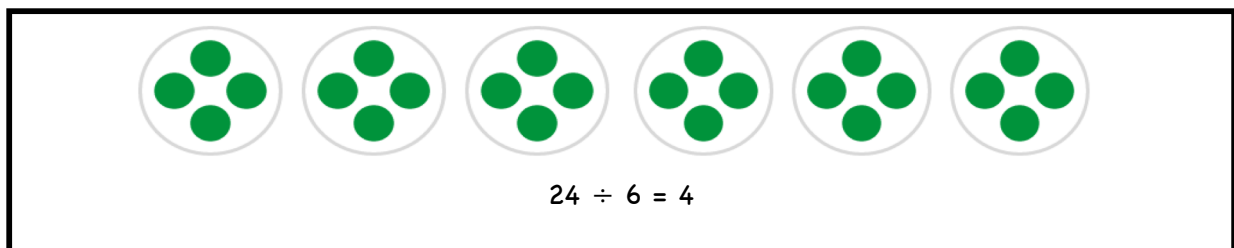


Figure 2.7: Division as repeated subtraction (grouping)

2.5.7. Division as Grouping (with remainder)

Division as grouping situations are defined in the same way as the repeated subtraction problems where an amount has to be put into equal sized groups and the size of the group is known. However, children are not always likely to experience neatly related equal groupings. Situations like this can be the focus for discussion as seven objects cannot be grouped exactly in twos and will involve objects that are left over (Anghileri, 2006b). Children are then encouraged to think about the patterns of numbers and develop some reason about the numbers involved. Incorporating questions *with remainder* introduce an element that children need to look at i.e. the mathematical solution to the problem so as to decide whether or not the solution is sensible in the context of the problem (Askew, 2004). These ideas can be associated with patterns on a number line when equal jumps are taken and not all numbers are 'landed on' (Anghileri, 2006b). Division situations like these require children to either round up or down providing a suitable answer thereafter. For example: *Jill has 45 eggs to take to the market. An egg box can hold 6 eggs. How many boxes will Jill need if she wants to take all the eggs to the market?* In this problem, $45 \div 6 = 7.5$ however it would not make sense to say that Jill would need $7 \frac{1}{2}$ boxes. To take all the eggs to the market, Jill would need 8 boxes (which would be the sensible answer to provide). The different classes of problems can be solved by using different representations which leads me to the next section on the role of representations in mathematics.

2.6 The role of representations and models in mathematics

The power of mathematics originates from the fact that we can represent real-life or concrete situations in mathematical language, pictures and symbols (Haylock & Cockburn, 2013). Representations form a significant part of mathematical competence. This is often because mathematical ideas are abstract, thus requiring representations to allow access to mathematics in a more meaningful way. Representations in mathematics can be both internal and external. Internal representations are mental mathematical ideas whereas external representations are creations for communicating mathematical ideas through language, symbols and pictures (Hiebert & Carpenter, 1992).

External representations in its different forms facilitates the learning of mathematics. Children make sense of problems through intuitively modelling the action and relations described in them (Carpenter et al., 1996). Young children are often encouraged to use

concrete representations that directly reflect or model a particular mathematical concept which is generally done with ease in the early years of schooling (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993). However, as they progress, children should be able to move from concrete to more symbolic representations. Harries & Barmby (2008) mention that having an understanding of a particular mathematics concept comes from being exposed to different representations and being able to move both within and between them. At the same time, representations are not self-evident, but rather they should be worked on and developed so that the essence of the representation is understood (Harries & Suggate, 2006). Research has shown that when learners interact with appropriate representations, their performances are enhanced (Ainsworth, 2006). Thus, it is important that children are active in developing and processing these representations so that mathematical thinking can occur.

In assisting children to represent multiplication and division, considerations around how the representation might show characteristics of the problem and aid the calculation process need to be thought of. Representations should be able to facilitate the process of mathematizing, exploring, reasoning and communicating so that a shift can occur from the real-world to a mathematical model which in turn allows for the mathematical solution to be obtained.

Representations that can be used towards finding the solution are part of mathematical modelling (Haylock, 2010). Van den Heuvel-Panhuizen (2003) defines a *model* to be a representation that learners use to represent a problem whereas a *strategy* is the process that one undertakes to arrive at the answer (correct or incorrect). Gravemeijer (1999) mentions that models are often used to ‘concretize’ formal abstract mathematics making it more accessible to learners. Haylock (2010) illustrates the process through which mathematical modelling can occur. This process involves four steps – (1) the *problem in the real world*, (2) the *mathematical model*, (3) the *solution in the real world* and (4) the *mathematical solution*.

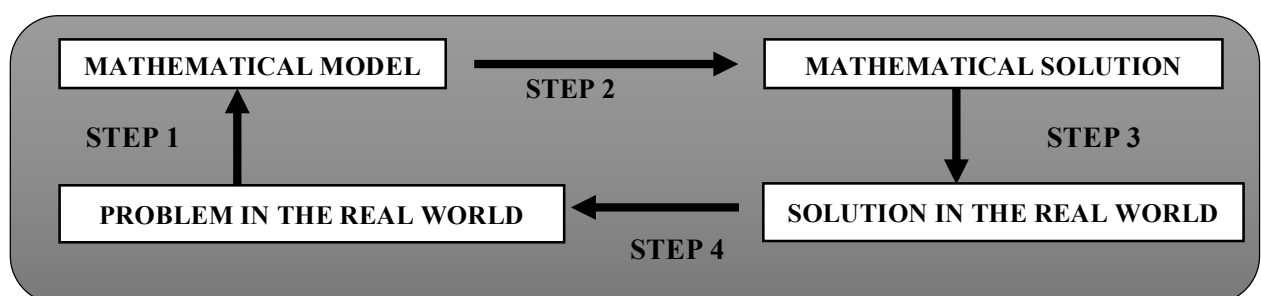


Figure 2.8: The process of mathematical modelling (Haylock, 2010)

Step One begins with the real-world problem which is translated into mathematical symbols (Haylock, 2010). In terms of RME which is discussed explicitly in chapter 3, this would be characterized as horizontal mathematization which represents key elements of a realistic situation in a mathematical model (Van den Heuvel-Panhuizen, 2000). For example, *66 children need to be transported to a sports match. The school has arranged for the parents to take the children by car. If the cars can only take 4 children, how many cars will be needed to transport all the children?* One way of translating this problem into a mathematical model would be $66 \div 4$ or a drawing which groups the 66 children into groups of 4. Step Two draws on vertical mathematization as it requires manipulating the mathematical model by means of written or mental calculations in order to find the solution (Haylock, 2010). The third step looks at interpreting the solution back into the real world. Therefore, based on the example provided 17 cars are needed to transport all the children. And lastly, step four, compares the result with the reality of the original situation. Therefore, even though the answer to the mathematical model is 16.5, it would be illogical to say that 16.5 cars are needed. An extra car would be needed to take the remaining two children.

The way in which children move from informal reasoning to more formal ways of reasoning is of importance. Gravemeijer (1999) states that as children move from using informal strategies to more formal ways of reasoning, a transition from *models of* situations to *models for* mathematical thinking occurs where over time, as children work more frequently with these *models for* (formal ways of reasoning), they will eventually become *tools* which they think with (Askew, 2012b). In the next section, I consider some multiplicative models and how learners may use these models to solve various problems.

2.7 Models and associated strategies for multiplication and division

In this section, I discuss key models for multiplication and division as well as some of the associated strategies that literature alludes to with regards to these models.

2.7.1. Diagrammatic Representations

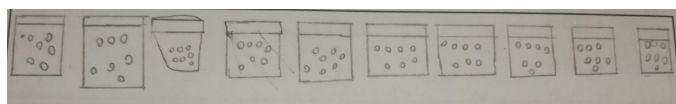
Arithmetic instruction for young children usually begins with concrete objects, progresses to ‘semi-concrete’ aids such as pictures and finally to ‘abstract’ symbols. Drawing diagrams can be useful for some children as it helps them better internalize and solve given problems.

When children employ their own models, they generally have a better success rate in solving problems and calculations.

In the early stages of learning, children tend to use informal solution strategies directly modelling problems (Verschaffel & De Corte, 1997). Carpenter, Fennema, Franke, Levi, & Empson, (1999) believe that direct modelling usually requires either physical or concrete representations of the problem. South African authors' (Ensor et al., 2009) work with foundation phase learners in the Cape Peninsula showed that young learners' attempts at problem solving generally started with using concrete representations which needed to progress as their mathematical thinking matured. They put forth the different stages of representations that learners pass through from concrete apparatus to the use of mathematical notation to undertake calculations with or without reference to empirical situations (Ensor et al., 2009). In terms of diagrammatic representations, these can take two forms; *iconic* or *indexical*. *Iconic* representations are realistic depictions of everyday contexts such as photographs, cartoons, or drawings whereas *indexical* representations use generic rather than realistic depictions of everyday contexts such as drawing of sticks, tallies, dots, circles and other shapes to represent everyday objects (Ensor et al., 2009). As learners progress through the grades, it is expected that these representations give rise to symbolic forms of representation moving learners to using more efficient strategies. Both of these types are exemplified in Figure 2.9:

Thabo planted 10 pots with magic beans. He planted 7 magic beans in each pot. How many beans did Thabo plant?

Iconic Model



Indexical Model

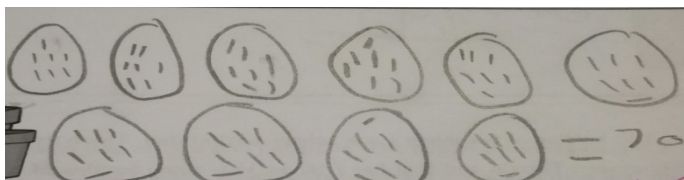


Figure 2.9: Examples of iconic and indexical representations

2.7.2. The Array Model

In order to establish the properties of multiplication, it is useful to give attention to the array model. Array models are important as they make the distributive and commutative properties of multiplication transparent (Haylock & Cockburn, 2013). Barmby, et al. (2009) looked extensively at the role of arrays in multiplication and how they support multiplicative reasoning. They explain how array models neatly represent the commutative nature of multiplication by re-orientating the array. The distributive nature of multiplication can also be easily shown using an array. In the array below, the horizontal spacing shows that 7×6 can be re-written as: $7 \times 6 = (5 + 2) \times 6 = (5 \times 6) + (2 \times 6)$ whereas the vertical spacing shows that $7 \times 6 = 7 \times (5 + 1) = (7 \times 5) + (7 \times 1)$. When considering both horizontal and vertical spacing, we can see that the seven times table could also be the $(5 + 2)$ times table because $7 \times 6 = (5 + 2) \times (5 + 1) = (5 \times 5) + (5 \times 1) + (2 \times 5) + (2 \times 1)$ (Barmby et al., 2009).

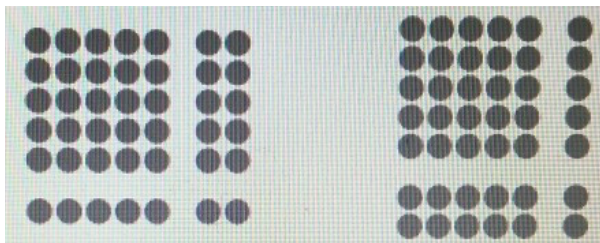


Figure 2.10: An array representation for (6×7) and (7×6)

Moreover, not only does the array model serve as a bridging tool to introduce algorithms for multi-digit multiplication but it also shows the connections between multiplication and division. For example, a child could be given twelve counters and challenged to find different ways of arranging them in a rectangular array. Discussions around the number of counters in each row and column could emerge drawing on the structure of equal sharing and grouping (Haylock & Cockburn, 2013). Additionally, if we want to find $42 \div 7$, we could set this up as an array with the value of one side missing and use known multiplication facts to build the value of the missing side. Therefore:

$$\begin{array}{c} ? \\ \boxed{42} \\ 7 \end{array}$$

Thus, it can be noted that the array not only supports understanding multiplication structures but it also supports division structures such as sharing and grouping.

2.7.3. Column Multiplication and Division

Column multiplication and division are algorithms commonly used in multi-digit multiplication and division. An algorithm is a general multi-step procedure that will produce an answer for a given class of problems or provide the result of a calculation (Fuson, 2003; Haylock, 2010). Column multiplication is a standard algorithm for multiplication and works with multiplying numbers in columns. *Long multiplication* is when two two-digit numbers are multiplied together (Haylock, 2010). Long multiplication is commonly taught in mathematics classrooms and is based on the distributive property of multiplication where one of the numbers are decomposed in to the sum of its tens and units and the multiplication by the other number is distributed over these. It is set out in the following way:

$$\begin{array}{r} 26 \\ \times 34 \\ \hline 780 \\ + 104 \\ \hline 884 \end{array}$$

← 26×30

← 26×4

(Multiplying by tens and then units)
(Haylock, 2010)

$$\begin{array}{r} 26 \\ \times 34 \\ \hline 104 \\ + 780 \\ \hline 884 \end{array}$$

← 26×4

← 26×30

(Multiplying by units and then tens)
(Ma, 1999)

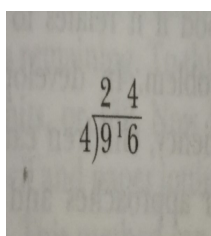
Figure 2.11: The column multiplication algorithm

Long multiplication algorithms like the ones presented above often lead to partial product calculation errors as learners seemingly forget to move the numbers over on each line (Ma, 1999). Earlier studies conducted in the United States (Ma, 1999) show that one of the reasons why learners tend to make these errors is because they do not really understand the reason behind lining up the partial products in the way the long multiplication algorithm requires. Moreover, learners may also overgeneralize the long multiplication rule by putting down a zero for multiplication by the units and tens. Hansen (2011) discusses that this type of error may occur when children have learnt a rule and then applied it inappropriately because they have not gained a sound understanding of the methods introduced to them.

In terms of column division, this could be done through using either short or long division. *Short division* is a compact standard algorithm for a division calculation generally involving a single-digit divisor. The divisor is divided into each digit, working from left to right, with any

remainders being transferred to the next column (Haylock, 2010, p.107). *Long division* on the other hand is slightly more complex because of the number of steps involved (Anghileri, 2006b). In the long division example below, we first divide, then multiply, subtract and bring down. Therefore, we first divide 15 by 43 to get the 2 as the tens part of the answer. The corresponding 30 (tens) is then placed under the 43 of 432. When 30 is subtracted from 43, we get an answer of 13 (which is the number of tens remaining). The two units are then brought down to make 132. 15 is then divided by 132 to get an answer of 8. A remainder of 12 is left therefore the solution to this calculation is 28 remainder 12. Overgeneralization errors can be noted in division algorithms when children begin dividing from right to left starting with the units (Hansen, 2011). Similar to the errors in multiplication, these errors tend to occur when children have a poor understanding of methods they have been introduced to. When children follow rules without understanding, errors occur because digits become the focus instead of the whole values of the numbers Anghileri (2006b).

Short Division



Long division

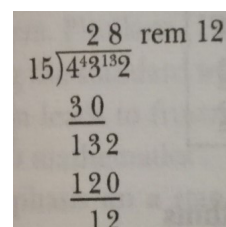


Figure 2.12: Short and long division algorithms

2.7.4. Ratio Models

To assist children in developing reasoning rather than blindly following procedures and algorithms which are rarely fully understood, it would be useful to introduce them to ratio models such as the *double number line* and *ratio t-table*. In this section, I discuss literature that advocates the discussion and use of both of these models and how they facilitate a better understanding of multiplicative problems.

2.7.4.1 Double Number Line

To understand mathematics, one must be able to interpret and generate a variety of representations one of which is the double number line. My rationale for beginning with the double number line in my study was because learners are already familiar with number lines and how they work from foundation phase. Van den Heuvel-Panhuizen (2002) provides a

good illustration of how the number line can be used to support learning since learners are familiar with the concept from as early as first grade (using bead strings to practice counting activities) progressing to an empty number line for addition and subtraction and thereafter using a double number line to support problems on ratio. Therefore, introducing the double number did not require a big shift for learners and could be easily assimilated and adapted.

The double number line serves as a *model of* a variety of contexts that should be reasonably accessible to students (Küchemann, Hodgen, & Brown, 2011a). The double number line can be used to promote proportional reasoning by supporting the coordination of two values, relying on familiar ideas of linear measurement (Orrill & Brown, 2012). It draws on ideas of multiplication as ‘stretching’ or ‘shrinking’ to include two quantities which are stretched or shrunk by the same factor to retain the fixed proportional relationship between quantities (Orrill & Brown, 2012). In working with whole numbers, learners can model multiplication of smaller number ranges by ‘skip counting’ which is reliable and efficient in arriving at the correct solution (Küchemann et al., 2011). As learners become familiar with these practices, they may not need to skip count if they can recall the answer immediately or use a quicker approach.

Küchemann et al. (2011b) argues that the double number line is attractive as it provides a gentle way of moving from an additive approach to multiplication and provides a neat way of representing multiplicative relations that have different quantities. Scales on a map are well known examples of double number lines with one variable showing distances on the map (in cm, for example) and the other, corresponding distances on the object depicted by the map (in km, for example). Therefore, in multiplication as scaling problems, the double number line can serve as a direct model of the situation.

Earlier studies conducted by Moss & Case (1999) and Corina, Zhao, Cobb, & McClain, (2004) showed positive results with implementing the double number line as it supported students’ reasoning on proportion especially on continuous quantities (Orrill & Brown, 2012). It is for these reasons that I worked with introducing the double number line with the intention of progressing to the ratio t-table thereafter. An example of the double number line is presented for the following problem:

Thabo is planting magic beans in 7 pots. He plants 3 magic beans in each pot. How many magic beans does Thabo plant altogether?

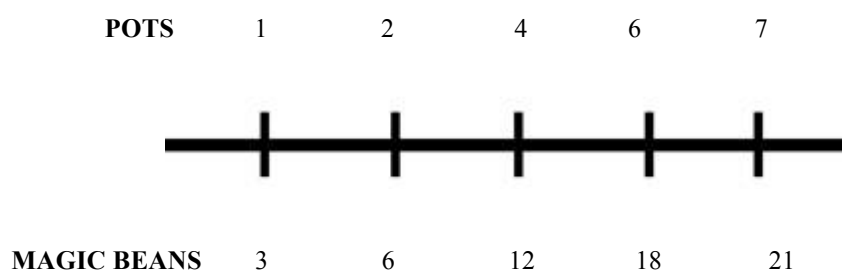


Figure 2.13: The Double Number Line

2.7.4.2. Ratio T-Table

In solving problems with rate and proportion, a ‘building up’ or ‘breaking down’ strategy can be very useful. In using a building up strategy to solve proportion problems, we could establish a ratio or use an already established ratio to find another ratio using addition (Lamon, 2012). If we are given a problem like: *There are 12 friends at an orange farm. They each collect 13 oranges in their bags. How many oranges do they have altogether?* The ratio that is given is that friend collects 13 oranges and we are required to establish how many oranges will be collected altogether. We could use the ‘building up’ idea to find the answer by working with the notion of doubling, therefore 2 friends will collect 26 oranges, 4 friends 52 oranges, 8 friends 104 oranges, and add another 52 more oranges to find the total number of oranges that 12 friends would collect (156). Similarly, in problems where the total and the number of groups are provided, we could use a ‘breaking down’ approach to find out how many items there will be in one group. This amount could be found using halving, therefore if given that 8 children equally share a bag of 72 sweets, how many sweets will each child get? we could say that if 8 children have 72 sweets, then 4 children will have 36 sweets, and 2 children 18 sweets which leaves 1 child with 9 sweets. The building up and breaking down strategy is one that children tend to use spontaneously and works in many situations as children build on their reasoning processes. Anghileri (2007) states that repeated doubling is very effective as children grasp the idea of doubling quite easily therefore both learners who are proficient and those who struggle with their multiplication tables may benefit. Similarly, Lamon (2012) argues that these reasoning up or down ideas encourage powerful and highly desirable ways of thinking which over time could lead to children deducing the desired

algorithm. Additionally, this method allows for strategic planning, number sense and confidence in problem solving (Lamon, 2012).

The idea of building up or breaking down can be represented using a horizontal or vertical arrangement - a *ratio table* which helps keep work organized allowing for visible patterns that can be easily analysed (Campbell, 2005; Lamon, 2012; Middleton & Van den Heuvel-Panhuizen, 1995). The ratio table is a tool that builds connections in equivalent situations in a way that helps students develop an understanding of rational numbers. The basic idea behind the ratio table is that one can generate equivalent ratios by successive manipulations of the number, as discussed earlier with breaking up or down, until one reaches the ratio which best fits the situation (Middleton & Van den Heuvel-Panhuizen, 1995). Figure 2.14 illustrates examples of the ratio t-table for the following problems:

- a. *There are 12 friends at an orange farm. They each collect 13 oranges in their bags.
How many oranges do they have altogether?*

Number of Friends	Number of Oranges
1	13
2	26
4	52
8	104
12	156

← Given in the problem (they each collect 13 oranges in their bags)

} Using doubles to build up to 12 friends

Number of Friends	Number of Oranges
1	13
10	130
2	26
12	156

← Given in the problem (they each collect 13 oranges in their bags)

} Breaking up the number 12 into 10 and 2

(a) 8 children are given a bag of 72 sweets. They share them out equally. How many sweets does each child get?

Children	Sweets
1	? (9)
2	18
4	36
8	72

Using halving to find the number of sweets that each child gets

Given in the problem (8 friends have a bag of 72 sweets)

Figure 2.14: Examples of the ratio t-table

In the examples above, the first ratio t-table displays the use of doubles to build up to finding how many oranges 12 friends will have collected whereas in the second ratio t-table, the number 12 has been broken up into 10 and 2 as these are relatively easy numbers to multiply. In the third ratio table, the problem is dealing with division; therefore, to get to the number of sweets that each child gets we can use halving to break down until we get to 1 child.

There are many ways in which learners could produce t-tables to get to the solution of the problem. Literature suggests it would be good to present these tables to the class and get learners to discuss how they worked the answer out. In cases where learners are not able to perfectly double and halve the numbers, they should be made aware that they can decompose the numbers in ways most easy for them. Learners should be encouraged to use multiplication and division of 2, 5 and 10 as these are easy numbers to work with (Lamon, 2012). During my work with the Grade 4 learners, these recommendations were kept in mind and encouraged all times.

2.7.5. Other models related to multiplication and division

Skip Counting and calculating without counting: Skip counting and calculating without counting relate to more formal ways of reasoning (models *for*) which are strategies based on particular models of situations. Skip counting refers to when one counts every n th number in a series (Frank, 1989). Skip counting can be a useful aid for learning multiplication as learners who are familiar and proficient with skip counting in numbers may find it easier to learn the concept of multiplication by those numbers. Carpenter et al. (1999) states that children are generally more proficient in skip-counting by certain numbers such as 2's and

5's as compared to counting in 7's. This may be because they have been rigorously taught how to skip count in those numbers in the earlier grades.

Contrarily, some learners may not need to count in any way to arrive at the correct solution but rather draw on known number facts. Ensor et al. (2009) mentions that when multiplying and dividing, it may be that the answer is achieved through memorized number facts such as times tables.

Compensation: According to Haylock (2010), compensation involves replacing a number in a calculation with an easier number closer to it and then compensating for it later. In division situations, some learners may use the compensation method when sharing out objects starting with easier number such as 2, 5 and 10.

Idiosyncratic Division: This model includes a distinct setup of the division problem which differs from standard division algorithms. Ruthven (1998) states that when deliberating pupils use of written recording, two distinct purposes emerge – (a) to augment working memory by recording key items of information and (b) to cue sequences of actions through schematising such information within a standard spatial configuration. The former identifies with informal solution strategies that are often idiosyncratic and gives little consideration to efficiency whereas the latter suggests a taught procedure that directs and organizes children's approaches (Anghileri, Beishuizen & van Putten, 2002).

2.8 Concluding Remarks

The literature reviewed points towards the importance of modelling different multiplicative situations from informal representations to more formal ones that incorporate mathematical symbols, algorithms and ratio modelling. It is equally important that children can make sense of the problems they are presented with so as to be able to solve them with efficiency.

As discussed in the earlier sections, teaching children procedures and algorithms do not result in them understanding the mathematics and they often erroneously solve problems using poorly understood methods that have been taught to them. Hence whilst algorithms may provide security of producing 'correct' answers, children's thinking remains blocked by this practice (Kamii & Livingstone, 1994). Instead, they are encouraged to think about the

arithmetic and create representations that help them understand the mathematics in a more meaningful way. Studies conducted by Caliandro (2000) found that children can invent their own methods for solving multi-digit multiplication and division problems without learning conventional algorithms that are so often taught to them. In helping learners construct models for solving multiplication and division problems, it would be worthwhile to explore and develop various models that are not based on traditional algorithms and draw on different strategies that learners could use to arrive at the solution. Therefore, exploring and developing different models and ways of working with models can be one attempt at achieving better performance in multiplicative word problems. In Chapter Three I discuss Realistic Mathematics Education as a theoretical framework of my study.

