MSc Research Report

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

Application of queueing theory in the bed allocation policy of a hospital intensive care unit

by **Vusani Masocha**
Student no: 789644

Supervisor: Dr HW Chipoyera

A research report submitted to the Faculty of Science, University of the Witwatersrand, in partial fulfillment of the requirements for the degree of Masters of Science (by Coursework and Research)

School of Statistics and Actuarial Science

October 1, 2021

DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Masters in Statistics at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

(Signature of candidate)________________________________

October 1, 2021

Abstract

An intensive care unit of a hospital receives critically ill patients at any given time. If no proper management of the flow of patients is done, a dire situation will be the order of day. This study has explored the problem of determining an optimal number of Intensive Care Unit (ICU) beds in such a way that a certain level of service set is achieved. The purpose of the research has been to assess the patients flow in and out of the ICU in order to identify the best queueing system.

This research work has found that patient inter-arrival times can be modelled using an exponential distribution although the lognormal distribution provides a better fit. Also the waiting times possibly follow an exponential distribution. On the basis of the Akaike Information Criterion and Bayes Information Criterion, it was found that the waiting times are best modelled using a Gamma distribution. The length of stay in the ICU by patients also can be modelled by an exponential distribution. The lognormal is however a better distribution at modelling the length of stay times. The two queueing models were compared. Another dimension explored in the analysis was to check for factors that impact on the length of stay of patients in an ICU. It was found that type of insurance and admission type (which is the nature of the illness) were the only two covariates that signficantly impacted on the length of stay. The research work winds up by giving recommendations, particularly on the need to replicate a similar study at an ICU in a health facility in South Africa.

Acknowledgments

The author wishes to express his profound gratitude to the following people:

1. Dr Honest Chipoyera, who is his supervisor, for his patience and guidance,

2. Dr David Rose for invaluable comments which helped shape the direction of this research work,

3. Professor Frank Beichelt who was his supervisor when he started working on this research project,

4. Dr Thulani Tshabala for the advise and guidance that he gave,

5. his family for their patience and relentless support.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 The South African health sector

According to Walther (2005), from 1994, a great deal of transition has been witnessed in the South African health care system. This has been due to changes in a combination of citizens socio-economic factors such as social status, financial status and transition of communities. As a result there have been some restrictions in the South African healthcare system. An example is that of failure to access some medical facilities such as the Intensive Care Unit (ICU) for certain citizens categories. Access to some medical facilities is not possible if one does not have medical insurance or their finances are deemed inadequate.

Belciug and Gorunescu (2015) state that diseases such as HIV/AIDS and tuberculosis alongside increasing poverty levels and population increases have put exerted enormous pressure on the South African medical sector. Mayosi and Benatar (2014) note that the global economic trend has a major influence on the provision of good health care service being offered in Africa as governments have low revenues and cannot channel enough resources to the health sector. The effect of this is that it has a direct impact on critically important hospital elements such as ICUs of hospitals when it comes to the allocation of key resources like hospital beds.

1.2 ICUs in South Africa

Chanques et al. (2006) say that an ICU is a department in a hospital which houses seriously ill patients who are monitored and treated by specially trained nurses and doctors throughout their stay there.

Mayosi and Benatar (2014) say that health professionals have the primary duty of ensuring the well-being of their patients through taking utmost care of them. Hospital management in conjunction with personnel in operations are the main people who make decisions in this regard. One of the critically important questions that health professionals need to address is: "Who gets priority in getting an ICU bed?" In the past, rules have been put in place in such a way that getting allocated a bed is made on the basis of rationality of the hospital staff (Sinuff et al., 2004). However the challenge is that staff are susceptible to human error caused by emotional issues and maladministration. A more rigid and yet flexible mathematical approach is therefore seen as being more objective, impartial, and is proposed to ensure that patients receive the fairest care possible.

Gorunescu (2011) says that allocation of resources in hospitals is on two levels: the first level deals with the organisation of public health care change while the second level addresses the specific criteria employed from day to day decision making with regards to utilisation of available resources (which are usually constrained) while dealing with a demand that often exceeds supply.

Remark 1 *This study endeavours to assess the effectiveness of bed allocation policies in hospital ICUs. While there exist methods augmented or supported* *by the ICU objective scores such as the Acute Physiology and Chronic Health Evaluation (APACHE) II mortality prediction score, the author is of the firm belief that a mathematical model can be helpful to improve decision making that will ensure maximum usage of beds and allocated space.*

1.3 ICU Patient Flow Modelling

An ICU can be modelled using a modified version of a standard queueing system illustrated in Figure 1.1. Patients arrive randomly and are admitted to the ICU ward, some are treated and exit, while some unfortunately die (Beichelt, 2006). Upon arrival, the patient either joins the queue and waits to be allocated a bed in the ICU or is turned away to some other health facility. Some patients desert the queue after some time and this element of deserting the queue is called *abandonment*. Therefore, an ICU can be modelled using a *waiting loss* system with finite waiting capacity for patients (Beichelt, 2006).

1.3.1 Resource constraints in ICUs

According to Vincent and Singer (2010), an ICU offers monitoring of critically ill patients as well as provide organ support coupled with intervention that cannot be delivered in a general ward.

(Haupt et al., 2003) discuss various ICU resource constraints that include skilled human personnel, equipment and documentation. They also mention beds as a critical resource and how ICUs are subjected to excessive pressure since every patient admitted to an ICU requires a bed but may not necessarily need all of the specialized services on offer such as radiographic scanning. The National Health Service (NHS) Confederation in 2004 reports that the number of beds in South African hospitals experienced a decline of 31% during the 1984 to 2004 period, whilst in that same period there was an increase in the number of in-patients (Haupt et al., 2003). The pressure

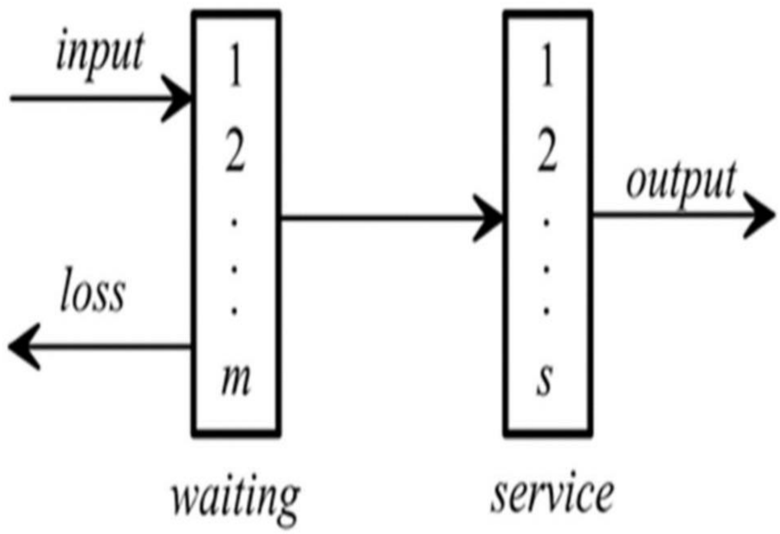

Figure 1.1: ICU waiting line

effectively exerted on hospitals, especially ICUs, is consequently unbearable and requires expertise in planning occupancy and distribution of resources in order to ensure acceptable levels of queueing time by patients awaiting treatment.

Another challenge is that it is possible that a patient may find a bed in an ICU, but the technology needed for his treatment is presently being used by other patients thus the patient is forced to wait (Lattimer, Brailsford, Turnbull, Tarnaras, Smith, George, Gerard and Maslin-Prothero (2004)).

1.4 Statement of the Problem

The rapid population growth that the Republic of South Africa has experienced[1] has heralded a myriad of challenges to many public service systems. The Department of Health has not been spared and is grappling with these challenges and this has had a knock-on effect on hospital ICUs which are vital elements of any hospital. The study by Belciug and Gorunescu (2015), notes that from 2009 to date the *usable bed capacity in hospitals*[2] increased from 65% to 77%. This clearly shows that pressure on hospitals in as far as the allocation of beds in ICUs is on the increase.

Luchetti (2013) states that judiciously and meticulously administering the process of patient admission and assignment to wards are important tasks in ensuring good management of hospital operations. Consequently, the allocation of beds plays a vital role in the operational functions of hospitals in ensuring that patients receive the best possible treatment. For the ICU to optimise the usage of these facilities, it is an imperative that an algorithm that effectively manages bed allocation in ICUs of hospitals is developed.

ICU beds are allocated to the most deserving patient candidates in terms of their clinical judgement which is invariably augmented by objective ICU scores such as the APACHE II mortality prediction score and also on a first come first serve basis. Allocation rate which is determined by the length of stay by patients in the ICU is challenge. Some patients stay longer depending on their ailments. It is expected that an application of Queueing Theory in the allocation of beds in an intensive care unit will most probably assist in ensuring a more effective allocation policy of beds in the ICU.

[1]South Africa's population size in the years 1994 and 2019 were 33.27 million and 58.78 million (Stats, 2020), respectively

[2]The percentage of beds occupied by patients out of the total number of beds available

1.5 Aim and Objectives

1.5.1 Aim

The aim of this study is to ascertain whether or not the use of Queueing Theory results in a better allocation policy of beds in a hospital ICU when compared to the current policy. The comparison is mainly based on reduced queueing time and improved ICU resource utilisation.
This study has to provide convincing answers to the following pertinent questions:

1. Will using queueing theory result in an improvement of the flow of patients in and out of an ICU?

2. Will the model improve bed allocation in an ICU compared to the current method used?

3. Will the outcomes/results of the study help maximise the use of resources at an ICU?

Remark 2 *Any policy that results in the time patients have to wait in a ward before being transfered to an ICU being minimised is a heartily welcome policy.*

1.5.2 Objectives

The objectives of this study, which will help realise the aim, are:

1. to explore the efficacy of different queuing models and identify the queuing model that best fits the ICU patient flow problem, and

2. to determine or ascertain the optimal number of beds in a hospital ICU; in calculating the optimal capacity, presumably the best possible way of doing so is through connecting the supply (as measured by the

arrival rate of patients at the ICU) and system service rate as measured by the average number of patients exiting the ICU per unit of time.

1.6 Assumptions of the models

This study makes the following assumptions:

- the number of patients arriving and queuing to be admitted at the ICU per unit of time and the number of patients leaving the ICU per unit of time are independent random variables,

- the time between the release of a serviced customer/patient from the ICU facility and the admission of a new patient from the waiting line is negligible,

- The number of nurses and doctors required in the ICU (as well as operational costs incurred by the hospital in running the ICU) is commensurate with the number of beds in the ICU,

- a patient admitted in the ICU only leaves the ICU as soon as (1) he/she is deemed stabilised (upon which he/she is transferred to an ordinary ward) or (2) upon his/her death,

- The time spent receiving service in the ICU is independent of the time spent waiting in the queue to be admitted to the ICU.

1.7 Relevance of the Study

According to Luchetti (2013), intensive care of patients admitted in hospital is probably the most expensive speciality in any medical facility. The limited number of beds is a frequently encountered problem and is considered as probably the toughest challenge for a hospital. Ross (2017) states that the concept of the allocation of medical resources on the basis of relative medical benefits is referred to as *triaging* (Lattimer et al., 2004). Triaging involves establishing the most appropriate disposition of a patient on the basis of the assessment of the patient's illness and medical emergency (Ross, 2017).

Skowronski (2001) has stated that the methods available to assist physicians faced with a dilemma when it comes to patient-selection include cost effectiveness and utility analysis. It is clear that resources used in ICUs are the critically important cost drivers and this does affect allocations and usage by patients to a great extent. According to Luchetti (2013), ICU decisions are not only made on the basis of providing medical care to patients; the decisions also take on board ethical, economic, social and legal considerations. Therefore, hospitals face a dilemma of whether to hire more medical staff or buy more equipment for ICU. This question is particularly important in these times of difficult global financial conditions where budgets are restricted. Too many beds result on average in wasteful expenditure and fewer beds than needed may have dire consequences, particularly in emergency situations. (Lanken et al., 1997) states that excessive pressure is presently exerted on hospitals due to increases in population as well as dwindling resources, making both the decision making and implications quite complex.

