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Conditional Symmetry Properties for Ordinary Differential Equations

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A PhD thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy.

Abstract

This work deals with conditional symmetries of ordinary differential equations (ODEs). We refine the definition of conditional symmetries of systems of ODEs in general and provide an algorithmic viewpoint to compute such symmetries subject to root differential equations. We prove a proposition which gives important and precise criteria as to when the derived higher-order system inherits the symmetries of the root system of ODEs. We firstly study the conditional symmetry properties of linear nth-order ($n \geq 3$) equations subject to root linear second-order equations. We consider these symmetries for simple scalar higher-order linear equations and then for arbitrary linear systems. We prove criteria when the derived scalar linear ODEs and even order linear system of ODEs inherit the symmetries of the root linear ODEs. There are special symmetries such as the homogeneity and solution symmetries which are inherited symmetries. We mention here the constant coefficient case as well which has translations of the independent variable symmetry inherited. Further we show that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves. This is even true if the system has no Lie point sym-

metries. Examples of second- and third-order ODEs which admit conditional symmetries but are devoid of Lie point symmetries are presented. We further give examples of the conditional symmetries of nonlinear third-order equations which are linearizable by second-order Lie linearizable equations. We demonstrate the inheritance of symmetries for such second-order equations. Finally we provide applications to classical and fluid mechanics.

Furthermore, we extend our idea of conditional symmetries to consider scalar second-order ODEs which have seven equivalence classes and possess three Lie point symmetries as root ODEs for the derived third-order ODEs. We show that the symmetries of scalar second-order ODEs considered as first integrals are preserved as Lie point symmetries of the corresponding third-order ODEs. They in general become conditional symmetries of the respective third-order ODEs when derived without substitution. Also we show how one can calculate the conditional symmetries of third-order nonlinear ODEs subject to second-order nonlinear ODEs which posses three point symmetries. We study the conditional $sl(2, R)$ algebra symmetry algebra criteria for scalar third-order ODEs, obtained after the differentiation of the scalar second-order ODE which admits $sl(2, R)$ symmetry algebra. We work out the criteria in terms of the coefficients involved in the second-order nonlinear ODE which has $sl(2, R)$ symmetry algebra. The Tressé invariants are used to give the $sl(2, R)$ symmetry algebra criteria for second-order ODEs. These criteria on the coefficients can be stated in terms of the coefficients of the derived third-order ODE.

We moreover investigate the fourth-order static Euler-Bernoulli ODE, where the elastic modulus and the area moment of inertia are constants and the applied

load is a function of the normal displacement. In the maximal case this has three symmetries. This corresponds to the negative fractional power law $y^{-5/3}$ and the equation has the nonsolvable algebra $sl(2,\mathbb{R})$. We determine new two- and three-parameter families of exact solutions when the equation has this symmetry algebra. These are shown to be conditional. This is studied via the symmetry classification of the three-parameter family of second-order ordinary differential equations that arises from the relationship among the Noether integrals. In addition, we present a complete symmetry classification of the second-order family of equations. Hence the admittance of $sl(2,\mathbb{R})$ remarkably allows for a three-parameter family of exact solutions for the static beam equation with load a fractional power law $y^{-5/3}$.

Finally, we discuss the integrability of the scalar fourth-order ordinary differential equations and obtain some missing canonical forms for fourth-order ODEs that admit four-dimensional Lie algebra. This is motivated by the work on the static Euler-Bernoulli beam ODE. It will be interesting to look at reductions of such ODEs conditionally when they possess higher symmetries.

List of Papers from the Thesis

1. Aeeman Fatima and F.M. Mahomed, Conditional symmetries for ordinary differential equations and applications, International Journal of Non-Linear Mechanics, 67 (2014) 95-105.

2. Aeeman Fatima, F.M. Mahomed and M. Ayub, A Note on Four-Dimensional Symmetry Algebras and Fourth-Order Ordinary Differential Equations, Journal of Applied Mathematics, Article ID 848163, Volume 2013.

3. Aeeman Fatima, F.M. Mahomed, F. D. Zaman and A.H. Bokhari, A note on the integrability of a remarkable static Euler-Bernoulli beam equation, Journal of Engineering Mathematics, 82 (2013) 101-108.

4. Aeeman Fatima and F.M. Mahomed, Conditional symmetries of nonlinear third-order ordinary differential equations subject to root second-order equations with three-dimensional Lie algebra (to be submitted).

Declaration

I declare that this is my own work except where due references have been made to the literature. It is being submitted for the degree of Doctor of Philosophy by thesis to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree to any other university.

Aemaan Fatima

Date: 28 January 2015

Dedication

To my parents,

my husband Taha, my daughter Ayaat and my supervisor Prof Fazal Mahomed.

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First and above all, I praise Allah, the almighty for providing me this opportunity and granting me the capability to proceed successfully. This thesis appears in its current form due to the assistance and guidance of several people. I would therefore like to offer my sincere thank you to all of them.

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Chapter 1

Aims and Objectives of the Thesis

1.1 Introduction

The concept of symmetry has fascinated many artists and scientists through many centuries, from the ancient Greeks to Kepler, in his efforts for determining the orbits of the planets, to Newton, who embodied in the laws of mechanics as a symmetry principle the equivalence of motion in different inertial frames (Galilean invariance), to Einstein, who generalized Galileos principle of relativity from mechanics to all the laws of physics.

In the nineteenth century a great advance arose when the Norwegian mathematician Sophus Lie began to investigate continuous groups of transformations

leaving differential equations invariant, creating what is now called the symmetry analysis of differential equations. The original aim of Lie was that of setting up a general theory for the integration of ordinary differential equations (ODEs) in a similar vein to that developed by E. Galois and N. Abel for algebraic equations (see [1] for a detailed historical analysis). Thus, symmetry analysis of differential equations was inaugurated and applied by Sophus Lie during the period 1872-1899 [2, 3]. This theory enables one to derive reductions and solutions of differential equations in a completely algorithmic way without appealing to special lucky guesses. Sophus Lie pointed out the main problem he faced with his theory in a foreword to his lecture Differentialgleichungen. The Lie approach to differential equations was not exploited for half a century and mainly the abstract theory of Lie groups grew (the term Lie group was coined in 1928 by Hermann Weyl). Further, it was the Russian school with L. V. Ovsiannikov [4] in the forefront that since 1960 began to exploit systematically the methods of symmetry analysis of differential equations in the explicit construction of solutions of many kinds of problems, even complicated, and mostly of mathematical physics.

The Lie classical theory of symmetries of differential equations is an inspiring source for various generalizations and extensions aiming to work out ways for obtaining the explicit solutions of differential equations [5]. The historical origin of group theory goes back to the seminal work of Evaristé Galois in 1831. He was the first who really gave the essence for the notion of algebraic solution of equations which are related to their group properties. Galois solved the age old problem of finding a formula to solve a polynomial equation of arbitrary order. Galois realized that the roots of the polynomials have symmetries and

these symmetries possess the group structure. He proved that there exists polynomial equations whose groups are not solvable and the associated equations are therefore not solvable by means of radicals. Inspired by Galois theory, Lie endeavored to do for differential equations what Galois had done for polynomial equations. Lie developed a highly algorithmic method for the reductions and solutions of differential equations. He studied the groups of continuous transformations in Cartesian space. Such groups depend on continuous parameters and consists of point transformations acting in the space of dependent and independent variables. Lie demonstrated that many techniques, which appear ad-hoc, for finding solutions of differential equations can be unified and duly extended by considering the symmetry structure of the differential equations. Today, the Lie symmetry approach to differential equations is widely and extensively applied in various fields of mathematics, mechanics, economics and theoretical physics and many of the results published in these areas justify that Lie's theory is indeed an efficient tool for solving many differential equations.

With the passage of time as more and more researchers tried to go forward with Lie's theory, certain limitations came to light in the Lie approach. Hence, the theory of Lie groups is not complete for the integrability of differential equations. In the case of ODEs, on the one hand a differential equation may admit a sufficient number of symmetry transformations but may not yet be solvable by quadratures via the classical Lie approach. A particular example of this is a variable coefficient linear second-order ODE which has eight Lie point symmetries but is not integrable by quadratures. On the other hand, an ODE or a system of ODEs has general solution in terms of quadratures without admitting any non-trivial symmetry generators [6, 7]. These insights have given rise to

extensions to different types of symmetries such as nonlocal, λ as well as other types of symmetries [8, 9, 10, 11, 12]. Scalar linear second-order ODEs are similar to each other. Characteristically, they can be converted to the simplest equation by invertible maps. In the earlier literature, much focus has been on the study of nonlinear differential equations and the procedures to reduce them to linear form. Reduction and exact solutions of ODEs were investigated by Lie in a systematic manner by use of his continuous groups of point transformations. He, in [5], obtained a complete classification in the complex domain of such scalar second-order ODEs in terms of their groups. In particular, Lie determined algebraic and invariant criteria of second-order ODEs that could be linearized. Furthermore, he discovered the class of scalar second-order ODEs that transform to linear form under point transformations. These are called Lie linearizable ODEs. Lie found criteria for linearizability and obtained the most general form being that of the cubically nonlinear second-order ODE in its first derivative which was linearizable via point transformations.

The linearization of ODEs of the second-order was extended to the third order in [13, 14] and [15, 16]. It was shown in the latter works that there are two classes of third-order ODEs that are linearizable by point transformations. It is possible to extend the linearization procedure for any order n with available algebraic computation packages. For $n = 4$ this was achieved in [17].

Lie algebraic properties of scalar linear nth-order ($n \geq 3$) ODEs were investigated in [18] as well as [19]. It was shown in [19] that there are three classes of such higher order ODEs, viz. $n + 1$, $n + 2$ and $n + 4$. This is in contrast to scalar linear second-order ODEs.

A significant extension of classical Lie symmetries for partial differential equations (PDEs) is that of non-classical or what is frequently called conditional symmetries [20, 21]. In the past two decades, the theoretical background of nonclassical symmetry has been widely investigated and non-classical symmetry techniques have been effectively applied to find the exact solutions of many partial differential equations from mathematical physics [22, 23, 24, 25, 26, 27, 28, 29]. Literature survey witness that considerable amount of work has been done on the study of conditional symmetry properties of PDEs. Conditional symmetries of PDEs have direct relation to side conditions imposed on the PDEs [30, 31]. Olver and Rosenau [30] showed that the construction of exact solutions to a PDE can be considered within the framework of a unifying approach of adding side conditions to the PDE and then solving the resultant overdetermined system for the solutions. An algorithmic approach to characterize weak symmetry groups of a given PDE is presented by Pucci and Saccomandi in [31]. The authors in [31] utilize the complete integrability condition of the system which is comprised of the PDE itself and the group characteristic equation. In this approach the notion of completeness of the system plays a pivotal role. There has been considerable amount of work on conditional symmetries of PDEs which pervade the literature. Our interest here is that of conditional symmetry of ODEs which has not enjoyed much interest. One of the initial investigations of the non-classical method to ODEs is due to Gaeta [32]. In this study [32], he discussed, inter alia, conditional constants of the motion as well as their relation to conditional symmetries for first-order dynamical systems of ODEs. In the definition of conditional symmetries proposed in [32], the invariant curve condition is appended as a side condition. We in our study consider higher-order ODE systems and

provide a definition in this context (see the next section). First integrals or constants of the motion are naturally studied in the formalisms of Hamiltonian and Lagrangian systems. The direct relationship between symmetries and first integrals for canonical Hamilton's equations first appeared in the work of Levi-Civita [33] (see also the interesting translation of the work [33] and its historical perspective in [34]) and for symmetries related to Lagrangian systems in the work of Noether [35]. Configurational invariants and second integrals were studied by Darboux, Poincaré, Painlevé, Hess amongst others (see, e.g. [36] for a review). In the paper by Sarlet et al. [37], configurational invariants were investigated in the context of Hamiltonian systems and a weak form of complete integrability. This notion is utilized as conditional constants of the motion in [32] and in the present treatment we formulate the definition of conditional first integrals in the context of higher-order ODEs under discussion.

Linearization plays a significant role in the theory of differential equations. First order ODEs can always be linearized by point transformations. Lie proved that the necessary and sufficient condition for a scalar nonlinear ODE to be linearizable is that it must have eight Lie point symmetries. He exploited the fact that all scalar linear second-order ODEs are equivalent under point transformations; that is every linearizable scalar ODE is reducible to the free particle equation. Lie took this approach no further but considerably later Chern [38] extended the analysis to a class of scalar third-order ODEs by using contact transformations. Mahomed and Leach [39] proved that for mth-order ODEs, $m \geq 3$, there was no unique class of linearizable ODEs. Instead there were three equivalence classes as remarked earlier.

Conditional classification of ODEs was proposed in [40] in which the authors provide definitions of conditional symmetry and linearizability of scalar ODEs subject to lower order root ODEs. They, in [41], discussed the invariant criteria for conditional linearizability of third-order equations subject to root second-order Lie linearizable ODEs. They have shown that certain gaps may be filled by the recent development of conditional linearizability for the deficiency of certain cases when the ODEs are not Lie linearizable. Differentiating the quadratically and cubically scalar semi-linear second order ODEs relative to the independent variable gives scalar third-order ODEs. By investigating the general class of scalar third-order ODEs that are conditionally linearizable one finds conditional linearizability criteria for scalar third-order ODEs subject to root second-order linearizable ODEs [41]. These classes were not subsumed in [15] nor in [16]. This in effect was a new type of linearizability. Further work on conditional linearization of fourth-order ODEs was performed in [42] as well as the conditional linearization of third-order systems in [43]. These were subject to Lie linearizable lower second-order ODEs.

The Euler-Bernoulli beam equations, both static and dynamic, have recently been under the spotlight from the Lie and Noether symmetry viewpoints. The dynamic beam fourth-order partial differential equation (PDE) [44]

$$\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 u}{\partial x^2}) + \mu\frac{\partial^2 u}{\partial t^2} = f, \tag{1.1}$$

where E is the elastic modulus, I the area moment of inertia, μ the lineal mass and f the applied load has been investigated algebraically for E, I, μ and f as some given constants or functions. The complete Lie symmetry group classification of the dynamic fourth-order Euler-Bernoulli (1.1), where the elastic

modulus and the area moment of inertia are constants and the applied load is a nonlinear function, was recently studied in [45]. It has also been studied from other perspectives (see e.g. [46]) Wafo Soh [47] solved the equivalence problem using symmetry for

$$\frac{\partial^2}{\partial x^2}(g(x)\frac{\partial^2 u}{\partial x^2}) + m(x)\frac{\partial^2 u}{\partial t^2} = 0,\ t > 0, 0 < x < L, \tag{1.2}$$

in which $g(x)$ is the flexural rigidity, $m(x) > 0$ the lineal mass density and $u(t, x)$ the transversal displacement at time t and position x. Özkaya and Pakdemirli [48], used the Lie group-theoretic approach to analyze the axially accelerating beam problem. For a simply supported beam, they found approximate solutions in the case of an exponentially decaying and harmonically varying beam. In [49], Bokhari et al. studied the static Euler-Bernoulli beam equation

$$\frac{d^4 y}{dx^4} = f(y), \tag{1.3}$$

where f is an arbitrary function of the transversal displacement, from both the Lie and Noether viewpoints. They arrived at an exceptional case, viz.

$$\frac{d^4 y}{dx^4} = \delta y^{-5/3}, \delta = \pm 1, \tag{1.4}$$

for which the nonsolvable algebra $sl(2, \mathbb{R})$ admitted by equation (1.4) is isomorphic to the Noether algebra with respect to the standard Lagrangian of (1.4)

$$L = \frac{1}{2}(y'')^2 + \frac{3}{2}\delta y^{-3/2}. \tag{1.5}$$

The generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y} \tag{1.6}$$

which are both Lie and Noether symmetries for (1.4) and corresponding to (1.5) span the Lie algebra $sl(2,\mathbb{R})$. It was found that the Lie route gives two reductions by use of the two-dimensional ideal generated by X_1 and X_2 given in equation (1.6) whereas the Noether route, which yielded three first integrals corresponding to each X_i in equation (1.6), with respect to the Lagrangian (1.5), provided a transparent way for integrability and a two-parameter family of exact solutions (see [49])

$$\int \frac{dy}{\pm\sqrt{\frac{1}{2}c_3 + c_4 y^{4/3}}} = x + c_5, \tag{1.7}$$

where c_4 and c_5 are constants and c_3 and c_4 are not independent but related by $27\delta + 2c_3c_4 = 0$. These solutions were consequences of the second-order ODE

$$3yy'' - 2y'^2 + x^2c_1 - 2xc_2 + c_3 = 0, \tag{1.8}$$

where the c_is are constants, which arises via the relationship among the Noether integrals.