Chapter Three: Theoretical Framework

3.1 Introduction

The focus of my study, *exploring multiplicative reasoning with Grade 4 learners through structured problem solving*, was broadly driven by an interest in how learners model multiplicative word problems. Models play a significant role in facilitating the learning of mathematics therefore I chose to use a *Realistic Mathematics Education* (RME) approach to inform my study as it provides a platform for thinking about how learners can be assisted in exploring and developing models to solve multiplicative word problems.

The RME framework provides a platform for conceptualising mathematical concepts and thinking about how learners can be assisted in exploring the construction of models to solve multiplicative word problems. Hence, I draw on *RME* for two reasons. Firstly, trends around the world show that for learners to grasp mathematics, there has to be a link between the mathematics taught and the world in which learners are in (van den Heuvel-Panhuizen, 2003). Word problems used in my study incorporated contexts that learners can relate to – real and imaginary. Second, to encourage reasoning, it is important that learners play a role in producing their own models to solve problems (Gravemeijer, 1999), an aspect which was strongly encouraged during the facilitation of intervention lessons. Following on from the literature detailed in the previous chapter, the main concept discussed in this chapter is the theory of RME and its related principles.

3.2 The theory of Realistic Mathematics Education (RME)

RME has its roots in Hans Freudenthal's interpretation of mathematics as a human activity which was founded in the early 1970's (Gravemeijer, 1994). To this end, Freudenthal emphasized the idea of *doing* mathematics, an activity predominantly consisting of organizing or mathematizing subject matter embedded in real contexts. Mathematics should be learnt by mathematizing subject matter from real contexts and their own activities rather than from the traditional view of mathematics being a ready-made system with general applicability (Gravemeijer, 1994). Real contexts that are commonly associated with RME can include contextual problems or mathematically authentic contexts that are relevant and real (Barnes, 2005). This does not mean that mathematics has to be real in every aspect but rather, the focus is internal - formulating something real in one's own mind. Reality can thus be

understood as a combination of interpretation and sensual experience which implies that mathematics, too, can become part of a person's reality (Gravemeijer & Terwel, 2000).

A central component of my study requires learners to mathematize problems.

Mathematization is a process that assists learners with mathematical approaches that are suitable to deal with everyday situations and is closely aligned with the idea of guided re-invention which is discussed a little later. There are two central components of mathematization that allow learners to see mathematics as both from 'reality' and 'mathematical matter' (Gravemeijer & Terwel, 2000). These are horizontal and vertical mathematization. *Horizontal mathematization* refers to learners using informal tools such as diagrams and pictures to solve problems. In my study, learners were initially allowed to explore problems using any model including visuals or diagrams. Informal models are important as they lay a foundation for developing sophisticated reasoning skills. *Vertical mathematization* on the other hand involves a process of reorganization whereby learners' informal models lead them to solving the problem using mathematical language finding shorter cuts and discovering connections between concepts and strategies (Van den Heuvel-Panhuizen, 2000). Hence, in this study, the way learners represented a problem (the model presented) was regarded as horizontal mathematization whereas the process carried out to solve the problem was considered as vertical mathematization. Moreover, adopting an RME framework was meant to provide learners with opportunities to set up models that could better organize and solve problem situations in 'real' contexts. Drawing on these concepts, it was also anticipated that different models would emerge as learners engage with the problems presented. The process of mathematization is illustrated in Figure 3.1 with dotted lines indicating horizontal mathematization and solid lines with arrows indicating vertical mathematization.

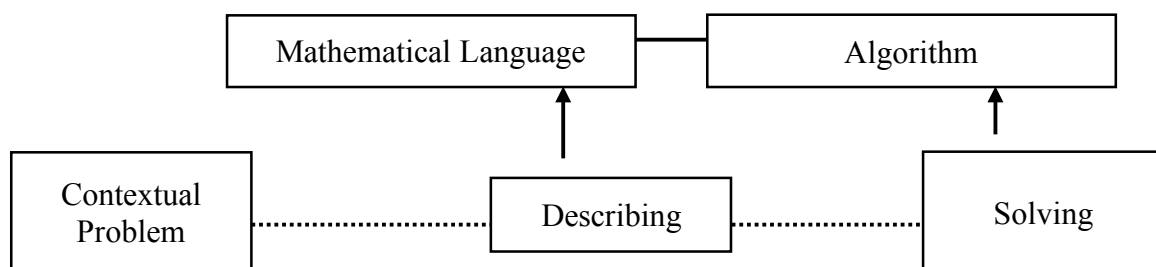


Figure 3.1: Representation of horizontal and vertical mathematization (Gravemeijer, 1994)

3.3 Instructional Design Principles of RME

The RME framework is characterized by three instructional design heuristics, namely:

- Guided re-invention
- Didactical phenomenology
- Emergent Modelling

3.3.1. Guided Re-invention

Mathematization involves using tasks as a tool to provide learners with an opportunity to reinvent mathematics through “*guided re-invention*”. Guided reinvention provides numerous ‘guided’ opportunities to ‘re-invent’ mathematics by *doing it* (Van den Heuvel-Panhuizen, 2000). It focuses on allowing learners to regard the knowledge they acquire as their own which they are solely responsible for (Gravemeijer & Terwel, 2000). Guided re-invention requires that well-chosen contextual problems are presented to learners so that they have the chance to develop informal, highly context-specific solution strategies which may then function as inventions for formalization and generalization, a process referred to as “progressive mathematising” (Doorman, 2001 as cited in Barnes, 2005; Gravemeijer, 1994). Teachers have a crucial role in how learners acquire mathematical knowledge through the activities provided. Throughout my intervention program, lessons began by learners solving the problems using any model they wanted to use after which different models used by the learners were elaborated and refined by the teacher with the aim of exploring and developing *horizontal* and *vertical mathematization*.

3.3.2. Didactical Phenomenology

The didactical phenomenology design heuristic calls for a phenomenological analysis (Gravemeijer, 2004). This means that in learning mathematics, one has to start from phenomena meaningful to the learner in order to stimulate the learning process. Mathematics education is often seen and taught as a closed system (Barnes, 2005) which requires organizing phenomena through the process of mathematization which is largely in favour of mathematics as a human activity (Gravemeijer, 1994). Thus, concepts taught need to be situated in learners’ informal mathematization which later develops into formal mathematics. Models are a key aspect of RME as they facilitate the process of mathematization

(Gravemeijer, 2004; van den Heuvel-Panhuizen, 2002; van den Heuvel-Panhuizen, 2003). Models not only link informal and formal mathematical understanding, but also have the power to raise children's level of mathematical understanding starting with phenomena which initially required organisation (Gravemeijer, 1999; 2004). Therefore, in order to determine the route of intervention, researchers are encouraged to analyse problem situations that may give rise to situation-specific procedures which in turn can be mathematized to arrive at conventional mathematical procedures (Gravemeijer, 2004).

3.3.3. Emergent Modelling

As discussed in the literature chapter, a model is a representation that helps children make sense of a problem situation (van den Heuvel-Panhuizen, 2003). Emergent modelling refers to the notion of a model that may come to the fore as a model of informal mathematical activity and over time may develop into a model for more formal mathematical reasoning (Gravemeijer, 2004). In emergent modelling, the 'model' and the conception of what is modelled co-evolve. Models that learners initially present gradually evolve over time to become *models for* more formal mathematical reasoning. In the process of emergent modelling, a series of sub-models make up 'the model' which develop from a *model-of* informal mathematical activity to a *model for* more formal mathematical reasoning (Gravemeijer, 1999; 2004).

The process from informal mathematical activity to formal mathematical reasoning can occur through four different levels of student activity. I draw on Gravemeijer (2004)'s explanation to explain these levels of student activity. The first level is *situational* where solutions depend on how to act in a particular task setting which is often 'out of school'. The second level, *referential*, concerns reasoning about such situations in the context of school tasks where situation-specific models are needed; thus, the model functions as a *model of* that activity. Next, with the assistance of the teacher, attention gradually shifts towards the mathematical relations involved which is referred to as the *general* level. And lastly, as the model starts to derive meaning from mathematical relations formed, the model begins to become a *model-for* more formal mathematical reasoning. Therefore, when students reach a formal level of mathematical activity that is no longer dependant on the support of a model, they have acquired new mathematical reality.

3.4 The principles of RME

At the heart of RME is that children “should learn mathematics by developing and applying mathematical concepts and tools in daily-life problem situations that make sense to them” (van den Heuvel-Panhuizen, 2003, p. 9). RME involves a number of core principles for teaching mathematics which were articulated by Treffers (1978) and over the years were reformulated. Teachers should therefore be aware of six guiding principles of RME: the *activity*, *reality*, *level*, *intertwinement*, *interaction* and *guided re-invention* principles.

The *activity principle* is based on the notion of students being active participants in the learning process. This principle also emphasizes the idea that mathematics is best learnt by *doing* it – an idea strongly reflected by both Freudenthal and Treffers (Van den Heuvel-Panhuizen & Drijvers, 2014).

The *reality principle* expresses the importance attached to the goal of mathematics education including students’ abilities to solve ‘real-life’ problems. It also stresses the importance of starting with meaningful problems so that learners may attach deeper meaning to the mathematical constructs they develop (Van den Heuvel-Panhuizen & Drijvers, 2014).

The third principle, *level principle*, underlines that learning mathematics means that students pass through various levels of understanding, from informal solutions to more schematized formal mathematical constructs (Van den Heuvel-Panhuizen & Drijvers, 2014). This principle is dependent on the models presented by mathematics students and considers the shifts from *models of* to *models for* situations (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen & Drijvers, 2014).

The *intertwinement principle* advocates that various mathematics topics should be integrated so that learners develop an integrated view of mathematics as well as the flexibility to link different sub-domains (Van den Heuvel-Panhuizen, 2000). For example; within the domain of multiplicative reasoning – number sense, mental arithmetic, estimation, and algorithms are taught in close connection to each other.

The *interactivity principle* of RME suggests learning mathematics is both an individual and social activity (Van den Heuvel-Panhuizen & Drijvers, 2014). Thus, RME is in favour of co-operative learning such as whole-class discussions, pair and group work. These interactions offer learners the opportunity to share ideas, models and inventions with their classmates

evoking better insight and reflection which could possibly lead to a better level of understanding.

Lastly, the *guidance principle* draws on Freudenthal's guided re-invention idea where teachers need to have a proactive role in how their students acquire mathematical knowledge so that positive shifts in understanding can occur (Van den Heuvel-Panhuizen & Drijvers, 2014). This principle implies the importance of the route to learning that is mapped out by teachers.

In sum, the principles discussed in this section highlight both teaching and learning principles necessary for acquiring and transmitting mathematical knowledge. The *reality*, *intertwinement* and *guidance principles* are related to teaching whereas the *activity*, *level* and *interaction principle* are related to learning. Therefore, drawing on these principles which are strongly embedded in RME was a useful guiding tool for my intervention lesson structure. The way in which key aspects of RME were embedded in the study's design and analysis are described in the next chapter.

Chapter Four: Methodology

4.1 Introduction

The importance of educational research is that it involves asking questions, collecting and analysing data to determine credible answers to the questions asked (Creswell, 2012). Knowledge created from such research can, in turn, have an influential impact on educational practices as it provides a better understanding of certain problems and issues. For Opie (2004) educational research is seen as a methodological process that adds on to one's own body of knowledge as well as those in a similar field. The focal point of educational research is 'knowledge growth' through the construction of insights in ones' area of study providing an important framework for research to be conducted. Educational research can thus be seen as a science which utilizes scientific approaches that test, explain, observe and make predictions. To gain a better sense of how learners reason multiplicatively, I chose to adopt a mixed method approach whereby both *qualitative* and *quantitative* data were collected providing a rich picture of my chosen research topic.

This chapter discusses the methods used to generate the findings of my study. It includes the design of my study, target population, sample selection, data collection methods and the sources of information used. Furthermore, the way in which data was analysed and the measures undertaken to ensure rigour throughout my study (validity and reliability) is discussed. And lastly, ethical considerations are reviewed.

4.1 Research Questions

In this section, I outline the research questions used in my study as well as a brief discussion on how I answered each question.

1. What kinds of multiplicative reasoning (models) are Grade 4 learners using prior to intervention, as seen in pre-test responses?
2. What changes, if any, are seen in overall performance, across the intervention and control group, in the post-test?
3. What kinds of differences in model use were associated with the shifts in performances?

In order to answer question one, the models used by learners in the pre-test were recorded and analysed. For question two, learners' overall test scores in the post-test were compared against scores in the pre-test in order to determine if any shifts were achieved. Both the intervention and control groups results were looked at to see if gains were stronger for learners who received intervention. These shifts were then carefully analysed and led to answering question three which sought to understand the differences in models associated with the shifts in performance. Learners' models used in the post-test were compared to those used in the pre-test with analysis and discussions on the shifts in both the intervention and control groups.

4.3 Research Design

A mixed method study combines characteristics of both quantitative and qualitative approaches to research. Quantitative methods emphasize objectivity in measuring and describing phenomena whereas qualitative methods inquire, collect data and analyse it in an inductive way establishing patterns or themes which describe and interpret the problem at hand (McMillan & Schumacher, 2010).

The type of mixed methods approach which I adopt in my study is called a *sequential explanatory design*. In sequential explanatory designs, quantitative data are collected and analysed followed by qualitative data collection and analysis (McMillan & Schumacher, 2010). For my study, qualitative methods such as document analysis and interviews were needed to elucidate the quantitative findings presented.

My research design adopts an experimental approach which is an important sub-classification of the quantitative design. Here, the researcher intervenes with a procedure that determines what participants will experience hence giving the researcher some control over what will happen with the participants by systematically withholding or imposing interventions (McMillan & Schumacher, 2010; Cohen & Manion, 1994).

My research involved testing (pre-and post-tests) with Grade 4 learners of which one group was provided intervention and the other group of learners were used as a control group. This approach draws on a *Quasi-Experimental* design as classes were not randomly organized or assigned to have different teachers however the choice of which classes received intervention and which were used as a control group was randomly chosen. A quasi-experimental design

allows control over the ‘who and to whom of measurement’ but lacks control over ‘the when and to whom of exposure’ (Cohen & Manion, 1994).

The group who received intervention were taught by myself, hence my role was to act as both a researcher and teacher. As a teacher, throughout the intervention program, my role was to aim for learners to gain an improved understanding of different modelling approaches to multiplicative word problems; whereas a researcher, my role was to carefully analyse whether learners had made any gains through the small-scale intervention program.

In my research study, *document analysis* was used to enhance and complement other forms of data collection painting a broader and more in-depth picture of the results of my study. Bowen (2009, p.27) defines document analysis as a “systematic procedure for reviewing or evaluating documents – both printed and electronic (computer based and internet transmitted) material”. Document analysis was conducted prior to interviewing learners as an understanding of how learners modelled the problems needed to be gauged before the interview process. An inductive approach was also adopted as trends in the data were interpreted. I focused extensively on the different models that learners presented in the tests, as this linked with the RME theoretical base of this study which argues that by modelling contextual problems that are real, children can use their informal knowledge to reinvent mathematics (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2002; Van den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen, 2000). Models were considered quantitatively from the document analysis with the details of their assembly probed in the follow-up interview sample so that a better understanding of how learners reasoned and solved problems could be gained. Interviews help validate and go deeper into the motivations of respondents and their reasons for responding the way they do (Cohen, Manion, & Morrison, 2011). Hence, this sequence of data collection and analysis comprised the sequential explanatory mixed methods approach taken up in this study.

4.4 Target population and Sample

The *target population* of my study was Grade 4 learners (approximately 9 to 11 year olds) in a suburban former Model C school in Johannesburg North. This school was chosen since I was a former teacher at the school making it easier to gain access for research to be done. It is a relatively privileged setting in the South African context. Thus, trialling the intervention in this relatively favourable setting was intentional as if the intervention had limitations in this

setting, it was likely to require substantial reworking in less privileged settings.

A Grade 4 level was chosen since it is the first year of the senior primary phase and commonly the year where a ‘dip’ in mathematics performance is seen (Department of Basic Education, 2014). The participants in my study were drawn from three classes of Grade 4 learners in the school of which two classes formed the intervention group and the remaining class formed the control group (with prior agreement with the school that this control class would get later access to the intervention materials if the results were favourable). There were 37 learner participants in the intervention group and 24 learner participants in the control, with all participants giving their informed consent to participating in this study.

Following an initial analysis of all the test scripts, a *random stratified sampling* of interviews was conducted. A probability sample was drawn from each class based on the results of the pre-test. Test scores were added up and ordered so that low, middle and high attaining learners were selected. This type of sampling was done by identifying the level of attainment and choosing a random sample within these groups with the size of each group determined by the researcher (Creswell, 2012; McMillan & Schumacher, 2010). 18 learners were chosen from the pre-test - six low, middle and high attaining learners from each class. However, three learners interviewed on the pre-test were not present at school on the day of the post-test thus I was left with 15 interviews from the pre-and post-test.

4.5 Data Collection

After receiving consent from the relevant authorities and participants, I collected and generated data using the aforementioned strategies. Different data collection instruments were reviewed and thought through before I settled on using, pre-and post-tests (developed through a piloting process) as well as interviews. A detailed discussion of the research instruments used in my study is provided below.

4.6 Research Instruments

4.6.1. Testing

Since my study was focused on learning related to multiplicative reasoning and involved a series of intervention lessons, using tests such as pilot tests, pre-and post-tests provided an appropriate route to answer my research questions. Tests are a common method of data collection in experimental research (Bertram & Christiansen, 2014). Researchers test learners

before intervention to ascertain learner's achievement (in my study it was to explore the models of multiplicative situations) and then test them again after the intervention.

The content of word problems in my tests and intervention lessons were drawn from Askew's (2004) *Big Book of Word Problems* – a set of materials developed in England that includes different categories and levels of word problems related to multiplicative reasoning. More details on each test instrument is provided below.

4.6.1.1 Pilot Tests

Pilot testing is a useful measure of trialling tests that one intends to use in their research study as it provides insight as to whether tests are appropriate and need any adjusting prior to being formally conducted (Bertram & Christiansen, 2014; Creswell, 2012; McMillan & Schumacher, 2010; Opie, 2004). In my study, one class of Grade 4 learners in the same school were asked to write a pilot test to help determine the level of multiplicative reasoning and word problems from Big Books by Askew (2004) to include in the pre- and post-tests of my study.

The pilot test was structured in such a way that it comprised of twenty questions spread across a range of problem types. Since my study was on exploring multiplicative reasoning through structured problem solving, problems were selected from five classifications of multiplicative word problems that have been detailed in the literature overview (Chapter 2) – (1) Multiplication as repeated addition (2) Multiplication as scale (3) Multiplication as rate (4) Division as sharing and (5) Division as (grouping) repeated subtraction. These categories of problems were informed by Askew (2004)'s explanation that there are three 'root' situations for multiplication: repeated addition, rate and scaling which are linked to two division models –sharing and grouping which all provide a helpful model for teaching. Additionally, two problems (one subtraction and one addition) were selected as buffer problems to gauge whether learners could select the appropriate operation. 'Bald' number calculation items were also inserted with the focus on whether learners could carry out multiplication and division procedures and identify inverse operations. Multiple choice items focused on identifying models or outcomes of procedures with the purpose of determining whether learners could select the correct model from the options provided were included. These kinds of questions assisted in understanding whether learners' difficulties lay in interpreting word problems or in their multiplicative calculation fluencies or perhaps in both.

The results of the pilot test indicated that the overall performance across all the questions in the test were below 50% with learner performances varying across the different problem types. Difficulties were noted in both the word problems and multiplicative calculation fluencies. The results from the pilot test led me to rethinking about some of the questions (especially with the number range) as well as including array type problems in the pre- and post-tests. Hence based on the results from the pilot test; the number range was adapted in some questions, contexts differed, fewer bald calculations were drafted and an extra category of problems (arrays) were included in the pre- and post-tests.

4.6.1.2. Pre- and post-tests

Pre- and post-tests were the main research instrument used in my study. These tests required careful scaffolding and thought which was informed from the results of the pilot. The test comprised of four multiplications and three division classes of word problems adapted from the Big Book of word problems. These classes were:

- | | |
|---|---|
| <i>a. Multiplication as Repeated Addition</i> | <i>e. Division as Sharing</i> |
| <i>b. Multiplication as Rate</i> | <i>f. Division as Grouping (repeated subtraction)</i> |
| <i>c. Multiplication as Scale</i> | <i>g. Division as Grouping (with remainder)</i> |
| <i>d. Multiplication as Array</i> | |

All the word problem chosen were drafted with an RME approach in mind. The contexts of the problems included situations that learners could relate to making the problems more realistic for them. All the problems that learners had to solve were set in contexts that were sufficiently ‘real’ to South African Grade 4 learners.

Superfluous information was also included in some of the questions to determine whether learners could select the appropriate quantities from irrelevant information. For learners to be good problem solvers, they should be able to identify and select appropriate quantities from irrelevant information. The way learners use numbers in problems containing superfluous information indicates whether they number grab when given irrelevant information.

Furthermore, like in the pilot test, I included one addition and one subtraction buffer question as well as three bald calculation questions and one multiple choice question. The number of bald questions were reduced in the main tests to four as compared to the eight questions in the pilot test (see Table 4.2).

In terms of the connections behind the numbers chosen, a variety of numbers were used which comprised of multiples of 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14 and 16. To avoid making the test very simple, I varied my number range to include single and multi-digit multiplication as well as division of three digit numbers by 1 digit.

The tests were also developed with the curriculum policy in mind therefore the problems incorporated contexts that learners were familiar with and could easily relate to. Moreover, the CAPS document emphasizes that contexts of word problems should incorporate social, environmental and economic contexts. Hence, some of the problems incorporated these contexts with items set with ‘nearness’ and familiarity to learners’ lives. This ties in with my theoretical framework of RME.

The kind of testing design approach that I used was a *Randomized Pre-test-Post-test Control Group Design* – figure 4.1 (McMillan & Schumacher, 2010). Learners were randomly classified into control and experimental groups with both groups writing the tests within the same period of time and intervention only given to the intervention group.

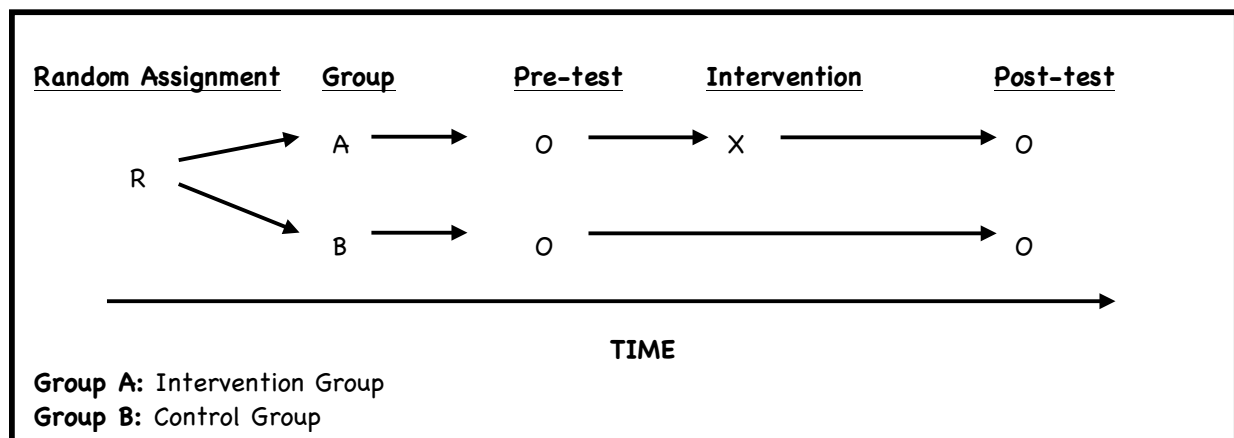


Figure 4.1: Randomized Pre-test-Posttest Control Group Design (McMillan & Schumacher, 2010)

The benefits of using a pre- and post-test were that I could ascertain learners’ achievement prior to intervention and then re-test them again after intervention (Bertram & Christiansen, 2014). Cohen et al. (2011) maintain that using a pre-test prior to an intervention program helps identify starting abilities and achievements in students as well as adds value to teaching and learning whereas a post-test assesses participants in an experiment after treatment (Creswell, 2012). Thus, post-tests are useful in determining whether the intervention lessons were at all beneficial. Table 4.1 and 4.2 provide summaries of the word problems and bald calculations asked in the tests.