Chapter 2

Literature Review

2.1 Queueing Theory

Waiting in a queue for service at banks, shops, hospitals, petrol stations and so forth, is of course an inevitable "nuisance". People cannot avoid waiting in queues whenever they seek a service but through careful planning, waiting times can be reduced to bearable levels. Numerous studies (e.g. Skowronski (2001), Gorunescu (2011)) on congestion and delays associated with waiting in a queue as well as intervention methods in order to reduce the waiting time have been conducted.

2.1.1 Kendall's notation

Kendall's notation is the universally accepted notation for characterising queuing models. According to Beichelt (2006), the notation takes the form $v/w/x/y/z$ where

v - stands for the probability distribution of the number of customers arriving to receive service per unit of time,

w - stands for the probability distribution of the times taken to service the customers,

x - stands for the number of servers available,

y - stands for the capacity of the queuing system,

z - stands for the queue service discipline.

Franzese, Fioroni, Botter, and de Freitas Filho (2009) weighs in to provide further clarity on codes used to describe incoming calls and their service times as follows:

Code to describe incoming calls.

M stands for "Markovian", implying that the time between two successive incoming customers follows an exponential distribution,

Codes to describe the service process.

M stands for "Markovian", implying that service times follow an exponential distribution,

G stands for "General Distribution",

Definition 2.1.1 *A queuing system is said to be in steady state if* $\{N(t), t \geq 0\}$ *is a stationary process.*

Remark 3 *The number of busy servers when the system is in steady state, S is a random variable such that $0 \leq S \leq s$ and*

$$S = \begin{cases} N(t), & \text{if } N(t) < s \\ \\ s, & N(t) \geq s \end{cases} \tag{2.1}$$

and the event that $S = s$ is equivalent to the event of an arriving customer not being immediately attended to and having a non-zero waiting time for them to receive service.

2.1.2 Notation

The key terminology and notation used in this study are presented in Table 2.1.

Table 2.1: Key Terminology and notation

Parameter	**Meaning**
λ	average number of arriving customers at the queueing system per unit of time i.e. the arrival rate or arrival intensity,
s	number of servers in the system (the servers are assumed to provide an identical service),
μ	service rate or service intensity of each of the servers i.e. average number of patients each server can serve per unit of time,
Z or LOS	random variable denoting the time taken to service a customer,
ρ	traffic intensity of each server : $\rho = \frac{\lambda}{\mu}$,
$N(t)$	total number of customers in the system at time t,
S	number of busy servers when the system is in the steady state,
L	average number of customers in the system when its in steady state,
W	average time spent in system by a customer (when in steady state),
$M/M/s/c$	Poisson arrivals, Exponential service, s servers and c waiting capacity or maximum queue size permissible.

2.2 Erlang models

The process of patients queuing and waiting to be admitted to an ICU, and then being admitted and spending time in the ICU up to the point they are discharged (when they either get better or die) is a typical example of a queuing system (Bhat, 2015).

The fitting of a queuing model involves the determination of queue attributes such as

- the arrival pattern of patients at the ICU; i.e. knowledge of the probability distribution of the number of arriving patients per unit of time,
- the mean number of patients each bed can serve per unit of time,
- the mean amount of time a patient spends in the ICU,
- the mean of the total number of patients (customers) in the system ,
- the average amount of time a patient waits to be admitted in the ICU,
- the probability distribution of the number of patients in the ICU.

As a subject, Queuing Theory evolved with the Erlang models. The most common Erlang models are the Erlang A, Erlang B and Erlang C models.

2.2.1 Erlang C Model

Erlang C is probably the most commonly used queuing model largely because of its tractable results and simplicity. For a queue with s identical servers, in terms of Kendall notation, the Erlang C model is characterised as the $M/M/s/\infty$ (or $M/M/s$) queuing system. The model assumes that the number of customers that can be in the system is anything from zero to infinite and no customer queue abandonment nor customer loss takes place. Thus, every customer will wait in the queue for some time until they receive service. It is only after they have received service (which takes some time as well), that the customer leaves the queue.

It needs to be noted that the Erlang C Model has two main assumptions. Firstly it assumes that clients wait in the queue as long as it takes. Secondly, it also assumes that clients arrive at a known average rate per unit of time and are served by a given number of servers[1] (Robbins et al., 2010).

[1] the number of servers remains the same over time

Important queuing metrics for the Erlang C Model with service rate μ and average number of customers arriving in the ICU, λ are calculated as follows:

1. The probability that an arriving patient has to wait, i.e. $P(S = s) = P(Wait > 0)$:

$$P(S = s) = 1 - \left(\sum_{i=0}^{s-1} \frac{\rho^i}{i!} \right) \div \left(\sum_{i=0}^{s-1} \frac{\rho^i}{i!} + \left(\frac{\rho^s}{s!} \right) \left(\frac{1}{1 - \frac{\rho}{s}} \right) \right) \qquad (2.2)$$

   where $\rho = \frac{\lambda}{\mu}$ (Robbins et al., 2010).

2. The mean waiting time for a patient to get to be admitted in the ICU, sometimes referred to as *Average Speed to Answer* (ASA) :

$$ASA = E(Wait) = P(S = s). \left(\frac{1}{s} \right) \cdot \left(\frac{1}{\mu} \right) \cdot \left(\frac{1}{1 - \frac{\rho}{s}} \right) \qquad (2.3)$$

Remark 4 *ASA is one of the critically important measures of the perfomance of a queuing system.*

Another important performance metric for a queuing system is its *service level* (SL).[2]. For example, the hospital management may report the service level as the percent of arriving ICU patients who are put on hold for less than T days. The service level metric can then be expressed as

$$\begin{aligned} SL = P(Wait < T) &= 1 - P(Wait > 0) \times P(Wait > T | Wait > 0) \\ &= 1 - P(S = s) \times e^{-s\mu(1-s\rho)T} \end{aligned} \qquad (2.4)$$

[2]for a call centre queuing system, the equivalent is called the Telephone Service Factor (TSF) - it is the fraction of those arriving calls which get to be serviced and for which the delay is below a given level

Abandonment Rate which is the proportion of customers that leave the queue prior to getting admission in the ICU is one of the queue performance matric.

Remark 5 *The assumption that a customer waits in the queue for as long as it takes to get their desired service may indeed be naive and is certainly a limitation in terms of the Erlang C model being applicable to many real-life queuing situations. Robbins et al. (2010) point out that even a low level of abandonment can dramatically upset the performance of a queuing system.*

Other problems noted in the application of the Erlang C Model are summarised below:

- Robbins et al. (2010) state that the assumption of the mean arrival rate being a known constant over time has been found questionable by many researchers and have made useful suggestions of how to handle the problem should it arise,

- the assumption of exponentially distributed service times has been found not to work well in a number of empirical studies. When the assumption breaks down, the resulting queuing model is characterised as the $M/G/s/\infty$ queueing model (where the G stands for "general" service time distribution). Gans, Koole and Mandelbaum (2003) mention that results for the $M/G/s/\infty$ models are intractable and queuing metric measures can only be approximated.,

- for queuing systems where a number of servers are serving customers in parallel, the assumption of the servers having the same mean service time may not be realistic as some servers would be more efficient compared to others. However, for the ICU queing system discussed in this research work, this assumption may not be subject to serious violations.

Remark 6 *Robbins et al. (2010) state that a good number of analyses of service time distributions have found favour with the lognormal distribution.*

2.2.2 Erlang A Model

Just like the Erlang C model, the Erlang A model assumes that the arrivals of patients follows a Poisson distribution and service times are exponentially distributed (Gans et al., 2003). The model addresses the short-comings of the Erlang C model that are discussed in Section 2.2.1.

In the Erlang A queuing system, each customer will patiently wait in the queue until their patience runs thin and at this point they leave the queue without having received service. The amount of time which a customer waits in the queue is referred to as "patience time". Customer patience times are assumed to be independent and identically distributed random variables following an exponential distribution with mean $\frac{1}{\theta}$ (Gans et al., 2003).

The Erlang A Model with service rate μ, average number of customers arriving in the ICU being λ and s servers is affected by the abandonment rate θ as follows:

1. As $\theta \to 0$ the Erlang A model approaches the Erlang C model $M/M/s/\infty$
2. As $\theta \to \infty$ the Erlang A model approaches the Erlang B model $M/M/s/c$

(Knessl and van Leeuwaarden, 2015)

2.2.3 Erlang B Model

Just like the Erlang A and Erlang C models, the Erlang B model assumes that the arrival stream of customers follows a Poisson distribution and service times are exponentially distributed (Franzese et al., 2009). Any customer arriving when all the servers are busy is blocked from joining the queue and there is no provision of waiting for service. In terms of the Kendall notation, the Erlang B model is characterised as the $M/M/s/s$ queuing system.

(Franzese et al., 2009).

In the case of an Erlang B model with arrival rate λ and service rate μ, the steady state probability that an arriving customer is blocked/rejected is

$$P(\text{block}) = \frac{\frac{\rho^s}{s!}}{\sum\limits_{i=0}^{s} \frac{\rho^i}{i!}} \tag{2.5}$$

where $\rho = \frac{\lambda}{\mu}$.

Remark 7 *Result 2.6 is referred to as the Erlang loss formula.*

Remark 8 *The mathematical function $B(s, \rho)$ is*

$$B(s, x) = \frac{\frac{x^s}{s!}}{\sum\limits_{i=0}^{s} \frac{x^i}{i!}} \tag{2.6}$$

is referred to as the Erlang loss formula.

2.3 Patients arrival process

The arrival of customers at a service facility is critically important when studying a queuing system. In particular, patients arrival at an ICU can either be a homogeneous poisson process (HPP) or non-homogeneous poisson process (NHPP).

2.3.1 Arrival process a HPP

Definition 2.3.1 *(Homogeneous Poisson process (HPP)) A counting process*
$\{N(t); t > 0\}$ is said to be a HPP having rate or intensity $\lambda > 0$, if

1. N(0) = 0,

2. The process has independent increments,

3. The number of events in any interval of length t is Poisson distributed with mean λt. Therefore $E[N(t)] = \lambda t$.

An inhomogeneous Poisson process, on the other hand, is a point process with whose intensity varies across its domain (which may be time or space). The main difference between a HPP and an NHPP is simply that, for the former process, the intensity is not time-dependent while for the latter process the intensity is time-dependent.

2.3.2 Arrival process a NHPP

When the arrival of patients vary significantly over time and location, it will be modelled as a non-homogeneous (also called inhomogeneous) Poisson process. The characterization of a non-homogeneous or in-homogeneous Poisson process is given in Definition 2.3.2 (this definition is taken from Kim and Whitt (2014)). Kim and Whitt (2014) notes that since the arrival rate in many service systems typically varies a lot over time of day, it is is prudent to model many such arrival processes as NHPP.

Definition 2.3.2 *[Non-homogeneous Poisson process] A counting process $\{N(t); t > 0\}$ is said to be a non-homogeneous Poisson process with intensity $\lambda(t)$; $t > 0$, if*

1. $N(0) = 0$,

2. $\{N(t); t > 0\}$ *has independent increments,*

3. $P[N(t + h) - N(t) > 2] = o(h)$,

4. $P[N(t + h) - N(t) = 1] = \lambda(t)h + o(h)$.

Remark 9 *The problem of an arrival process being NHPP as opposed to HPP presents some challenges which are usually circumvented by treating the process as having a piecewise constant arrival intensity function. This is discussed in Section 3.3.*

2.4 Other related studies

Some studies have observed that there are other aspects that need investigation when modelling patients admission at ICUs These include:

- Griffiths, Price-Lloyd, Smithies and Williams (2006) has used a queue theoretic approach in the studies of activities at a major teaching hospital. This approach included modelling the characteristics of the patients, admission source and type of surgery which have an effect on the duration of stay of patients in the ICU.