The integrability of scalar ordinary differential equations (ODEs) by use of the Lie symmetry method depends on their symmetry Lie algebra if the Lie algebra is solvable and of sufficient dimension. There exists two different approaches to the integrability of differential equations using Lie point symmetries. One is the direct method in which Lie point symmetries are utilized to perform integrability by successive reduction of order of the equation using ideals of the algebra. The other approach is the canonical form method if the equations are classified into different types according to the canonical forms of the corresponding Lie algebra. Lie [50] classified scalar second-order ODEs into four types on the basis of their admitted two-dimensional Lie algebras and also performed integration of the representative equations corresponding to the canonical forms of the

two-dimensional symmetry algebra. Therefore scalar second-order ODEs can be integrated by using canonical variables which map the symmetry generators to Lie's canonical forms. Also here one can use successive reduction of order by using ideals of the symmetry algebra (see, e.g. Olver [51]). The Noether equivalence approach for Lagrangians corresponding to scalar second-order ODEs is discussed in Kara et al. [52]. The canonical forms for scalar third-order ODEs that admit three symmetries were obtained by Mahomed and Leach [53]. Then Nucci and Ibragimov [54] provided the integrability of these canonical forms. In their paper Cerquetelli et al. [55] constructed realizations in the plane of four-dimensional Lie algebras listed by Patera and Winternitz [56]. Moreover, the classification of subalgebras of all real Lie algebras of dimension ≤ 4 were discussed in [56]. The construction of Cerquetelli et al. [55] was based on the three-dimensional subalgebras provided in [56]. They invoked the realizations of three-dimensional Lie algebras in the plane derived earlier by Mahomed and Leach [57] for this purpose. They then determined the fourth-order ODEs admitting the realizations of the obtained four-dimensional algebras as their Lie symmetry algebra. Finally they provided the route to integration of the classified fourth-order ODEs. However, the derived realizations of four-dimensional Lie algebras need supplementation in the light of recent work by Popovych et al. [58]. Recently these authors constructed a complete set of inequivalent realizations of real Lie algebras of dimension not greater than four in vector fields in the space of an arbitrary (finite) numbers of variables. We use the results of [58] to complete the classification of fourth-order ODEs in terms of their four-dimensional algebras presented in [55]. Apart from scalar fourth-order ODEs arising in the symmetry reductions of partial differential equations such as the

linear wave equation in an inhomogeneous medium (see [59]), they occur prominently as model equations in the form of the static Euler-Bernoulli beam (see, e.g. [60]) and Emden-Fowler equations. Such equations have been investigated for symmetry properties in [49, 61]. Also their conditional symmetries become of importance in future works. For a special case of a fourth-order ODE, viz. the static Euler-Bernoulli beam equation the Noether reductions allow for conditional integrability of the equation.

1.2 Outline of thesis

A detail outline of the thesis is as follows.

In Chapter 1, we investigate the conditional symmetries of ODEs when the ODEs under study have insufficient number of Lie point symmetries. These are useful for integration. We have shown by means of an algorithm as to how one calculates the conditional symmetries of ODEs subject to lower order ODEs. Many simple examples as well as the properties of scalar linear and linear systems of ODEs are considered for their conditional symmetries. We explain how the symmetries of the root linear ODE are inherited by the derived ODE by means of a four propositions. Moreover we prove that if a system of ODEs, has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves. We also investigate third-order nonlinear ODEs which admit eight conditional symmetries subject to a Lie linearizable second-order ODE. Finally we present

applications of conditional symmetries to ODEs that arise in mechanics and obtain new solutions. The findings of this chapter have been published in the "International Journal of Non-Linear Mechanics, 67 (2014) 95-105" [62].

The purpose of Chapter 2 is to venture further in the regime of conditional symmetries of ODEs. This chapter continues the study which was carried out in the previous chapter for scalar third-order ODEs subject to second-order ODEs which have seven equivalence classes and possess three-dimensional Lie algebra. We show that all the Lie point symmetries of the second-order ODEs when derived or taken as first integrals are preserved but not necessarily as inherited symmetries as they may be preserved as conditional symmetries as well. Further we discuss when the derived nonlinear ODEs have less point symmetries (as inherited symmetries) than the root ODEs. Finally, we examine the conditional $sl(2, R)$ criteria for a scalar third-order ODE subject to the root ODE having $sl(2, R)$ symmetry and workout the values of the coefficients involved in the second-order root equation carried over to the third-order ODE. The contents of this chapter have been written up in the form of a paper and intend to be submitted for publication [63].

In Chapter 3, we perform the complete Lie group classification of the second-order ODE (1.8) that results from the reduction of our static fourth-order Euler-Bernoulli beam ordinary differential equation (1.4). We obtain all the Lie reductions and integrability of the static beam equation (1.4) when equation (1.4) admits two symmetries. We obtain new two- and three-parameter families of exact solutions for the beam problem. This is studied via the symmetry classification of the three-parameter family of second-order ordinary differential equa-

tions. The results presented in Chapters 3 have been published in the "Journal of Engineering Mathematics, 82 (2013) 101-108" [64]. Herein the reduced ODE have symmetries which are conditional symmetries of the beam ODE.

In Chapter 4, we show that the results on realizations of four-dimensional algebras in the plane given in [55] are a special case of the corresponding set of realizations given in [58]. We make a comparison of the lists of realizations given in [55] and [58]. It should be remarked that in general a result of classification of realizations may contain errors of two types, viz.

1. Missing of some inequivalent cases.
2. Mutually equivalent cases.

In the comparison made in this chapter, we show that various cases have been missing in the results of [55]. We found that there are some other cases which can be combined in a compact form and also some arbitrary parameters and functions need to be modified according to the results of [58] related to realizations in the plane. The contents of this chapter have been published in "Journal of Applied Mathematics, Article ID 848163, Volume 2013" [65]. This work is also related to the previous chapter and further study could be the reductions of maximally symmetric equations in which the reduced equations have symmetries that are conditional symmetries of the fourth-order ODE.

Finally, some concluding remarks and future works are summarized in Chapter 5.

Chapter 2

Conditional Symmetries for Ordinary Differential Equations and Applications

2.1 Introduction

In this chapter, we refine the definition of conditional symmetries of ordinary differential equations (ODEs) and provide an algorithm to compute such symmetries subject to root ODEs. These are considered for arbitrary systems of ODEs. A proposition is proved which provides essential criteria as to when the point symmetries of the root system of ODEs are inherited by the derived higher-order system. We provide illustrative examples and then investigate the

conditional symmetry properties of linear nth-order ($n \geq 3$) equations subject to root linear second-order equations. First this is considered for simple linear equations and then for arbitrary linear systems. We prove criteria when the symmetries of the root linear ODEs are inherited by the derived scalar linear ODEs and even order linear system of ODEs. Furthermore, we show that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves. Moreover, we give examples of the conditional symmetries of nonlinear third-order equations which are linearizable by second-order Lie linearizable equations. Applications to classical and fluid mechanics are presented.

2.2 Conditional symmetries: definitions and algorithm

We firstly consider some preliminaries before proceeding to the notion of conditional symmetries of arbitrary systems of ODEs.

Consider the system of nth-order ODEs of the form

$$y_{\alpha}^{(n)} = f_{\alpha}(x, y, y^{(1)}, ..., y^{(n-1)}), n \geq 1, \alpha = 1, \ldots, \tilde{p}, \tag{2.1}$$

where x is the independent variable, $y = (y_1, \ldots, y_p)$ the dependent variable and $y^{(1)}, y^{(2)}..., y^{(n)}$ denote the derivatives of y with respect to x up to order n. We may also say that the system (2.1) is of the *order* $n\tilde{p}$ which then directly gives

the number of essential constants that arises if (2.1) has general solution.

We now have the following notion of compatibility which is much more familiar in the setting for PDEs as in [31, 51].

If every solution of the mth-order ODE

$$y_\beta^{(m)} = g_\beta(x, y, y^{(1)}, ..., y^{(m-1)}),\ m \leq n, \beta = 1, \ldots, q \leq \tilde{p}, \tag{2.2}$$

is also a solution of the nth-order ODE (2.1), then the mth-order ODE (2.2) is said to be compatible to the nth-order ODE (2.1).

For compatible systems of ODEs, the number of essential constants for the appended side condition system is mq which is in general lower than that of the original ODE system of order n which is $n\tilde{p}$. This reduction in the number of essential constants will be seen in the examples considered in the sequel.

Furthermore, it is well-known that a first integral of the system (2.1) is a function $I(x, y, y^{(1)}, ..., y^{(k)})$, $k \leq n$, which satisfies the relation

$$D_x I = 0, \tag{2.3}$$

where $D_x \equiv d/dx$ is the total derivative operator with respect to x, on the solutions of the equations (2.1).

Following Sarlet et al. [37], who stated this definition for a system of two second-order ODEs, we have the notion of conditional first integrals for an arbitrary system of ODEs. Note that in [37] and before, the terminology configurational invariant was in vogue (presumably due to the advent of conditional symmetries

appearing later) and in [32], Gaeta used the notion of conditional constants of the motion for investigating dynamical systems. Here the context is higher-order ODEs. We therefore have the definition below.

Definition 1. A function $J(x, y, y^{(1)}, ..., y^{(k)})$, $k \leq n$, is called a conditional first integral of the system of ODEs (2.1) with respect to the first integral $I(x, y, y^{(1)}, ..., y^{(k)}) = C$, $k \leq n$, C a constant, if the restriction of J to a certain fixed surface obtained by setting $C = C_0$, i.e. $J|_{C=C_0}$ is a first integral of the corresponding reduced system.

It is important to remark that the number of essential constants for our system reduces by one if we have a proper conditional first integral of the system. The examples that ensue will amply illustrate this fact.

We start here by giving a precise definition of conditional symmetries. This is a natural extension of that given in the work [40]. Also this in general differs from the work [32] who require only invariant curve conditions.

Definition 2. An nth-order system of ODEs $n \geq 1$ (2.1) is conditionally classifiable by a symmetry algebra $\mathcal{A}$ with respect to an mth-order system of ODEs $m \leq n$ (2.2) called the root ODEs if and only if the nth-order system of ODEs jointly with the mth order system of ODEs forms an overdetermined compatible system (so the solutions of the nth-order system of ODEs reduce to the solutions of the mth-order system of ODEs) and the mth-order system of ODEs has symmetry algebra $\mathcal{A}$ which is the conditional symmetry algebra of the nth-order system.

Now we present an algorithm for computing the conditional point symmetries of the system (2.1) subject to the root system (2.2).

Let X be the vector field of dependent and independent variables given by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta_\alpha(x, y)\frac{\partial}{\partial y_\alpha}, \tag{2.4}$$

where ξ and η_α are the coefficient functions of the vector field X and the summation convention applies on α from 1 to p.

Suppose that the vector field X is a conditional symmetry generator of system (2.1) subject to an mth-order root system of ODEs (2.2). Then the conditional symmetry condition

$$X^{[n]}[y_\alpha^{(n)} - f_\alpha(x, y, y^{(1)}, ..., y^{(n-1)})]|_{y_\alpha^{(n)} - f_\alpha = 0, y_\beta^{(m)} - g_\beta = 0} = 0, \tag{2.5}$$

holds, where $X^{[n]}$ denotes the nth prolongation of the generator X as

$$X^{[n]} = X + \sum_{j=1}^{n} \zeta_j^\alpha \frac{\partial}{\partial y_\alpha^{(j)}} \tag{2.6}$$

in which

$$\zeta_j^\alpha = D_x(\zeta_{j-1}^\alpha) - y_\alpha^{(j)} D_x(\xi),\ j = 1, \ldots, n,\ \zeta_0^\alpha = \eta^\alpha, \tag{2.7}$$

and D_x is again the usual total differentiation operator.

The conditional symmetry approach proposed for a given system of ODEs is a technique that allows us to construct particular compatible side conditions to the system to enable one to obtain the construction of exact solutions of the system if there is a sufficient number of conditional symmetries. The number of conditional symmetries required for a solution of a nonlinear ODE system is the

same as for the Lie approach as we seek reduction and solution to the compatible root equation (see also [40]). The exact solutions that arise in this approach has in general fewer or for many equations that occur in applications no essential constants when compared to the original system. This is seen in the examples presented below and in the section on physical applications. In relation to this, we mention that the conditional linearization of classes of third-order ODEs subject to Lie linearizable second-order ODEs gives rise to two essential constants for the original classes [41] as the root ODEs are of order two for which we know the general solutions. If the side condition is of one order (order in the sense of essential constants) less than the original system, then first or conditional first integrals are the appropriate side conditions. We mention these when they arise in the examples considered.

It is now opportune to provide examples of conditional symmetries for second- and third-order ODEs.

1. Consider firstly the scalar second-order ODE

$$y'' = B + 3Ax^2y^{-1} - ABx^4y^{-2} - A^2Bx^6y^{-3}, \tag{2.8}$$

where A and B are constants. By using the Lie symmetry condition

$$X^{(2)}(y'' - B - 3Ax^2y^{-1} + ABx^4y^{-2} + A^2Bx^6y^{-3}) = 0, \tag{2.9}$$

whenever (2.8) holds, we obtain the following determining equation

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y^3 \\
&+\eta_y(B + 3Ax^2y^{-1} - ABx^4y^{-2} - A^2Bx^6y^{-3}) \\
&-3\xi_y y'(B + 3Ax^2y^{-1} - ABx^4y^{-2} - A^2Bx^6y^{-3})
\end{aligned}$$

$$-6Ax\xi y^{-1} + 4\xi ABx^3y^{-2} + 6\xi A^2x^5y^{-3}$$
$$+3\eta Ax^2y^{-2} - 4\eta ABx^4y^{-3} - 3\xi A^2x^6y^{-4} = 0. \tag{2.10}$$

Separation of (2.10) by powers of y', as ξ and η are independent of the derivatives of y, leads to the overdetermined system of linear homogeneous partial differential equations

$$\xi_{yy} = 0, \tag{2.11}$$
$$\eta_{yy} - 2\xi_{xy} = 0, \tag{2.12}$$
$$2\eta_{xy} - \xi_{xx} - 3\xi_y B - 9Ax^2\xi_y y^{-1}$$
$$-3ABx^4y^{-2}\xi_y + 3A^2x^6y^{-3}\xi_y = 0, \tag{2.13}$$

$$\eta_{xx} + \eta_y(B + 3Ax^2y^{-1} - ABx^4y^{-2} - A^2Bx^6y^{-3})$$
$$+3Ax^2y^{-1} - ABx^4y^{-2} - A^2Bx^6y^{-3} - 2\xi_x(B + 3Ax^2y^{-1}$$
$$-ABx^4y^{-2} - A^2Bx^6y^{-3}) - 6Ax\xi y^{-1} + 4\xi ABx^3y^{-2}$$
$$+6\xi A^2x^5y^{-3} + 3\eta Ax^2y^{-2} - 4\eta ABx^4y^{-3} - 3\xi A^2x^6 = 0. \tag{2.14}$$

By solving the above system (2.11)-(2.14), we find the trivial solution

$$\xi = 0,\ \eta = 0. \tag{2.15}$$

Hence the ODE (2.8) has no Lie point symmetry and we consequently cannot use the Lie reduction technique to solve it.

We now compute the conditional symmetries of the second-order ODE (2.8) subject to the nonlinear Abel first-order ODE

$$y' = Bx + Ax^3y^{-1}. \tag{2.16}$$

That is the conditional symmetry condition is

$$X^{[2]}(y'' - B - 3Ax^2y^{-1} + ABx^4y^{-2} + A^2Bx^6y^{-3})_{(2.8),(2.16)} = 0. \tag{2.17}$$

If we assume that

$$\xi = \alpha(x)\ \eta = \beta(x)y, \tag{2.18}$$

since a symmetry of (2.16) is of this form, then the determining equations (2.12)-(2.15) after split in powers of y result in

$$\begin{aligned} -2x\beta + 2\alpha' x + 6\alpha &= 0, & (2.19)\\ -3x\beta + 2\alpha' x + 4\alpha &= 0, & (2.20)\\ x^2(2\beta' - \alpha'') + 6x\beta - 6\alpha' x - 6\alpha &= 0, & (2.21)\\ x(2\beta' - \alpha'') + \beta - 2\alpha' &= 0. & (2.22) \end{aligned}$$

After we solve the system (2.19)-(2.22) for α and β, we obtain

$$X = x\partial_x + 2y\partial_y, \tag{2.23}$$

which is now the conditional symmetry of our nonlinear ODE (2.8) subject to the root equation (2.16). By using this symmetry (2.23) we can construct a one-parameter family of exact solutions of (2.8). Here we see that the root ODE (2.16) is a conditional first integral of our equation (2.8). Thus a reduction occurs in the number of essential constants from two to one therefore yielding a one-parameter family of exact solutions.