CLASSIFICATION OF PROBLEMS	QUESTION NUMBER	QUESTION
Multiplication as Repeated Addition	7	Sam loves collecting bottle tops and straws. She collected 11 bottle tops on Tuesday, 11 bottle tops on Wednesday, 15 straws on Thursday and 11 bottle tops on Friday. How many bottle tops will Sam have collected?
Multiplication as Rate	2	Thabo planted 10 pots with magic beans. He planted 7 magic beans in each pot. How many beans did Thabo plant?
	11	There are 12 friends at an orange farm. They each collect 13 oranges in their bags. How many oranges do they have altogether?
Multiplication as Scale	9	Dexter has a magic bag of rice and potatoes. The bag of potatoes weighs 8kg and the bag of rice weighs 7kg. Dexter pours a growing potion and the bag of rice becomes 4 times as heavy. How much does Dexter's bag of rice now weigh?
	13	Leti and Ayesha have string. Leti's string is 14 metres long and Ayesha's string is 7 times as long. How long is Ayesha's string?
Multiplication as Array	10	Thulelah is packing rows of pies in a tray. The tray can fit 6 pies across and 3 pies down. How many pies can Thulelah pack in the tray?
	14	Lerato is tiling a school hall floor that is 16 metres long and 14 metres wide. How many tiles will Lerato need to tile the school hall if the tiles are 1m by 1m?
Division as Sharing	3	9 friends were given a bag of 72 sweets. They shared them out equally. How many sweets did they each get?
	6	Thando makes 120 bottles of lemonade for a school market day. The bottles of lemonade are shared equally between Thando and 5 of her friends. How many bottles of lemonade do they each get to sell?
Division as Repeated Subtraction	8	Mary has 96 biscuits that she wants to put in boxes. 12 biscuits can fit into one box. How many boxes can Mary fill?
	15	Sarah is packing T-Shirts onto shelves in a clothing store. One shelf can hold 44 T-Shirts. There are 88 T-shirts that need to be packed. How many shelves would be needed to pack all the T-shirts away?
Division as Grouping (with remainder)	12	66 children need to be transported to a sports match. The school has arranged for parents to take the children by car. If the cars can only take 4 children, how many cars will be needed to transport all the children?
	16	Jill has 45 eggs to take to the market. An egg box can hold 6 eggs. How many boxes will Jill need if she wants to take all the eggs to the market?

Table 4.1: Summary of word problems asked in the pre-and post-test

TYPE OF QUESTION	QUESTION NUMBER	CALCULATION
Multiple-Choice	17	37×9
Multiplication	18	$6 \times _ = 24$
Division	19	$46 \div 2 = _$
Equivalence	20	$2 \times 24 = 4 \times _$

Table 4.2: Summary of bald calculations asked in the pre- and post-test

4.6.2 Interviews

From an interpretivist perspective, interviews are a useful method of exploring and describing perceptions and understandings that are unique to people (Bertram & Christiansen, 2014). In qualitative research, it provides useful information when one cannot directly observe participants. It also permits participants to describe detailed information with the interviewer having better control over the types of information received (Creswell, 2012). The purpose of my interviews was to gain a better sense of how learners were thinking about and solving problems in the tests.

The interviews which I conducted were *semi-structured*, i.e. a set of predetermined questions were drafted with flexibility to probe and expand on the learners' responses seen on their test scripts. Three questions were used in each interview with expansions and adaptations being made depending on what learners did in the tests and how they responded during the interview. An example of the types of questions asked are: (a) *Can you talk me through what you have done here? I'd like to understand why you drew or wrote that?* (b) *I can't see how you worked this out? Can you describe the thinking that got you to this answer?* (c) *That must have taken you quite some time to work out, can you think of a quicker way to answer the question or an easier way of setting out your problem?* Semi-structured questions allow leeway from a pre-arranged text and allow the interviewer to change the wording of questions or the order in which they are asked (Opie, 2004).

4.6.3. Intervention Lessons

As mentioned earlier in the chapter, my research approach was broadly structured around an intervention based program drawn from the materials, pedagogical approach and lesson structure suggested in Askew (2004)'s Big Book of word problems relating to multiplication and division. In the teacher guides provided with the Big Books, Askew (2004) advocates a supportive classroom environment when word problems are taught. Teachers need to be able

to support learners in making sense of various problems through an exploratory approach whereby learners can advance their own models of the problems as well as gain an understanding of their peers' approaches in the classroom. To help learners focus on the mathematics in the word problems, Askew (2004) devises a four-part sequence to the lessons, which was followed by my intervention with the learners. These are:

1. Solving the Big Book problems
2. Linking up the problems
3. Follow-up problems
4. Wrap-up

In the **first part**, learners were required to solve three problems, one at a time either in pairs or individually. Between each problem, a class discussion was conducted on the ways in which the problems were interpreted and modelled. Learners were called up to the board to illustrate and explain how they solved the problems with the aim of having discussions around how to use different models. Particular emphasis in my intervention lessons was placed on building up to ratio models, given that the literature suggested the usefulness of these models in understanding multiplicative structures. Effective teaching encompasses direct teaching that makes effective use of unexpected and unforeseen opportunities for children's learning which includes both instructive and constructive approaches to learning (Kortenkamp, Brandt, Benz, Krummheuer, Ladel, & Vogel et al., 2013).

Therefore, in the context of the first problem, if any learner presented a skip counting model they were called up to the board with the intention of getting learners to think about and adapt the model in a way that related to the ratio model with the double number line as a starting point. Learners were initially shown the double number line to help them see the relationship between the variables in the problems. The double number line was first explored as learners in Grade 4 are generally exposed to the number line from addition and subtraction making it easier for them to make sense of. Once learners could make appropriate sense of the relationship of numbers in the double number line, the ratio t-table model was introduced and worked with throughout all the lessons.

In the scaling problems, scale diagrams were set up in addition to the t-tables so that learners would not only understand the language of scaling but also appreciate the difference between a change situation where change is the result of scaling (the original length was x times as

long) and a change situation where change is the result of addition (the original length was made x metres long). Similarly, in the array type problems, array diagrams were set up so that learners could picture the number of row and items per row gaining a better sense of the context of the problem and how the information could be represented in a ratio t-table. These approaches are reaffirmed by Askew (2004, p.7) who mentions that teachers' concentration should be on supporting learners' reasoning through "clarifying questions, setting up models, pictures or diagrams" which if learners' interact with appropriately, their performances may be enhanced (Ainsworth, 2006).

In the **second part** of the lesson structure, learners were required to link up the problems through finding common mathematical structures that underline each of the three problems. This was so that learners could classify the problem types according to the root situations through guidance from the teacher and so that they could see patterns of numbers particularly 'halves', 'doubles' and 'multiplying by 10 and/or 5'. Seeing these patterns was aimed at developing learners' number sense as part of a route to better performance in multiplicative situations.

In the **third part**, follow up problems were introduced for learners to work more independently on the ideas introduced in the lesson with the primary aim being to consolidate the concepts taught in the lessons. (Askew, 2004). And lastly, the **fourth part** was a wrap-up of the lesson whereby the class came back together to explore the nature of the problems and reinforce key underlying ideas (Askew, 2004).

My intervention program thus aligned with the tenets of RME as models and/or strategies were not imposed on learners but rather they were encouraged to reflect, compare and try out different modelling approaches with the aim of finding an approach that they could take up.

The pre- and post-tests were conducted ten weeks apart. Intervention lessons were approximately an hour long and spread over eight weeks with one lesson per week to incorporate and introduce different multiplicative word problems to learners. Table 4.3 provides a summary of the objective(s) of each intervention lesson.

INTERVENTION LESSON	OBJECTIVE(S) OF THE LESSON
1	- Recognize 'multiplication as rate' problems - Use the double number line to solve problems
2	- Recognize 'multiplication as repeated addition' problems - Use the ratio table to record information
3	- Recognize 'multiplication as scale' problems - Use the ratio table to solve scaling problems - Use and apply knowledge of multiplication bonds
4	- Recognize 'multiplication as array' problems - Use the ratio table to record information
5	- Recognize 'division as repeated subtraction' problems - Use the ratio table to record information and find the answer to the problem
6	- Recognize 'division as sharing' problems - Identify the relationship between multiplication and division
7	- Recognize 'division as grouping' problems - Deciding whether to round up or down an answer
8	- Recognize different multiplicative problems (a mix of multiplication and division) - Use the ratio table to record information and find the answer to the problem

Table 4.3: Summary of the objective(s) of intervention lessons taught

Word problems were typically presented on A3 sheets with learners' models presented alongside the problems. An emphasis was placed broadly on providing learners the chance to evaluate emerging models with my support. This approach together with the lesson structure was to assist the children in critically thinking about the mathematics involved as well as possibly moving them from being novice to expert problem solvers.

All the intervention lessons were videotaped to provide a record of the way the intervention lessons were conducted and to assist me in describing what occurred during the lessons. A copy of the lesson plans to intervention lesson 1 and 2 is provided in Appendix A.

4.7 Data Analysis

4.7.1 Analysis and presentation of tests

Both the pre-and post-test were marked and analysed by myself in this study. Broadly recorded, a *summed* and *difference score* approach was used to assist in analysing the results

from the tests. Each question was assigned a score of either 1 for a correct answer or 0 for an incorrect answer. *Summed scores* focused on the total scores added over several questions (Creswell, 2012). Scores were then used to develop an overall test score that could compare the pre-and post-test. *Difference scores* assisted in determining whether any shifts have been made from the pre-to post-test as they represent a change or difference for individuals (Creswell, 2012). All but one multiplicative word problems were analysed with scores being represented on bar graphs so that shifts in the overall performances could be seen across both the control and intervention groups. The problem omitted from the analysis was a multiplication as rate question. This question was included in the tests, but its wording, which included a yes or no option coupled with a request for an explanation was poorly understood, often eliciting a yes or no answer which made it difficult to interpret in terms of awareness and/or use of models.

To gain insight to the models that learners used prior and post intervention, a set of codes were developed according to the literature base on widely used models, and included the new ones that were introduced in the intervention lessons so that inferences could be made as to whether any gains were made. An *open coding* approach was adopted whereby learners models of the problems were labelled, defined and categories developed based on their properties and dimensions (Khandkar, 2009). Creswell (2012) views this process as an inductive process of narrowing data into few themes. Models were coded and listed in hierarchical order ranging from informal intuitive strategies to more formal symbolic models such as horizontal number sentences, column and ratio models. A description of the codes used for the multiplication and division problems in the tests are provided in Table 4.4 (a) and (b).

CODES	DESCRIPTION
No Model (Incorrect)	Learners who presented an incorrect answer but no model to the problem.
Incorrect Model	A completely incorrect model to the situation in the problem. All the incorrect models presented by learners resulted in an incorrect answer.
Diagrammatic Representation	For this category, learners displayed a diagrammatic model which was either an iconic or an indexical representation of the multiplication problems. <i>An iconic representation</i> included images of everyday contexts – generic rather than realistic depiction of everyday contexts with apparatuses including drawings of some sort (Ensor et al., 2009). <i>Indexical representations</i> indexes everyday contexts – generic rather than realistic depictions of everyday contexts with apparatuses featuring drawings of sticks, tallies, dots, circles and other shapes to depict everyday objects (Ensor et al., 2009).
Symbolic Repeated Addition	Symbolic models of repeated addition which were either in the form of a horizontal calculation sentence and/or as a column addition model.
Horizontal multiplication sentence or Column Multiplication	Models of multiplication which were either in the form of a horizontal calculation sentence and/or as a column multiplication algorithm. In questions that required doing multiplication with two-digit by two-digit numbers, learners who presented a long multiplication column algorithm were also coded under this category.
Ratio Models	Ratio models were introduced to learners in the intervention lessons with the aim of getting them to identify the relationship between the numbers in the problem as well as foster multiplicative reasoning. This code was created for learners who presented either a double number line and/or a ratio t-table. The double number line enables children to make links between numbers, developing a powerful understanding of multiplication (Küchemann, Hodgen, & Brown, 2011a). Whereas the ratio t-table covers all situations where there is a direct proportion involved (Greer, 1992).
No Model (Correct)	Learners who presented a correct answer without a model to the problem. It was assumed that learners either calculated the answer mentally or another on sheet of paper other than the test.

Table 4.4 (a): Description of codes for multiplication word problems

CODES	DESCRIPTION
No Model (Incorrect)	The description to this code is the same as the multiplication description where learners presented no model and just an incorrect answer.
Incorrect Model	A completely incorrect model to the situation in the problem. All the incorrect models presented by learners resulted in an incorrect answer.
Diagrammatic Representation	A diagrammatic model (either iconic including images of everyday contexts or indexical including indexes of everyday contexts).
Idiosyncratic division model	A distinct setup of the division problem which differed from the standard algorithms.
Horizontal division sentence or Traditional Division	Models of division either in the form of a horizontal calculation sentence and/or as a standard division model.
Ratio Models	Like with the multiplication problems, this code was created for learners who presented either a double number line and/or ratio table. According to Greer (1992, p.283), multiplication and division problems can be seen as special cases of rule-of-three problems (i.e. problems involving two equal ratios) in which one of the terms is 1; depending on which of the three remaining terms is unknown, the problem is solved by multiplication or division of one type or the other.
Other	Learners who presented either a skip counting model, a compensation model or a column addition or subtraction model.

Table 4.4 (b): Description of codes for division word problems

The codes developed were presented in tables using excel so comparisons from the pre- to post-test could be made. Codes in the tables were hierarchically organized with no model (incorrect) being the lowest cognitive model presented and the symbolic models (horizontal number sentences, column and ratio models) being more advanced efficient models. As detailed in the literature, early studies (Anghileri, 1992; Kouba, 1989; Mulligan and Micheltmore, 1997) have shown that children's solution strategies generally begin with direct modelling and unitary counting, progress' to skip counting, double counting and repeated addition or subtraction, and then progress' to the use of known multiplication and division facts (Downton, 2008).

The frequency of the model uses and the number of correct answers in both tests across all the categories of problems were shown by means of tables. Tables are a convenient method of displaying data and helps summarize large amounts of data in a small amount of space (Creswell, 2012; Opie, 2004). An example of one table used is shown below with figures in the brackets denoting the percentages of the frequency of models used and the percentages of correct answers per model in both tests. Percentages shown were calculated in relation to the whole group.

TABLE SUMMARY OF MULTIPLICATION AS SCALE QUESTIONS (INTERVENTION GROUP; n= 37)									
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS					
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Symbolic Repeated Addition	Horizontal Number Sentence or Column Multiplication	No model Correct
9 Dexter has a magic bag if rice and potatoes. The bag of potatoes weighs 8kg and the bag of rice weighs 7kg. Dexter pours a growing potion and the bag of rice becomes 4 times as heavy. How much does Dexter's bag of rice now weigh?	7 ✖ 4	PRE-TEST	FREQUENCY	5 (14%)	24 (65%)	0 (0%)	2 (5%)	5 (14%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	2 (5%)	3 (8%)	0 (0%)
		POST-TEST	FREQUENCY	0 (0%)	18 (49%)	0 (0%)	4 (11%)	18 (49%)	2 (5%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	3 (8%)	18 (49%)	2 (5%)
13 Leti and Ayesha have string. Leti's string is 14 metres long and Ayesha's string is 7 times as long. How long is Ayesha's string?	14 ✖ 7	PRE-TEST	FREQUENCY	8 (22%)	14 (38%)	1 (3%)	0 (0%)	14 (38%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	0 (0%)	6 (16%)	0 (0%)
		POST-TEST	FREQUENCY	0 (0%)	6 (16%)	0 (0%)	3 (8%)	22 (59%)	5 (14%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (3%)	19 (51%)	2 (5%)

Table 4.5: Example of one table summary of how questions were coded

Table summaries were created for both the control and intervention groups with bar graphs presented to show summaries of learner performances in the pre- and post-tests as well as any shifts that occurred from one test to another. In my analysis, I discuss the overall performance of both groups however I do not go in-depth into the results of the control group but rather draw on comparisons between the groups where needed. The main reason for this was because there were fewer shifts in the model use of the control group between pre- and post-tests which is predictable given that they did not have access to the intervention focused on key multiplicative models.

4.7.2. *Analysis of Interviews*

All the interviews conducted were audio-recorded to provide an accurate record of the conversation between the interviewer and interviewee (Cohen et al., 2011; Creswell, 2012; McMillan & Schumacher, 2010; Opie, 2004). Notes were carefully taken during the interviews so that central facts could be noted. Written work by learners was also collated so that I could see how some of the problems were solved (in instances where no models and/or strategies were shown). Notes were also taken based on what learners were pointing to or if any hand gestures were used. Since my interview data was used as a secondary source to elaborate on the quantitative data, the audio-recorded interviews were not transcribed in detail and transcription notes taken in the interviews were instead used. These notes were sufficient in assisting me to enhance my discussion on the model's learners presented.

4.8 Rigor

In any study, there needs to be a set of criteria which determines the quality of the research being conducted and/or presented. This set of criteria includes *validity* (or internal validity), *reliability* and *generalizability* (or external validity). Drawing on the work of different authors, these criteria can be broadly described as follows: validity looks at the credibility of data against what the researcher claims in the study i.e. the truthfulness of one's study (Bell, 2005); *reliability* is the extent to which a test or procedure produces similar results under constant conditions on all occasions (Bell, 2005); and *generalizability* looks at the extent to which the results of one study can be used as knowledge about other populations (McMillan & Schumacher, 2010). These criteria are thus important concerns for those interested in the outcomes of the research study and should be carefully considered.

4.8.1. *Validity*

In terms of *validity* – data itself does not ensure validity. Rather validity relies on the relationship between the claim and the data gathering process which measures what it is truly supposed to measure (Opie, 2004; Wellington, 2000).

Adopting an experimental approach, required consideration of certain risk factors. Pre-and post-tests pose potential threats of history, testing, instrumentation and regression. With this in mind, data instruments were carefully thought through to minimize these potential threats. To ensure validity in the data collection process, tests administered to the Grade 4 learners were level appropriate, non-biased (contexts were relevant to South African

children), non-discriminatory and neither very long nor too short (Cohen et al., 2011). Since the focus of my study was on exploring mathematical models that learners presented to word problems, I tried to ensure that questions were clearly worded with varied number ranges. Pilot tests were also administered prior to the actual tests being administered to ascertain whether the questions were grade appropriate thus ensuring validity.

Moreover, the use of the same test for the pre- and post-test also posed a threat to validity. The decision to use the same questions in both tests allowed for direct measurement of the effectiveness of the intervention to see if learner performances improved. To ensure that the marks did not increase based on practice-effect, the tests were written ten weeks apart and learners were not informed that the content in both tests would be the same.

From a qualitative research approach, there are some limitations and validity issues which need to be factored in. In this study, I was the only researcher which meant that the quality of my study was largely dependent on my own skills opening room for bias and subjectivity. Qualitative research can be superficial when information is presented according to themes or categories with a thin interpretation from the researcher (Henning, Van Rensburg, & Smith, 2004). However, to avoid this, my interpretation of the categories is explained in detail, and were discussed with my supervisors and the broader research team which I am in. Moreover, to enhance the accuracy of my study, 'triangulation' was applied by using different types of data and methods of data collection. Creswell (2012) states that triangulation affirms accuracy in research as it draws on multiple sources of information encouraging researchers to develop a report that is both accurate and reliable.

4.8.2. Reliability

Reliability in one's research is vital as it ensures confidence and credibility. Reliability is synonymous with consistency and replicability over time, instruments and groups of respondents (Cohen et al., 2011).

Since I used the test/re-test method with three classes in the same school with learners of a similar age, I expected the results to be comparable across the classes. Using the test/re-test method required some care in terms of the timing between the tests thus the pre- and post-test were conducted about eight weeks apart so that the children could not rely on what they had remembered (Cohen et al., 2011). Furthermore, clear instructions were provided in the tests, and questions were clearly worded to avoid any misinterpretations. Test anxiety was avoided

by reassuring learners that the tests were not for marks and the tests were also administered by myself to ensure that the data was collected under similar circumstances with all the classes.

Moreover, because reliability is the extent to which a test, measure or instrument can be repeated with the same or similar group of respondents and still produce the same (or similar) results (Bertram & Christiansen, 2014), it would be incorrect to claim that my study was 100% reliable as it was difficult to replicate all the conditions of the small-scale study, however, a string of prior results using an intervention-lesson model, indicate that such models show promise.

Classroom-based studies can also be difficult because there will never fully be constant conditions. Attrition within the groups of learners occurred as there were learners who sat for the pre-test but were absent for the post-test or vice-versa. Matched comparisons of the pre- and post-tests analysis were based on the learners who wrote both tests.

And finally, since my classroom dataset was relatively small-scale, the dependability of the claims made in my study are limited. However, it can be seen that test instruments were valid and reliable, making it sufficient to claim that this study and the results thereof meet the requirement of validity enabling me to tell a story of the models presented by Grade 4 learners in multiplicative word problem situations both before and after intervention.

4.8.3. Generalisability

Although there is no need to provide results which can be generalized, the findings from one's study can have important implications for personal practitioners or others working in similar areas (Opie, 2004). The aim of my study was not to make broad claims or generalizations but rather to gain insight into the way Grade 4 learners reasoned multiplicatively in word problems. Therefore, findings are based more on the relatability thereof i.e. how can it be related to what is happening in other classrooms or the degree to which a teacher teaching Grade 4 mathematics can use the outcomes of this study to inform his/her own practices.

4.9 Ethical Considerations

To ensure the safety of participants in research, gaining an ethics clearance is crucial (Bell, 2005). Researchers are ethically responsible for protecting the rights and welfare of the

subjects who participate in the study thus no research can be led without an ethics clearance being sought (McMillan & Schumacher, 2010).

Before initiating this research study, I applied and received clearance from the Human Research Ethical Committee. This application was put through in early 2016 and protocol number **2016ECE010M** was provided. The ethics clearance letter is provided in Appendix B.

In addition to the universities ethics committee, an application was also submitted to the Department of Education informing them about the nature of my study. The school governing body, principal, parents and learners were also informed about this study and permission to conduct my research was sought from all the relevant parties. Official letters describing the nature of my study was provided to these parties with learners in the school being verbally told what their participation would entail. Since this study was conducted during school hours, relevant parties were informed that the lessons and tests would be aligned to the CAPS ensuring that learners would not lose out on too much of teaching time.

Learners and parents were informed that participation was voluntary and that learners could withdraw at any time without any penalty. It was also made explicit that no monetary rewards would be given for participating. Moreover, learners were affirmed that the information gained in my study will be kept safely for a five-year period before being destroyed.

Learners' names were also kept anonymous to protect their identity. Pseudonyms were used in instances where names were needed. All this information was delivered to learners and parents to examine at home before completing and returning the consent forms. Copies of the information letters and consent forms can be found in Appendix C and D.

Finally, the use of a quasi-experimental design poses specific ethical considerations that need addressing. The use of a control group raised the issue of fairness with regards to gaining access to the same lessons. Prior to conducting the study, the learners in the control group were informed that should the intervention show positive gains, the same lessons would be conducted with them by myself or their maths teacher with all the lessons being readily available to the teacher. Therefore, since the results of the intervention group showed positive gains, all lesson plans that were used for the intervention group were passed on to the Grade 4 teacher with explanation and assistance provided by me so that the teacher could conduct the same lessons with the control group.

4.10 Conclusion

Based on the methodology discussed in this chapter, it is evident that a combination of research instruments was used such as testing (pilots, pre- and post-tests) and interviews. Using an experimental and exploratory approach was effective in collecting text-rich data which are analysed in the next chapter.

Tests and intervention lessons were the main aspects to consider especially in determining whether learners were able to develop better multiplicative reasoning over a period of time. The next chapter will indicate whether and to what extent the intervention lessons played a role in learning multiplication and division. Intervention lessons required careful planning which aligned with the needs of the participants. Intervention that is carefully thought through and purposefully directed, should result in positive experiences for the learners. In this study, intervention lessons were aimed at helping Grade 4 learners to better their understanding of multiplicative concepts in mathematics with tests being administered prior and post-intervention. In the next chapter, I present a discussion on the findings of this study.

Chapter Five: Analysis and Findings

5.1. Introduction

The purpose of my study was to explore and develop multiplicative reasoning with Grade 4 learners through a small-scale intervention focused on lesson sequence based problem solving. Multiplicative reasoning was explored by examining the models and strategies used by learners across a range of multiplicative word problems (multiplication and division).

The aim of this chapter is to present the findings of my study in relation to the models and strategies used based on the results from a pre-and post-test as well as interviews that were conducted immediately after tests were written. Learner responses on the pre-test were analysed to understand the kinds of model's learners presented for solving multiplicative word problems prior to a small-scale intervention. During the series of intervention lessons, learners were introduced to ratio models with the aim of developing multiplicative reasoning. The post-test results were then analysed to determine the shifts (if any) in the use of models and the gains (if any) achieved in learner performance. A control group was used to assist in determining whether the intervention program was beneficial to learners in showing differential effects in terms of overall performance, and modelling approaches. Interviews conducted immediately after the tests provided insight into how certain questions were answered including the reasons for learners' choices of the models used in the tests.