- Hershy, Weiss and Cohen (1981) has used a semi-Markov process in the studies of a stochastic service network model that is applicable to hospital facilities and found a better fit when a linear model is used in conjunction with the Erlang C model,

- Gorunescu (2011) has researched on the problem of an optimal number of beds in the case of a hospital which has a policy of turning away some patients once the numbers have reached a certain level. In the research, queueing theory is used to propose a way of saving costs in a hospital by balancing between the beds occupied and the rejection of patients.

2.5 Queueing theory in health facilities

The Markov queuing strategy has been shown to work effectively in hospitals. Wu, Jingna, Chen, Bo, Wu, Danping, Wang, Jianqiang, Peng, Xiaodong, Xu and Xia (2020) report that an optimal allocation of beds in hospitals can significantly improve the utilisation of resources in general which result in patients being satisfied with the service. A study done in a hospital in Ghana indicated that the optimum system performance of the hospital can be achieved when Queueing Theory is applied (Afrane and Appah (2014)). In a study in Nigeria, Queueing Theory was found to solve the problem of turning patients away (Kembe, Agada and Owuna (2014)).

Chapter 3

Methodology

3.1 Introduction

This chapter gives a preamble of the data and methodology used in this study. The methods encompass a mix of graphical methods, Descriptive Statistics and fitting probability distributions.

3.2 Data

The dataset utilised in the current study was obtained from a large hospital in the United States of America (USA)[1]. It was collected between 2001 and 2012. During this period, 58 977 patients were admitted in the ICU of the hospital. The ICU had 70 beds and it was operational for 24 hours. About 8 110 patients had missing data. The data were stored in a database with coded values for variables such as gender, religion, age, ethnicity and the diagnosis at the time of admission. The missing values were estimated using k nearest neighbors (knn) that are similar to the missing value. This is done

[1]The researcher struggled to get a dataset from South African hospitals and opted for the secondary data from America

using the Visualization and Imputation of Missing (VIM) package.

A description of the variables used in the analysis is presented in Table 3.1.

Table 3.1: Description of the variables

Variable	**Desrciption**
length of stay (LOS)	Length of stay of a patient in the ICU (in days)
Waiting time	Length of time (in days) it takes a patient to be admitted in the ICU from the time he/she is referred to the ICU
Gender	Gender of the patient
Age	Patients' age at their last birthday (ranged from 0 to 89 years)
Admission type	Patients either admitted under emergency, elective, agent etc
Admission location	Patients were referrals from the clinic, from ermegency room, transfer from other hospitals etc
Insurance	Types of medical aid used by patients which included medicare, private etc
Religion	Patient's religion (christianity, muslim etc)
Marital status	Marital status of patient (e.g.)married, divorced etc)
Ethnicity	Ethnicity (e.g.blacks, whites, indians etc)

3.3 Addressing the problem of an arrival process that is an NHPP

The approach recommended in the event that the patient arrival process is NHPP is similar to that discussed by Brown et al. (2005), i.e. the arrival intensity function is treated as piece-wise constant over time. Each day is broken into short and equal segments (of length L) so that the arrival rate in a segment may reasonably be assumed to be constant. Defining $\tau_{ij},\ j = 1, \cdots, J(i)$ as the j^{th} ordered arrival time in the i^{th} segment, it can be shown that the transformed variables V_{ij} are are independent exponentially distributed random variables where $J(i)$ is the total number of arrivals in the i^{th} segment.

$$V_{ij} = (J(i) + 1 - j)\left(-log\left(\frac{L - \tau_{ij}}{L - \tau_{ij} - 1}\right)\right) \tag{3.1}$$

where $j = 1, \cdots, J(i)$.

3.3.1 Stationarity for arrival of patients process

Stationarity in the patient arrival process is checked using the augmented Dickey- Fuller test (ADF). The adf.test() from the tseries package in the R statistical software is used in conducting the Dickey - Fuller test.

3.4 Fitting probability distributions

The exponential distribution is commonly fitted when dealing with queueing models. The atkinson.exp.test function from the package exptest in R is used to test whether or not inter-arrival times or length of stay times follow an exponential distribution.

As noted in Remark 6 (in Chapter 2), the lognormal distribution is a potential candidate probability distribution for modelling inter-arrival times, waiting times and length of stay in ICU times. To check if its ideal, one procedure involves calculating the natural logarithms of each of the transformed times as described below. If each time T_i , $i = 1, \cdots, n$ is hypothesized to follow a lognormal distribution, then the transformed values V_i:

$$V_i = \frac{\log T_i - \mu}{\sigma} \tag{3.2}$$

are normally distributed. If indeed the values V_i are normally distributed, a QQ plot as well as a histogram can be used to visually check if indeed this is the case. To rigorously check if the transformed values follow a normal distribution, the Lilliefors test is one test that can be used for the purpose.

Skewness and kurtosis plots are used to identify suitable probability distributions from an array of distributions that includes the exponential, gamma, Weibull and Beta distribution. In a skewness-kurtosis plot the observed results are represented by a single point. Distributions such as normal, uniform and logistic are represented by single points. For instance, the normal distribution has kurtosis equal to 3 and skewness equal to 0 and therefore the point (3,0) would represent the normal distribution. The exponential distribution is also represented by a point with skewness of 2 and kurtosis of 6 while the gamma and lognormal distributions have kurtosis-skewness values represented by lines as they have varying skewness and kurtosis which depend on the values of their underlying parameters. The beta distribution has kurtosis- skewness values falling in a band because skewness and kurtosis depends on the values taken by parameters denoted by α and β.

A histogram of the dataset with fitted probability distributions superimposed as well as the empirical cumulative distribution functions are also good tools to use when visually inspecting the suitability of different distributions. The goodness of fit statistics (which are the Akaike Information Criterion (AIC) and the Bayes Information Criterion (BIC)) give a good basis of comparison of the efficacy of the distributions.

3.5 The M/M/s queueing model

According to Ross (2014) the fundamental queueing metrics of the queueing model follow:

1. The mean number of patients waiting to be admitted into the ICU, L_q:

$$L_q = \frac{P_0 \left(\rho\right)^s \frac{\rho}{s}}{c! \left((1 - \frac{\rho}{s}\right)^2} \tag{3.3}$$

2. The probability that there are no patients in the ICU, P_0:

$$P_0 = \left[\sum_{m=0}^{s-1} \frac{\rho^m}{m!} + \frac{\rho^s}{s!(1 - \frac{\rho}{s})}\right]^{-1} \tag{3.4}$$

3. The mean of time spent waiting to be admitted into the ICU by patients, W_q:

$$W_q = \frac{L_q}{\lambda} \tag{3.5}$$

4. The mean number of patients in the ICU, L:

$$L = \rho \left[1 + \frac{s}{(s - \rho)^2} \times P_s\right] \tag{3.6}$$

where P_s:

$$P_s = \frac{\rho^s}{s!} P_o \tag{3.7}$$

5. The average time spent in the system by an arbitrary customer is given by W:

$$W = \frac{s}{\mu(s - \rho)^2} \times P_s \tag{3.8}$$

3.6 The G/G/s queueing model

If any one of the assumptions of the M/M/s model is violated, it is usually the case that the G/G/s model may be fitted and the queueing metrics obtained are compared to those of the M/M/s model. The mean waiting time for this model, $W_q^{G/G/s}$:

$$W_q^{G/G/s} \approx W_q^{M/M/s} \frac{C_a^2 + C_s^2}{2} \tag{3.9}$$

where $W_q^{M/M/s}$ denotes the waiting time of $M/M/s$ and C_a^2 is the coefficient of variation squared for inter-arrival times and C_s^2 is coefficient of variation squared for service time.

3.7 Cox Proportional Hazard regression mode

The relationship between the survival time of patients with other variables can be modelled using a regression model called Cox Proportional Hazard regression model (Coxph). A function coxph [2] is used to test the effect of all the variables in the dataset on the length of stay of patients in the ICU.

[2]regression model for exploring relationship between the survival time of patients and one or more explonatory variables

Chapter 4

Results

4.1 Introduction

Chapter 4 presents the results of this study. The analysis was done using the R statistical software.

4.2 Descriptive statistics of the data

From the data, it is observed that

- 32 950 (55.9%) of the patients admitted were female.
- Most patients admitted in the ICU were from the emergency room (39%) and several clinics (18%).
- About 43% of patients admitted in the ICU used Medicare, while 34% used private insurance.
- Elective[1] and new-born[2] types of admission had the same number of patients which approximates to 8 000 patients.

[1] Admission date is known in advance
[2] Babies who need to be admitted into the ICU soon after their birth

- The majority of patients admitted in the ICU were either married (39.9%) or single parents (22.1%).
- The reasons why patients were admitted in the ICU was not given.
- 8 110 patients had at least one of waiting time and inter-arrival time missing. The missing data values were estimated using knn from the VIM package as discussed in Chapter 3.

4.3 Modelling inter-arrival times at the ICU

The inter-arrival times of patients at the ICU ranged from 0 to 8 days with a median of 0.07 days. All inter-arrival times in excess of 0.45 days were found to be outliers. The modal value for inter-arrival times was found to be 0 days; 26 660 inter-arrival times had 0 days.

Tests for stationarity in the data (using the Augmented Dickey-Fuller test) were conducted as discussed in Section 3.3.1. The p-value (see Appendix 2) in the test for stationarity was found to be 0.01. This is indicative of the fact that the inter-arrival times exhibit stationarity.

The data were also checked to see whether or not the arrival process is a NHPP. The Kolmogorov-Smirnov test (discussed in Section 3.3) was used on different time segments of the data; in each case, the p-value was found to be 0.9956(see Appendix 2) and this leads to the conclusion that the arrival times are a NHPP.

A test of hypothesis to check whether or not the inter-arrival times of patients at the ICU follow an exponential distribution was conducted. The test selected for the purpose uses the atkinson.exp.test function from the package exptest in R. The Atkinson test for exponentiality is based on the following statistic:

$$T_n(p) = \sqrt{n} \left| \frac{\left(n^{-1} \sum_{i=1}^{n} X_i^p\right)^{1/p}}{\overline{X}} - (\Gamma(1+p))^{\frac{1}{p}} \right|.$$

The statistic is asymptotically normal: $T_n(p) \to |N(0, \sigma^2(p))|$, where

$$\sigma^2(p) = (\Gamma(1+p))^{\frac{2}{p}} \left(-1 - \frac{1}{p^2} + \frac{\Gamma(1+2p)}{p^2 \Gamma^2(1+p)} \right).$$

The p-value (see Appendix 2) was found to be 0.1539. One thus fails to reject the hypothesis that the inter-arrival times of patients at the ICU can be modeled using an exponential distribution.

Figure 4.1 gives the histogram and boxplot of inter-arrival times of patients visiting the ICU. Apparently the data are skewed to the right. This observation points to potential candidate probability distributions for modelling inter-arrival times being any of the following: Exponential distribution, a Weibull distribution, Lognormal distribution among others.