We proceed further to a simpler equation by considering a linear equation in order to understand the behavior of such symmetries.

2. We work out the conditional symmetries of the simplest third-order ODE, viz.

$$y''' = 0, \tag{2.24}$$

subject to the free particle equation

$$y'' = 0. \tag{2.25}$$

Here the determining equation is

$$\zeta_3|_{y'''=0=y''} = 0. \tag{2.26}$$

The expansion of (2.26) gives rise to

$$\begin{aligned} &\eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy}) \\ &y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 = 0. \end{aligned} \tag{2.27}$$

Separation of (2.27) and then solving the resultant linear homogeneous system yields 15 conditional symmetries which are

$$\begin{aligned} X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, \\ X_5 = y\partial_x, X_6 = x\partial_y, X_7 = x^2\partial_x, X_8 = xy\partial_y, \\ X_9 = xy\partial_x, X_{10} = y^2\partial_y, X_{11} = y^2\partial_x, X_{12} = x^2\partial_y, \\ X_{13} = xy^2\partial_x + y^3\partial_y, X_{14} = x^2y\partial_x + xy^2\partial_y, X_{15} = x^3\partial_x + x^2y\partial_y. \end{aligned} \tag{2.28}$$

We note that these conditional symmetries contain both the symmetries of the free particle equation (2.25) (these are X_1 to X_6, $X_7 + X_8$ and $X_9 + X_{10}$) as well as those of the simplest third-order ODE (2.24) (these are X_1 to X_4, X_6, X_{12} and $X_7 + 2X_8$). This should be the case as conditional symmetries are more general

than Lie point symmetries as in the case of PDEs. Moreover, it should be seen that X_{13}, X_{14} and X_{15} are not symmetries of (2.25) nor of (2.24). These are in addition. Further, the symmetries (2.28) do not close under the Lie bracket and therefore do not form a Lie algebra. One can observe that by merely computing $[X_7, X_{12}] = 2x^3\partial_y$ which is not a conditional symmetry of (2.24). Also of interest is that X_{13}, X_{14} and X_{15} form an Abelian three-dimensional Lie algebra.

In general we have that the free particle symmetries are conditional symmetries of the third-order ODE (2.24). The question arises as to which symmetries of the free particle equation are inherited by the derived third-order ODE as point symmetries and which remain as proper conditional symmetries.

We focus now on the symmetries of the free particle ODE (2.25) that are inherited by the third-order ODE (2.24). It is seen that the generators X_1, X_2, X_3, X_4 and X_6 given in (2.28) are inherited by (2.24). The remaining generators X_5, $X_7 + X_8$ and $X_9 + X_{10}$ of the free particle equation (2.25) thus become the proper conditional symmetries of the third-order ODE (2.24). Also note that the solution symmetry X_{12} and the symmetry $X_7 + 2X_8$ of equation $y''' = 0$ given in (2.28) do not come from the symmetries of the equation $y'' = 0$. In this example the conditional first integral is $y'' = 0$ which restricts the number of essential constants to two of the original third-order ODE $y''' = 0$ for solution purposes.

3. Next we consider the nonlinear third-order ODE

$$y''' + y'^3 + 3xy'^2y'' - \frac{2}{x}y' + \frac{2}{x}y'' = 0, \tag{2.29}$$

which is of the class considered in [40]. This ODE (2.29) is also conditionally

linearizable subject to the root Lie linearizable ODE $y'' + xy'^3 + 2y'/x = 0$ [41]. The determining equations for the symmetries of the ODE (2.29) are

$$\begin{aligned} \xi_y = 0,\ \eta_{yy} = 0,\ \eta_x &= 0, \\ 4\xi - 4x\xi_x - 2x^2\xi_{xx} - x^3\xi_{xxx} &= 0, \\ -2\eta_y + 3x\xi_{xx} &= 0, \\ 2\xi - 2x\xi_x + 3x^2\xi_{xx} &= 0, \\ \xi - x\xi_x + 2x\eta_y &= 0. \end{aligned} \tag{2.30}$$

The solution of the determining system (2.30) gives

$$\begin{aligned} Y_1 &= x\partial_x, \\ Y_2 &= \partial_y. \end{aligned} \tag{2.31}$$

One can at most perform a double reduction of the ODE (2.29) by using the Lie reduction approach. However, we cannot solve (2.29) completely by using the Lie approach.

The conditional symmetries of (2.29) subject to the second-order ODE

$$y'' + xy'^3 + \frac{2}{x}y' = 0, \tag{2.32}$$

are

$$\begin{aligned} X_1 = x^2\cos y\partial_x,\ X_2 = x^2\sin y\partial_x,\ X_3 = \sin y\partial_x + \frac{1}{x}\cos y\partial_y, \\ X_4 = -\cos y\partial_x + \frac{1}{x}\sin y\partial_y,\ X_5 = x\cos 2y\partial_x - \sin 2y\partial_y, \\ X_6 = x\sin 2y\partial_x + \cos 2y\partial_y,\ X_7 = x\partial_x,\ X_8 = \partial_y. \end{aligned} \tag{2.33}$$

We see that only $X_7 = Y_1$ and $X_8 = Y_2$ are inherited symmetries of (2.29). This is quite distinct from that of the linear example considered in Example 2. The

rest are proper conditional symmetries. Here the conditional first integral is of order two given by (2.32).

In order to understand when the symmetry of the base scalar *root equation* is inherited by the derived equation, we therefore prove the following proposition.

Proposition 1.
Consider the system of nth ODEs

$$E_\alpha(x, y, y', y'', ..., y^{(n)}) = 0,\ \alpha = 1, \ldots, \tilde{p}, \tag{2.34}$$

where x is the independent variable, $y = (y_1, \ldots, y_p)$ the dependent variable and $y', y'', ..., y^{(n)}$ denote the first, second and up to nth order derivatives of y with respect to x. If the generator (2.4) is a Lie point symmetry generator of the ODE (2.34), then

$$X^{[n+1]} D_x E_\alpha = D_x(\lambda_\alpha^\beta) E_\beta + \lambda_a^\beta D_x E_\beta - D_x \xi D_x E_\alpha, \tag{2.35}$$

holds. Thus X is a symmetry of the derived ODE $D_x E_\alpha = 0$ for each α if λ_α^β are constants. Moreover if λ_α^β are constants, then one has the following relation for the kth derived ODEs $D_x^k E_\alpha = 0$, $\alpha = 1, \ldots, \tilde{p}$,

$$X^{[n+k]} D_x^k E = -\sum_{i=0}^{k-1} D_x^{k-i} \xi \frac{k!}{(k-i)!i!} D_x^{i+1} E_\alpha + \lambda_\alpha^\beta D_x^k E_\beta. \tag{2.36}$$

Furthermore, for λ_α^β constants and $D_x^2 \xi = 0$, we obtain

$$X^{[n+k]} D_x^k E_\alpha = \lambda_\alpha^\beta D_x^k E_\beta - k D_x \xi D_x^k E_\alpha. \tag{2.37}$$

This relation (2.37) implies that for the derived kth-order ODEs $D_x^k E_\alpha = 0$, for each α, the symmetry X is inherited from $E_\alpha = 0$ provided that λ_α^β are constants and ξ is linear in x.

The proof of the first part follows from the identity $X^{[n+1]}D_x = D_xX^{[n]} - D_x\xi D_x$ which acts on a differential function of order n. It is the case that

$$X^{[n+1]}D_xE_\alpha|_{D_xE_\alpha=0} = D_x(\lambda_\alpha^\beta)E_\beta|_{D_xE_\alpha=0} + (\lambda_\alpha^\beta D_xE_\beta)|_{D_xE_\alpha=0} - D_x\xi D_xE_\alpha|_{D_xE_\alpha=0}. \tag{2.38}$$

Thus $D_x\lambda_\alpha^\beta = 0$ implies that λ_α^β are constants, meaning that X is an inherited symmetry of the derived ODEs $D_xE_\alpha = 0$ for each α. Moreover, the second part follows from induction. Certainly, it applies for $k = 1$ as

$$X^{[n+1]}D_xE_\alpha = \lambda_\alpha^\beta D_xE_\beta - D_x\xi D_xE_\alpha. \tag{2.39}$$

For $k = 2$, we have

$$X^{[n+2]}D_x^2E_\alpha = -(D_x^2\xi)D_xE_\alpha + \lambda_\alpha^\beta D_x^2E_\beta - 2D_x\xi D_x^2E_\alpha. \tag{2.40}$$

The rest follows by induction for if it is true for $k = m$, then for $k = m + 1$ one has

$$X^{[m+n+1]}D_x^{m+1}E_\alpha = D_x(X^{[m+n]}D_x^mE_\alpha) - D_x\xi D_x^{m+1}E_\alpha,$$

which results in

$$\begin{aligned} X^{[n+m+]}D_x^{m+1}E_\alpha &= \lambda_\alpha^\beta D_x^{m+1}E_\beta - \sum_{i=0}^{m-1} D_x^{m+1-i}\xi\frac{m!}{(m-i)!i!}D_x^{i+1}E_\alpha \\ &\quad - \sum_{i=1}^{m} D_x^{m+1-i}\xi\frac{m!}{(m-i+1)!(i-1)!}D_x^{i+1}E_\alpha - D_x\xi D_x^{m+1}E_\alpha \\ &= \lambda_\alpha^\beta D_x^{m+1}E_\beta - \sum_{i=0}^{m} D_x^{m+1-i}\xi\frac{(m+1)!}{(m+1-i)!i!}D_x^{m-i+1}\xi D_x^{i+1}E_\alpha, \end{aligned} \tag{2.41}$$

since $\frac{m!}{(m-i)!i!} + \frac{m!}{(m-i+1)!(i-1)!} = \frac{(m+1)!}{(m-i+1)!i!}$. The final part is a consequence of the fact that if ξ is linear in x, then $D_x^2\xi = 0$ for $k \geq 2$ in (2.36). This directly gives (2.37).

We now provide applications of Proposition 1.

4. The nonlinear equation which is of the class in [40]

$$yy''' + 3y'y'' = 0, \tag{2.42}$$

subject to the Lie linearizable ODE

$$yy'' + y'^2 = 0, \tag{2.43}$$

has conditional symmetries

$$\begin{aligned} X_1 = \partial_x,\ X_2 = \frac{1}{y}\partial_y,\ X_3 = x\partial_x, \\ X_4 = y\partial_y,\ X_5 = y^2\partial_x,\ X_6 = \frac{x}{y}\partial_y, \\ X_7 = x^2\partial_x + \frac{1}{2}xy\partial_y,\ X_8 = xy^2\partial_x + \frac{1}{2}y^3\partial_y. \end{aligned} \tag{2.44}$$

Clearly X_1 is a inherited symmetry of our ODE (2.42).

Now consider the operator $X_2 = \frac{1}{y}\partial_y$ with its second prolongation

$$X_2^{[2]} = \frac{1}{y}\partial_y - \frac{1}{y^2}y'\partial y' + (-\frac{1}{y^2}y'' + \frac{2}{y^3}y'^2)\partial y'', \tag{2.45}$$

Operating $X_2^{[2]}$ on ODE (2.43), we find

$$X_2^{[2]}(yy'' + y'^2) = 0, \tag{2.46}$$

on the solution; hence X_2 by Proposition 1 is also a symmetry of the ODE (2.42) with $\lambda = 0$. Consider the generator $X_3 = x\partial_x$ with its second-prolongation

$$X_3^{[2]} = x\partial_x - y'\partial y' - 2y''\partial y''. \tag{2.47}$$

Applying $X_3^{[2]}$ on ODE (2.43), results in

$$X_3^{[2]}(yy'' + y'^2) = -2(yy'' + y'^2) = 0, \tag{2.48}$$

where we have $\lambda = -2$. Thus X_3 is an inherited symmetry by Proposition 1.

Now consider $X_4 = y\partial_y$, with

$$X_4^{[2]} = y\partial_y + y'\partial y' + y''\partial y'', \tag{2.49}$$

Making use of $X_4^{[2]}$ on ODE (2.43), we deduce

$$X_4^{[2]}(yy'' + y'^2) = 2(yy'' + y'^2) = 0, \tag{2.50}$$

with $\lambda = 2$. Hence X_4 is a inherited symmetry by Proposition 1.

Now for operator $X_5 = y^2\partial_x$, we have

$$X_5^{[2]} = y^2\partial_x - 2yy'^2\partial y' + (-2y'^3 - 4yy'y'' - 2yy'y'')\partial y''. \tag{2.51}$$

Operating $X_5^{[2]}$ on ODE (2.43), we get

$$X_5^{[2]}(yy'' + y'^2) = -6yy'(yy'' + y'^2) = 0, \tag{2.52}$$

where $\lambda = -6yy'$, so that X_5 is a proper conditional symmetry.

Consider the symmetry $X_6 = \frac{x}{y}\partial_y$ with second prolongation given by

$$X_6^{[2]} = \frac{x}{y}\partial_y - (\frac{1}{y} - \frac{x}{y^2}y')\partial y' + (\frac{x}{y})''\partial y''. \tag{2.53}$$

Applying $X_6^{[2]}$ on ODE (2.43), we find

$$X_6^{[2]}(yy'' + y'^2) = 0. \tag{2.54}$$

with $\lambda = 0$. This symmetry is inherited as well.

Now consider $X_7 = x^2\partial_x + \frac{1}{2}xy\partial_y$ with second prolongation:

$$X_7^{[2]} = 2x^2\partial_x + xy\partial_y + (y - 3xy')\partial y' + (-2y' - 7xy'')\partial y''. \tag{2.55}$$

We have

$$X_7^{[2]}(yy'' + y'^2) = -6x(yy'' + y'^2) = 0, \tag{2.56}$$

where $\lambda = -6x$; this implies that X_7 is a proper conditional symmetry.

Finally, we consider $X_8 = xy^2\partial_x + \frac{1}{2}y^3\partial_y$ with

$$X_8^{[2]} = xy^2\partial_x + \frac{1}{2}y^3\partial_y + (\frac{y^2y'}{2} - 2xyy'^2)\partial y' - (\frac{y^2y''}{2} + yy'^2 + 2xy'^3 + 6xyy'y'')\partial y''. \tag{2.57}$$

Now with the use of $X_8^{[2]}$ on ODE (2.43), we determine

$$X_8^{[2]}(yy'' + y'^2) = -6xyy'(yy'' + y'^2) = 0, \tag{2.58}$$

which in this case yields $\lambda = -6xyy'$. Thus X_8 is a proper conditional symmetry.

Therefore the inherited symmetries of (2.42) are X_1 to X_4 and X_6. Here five symmetries are inherited whereas in Example 3. we had only two inherited symmetries. These symmetries are also the inherited symmetries for any derived ODE of (2.43) as ξ is linear in x.

A few examples on systems of two ODEs are now given.

5. The variable coefficient linear system

$$\begin{aligned} E_1 &= x'' + tx = 0, \\ E_2 &= y'' + ty = 0, \end{aligned} \tag{2.59}$$

obviously admits the scaling symmetries $X_1 = x\partial_x$ and $X_2 = y\partial_y$. According to Proposition 1 these are inherited symmetries of any derived even order system $D_x^k E_\alpha = 0$, $\alpha = 1, 2$, $k \geq 1$. In fact we see that for X_1, the λs are $\lambda_1^1 = 1$ and $\lambda_\beta^\alpha = 0$ for the other values. Similarly one can obtain $\lambda_2^2 = 1$ for X_2 with the rest zero.