Data were gathered from 61 learners (across the intervention and control groups) which included pre-and post-tests and audio-recorded interviews with a sample of 15 learners. These results were then quantitatively analysed and presented by means of bar graphs and tables. Models were coded to assist in analysing and making inferences of the shifts noticed.

At the beginning of this chapter, I present the overall results of my study. Thereafter, table summaries and graphs are provided to show detailed analysis and discussion of the models used in the pre-and post-test with the intervention group learners in specific categories of problems. I then summarise the shifts in model use and the success rates associated with the different models as well as provide insight into the performances of the bald calculation questions so that a broader sense of the impact of the intervention lesson sequence could be determined.

5.2. Overall average performance in the pre-and post-test in the intervention and control groups

GROUP	NUMBER OF LEARNERS PER GROUP (n)	PRE-TEST MEAN (%)	POST-TEST MEAN (%)	GAINS (%)
Intervention	37	32,05	52,90	+ 20.85
Control	24	38,69	43,45	+ 4.76

Table 5.1: Overall average performance in the pre-and post-test (percentages based on 13 items)

Table 5.1 shows the overall mean performance score across the pre-and post-tests that were written by learners who formed part of both the intervention and control group. The results from the pre-test conducted reveal that learners in both groups initially scored poorly (below 40%) with learners in the intervention group performing more poorly than those in the control group. However, the post-test results indicate substantial change for the intervention group, with a much smaller gain in the control group. The results of the intervention group show that a 20.85%-point gain was achieved with the average score of the group standing at 52.90% in the post-test. Learners who were part of the control group had average test scores which were relatively similar across both tests with a gain of approximately 5% points. This gain was significantly smaller than the gain seen in the intervention group.

Therefore, the differences in gains achieved suggested that the intervention model and content of the lesson sequences were successful, thus making it useful to look in more detail at the nature and extent of differences in ways of working on problem-solving that could underlie the different shifts in performance.

In the following section, I compare learners' gains in the pre-and post-test across the various multiplicative word problem categories for the intervention and control groups.

5.3. Outline of learner performance in the intervention and control group

In this section I outline learner performances in the intervention and control group across the pre-and post-test (Figure 5.1 (a)) and indicate learner gains across both tests in the control and intervention groups (Figure 5.1 (b)).

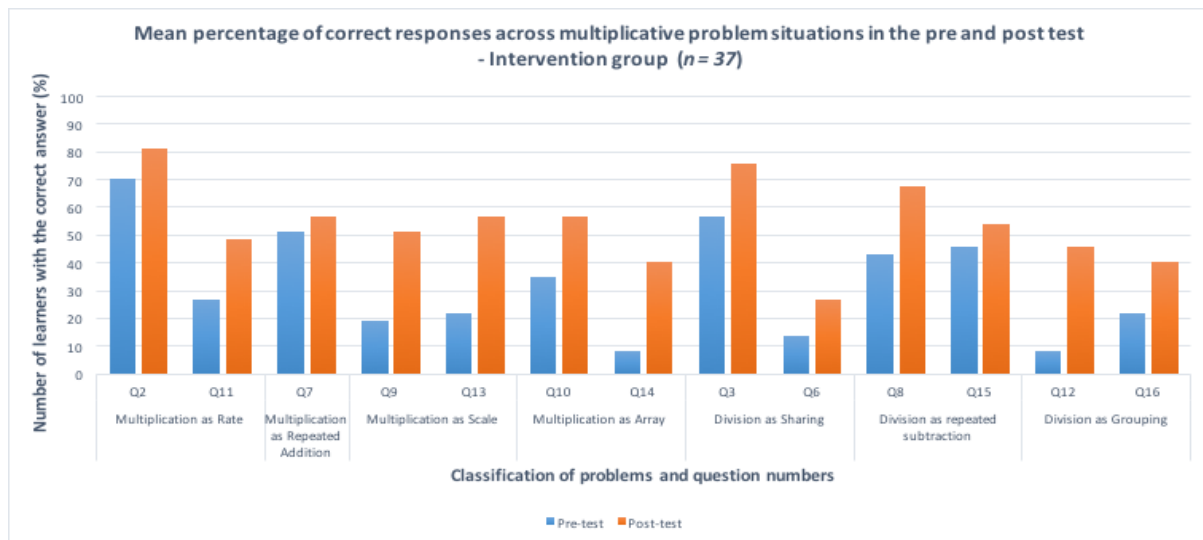


Figure 5.1(a): Overall performance in the pre-and post-test across different multiplicative problems (Intervention Group)

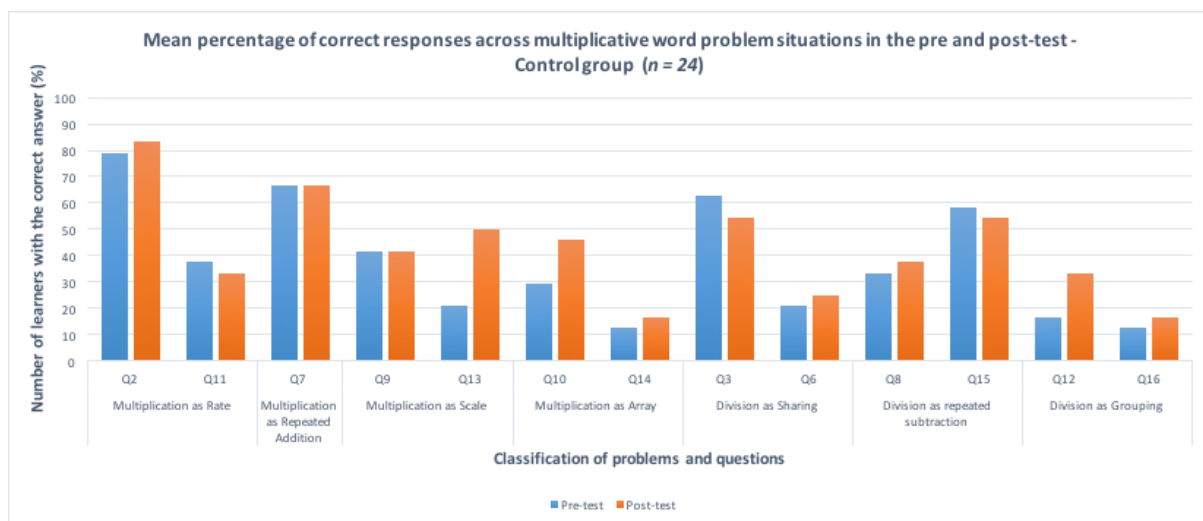


Figure 5.1(a): Overall performance in the pre-and post-test across different multiplicative problems (Control Group)

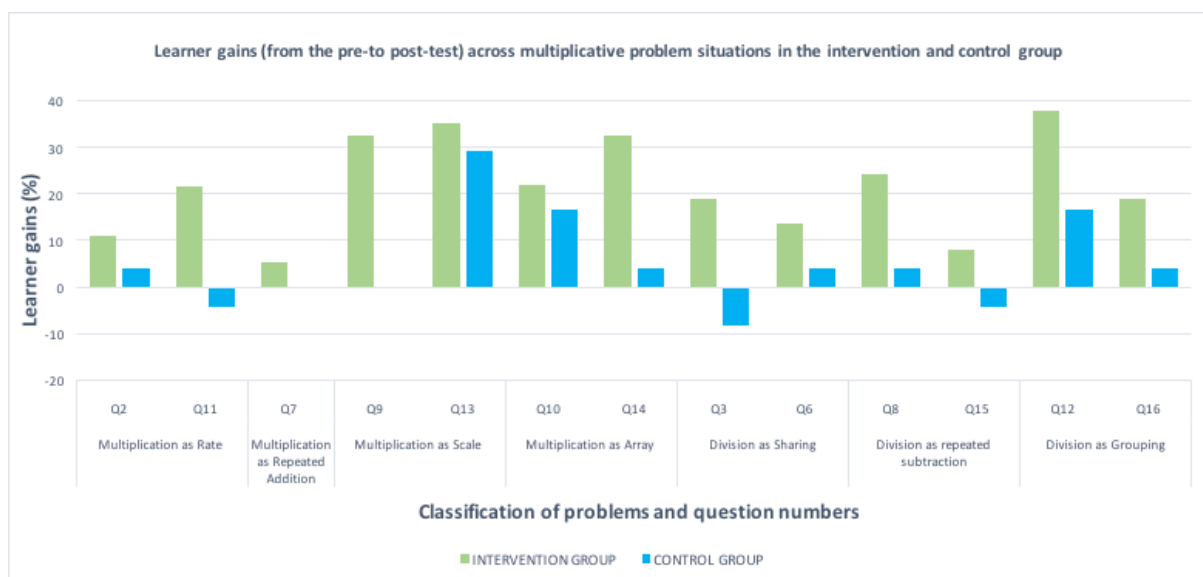


Figure 5.1(b): Learner gains (from pre-to post-test) across multiplicative problem situations in the intervention and control groups

Figure 5.1(a) presents an overview of learners' performance across specific multiplicative word problems asked in both the pre-and post-test for the intervention and control group. In the pre-test with both groups, the scale, array and division as grouping questions were poorly answered. However, in the post-test, learners who received intervention showed stronger gains in these categories than the learners in the control group.

To show the gains achieved across all the problem types in both groups, a third bar graph (Figure 5.1 (b)) was created. The *y-axis* illustrates learner *gains* as a percentage point increase across the classification of word problems and the question numbers. The bars signify specific gains achieved in each question with green bars representing the gains in the intervention group and the blue bars representing the gains achieved in the control group (including negative gains).

Figure 5.1(b) illustrates that within the ***intervention group***, substantial gains were noted in the *multiplication as scale*, *multiplication as array*, *division as sharing* and *division as grouping* questions. Questions 8 (24%), 9 (32%), 11 (22%), 12 (38%), 13 (35%) and question 14 (32%) all yielded an increase of over 20% points from the pre-to post-test. Question 12, with the largest gain, was a division as grouping item *with a remainder* requiring learners to provide a suitable, rounded answer rather than the calculation answer. The scaling questions (question 9 and 13) also produced strong gains of over 30% points indicating a better performance and sense making of these problems.

For the ***control group***, although a small overall increase in the general test score was noted (see Table 5.1), shifts in individual questions portray a different picture. Questions 3, 11 and 15 saw a drop from the pre-to post-test with learner performance decreasing the most in question 3 (-8% points). No gains were noted in questions 7 and 9 whilst questions 2, 6, 8, 13, 14 and 16 showed small gains of about 4%. Questions 10 and 12 showed positive gains of approximately 17%.

The stronger overall gains in the intervention group across questions suggests that this group benefitted from the lessons with substantial gains being made on most of the items. A focus on different modelling approaches (i.e. ratio models) in the intervention lessons that allowed learners to explore and develop different ways of multiplicative reasoning in the post-test compared to the pre-test could be associated with the improved performance for the

intervention group. This association is investigated further in the study of models used across the pre- and post-tests by the intervention group with comparisons being made to the control group where appropriate.

My main focus was on the areas where substantially large gains were achieved so that I could gauge the usefulness of the intervention structure. In my analysis that follows, I concentrate particularly on five multiplicative categories where substantial gains were made by the intervention group and draw on comparisons of the control group where needed. These are: (i) multiplication as scale, (ii) multiplication as array, (iii) division as sharing, (iv) division as repeated subtraction and (v) division as grouping questions. The aim is to understand the underpinning to these strong gains with the focus, as the RME theory suggests, on examining learners' model use. I thereafter look more broadly at the other categories of questions. A comprehensive table summary of the results from the control group can be found in Appendix E.

5.4. Classes of word problems with substantial gains

In this section, I present analyses and discussions of the models used by students in the intervention group for each category of problems. The results are shown by means of tables with codes for various models. The frequency of the models used and the percentage of correct answers are presented alongside. Percentages indicated in the tables are rounded off to the nearest whole number.

Broad findings from my analyses suggest that learners in the intervention group gained a better sense of problems in the post-test which were initially poorly answered in the pre-test. There was also an emerging take up of ratio models (double number lines and t-tables), models that were widely discussed in the intervention lessons and not used at all in the pre-test. Given that in the control group, no ratio models emerged in the post-test and sense making did not improve as much as it did in the intervention group, these results point to both a lack of use of the ratio model in the normal classroom experiences of these learners, and the usefulness of this model for increased success in multiplicative problems.

5.4.1. Multiplication as Scale

Askew (2004) describes ‘multiplication as scaling’ problems as being made up of a continuous quantity that is increased in size by a scale factor. Table 5.2 provides a summary of the models presented by the intervention group in the multiplication as scale questions.

TABLE SUMMARY OF MULTIPLICATION AS SCALE QUESTIONS (INTERVENTION GROUP; $n = 37$)										
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS						
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Symbolic Repeated Addition	Horizontal Number Sentence or Column Multiplication	Ratio Model	No model Correct
9 Dexter has a magic bag of rice and potatoes. The bag of potatoes weighs 8kg and the bag of rice weighs 7kg. Dexter pours a growing potion and the bag of rice becomes 4 times as heavy. How much does Dexter's bag of rice now weigh?	7 ✖ 4	PRE-TEST	FREQUENCY	5 (14%)	24 (65%)	0 (0%)	2 (5%)	5 (14%)	0 (0%)	1 (3%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	2 (5%)	3 (8%)	0 (0%)	1 (3%)
		POST-TEST	FREQUENCY	0 (0%)	18 (49%)	0 (0%)	4 (11%)	18 (49%)	2 (5%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	3 (8%)	18 (49%)	2 (5%)	0 (0%)
13 Leti and Ayesha have string. Leti's string is 14 metres long and Ayesha's string is 7 times as long. How long is Ayesha's string?	14 ✖ 7	PRE-TEST	FREQUENCY	8 (22%)	14 (38%)	1 (3%)	0 (0%)	14 (38%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	0 (0%)	6 (16%)	0 (0%)	0 (0%)
		POST-TEST	FREQUENCY	0 (0%)	6 (16%)	0 (0%)	3 (8%)	22 (59%)	5 (14%)	1 (3%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (3%)	19 (51%)	2 (5%)	1 (3%)

Table 5.2: Table summary of the models presented by the intervention group for the multiplication as scale questions in the pre-and post-test

The overall pre-test results for the scaling questions suggest that learners did not perform well in these questions. 16% of the of the intervention group got the solution to question 9 and 12 correct in the pre-test. Most learners who were not successful in solving the scaling questions, presented an incorrect model. Table 5.2 reveals that 65% and 38% of the intervention group presented an incorrect model whereas 14% and 38% of the learners presented a symbolic multiplication model for questions 9 and 13 respectively.

Incorrect models were largely associated with additive models demonstrating difficulties with translating the problem efficiently at a horizontal mathematization level. Learners defaulted to using an additive operation as they failed to make appropriate sense of the ‘scaling’ situations. In question 9, incorrect models were typically presented as either $(8 + 4)$, $(7 + 4)$, $(8 + 7)$ or $(8 + 7 + 4)$. Learners who presented these models also did not select the correct numbers in the problem. The superfluous information presented in question 9 contributed to 20% of the learner’s ‘number grabbing’ (Hegarty et al., 1995), indicating problems at a horizontal mathematization level as learners incorrectly selected numbers making poor sense of the problem.

As with question 9, incorrect models in question 13 were also associated with additive models $(14 + 7)$. Whilst question 13 did not contain superfluous information, learners

struggled to understand the scaled nature of the relationship between the numbers in the problems - Ayesha's string was **seven times as long**. Nunes & Bryant (2009) mention that problems which involve relations are more difficult than those that involve quantities. This is also reaffirmed by Van Dooren, De Bock, & Verschaffel (2010) who explain that additive errors like these are typical in younger children with limited instructional experience with multiplicative relations in proportional situations. The common interpretation of both scaling questions was that the bag of rice became 4 kilograms heavier and the string was 7 metres longer.

Moving on to appropriate models, learners who presented an appropriate multiplication model either presented a horizontal calculation sentence of 8×4 (*question 9*) and 14×7 (*question 13*) and/or a column multiplication model of:

$$\begin{array}{r} 8 \\ \times 4 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{r} 14 \\ \times 7 \\ \hline \end{array}$$

Whilst these models were correct models of the situations, the success rate was not good especially in question 13 where less than half of the learners who presented a symbolic multiplication model got the answer correct (16%) in the pre-test. Here, vertical mathematization problems were evident as learners inaccurately calculated 14×7 . There is little evidence in the test scripts to suggest why these errors occurred. However, based on interview data, when learners were asked to talk through what they had done in question 13, learners who were interviewed said that they “*plussed 7 each time*” or “*added 14 each time*” in their minds to get to the answer. Learners also said that this (‘plussing in their minds’) was easier because they did not know their seven times table. Careless calculation errors were also seen when learners tried to add 14 or 7 repeatedly. The popularity of repeated addition either as a tool to work out the answer to the symbolic multiplication model or as a model on its own suggests a lack of fluency with times tables and calculation errors occurred when trying to repeatedly add in their minds.

The post-test data on the other hand indicated considerable gains in both scaling questions (+30% points). Approximately 60% of the learners got the solution to both questions correct. The post-test data revealed a decline in the number of incorrect models from 65% – to 49% in

question 9 and from 38% – to 16% in question 13 as well as large increases in the use of horizontal and/or column multiplication models, with high proportions using these successfully. A take up of ratio modelling was also seen in the post-test data with a small number of learners (5% - question 9 and 14% - question 13) presenting ratio models.

During the intervention lessons, awareness was directed to the nature of the scaling questions and specific language such as *scale factor*, *so many times bigger than* (*longer than*, *heavier than*, *as many as*) was emphasized and developed. The shifts made from the pre-to post-test suggest that this emphasis was useful, with limited changes in the control group on these items backing this interpretation (*refer to Appendix E*). Haylock & Cockburn (2013) emphasise the importance of scaling structures as they form an important network of connections for multiplication especially as learners move into mathematical topics that involve the notion of scaling such as percentages, scale drawings or map work.

Furthermore, whilst traditional symbolic multiplication models (horizontal and/or column multiplication) was not explicitly taught in the intervention lessons, learners tended to resort to using this model with more efficiency as all the learners who presented these models for question 9 had a correct solution and 19 out of 22 learners in question 13 got the solution correct. These changes indicate that after intervention, learners had gained better fluency with the models they were already familiar with and based on what was taught during the lessons, familiar models were better internalised and successfully used.

In terms of the ratio models, t-tables and double number lines were also presented by learners in the post-test. The double number line provides a tabular way of representing and embodying multiplication as scaling problems enabling learners to make links between the variables as well as the multiple patterns within the variables, developing a richer and more powerful understanding of multiplication (Küchemann et al., 2011). The double number line is also a useful model to use since it directly models scale situations however, not many learners resorted to using these models for the scaling items. Two learners presented a ratio model in question 9 whereas five learners presented this model in question 13 with only two learners getting the solution correct. Although the number of learners who presented ratio models was low, it was interesting to see where learners went wrong. The example in *Figure 5.2* shows how the double number line was used to model question 13. In this example, the child displays an understanding of the double number line at a horizontal mathematization

level with a correct model of the situation set up. However, at a vertical mathematization level from 7×7 , inaccuracies in the learner's calculation emerge. Difficulties are seen with the fluency of skip counting in multiples of 7 and as a result, the child gets to an incorrect answer of $7 \times 8 = 97$. This presentation of the double number line (ratio model) can be also a tedious process especially in cases with a larger number range.

13. Leti and Ayesha have string. Leti's string is 14 meters long and Ayesha's string is 7 times as long. How long is Ayesha's string?

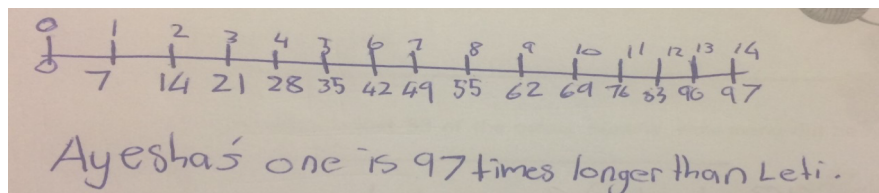


Figure 5.2: An illustration of a double number line (ratio model) in the post-test

5.4.2. *Multiplication as Array*

To ensure learners understand the commutativity principle of multiplication, array type problems were included. Arrays provide children the opportunity to arrange and restructure sets of objects which allow them to explore the commutativity rule of multiplication (Anghileri, 2006b; Barmby et al., 2009; Greer, 1992; Haylock & Cockburn, 2013; Hurst, 2015; Hurst & Hurrell, 2014; Siemon et al., 2005).

In this group of problems, the overall performance was low in the pre-test across both groups of learners. In the intervention group 34% and 8% got the solutions to question 10 and 14 correct whereas similarly, in the control group, 34% and 12% got the solutions to these questions correct. Table 5.3 provides a summary of the models presented across the array questions by learners in the intervention group.

TABLE SUMMARY OF MULTIPLICATION AS ARRAY QUESTIONS (INTERVENTION GROUP; $n = 37$)									
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS					
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Symbolic Repeated Addition	Horizontal Number Sentence or Column Multiplication	No model Correct
10 Thuleleh is packing rows of pies in a tray. The tray can fit 6 pies across and 3 pies down. How many pies can Thuleleh pack in the tray?	6 * 3	PRE-TEST	FREQUENCY	5 (14%)	18 (49%)	3 (8%)	0 (0%)	9 (24%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	2 (5%)	0 (0%)	9 (24%)	0 (0%)
		POST-TEST	FREQUENCY	5 (14%)	9 (24%)	4 (11%)	0 (0%)	16 (43%)	8 (22%)
			CORRECT ANSWER	0 (0%)	0 (0%)	4 (11%)	0 (0%)	16 (43%)	8 (22%)
14 Lerato is tiling a school hall floor that is 16 metres long and 14 metres wide. How many tiles will Lerato need to tile the school hall if the tiles are 1m by 1m?	16 * 14	PRE-TEST	FREQUENCY	5 (14%)	26 (70%)	0 (0%)	0 (0%)	6 (16%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	0 (0%)	3 (8%)	0 (0%)
		POST-TEST	FREQUENCY	4 (11%)	10 (27%)	4 (11%)	0 (0%)	20 (54%)	5 (14%)
			CORRECT ANSWER	0 (0%)	0 (0%)	3 (8%)	0 (0%)	14 (38%)	2 (5%)

Table 5.3: Table summary of the models presented by the intervention group for the multiplication as array questions in the pre-and post-test

The results in table 5.3 reveal that in the intervention group, there were a large number of learners who presented an incorrect model across both problems (49% in question 10 and 70% in question 14). Horizontal number sentences and/or column multiplication models were common with 24% and 16% of the group presenting these models in contrast to more limited diagrammatic modelling with (8% and 0% in question 10 and 14).

The high number of incorrect models indicate difficulties at a horizontal mathematization level as learners struggled to translate the problem situations correctly. Additive models, once again dominated the incorrect model category with 41% of the learners defaulting to presenting models of $6 + 3 = 9$ in question 10 and 50% of the group presented models of $14 + 16 = 30$ in question 14. Since most learners either modelled the problem using an incorrect additive model and very few (question 10) or none (question 14) presented an array model, a general gap in awareness of the array structure was noted.

Outhred & Mitchelmore (2004) state that difficulties in representing arrays are often a consequence of limited conceptions of array structure which can also be seen in the limited use of direct modelling (diagrammatic representations – array). This was noted with the intervention group learners as only 3 learners presented a diagrammatic representation model in question 10 whilst none of the learners presented a diagrammatic representation in question 14. Students' drawings of array models reveal their mental representations which indicates their understanding of the problem (Bensur & Eliot, 1993 as cited in Outhred & Mitchelmore, 2004). For question 10, learners did not contextualize the problem fully to understand that the entire tray was filled with pies and not just the outside. Similarly, for question 14, learners did not consider the entire hall floor but rather only took into

consideration the two sides i.e. the hall being 16 meters long and 14 meters wide. A large number of learners thus went straight into presenting an additive algorithm suggesting that they too did not contextualize the problem efficiently. Amongst the learners who presented a diagrammatic representation, one child drew a partial array showing that he only took into consideration the number of pies on two sides of the tray (see figure 5.3).