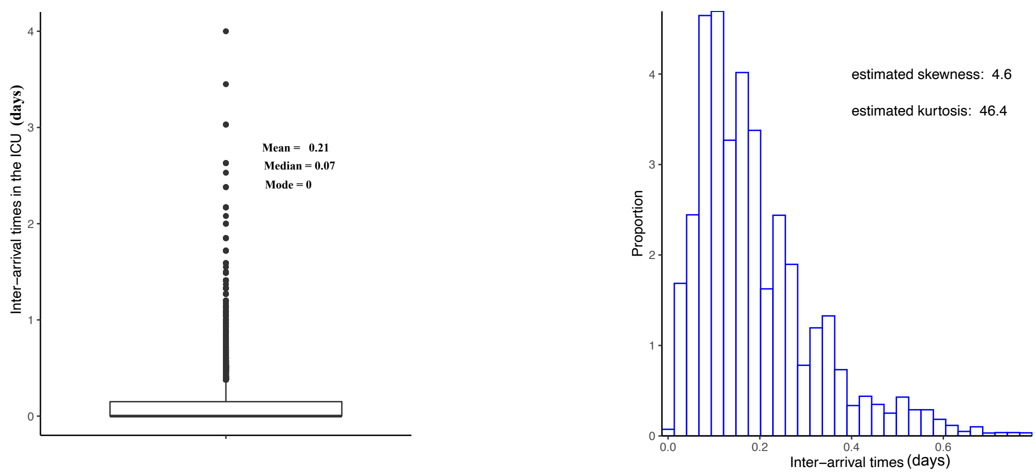

Figure 4.1: Inter-arrival times of patients

Figure 4.2 gives a histogram of logarithms of standardised inter-arrival times of patients arriving at the ICU and the Q-Q plot to help check for normality in the data. Both plots are indicative of the fact that inter-arrival times possibly follow a Lognormal distribution. The Lilliefors (Kolmogorov-Smirnov) test was conducted to test if the logarithms of standardised inter-arrival times of patients into the ICU was normally distributed. The p-value was found to be 0.678; thus, it can be concluded that the log values are normally distributed and the hypothesis of waiting times following a Lognormal distribution cannot be rejected.

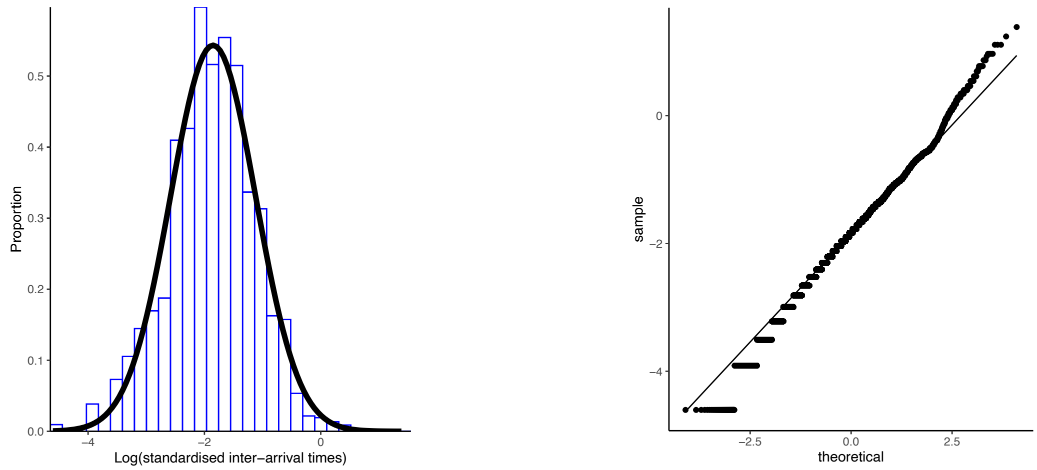

Figure 4.2: Histogram of Logarithm of standardised inter-arrival times (days) and Q-Q plot

An investigation of other possible candidate distributions for modeling the data was done. Figure 4.3 is a skewness-kurtosis plot of inter-arrival times of patients coming to the ICU. The blue dot is for the patients inter-arrival times at the ICU. Since the kurtosis-skewness curve of the Lognormal distribution passes very close to the blue dot, the conclusion arrived at is that the inter-arrival times may be adequately modelled using a Lognormal distribution.

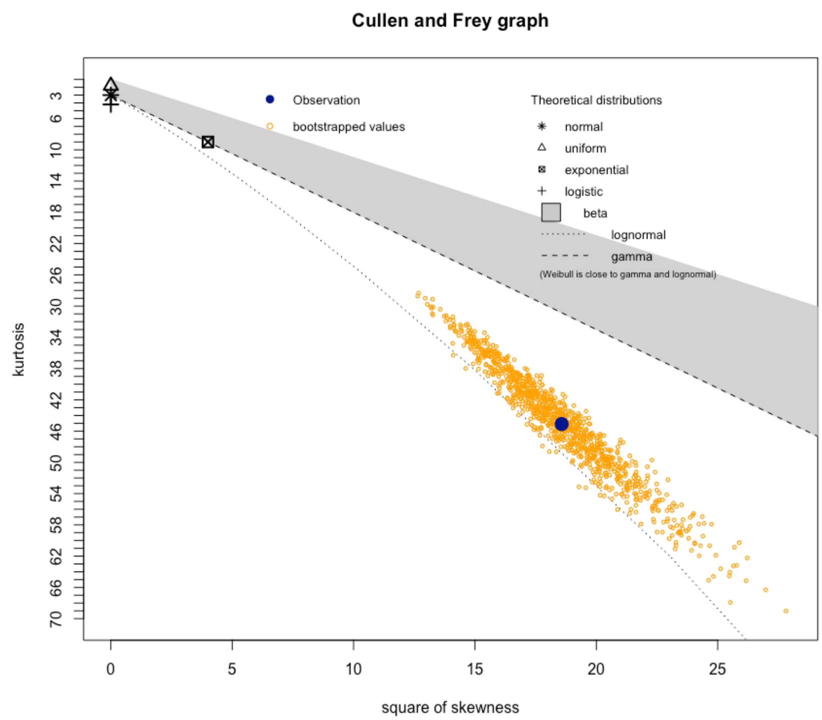

Figure 4.3: Skewness-kurtosis plot of inter arrival times (days)

Figure 4.4 is a histogram of inter-arrival times of patients coming to the ICU with the four fitted distributions superimposed on it. Arguably, the conclusion one draws is that the Gamma, Lognormal and Weibull distributions all provide a good fit for the data.

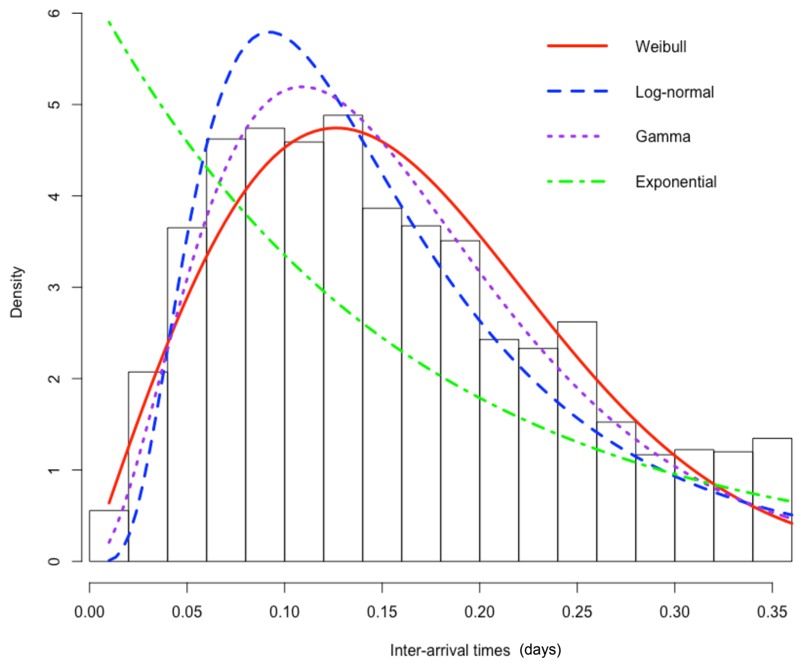

Figure 4.4: Histogram of inter-arrival times (days) in the ICU with fitted distributions

Figure 4.5 is a plot of emperical cumulative distribution of the fitted distributions superimposed on it. In comparison to the exponential and lognormal distributions, the Gamma and Weibull distributions appear to be the better fitting distribution for the inter-arrival times.

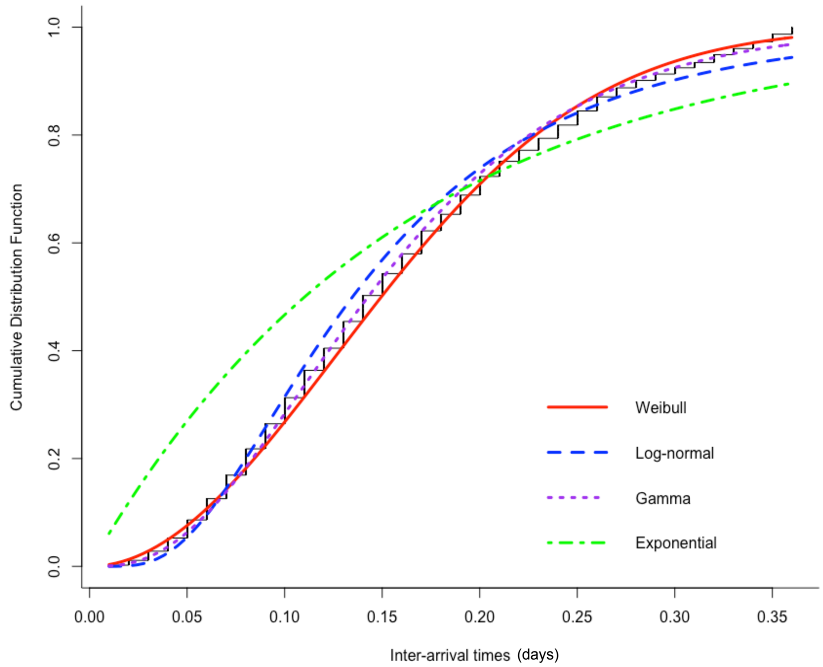

Figure 4.5: Emperical cumulative distribution of inter-arrival times (days) in the ICU with fitted distributions

The Akaike Information Criterion (AIC) and the Bayes Information Criterion (BIC) are used as measures to compare how well different probability distributions fit a dataset. Table 4.1 gives the results for AIC and BIC for the distributions under the spotlight. Since the Lognormal distribution has the smallest values for both AIC and BIC, one concludes Lognormal distribution provides the best fit for the inter-arrival times.

Remark 10 *It apparently is the case that all the four distributions discussed in Table 4.1 can be used to model the inter-arrival times of patients at the ICU of the hospital.*

Table 4.1: Goodness-of-fit AIC and BIC

Distribution type	**AIC**	**BIC**
Lognormal	36056.23	36048.25
Gamma	45856.36	45840.41
Weibull	47764.91	47748.95
Exponential	48038.16	48022.21

4.4 Modelling waiting times at the ICU

Waiting time (in days) is the length of time it takes for a patient to get admitted into the ICU. Some patients unfortunately die after admission into the ICU and some patients get transferred upon admission. Figure 4.6 is a histogram of the waiting times of patients waiting for admission into the ICU. Apparently the data are skewed to the right. This observation points to potential candidate probability distributions for modelling waiting times being any of the following: Exponential distribution, a Weibull distribution, Lognormal distribution among others.

A test of hypothesis to check whether or waiting times of patients at the ICU follow and exponential distribution was conducted using the atkinson.exp.test function from the package exptest. The p-value in the test was found to be 0.2716. This indicates that the waiting times of patients to be admitted into the ICU can be modelled using exponential distribution.