6. The nonlinear system (see [66])

$$\begin{aligned} E_1 = x'' - y'^2 = 0, \\ E_2 = y'' = 0, \end{aligned} \tag{2.60}$$

admits the symmetries

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = y\partial_x,\ X_4 = y\partial_t. \tag{2.61}$$

By Proposition 1, the symmetry X_1, X_2 and X_3 are inherited by any derived system $D_x^k E_\alpha = 0$, $\alpha = 1, 2$, $k \geq 1$. For the first two it is evident and for X_3 we have that the nonzero λ is $\lambda_1^2 = 1$ which is constant. In the case of X_4 we have that $\lambda_1^1 = -2y'$, $\lambda_1^2 = -x'$, $\lambda_2^1 = 0$ and $\lambda_2^2 = -3y'$. Hence not all λ_a^βs are constant and X_4 is a conditional symmetry of the derived system (2.60).

6. Finally for the nonlinear Newtonian system (see [66])

$$\begin{aligned} E_1 = x'' - x'^2 - y'^2 = 0, \\ E_2 = y'' - 2x'y' = 0, \end{aligned} \tag{2.62}$$

the admitted symmetries are

$$X_1 = \partial_t,\ X_2 = \exp -(x+y)\partial_t,\ X_3 = \exp -(x-y)\partial_t,\ X_4 = t\partial_t. \tag{2.63}$$

The symmetry X_1 is clearly inherited and so is X_4 since $\lambda_1^1 = -2$ and $\lambda_2^2 = -2$ with the rest zero. Any derived system of (2.62) also has inherited symmetries X_1 and X_4. These follow from Proposition 1. However, X_2 and X_3 are conditional symmetries.

2.3 Integrable equations admitting conditional symmetries

It is almost a folklore even nowadays that any ODE integrable by quadratures has nontrivial symmetries. This was dispelled in the two telling examples provided in [6, 7]. The second paper was on a system of two second-order ODEs which we focus on later. The example we first draw attention to is from [6]. In [6] the scalar second-order ODE

$$y'' = y^{-1}y'^2 + pg(x)y^p y' + g'(x)y^{p+1},\ y > 0, \tag{2.64}$$

where $p \neq 0$ is a constant and $g \neq 0$ an arbitrary function of x, was shown to be completely integrable by quadratures but devoid of point symmetry.

Here we demonstrate that (2.64) has a conditional symmetry subject to the Bernoulli equation

$$y' - Cy = g(x)y^{p+1}, \tag{2.65}$$

where C is an arbitrary constant.

We assume that $\xi = 0$, $\eta = \alpha(x)y^{p+1}$ since a symmetry of (2.65) is of this form. The substitution of this form into the determining equation (we invoke (2.5)) on

the solutions of (2.64) and (2.65) gives after separation with respect to separate powers of y that $\alpha(x) = \exp(-Cpx)$. Thus a conditional symmetry of (2.64) subject to (2.65) is

$$X = \exp(-Cpx)y^{p+1}\partial_y. \tag{2.66}$$

In fact the conditional symmetry (2.66) results in the complete integrability by quadratures of the ODE (2.64) for arbitrary $g(x)$ and $p \neq 0$. Here the Bernoulli equation (2.65) is a first integral of the original second-order ODE (2.64). Thus the number of essential constants in the solution is two which results in the general solution.

In fact we can go a step further and construct what appears to be the first higher-order ODE which is integrable by quadratures but has no point symmetry.

We utilize (2.65) in the form

$$y^{-1}y' - Kg(x)y^p = C, \tag{2.67}$$

where C and K are arbitrary constants. Then the corresponding second-order ODE after differentiation is

$$y^{-1}y'' = y^{-2}y'^2 + pg(x)Ky^{p-1}y' + g'(x)Ky^p,\ y > 0, \tag{2.68}$$

Utilizing (2.68) as a first integral for the third-order ODE, we determine

$$\begin{gathered} pg(x)y^{p-2}y'y''' + g'(x)y^{p-1}y''' - pg(x)y^{p-2}y''^2 - pg(x)y^{p-3}y'^2y'' - p^2g(x)y^{p-3}y'^2y'' \\ -3g'(x)y^{p-2}y'y'' - 2pg'(x)y^{p-2}y'y'' - g''y^{p-1}y'' + pg(x)y^{p-4}y'^4 + p^2g(x)y^{p-4}y'^4 \\ +2pg'(x)y^{p-3}y'^3 + 2g'(x)y^{p-3}y'^3 + g''(x)y^{p-2}y'^2 = 0. \end{gathered} \tag{2.69}$$

This ODE (2.69) subject to (2.67) admits the conditional symmetry (2.66). Here the compatible side condition (2.67) has two constants K and C, and its integrability then provides a further constant yielding in all three required for complete integrability.

We are in a position to prove the following proposition.

Proposition 2.
If a scalar nth-order, $n \geq 2$, ODE of the form (2.34) is completely integrable by quadratures, then it admits a conditional symmetry subject to the first-order ODE related to the invariant curve condition which arises from the known solution curves.

The proof of this follows at once. Indeed, since the nth-order ODE of the form (2.34) is completely integrable, it has integral curves $\phi(x, y, C_1, \ldots, C_n) = 0$ in which C_i, $i = 1, \ldots, n$ are n arbitrary constants (these n constants result in n first integrals). We solve for one of the C_is, say C_n, to deduce $C_n = \Phi(x, y, C_1, \ldots, C_{n-1})$. The total differentiation of this, viz. $D_x\Phi(x, y, C_1, \ldots, C_{n-1}) = 0$ yields

$$\Phi_x + y'\Phi_y = 0, \Phi_y \neq 0, \tag{2.70}$$

which is the invariant curve condition. The condition (2.70) then gives the conditional symmetry

$$X = \partial_x - \frac{\Phi_x}{\Phi_y}\partial_y, \tag{2.71}$$

of our nth-order ODE (2.34) subject to (2.70).

In the light of Proposition 2, we can now re-look at the example with which we

started. For the second-order ODE (2.64), a conditional symmetry is

$$X = \partial_x - (Cy + g(x)y^{p+1})\partial_y, \tag{2.72}$$

which is subject to the invariant curve condition (2.65). Similar considerations apply to the third-order ODE (2.69).

An ODE may not be completely integrable such as the one given in Example 1 of the previous section in which the second-order ODE has a one-parameter family of exact solutions. In this event, one can weaken Proposition 2 and write the integral curves as $\phi(x, y) = 0$ or $\phi(x, y, C_1, \ldots, C_r) = 0$, where r ranges from 1 to $r < n$. We can state a weaken form of Proposition 2 as follows.

Proposition 3.
If a scalar nth-order, $n \geq 2$, ODE of the form (2.34) has exact solutions $\phi(x, y) = 0$ or $\phi(x, y, C_1, \ldots, C_r) = 0$, where r ranges from 1 to $r < n$, then it admits a conditional symmetry subject to the first-order ODE related to the invariant curve condition which arises from the known solution curves.

The proof of this is analogous to that of Proposition 2. In this case there are r first integrals resulting in r essential constants or no first integral thereby giving an exact solution with no arbitrary constants – the side condition is the invariant condition which is compatible.

2.4 Conditional symmetries of scalar linear ODEs

Now we generalize the above analysis of the free particle ODE to scalar nth-order linear ODEs. We begin our discussion with simple linear ODEs.

Consider the simplest $(n-1)$th order ODE

$$y^{(n-1)} = 0,\ n \geq 4. \tag{2.73}$$

The above equation (2.73), as is well-known, has $(n-1)+4 = n+3$ point symmetries for $n \geq 4$ which are (see, e.g. [19])

$$\begin{aligned} &X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x,\ X_4 = y\partial_y,\ X_5 = x\partial_y, \\ &X_i = x^{i-4}\partial_y,\ i = 6, \ldots, n+2,\ X_{n+3} = x^2\partial_x + (n-2)xy\partial_y. \end{aligned} \tag{2.74}$$

The derived ODE

$$y^{(n)} = 0, \tag{2.75}$$

has $n+4$ symmetries. The operators $X_1, \ldots, X_{n+2}$ of (2.74) are the inherited symmetries of the nth-order ODE (2.75) and X_{n+3} of (2.74) is proper conditional. This can be verified by Proposition 1 as only the translations, uniform scalings as well as solution symmetries of (2.73) have λ constant. Thus these symmetries are inherited by the simplest derived higher-order linear ODE. If one looks for the inherited symmetries that arise from the free particle equation to any subsequent order equation $y^{(s)} = 0$, $s \geq 3$, then these are $X_1, \ldots, X_5$. However, if we desire the inherited symmetries from $y^{(p)} = 0$, $p \geq 3$ to $y^{(q)} = 0$, $q \geq 4$, then these are translations, uniform scalings and solution symmetries up to $X_{p+3} = x^{(p-1)}\partial_y$. The symmetry X_{p+4} is a proper conditional symmetry given as

$$X_{p+4} = x^2\partial_x + (p-1)xy\partial_y. \tag{2.76}$$

We now consider how the symmetries of scalar linear second-order ODEs with variable coefficients are inherited by the derived compatible higher-order linear ODEs. The linear second-order ODE

$$y'' + a(x)y' + b(x)y = 0, \tag{2.77}$$

has the point symmetries

$$\begin{aligned}
X_1 &= \alpha_1 y\partial_x + (\alpha_1^{'} - \alpha_1 a)y^2\partial_y, \\
X_2 &= \alpha_1 y\partial_x + (\alpha_2^{'} - \alpha_2 a)y^2\partial_y, \\
X_3 &= \beta_1\partial_x + \frac{(\beta_1^{'} - \beta_1 a)}{2}\partial_y, \\
X_4 &= \beta_2\partial_x + \frac{(\beta_2^{'} - \beta_2 a)}{2}\partial_y, \\
X_5 &= \beta_3\partial_x + \frac{(\beta_3^{'} - \beta_3 a)}{2}\partial_y, \\
X_6 &= \delta_1\partial_y,\ X_7 = \delta_2\partial_y,\ X_8 = y\partial_y.
\end{aligned} \tag{2.78}$$

where the α_i, β_i and δ_i are independent solutions of

$$\begin{aligned}
\alpha'' - (a\alpha)' + b\alpha &= 0, \\
\beta''' - (2a' + a^2 - 4b)\beta' - \frac{1}{2}(2a' + a^2 - 4b)'\beta &= 0, \\
\delta'' + a\delta' + b\delta &= 0.
\end{aligned} \tag{2.79}$$

It transpires by Proposition 1, that any derived ODE from (2.77) of third- or higher-order inherits in general the symmetries X_6, X_7, X_8 only for non-constant coefficients. If the coefficients of (2.77) are constants then a further symmetry $X = \partial_x$ (which comes from one of X_3 to X_5) is inherited.

We consider a particular example by putting $a = 0$ and $b = -1$ in ODE (2.77). Then the ODE is

$$y'' - y = 0. \tag{2.80}$$

Thus the symmetries X_3, X_4 and X_5 of ODE (2.80) are (these come from the corresponding symmetries in (2.78))

$$\begin{aligned} X_1 &= e^x y\partial_x + e^x y^2\partial_y,\ X_2 = e^{-x} y\partial_x - e^{-x} xy^2\partial_y,\ X_3 = \partial_x, \\ X_4 &= e^{2x}\partial_x + e^{2x} y\partial_y,\ X_5 = e^{-2x}\partial_x - e^{-2x} y\partial_y \\ X_6 &= e^x\partial_y,\ X_7 = e^{-x}\partial_y,\ X_8 = y\partial_y. \end{aligned} \tag{2.81}$$

The only inherited symmetries to the derived ODE are X_3, $X_6, \ldots, X_8$. Those of X_1, X_2, X_4 and X_5 are proper conditional symmetries of the derived equation. In fact if one computes λ for X_4 and X_5, one gets $\lambda = -3e^{2x}$ and $\lambda = 3e^{2x}$, respectively. Hence these are not inherited symmetries by Proposition 1.

For the simple harmonic oscillator equation

$$y'' + y = 0, \tag{2.82}$$

again the inherited symmetries to any higher-order derived ODE are translations in x, uniform scalings in y, and the two solution symmetries $\sin x\partial_y$ and $\cos x\partial_y$.

We may as well investigate what happens to the inherited symmetries if we substitute the highest derivative of the root third-order ODE in the derived ODE. Consider the first derived ODE of (2.77), viz.

$$y''' + ay'' + (a' + b)y' + b'y = 0. \tag{2.83}$$

Now we substitute the value for y'' from (2.77) into this ODE (2.83). This results in

$$y''' + (a' + b - a^2)y' + (b' - ab)y = 0. \tag{2.84}$$

This equation (2.84) inherits the solution and homogeneity symmetry X_6, X_7 and X_8 as given in (2.78) of the second-order ODE (2.77) although Proposition

1 does not apply. The reason being that δ_1 and δ_2 of X_6 and X_7 solves (2.84) as well as X_8 is clearly a symmetry of (2.84).

We now consider higher-order variable coefficient linear nth-order ($n \geq 3$) ODEs of the homogeneous type

$$E \equiv y^{(n)} + \sum_{i=0}^{n-2} a_i(x) y^{(i)} = 0. \tag{2.85}$$

We content ourselves with this form as the treatment of the linear ODE without the second highest derivative coefficient annulled is the same. We thus use the canonical form (2.85) as it has received full treatment for its symmetry properties in Mahomed and Leach [19]. Equation has point symmetries [19]

$$\begin{aligned} X_1 &= y\partial_y, \\ X_{i+1} &= \alpha_i(x)\partial_y,\ i = 1, \ldots, n, \qquad (2.86) \\ X_j &= \beta(x)_j\partial_x + \frac{n-1}{2}\beta_j'(x) y\partial_y,\ j = 1, 2, 3, \qquad (2.87) \end{aligned}$$

where α_i are n linearly independent solutions of (2.85) and β_j solves

$$\frac{(n+1)!}{(n-2)!4!}\beta''' + \beta' a_{n-2} + \frac{1}{2}\beta a_{n-2}' = 0. \tag{2.88}$$

Any derived linear equation of (2.85), viz. $D_x^m E = 0$ inherits the symmetries (2.86) as Proposition 1 applies. The symmetries (2.87) become proper conditional symmetries of $D_x^m E = 0$.

Now the differentiation of (2.85) twice, $D_x^2 E = 0$, and then the substitution of the value of $y^{(n)}$ from (2.85) into $D_x^2 E = 0$ gives

$$D_x^2 E\big|_{y^{(n)} = -\sum_{i=0}^{n-2} a_i(x) y^{(i)}} = 0. \tag{2.89}$$

The linear ODE (2.89) inherits the symmetries (2.86) of the root equation (2.85) which comprise the scaling and solution symmetries. Here Proposition 1 does not apply. This result though follows from the fact that the ODE (2.85) is compatible with the ODE (2.89) and the solutions of (2.85) are the solutions of (2.89) meaning that the solution symmetries in (2.86) are inherited by the ODE (2.89). Also clearly (2.89) admits the uniform dilation symmetry X_1 of (2.86).

The above argument is valid and can be extended to an arbitrary order derived ODE subject to appropriate substitutions of (2.85) at each stage in the derived ODE.

In the case of constant coefficient second- and higher-order linear ODEs (2.77) and (2.85), the derived ODE and derived ODE after substitution of the root equation inherits in addition to the solution and homogeneity symmetries, the translation symmetry viz. ∂_x.

One thus has the following proposition.

Proposition 4.
Any derived scalar linear ODE $D_x^m E = 0$, $m \geq 1$, with or without substitution of the root linear nth-order ($n \geq 2$) ODE of the homogeneous type $E \equiv y^{(n)} + \sum_{i=0}^{n-1} a_i(x) y^{(i)} = 0$ inherits the homogeneous symmetry $y\partial_y$ and n solution symmetries $\alpha_i(x)\partial_y$, $i = 1, \ldots, n$, where α_i solves $E = 0$. Further if the coefficients a_i are constants, then in addition the translation symmetry ∂_x is inherited by the derived ODE. The remaining symmetries are proper conditional symmetries of the derived ODE.

2.5 Conditional symmetries of systems

Symmetries and integrability for systems of second-order ODEs were considered recently in [67]. In general we have that the symmetries of the root system are conditional symmetries of the derived higher-order system of ODEs. The question arises as to which symmetries of the root system are inherited by the derived system of ODEs as point symmetries and which remain as proper conditional symmetries.