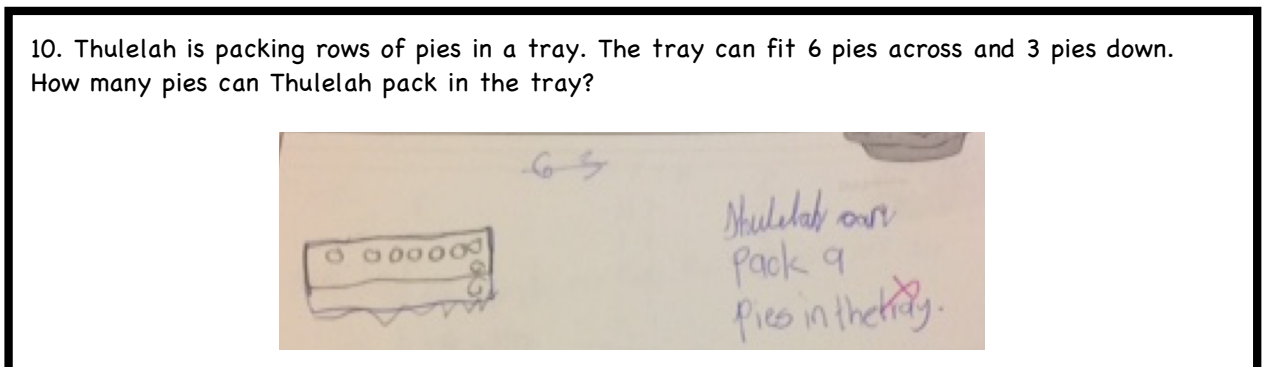


Figure 5.3: An illustration of a partial array model

The pre-test data in the table above also indicated that almost a quarter of the intervention group presented horizontal and/column multiplication models in the array problems. Learners who presented these models either presented a horizontal calculation sentence of 6×3 and 16×14 or they presented a column multiplication model of:

$$\begin{array}{r} 6 \\ \times 3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 16 \\ \times 14 \\ \hline \end{array}$$

The success rate amongst learners who used these models was good as all the learners who presented these models in question 10 got the solution correct. However, the same results were not seen in question 14 (where a higher number range was present), as the success rate amongst learners who presented horizontal and/column multiplication models stood at 8%. Although correct models of the problems were presented, the number range in question 14 (involving two-digit by two-digit multiplication) posed challenges for learners indicating difficulties at a vertical mathematization level. Common errors associated with the column multiplication model presented in question 14 were seen in the way learners worked through the calculation. 8% of the learners failed to put in the place holder when multiplying the tens by the four units. This ties in with Ma (1999) who explains that most learners tend to forget to move numbers over on to subsequent lines and they often leave out the place value holder.

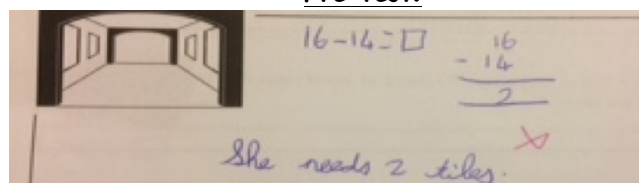
Haylock (2010) also mentions that calculations within the column model can be difficult for children and thus open room for errors.

Post-intervention results for question 14 on the other hand indicated that the number of incorrect models decreased significantly from 70% (pre-test) to 27% (post-test). Across both array problems, horizontal and/or column multiplication models were more frequently presented with a better success rate in the post-test than in the pre-test – question 10 produced a 100% success rate whilst question 14 produced a 70% success rate. Eight and five learners presented ratio models in the post-test for question 10 and 14 respectively with a better success rate noted in question 10 than in question 14. The higher success rate in question 10 suggested that in the array problem with one-digit by one-digit, learners were able to make better sense of the problem hence performing better at both a horizontal and vertical mathematization level. The same success rate was not achieved in the control group as limited (question 10) or no gains (question 14) were noted.

Similar to the scaling category, the frequency of horizontal number sentences and/or column multiplication models show that learners expanded their use of familiar models, and the increase in the success rates with these models suggested that learners could better internalise these models after being exposed to different modelling approaches. The number of learners who presented ratio models for these questions was relatively low (22% in question 10 and 14% in question 14). However, upon looking at individual performances, it was interesting to see the performance of a learner who could not solve the problem correctly in the pre-test, use a ratio model in the post-test with a correct solution to the problem. Figure 5.4 shows an illustration of such a shift in reasoning and performance.

14. Lerato is tiling a school hall floor that is 16 metres long and 14 metres wide. How many tiles will Lerato need to tile the school hall if the tiles are 1m by 1m?

Pre-test:



Post-test:

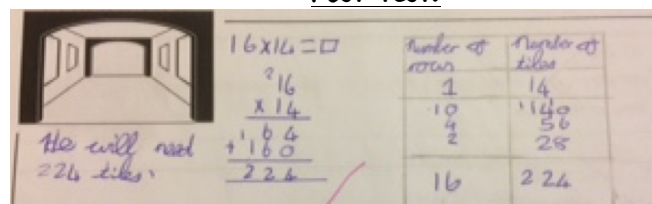


Figure 5.4: An illustration of a one learners modelling of question 14 in the pre-and post-test

The example in figure 5.4 shows that prior to intervention, the child did not make any sense of the problem in question 14 as an incorrect model was presented. However, in the post-test there appeared to be an increased understanding with the learner presenting horizontal and column multiplication models as well as a ratio t-table model. In the ratio t-table drawn up by the learner, she has decomposed 16 into ‘10’, ‘4’ and ‘2’ to make it easier to multiply. It is unclear from the test script as to which model the child presented first yet what is clear in this instance is a broadened sense making of the problem in horizontal mathematization terms with fluent and flexible use of models in vertical mathematization.

Overall, in the array problems, it was evident that prior to intervention, additive bias was common with partial readings of the situations and difficulties were also noted in the multiplication of larger numbers. However, after intervention, these difficulties were remediated as more learners presented appropriate models and solutions to the problems.

5.4.3. Division as Sharing

Division as sharing is when a collection of objects is to be divided or partitioned into an equal number of groups (Askew, 2004; Carpenter et al., 1999; Fischbein et al., 1985; Haylock & Cockburn, 2013). In this group of problems, the overall results for the division as sharing problems indicated strong gains of over 10% in the intervention group whereas the

performance of the control group either decreased (question 3) or showed slight improvement of about 4% (in question 6) – see figure 5.1 (a). It is therefore of interest to look in depth as to why and where these gains occurred in the intervention group with a closer focus on how learners modelled the sharing problems.

TABLE SUMMARY FOR DIVISION AS SHARING QUESTIONS (INTERVENTION GROUP; n=37)										
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS						
				No model Incorrect	Incorrect Model	Model of $120 \div 5$	Diagrammatic Representation	Idiosyncratic Division	Horizontal number sentence or Traditional Division	Ratio Model
3 9 friends were given a bag of 72 sweets. They shared them out equally. How many sweets did they each get?	$72 \div 9$	PRE-TEST	FREQUENCY	0 (0%)	7 (19%)	0 (0%)	12 (32%)	2 (5%)	34 (92%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	8 (22%)	2 (5%)	26 (70%)	0 (0%)
		POST-TEST	FREQUENCY	0 (0%)	4 (11%)	0 (0%)	6 (16%)	4 (11%)	33 (89%)	16 (43%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	4 (11%)	3 (8%)	28 (76%)	14 (38%)
6 Thando makes 120 bottles of lemonade for a school market day. The bottles of lemonade are shared equally between thando and 5 of her friends. How many bottles of lemonade do they each get to sell?	$120 \div 6$	PRE-TEST	FREQUENCY	2 (5%)	28 (76%)	25 (68%)	2 (5%)	2 (5%)	8 (22%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	10 (27%)	1 (3%)	0 (0%)	6 (16%)	0 (0%)
		POST-TEST	FREQUENCY	1 (3%)	18 (49%)	14 (38%)	3 (8%)	3 (8%)	22 (59%)	11 (30%)
			CORRECT ANSWER	0 (0%)	0 (0%)	5 (14%)	1 (3%)	2 (5%)	11 (30%)	4 (11%)

Table 5.4: Table summary of the models presented by the intervention group for the division as sharing questions in the pre-and post-test

Pre-test results of the sharing problems portray that the overall performance in question 3 was better than the performance in question 6 with most learners in question 3 being able to make appropriate sense of the problem (70% correct). The pre-test data indicates that symbolic division models in the form of a horizontal number sentence and/or traditional division algorithm dominated. 92% of the intervention group presented these models in question 3 whereas in question 6, 68% presented an incorrect division model of $120 \div 5$ and 22% presented the correct horizontal number sentence and/or traditional division model of the situation.

A key difference in the sharing problems related to the wording which may have caused learners to perform differently in both questions. Both questions in this category were partitive division problems whereby the *dividend* was shared equally between a certain number of recipients and the size of the portion (*quotient*) varied on the number of recipients (the *divisor*) (Squire & Bryant, 2002). However, the differences in the questions were that in question 3, the divisor was presented explicitly whereas in question 6 learners had to determine the divisor from the situational information provided. As a result, 68% of the intervention group interpreted the divisor to be 5 instead of 6.

Due to the high number of learners who incorrectly interpreted question 6, a code for *models of $120 \div 5$* was created with the intention of determining whether learners' issues laid primarily within the model set-up or the division calculation. Therefore, if learners had correctly calculated $120 \div 5$, difficulties would not have been in the way they calculated the problem but rather in the interpretation of the question. The interpretation of the divisor in question 6 appeared to be a contributory factor to the differences in learner's performance as many learners did not consider Thando to be one of the friends who sold lemonade. Upon closer examination of the model of $120 \div 5$, it was found that about 30% of the learners got the answer correct which suggests that for these learners, had they interpreted the problem correctly, they would have probably got the answer correct. The remainder of learners had difficulty with both interpreting the question and in calculating the problem as they presented a model of $120 \div 5$ but calculated the answer incorrectly, or they defaulted to using subtraction of $120 - 5$. These models and strategies by the learners indicated difficulties at both a horizontal and vertical mathematization level indicating the need for intervention so that better reasoning, sense making and mathematization could be achieved.

In question 3, similar patterns emerged with the horizontal number sentence and/or traditional division model. Whilst these models were correct models of the situation, 20% of the group got the solution incorrect indicating difficulties in vertical mathematization. Research centred on formal methods of division have found that difficulties in division may arise as more formal structures of written methods make learners more prone to errors Anghileri (2001).

Moreover, 25 and 6 learners in question 3 and 6 presented horizontal number sentences with 16 and 2 learners only providing an answer. This made it difficult to see how learners moved from the model to the answer. Interview data was brought in here to gain a better sense of how learners worked through their models. The data suggested that learners were calculating $72 \div 9$ and $120 \div 6$ in their minds which may have caused calculation errors. Below is an excerpt from an interview with one of the learners.

Interviewer: *Can you talk me through what you have done here (in question 3)? I'd like to understand why you wrote a number sentence?*

Learner: *I kind of divided in my mind, I don't know how to explain but I just wrote the number sentence.*

Moving on to the post-test results, the data reveal an improvement of 19% in question 3 whereas in question 6 the gains almost doubled from 13% in the pre-test to 27% in the post-test. After having gone through intervention, the number of learners who modelled the problem $120 \div 5$ decreased from 68% in the pre-test to 38% in the post-test with more learners displaying a correct horizontal number sentence and/or traditional division model in question 6 (from 22% in the pre-test to 60% in the post-test). Moreover, 43% of the intervention group used a ratio model (t-table or double number line) for question 3 whilst 30% used the model for question 6. The success rate of the ratio model in question 3 was good with 88% getting the answer correct whereas in question 6, only 36% got the answer correct.

The post-test data indicate an emergent use of ratio models which were introduced and developed during the intervention lessons. Variation in the success rate of the ratio models may have been due to the number range - question 3 required a division calculation of two-digits by one-digit which may have been easier than dividing a three-digit number by one-digit in question 6. Finally, although there appeared to be take-up of the ratio model in the post-test data, the overriding model was still either a horizontal number sentence and/or traditional division model across both questions suggesting that most learners opted to use models they were familiar with.

5.4.4. *Division as Repeated Subtraction*

In situations where an amount has to be put into equal sized groups and the size of the group is provided, we call this division as repeated subtraction (Askew, 2004). In the division as repeated subtraction problems, the overall performance in the pre-test indicates similar results across both questions 8 and 15 with 44% of the intervention group getting the answers correct (*see figure 5.1(a)*). Table 5.5 provides more detail on the specific model use in these questions.

TABLE SUMMARY FOR DIVISION AS REPEATED SUBTRACTION QUESTIONS (INTERVENTION GROUP; n=37)										
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS						
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Idiosyncratic Division	Horizontal number sentence or Traditional Division	Ratio Model	No model Correct
8 Mary has 96 biscuits that she wants to put into boxes. 12 biscuits can fit into one box. How many boxes can Mary fill?	$96 \div 12$	PRE-TEST	FREQUENCY	5 (14%)	13 (35%)	3 (8%)	1 (3%)	22 (59%)	0 (0%)	2 (5%)
			CORRECT ANSWER	0 (0%)	0 (0%)	3 (8%)	1 (3%)	19 (51%)	0 (0%)	2 (5%)
		POST-TEST	FREQUENCY	3 (8%)	6 (16%)	1 (3%)	2 (5%)	22 (59%)	22 (59%)	1 (3%)
			CORRECT ANSWER	0 (0%)	0 (0%)	1 (3%)	2 (5%)	18 (49%)	17 (46%)	1 (3%)
15 Sarah is packing T-shirts onto shelves in a clothing store. One shelf can hold 44 t-shirts. There are 88 T-shirts that need to be packed. How many shelves would needed to pack all the T-shirts away?	$88 \div 44$	PRE-TEST	FREQUENCY	5 (14%)	10 (27%)	0 (0%)	1 (3%)	16 (43%)	0 (0%)	4 (11%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (3%)	12 (32%)	0 (0%)	4 (11%)
		POST-TEST	FREQUENCY	2 (5%)	9 (24%)	1 (3%)	2 (5%)	19 (51%)	10 (27%)	3 (8%)
			CORRECT ANSWER	0 (0%)	0 (0%)	1 (3%)	2 (5%)	15 (41%)	8 (22%)	3 (8%)

Table 5.5: Table summary of the models presented by the intervention group for the division as repeated subtraction questions in the pre-and post-test

The pre-test results presented above suggest that learners' use of models across both questions appeared to be dominated by horizontal number sentences and/or traditional division models as almost 60% and 43% of the group presented this model for question 8 and 15. The success rate with these models in the pre-test was relatively good with 19 out of the 22 learners (question 8) and 12 out of the 16 learners (question 15) getting the answers correct. About 30% of the intervention group also presented incorrect models across both repeated subtraction problems in the pre-test.

The high success rate across the questions with learners who presented more symbolic division models (horizontal and/or traditional division) imply that these learners could make adequate sense and calculate the problem correctly. However, there were still about half the class that did not get the correct solution. *Figure 5.5* illustrates how one learner presented a correct model to question 15 but calculated the solution incorrectly.

15. Sarah is packing T-shirts onto shelves in a clothing store. One shelf can hold 44 T-shirts. There are 88 T-shirts that need to be packed. How many shelves would be needed to pack all the T-shirts away?

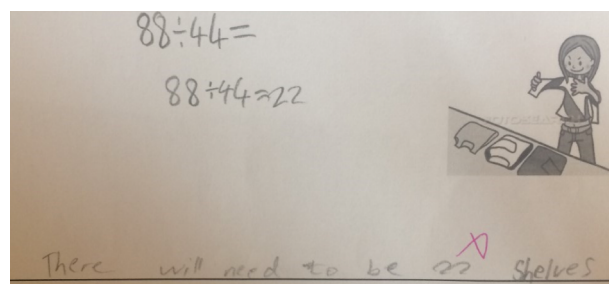


Figure 5.5: Learners examples of different incorrect models presented in the repeated subtraction questions in the pre-test

The example in figure 5.5 shows how the child has made an error at a vertical mathematization level with $88 \div 44$. It appears as if he divided the units separately and thereafter the tens. The child's working out also suggests an awareness that 4 is half of 8 and has thus applied the 'rule' to the digits individually. Evidence points towards the fact that a learned rule has been applied inappropriately and there may be a possibility that the learner does not have a solid understanding of the method used (Hansen, 2011). Moreover, the learner's solution strategy also indicates a possible lack of number sense because if the meaning of division ($88 \div 44$) was well understood, the child would have known that the answer to $88 \div 44$ cannot be 22. Anghileri (2006b) states that when learners are faced with division with two-digit divisors, number sense plays an important role as they should select an approach that best suits the numbers involved. Therefore, had this learner had sound number sense, one way of solving the problem would be to notice that 44 is half of 88 hence two lots of 44 make 88 or two shelves are needed.

Additionally, learners who presented an incorrect model also did not make adequate sense of the problems and had thus resorted to using either addition, subtraction or multiplication showing no understanding of the problem. When children do not have a sound conceptual understanding of multiplication and division, they tend to revert or default to more familiar concepts of addition and subtraction (Hansen, 2011).

After having gone through intervention, the data set reveals a decrease in incorrect models with question 8 dropping by about 20% points and a decrease in the presence of no models (incorrect) across both questions. The use of ratio models was also seen in the post-test data with a varied frequency in each question. About 60% of the intervention group presented a ratio model for question 8 whereas 27% of the group presented the model for questions 15.

Although the frequency of the ratio model was lower in question 15 than it was in question 8, those who presented these models had a strong success rate. 17 out of the 22 learners in question 8 got the solution correct whereas 8 out of the 10 learners got the solution to question 15 correct. This suggests that learners who presented a ratio model, could make better sense and reasoning of the problems through t-tables developed during intervention. An example of one of the learners who presented a ratio model for question 8 is shown in Figure 5.6.

8. Mary has 96 biscuits that she wants to put into boxes. 12 biscuits can fit into one box. How many boxes can Mary fill?

$96 \div 12 = \square$

boxes	biscuits
1	12
8	96

She can fill 8 boxes.



Figure 5.6: Example of one learners use of a ratio model in the post-test

The model in *figure 5.6* exemplifies a learner's presentation of the ratio model to question 8. This example shows that at a horizontal mathematization level, the learner could model the problem in two ways, using – a horizontal calculation sentence and a ratio t-table. It is also evident that the child has understood the problem as both a correct model and solution are given. Conversely, no evidence on the script is shown to explain how the learner used the ratio t-table to get to solution. The child may have known the answer as a recalled fact. This was verified in the learner's interview, as follows:

Interviewer: *Can you talk me through what you have done here?*

Learner: *For this question, I used 96 divided by 12. For 12 biscuits, you can get 1 box. So, for 8 boxes there are 96 biscuits. So, we could either say 'something' times 12 equals 96 or 96 divided by 12 equals 8. And the method I used was the ratio table.*

The response given by the child shows how the learner has thought about the problem. The learner has thought about the problem from both a multiplication and division perspective knowing that 8 times 12 equals 96. It is evident that the child understands the multiplicative structure of the problem and can show this using a ratio t-table. The ratio t-table is thus an effective model as it provides a better understanding of the relationship between numbers as compared to more symbolic traditional models.

5.4.5. Division as Grouping (with remainder)

Similar to the repeated subtraction problems, this category of problems provides the total number of objects and the number of objects in each group with the number of groups being

the unknown variable (Carpenter et al., 1999). Learners are required to look at the mathematical solution to the problem and decide whether or not the solution is sensible in the context of the problems (Askew, 2004). The overall performance in this category of problems reveal that learners initially performed poorly in these questions (*see figure 5.1 (a)*). Table 5.6 provides more detail on the different modelling approaches and success rates of this category of problems.

TABLE SUMMARY FOR DIVISION AS GROUPING QUESTIONS (INTERVENTION GROUP; n= 37)												
QUESTIONS	CALCULATION (BARE PROBLEMS)	MODELS										
				No model Incorrect	Unsuitable Answer	Incorrect Model	Diagrammatic Representation	Idiosyncratic Division	Horizontal number sentence or Traditional Division	Ratio Model	No model Correct	Other
12 66 children need to be transported to a sports match. The school has arranged for parents to take the children by car. If the cars can only take 4 children, how many cars will be needed to transport all the children?	$66 \div 4$	PRE-TEST	FREQUENCY	9 (24%)	14 (38%)	11 (30%)	1 (3%)	3 (8%)	21 (57%)	0 (0%)	0 (0%)	1 (3%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	4 (11%)	0 (0%)	0 (0%)	0 (0%)
		POST-TEST	FREQUENCY	2 (5%)	9 (24%)	7 (19%)	1 (3%)	3 (8%)	23 (62%)	18 (49%)	1 (3%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (3%)	1 (3%)	16 (43%)	12 (32%)	1 (3%)	0 (0%)
16 Jill has 45 eggs to take to the market. An egg box can hold 6 eggs. How many boxes will Jill need of she wants to take all the eggs to the market?	$45 \div 6$	PRE-TEST	FREQUENCY	6 (16%)	27 (73%)	13 (35%)	2 (5%)	0 (0%)	23 (62%)	0 (0%)	1 (3%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (3%)	0 (0%)	9 (24%)	0 (0%)	1 (3%)	0 (0%)
		POST-TEST	FREQUENCY	4 (11%)	20 (54%)	12 (32%)	0 (0%)	1 (3%)	19 (51%)	12 (32%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (3%)	12 (32%)	10 (27%)	0 (0%)	0 (0%)

Table 5.6: Table summary of the models presented by the intervention group for the division as grouping questions in the pre-and post-test

The pre-test data reveals that 38% of the intervention group provided an unsuitable answer to question 12 whereas 73% did so in question 16. About 30% of the group presented an incorrect model of both situations whilst 57% and 62% of the group presented a horizontal number sentence and/or a traditional division model for questions 12 and 16 respectively. Whilst it is evident that the horizontal number sentence and/or traditional division models were popular, the percentage of correct answers was poor with only 11% and 24% getting the solutions to questions 12 and 16 correct.

Sense making in both questions was relatively poor in the pre-test as the percentage of learners who presented an incorrect model and/or unsuitable answer was high (38% in question 12 and 73% in question 16 presented an unsuitable answer; 30% and 35% presented an incorrect model in question 12 and 16). Learners who presented incorrect models tended to default to using addition or subtraction. Figure 5.7 illustrates some of the models which learners presented. A short discussion on these illustrations is provided thereafter.

12. 66 children need to be transported to a sports match. The school has arranged for parents to take the children by car. If the cars can only take 4 children, how many cars will be needed to transport all the children?

$$66 - 4 = \square$$

$$\begin{array}{r} 66 \\ - 4 \\ \hline 62 \end{array}$$

62 ~~x~~ children

16. Jill has 45 eggs to take to the market. An egg box can hold 6 eggs. How many boxes will Jill need if she wants to take all the eggs to the market?

$$45 + 6 = \square$$

$$\begin{array}{r} 45 \\ + 6 \\ \hline 51 \end{array}$$

51 ~~x~~

Figure 5.7: Learners examples of different incorrect models presented in the grouping questions in the pre-test

In question 12, the learner displays a partial understanding of the problem but does not provide any other evidence of her working out except for subtracting 4 from 66. Whilst this indicates a possible attempt at repeated subtraction, the child failed to carry through the process of repeatedly subtracting 4 from 66 until all the children were accounted for. The learner displays a lack of number sense as she has not contextualised the situation to help her get to the correct solution.

In question 16, this learner has presented an incorrect additive model of $45 + 6 = 51$. The interview data for this learner does not provide any insight as to why he may have thought about the problem as addition so one can assume that the child has thought about the problem as Jill taking 6 more eggs. This may have been triggered by the wording of the problem: *How many boxes would she need to take **all** the eggs to the market?* The word **all** may have prompted the child to add the numbers in the problem without making sense of the situation.

On the other hand, whilst most the group (57% in question 12 and 62% in question 16) presented either a horizontal number sentence and/or a traditional division model; the success rate did not correspond to the frequency of the model. Anghileri (2006a) argues that although traditional algorithms have been to direct and organize children's approaches, it is prone to

errors. Children tend to make inappropriate adaptations to the methods they learn as they try to reconstruct a procedure they have not understood and cannot remember (Ruthven and Chaplin, 1998 as cited in Anghileri, 2006a). Moreover, the questions in this category required learners to ‘round up’ their answers which meant that the results needed to be interpreted as an answer to the problem rather than as a numerical solution (Anghileri, 2006a).

Moving on to the post-test results, the data suggests that the intervention lessons were effective as gains were achieved in both questions – 38% improvement in question 12 and a 19% improvement in question 16. These gains were stronger than those of the control group where less pronounced shifts were seen. The data reveals that there was take up of the ratio model with 49% and 32% presenting this model for questions 12 and 16. A moderate to strong success rate was noted with the ratio model as 67% and 83% of the learners got the solution to question 12 and 16 correct. This model was completely unfamiliar to learners prior to intervention therefore it was good to see positive changes emerge in the use of the ratio model. The example in figure 5.8 illustrates how one learner shifted from presenting a diagrammatic representation in the pre-test to a more formal model (t-table) in the post-test.