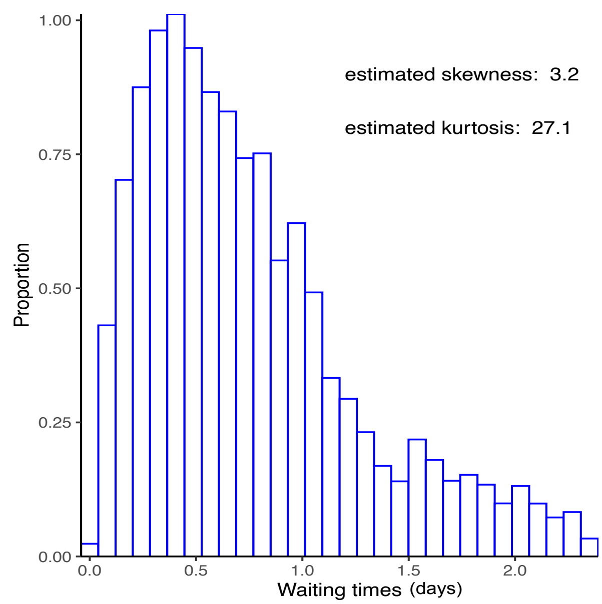

Figure 4.6: Histogram of the waiting times (days) in the ICU

In checking the suitability of the lognormal distribution to model the data, Figure 4.7, which is a histogram of the logarithm of standardised waiting times of patients at the ICU was constructed. The histogram is indicative of the fact that waiting times possibly follow a Lognormal distribution. The Lilliefors (Kolmogorov-Smirnov) test was conducted to test if the logarithms of standardised waiting times of patients into the ICU was normally distributed. The p-value was found to be 0.4329 (see Appendix 4); thus, it can be concluded that the log values are normally distributed and the hypothesis of waiting times following a Lognormal distribution cannot be rejected.

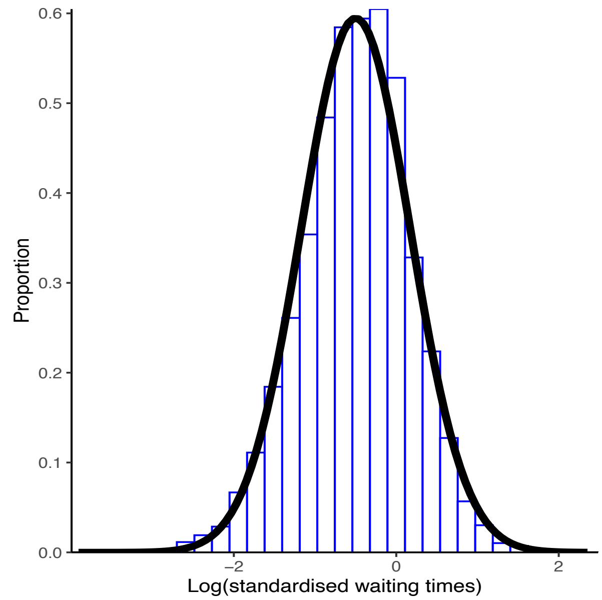

Figure 4.7: Histogram of logarithm waiting times (days) in the ICU

An investigation of the suitability of other distributions for modelling waiting times was conducted. Figure 4.8 is a kurtosis-skewness plot of patients waiting to be admitted into the ICU. It is observed that the blue dot for the data is contained in band for the Beta distribution (grey band) and the gamma distribution curve passes close to the blue dot. This observation suggests that waiting times may adequately modelled using the Gamma or Beta distribution.

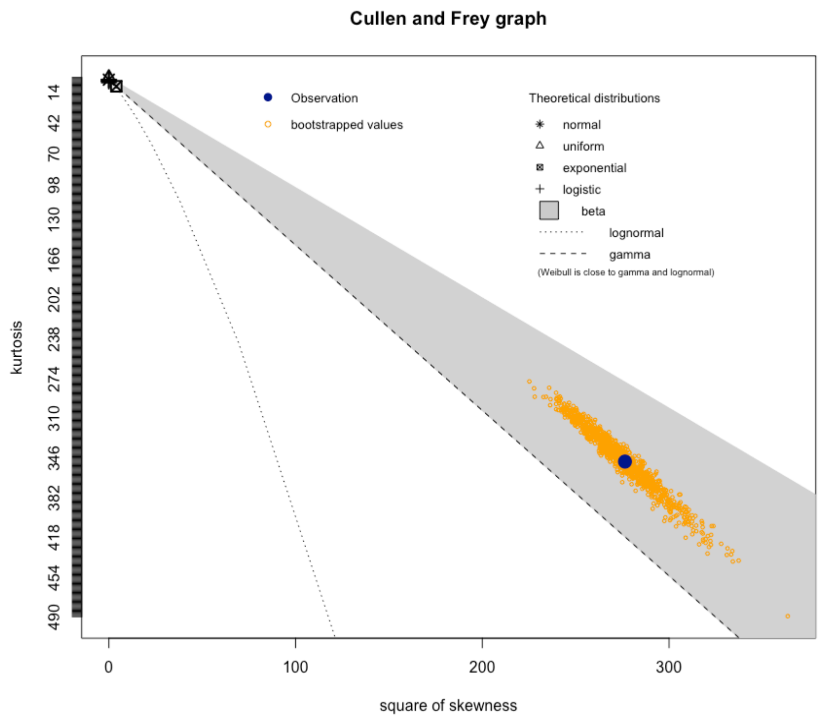

Figure 4.8: Skewness-kurtosis plot of waiting times (days)

Figure 4.9 is a histogram of waiting times of patients being admitted into the ICU with fitted probability distributions superimposed on it. From the figure one can suspect that the Weibull and Gamma distributions all provide a good fit for the waiting times.

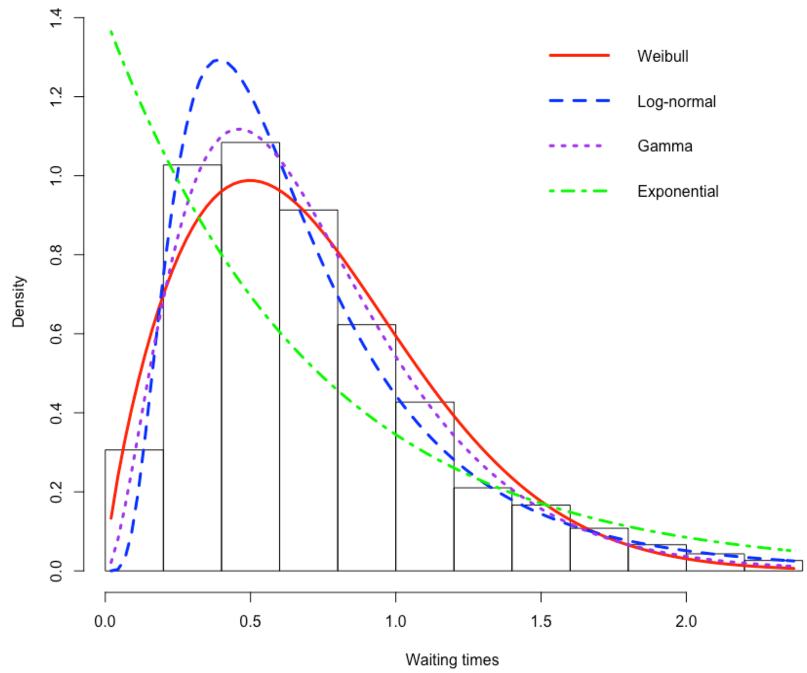

Figure 4.9: Histogram of waiting times (days) with fitted distributions

Figure 4.10 is a plot of the empirical cumulative distribution of waiting times with the fitted distributions superimposed on it. Arguably, Weibull and Gamma distributions all provide a good fit for the waiting times.

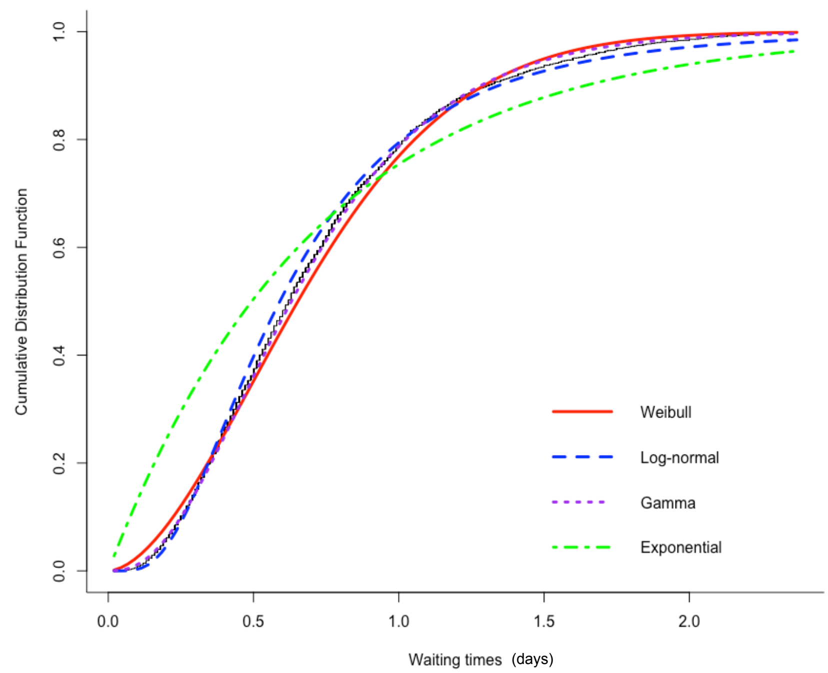

Figure 4.10: Emperical cumulative distribution of waiting times (days) with fitted distributions

Table 4.2 gives the goodness of fit results for the fitted distributions on waiting times. Since the Gamma distribution has the smallest values for both AIC and BIC, one concludes that the waiting times follow a Gamma distribution.

Remark 11 *All four probability distributions apparently model the data adequately, albeit with different efficacy levels.*

Table 4.2: Goodness-of-fit AIC and BIC

Distribution type	**AIC**	**BIC**
Gamma	55472.79	55490.35
Weibull	56104.52	56122.08
Lognormal	58602.80	58620.36
Exponential	68851.72	68860.50

4.5 Modelling length of stay at the ICU

The lengths of stay (LOS) in the ICU ranged from 0 to 280 days with a median of 6.46 days. All the number of days in excess of 23.88 days were found to be outliers. The modal value for LOS in the ICU was found to be 4 days with a frequency of 3 506 patients.

A hypothesis test to check whether or not LOS times follow an exponential distribution was conducted, again using the atkinson.exp.test function from the package exptest. The p-value (see Appendix 3) in the test for exponentiality was found to be 0.2176. This indicates that the LOS of patients in the ICU can be modelled adequately using exponential distribution.

Figure 4.11 gives the histogram and boxplot of times patients have stayed in the ICU. The distribution of length of stay in the ICU is skewed to the right suggesting time spent in the ICU may possibly be exponentially distributed or following any of the following distributions: Weibull, Lognormal, etc.

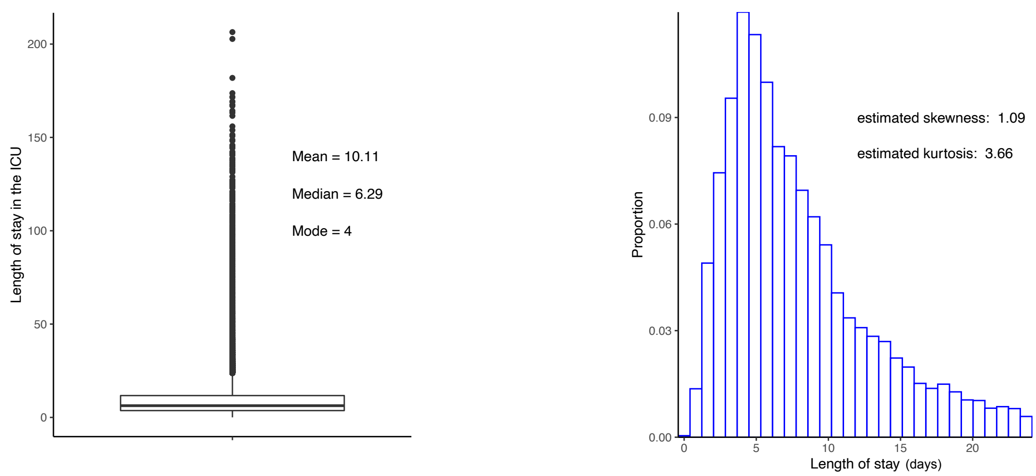

Figure 4.11: Length of stay (days) of patients

In an effort to check for the suitability of the Lognormal distribution for modelling the time spent in the hospital by patients, Figure 4.12 which is a histogram of logarithms of time spent by patients in the ICU was constructed. The histogram gives the impression that the lengths of stay can possibly be adequately modelled by a Lognormal distribution. The Lilliefors (Kolmogorov-Smirnov) test was conducted to test if the logarithms of standardised service times of patients in the ICU was normally distributed. The p-value (see Appendix 3) was found to be 0.248; thus, it can be concluded that the hypothesis of stay times following a Lognormal distribution may not be rejected.