We commence our initial investigation by considering the simplest system of two second-order ODEs, viz.

$$\begin{aligned} x'' &= 0, \\ y'' &= 0, \end{aligned} \tag{2.90}$$

where the prime denotes differentiation with respect to t. This system (2.90) has the $sl(4,R)$ symmetry algebra with well-known point symmetries

$$\begin{aligned} X_1 &= \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = t\partial_t,\ X_5 = x\partial_x, \\ X_6 &= y\partial_y,\ X_7 = t\partial_x,\ X_8 = t\partial_y,\ X_9 = x\partial_t,\ X_{10} = y\partial_t, \\ X_{11} &= y\partial_x,\ X_{12} = x\partial_y,\ X_{13} = t^2\partial_t + tx\partial_x + ty\partial_y, \\ X_{14} &= tx\partial_t + x^2\partial_x + xy\partial_y,\ X_{15} = ty\partial_t + xy\partial_x + y^2\partial_y. \end{aligned} \tag{2.91}$$

It easy to deduce that the derived system

$$\begin{aligned} x''' &= 0, \\ y''' &= 0, \end{aligned} \tag{2.92}$$

has the inherited symmetries X_1 to X_8 as well as X_{11} and X_{12}. This means that 10 symmetries are inherited. These are comprised of 3 translations, 3 homogeneous, 2 solution and two symmetries of mixed type in the space variables, not time t, viz. X_{11} and X_{12}. We remind the reader that upper bounds for symmetry algebras for higher-order systems were given in [68].

One can extend the above argument to a system of more than two second-order ODEs. To begin with we consider a system of three equations. In this case we have 4 translations, 4 homogeneous, 3 solution and 6 symmetries of mixed type in the space variables. One can continue in this vein and we have the following proposition.

Proposition 5.
For a system of n, $n \geq 2$, free particle equations which admits the symmetry algebra $sl(n+2, R)$, the derived even order system inherits $n^2 + 2n + 2$ of the point symmetries of the free particle system comprising $n+1$ translations, $n+1$ homogeneous, n solution and $n(n-1)$ symmetries of mixed type in the space variables. The remaining $2n + 1$ symmetries are proper conditional symmetries of the derived system.

The proof is constructive and follows easily. The free particle system has $n^2 + 4n + 3$ point symmetries. Of these, n are the symmetries $y_i\partial_t$, $i = 1, \ldots, n$, if we label the space variables as y_i, and $n + 1$ are the true projective symmetries which involve all the variables. These constitute $2n+1$ symmetries which one can check are not inherited by the derived system and are conditional symmetries. Therefore $n^2 + 2n + 2$ are the inherited symmetries.

Consider now the nth-order system of ODEs

$$y^{(n)} + \sum_{i=0}^{n-1} A_i(t) y^{(i)} = \mathbf{0}, \tag{2.93}$$

where the A_is are $m \times m$ time-dependent matrices and $y = (y_1, \ldots, y_m)^T$.

We now have the following result which is a natural extension of Proposition 4.

Proposition 6.
Any derived linear even order system of ODEs $D_x^p E = \mathbf{0}$, $p \geq 1$, with or without substitution of the root linear nth-order ($n \geq 2$) ODE of the homogeneous type $E \equiv y^{(n)} + \sum_{i=0}^{n-1} A_i(t) y^{(i)} = \mathbf{0}$ inherits the m homogeneous symmetries $y_i \partial_{y_i}$, $i = 1, \ldots, m$ and nm solution symmetries $\alpha_i(x) \partial_{y_j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where α_i solves $E = \mathbf{0}$. Further if the coefficients A_i are constants, then in addition the translation symmetry ∂_t is inherited by the derived system of ODEs.

It will also be of interest for further study to investigate inherited symmetries of derived systems of linear ODEs which are of mixed type not considered above.

Before we terminate this section, we state an analogous result as for scalar nth-order ODEs for systems regarding integrable equations.

Proposition 7.
If an nth-order, $n \geq 2$, system of ODEs $E_\alpha(x, y, \ldots, y^{(n)}) = \mathbf{0}$, $\alpha = 1, \ldots, m$, where $y = (y_1, \ldots, y_m)^T$, has exact solutions $\phi_\alpha(x, y) = 0$ or $\phi_\alpha(x, y, C_1, \ldots, C_r) = 0$, where r ranges from 1 to $r \leq mn$, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves.

The proof is similar to that of Proposition 2 except that we have ϕ_α for each α. In this context there are r first integrals yielding r essential constants in the solution or no first integral just giving a particular solution devoid of any essential constants.

As an example, we take the system of two ODEs considered in the work [7] which drew attention to the misunderstanding that a system integrable by quadratures must have a nontrivial symmetry group. It was demonstrated in [7] that the system

$$\begin{aligned} x'' &= c(x),\ ' = d/dt, \\ y'' &= (f(t,x,x')y)' + g(t,x,x'), \end{aligned} \tag{2.94}$$

is integrable by quadratures but in general has a trivial symmetry group. In [7], the explicit forms for $f = \lambda(t)x'^m$ and $g = \mu(t)x'^{n+m}$ was utilized to construct the trivial group.

The system (2.94) for arbitrary c, f and g has conditional symmetry

$$\begin{aligned} X = \partial_t + s(x)\partial_x + \left(yf\left(\int s^{-1}(x)dx - c_2, x, s(x)\right)\right. \\ \left. + \int g\left(\int s^{-1}(x)dx - c_2, x, s(x)\right)s^{-1}(x)dx + c_3\right)\partial_y, \end{aligned} \tag{2.95}$$

where $s(x) = \pm\sqrt{2\int c(x)dx + 2c_1}$ with c_1, c_2 and c_3 arbitrary constants, subject to the invariant curve conditions

$$\begin{aligned} x' &= s(x), \\ y' &= f(s^{-1}(x)dx - c_2, x, s(x))y \\ &+ g(\int s^{-1}(x)dx - c_2, x, s(x))s^{-1}(x)dx + c_3. \end{aligned} \tag{2.96}$$

The invariant curve conditions have three arbitrary constants and their integrability gives one further constant.

2.6 Conditional symmetries subject to linearizable ODEs

Conditional linearizability of scalar third-order ODEs subject to Lie linearizable second-order ODEs were completely invariantly characterized in terms of the coefficients of the equation in the papers [40, 41] wherein the derived third-order ODE before and after substitution of the second derivative were investigated. Here we briefly focus on the conditional symmetries of two conditionally linearizable equations subject to second-order Lie linearizable root ODEs.

1. The nonlinear third-order ODE which is of class given in [40]

$$y^2y''' - 3y'^3 = 0, \tag{2.97}$$

subject to the linearizable second-order ODE (2.43) has conditional symmetries (2.44).

Here clearly X_1 is a inherited symmetry of (2.97).

Now consider the symmetry X_2 and its third prolongation and operating it on ODE (2.97), we find

$$X_2^{[3]}(y^2y''' - 3y'^3) = y'''y^2 + 3y'^3 + 6yy'y'', \tag{2.98}$$

Now putting $y'' = \frac{y'^2}{y}$ in (2.98), we obtain

$$X_2^{[3]}(y^2y''' - 3y'^3) = y'''y^2 - 3y'^3 = 0 \tag{2.99}$$

on solutions of the equation. Thus X_2 is a proper conditional symmetry when we substitute the value of y''.

Consider the generator X_3 with its third prolongation and operating it on ODE (2.97), we get

$$X_3^{[3]}(y^2y''' - 3y'^3) = -3(y^2y''' - 3y'^3) = 0. \tag{2.100}$$

Hence X_3 is an inherited symmetry of ODE (2.97).

The symmetry X_4 is also an inherited symmetry for ODE (2.97).

Since $X_5 = y^2\partial_x$ is a conditional symmetry by Proposition 1 with $\lambda = -6yy'$, thus it would also remain a conditional symmetry for a third-order ODE (2.97) after substitution of y''.

Consider now the symmetry $X_6 = \frac{x}{y}\partial_y$ with its third prolongation and operating on ODE (2.97) we find

$$X_6^{[3]}(y^2y''' - 3y'^3) = xy''' - 3y'' - 3\frac{y'^2}{y} + 3\frac{xy'^3}{y^2} + 6\frac{xy'y''}{y}. \tag{2.101}$$

Now putting the value of y'' in (2.101), we deduce

$$X_6^{[3]}(y^2y''' - 3y'^3) = xy''' - \frac{3xy'^2}{y^2} = \frac{x}{y^2}(y^2y''' - 3y'^3) = 0. \tag{2.102}$$

Hence X_6 is a proper conditional symmetry of a third-order ODE (2.97) after substitution.

Now for $X_7 = x^2\partial_x + \frac{1}{2}xy\partial_y$, we have found that $\lambda = -6x$. Thus X_7 is a conditional symmetry for the third-order equation before and hence it remains the same for the third-order equation (2.97) after substitution of y''.

Finally we have $X_8 = xy^2\partial_x + \frac{1}{2}y^3\partial_y$ with $\lambda = -6xyy'$. Here X_8 is also a conditional symmetry of the third-order equation before and it remains the same for the third-order equation (2.97) by substituting the value of y''. Here we see that only X_1, X_3 and X_4 are inherited symmetries.

2. Another example given by

$$y''' - \frac{6}{x^2}y' - 7y'^3 - 3x^2y'^5 = 0, \tag{2.103}$$

is of the class given in [40] and has conditional symmetries (2.33) subject to the linearizable root equation (2.32). We have the same symmetries X_7 and X_8 of (2.33) as inherited symmetries.

We have seen that the inherited symmetries are different for linear and nonlinear ODEs. That is, a derived linear second-order equation has more inherited symmetries than a conditionally linearizable equation as in (2.103). Moreover, we see from the above two examples that conditional symmetries of the derived ODE before and after substitution need not be the same. In Example 2. above the inherited symmetries are the same as that of Example 3. given in Section 2. The same does not apply for Example 1. above which has X_1, X_3 and X_4 as inherited symmetries and in Example 4. of Section 2, we have that X_1 to X_4 and X_6 are inherited symmetries.

2.7 Physical applications

It is well-known that conditional symmetries of partial differential equations are of paramount importance as they unearth physical relevant solutions. Recently this has been applied to non-Newtonian fluid flow problems [69, 70].

Here we demonstrate the utility of conditional symmetries for ODEs that arise in the non-Newtonian fluid flow of a third grade fluid as well as in classical mechanics.

1. The third-order nonlinear ODE for the steady-state solution $u = f(y)$ of the unsteady flow of a third grade fluid through a porous plate without a modified pressure gradient is (see Aziz and Mahomed [69])

$$\mu_* f'' - \alpha_* W f''' + 3\gamma f'^2 f'' - \gamma_* f f'^2 + W f' - f/K_* = 0, \, ' = d/dy, \tag{2.104}$$

where the relevant constants in (2.104) are the parameters of the fluid problem. The boundary conditions are

$$f(0) = l_2, \, l_2 = \text{const.}, \, f(\infty) = 0, \, f'(\infty) = 0. \tag{2.105}$$

We show here that the physical solution of (2.104) subject to (2.105) arises from a conditional symmetry. Indeed, the conditional symmetry of the ODE (2.104) subject to the first-order linear ODE

$$f' + \frac{\sqrt{\gamma_*}}{\sqrt{3\gamma}} f = 0, \tag{2.106}$$

is

$$X = \partial_y - \frac{\sqrt{\gamma_*}}{\sqrt{3\gamma}} f \partial_f. \tag{2.107}$$

The solution of the corresponding characteristics system of (2.107) results in the solution of (2.104) as

$$u = f(y) = l_2 \exp(-\frac{\sqrt{\gamma_*}}{\sqrt{3\gamma}}y), \tag{2.108}$$

which is precisely of the form given in [69]. The substitution of this solution (2.108) into the ODE (2.104) gives the parameters constraint

$$\frac{\gamma_*}{3\gamma}\mu_* + \alpha W\frac{\sqrt{\gamma_*}}{\sqrt{3\gamma}}\frac{\gamma_*}{3\gamma} - W\frac{\sqrt{\gamma_*}}{\sqrt{3\gamma}} - \frac{1}{K_*} = 0, \tag{2.109}$$

as is the case [69]. In this example the one constant in the first-order ODE (2.106) which is a compatible side condition satisfies the boundary conditions with a further constraint on the parameters as given in (2.109).

2. We consider the modified Emden equation

$$y'' + (k + \frac{n-1}{n+3}x)^{-1}y' + y^n = 0,\ n \neq 0, 1, -1, -3, \tag{2.110}$$

where k is an arbitrary constant. This ODE (2.110) is contained in the general class

$$y'' + a(x)y' + y^n = 0, \tag{2.111}$$

where $a(x)$ is an arbitrary function, which has been studied by various authors, notably Leach [71], Berkovich [72] recently Mahomed and Momoniat [73]. Also extensions have been investigated [74]. Leach [71] investigated the first integrals of (2.111) via the Noether theorem. Berkovich [72] showed that one can reduce the equation (2.111) to an autonomous canonical form by means of the Kummer-Liouville transformation

$$y = v(x)z, \quad dt = u(x)dx. \tag{2.112}$$

In [72], the point symmetry properties and exact solutions of the particular form

$$y = \rho v(x), \tag{2.113}$$

for constant ρ which satisfies a polynomial equation were also determined. The recent work [73] focused on a special case of (2.111), viz.

$$y'' + \frac{3x}{k+x^2}y' + y^3 = 0,\ k = \text{const.} \tag{2.114}$$

The authors of [73] then found the new general solution of (2.114) utilizing a symmetry of a first integral of (2.114).

Note that our ODE (2.110) has particular solutions of the form (2.113) as is easily constructible from the results of Berkovich [72]. These are

$$y = (\frac{2}{n+3})^{\frac{2}{n-1}}(k + \frac{n-1}{n+3}x)^{\frac{-2}{n-1}}. \tag{2.115}$$

We now obtain the new general solution of (2.110) by using a conditional symmetry of (2.110) subject to the first-order ODE

$$\begin{aligned}&\tfrac{1}{2}(k + \tfrac{n-1}{n+3}x)^{\frac{2(n+1)}{n-1}}y'^2 + \tfrac{2}{n+3}(k + \tfrac{n-1}{n+3}x)^{\frac{n+3}{n-1}}yy'\\ &+\tfrac{1}{n+1}(k + \tfrac{n-1}{n+3}x)^{\frac{2(n+1)}{n-1}}y^{n+1} = C,\ C = \text{const.}\end{aligned} \tag{2.116}$$

This is easily deduced and we have

$$X = \frac{n+3}{2}(k + \frac{n-1}{n+3}x)\partial_x - y\partial_y. \tag{2.117}$$

We thus have an invariant of this operator (2.117)

$$u = y(k + \frac{n-1}{n+3}x)^{\frac{2}{n-1}}. \tag{2.118}$$

This invariant (2.118) after substitution in the ODE (2.116) results in the separable ODE

$$(k+\frac{n-1}{n+3}x)u'=\pm\sqrt{2C+\frac{4u^2}{(n+3)^2}-\frac{2}{n+1}u^{n+1}}, \tag{2.119}$$

which is integrable by quadratures. Thus the new general solution of (2.110) is given by (2.119). Note here that the first-order ODE (2.116) is a first integral which has the arbitrary constant C in it. The solution of this then gives the complete integrability of our modified Emden equation (2.110).

3. The fifth-order nonlinear ODE for the steady-state solution $u = h(y)$ of the unsteady flow of a fourth grade fluid over an infinite moved plate with suction/injection without a modified pressure gradient is (see Aziz and Mahomed [70])

$$\begin{aligned} W_0h'+h''-\alpha_1W_0h'''+\beta_1W_0^2h''''+\beta(h')^2h'' \\ -\gamma W_0^3h''''''-2\Gamma W_0\frac{d}{dy}[(h')^2h'']=0, \end{aligned} \tag{2.120}$$

where the relevant constants in (2.120) are the parameters of the fluid problem.

We show here that the physical solution of fifth-order ODE (2.120) arises from a conditional symmetry. Indeed, the conditional symmetry of the ODE (2.120) subject to the first-order linear ODE

$$\frac{dh}{dy}-\frac{\beta}{3\Gamma W_0}h=0, \tag{2.121}$$

is

$$X=\frac{\partial}{\partial y}+\frac{\beta}{3\Gamma W_0}h\frac{\partial}{\partial h}. \tag{2.122}$$

By solving the corresponding characteristics system of (2.122), the solution of the ODE (2.120) is

$$u = h(y) = \exp(\frac{\beta y}{3\Gamma W_0}), \tag{2.123}$$

which is precisely of the form mentioned in [70]. The substitution of this solution (2.123) into the ODE (2.120) gives the constraint on the physical parameters

$$-\frac{\beta}{3\Gamma} = \frac{\beta^2}{(3\Gamma)^2(W_0)^2} - \frac{\alpha_1\beta^3}{(3\Gamma)^3(W_0)^2} + \frac{\beta_1\beta^4}{(3\Gamma)^4(W_0)^2} - \frac{\gamma\beta^5}{(3\Gamma)^5(W_0)^2}. \tag{2.124}$$

as given in [70]. Similar to Example 1. above, we have a first-order side condition which yields one arbitrary constant and the solution of this satisfies the boundary conditions with the above constraint on the parameters.