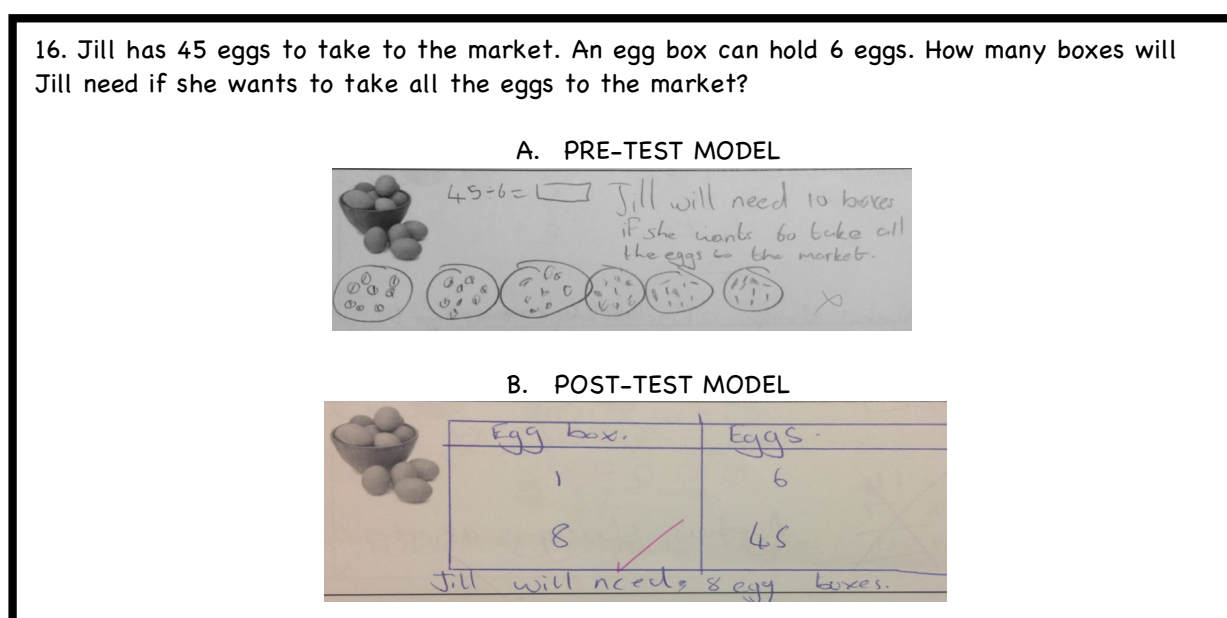


Figure 5.8: Example of the shift in models presented in the pre-and post-test for question 16

The models presented above are indicative of how a learner solved the problem in question 16 prior and post intervention. In the pre-test script, two models are shown - horizontal calculation sentence and a diagrammatic representation (indexical). At a horizontal mathematization level, the learner has modelled the problem correctly using a horizontal number sentence and has worked through the problem using a diagrammatic representation.

However, the diagrammatic representation model is partially incorrect. Firstly, the diagram drawn does not directly model the situation presented – instead of the learner drawing boxes and placing 6 eggs into boxes (quotitive division), the learner has drawn 6 circles and shared out the 45 eggs (partitive division). And secondly, the diagram drawn is also inaccurate as the learner has a different amount of ‘eggs’ (tally marks) in each circle. The learner has more tally marks than required which results in different amounts within each circle. Whilst tallying and sharing like the one presented above display an understanding and the way the child thinks of the problem, it is not always sustainable without errors (Anghileri, 2001). In the post-test, the same child has shown a shift in her modelling through presenting a ratio table as was developed during intervention. Interview data supports that this learner calculated the answer in her mind going back to the question to put down a final answer of 8.

The intervention lessons also allowed for better sense making which can be seen in the double number line presented in question 16 by one of the learners (Figure 5.9). In this model, the top row shows the learner counting in sixes whereas the bottom row shows unit counting in ones. The learner has counted accurately up to 42 but then writes 45 as the next number. There are 45 eggs that need to be taken to the market thus 8 boxes would be needed. Although the learner has unit counted in the bottom row, she still displays sound number sense as she was able to make a calculated estimation and round off correctly. The double number line provides a neat way of representing multiplicative relations building on multiplicative reasoning (Küchemann et al., 2011a).

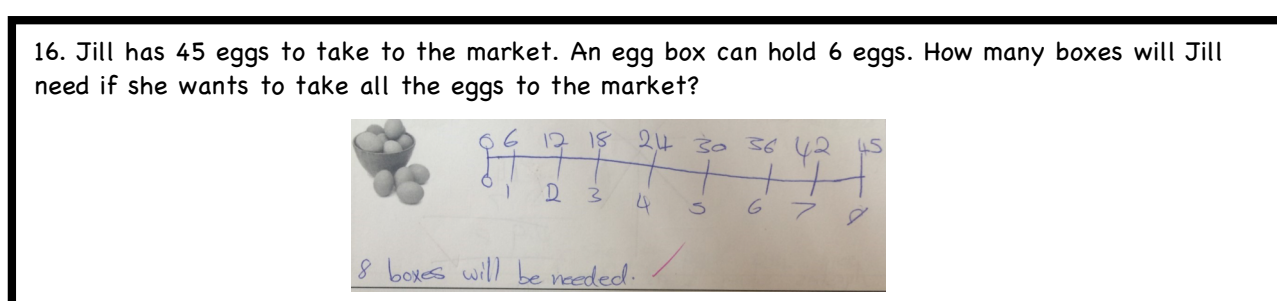


Figure 5.9: An example of a learner modelling the problem in Question 16 in the post-test using a double number line

5.4.6. *Idiosyncratic Division*

Throughout the division problems, a model which surfaced was the idiosyncratic division model. This model included a distinct setup of the division problem which differs from the standard, known division algorithms. Anghileri et al. (2002) draw on the work of Ruthven

(1998) who states that when considering pupils use of written recording, two distinct purposes emerge – (a) to augment working memory by recording key items of information and (b) to cue sequences of actions through schematising such information within a standard spatial configuration. The former identifies with informal solution strategies that are often idiosyncratic and gives little consideration to efficiency whilst the latter indicates a taught procedure that directs children’s approaches (Anghileri et al., 2002).

A typical example of an idiosyncratic division model is as follows:

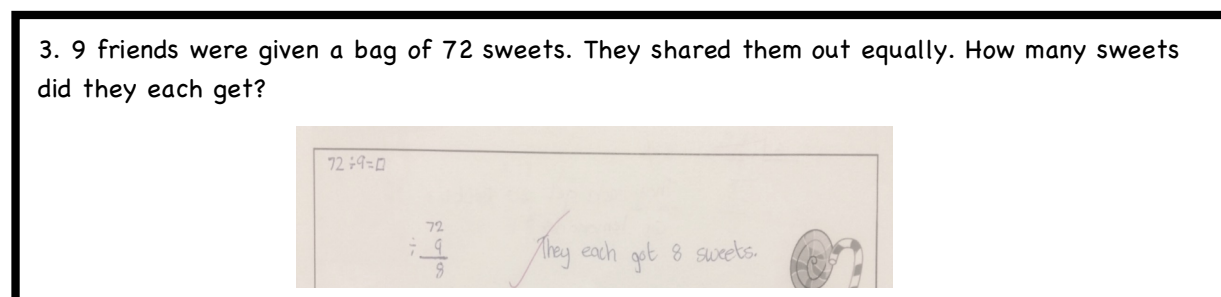


Figure 5.10: An idiosyncratic model used for a division as sharing problem (Question 3)

The model above suggests that learners could be calculating this problem in one of two ways, either by asking themselves ‘what times 9’ equals to 72 or they could be using division and calculating directly – 72 divided by 9 equals 8. A notable number of learners in both the intervention and control group presented this model in the pre-test implying that they may have been taught or shown this model at some point by their teacher.

The presence of idiosyncratic division was more common amongst learners in the control group than it was with learners in the intervention group. On average, 12% of the control group presented an idiosyncratic model across all the division problems whereas only about 5% of the intervention group presented this model. The success rate with this model varied in individual questions with a poor overall success rate - only about 3% of those who used the model got the solution correct. These results show that although learners were presenting this model, it was inefficient as in most cases they failed to arrive at the correct solution.

5.4.7. Other classes of word problems

5.4.7.1. Multiplication as Rate

In this category of questions, questions 2 and 11 were analysed. The overall results for the rate questions indicate that larger gains of over 10% and 20% in question 2 and 11 were

noted in the intervention group whereas limited changes and a slight drop in performance (4% gain in question 2 and a 4% decline in question 11) was seen in the control group. Table 5.7 provides a summary of the models and performance in the rate questions with the intervention group.

TABLE SUMMARY OF MULTIPLICATION AS RATE QUESTIONS (INTERVENTION GROUP; n=37)									
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS					
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Symbolic Repeated Addition	Horizontal Number Sentence or Column Multiplication	No model Correct
2 Thabo planted 10 pots with magic beans. He planted 7 magic beans in each pot. How many beans did Thabo plant?	10 × 7	PRE-TEST	FREQUENCY	0 (0%)	9 (24%)	6 (16%)	1 (3%)	24 (65%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	5 (14%)	1 (3%)	23 (62%)	0 (0%)
		POST-TEST	FREQUENCY	0 (0%)	5 (14%)	5 (14%)	2 (5%)	23 (62%)	25 (68%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	3 (8%)	12 (32%)	6 (16%)
11 There are 12 friends at an orange farm. They each collect 13 oranges in their bags. How many oranges do they have altogether?	12 × 13	PRE-TEST	FREQUENCY	3 (8)	12 (32%)	1 (3%)	1 (3%)	19 (51%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (3%)	7 (19%)	0 (0%)
		POST-TEST	FREQUENCY	3 (8)	5 (14%)	1 (3%)	4 (11%)	16 (43%)	11 (30%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	3 (8%)	12 (32%)	6 (16%)

Table 5.7: Table summary of the models presented by the intervention group for the multiplication as rate questions in the pre-and post-test

In the pre-test, learners performed better in question 2 than they did in question 11. 65% of the intervention group presented a horizontal number sentence and/or column multiplication model of which all except one learner got the correct solution. In question 11, 51% of the group presented a horizontal number sentence and/or column multiplication model with only 19% of the learners getting the answer correct. The variation in the success rate of these questions can be associated with the number range in the questions. Question two required learners to multiply a two-digit by one-digit number with one of the numbers being 10 and question 11 required a multiplication calculation of two-digit by two-digit numbers. Multiplying by 10 was relatively easy for the majority of learners as they often know the answer as a known multiplication fact (Mulligan & Mitchelmore, 1997; Kouba, 1989). Multiplication of two-digit by two-digit numbers, in question 11, presented challenges for learners' as errors were commonly associated to the long multiplication algorithm. An

example of two learner's models and procedures from the pre-test data for question 11 is shown in figure 5.11.

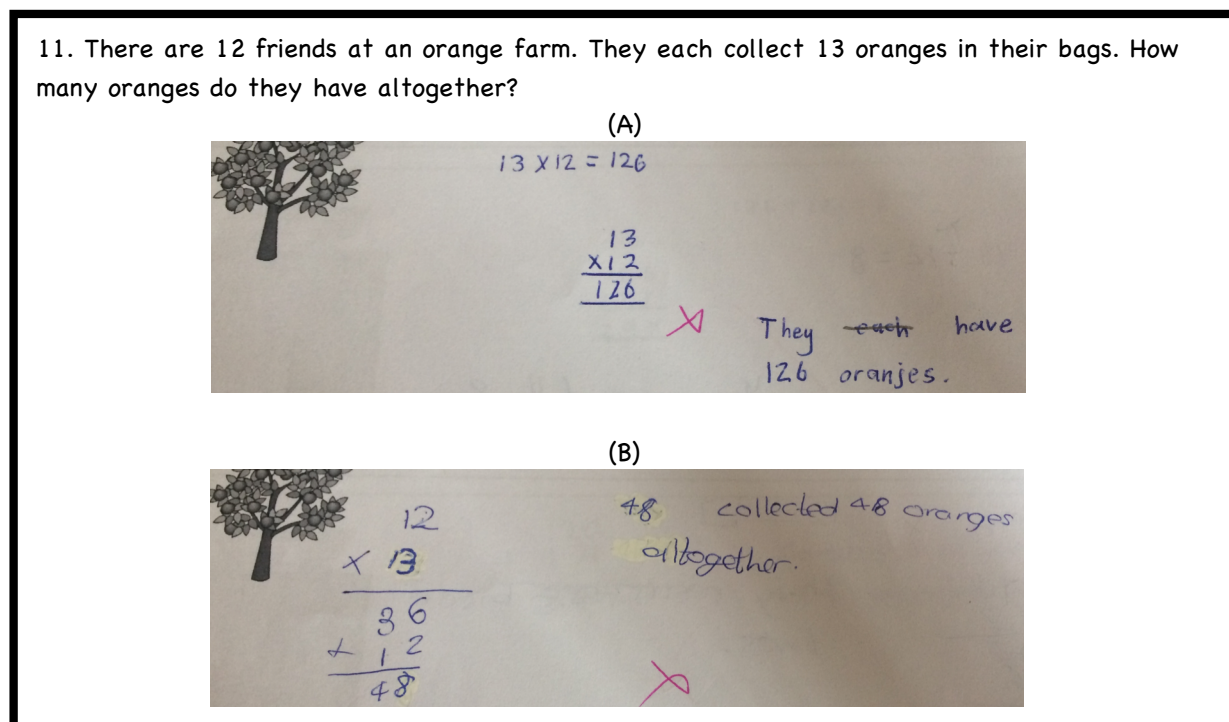


Figure 5.11: Examples of two different learner's models and procedures to question 11 in the pre-test

Figure 5.11 shows that both learners presented appropriate multiplication models of the problem, however difficulties are shown in the way these learners worked with the models at a vertical mathematization level. Learner A has multiplied the units (2, 3) by each other and thereafter has multiplied the '1' in the tens column of 13 by 12 to get an answer of 126. Learner B has attempted a long multiplication algorithm however, when multiplying 12 by 10, he failed to put the zero-place holder in. He has treated the tens digit multiplication as if it were a unit's digit multiplication and as a result misunderstands that the '1' in 13 is '1' ten or 10 and not just '1'. Number operations errors like the ones above tend to occur because the child uses standard written methods which are not fully understood (Hansen, 2011).

The post-test data on the other hand reveals a pronounced shift in the choice of ratio models with about 68% and 30% of the group using this model for question 2 and 11. A high prevalence of horizontal number sentences and/or column multiplication modelling was also noted with about 62% and 32% of the learners presenting this model in question 2 and 11 respectively.

The findings above suggest that as learners were introduced to the double number line and ratio t-table, it could have been made easier to understand the relationship between the numbers in context. The rate questions in the tests had an implicit ratio (Askew, 2004) – *he planted 7 pots in each pot, they each collected 13 oranges in their bags* therefore, 62% used a ratio model. Teppo & van den Heuvel-Panhuizen (2014) explain that a number line model is part of a representational system organized to highlight patterns of relations among numbers and operations with the double number line displaying multiplicative relationships.

Like in the other categories of problems, although horizontal number sentences and/or column multiplication models were not explicitly taught, the model still dominated. These symbolic models noted gains in efficiency of the model with an improved success rate. A possible explanation to these gains could be due to the way in which the ratio models were shown to learners in the intervention lessons. Learners were shown how to decompose numbers (especially larger numbers such as 12 and 13) making it easier to multiply hence these strategies could have fed into the way learners solved the rate problems especially with the larger number range in question 11.

5.4.7.2. Multiplication as Repeated Addition

In this category, learner performance (correct responses) with both groups stood between 50% and 60%. However, shifts in the post-test were only seen with the intervention group (5%) whereas no gains were seen in the control group. In this category, only one explicit repeated addition problem was asked which contained superfluous information. Table 5.8 displays the model frequency and success rates for the repeated addition problem.

TABLE SUMMARY OF MULTIPLICATION AS REPEATED ADDITION QUESTIONS (INTERVENTION GROUP; n= 37)										
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS						
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Symbolic Repeated Addition	Horizontal Number Sentence or Column Multiplication	Ratio Model	No model Correct
7 Sam loves collecting bottle tops and straws. She collected 11 bottle tops on Tuesday, 11 bottle tops on Wednesday, 15 straws on Thursday and 11 bottle tops on Friday. How many bottle tops will Sam have collected?	10 ✖ 7	PRE-TEST	FREQUENCY	2 (5%)	16 (43%)	0 (0%)	14 (38%)	8 (22%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	14 (38%)	8 (22%)	0 (0%)	0 (0%)
		POST-TEST	FREQUENCY	2 (5%)	14 (38%)	1 (3%)	15 (41%)	5 (14%)	2 (5%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	1 (3%)	14 (38%)	5 (14%)	2 (5%)	0 (0%)

Table 5.8: Table summary of the models presented by the intervention group for the multiplication as repeated addition question in the pre-and post-test

From the table above, 38% of the intervention group presented a symbolic repeated addition model with the correct solution however the dominant model which featured in the pre-test was an incorrect model with 43% of the group incorrectly selecting numbers from the problem situation.

The superfluous information may have baffled learners hence provoking them to make use of all the numbers in the problem modelling $11 + 11 + 11 + 15 = 48$. Hegarty et al. (1995) mention that one of the causes of erroneous solutions to word problems is *number grabbing*. The information in question seven included both bottle tops and straws however the question only asked about the total number of bottle tops collected. Hence, the superfluous information in the question was on the ‘straws’ which many learners included as part of their calculation indicating problems at a horizontal mathematization level. More so, learners are unable to solve problems like these because the superfluous information may have caused an interference in the child’s ability to transform the problem into a calculation (Hansen, 2011). Contrastingly, the learners who presented repeated addition models used an intuitive calculation strategy which is an advance on direct counting but is not very abstract as learners still rely on known number facts (Ensor et al., 2009).

The post-test results reveal that the ratio t-table model was not favoured in the repeated addition problems since only two learners presented this model. Learners appeared to be reluctant to use the t-table and relied mainly on symbolic repeated addition and multiplication models, still directly modelling the problem as repeated addition.

The next section looks at the overall shifts and progression of models and their respective success rate across both groups.

5.5. Summarizing shifts in models in the multiplicative problems

In this section, I summarise the shifts in models presented and their respective success rate across the multiplicative questions that occurred after the small-scale intervention program. These shifts are displayed by means of bar graphs to make it clearer for the reader. *Figure 5.12 (a)* displays the overall shifts in models used across all the multiplication questions from the pre- to post-test with both groups of learners (intervention and control). *Figure 5.12 (b)* displays the overall shifts in the success rates of the models used across the multiplication

questions from the pre- to post-test with both groups. The bars represented in the graph show the average use of the models and their success rate. Averages were calculated by adding the frequencies of each model and dividing it by seven (multiplication) questions. For the overall success rate, I looked at what fraction of learners using each particular model got the solution to the problems correct. Based on these figure, I then calculated the average success rate per model. The same method was used for the overall shifts in models and success rate per model for the division problems with the averages calculated based on six questions. These graphs are shown in *figure 5.13 (a)* and *(b)*. The bars on all the graphs are indicative of the results from the intervention and control groups. The green and orange bars represent the pre-and post-test results of the intervention group whereas the yellow and blue bars represent the pre-and post-test results of the control group.

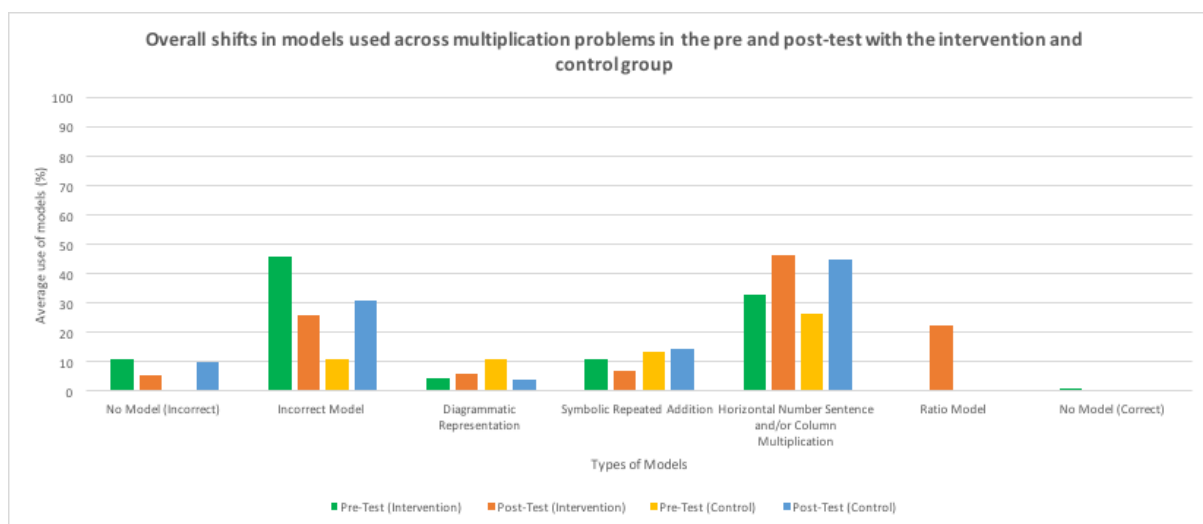


Figure 5.12 (a): Shifts in models used across multiplication problems in the pre-and post-test for the intervention and control group

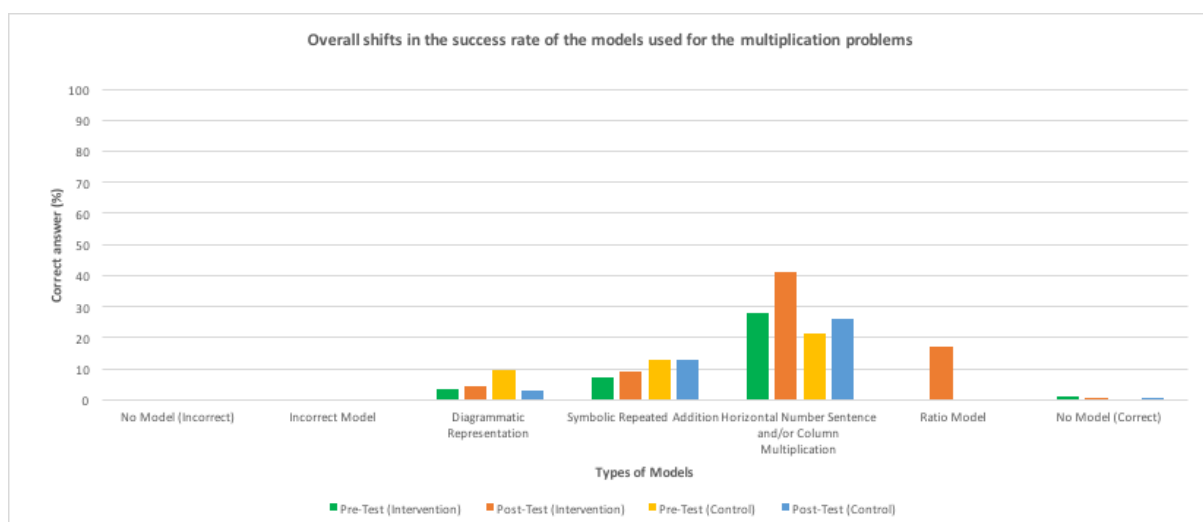


Figure 5.12 (b): Shifts in the success rate of models used across multiplication problems in the pre-and post-test for the intervention and control group

Figure 5.12 (a) indicates that ‘no model (incorrect)’, *incorrect models*, and *symbolic repeated addition* model selection reduced with learners who had intervention. About 20% of the learners in the intervention group no longer presented an incorrect model indicating that there were less learners who found difficulty in modelling the problems. However, in comparison, the control group showed an overall increase in incorrect models with the frequency of the model increasing by a little over 20%. The shifts in results of the incorrect models in both groups of learners suggest that learners who received intervention could make better sense of the problems post-intervention when compared to their counterparts in the control group.

Another notable change in model use was that of the horizontal number sentence and/or a column multiplication model where a 14% increase from the pre-to post-test was witnessed in the intervention group. Although the intervention group learners were not encouraged to use traditional algorithms, they presented the model with more fluency in the post-test as a 13% increase was achieved. As mentioned in the earlier part of this chapter, the work done in the intervention lessons, may have fed into the way learners worked with their existing models hence they had better success rates with model’s familiar to them. In the control group, an increase in the use and the success rate of the horizontal number sentences and/or column multiplication models was noted, however, this shift was small when compared to the intervention group.

Moreover, new models (double number line and ratio t-table) also emerged in the data of the intervention group. The absence of use of the ratio model in the pre-test for both groups suggested that the model was unfamiliar to learners prior to the intervention; yet about 20% the intervention group adopted this model in the post-test with a 17% success rate. No ratio models were presented by the learners in the control group.

Similar shifts were note for the division problems with the intervention group (see figure 5.13 (a) and (b)). The percentage of incorrect models decreased by 12% from the pre- to post-test in the intervention group whereas a small limited change was noticed in the control group. The frequency of a horizontal number sentence and/or traditional division model increased slightly in the post-test data of the intervention group (6%), whilst a more notable shift was seen in the ratio model which had a frequency of approximately 40%. The success rate with the ratio model was also good as 30% of the intervention group got the solution correct.