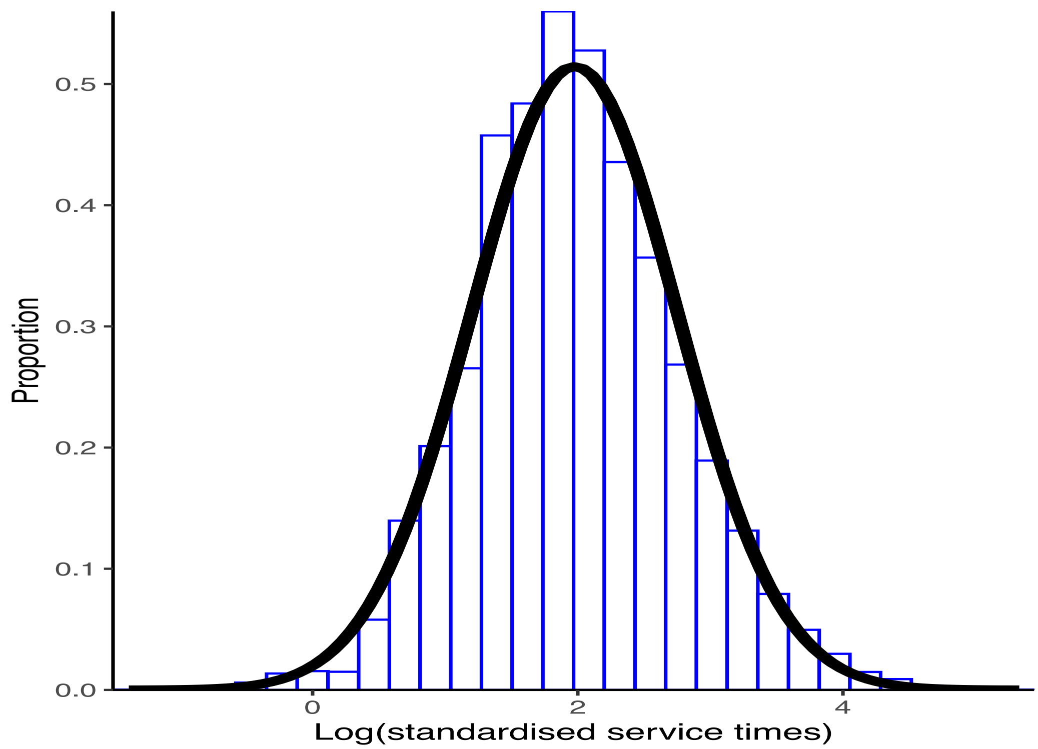

Figure 4.12: Histogram of the logarithm of standardised service time (days)

An investigation of the suitability of other distributions for modelling time spent in the ICU was conducted. Figure 4.13 is a skewness-kurtosis plot of length of stay of patients in the ICU. Arguably, the conclusion one draws is that the Gamma and Weibull distributions both appear to provide a good fit for the data.

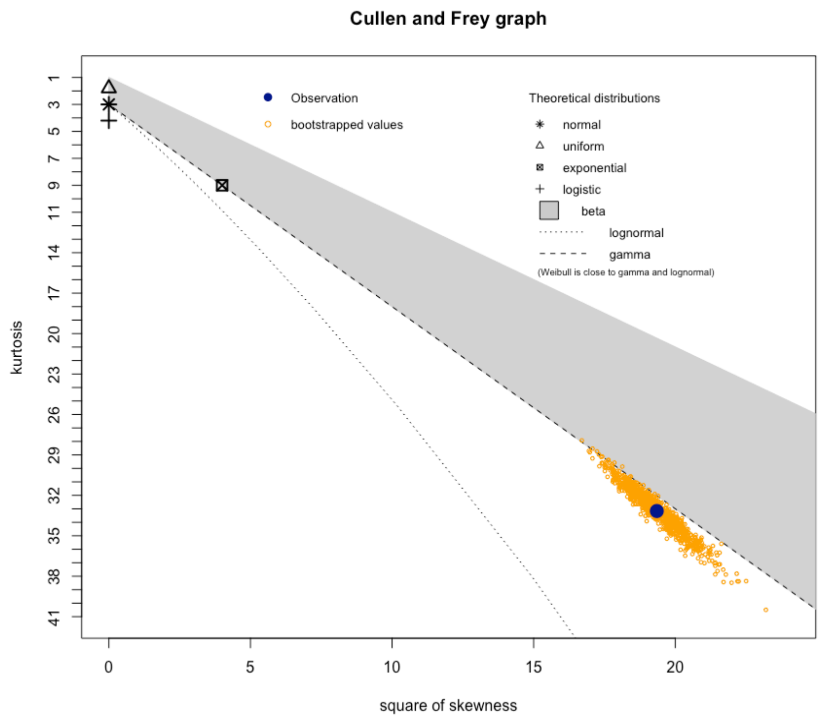

Figure 4.13: Skewness-kurtosis plot of length of stay (days)

Figure 4.14 is a histrogram of times spent by patients in the ICU with fitted probability distributions superimposed on it. The Gamma and Lognormal distributions both provide a good fit for the LOS in the ICU.

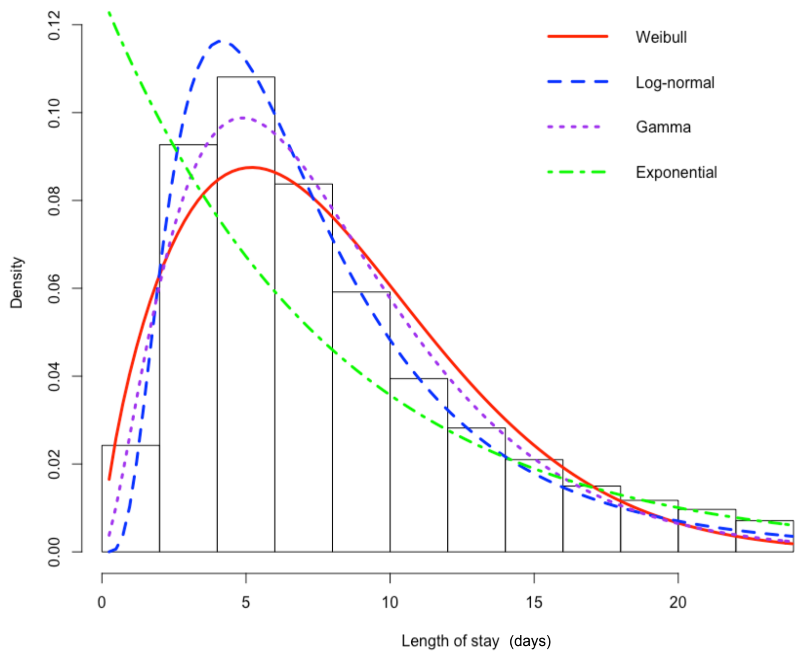

Figure 4.14: Histogram of length of stay (days) with fitted distributions

Figure 4.15 is a plot of the emperical cumulative distribution of time spent by patients in the ICU with fitted probability distributions superimposed on it. One may conclude that Gamma and Weibull distributions both provide a good fit for the LOS in the ICU.

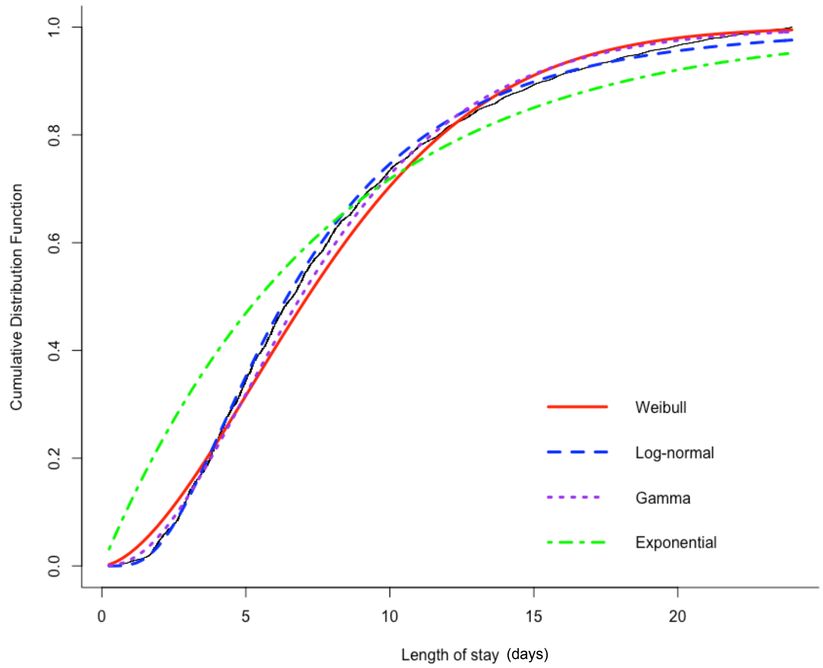

Figure 4.15: Emperical cumulative distribution of length of stay (days) with fitted distributions

Table 4.3 gives the results for AIC and BIC for the distributions under the spotlight for the LOS in the ICU. Since the Gamma distribution has the smallest values for both AIC and BIC, one concludes that the LOS follow a Gamma distribution.

Table 4.3: Goodness-of-fit AIC and BIC

Distribution type	**AIC**	**BIC**
Gamma	152053.3	152069.5
Weibull	155853.7	155869.9
Lognormal	157721.0	157737.2
Exponential	159486.4	159494.5

4.6 Estimation of patient queue abandonment

In effort to estimate the probability distribution of the time it takes a patient to wait in the queue and then abandon the queue after feeling exasperated, all patients whose length of stay was recorded to be 0 were assumed to have left the queue before admission. Information about deaths that could have possibly occurred was not available in the dataset. Figure 4.16 is the estimated survival function of patients that left the system without being admitted into the ICU. The plot is typically an exponential decay curve suggesting that the time to abandonment may follow an exponential distribution.

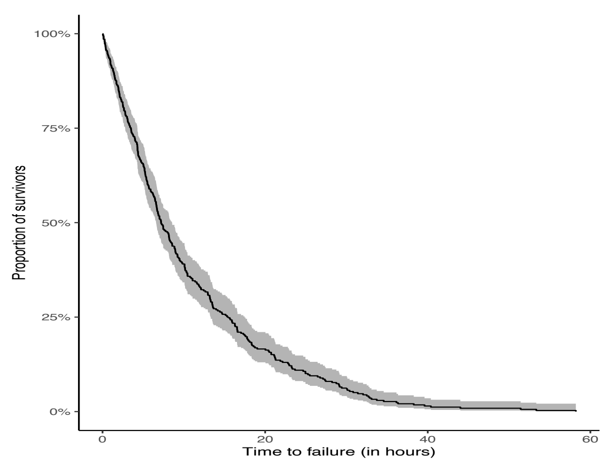

Figure 4.16: Survival function of abondonment

4.7 Fundamentals of the Queueing System - modelling with M/M/70 and G/G/70

4.7.1 Modelling with M/M/70

The mean arrival rate of patients at the ICU was found to be $\lambda = 6.5$ patients per day while the average length of stay in the ICU was 10.11 days which yields a mean service rate of $\mu = 1/10.11 = 0.099$ patients per day. The ICU has $s = 70$ beds with a waiting capacity of 20 patients resulting in traffic intensity being

$$\rho = \frac{\lambda}{\mu} = \frac{6.5}{0.099} = 65.72 \tag{4.1}$$

The queue_step function from queuecomputer package in R software was used to calculate the following parameters:

- Mean waiting time $W_q = 1.14$ days
- Average time ICU takes to serve a patient = 3.56 days
- Utilisation factor = 0.93
- Mean number of patients waiting to be admitted in the ICU $L_q = 7.4$
- Mean number of patients in the ICU = 67. The mean bed occupancy rate, which is the mean of the proportion of beds occupied by patients:

  $$\frac{\text{L}}{s} \times 100 = \frac{67}{70} \times 100 = 95.7\%$$

  This much higher than the recommended of 75% which is generally deemed satisfactory in the health sector.