2.8 Concluding remarks

Conditional symmetries of ODEs are of paramount importance when the ODEs under investigation have insufficient number of Lie point symmetries. These are useful for integration as we have seen.

In this chapter, we have shown by means of an algorithm as to how one calculates the conditional symmetries of ODEs subject to lower order ODEs. Many simple examples as well as the properties of scalar linear and linear systems of ODEs were considered for their conditional symmetries. We explained how the symmetries of the root ODE are inherited by the derived ODE by means of a four propositions. Moreover we have proved that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs

related to the invariant curve conditions which arises from the known solution curves.

We have also investigated third-order non-linear ODEs which admit eight conditional symmetries subject to a Lie linearizable second-order ODEs.

Finally we presented applications of conditional symmetries to ODEs that arise in mechanics and have obtained new solutions.

Chapter 3

Conditional Symmetries of Nonlinear Third-order ODEs subject to Second-order Equations with Three-Dimensional Algebra

3.1 Introduction

The present chapter continues the research which was performed in Chapter 2. Here we take as our base, scalar second-order ordinary differential equations

(ODEs) which have seven equivalence classes with each class possessing three Lie point symmetries. We show when scalar second-order ODEs taken as first integrals are preserved as Lie point symmetries and conditional symmetries of the derived third-order ODE. Furthermore, we show how one can calculate the conditional symmetries of third-order non-linear ODEs subject to second-order nonlinear ODEs which admit point symmetries. Thus scalar nonlinear third-order ODEs are considered for their conditional symmetries subject to second-order root ODEs having symmetry. Finally, we consider the conditional $sl(2, \mathbb{R})$ criteria for the scalar third-order ODE, obtained after the differentiation of the scalar second-order ODE which admits the $sl(2, \mathbb{R})$ symmetry algebra. We work out the criteria of the derived third-order ODE in terms of the coefficients involved in the second-order nonlinear ODE which has $sl(2, \mathbb{R})$. These are carried out by use of the Tressé invariants which give the $sl(2, \mathbb{R})$ symmetry criteria for second-order ODEs admitting such a symmetry algebra.

We commence by giving a precise definition of conditional symmetries (see Chapter 2 and [62]) adapted to third-order ODEs.

Definition 3. A nonlinear third-order ODE is conditionally classifiable by a symmetry algebra $\mathcal{A}$ with respect to a nonlinear second-order ODE called the root ODE if and only if the third-order ODE jointly with the second-order ODE forms an overdetermined compatible system (so the solutions of the third-order ODE reduces to the solutions of the second-order ODE) and the second-order ODE has symmetry algebra $\mathcal{A}$ which is the conditional symmetry algebra of the nonlinear third-order ODE.

We now present the algorithm for computing the conditional symmetries of a scalar third-order ODE of the form (these can be derived from Chapter 2 in which we stated it in general)

$$y''' = f(x, y, y', y''), \tag{3.1}$$

where x is the independent variable, y the dependent variable and y', y'' denote the derivatives of y with respect to x.

Let X be the vector field of dependent and independent variables given by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{3.2}$$

where ξ and η are the coefficient functions of the vector field X.

Suppose that the vector field X is a conditional symmetry generator of equation (3.1) subject to a nonlinear second-order ODE

$$y'' = g(x, y, y'). \tag{3.3}$$

Then the conditional symmetry condition

$$X^{[3]}[y''' - f(x, y, y', y'')]|_{y'''-f=0, y''-g=0} = 0, \tag{3.4}$$

holds. Here $X^{[3]}$ denotes the 3rd prolongation of the generator X given by

$$X^{[3]} = X + \sum_{j=1}^{3} \zeta_j \frac{\partial}{\partial y^{(j)}}, \tag{3.5}$$

in which

$$\zeta_j = D_x(\zeta_{j-1}) - y^{(j)} D_x(\xi),\ j = 1, 2, 3,\ \zeta_0 = \eta \tag{3.6}$$

and D_x is the usual total differentiation operator with respect to x.

3.2 Examples of conditional symmetries for third-order ODEs

In this section, we present two examples of third-order ODEs. One is conditionally linearizable with respect to a Lie linearizable second-order ODE. The other has conditional $sl(2,\mathbb{R})$ symmetry subject to an Ermakov equation which admits the $sl(2,\mathbb{R})$ algebra.

1. First we consider the example of differentiating the linearizable equation [41]

$$y'' + xy'^3 + \frac{2}{x}y' = 0, \tag{3.7}$$

which results in

$$y''' - 3x^2y'^5 - 7y'^3 - \frac{6}{x^2}y' = 0. \tag{3.8}$$

We take $c = 0$, $d = 0$, $g = f - 2b = 0$ and $h = a - 2e = \frac{2}{x}$ in the class considered in [41] and select $b = 0$ and $e = -\frac{1}{x}$. Then $f = 0$ and $a = 0$. It is easily verified that the conditions are met and that it is not in any of the classes of [15, 16, 75] and cannot be linearized according to the methods discussed in these. Equation (3.7) reduces to the simplest linear form $\frac{d^2v}{d^2u} = 0$ by use of the transformation $u = x\cos y$ and $v = x\sin y$. The solution is given in terms of two arbitrary constants. Hence the third-order ODE (3.8) is conditionally linearizable by a point transformation in u and v with respect to the second-order ODE (3.7).

2. Consider the second-order Ermakov equation [76]

$$y'' + \alpha y^{-3} = 0. \tag{3.9}$$

The above equation has the following symmetry generators (which form the $sl(2,\mathbb{R})$ algebra)

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= x\partial_x + \frac{1}{2}y\partial_y, \\ X_3 &= x^2\partial_x + xy\partial_y. \end{aligned} \tag{3.10}$$

Now differentiating (3.9) we obtain

$$y^4 y''' - 3\alpha y' = 0. \tag{3.11}$$

It is obvious that X_1 is the inherited symmetry of (3.11). Now consider the generator X_2 with its second prolongation

$$X_2^{[2]} = x\partial_x + y\partial_y - \frac{1}{2}y'\partial_{y'} - \frac{3}{2}y''\partial_{y''}. \tag{3.12}$$

Operating (3.12) on (3.9), we deduce

$$X_2^{[2]}[y'' + \alpha y^{-3}] = -\frac{3}{2}(y'' + \alpha y^{-3}), \tag{3.13}$$

with $\lambda = -(3/2)$ (see Proposition 1 of Chapter 2 and also [62]). Hence X_2 is an inherited symmetry generator of (3.11) by Proposition 1. Now consider the generator X_3 with its prolongation

$$X_3^{[2]} = x^2\partial_x + xy\partial_y + (y - xy')\partial_{y'} - 3xy''\partial_{y''}. \tag{3.14}$$

By making use of (3.14) on ODE (3.9), we easily obtain

$$X_3^{[2]}[y'' + \alpha y^{-3}] = -3x[y'' + \alpha y^{-3}], \tag{3.15}$$

and this results in $\lambda = -3x$. This shows that by using $X_3^{[2]}$ on (3.9) we are able to determine X_3 as the symmetry generator of (3.11); hence X_3 is a proper conditional symmetry generator of (3.11) with $\lambda = -3x$ by Proposition 1.

Now by eliminating α from (3.9) we arrive at

$$yy''' + 3y'y'' = 0. \tag{3.16}$$

Operating (3.12) on (3.16), we find

$$X_2^{[3]}[yy''' + 3y'y'']_{(3.16)} = 0 \Rightarrow -2[yy''' + 3y'y'']_{(3.16)} = 0, \tag{3.17}$$

this is an inherited symmetry generator of (3.11). Note that Proposition 1 does not apply as we have substituted α. Making use of (3.14) on ODE (3.16), we obtain

$$X_3^{[3]}[yy''' + 3y'y'']_{(3.16)} = 0. \tag{3.18}$$

which gives

$$[-3x(yy''' + 3y'y'')]_{(3.16)} = 0, \tag{3.19}$$

and thus X_3 is a proper symmetry generator of ODE (3.11).

3.3 Symmetries of first integrals of third-order ODEs

In this section, we consider scalar second-order ODEs in their canonical forms, taken as first integrals of the respective third-order ODEs, which admit three symmetry generators. We see that all the symmetries of the second-order ODEs considered as first integrals are inherited as point symmetries of the third-order ODEs.

Consider the following table of second-order ODEs in canonical form that possess Lie point symmetries [77].

Table 1

Lie group classification of scalar second-order equations in the real plane

$p=\partial/\partial x$ and $q=\partial/\partial y$		
Algebra	Canonical forms of generators	Representative equations
L_1	$X_1=p$	$y''=g(y,y')$
$L^I_{2;1}$	$X_1=p, X_2=q$	$y''=g(y')$
$L^{II}_{2;1}$	$X_1=q, X_2=xp+yq$	$xy''=g(y')$
$L^I_{3;3}$	$X_1=p, X_2=q, X_3=xp+(x+y)q$	$y''=Ae^{-y'}$
$L^I_{3;6}$	$X_1=p, X_2=q, X_3=xp+ayq$	$y''=Ay'(a-2)/(a-1)$
$L^I_{3;7}$	$X_1=p, X_2=q, X_3=(bx+y)p+(by-x)q$	$y''=A(1+y'^2)^{\frac{3}{2}}e^{b\arctan y'}$
$L^I_{3;8}$	$X_1=q, X_2=xp+yq, X_3=2xyp+y^2q$	$xy''=Ay'^3-\frac{1}{2}y'$
$L^{II}_{3;8}$	$X_1=q, X_2=xp+yq, X_3=2xyp+(y^2+x^2)q$	$xy''=y'+y'^3+A(1+y'^2)^{\frac{3}{2}}$
$L^{III}_{3;8}$	$X_1=q, X_2=xp+yq, X_3=2xyp+(y^2-x^2)q$	$xy''=y'-y'^3+A(1-y'^2)^{\frac{3}{2}}$
$L_{3;9}$	$X_1=(1+x^2)p+xyq, X_2=xyp+(1+y^2)q,$ $X_3=yp-xq$	$y''=A[\frac{1+y'+(y-xy')^2}{1+x^2+y^2}]^{3/2}$

We briefly review the following definitions and results of [78] which shows that the symmetry of a first integral is the symmetry of the equation itself.

Definition 4 (see [78]).
The differential $(n-1)$-form

$$w=T^i(x,u,u_1,...,u_{r-1})\frac{\partial}{\partial x^i}\rfloor(dx^1\wedge...\wedge dx^n) \tag{3.20}$$

is called a conserved form of

$$E^{\beta}(x, u, u_{(1)}, ..., u_{(r)}) = 0, \beta = 1, ..., p \leq m, \tag{3.21}$$

if

$$Dw = 0, \tag{3.22}$$

is satisfied on the manifold in the space variables $x, u, u_{(1)}, ..., u_{(r)}$ defined by the system, where D is the total exterior derivative.

Theorem 1 (see [78]).
Suppose that

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + ..., \tag{3.23}$$

where $\xi^i (i = 1, 2, ..., n)$ and $\eta^{\alpha} (\alpha = 1, 2, ..., m)$ are differential functions and the additional coefficients are $\zeta_i^{\alpha} = D_i(\eta^{\alpha}) - u_j^{\alpha} D(\xi^j)$, $\zeta_{i_1 ... i_s}^{\alpha} = D_{i_s}(\zeta_{i_1 ... i_{s-1}}^{\alpha}) - u_{j i_1 ... i_{s-1}}^{\alpha} D_{i_s}(\xi^j)$, $s > 1$.

The operator X is a Lie-Bäcklund symmetry generator of the system (3.21) such that the conserved form w of (3.21), given by (3.20) is invariant under X. Then

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i), i = 1, ..., n, \tag{3.24}$$

where $D_i = \xi^i \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + ...$ is the total derivative with respect to x^i

Note.
For any differential function f, $Df = D_i f dx^i$ and for any k-form $w = f_{i_1 i_2 ... i_k} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$, $Dw = Df_{i_1 i_2 ... i_k} dx^j \wedge dx^{j_1} \wedge dx^{j_2} \wedge ... \wedge dx^{j_k}$.

The above Theorem 1, in the case of ODEs, shows that if we consider a second-order ODE considered as a first integral $A = I(x, y, y', y'')$, with three symmetries, of a third-order ODE and take $D_x A = 0$ on $A = I(x, y, y', y'')$, we obtain the same symmetries of the third-order ODE.

We focus on Table I.

Consider the representative ODE for $L^I_{3;3}$ from Table 1. The canonical equation is

$$y'' = Ae^{-y'}, \tag{3.25}$$

and has the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + (x + y)\partial_y. \tag{3.26}$$

The derived ODE ((3.25) is taken as a first integral) corresponding to (3.25) is [79]

$$y''' + y''^2 = 0. \tag{3.27}$$

It is obvious that X_1 and X_2 are the symmetries of (3.27). Now consider the generator X_3 with its third prolongation

$$X_3^{[3]} = x\partial_x + (x + y)\partial_y, +\partial_{y'} - y''\partial_{y''} - 2y'''\partial_{y'''}. \tag{3.28}$$

Operating (3.28) on (3.27), results in

$$X_3^{[3]}[y''' + y''^2]_{(3.27)} = 0 \Rightarrow -2[y''' + y'^2]_{(3.27)} = 0. \tag{3.29}$$

Thus X_3 is a symmetry generator of (3.27). Hence the third-order ODE corresponding to (3.25) has the same symmetries as the second-order ODE.

Now for $L^I_{3;6}$ from Table I, the equation is

$$y'' = Ay'^{(a-2)/(a-1)}, a = 0, \frac{1}{2}, 2, \tag{3.30}$$

which has the following symmetry generators

$$X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + ay\partial_y. \tag{3.31}$$

This ODE (3.30) corresponds as a first integral to the ODE [79]

$$y'''y' - (\frac{a-2}{a-1})y''^2 = 0. \tag{3.32}$$

We know that X_1 and X_2 are the inherited symmetries of ODE (3.32). We consider the symmetry generator X_3 with its third prolongation

$$X_3^{[3]} = x\partial_x + ay\partial_y + (a-1)y'\partial_{y'} + (a-2)y''\partial_{y''} + (a-3)y'''\partial_{y'''}. \tag{3.33}$$

Operating (3.33) on (3.31), we find

$$X_3^{[3]}[y'''y' - (\frac{a-2}{a-1})y''^2]_{(3.32)} = 0. \tag{3.34}$$

Thus X_3 is also an inherited symmetry generator of (3.32). Hence the third-order ODE corresponding to (3.30) has the same symmetries as the second-order ODE which is its first integral.

For $L^I_{3;7}$, the equation is given as

$$y'' = A(1 + y'^2)^{\frac{3}{2}} e^{b \arctan y'}, \tag{3.35}$$

and has the following operators

$$X_1 = \partial_x, X_2 = \partial_2, X_3 = (bx + y)\partial_x + (by - x)\partial_y. \tag{3.36}$$

This equation as first integral (3.35) corresponds to [79]

$$y''' - \frac{(3y' + b)y''^2}{(1 + y'^2)} = 0. \tag{3.37}$$

Here X_1 and X_2 are obviously the symmetry generators of ODE (3.37). Consider X_3 and its third prolongation

$$X_3^{[3]} = (bx+y)\partial_x+(by-x)\partial_y-(1+y^{'2})\partial_{y'}-(b+3y')y''\partial_{y''}-[3y^{''2}+2(b+2y')y''']\partial_{y'''}, \tag{3.38}$$

such that

$$X_3^{[3}[y''' - \frac{(3y' + b)y''^2}{(1 + y'^2)}]_{(3.37)} = 0. \tag{3.39}$$

This implies that X_3 is a proper symmetry generator of (3.37) with $X_3^{[3]}E = \lambda E = 2(b + 3y')E$, where E is the ODE (3.37). Here note that Proposition 1 does not apply as we have substituted the value of A after differentiation.