The emergence of the ratio models and its respective success rates in the multiplication and division problems show that this model was effective even though it was explored and developed over a short period of time. Moreover, the intervention group learners also performed better using models they were familiar with. The work done with ratio modelling appeared to give learners a broader understanding of multiplication and division as a complete structure rather than viewing them as isolated concepts. Hence gains in the overall performance ascertain that the intervention program was useful in developing multiplicative reasoning with the Grade 4 learners.

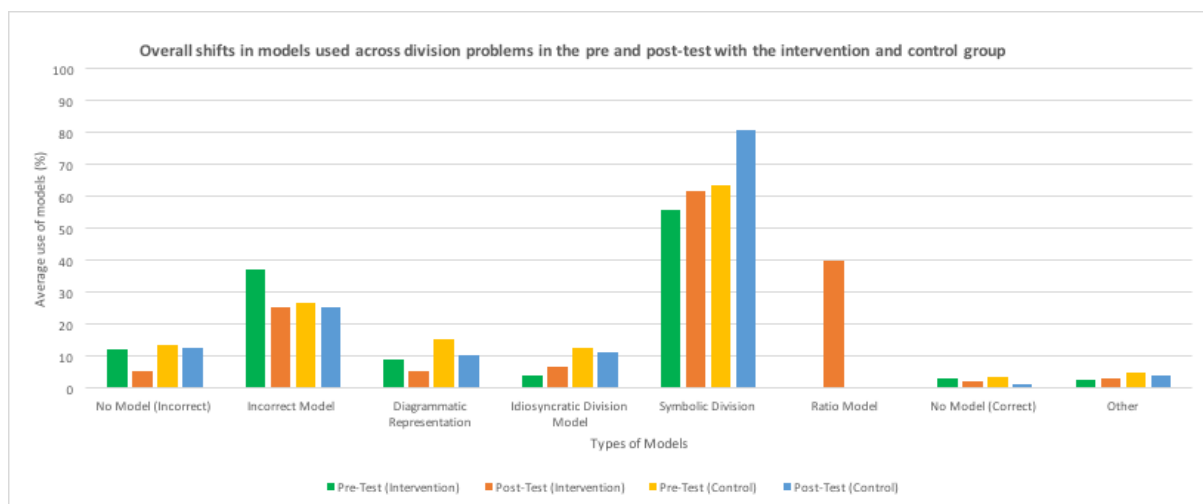


Figure 5.13 (a): Shifts in models used across division problems in the pre-and post-test for the intervention and control group

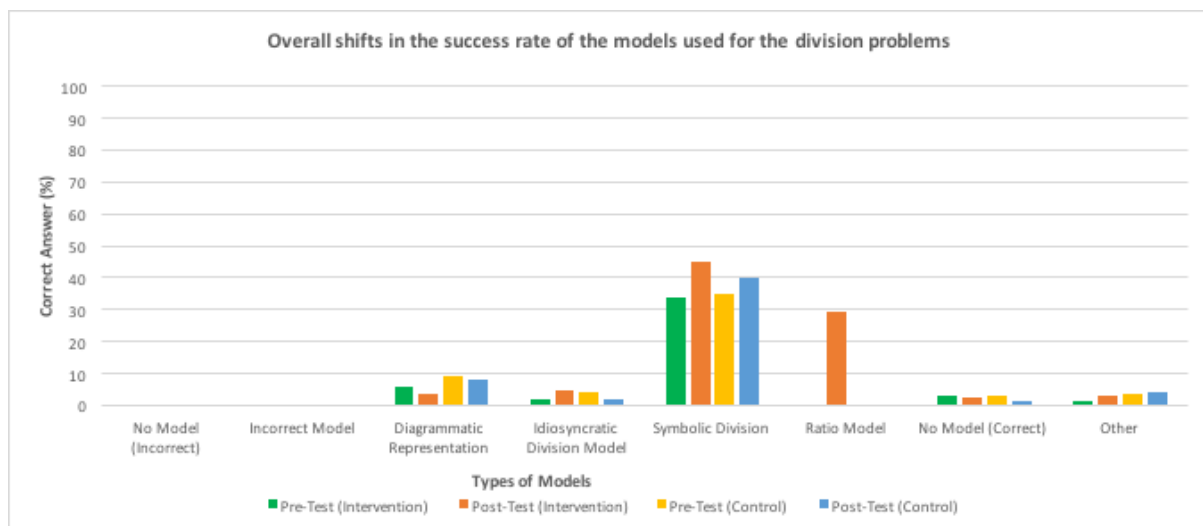


Figure 5.13 (b): Shifts in the success rate of the models used across division problems in the pre-and post-test for the intervention and control group

In the last section, I consider whether any gains were made in the bald questions as a result of the work conducted in the intervention lessons with concluding remarks made thereafter.

5.6. Gains across bald questions

The pre- and post-tests included buffer questions which consisted of one addition and one subtraction problem. They also included three bald calculations which comprised of one multiplication, one division and one equivalence calculation. One multiple-choice question was also inserted. The results for the intervention group showed that learner performances improved by a small percentage across all the multiplicative reasoning bald questions whereas the control groups performance gains varied per question with a negative overall gain across all the questions (see table 5.9).

	Multiplication		Division	Multiplication with Equivalence	Mean Gain (%)
	Q17	Q18	Q19	Q20	
Intervention Group	5,4%	5,4%	10,0%	5,4%	6,6%
Control Group	1,8%	0,3%	10,0%	-20,0%	-1,9%

***Table 5.9:** Individual and average percentage point gains of multiplicative bald questions for the intervention and control groups*

The mean gain across the multiplicative reasoning questions for the intervention group showed a positive shift of 6,6% whereas the mean gain across the items for the control group was negative 1,9%. These figures can be compared to the patterns noted earlier with the word problem items where similar downward change was noted in some of the questions for the control group. The performance of the control group in question 20 was notably lower than the rest of the questions, with a decline in performance from pre-to post-test, -20% (5 more learners getting the solution incorrect). The calculation involved an equivalence structure requiring learners to find $2 \times 24 = 4 \times \underline{\hspace{1cm}}$. Responses suggested that these five learners appeared to have guessed the answer in the post-test using numbers which had no relationship to the 2×24 .

The overall pattern of the results from these items indicate that learners in the intervention group performed better in the bald calculations in the post-test than the control group whose performance showed more limited gains or a downward shift in performance. These differences suggest that the intervention on supporting multiplicative reasoning had some positive impact beyond word problems into broader multiplication calculations and structures

as well. However, since these gains were small, attention to calculation strategies for multiplication and division is still needed with considerations to include more of these calculations in intervention lessons.

In conclusion, the data presented and discussed in this chapter answers the three research questions which guided this study. To answer question one (what kinds of multiplicative reasoning (models) are Grade 4 learners using prior to intervention, as seen in pre-test responses?): the data shows that prior to any intervention, learners were predominantly using column models to solve the problems in the pre-test. These models were not associated with a high success rate suggesting the need for intervention. After intervention, the data showed improvements in performance with the intervention group yielding gains of 20.85% compared to the control group who saw a much smaller gain of 4.76%. These results answer research question two which sought to find what changes, if any, are seen in the overall performance, across the intervention and control group in the post-test. The gains achieved with the intervention group led to further discussion on the kinds of differences in model use that were associated with the shifts in performance (research question three). The data presented showed that after intervention, there was an emergence of use of ratio-oriented modelling which had not been previously used as well as a higher success rate with familiar models such as the column algorithm.

Across all the data gathered and analysed in this study, evidence suggests that teaching through a problem-solving approach using models that connect multiplication and division concepts is useful and worthwhile for enhancing learners understanding of multiplicative concepts.

Chapter Six: Conclusion

6.1 Introduction

In this chapter I present my concluding remarks by highlighting interesting key findings of my study, noting the limitations of the study and offering recommendations for future research in similar fields of study that may arise.

The initial motivation of my study was sparked by the fact that multiplication and division are often challenging areas in mathematics. For a number of years, South African school learners' have been performing poorly in these areas with performances dipping in Grade 4 (Department of Basic Education, 2012, 2013, 2014; Schollar, 2008). Learners are tasked with understanding various meanings of multiplication and division as well as identifying and using inverse operations to solve a range of problems which are often challenging for them. As a primary mathematics school teacher, I have noticed that multiplication and division are often taught narrowly as repeated addition and subtraction and through traditional methods presenting limited opportunities for understanding. It was due to these issues that I decided to explore multiplicative reasoning with Grade 4 learners through structured problem solving.

Since models were an essential part of my study, Realistic Mathematics Education (RME) was used as the theoretical framework. The RME framework provided insight into how learners can be guided in constructing models for word problems as well as stress the importance of connecting mathematics to learner's reality. For learners to grasp mathematics, there needs to be a link between the mathematics taught and the world in which learners are in (Van den Heuvel-Panhuizen, 2003). Therefore, incorporating problems with contexts that learners could relate to played an important role in the sense making process.

The focus of my study was three-fold; (i) to explore what kinds of multiplicative reasoning (models) learners are using prior to an intervention, (ii) to identify the shifts in the overall performance across the intervention and control groups and (iii) to determine what kinds of differences in model use were associated with the shifts in performance.

6.2 Key findings

The data gathered from my study reveal interesting findings with regards to multiplicative reasoning. Pre-test data reveal that initially, Grade 4 learners presented limited models which

comprised mainly of symbolic horizontal number sentences and column modelling which generally had a low success rate in the pre-test. However, after a series of intervention lessons, learners began to solve the problems using a broader range of models with ratio models being taken up. Horizontal number sentences and column models still dominated in the post-test; however, the success rate associated with these models improved suggesting that the work done during the intervention may have fed into learners being better able to solve the problems using models they were familiar with. The same could not be said about the control group as learners in the control group showed no changes with regards to model use and the gains made in the overall performance reflected a much smaller increase than that of the intervention group. Sense making of problems remained relatively the same in the control group thus not much improvement was seen with these learners.

The comparison of the overall performances in both groups suggest that the intervention lessons played an important role in advancing learners at both horizontal and vertical mathematization levels.

The main themes that emerged from the data were (a) additive bias, (b) number range, (c) number grabbing and (d) emerging take up of the ratio model.

Additive bias was common across all the questions and seen in the setting up of incorrect models. Learners who defaulted to addition failed to see the multiplicative nature of the relationship between the quantities in the problems (Hart, 1981).

In terms of the **number range**, an in-depth discussion in the early sections of Chapter 5 indicates that this was one of the reasons as to why different results and performances were noted. Questions which included a larger number range such as questions 6, 11 and 14 tended to be more challenging for learners to solve. Question 6 involved division of a three-digit by one-digit division whereas questions 11 and 14 involved two-digit by two-digit multiplication. Errors were commonly linked to the long multiplication algorithm with learners finding difficulty in using the model to find the solution. Learners were using this algorithm incorrectly, multiplying digits in ways that were not mathematically correct and ignoring the idea of place value (Hansen, 2011).

Another theme which emerged in the questions that contained superfluous information was **number grabbing** whereby learners had inappropriately selected information from the

problems utilizing all the numbers provided (Cook, 2006; Littlefield & Rieser, 1993; Hegarty et al., 1995). Two questions (question 7 and 9) contained irrelevant information to determine whether learners could appropriately select information in the text. The findings indicated that a high number of learners in both groups ‘number grabbed’ in the pre-test. However, post- intervention, learners in the intervention group presented fewer incorrect models (number grabbing) suggesting that they could better discriminate between critical and superfluous information in the text, contrasting with the control group where learners’ difficulties in discriminating between irrelevant and relevant information with number grabbing remained the same. Norvedt (2008) states that solving problems is not a linear process and demands are placed on both comprehension and solving strategies especially in discerning between relevant and irrelevant information. Studies from as early as the 80’s (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1980) have shown that when problems were presented in a format that differed from the standard format (i.e. when they required more than one step to solve or when they contained extraneous or insufficient information), students had a great deal of difficulty reaching correct solutions (Littlefield & Rieser, 1993). Thus, intervention lessons assisted in highlighting key aspects of the problems that were of importance to solving the problems correctly.

Lastly, the findings from my results revealed that at least a third of the intervention group learners had taken up a **ratio model** in the post-test whereas the control group did not present any of these models in both tests. Since the aim of my study was to explore and develop multiplicative reasoning, I focused on developing meaning through models with the initial starting point resting in learners own ways (be it formal or informal) of modelling the problems. Learners’ models were talked through and discussed before new models were introduced to them. The double number line was introduced as an initial tool to later introduce the ratio t-table. This was because number lines are a representation that learners are familiar with especially if they work with them in addition and subtraction in the earlier years (grades 1 to 3). Küchemann et al. (2011) iterate that the double number line is an important model *for* and model *of* multiplication as it provides a neat representation of the multiplicative relations that exist. In describing RME, Van den Heuvel-Panhuizen (2002) posits that one way of supporting problems on ratio is through a double number line.

Learners in the intervention group generally understood this model quite well and as a group, different ways of skip counting on the double line were explored. However, although the

double number line was well adopted, not many learners used shorter skip counting approaches such as doubling and halving, when working out the answers suggesting the need for more practice and awareness of working with the model.

Ratio t-tables were introduced after the double number line so that links could be made between the ratio t-table and the double number line in terms of looking at the relationship between two quantities. Middleton & Van den Heuvel-Panhuizen (1995) mention that one way of assisting learners develop strategies for solving proportion is using ratio tables. Therefore, ratio tables encouraged the use of number strategies such as halving, doubling, multiplying by 10, and so on (Dole, 2008). Ratio t-tables also allowed learners to notice and understand the inverse nature of multiplication and division which provided them with the opportunity to work more flexibly with and/or between both operations. Finally, although the use of ratio modelling was not the dominant model adopted in the post-test with the intervention group, the emerging take-up of the model and its corresponding success rate suggest that the work done with learners in the intervention lessons may have fed into how they worked with familiar models leading to the improved performance and stronger gains with these learners.

6.3 Limitations

Very seldom are research studies conducted without any limitations to it and as with any novice research, this study too has its limitations. Two main limitations which I encountered were time and the data set.

This study took place over eight weeks with one to two lessons per week which was not enough time to work rigorously with learners so that larger gains could have been made. Due to time constraints, only ratio models and the array model (in the array questions) were explored. However, had there been more time and lessons allocated, a wider range of models could have been explored and developed. Additionally, due to compulsory school programs and assessments, lessons needed to be worked around and rescheduled – which sometimes resulted in the intervention lessons being slightly shorter.

With regards to my data set, the number of learners used for this study was relatively small. Only one school with a maximum of 61 learners was used. Thus, more generalized claims could not be made.

Moreover, after having completed my data collection, a thought crossed my mind about the effectiveness of the control group. Although my initial idea was to use a control group to judge the effectiveness of intervention with learners, it may have been more useful if the control group had been taught multiplicative structures traditionally by their teacher with subsequent comparisons made between methods of teaching in addition to the overall performance.

Overall, although strong claims could not be made from one small scale study, this study provides an important lens and enough to signal the role that different modelling approaches can play in contextualizing problems through an RME approach developing learners at both a horizontal and vertical mathematization level.

6.4 Recommendations for future research

The findings from this study provide good motivation for future research as to how different modelling approaches (particularly ratio t-tables and double number lines) can be fit into the teaching and learning of multiplicative structures. The evidence suggests that the skills and sense making learnt through different models can provide learners with the opportunity to reflect and express mathematical ideas more meaningfully. Further, the findings point to modelling as a tool for solving word problems, that supports performance and sense making on multiplication and division problems.

The results from my small-scale study add to the evidence base for this kind of intervention model across the intermediate phase with the use of a comparative control group. A broader study could reveal more far-reaching results on the shifts with regards to modelling approaches in multiplication and division. Broader studies could also result in more in-depth understanding of multiplicative concepts with regards to the models they are exposed to. A more extended period could allow learners to be frequently exposed to use introduced models which may make them more confident doers of multiplication and division.

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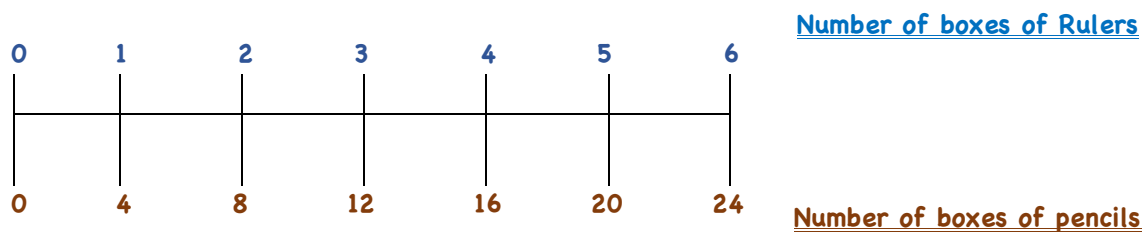
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APPENDICES

APPENDIX A: Example of Intervention Lesson 1 and 2

Objectives:
- Recognize 'Multiplication as Rate" problems - Use double number line to solve problems
Activity:
1. Count (Approximately 5 mins) - Count on in multiples of 3's, 4's and 5's - Be innovative in the way in which counting on is practiced so that it is not merely rote choral chanting 2. Pair/Whole Class Discussion - Put Problem (1) up on the board. Read through the problem with learners. Encourage learners to use any method or strategy they would like to use to solve the problem. <u>PROBLEM 1:</u> <i>Sam orders 4 boxes of pencils for every box of rulers. If Sam orders 6 boxes of rulers, how many boxes of pencils does she order?</i> - Ask the children to work individually or with a partner to find the solution. - Walk around and note the different methods used. - Particularly look out for children who may have used a number line. - If none of the children have used a number line, then introduce the model to them. We could show this problem on a double line: - Ask learners if they have worked with or seen a double number line before. - If they have, get them to briefly explain it to the class. - Thereafter, begin draw the double number line on the board



Explain to learners that the top row represents the number of boxes of rulers whereas the bottom row of numbers represents the number of boxes of pencils.

Therefore:

For 1 box of rulers, we have 4 boxes of pencils – this is what is told to us in the problem

For 2 boxes of rulers, we'd have 8 boxes of pencils

For 3 boxes of rulers, we'd have 12 boxes of pencils

For 4 boxes of rulers, we'd have 16 boxes of pencils

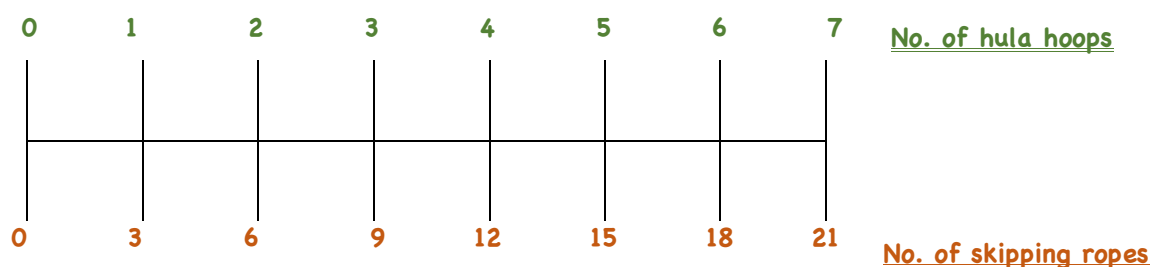
For 5 boxes of rulers, we'd have 20 boxes of pencils

For 6 boxes of rulers, we'd have 24 boxes of pencils

PROBLEM 2:

Caitlin is ordering skipping ropes and hula hoops. She orders 3 skipping ropes for every 1 hula hoop. If Caitlin ordered 7 hula hoops, how many skipping ropes will she have?

- Ask learners to solve this problem using the double number line (learners can work in pairs if needs be).
- Call learners up to show how they have modelled the problem.



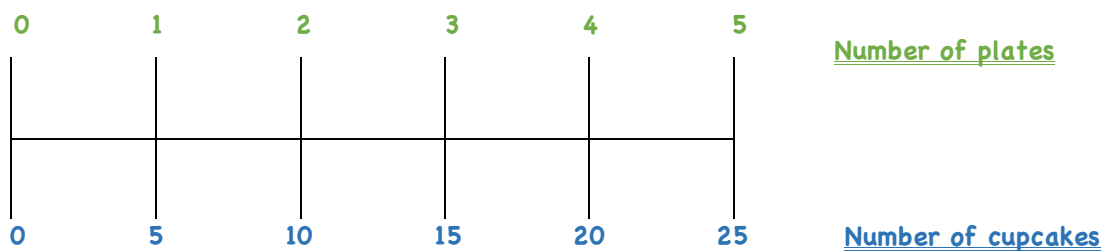
- Talk through the relationship between the numbers on both lines (the problem states that we have hula hoops and skipping ropes – for 1 hula hoop there are 3 skipping ropes)
- Show learners on the number line that for 2 hula hoops ordered, we'd have 6 skipping ropes; and so on...)
- Because Caitlin only ordered 7 hula hoops, we do not need to any further.

- Therefore, for 7 hula hoops, Caitlin orders 21 skipping ropes.

PROBLEM 3:

Thulela is packing cupcakes onto plates. In each plate, she packs 5 cupcakes. If Thulelah has 5 plates, how many cupcakes does she pack?

- Ask learners to draw a double number line for this problem
- Treat it the same as problem 2
- Re-inforce how the double number line works



Follow-up Problems:

- Children are expected and should be encouraged to use the double number line

Follow-up problems:

- A. Jack and Hassan are swapping stickers for magic beans. Jack swops 5 magic beans for every one sticker that Hassan gives him. If Hassan gives Jack 6 stickers, how many magic beans will Jack give Hassen?
- B. Miss Tommy is giving her class paint and paintbrushes for a task they have to do. For one paintbrush, the children get 3 colours of paint. If Miss Tommy gives the children 10 paintbrushes, how many colours of paint do they get?

Wrap-Up:

- Go over one or both of the follow up problems
- Work on representing the information on a number line
- Give learners a sheet with two problems to practice for homework

INTERVENTION LESSON 2

Objectives:

- Recognize 'Multiplication as Repeated Addition' problems
- Use the ratio table to record information

Activity:

3. Count (Approximately 5 mins)

- Count on in multiples of 2's, 4's and 5's
- Be innovative in the way in which counting on is practiced so that it is not merely rote choral chanting

4. Pair/Whole Class Discussion

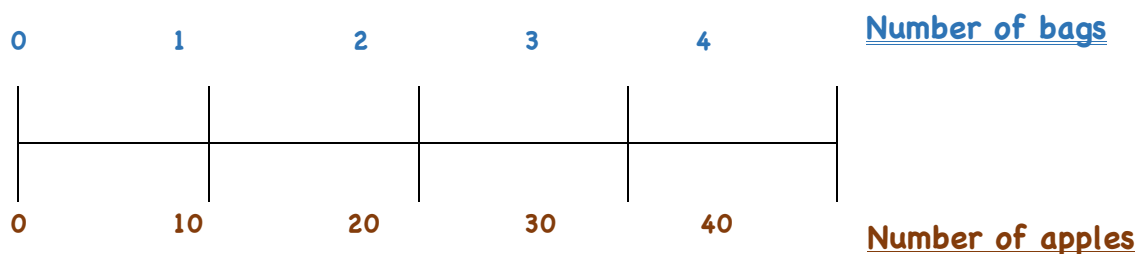
- Put Problem (1) up on the board.
Read through the problem with learners.

PROBLEM 1:

Sonny buys 4 bags of apples at the grocery store. The apples come 10 in a bag. How many apples to they have in total?

- Ask the children to work individually or with a partner to find the solution.
- Walk around and note the different methods used – since the double number line was introduced in the previous lesson, some learners may use that to solve the problem
- Some learners may also use repeated addition
- Particularly look out for children who have used the double number line, add together 4 lots of 10 and those who carry out multiplication of '4 multiplied by 10'.
- Work with the children to record these approaches:

a. Number Line



b. $10 + 10 + 10 + 10 = 40$ AND

c. $10 \times 4 = 40$

- Since there were 4 bags of 10 apples each, we could write this as
 $10 + 10 + 10 + 10 = 40$ however, it is a longer way of solving the problem.

- Another way of making sense of the problem using the double number line would be to use a table.
- Explain how to set up the table and how it works.
 - We have two things to consider (the number of bags and the number of apples)
 - Let's first look at one bag: In one bag, there are ten apples
 - If we read down our table, we can see that the 1 becomes times bigger (1×4) so likewise the 10 needs to be multiplied by 4.
 - In four bags, we have 'ten times' more apples therefore $10 \times 4 = 40$

Number of bags	Number of apples
1	10
2	20
3	30
4	40

Number of bags	Number of apples
1	10
4	?

PROBLEM 2:

Nomsa is packing shoes into boxes. She packs 6 shoes in one box. If Nomsa packs 6 boxes, how many shoes has she packed?

- Ask learners to draw up a table for Problem 2. They may draw a double number line to assist them

Number of boxes	Number of shoes
1	6
6	?