  Using R, we obtained the following:

  $$P_o = 0.43, L_q = 7.4063 \text{ and } W_q = 1.139 \text{ days} = 27.346 \text{ hours}$$

When it comes to approximating the number of beds needed to achieve a desired service level (in terms of maximum of mean waiting time), the graph in Figure 4.17 is used. In constructing the plot, one makes the assumption that P_s in the expression for mean waiting time is robust as s increases from the current value of 70. To ensure a mean waiting time of 4.8 hours = 0.2 days is achieved, it can be seen that at least 73 beds would be needed.

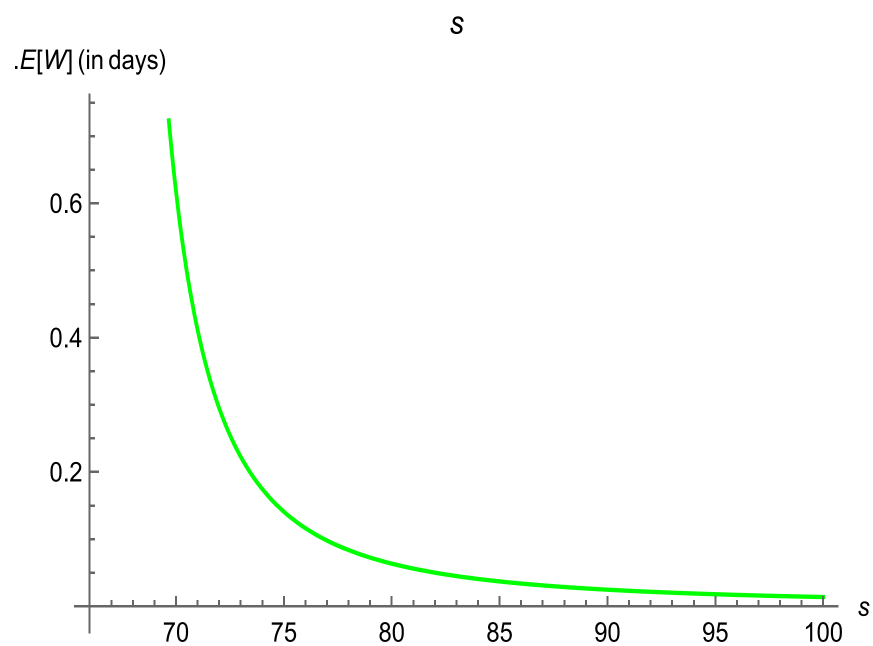

Figure 4.17: Plot of mean waiting time versus number of ICU beds

4.7.2 Modelling with G/G/70

As discussed in Section 3.6, the mean waiting time in the queue for the G/G/s where $s = 70$ model, $W_q^{G/G/s}$:

$$W_q^{G/G/s} \approx W_q^{M/M/s} \left(\frac{C_a^2 + C_s^2}{2} \right) \approx 1.14 \times \frac{0.98 + 1.01}{2} = 1.13 \text{ days} = 27.15 \text{ hours}$$

where $W_q^{M/M/s}$ denotes the waiting time of $M/M/c$ and C_a^2 is the coefficient of variation squared for inter-arrivals and C_s^2 is coefficient of variation squared for service time.

$$C_a^2 = \frac{0.043}{0.21^2} = 0.975$$

$$C_s^2 = \frac{102.98}{10.11^2} = 1.01$$

$$W_q^{G/G/s} = 1.14 \times \frac{0.98 + 1.01}{2} = 1.13 \text{ days} = 27.15 \text{ hours}$$

Thus the model estimates that the patients wait an average of 1.06 days before they are admitted into the ICU.

The mean number of patients waiting to be admitted into the ICU for the estimated model $L_q^{G/G/s}$ as follows:

$$L_q^{G/G/c} = W_q^{G/G/s} \times \lambda \tag{4.2}$$

$$L_q^{G/G/c} = 0.42 \times 6.5 = 2.73$$

Thus the model estimates that on average there are about 3 patients waiting to be admitted into the ICU.

The results in Table 4.4 show that the values for L_q are markedly different while the values for W_q are approximately equal. From logical reasoning, higher values of L_q generally would translate to higher values of W_q. It appears therefore that the M/M/s model is tending to over-estimate the value of L_q.

Table 4.4

Table 4.4: Models results comparison

Variable	**G/G/70**	**M/M/70**
L_q	2.73	7.4063
W_q	27.15	27.346

4.8 The effect of variable against length of stay of patients at ICU

The effect of variable against length of stay of patients at ICU were conducted using coxph as discussed in Section 3.7. Table 4.5 gives the results for the p-values for each variable against the time spent by patients in the ICU. The p-values for insurance type and admission type are both found to be less than 0.05 indicating that both variables significantly impact the length of stay. The patient's stay is determined by affordability (type of insurance) and type of sickness that brought him into the ICU (admission type).

Table 4.5: Cox proportional harzads model

Variable	$Pr(>\|z\|)$
Insurance	3.5×10^{-3}
Admission type	0.0043
Age	0.2556
Admission location	0.475
Gender	0.8795
Religion	0.3465
Marital status	0.4634
Ethnicity	0.9934

This concludes that the length of stay in the ICU is not only affected by severity of the illness but some variables like insurance type and admission type.

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

Chapter 4 presented the results of the analysis of the ICU data. It has been observed that the inter-arrival times possibly follow an exponential distribution although the AIC and BIC values for the exponential distribution are the highest for the four distributions investigated. A consequence of this observation is that the number of patients arriving at the ICU following a Poisson distribution cannot be ruled out.

In Section 4.4, an investigation of the probability distributions to model waiting times of patients coming to the ICU was done. The two most suitable distributions, judging by AIC and BIC values, were the Gamma distribution and Weibull distribution. The lognormal distribution and the exponetial distribution were not rejected as candidate distributions for modelling the data but their AIC and BIC values were comparatively higher.

Judging by the AIC and BIC values, the length of stay of patients in the ICU was best modelled by a gamma distribution. Again the hypothesis of length of times following an exponential distribution was not rejected and neither was the hypothesis of the times following a lognormal distribution.

An attempt to examine the problem of queue abandonment by patients who get to feel exasperated was done. The conclusion from the results gotten was that the length of time it takes for a patient to eventually decide to abandon the queue is an exponential distribution.

A comparison of the queuing metrics of the $M/M/70$ and the $G/G/70$ models was done. The conclusion drawn is that the $M/M/70$ tends to over-estimate the value of L_q. The two models, however, had comparable values of W_q (about 27 hours).

An application of the Cox Proportional Hazard model revealed that the insurance type and admission type were the only covariates that significantly impacted on length of stay in the ICU.

In the study, one criterion for deciding on an ideal number of ICU beds was explored. For a waiting time level, one can easily estimate the number of beds necessary.

5.2 Recommendations

In a quest to better understand the patient arrival process and service process at the ICU, the author believes that a dataset with less incidence of missingness would be useful. Another important aspect is that of the need to carefully monitor patient queue abandonment by collecting data on patients who abandon the queue as well as reasons why they abandon the queue. If that is done, the Erlang A model can be fitted and its performance in comparison to the models whose results are discussed in Chapter 4 can be assessed. A similar study needs to be replicated with South African data.

Appendices

Appendix 1

Table A1: Specimen of the data used in the study

adm_id	gender	age	LOSdays	admit_type	admit_locatic	AdmitDiagnc	insurance	religion	marital_status	ethnicity	inter_arrival_times	waiting_time
100001	F	35	6.17	EMERGENCY	CLINIC REFER	DIABETIC KE	Private	PROTESTAN	DIVORCED	WHITE	0.16	0.65
100003	M	59	4.04	EMERGENCY	EMERGENCY	UPPER GI BL	Private	NOT SPECIFI	SINGLE	WHITE	0.25	1.24
100006	F	48	12.04	EMERGENCY	EMERGENCY	COPD FLARE	Private	NOT SPECIFI	SINGLE	BLACK/AFRICA	0	0.33
100007	F	73	7.29	EMERGENCY	EMERGENCY	BOWEL OBS	Private	JEWISH	MARRIED	WHITE	0.41	0.96
100009	M	60	4.88	EMERGENCY	TRANSFER FI	CORONARY /	Private	CATHOLIC	MARRIED	WHITE	0	2.05
100010	F	54	4.38	ELECTIVE	PHYS REFERR	RENAL MASS	Private	EPISCOPALIA	MARRIED	WHITE	0.23	0.91
100011	M	21	14.38	EMERGENCY	CLINIC REFER	MOTOR VEH	Medicaid	NOT SPECIFI	SINGLE	HISPANIC OR L	0.07	0.21
100012	M	67	10.08	EMERGENCY	TRANSFER FI	CORONARY /	Medicare	CATHOLIC	MARRIED	WHITE	0.1	0.6
100014	F	49	0.63	ELECTIVE	PHYS REFERR	RIGHT SHOU	Medicaid	CATHOLIC	SINGLE	WHITE	0	4.76
100016	M	55	6.17	EMERGENCY	CLINIC REFER	PNEUMONIA	Medicare	PROTESTAN	SINGLE	WHITE	0	0.49
100017	M	27	0.67	URGENT	TRANSFER FI	OVERDOSE	Medicaid	CATHOLIC	SINGLE	UNKNOWN/N	0	2.99
100018	M	55	8.25	ELECTIVE	PHYS REFERR	HERNIATED I	Private	PROTESTAN	MARRIED	WHITE	0.12	0.48
100019	M	27	3.17	ELECTIVE	PHYS REFERR	AORTIC VALVE DISEASE\AORTIC VALVE REPLACEMENT						
100020	M	58	10.58	EMERGENCY	EMERGENCY	HYPONATRE	Private	CATHOLIC	MARRIED	WHITE	0.09	0.47
100021	M	54	60	EMERGENCY	EMERGENCY	BILATERAL ANKLE FX						
100023	M	0	2.33	NEWBORN	PHYS REFERR	NEWBORN	Private	CHRISTIAN S	NA	WHITE	0	1.72
100024	M	71	6.33	ELECTIVE	PHYS REFERR	CORONARY /	Medicare	NOT SPECIFI	MARRIED	UNKNOWN/N	0.16	0.63
100025	M	0	2.58	NEWBORN	PHYS REFERR	NEWBORN	Private	JEWISH	NA	WHITE	0	1.94
100028	F	72	6.88	EMERGENCY	CLINIC REFER	CHOLANGITI	Medicare	CATHOLIC	SINGLE	WHITE	0.15	0.58
100029	F	0	15	NEWBORN	CLINIC REFER	NEWBORN	Private	NOT SPECIFI	NA	WHITE	0	0.27
100030	M	33	15.88	EMERGENCY	EMERGENCY	PNEUMONIA	Private	PROTESTAN	MARRIED	HISPANIC OR L	0	0.25
100031	F	81	13.17	ELECTIVE	PHYS REFERR	AORTIC ASCE	Medicare	CATHOLIC	MARRIED	WHITE	0	0.61
100033	M	22	1.46	EMERGENCY	EMERGENCY	MEDIASTINA	Private	CATHOLIC	SINGLE	WHITE	0.68	2.05
100034	M	68	3.75	ELECTIVE	PHYS REFERR	CORONARY /	Medicare	NOT SPECIFI	MARRIED	WHITE	0.27	1.33
100035	M	36	25.29	EMERGENCY	CLINIC REFER	POST ARRES	Medicaid	NOT SPECIFI	SINGLE	HISPANIC OR L	0.04	0.2
100036	F	82	10.04	EMERGENCY	TRANSFER FI	CHF	Medicare	CATHOLIC	SINGLE	WHITE	0.1	0.5
100037	M	58	46.71	EMERGENCY	EMERGENCY	WEAKNESS	Private	PROTESTAN	MARRIED	WHITE	0.04	0.13
100038	F	57	1.83	EMERGENCY	EMERGENCY	CHEST PAIN	Medicaid	CATHOLIC	SINGLE	HISPANIC OR L	0.55	2.19
100039	F	38	28.58	EMERGENCY	CLINIC REFER	ABDOMINAL	Government	CATHOLIC	SINGLE	BLACK/AFRICA	0.03	0.17
100040	M	30	2.88	EMERGENCY	CLINIC REFER	BLUNT TRAU	Private	NOT SPECIFI	MARRIED	BLACK/AFRICA	0.35	1.39
100041	M	64	4.25	ELECTIVE	PHYS REFERR	CORONARY /	Private	UNOBTAINA	MARRIED	WHITE	0	1.18
100044	M	0	32.79	NEWBORN	PHYS REFERR	NEWBORN	Government	NOT SPECIFI	NA	WHITE	0	0.18
100045	F	69	10	EMERGENCY	EMERGENCY	CHANGE IN I	Medicare	CATHOLIC	WIDOWED	WHITE	0.2	0.3
100046	F	66	5.67	EMERGENCY	EMERGENCY	ALCOHOL WI	Private	CATHOLIC	WIDOWED	WHITE	0	0.71
100047	M	56	6.71	EMERGENCY	EMERGENCY	HYPOTHERMIA						
100050	M	69	6	ELECTIVE	PHYS REFERR	CORONARY /	Medicare	CATHOLIC	MARRIED	WHITE	0.17	1
100052	F	44	2	EMERGENCY	CLINIC REFER	MAMMARY I	Private	CATHOLIC	DIVORCED	UNKNOWN/N	0.5	1.5
100053	M	56	4.88	EMERGENCY	EMERGENCY	HYPOTENSIC	Private	NOT SPECIFI	SINGLE	WHITE	0	0.41
100055	M	0	4.21	NEWBORN	PHYS REFERR	NEWBORN	Private	CATHOLIC	NA	WHITE	0	1.66
100058	F	56	17.67	EMERGENCY	TRANSFER FI	INTRACRANI	Private	UNOBTAINA	MARRIED	WHITE	0	0.34
100059	M	68	17.71	EMERGENCY	TRANSFER FI	CORONARY /	Medicare	CATHOLIC	MARRIED	WHITE	0	0.45
100060	F	48	4.46	EMERGENCY	EMERGENCY	OVERDOSE	Medicare	CATHOLIC	DIVORCED	WHITE	0	0.67
100061	F	62	1.88	EMERGENCY	TRANSFER FI	CONGESTIVE HEART FAILURE						
100062	F	0	6.96	NEWBORN	HMO REFERR	NEWBORN	Private	UNOBTAINA	NA	UNKNOWN/N	0	0.43
100063	M	46	2.46	EMERGENCY	EMERGENCY	SUICIDAL IDEATION						
100065	M	59	3.79	EMERGENCY	EMERGENCY	DIABETIC KE	Private	NOT SPECIFI	MARRIED	BLACK/AFRICA	0.26	0.79
100066	F	75	2.04	EMERGENCY	CLINIC REFER	NEW ONSET OF EXERTIONAL CHEST PAIN						
100068	F	74	15.04	EMERGENCY	EMERGENCY	CHEST PAIN	Medicare	PROTESTAN	WIDOWED	BLACK/AFRICA	0	0.4