For $L_{3;8}^I$, the equation is

$$xy'' = Ay^{'3} - \frac{1}{2}y'. \tag{3.40}$$

The above equation has the following symmetries

$$X_1 = \partial_y, X_2 = x\partial_x + y\partial_y, X_3 = 2xy\partial_x + y^2\partial_y. \tag{3.41}$$

The ODE (3.40) is a first integral of

$$y'y''' - 3y''^2 = 0. \tag{3.42}$$

It is obvious that X_1 is the symmetry generator of (3.42). Now consider the generator X_2 with its prolongation

$$X_2^{[3]} = x\partial_x + y\partial_y - y''\partial_{y''} - 2y'''\partial_{y'''}. \tag{3.43}$$

Making use of (3.43) on ODE (3.42), we obtain

$$X_2^{[3]}[y'y''' - 3y''^2]_{(3.42)} = 0. \tag{3.44}$$

This easily shows that X_2 is also an inherited symmetry generator of (3.42). Finally we consider the generator X_3 and its prolongation

$$\begin{aligned} X_3^{[3]} &= 2xy\partial_x + y^2\partial_y - 2xy'^2\partial_{y'} \\ &\quad -(2y'^2 + 2(y + 3xy')y'')\partial_{y''} \\ &+[(-6y''(2y' + xy'') - 4(y + 2xy')y''')]\partial_{y'''}, \end{aligned} \tag{3.45}$$

which on applying to (3.42) gives

$$X_3^{[3]}[y'y''' - 3y''^2]_{(3.30)} = [(-10xy' - 4y)(y'y''' - 3y''^2)]_{(3.42)} = 0, \tag{3.46}$$

with $\lambda = -(10xy' + 4y)$ in $X_3^{[3]}E = \lambda E$ which is zero on the equation $E = 0$, viz. equation (3.42). Hence we deduce that X_3 is a symmetry generator of (3.42) and here too Proposition 1 does not apply.

For $L_{3;8}^{II}$ the ODE is

$$xy'' = y' + y'^3 + A(1 + y'^2)^{\frac{3}{2}}. \tag{3.47}$$

The above equation has the following symmetries

$$X_1 = \partial_y, X_2 = x\partial_x + y\partial_y, X_3 = 2xy\partial_x + (y^2 - x^2)\partial_y. \tag{3.48}$$

Equation (3.47) is a first integral of

$$y''' + y'''y'^2 - 3y'y''^2 = 0. \tag{3.49}$$

Obviously the operator X_1 is the symmetry generator of (3.49). Now we consider the operator X_2 which has the prolongation given in (3.43). Making use of (3.43)

and operating it on (3.49) results in

$$X_2^{[3]}[y''' + y'''y'^2 - 3y'y''^2]_{(3.49)} = 0. \tag{3.50}$$

Here we see that X_2 is also an inherited symmetry generator of (3.49). Consider now the generator X_3 and its prolongation

$$\begin{aligned} X_3^{[3]} = 2xy\partial_x + (y^2 - x^2)\partial_y - 2x(1 + y'^2)\partial_{y'} \\ -2(1 + y'^2 + (y + 3xy')y'')\partial_{y''} \\ -[(-6y''(2y' + xy'') - 4(y + 2xy')y''')]\partial_{y'''}. \end{aligned} \tag{3.51}$$

Using the operator $X_3^{[3]}$ on (3.49) yields

$$X_3^{[3]}[y''' + y'''y'^2 - 3y'y''^2]_{(3.37)} = 0, \tag{3.52}$$

where $\lambda = -4y - 12xy'$ in $X_3^{[3]}E = \lambda E$. This clearly shows that X_3 is a proper symmetry generator of (3.49).

Finally for the third $L_{3;8}^{III}$ realization, the equation is

$$xy'' = y' - y'^3 + A(1 - y')^{\frac{3}{2}}, \tag{3.53}$$

which has the following generators

$$X_1 = \partial_y, X_2 = x\partial_x + y\partial_y, X_3 = 2xy\partial_x + (y^2 + x^2)\partial_y. \tag{3.54}$$

Equation (3.53) is a first integral of the ODE given by

$$y''' - y'''y'^2 - 3y'y''^2 = 0. \tag{3.55}$$

Evidently X_1 is the symmetry generator of (3.55). Now X_2 has its prolongation given in equation (3.43). Invoking (3.43) on (3.55) we deduce

$$X_2^{[3]}[y''' + y'''y'^2 - 3y'y''^2]_{(3.49)} = 0. \tag{3.56}$$

Here we have $\lambda = -2$ in $X_2^{[3]}E = \lambda E$; this shows that X_2 is also an inherited symmetry generator of (3.56). Finally, we consider X_3 with

$$\begin{aligned} X_3^{[3]} = 2xy\partial_x + (y^2 + x^2)\partial_y - 2x(y'^2 - 1)\partial_{y'} \\ -2(-1 + y'^2 + (y + 3xy')y'')\partial_{y''} \\ -[(6y''(2y' + xy'') - 4(y + 2xy')y''')]\partial_{y'''}. \end{aligned} \tag{3.57}$$

Applying (3.57) on (3.55), we find

$$X_3^{[3]}[y''' - y'''y'^2 - 3y'y''^2]_{(3.55)} = 0. \tag{3.58}$$

Here $\lambda = -4y - 12xy'$ in $X_3^{[3]}E = \lambda E$ and thus X_3 is a proper symmetry symmetry generator of ODE (3.55).

In the last case when the ODE admits the $L_{3;9}$ algebra (the rotation symmetries constitute this algebra), the equation is

$$y'' = A[\frac{1 + y' + (y - xy')^2}{1 + x^2 + y^2}]^{3/2}. \tag{3.59}$$

If we write this as a first integral, we have

$$I = y''[\frac{1 + y' + (y - xy')^2}{1 + x^2 + y^2}]^{-3/2}. \tag{3.60}$$

After straightforward but tedious calculations, we find that all the symmetries X_1, X_2 and X_3 are symmetries of I. The derived equation after substitution of A is

$$y''' = \frac{3}{2}y''[\frac{1 + y' + (y - xy')^2}{1 + x^2 + y^2}]^{-1}D_x[\frac{1 + y' + (y - xy')^2}{1 + x^2 + y^2}], \tag{3.61}$$

by Theorem 1. Therefore our derived ODE has the three rotation symmetries.

Hence we can state a proposition as follows.

Proposition 8: The scalar second-order ordinary differential equations $y'' = f(x, y, y'')$ in the real plane with three symmetries as in Table I taken as first integrals have their point symmetries inherited for the representative equations.

We have seen that in all cases X_1, X_2 and X_3 remain as inherited symmetries of the third-order ODEs with corresponding first integrals as nonlinear second-order root ODEs having the same symmetries.

3.4 Conditionals symmetries subject to nonlinear second-order ODEs

Conditional linearizability of scalar third-order and fourth-order ODEs subject to Lie linearizable second-order ODEs were completely and invariantly characterized in terms of the coefficients of the root equation in the papers [41, 42] wherein the derived third-order ODE before and after substitution of the second derivative were investigated. Here we briefly focus on the conditional symmetries of nonlinear third-order ODEs subject to second-order nonlinear root ODEs.

Again by considering $L^I_{3;3}$, on differentiating (3.25) with respect to x with $A \neq 0$, yields

$$y''' + Ay''e^{-y'} = 0. \tag{3.62}$$

The conditional symmetries are given in (3.26). Here clearly X_1 and X_2 are inherited symmetries of (3.62). Finally we consider the generator X_3 with its second prolongation as can be deduced from (3.28) and apply (3.28) on (3.25).

This gives rise to

$$X_3^{[2]}[y'' - Ae^{-y'}] = -(y'' - Ae^{-y'}). \tag{3.63}$$

Here we have $\lambda = -1$ and hence X_3 is also an inherited symmetry by Proposition 1 (see also [62]).

For $L_{3;6}^I$ we again consider (3.30). The third-order ODE is

$$y''' - (\frac{a-2}{a-1})Ay''y'^{\frac{1}{1-a}} = 0, \tag{3.64}$$

with conditional symmetry generators given in (3.31). Here again X_1 and X_2 are inherited symmetries of the nonlinear ODE (3.64). Finally consider the symmetry X_3 with prolongation given in (3.33). Invoking (3.33) on (3.30) we get

$$X_3^{[2]}[y'' - Ay'^{\frac{a-2}{a-1}}] = (a-2)[y'' - Ay'^{\frac{a-2}{a-1}}] = 0. \tag{3.65}$$

We have that $\lambda = a - 2$ and thus X_3 is also an inherited symmetry of (3.64) by Proposition 1, [62].

For $L_{3;7}^I$, the ODE (3.35) upon differentiation with $A \neq 0$ yields

$$y''' - Ay''(3y' + b)(1 + y'^2)^{\frac{1}{2}} e^{b \arctan y'} = 0. \tag{3.66}$$

The conditional symmetry generators are given in (3.36). Obviously X_1 and X_2 are the inherited symmetry generators of (3.66). Finally we consider X_3 and its prolongation given in (3.38) and applying it on (3.35) we have

$$X_3^{[2]}[y'' - A(1 + y'^2)^{\frac{1}{2}} e^{b \arctan y'}] = -(b + 3y')[y'' - A(1 + y'^2)^{\frac{1}{2}} e^{b \arctan y'}], \tag{3.67}$$

with $\lambda = -(3y' + b)$, so X_3 is a proper conditional symmetry of (3.66).

For $L^{I}_{3;8}$, we consider (3.40) and differentiating it with respect to x with $A \neq 0$ gives

$$xy''' + \frac{3}{2}y'' - 3Ay''y'^{2} = 0, \tag{3.68}$$

with conditional symmetry generators presented in (3.41). Now operating the second prolongation of X_2 which is given in (3.43) on (3.40) results in

$$X_2^{[3]}[xy'' - Ay'^{3} + \frac{1}{2}y'] = -1[xy'' - Ay'^{3} + \frac{1}{2}y'], \tag{3.69}$$

which shows that X_2 is an inherited symmetry generator of ODE (3.68) with $\lambda = -1$. Finally, we consider X_3 and with its prolongation given in (3.45) and applying it on (3.40) results in

$$X_3^{[2]}[xy'' - Ay'^{3} + \frac{1}{2}y'] = -2(y + 3xy')[xy'' - Ay'^{3} + \frac{1}{2}y'], \tag{3.70}$$

with $\lambda = -2(y + 3xy')$ which shows that X_3 is a proper conditional symmetry generator of (3.40).

For $L^{II}_{3;8}$, the third-order ODE of (3.47) with $A \neq 0$ is given by

$$xy''' + 3y''y'^{2} - 3Ay'y''(1 + y'^{2})^{\frac{1}{2}} = 0. \tag{3.71}$$

Again we consider X_2 and its prolongation given in (3.43) and operating it on (3.47) yields

$$X_2^{[2]}[xy'' - y' - y'^{3} - A'(1 + y'^{2})^{\frac{1}{2}}] = -[xy'' - y' - y'^{3} - A'(1 + y'^{2})^{\frac{1}{2}}], \tag{3.72}$$

with $\lambda = -1$ which shows that X_2 is an inherited symmetry generator of (3.71). Finally, we consider X_3 and its prolongation given in (3.51) and now apply this on (3.47) to find

$$X_3^{[2]}[xy''-y'-y'^{3}-A'(1+y'^{2})^{\frac{1}{2}}] = -2(y+3xy')[xy''-y'-y'^{3}-A'(1+y'^{2})^{\frac{1}{2}}], \tag{3.73}$$

with $\lambda = -2(y + 3xy')$. This shows that X_3 is a proper conditional symmetry generator of (3.71).

For $L_{3;8}^{III}$, the third-order ODE corresponding to (3.53) with $A \neq 0$ is

$$xy''' + 3y''y'^2 + 3Ay'y''(1 - y'^2)^{\frac{1}{2}} = 0. \tag{3.74}$$

For the operator X_2 and its prolongation given in (3.56) we have

$$X_2^{[2]}[xy'' - y' + y'^3 - A(1 - y'^2)^{\frac{1}{2}}] = -[xy'' - y' + y'^3 - A(1 - y'^2)^{\frac{1}{2}}], \tag{3.75}$$

with $\lambda = -1$, thereby showing that X_2 is an inherited symmetry generator of (3.74). Now for X_3 and its prolongation given in (3.57) we have

$$X_3^{[2]}[xy''-y'+y'^3-A'(1-y'^2)^{\frac{1}{2}}] = -2(y+3xy')[xy''-y'+y'^3-A'(1-y'^2)^{\frac{1}{2}}], \tag{3.76}$$

with $\lambda = -2(y+3xy')$. Therefore X_3 is a proper conditional symmetry of (3.74) when we totally differentiate (3.53) by setting $A \neq 0$.

For $L_{3;9}$, the derived third-order ODE inherits all the symmetries as conditional symmetries. We see this as follows. The derived ODE of (3.59) for the $L_{3;9}$ realization is

$$y''' = \frac{3}{2}A[\frac{1 + y' + (y - xy')^2}{1 + x^2 + y^2}]^{1/2} D_x[\frac{1 + y' + (y - xy')^2}{1 + x^2 + y^2}]. \tag{3.77}$$

This representative second-order ODE (3.59) has the three symmetries as given in Table I which are

$$X_1 = (1 + x^2)\partial_x + xy\partial_y,\ X_2 = xy\partial_x + (1 + y^2)\partial_y,\ X_3 = y\partial_x - x\partial_y. \tag{3.78}$$

For each symmetry in (3.78) we check if it is inherited by the derived ODE (3.77). In the case X_1 we have that its second prolongation is

$$X_1^{[2]} = X_1 + (y - xy')\partial_{y'} - 3xy''\partial_{y''}.$$

Acting with this on (3.59) we find that $\lambda = -3x$ and thus X_1 is a proper conditional symmetry of (3.77). Likewise for X_2 we have its second prolongation given by

$$X_2^{[2]} = X_2 + (yy' - xy'^2)\partial_{y'} - 3xy'y''\partial_{y''}$$

and this results in $\lambda = -3xy'$ implying that X_2 is a proper conditional symmetry of our derived third-order ODE (3.77). Lastly the second prolongation of X_3 is

$$X_3^{[2]} = X_3 - (1 + y'^2)\partial_{y'} - 3y'y''\partial_{y''}.$$

Here we see that $\lambda = -3y'$. Thus X_3 is a proper conditional symmetry of (3.77). In this final case we see that all symmetries are proper conditional symmetries of (3.77).

Here we have seen that in all cases except $L_{3;9}$, X_1 and X_2 remained as inherited symmetries and the derived nonlinear ODEs have less inherited symmetries than the root ODEs. Whereas X_3 is the inherited symmetry for $L^I_{3;3}$ and $L^I_{3;6}$, it is a proper conditional symmetry for all the remaining cases except $L_{3;9}$. For $L_{3;9}$ we have that all symmetries are proper conditional symmetries.

A similar proposition to that of Proposition 8 applies here too.

Proposition 9: The scalar second-order ODEs in the real plane in Table I with three symmetries which do not have $L_{3;9}$ symmetry taken as representative equations when derived to third-order ODEs have their symmetries inherited for $L^I_{3;3}$ and $L^I_{3;6}$ only. For the other realizations except $L_{3;9}$, the third symmetry X_3 is a proper conditional symmetry of the third-order equation and hence the Lie point symmetries reduce in number from three to two in these cases. For the

representative equation of $L_{3;9}$, all symmetries become conditional symmetries of the derived third-order ODE.

3.5 Conditional invariant criteria of third-order equation with conditional $sl(2, \mathbb{R})$ symmetry

Here we consider the conditional $sl(2, \mathbb{R})$ criteria for a class of scalar third-order ODEs, obtained after the differentiation of scalar second-order ODEs cubic in the first derivative which admit $sl(2, \mathbb{R})$ symmetry algebra. We work out the criteria for the third-order ODE in terms of the coefficients of the root second-order ODE with $sl(2, \mathbb{R})$ symmetry algebra in the same manner as investigated in [41] for linearizablity. The second-order ODE to be differentiated is cubic in y' with arbitrary coefficients which satisfy $I_1 = 0$ and $I_2 \neq 0$ for some terms and thus the third-order ODE have these coefficients appearing in some combinations.