Number of boxes	Number of shoes
1	6
2	12
3	18
4	24
5	30
6	36

- Talk through the relationship between the numbers in the columns – the 1 becomes 6 times bigger therefore 6 needs to become 6 times bigger too: $6 \times 6 = 36$
- We can see above in the first table that to write down all the number of boxes is quite tedious – so we can ask ourselves if there is an easier way to do this.

- One way can be:

Number of boxes	Number of shoes
1	6
2	12
4	24
6	36

Number of boxes	Number of shoes
1	6
3	18
6	36

The completed tables should be left on the board as prompts

PROBLEM 3:

Donald Duck buys 15 packets of lollipops from the grocery store. There are 4 lollipops in a packet. How many lollipops does Donald Duck have in total?

- Learners are to draw a table for problem 3
- Thereafter get learners to come up and show their work on the board
- Reinforce the table – the different elements and how we could solve the problem using the table.

Number of packets	Number of lollipops
1	4
2	8
3	12
4	16
5	20
6	24
7	28
8	32
9	36
10	40
11	44
12	48
13	52
14	56
15	60

Since this table is long to do, we could use numbers that will make it easier for us:

We could ask ourselves what numbers make up 15

(I'm going to use 10 and 5 because those are easy numbers to multiply)

Therefore:

$10 \times 4 = 40$ and $5 \times 4 = 20$; 40 plus 20 = 60 SO in 15 packets there will be 60 lollipops

Number of packets	Number of lollipops
10	40
5	20
<u>15</u>	<u>60</u>

$$\therefore 15 \times 4 = 60$$

Follow-up Problem:

Follow-up problem:

Ani wanted to buy bags of marbles so that she could compete with her friends. Ani buys 4 bags of marbles. How many marbles does she have if one bag has 12 marbles?

- Ask learners to draw a table for this problem on their own
- Thereafter discuss learner's solutions.

Number of bags	Number of marbles
1	12
4	?

Wrap-Up:

- Go over the follow up problem reinforcing the ratio table

APPENDIX B: Ethics Clearance Letter

Wits School of Education



27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

6 June 2016

Protocol Number: 2016ECE010M

Student number: 442004

Dear Sameera Hansa

Application for ethics clearance: Master of Science

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Exploring multiplicative reasoning with a Grade 4 class through structured problem solving

The committee recently met and I am pleased to inform you that **clearance was granted**.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

A handwritten signature in black ink, appearing to read "M Mabete".

Wits School of Education

011 717-3416

cc Supervisor - Professor Mike Askew and Professor Hamsa Venkatakrishnan

APPENDIX C: Information Letters



LETTER TO THE PRINCIPAL, SGB Chair

June 2016

Dear Cheryl Donald

My name is *Sameera Hansa* and I am a *Master's of Science in Mathematics Education student* in the School of Education at the University of the Witwatersrand. I am doing research on *Exploring multiplicative reasoning with Grade 4 learners through structured problem solving*.

My research will focus on the strategies that Grade 4 learners use when working with multiplicative word problems. My aim is to investigate this issue through the design and implementation of a small-scale intervention program.

This study will comprise of tests (a pre-test and post-test) as well as intervention lessons and interviews with learners. Both the pre and post-test will be carefully administered and analyzed. These tests will analyze learner's strategies of working with multiplicative word problems. Additionally, tests written by learners will not be used as part of their normal assessments and will only be used for research purposes. In between the pre-test and post-test, I will be conducting an intervention program which will be made up of 6-8 lessons. The intervention lessons will equip learners with different reasoning skills of working with multiplicative problems.

After every intervention lesson, I will collect samples of learners work and the lesson design. Part of my research will comprise of me using learner's tests, written work, interviews, audio and video recordings. Interviews and audio recordings will be conducted during school time and will be conducted privately for no longer than 10 minutes. These interviews will be done with learners from the sample group (6 learners from each class) and will only be done should the need arise to gain further insight into the models and strategies that learners use. Video-recordings will be done during the intervention lessons and will only video learners approach to the given problems. The focus of my analysis will be on the reasoning that learners use when working with multiplicative word problems. Thereafter, I will conduct a post-test which will be further analyzed. This study will take place during the third term and in order to ensure that teaching and learning is not disrupted, all lessons and tests will be aligned with CAPS.

A control class will be used to determine the effectiveness of my intervention program. The control group will be required to write the pre-and posttest but will not be given the intervention lessons at the same time as the other two classes. However, should the intervention lessons prove to be useful and effective, I will then conduct the same lessons with the control class or work with the relevant teacher so that she/he could conduct the same lessons with her class. Lesson designs from the intervention program will be readily available to the school and relevant teachers.

The reason why I have chosen your school is because I was a former Mathematics teacher at the school with a great interest in improving mathematics education. I believe that this study will help benefit me as a teacher which will in turn benefit children in general. Therefore, I am inviting the school to participate in this research.

The research participants (learners) will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in participating in this study. The participants will not be paid for this study.

The names of the research participants and identity of the school will be kept confidential at all times and in all academic writing about the study. Pseudonyms will be used in the place of the learners' names and of the school. Privacy will be maintained in all published and written data resulting from the study. The findings from this study may be used in academic journals and conferences. All research data will be kept safely under lock and key at my home and on my personal computer in a separate folder which requires a password. All research data will be destroyed between 3-5 years after completion of the project. All raw research data will be deleted and shredded. Please let me know if you require any further information. I look forward to your response as soon as is convenient.

Yours sincerely,

A handwritten signature in black ink, appearing to read 'Sameera'.

NAME: Sameera Hansa

EMAIL: sameera2802@hotmail.com or sameera.hansa@students.wits.ac.za

MOBILE NUMBER: 081 300 8541



INFORMATION SHEET PARENTS

June 2016

Dear Parent

My name is *Sameera Hansa* and I am a *Master's of Science in Mathematics Education student* in the School of Education at the University of the Witwatersrand.

I am doing research on *Exploring multiplicative reasoning with Grade 4 learners through structured problem solving*.

My research will focus on the multiplicative reasoning that Grade 4 learners use when working with multiplicative word problems. My aim is to investigate this issue through the design and implementation of a small-scale intervention program.

This study will comprise of a pre-test and post-test. Both tests will be carefully administered and analyzed. These tests will analyze learner's strategies of working with multiplicative word problems. Additionally, tests will not be for marks or part of learner's normal assessments. Rather, they will only be used for research purposes.

In between the pre-test and post-test, I will be conducting an intervention program which will be made up of 6-8 lessons. The intervention lessons will equip learners with different models and strategies of working with multiplicative word problems. After every intervention lesson, I will collect samples of learners work and the lesson design. The focus of my analysis will be on multiplicative reasoning that learners use when dealing with word problems. Thereafter, I will conduct a post-test which will be further analyzed. This study will take place during the third term. In order to ensure that teaching is not disrupted, all lessons and tests will be aligned with CAPS.

The reason why I have chosen your child's class is because I feel passionate about improving Mathematics Education and would like to support your child together with other children so that they can improve their understanding and performance in multiplicative problems. Further I would like to develop myself through this process for the benefit of the learners.

I was wondering whether you would mind if your child participates in this study. Your child will need to complete a pre-test and post-test on multiplicative word problems as well as possibly answer a few short questions in the form of an interview which will be audio taped. Not all learners will be audio-taped. Learner's approaches to word problems will be videotaped so that I could gain an in-depth understanding of their multiplicative reasoning practices. I will also take photographs only of your child's work. No photographs will be taken of the children themselves. Interviews will be done at school privately and will not last longer than 10 minutes.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

The findings from this study may be used in academic journals and at conferences. However, your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study. Pseudonyms will be used in the place of his/her name. All research data will be kept safely under lock and key at my home and on my personal laptop in a separate folder which will require a password. All raw research data will be destroyed between 3-5 years after completion of the project. Research data will be deleted and shredded.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

NAME: Sameera Hansa

EMAIL: sameera2802@hotmail.com or sameera.hansa@students.wits.ac.za

MOBILE NUMBER: 081 300 8541



INFORMATION SHEET - LEARNERS

June 2016

Dear Learner

My name is **Sameera Hansa** and I am a *Master's of Science in Mathematics Education student* in the School of Education at the University of the Witwatersrand.

I am doing research on *Exploring multiplicative reasoning with Grade 4 learners through structured problem solving*.

I am interested in the methods that Grade 4 learners use when working with multiplication and division word problems. I would like to investigate this problem through providing you with a program of six – eight lessons where different ways of working with word problems will be shown to you. After every lesson, I will collect samples of your work. I am interested in looking at the different methods that you use when doing word problems. Before and after the lessons, you will need to write a test on word problems. These tests will not be for marks but will only be used for my research. This study will take place during the third term.

I was wondering whether you would mind if I invited you to participate in this research.

I need your help with participating in completing the tests that I give to you as well as possibly answering questions in the form of an interview which will be audio taped. These interviews will help me gain a better understanding of some of work. Lessons that I do will be videotaped just so that I can have a better understanding of the methods you use to solve word problems. I will also take photographs of your work. Interviews will be done at school privately and will not last longer than 10 minutes.

Remember, though there is a test, it is not for marks and it is completely voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and you will not be advantaged or disadvantaged in any way. Also, no money will be paid to you for participating. I will not be using your own name but I will make one up so no one can identify you. All information about you will be kept private in all my writing about the study. I will only be using your work for my university research project. The findings from my research may be used in journals and at conferences.

Also, all collected information will be stored safely and privately. All data used for my project will be deleted and shredded between 3-5 years after I have completed my project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is entirely your decision to join me in the study.

I look forward to working with you.

Please feel free to contact me if you have any questions.

Yours Sincerely,

NAME: Sameera Hansa

EMAIL: sameera2802@hotmail.com or sameera.hansa@students.wits.ac.za

MOBILE NUMBER: 081 300 8541



INFORMATION SHEET – LEARNERS (Control Group)

June 2016

Dear Learner

My name is **Sameera Hansa** and I am a *Master's of Science in Mathematics Education student* in the School of Education at the University of the Witwatersrand.

I am doing research on *Exploring multiplicative reasoning with Grade 4 learners through structured problem solving*.

I am interested in the methods that Grade 4 learners use when working with multiplication and division word problems. I am interested in looking at the different methods that you use when doing word problems. In order for me to do this, you will need to write two tests on word problems. These tests will not be for marks but will only be used for my research. This study will take place during the third term.

I was wondering whether you would mind if I invited you to participate in this research. I need your help with participating in completing the tests that I give to you as well as possibly answering questions in the form of an interview which will be audio taped. These interviews will help me gain a better understanding of some of work. I will also take photographs of your work. Interviews will be done at school privately and will not last longer than 10 minutes.

Remember, though there is a test, it is not for marks and it is completely voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and you will not be advantaged or disadvantaged in any way. Also, no money will be paid to you for participating. I will not be using your own name but I will make one up so no one can identify you. All information about you will be kept private in all my writing about the study. I will only be using your work for my university research project. The findings from my research may be used in journals and at conferences.

Also, all collected information will be stored safely and privately. All data used for my project will be deleted and shredded between 3-5 years after I have completed my project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is entirely your decision to join me in the study.

I look forward to working with you.

Please feel free to contact me if you have any questions.

Yours Sincerely,

NAME: Sameera Hansa

EMAIL: sameera2802@hotmail.com or sameera.hansa@students.wits.ac.za

MOBILE NUMBER: 081 300 8541

APPENDIX D: Consent Forms



Parent's Consent Form

Kindly fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:

Exploring multiplicative reasoning with Grade 4 learners through structured problem solving

I, _____ the parent of _____

Permission to review/collect documents/artifacts

I agree that my child's (Pre-Test, Post-Test and Written work) can be used for this study only.

Circle one

YES/NO

Permission to observe my child in class

I agree that my child may be observed in class.

YES/NO

Permission for test

I agree to my child writing write test for this study.

YES/NO

Permission to be videotaped

I agree my child may be videotaped in class.

YES/NO

I know that the videotapes will be used for this project only.

YES/NO

Permission to be audio taped

I agree that my child may be audio taped during interview or observations.

YES/NO

I know that the audiotapes will be used for this project only

YES/NO

Permission to be interviewed

I agree that my child may be interviewed for this study.

YES/NO

I know that he/she can stop the interview at any time and doesn't have to answer all the questions asked.

YES/NO

Permission to be photographed

I agree that my child's work may be photographed during the study

YES/NO

I know that I can stop this permission at anytime

YES/NO

I know that the photos will be used for this project only

YES/NO

Informed Consent

I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audio taped, videotaped and/or photographed
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign _____ Date _____



Learner Consent Form

Please fill in the reply slip below if you agree to participate in my study called:

Exploring multiplicative reasoning with Grade 4 learners through structured problem solving

My name is: _____

Permission to review/collect documents/artifacts

I agree that (Pre-Test, Post-Test, your written work) can be used for this study only.

Circle one

YES/NO

Permission for test

I agree to write a test for this study.

YES/NO

Permission to observe you in class

I agree to be observed in class.

YES/NO

Permission to be audio taped

I agree to be audio taped during the interview or observation lesson

YES/NO

I know that the audiotapes will be used for this project only

YES/NO

Permission to be interviewed

I agree to be interviewed for this study.

YES/NO

I know that I can stop the interview at any time and don't have to answer all the questions asked.

YES/NO

Permission to be photographed

I agree for my work to be photographed during the study

YES/NO

I know that I can stop this permission at any time

YES/NO

I know that the photos will be used for this project only

YES/NO

Permission to be videotaped

I agree to be videotaped in class.

YES/NO

I know that the videotapes will be used for this project only.

YES/NO

Informed Consent

I understand that:

- My name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audio taped, videotaped and/or photographed.
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign _____ Date _____



Teacher Consent Form

Please fill in the reply slip below if you agree to participate in my study called:

Exploring multiplicative reasoning with Grade 4 learners through structured problem solving

My name is: _____

Permission to give learners pre-and posttest

I agree that you can give learners the pre-and post-test

Circle one

YES/NO

Permission for intervention lessons

I agree to have 6-8 lessons for the intervention program with my class of Grade 4 learners

YES/NO

Informed Consent

I understand that:

- My name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I can withdraw from the study at any time.
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign _____ Date _____



Learner Consent Form (Control Group)

Please fill in the reply slip below if you agree to participate in my study called:

Exploring multiplicative reasoning with Grade 4 learners through structured problem solving

My name is: _____

Permission to review/collect documents/artifacts

I agree that (Pre-Test and Post-Test) can be used for this study only.

Circle one

YES/NO

Permission for test

I agree to write a test for this study.

YES/NO

Permission to be audio taped

I agree to be audio taped during the interview

YES/NO

I know that the audiotapes will be used for this project only

YES/NO

Permission to be interviewed

I agree to be interviewed for this study.

YES/NO

I know that I can stop the interview at any time and don't have to answer all the questions asked.

YES/NO

Permission to be photographed

I agree for my work to be photographed during the study

YES/NO

I know that I can stop this permission at any time

YES/NO

I know that the photos will be used for this project only

YES/NO

Informed Consent

I understand that:

- My name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audio taped or photographed.
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign _____ Date _____

APPENDIX E: Table summaries of multiplication and division word problems for control group

TABLE SUMMARY OF MULTIPLICATION QUESTIONS (CONTROL GROUP; n=24)											
CLASSIFICATION OF PROBLEMS	QUESTIONS	CALCULATION (BARE PROBLEMS)	MODELS								
					No model Incorrect	Incorrect Model	Diagrammatic Representation	Symbolic Repeated Addition	Horizontal Number Sentence or Column Multiplication	Ratio Model	No model Correct
MULTIPLICATION AS RATE	2 Thabo planted 10 pots with magic beans. He planted 7 magic beans in each pot. How many beans did Thabo plant?	10 × 7	PRE-TEST	FREQUENCY	0 (0%)	4 (17%)	8 (33%)	2 (8%)	13 (54%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	8 (33%)	2 (8%)	12 (50%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	0 (0%)	3 (13%)	3 (13%)	3 (13%)	17 (71%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	3 (13%)	3 (13%)	17 (71%)	0 (0%)	0 (0%)
	11 There are 12 friends at an orange farm. They each collect 13 oranges in their bags. How many oranges do they have altogether?	12 × 13	PRE-TEST	FREQUENCY	1 (4%)	6 (25%)	3 (13%)	3 (13%)	13 (54%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	1 (4%)	1 (4%)	6 (25%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	1 (4%)	5 (21%)	2 (8%)	1 (4%)	15 (63%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (4%)	8 (33%)	0 (0%)	0 (0%)
MULTIPLICATION AS REPEATED ADDITION	7 Sam loves collecting bottle tops and straws. She collected 11 bottle tops on Tuesday, 11 bottle tops on Wednesday, 15 straws on Thursday and 11 bottle tops on Friday. How many bottle tops will Sam have collected?	10 × 7	PRE-TEST	FREQUENCY	0 (0%)	8 (33%)	0 (0%)	14 (58%)	2 (8%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	14 (58%)	2 (8%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	0 (0%)	8 (33%)	0 (0%)	14 (58%)	2 (8%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	14 (58%)	2 (8%)	0 (0%)	0 (0%)
MULTIPLICATION AS SCALE	9 Dexter has a magic bag of rice and potatoes. The bag of potatoes weighs 8kg and the bag of rice weighs 7kg. Dexter pours a growing potion and the bag of rice becomes 4 times as heavy. How much does Dexter's bag of rice now weigh?	7 × 4	PRE-TEST	FREQUENCY	3 (13%)	9 (38%)	2 (8%)	3 (13%)	7 (29%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	2 (8%)	3 (13%)	5 (21%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	5 (21%)	7 (29%)	0 (0%)	2 (8%)	10 (42%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (4%)	9 (38%)	0 (0%)	0 (0%)
	13 Leti and Ayesha have string. Leti's string is 14 metres long and Ayesha's string is 7 times as long. How long is Ayesha's string?	14 × 7	PRE-TEST	FREQUENCY	4 (17%)	10 (42%)	1 (4%)	3 (13%)	8 (33%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	1 (4%)	1 (4%)	5 (21%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	2 (8%)	7 (29%)	0 (0%)	1 (4%)	15 (63%)	0 (0%)	1 (4%)
				CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (4%)	10 (42%)	0 (0%)	1 (4%)
MULTIPLICATION AS ARRAY	10 Thulelah is packing rows of pies in a tray. The tray can fit 6 pies across and 3 pies down. How many pies can Thulelah pack in the tray?	6 × 3	PRE-TEST	FREQUENCY	2 (8%)	10 (42%)	6 (25%)	2 (8%)	6 (25%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	3 (13%)	1 (4%)	4 (17%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	3 (13%)	9 (38%)	2 (8%)	2 (8%)	9 (38%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	2 (8%)	2 (8%)	8 (33%)	0 (0%)	0 (0%)
	14 Lerato is tiling a school hall floor that is 16 metres long and 14 metres wide. How many tiles will Lerato need to tile the school hall if the tiles are 1m by 1m?	16 × 14	PRE-TEST	FREQUENCY	5 (21%)	15 (63%)	1 (4%)	0 (0%)	4 (17%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	1 (4%)	0 (0%)	2 (8%)	0 (0%)	0 (0%)
			POST-TEST	FREQUENCY	6 (25%)	13 (54%)	0 (0%)	0 (0%)	7 (29%)	0 (0%)	0 (0%)
				CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	0 (0%)	3 (13%)	0 (0%)	0 (0%)

TABLE SUMMARY FOR DIVISION AS SHARING QUESTIONS (CONTROL GROUP; n=24)												
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS								
				No model Incorrect	Incorrect Model	Model of $120 \div 5$	Diagrammatic Representation	Idiosyncratic Division	Horizontal number sentence or Traditional Division	Ratio Model	No model Correct	Other
3 9 friends were given a bag of 72 sweets. They shared them put equally. How many sweets did they each get?	$72 \div 9$	PRE-TEST	FREQUENCY	0 (0%)	3 (13)	0 (0%)	9 (38%)	6 (25%)	21 (88%)	0 (0%)	0 (0%)	2 (8%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	9 (38%)	6 (25%)	21 (88%)	0 (0%)	0 (0%)	2 (8%)
		POST-TEST	FREQUENCY	0 (0%)	7 (29%)	0 (0%)	6 (25%)	3 (13%)	17 (71%)	0 (0%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	5 (21%)	1 (4%)	12 (50%)	0 (0%)	0 (0%)	0 (0%)
6 Thando makes 120 bottles of lemonade for a school market day. The bottles of lemonade are shared equally between thando and 5 of her friends. How many bottles of lemonade do they each get to sell?	$120 \div 6$	PRE-TEST	FREQUENCY	2 (5%)	14 (58%)	12 (50%)	4 (17%)	1 (4%)	7 (29%)	0 (0%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	5 (21%)	3 (13%)	0 (0%)	3 (13%)	0 (0%)	0 (0%)	0 (0%)
		POST-TEST	FREQUENCY	1 (4%)	14 (58%)	10 (42%)	2 (8%)	3 (13%)	16 (67%)	0 (0%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	5 (14%)	1 (3%)	0 (0%)	13 (54%)	0 (0%)	0 (0%)	0 (0%)
TABLE SUMMARY FOR DIVISION AS GROUPING QUESTIONS (CONTROL GROUP; n= 24)												
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS								
				No model Incorrect	Unsuitable Answer	Incorrect Model	Diagrammatic Representation	Idiosyncratic Division	Horizontal number sentence or Traditional Division	Ratio Model	No model Correct	Other
12 66 children need to be transported to a sports match. The school has arranged for parents to take the children by car. If the cars can only take 4 children, how many cars will be needed to transport all the children?	$66 \div 4$	PRE-TEST	FREQUENCY	4 (15%)	15 (63%)	5 (21%)	3 (13%)	2 (8%)	18 (75%)	0 (0%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	3 (13%)	0 (0%)	4 (17%)	0 (0%)	0 (0%)	0 (0%)
		POST-TEST	FREQUENCY	4 (17%)	8 (33%)	4 (17%)	3 (13%)	2 (8%)	24 (100%)	0 (0%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	2 (8%)	0 (0%)	12 (50%)	0 (0%)	0 (0%)	0 (0%)
16 Jill has 45 eggs to take to the market. An egg box can hold 6 eggs. How many boxes will Jill need of she wants to take all the eggs to the market?	$45 \div 6$	PRE-TEST	FREQUENCY	7 (29%)	19 (79%)	3 (13%)	3 (13%)	3 (13%)	13 (54%)	0 (0%)	1 (4%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (4%)	0 (0%)	1 (4%)	0 (0%)	1 (4%)	0 (0%)
		POST-TEST	FREQUENCY	6 (25%)	12 (50%)	3 (13%)	1 (4%)	2 (8%)	22 (92%)	0 (0%)	0 (0%)	0 (0%)
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (4%)	0 (0%)	9 (38%)	0 (0%)	0 (0%)	0 (0%)
TABLE SUMMARY FOR DIVISION AS REPEATED SUBTRACTION QUESTIONS (CONTROL GROUP; n= 24)												
QUESTIONS	CALCULATION (BARE PROBLEMS)			MODELS								
				No model Incorrect	Incorrect Model	Diagrammatic Representation	Idiosyncratic Division	Horizontal number sentence or Traditional Division	Ratio Model	No model Correct	Other	
8 Mary has 96 biscuits that she wants to put into boxes. 12 biscuits can fit into one box. How many boxes can Mary fill?	$96 \div 12$	PRE-TEST	FREQUENCY	3 (13%)	7 (29%)	2 (8%)	3 (13%)	16 (67%)	0 (0%)	2 (8%)	3 (13%)	
			CORRECT ANSWER	0 (0%)	0 (0%)	0 (0%)	1 (4%)	10 (42%)	0 (0%)	2 (8%)	2 (8%)	
		POST-TEST	FREQUENCY	3 (13%)	3 (13%)	2 (8%)	3 (13%)	21 (88%)	0 (0%)	1 (4%)	2 (8%)	
			CORRECT ANSWER	0 (0%)	0 (0%)	2 (8%)	0 (0%)	10 (42%)	0 (0%)	1 (4%)	2 (8%)	
15 Sarah is packing T-shirts onto shelves in a clothing store. One shelf can hold 44 t-shirts. There are 88 T-shirts that need to be packed. How many shelves would needed to pack all the T-shirts away?	$88 \div 44$	PRE-TEST	FREQUENCY	3 (13%)	6 (25%)	1 (4%)	3 (13%)	16 (67%)	0 (0%)	2 (8%)	2 (8%)	
			CORRECT ANSWER	0 (0%)	0 (0%)	1 (4%)	3 (13%)	15 (63%)	0 (0%)	2 (8%)	1 (4%)	
		POST-TEST	FREQUENCY	4 (17%)	5 (21%)	1 (4%)	3 (13%)	16 (67%)	0 (0%)	1 (4%)	4 (17%)	
			CORRECT ANSWER	0 (0%)	0 (0%)	1 (4%)	2 (8%)	15 (63%)	0 (0%)	1 (4%)	2 (8%)	