Appendix 2

R output

r_part.R

user

2021-08-14

```
# Tests -----------------------------------------------------------
library(tseries)

## Warning: package 'tseries' was built under R version 4.1.1

## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo

adf.test(na.omit(research_data$inter_arrival_times))

## Warning in adf.test(na.omit(research_data$inter_arrival_times)): p-value s
maller
## than printed p-value

##
##  Augmented Dickey-Fuller Test
##
## data:  na.omit(research_data$inter_arrival_times)
## Dickey-Fuller = -36.576, Lag order = 37, p-value = 0.01
## alternative hypothesis: stationary

library(exptest)

atkinson.exp.test(inter_arrival_times)

##
##  Atkinson test for exponentiality
##
## data:  inter_arrival_times
## T = 0.0076583, p-value = 0.1539

library(dgof)

##
## Attaching package: 'dgof'

## The following object is masked from 'package:stats':
##
##     ks.test

ks.test(sample1$inter_arrival_times, sample2$inter_arrival_times)
```

Appendix 3

R output

```
## Warning in ks.test(sample1$inter_arrival_times,
## sample2$inter_arrival_times : cannot compute correct p-values with ties
##
##  Two-sample Kolmogorov-Smirnov test
##
## data:  sample1$inter_arrival_times and sample2$inter_arrival_times
## D = 0.010934, p-value = 0.9956
## alternative hypothesis: two-sided


atkinson.exp.test(research_data$LOSdays)
##
##  Atkinson test for exponentiality
##
## data:  LOSdays
## T = 0.0080128, p-value = 0.2176
##
##  Augmented Dickey-Fuller Test
##
## data:  na.omit(research_data$LOSdays)
## Dickey-Fuller = -36.859, Lag order = 37, p-value = 0.01
## alternative hypothesis: stationary
library(dgof)
ks.test(sample1LOSdays, sample2$LOSdays)
## Warning in ks.test(sample1$LOSdays, sample2$LOSdays :
## cannot compute correct p-values with ties
##
##  Two-sample Kolmogorov-Smirnov test
##
## data:  sample1$LOSdays and sample2$LOSdays
## D = 0.01973, p-value = 0.248
## alternative hypothesis: two-sided
library(tseries)
adf.test(na.omit(research_data$waiting_time))
## Warning in adf.test(na.omit(research_data$waiting_time)): p-value smaller
than
## printed p-value
##
##  Augmented Dickey-Fuller Test
##
## data:  na.omit(research_data$waiting_time)
```

Appendix 4

R output

```
## Dickey-Fuller = -36.047, Lag order = 37, p-value = 0.01
## alternative hypothesis: stationary

library(exptest)
atkinson.exp.test(research_data$waiting_time)

##
##  Atkinson test for exponentiality
##
## data:  waiting_time
## T = 0.0042901, p-value = 0.2716

library(dgof)
ks.test(sample1$waiting_time, sample2$waiting_time)

## Warning in ks.test(sample1$waiting_time,
## sample2$ waiting_time, : cannot compute correct p-values with ties

##
##  Two-sample Kolmogorov-Smirnov test
##
## data:  sample1$waiting_time and sample2$waiting_time
## D = 0.024267, p-value = 0.4329
## alternative hypothesis: two-sided

library(KSgeneral)

## Warning: package 'KSgeneral' was built under R version 4.1.1
```

References

Afrane, S. and Appah, A. (2014). Queuing theory and the management of waiting-time in hospitals: The case of anglo gold ashanti hospital in ghana. *International Journal of Academic Research in Business and Social Sciences*, 4(2):34.

Beichelt, F. (2006). *Stochastic processes in science, engineering and finance*. Chapman and Hall/CRC New York.

Belciug, S. and Gorunescu, F. (2015). Improving hospital bed occupancy and resource utilization through queuing modeling and evolutionary computation. *Journal of Biomedical Informatics*, 53:261–269.

Bhat, U. N. (2015). *An Introduction to Queueing Theory: modeling and analysis in applications*. Birkhäuser Dallas U.S.A.

Brown, L., Gans, N., Mandelbaum, A., Sakov, A., Shen, H., Zeltyn, S., and Zhao, L. (2005). Statistical analysis of a telephone call center: A queueing-science perspective. *Journal of the American Statistical Association*, 100(469):36–50.

Chanques, G., Jaber, S., Barbotte, E., Violet, S., Sebbane, M., Perrigault, P. F., Mann, C., Lefrant, J. Y., and Eledjam, J. J. (2006). Impact of systematic evaluation of pain and agitation in an intensive care unit. *Critical Care Medicine*, 34(6):1691–1699.

Franzese, L. A. G., Fioroni, M. M., Botter, R. C., and de Freitas Filho, P. J. (2009). Comparison of call center models. In *Proceedings of the 2009 Winter Simulation Conference (WSC)*, pages 2963–2970. IEEE.

Gans, N., Koole, G., and Mandelbaum, A. (2003). Telephone call centers: Tutorial, review, and research prospects. *Manufacturing & Service Operations Management*, 5(2):79–141.

Gorunescu, F. (2011). *Data Mining: Concepts, models and techniques*, volume 12. Springer Science & Business Media Berlin.

Haupt, M. T., Bekes, C. E., Brilli, R. J., Carl, L. C., Gray, A. W., Jastremski, M. S., Naylor, D. F., Spevetz, A., Wedel, S. K., and Horst, M. (2003). Guidelines on critical care services and personnel: recommendations based on a system of categorization of three levels of care. *Critical Care Medicine*, 31(11):2677–2683.

Kembe, M., Agada, P., and Owuna, D. (2014). A queuing model for hospital bed occupancy management: a case study. *International Journal of Computational and Theoretical Statistics*, 1(01).

Kim, S. H. and Whitt, W. (2014). Choosing arrival process models for service systems: Tests of a nonhomogeneous poisson process. *Naval Research Logistics (NRL)*, 61(1):66–90.

Knessl, C. and van Leeuwaarden, J. S. (2015). Transient analysis of the Erlang A model. *Mathematical Methods of Operations Research*, 82(2):143–173.

Lanken, P. N., Terry, P. B., Adler, D. C., Brooks Brunn, J. A., Crawford, S. W., Danis, M., Fedullo, A. J., Gottlieb, J. E., Hansen Flaschen, J., Kollef, M. H., et al. (1997). Fair allocation of intensive care unit resources. *American Journal of Respiratory and Critical Care Medicine*, 156(4 I):1282–1301.

Lattimer, V., Brailsford, S., Turnbull, J., Tarnaras, P., and Smith, H. (2004). Reviewing emergency care systems i: insights from system dynamics modelling. *Emergency Medicine Journal*, 21(6):685–691.

Luchetti, M. (2013). Intensive care resource allocation: when difficult choices have to be made. *British Journal of Medical Practitioners*, 6(4):4–6.

Mayosi, B. M. and Benatar, S. R. (2014). Health and health care in South Africa — 20 years after Mandela. *New England Journal of Medicine*, 371(14):1344–1353.

Robbins, T. R., Medeiros, D. J., and Harrison, T. P. (2010). Does the Erlang C model fit in real call centers? In *Proceedings of the 2010 Winter Simulation Conference*, pages 2853–2864. IEEE.

Ross, S. M. (2014). *Introduction to Probability Models*. Academic press Los Angeles California.

Ross, S. M. (2017). *Introductory Statistics*. Academic Press London.

Sinuff, T., Kahnamoui, K., Cook, D. J., Luce, J. M., Levy, M. M., et al. (2004). Rationing Critical Care Beds: a systematic review. *Critical Care Medicine*, 32(7):1588–1597.

Skowronski, G. A. (2001). Bed rationing and allocation in the intensive care unit. *Current Opinion in Critical Care*, 7(6):480–484.

Stats, S. (2020). Statistics South Africa. *Formal census*.

Vincent, J. L. and Singer, M. (2010). Critical care: advances and future perspectives. *The Lancet*, 376(9749):1354–1361.

Walther, F. J. (2005). Withholding treatment, withdrawing treatment, and palliative care in the neonatal intensive care unit. *Early Human Development*, 81(12):965–972.

Wu, J., Chen, B., Wu, D., Wang, J., Peng, X., and Xu, X. (2020). Optimization of markov queuing model in hospital bed resource allocation. *Journal of Healthcare Engineering*, 2020.