The scalar second-order ODE has the normal form

$$y'' = f(x, y, y'), \tag{3.79}$$

with the right hand side of (3.79) having the cubic in the first derivative form as

$$y'' = A(x, y)y'^3 + B(x, y)y'^2 + C(x, y)y' + D(x, y). \tag{3.80}$$

The Tressé relative invariants

$$I_1 = f_{y'y'y'y'}, \tag{3.81}$$

and

$$I_2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f_{y'y'} - 4\frac{\mathrm{d}}{\mathrm{d}x} f_{y'y} - 3f_y f_{y'y'} + 6f_{yy} + f'_y(f_{y'y} - \frac{\mathrm{d}}{\mathrm{d}x} f_{y'y'}), \tag{3.82}$$

both vanish identically for (3.79) when we linearize by point transformation. Then we obtain the coefficients A to D satisfying the following two invariant conditions (the Lie-Tressé conditions, see [77])

$$3A_{xx} + A_x C - 3A_y D + 3AC_x + C_{yy} - 6ADy + BC_y - 2BB_x - 2Bxy = 0$$

$$6A_x D - 3B_y D + 3AD_x + B_{xx} - 2BD_y + 3D_{yy} + 2CC_y - CB_x = 0. \tag{3.83}$$

Now we obtain the conditional $sl(2, \mathbb{R})$ criteria for a nonlinear third-order ODE with respect to a second-order ODE which admits $sl(2, \mathbb{R})$ symmetry. We first do this with respect to the canonical equation

$$xy'' = \alpha y'^3 - \frac{1}{2},\ \alpha \neq 0, \tag{3.84}$$

which has $sl(2, \mathbb{R})$ symmetry algebra. The relative invariant I_2 does not vanish for (3.84) and is $I_2 = 6\alpha/x^3$. The invariant criteria for a cubic in first derivative second-order ODE of the form (3.80) to be transformable by point transformation to the canonical equation (3.84) is that $I_1 = 0$ and $I_2 = 6\alpha/x^3$. That is we have (this is after some calculations on using the form (3.80))

$$3A_{xx} + A_x C - 3A_y D + 3AC_x + C_{yy} - 6ADy + BC_y - 2BB_x - 2Bxy = 6\alpha/x^3$$

$$6A_x D - 3B_y D + 3AD_x + B_{xx} - 2BD_y + 3D_{yy} + 2CC_y - CB_x = 0. \tag{3.85}$$

Differentiating (3.80) we find the following equation

$$y''' = A_x y'^3 + B_x y'^2 + C_x y' + D_x + y'(A_y y'^3 + B_y y'^2 + Cy' + D_y) + y''(3Ay'^2 + 2By' + C). \tag{3.86}$$

By invoking (3.80) into (3.86) we have

$$y''' = (3A^2)y'^5 + (5AB + A_y)y'^4 + (4AC + 2B^2 + A_x + B_y)y'^3 +$$
$$(3AD + 3BC + B_x + C_y)y'^2 + (2BD + C^2 + C_x + D_y)y' + (D_x + DC). \quad (3.87)$$

From (3.87) it follows that the most general form of the third-order equation is

$$y''' = \alpha y'^5 + \beta y'^4 + \gamma y'^3 + \delta y'^2 + \epsilon y' + \phi. \quad (3.88)$$

Now the comparison of (3.87) and (3.88) results in the conditions

$$\alpha = 3A^2, \quad (3.89)$$

$$\beta = 5AB + A_y, \quad (3.90)$$

$$\gamma = (4AC + 2B^2 + A_x + B_y), \quad (3.91)$$

$$\delta = (3AD + 3BC + B_x + C_y), \quad (3.92)$$

$$\epsilon = C^2 + C_x + D_y), \quad (3.93)$$

$$\phi = (D_x + DC). \quad (3.94)$$

The inversion of the first four equations gives the definitions for the four coefficients in (3.80) and the next two gives consistency constraint conditions:

$$A = \sqrt{\alpha/3}, \quad (3.95)$$

$$B = (\beta - A_y)/5A(A \neq 0), \quad (3.96)$$

$$C = (\gamma - 2B^2 - A_x - B_y)/4A(A \neq 0), \quad (3.97)$$

$$D = (\delta - 3BC - B_x - C_y)/3A(A \neq 0), \quad (3.98)$$

$$\epsilon = 2BD + C^2 + C_x + D_y), \quad (3.99)$$

$$\phi = (D_x + DC). \tag{3.100}$$

Now we are ready to state the following proposition.

Proposition 10: Equation (3.87) has conditional $sl(2, \mathbb{R})$ symmetry algebra with respect to the second-order ODE (3.80) which has $sl(2, \mathbb{R})$ symmetry algebra, where the coefficients are given by (3.95)-(3.100), $A \neq 0$.

As an example we consider (3.40) or we can write it as

$$y'' = \frac{\alpha}{x} - \frac{1}{2x}y'. \tag{3.101}$$

We have that I_1=0 and $I_2 = (\frac{12\alpha}{x^3})y'$ with the third-order derivative in the form of powers of y' given as

$$y''' = (\frac{3\alpha^2}{x^2})y'^5 - (\frac{3\alpha}{x^2})y'^3 + (\frac{3}{4x^2})y'. \tag{3.102}$$

Upon comparing (3.102) with (3.87), we have the following values of the coefficients $A = \frac{\alpha}{x}$, $B = 0$, $C = -\frac{1}{2x}$ and $D = \frac{1}{4\alpha x}$. Therefore, the third-order ODE (3.102) has conditional $sl(2, \mathbb{R})$ symmetry subject to the root ODE (3.101).

Now we consider the from (3.86) which has the y'' term (that is we do not replace y'' by the right hand side of equation (3.80)). We have

$$y''' = A_y y'^4 + (A_x + B_y)y'^3 + (B_x + C_y)y'^2 + (C_x + D_y)y' + D_x + y''(3Ay'^2 + 2By' + C). \tag{3.103}$$

The most general form corresponding to (3.103)is given as

$$y''' = b_4 y'^4 + (A_x + b_3)y'^3 + b_2 y'^2 + b_1 y' + b_0 + y''(a_2 y'^2 + a_1 y' + a_0). \tag{3.104}$$

comparing (3.104) with (3.103) we have the following values of the coefficients

$$A = \frac{a_2}{3}, \tag{3.105}$$

$$B = \frac{a_1}{2}, \tag{3.106}$$

$$C = a_0, \tag{3.107}$$

$$b_4 = \frac{a_{2y}}{3}, \tag{3.108}$$

$$b_3 = \frac{a_{2x}}{3} + \frac{a_{1y}}{2}, \tag{3.109}$$

$$b_2 = \frac{a_{1x}}{2} + a_{0y}, \tag{3.110}$$

$$D = \int b_0 dx + k(y) = \int (b_1 - a_{0x}) dy + l(x). \tag{3.111}$$

As an example we consider (3.101) We have that I_1=0 and $I_2 = (\frac{12\alpha}{x^3})y'$. The third-order derivative is given as

$$y''' = (\frac{3\alpha}{x} - \frac{3}{2x})y''. \tag{3.112}$$

Upon comparing (3.104) with (3.103), we have the following values of the coefficients $A = \frac{\alpha}{x}$, $B = 0$, $C = -\frac{3}{2x}$ and $D = -\frac{3y}{2}$. Therefore, the third-order ODE (3.112) has conditional $sl(2, \mathbb{R})$ symmetry subject to the root ODE (3.101).

We mention the following proposition.

Proposition 11: Equation (3.86) has conditional $sl(2, \mathbb{R})$ symmetry algebra with respect to the second-order ODE (3.80) with $sl(2, \mathbb{R})$ symmetry algebra, where the coefficients are given by (3.105)-(3.111), $A \neq 0$.

3.6 Concluding remarks

In this chapter, we have shown by means of an algorithm as to how one can calculates the conditional symmetries of nonlinear third-order ODEs subject to nonlinear second-order ODEs (this is adapted from the Chapter 2). Few simple examples of scalar nonlinear third-order ODEs were considered for their conditional symmetries.

We have also investigated that all the symmetries of the second-order ODEs when taken as first integrals are preserved as inherited or more generally as conditional symmetries. Furthermore, we have shown that the derived nonlinear ODEs after substitution have in general less inherited symmetries as compared to that of the root nonlinear second-order ODEs.

Finally, we embarked on the conditional criteria for $sl(2, \mathbb{R})$ symmetry algebra of a class of scalar nonlinear third-order ODEs, obtained after the differentiation of the scalar second-order cubic in the first derivatives ODEs which have $sl(2, \mathbb{R})$ symmetry. We worked out the criteria on the coefficients that enable conditional $sl(2, \mathbb{R})$ symmetry algebra for such third-order ODEs subject to the root second-order nonlinear ODEs with $sl(2, \mathbb{R})$ symmetry algebra.

Chapter 4

Integrability of a Remarkable Static Euler-Bernoulli Beam Equation

4.1 Introduction

The contents of this chapter is about Lie and Noether symmetry analysis of the fourth-order static Euler-Bernoulli ordinary differential equation (ODE), where the elastic modulus, the area moment of inertia are constants and the applied load is a function of the normal displacement, gives rise to the negative fractional power law $y^{-5/3}$ which has the nonsolvable algebra $sl(2,\mathbb{R})$. We first determine the inherited symmetries of the three-parameter family of second-

order ordinary differential equations which arises from the relationship among the Noether integrals of the fourth-order beam equation. In addition we present a complete symmetry classification of the second-order family of equations. We obtain new two- and three-parameter families of exact solutions for this problem. This is studied via the symmetry classification of the three-parameter family of second-order ordinary differential equations. Hence the admittance of $sl(2,\mathbb{R})$ remarkably allows for a three-parameter family of exact solutions for the static beam equation with load a fractional power law $y^{-5/3}$. The symmetries of the reduced ODEs are the conditional symmetries of the fourth-order beam ODE.

4.2 Inherited and Lie symmetries

The first integrals of the ODE (1.4) by use of the Noether theorem with respect to the Lagrangian in (1.5) are ([49])

$$\begin{aligned} I_1 &= \frac{1}{2}(y'')^2 - \frac{3}{2}\delta y^{-2/3} - y'y''', \\ I_2 &= \frac{1}{2}x(y'')^2 - \frac{3}{2}x\delta y^{-2/3} - xy'y''' + \frac{3}{2}yy''' - \frac{1}{2}y'y'', \\ I_3 &= \frac{1}{2}x^2(y'')^2 - \frac{3}{2}x^2\delta y^{-2/3} - x^2y'y''' + 3xyy''' - xy'y'' - 3yy'' + 2y'^2, \end{aligned} \tag{4.1}$$

for the symmetries (1.6) X_1, X_2 and X_3 respectively. The integrals I_2 and I_3 can be written in terms of I_1 as (see [49])

$$\begin{aligned} I_2 &= xI_1 + \frac{3}{2}yy''' - \frac{1}{2}y'y'', \\ I_3 &= x^2I_1 + 3xyy''' - xy'y'' - 3yy'' + 2y'^2. \end{aligned} \tag{4.2}$$

If one sets $I_1 = c_1$, $I_2 = c_2$ and $I_3 = c_3$ and eliminate y''' one immediately arrives at (1.8). Now since equation (1.8) arises from the static fourth-order Euler-Bernoulli beam equation (1.4), via the first integrals (4.1), it is likely to inherit some symmetries of the parent equation (1.4). If we take a linear combination of the X_is of (1.6) in the form $Y = a_1X_1 + a_2X_2 + a_3X_3$, then we see that (1.8) admits Y if and only if the constants are constrained by the following relations

$$\begin{aligned} 2a_3c_2 + c_1a_2 &= 0, \\ a_1c_1 - a_3c_3 &= 0, \\ a_2c_3 + 2a_1c_2 &= 0. \end{aligned} \tag{4.3}$$

The following cases occur. (i) $c_1 \neq 0$.

In this case we have one symmetry generator

$$Y = X_3 + \frac{c_3}{c_1}X_1 - 2\frac{c_2}{c_1}X_2, \tag{4.4}$$

for equation (1.8). (ii) $c_1 = 0, c_2 \neq 0$.

Here we find

$$Y = X_2 - \frac{c_3}{2c_2}X_1, \tag{4.5}$$

as admitted by equation (1.8). (iii) $c_1 = c_2 = 0$.

We deduce that

$$Y = X_1. \tag{4.6}$$

is the admitted operator for equation (1.8). (iv) $c_1 = c_2 = c_3 = 0$.

In this last case all the three symmetries of equation (1.6) are inherited by equation (1.8). The admittance of a single symmetry gives in general one reduction of the ODE (1.8). The case (iv) for which all the c_is are zero does not yield a nontrivial solution of our beam ODE (1.4). This is easily seen by differentiation

of (1.8) and substitution of the beam equation. Moreover, there could be more symmetries of equation (1.8) not inherited from equation (1.4). Therefore we next analyze the ODE (1.8) for further symmetries. In fact we see that it is rich of symmetries. We investigate equation (1.8) for Lie point symmetries (see [77, 80, 81] for symmetry properties of scalar ODEs). The operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{4.7}$$

is a generator of symmetry of ODE (1.8) if the symmetry condition

$$X^{[2]}(3yy'' - 2y'^2 + x^2c_1 - 2xc_2 + c_3)|_{(1.8)} = 0, \tag{4.8}$$

is satisfied, where

$$X^{[2]} = X + \zeta_x\frac{\partial}{\partial y}' + \zeta_{xx}\frac{\partial}{\partial y''}, \tag{4.9}$$

in which

$$\begin{aligned} \zeta_x &= D_x\eta - y'D_x\xi, \\ \zeta_{xx} &= D_x\zeta_x - y''D_x\xi, \end{aligned} \tag{4.10}$$

with

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\cdots. \tag{4.11}$$

The determining equations that arise from (4.8) easily yield

$$\begin{aligned} \xi &= b_1 + b_2x + b_3x^2, \\ \eta &= (3b_3x + d_1)y, \end{aligned} \tag{4.12}$$

where the constants d_1 and b_i are further constrained by the relations

$$\begin{aligned} b_3c_2 + 2c_1b_2 - c_1d_1 = 0, \\ -b_3c_3 - 3b_2c_2 + 2d_1c_2 + b_1c_1 = 0, \\ -d_1c_3 + b_2c_3 - b_1c_2 = 0. \end{aligned} \tag{4.13}$$

The analysis of (4.13) after much algebra gives rise to the following cases.

I. Single symmetry cases

(a) $c_1 \neq 0$, $c_2^2 \neq c_1 c_3$.

We arrive at

$$X = (\frac{c_3}{c_1} - 2\frac{c_2}{c_1}x + x^2)\frac{\partial}{\partial x} + 3(xy - \frac{c_2}{c_1}y)\frac{\partial}{\partial y}. \tag{4.14}$$

(b) $c_1 = 0$, $c_2 \neq 0$.

Here we obtain

$$X = (-\frac{1}{2}\frac{c_3}{c_2}x + x)\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y}. \tag{4.15}$$

We observe that these are exactly the cases (i) and (ii), viz. (4.5) and (4.6) of the inherited symmetries.

II. Two symmetry cases

(d) $c_1 \neq 0$, $c_2 \neq 0$, $c_2^2 = c_1 c_3$.

Here we have

$$\begin{aligned} X_1 &= (-\frac{c_2}{c_1} + 2x - \frac{c_1}{c_2}x^2)\frac{\partial}{\partial x} + 3(y - \frac{c_1}{c_2}xy)\frac{\partial}{\partial y}, \\ X_2 &= (\frac{c_2}{c_1} - x)\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}. \end{aligned} \tag{4.16}$$

(e) $c_1 = 0$, $c_2 = 0$, $c_3 \neq 0$.

This case gives

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned} \tag{4.17}$$

It should be noted that for the inherited symmetries given in case (iii) as equation (4.6), X_2 does not appear. As a matter of fact this case implicitly occurs in ([49]) in the sense that its integrability was discussed there without referring to

its symmetries which are mentioned above. (f) $c_2 = c_3 = 0$.

Here one has

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \\ X_2 &= x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}. \end{aligned} \tag{4.18}$$

III. Three symmetry case.

(g) $c_1 = c_2 = c_3 = 0$.

Equation (1.8) admits all three symmetries (1.6). However, as mentioned in the discussion on inherited symmetries, this case does not result in a nontrivial solution. In summary, we have two single symmetry cases which are also inherited symmetries from the beam ODE. In addition we get three two symmetry cases which were not obtained as inherited symmetries before. One of these, viz. II. (e) was indirectly utilized in ([49]) without reference to the symmetries. The three symmetry case which occurs in the inherited symmetries too, does not furnish new solutions as pointed out. In the next section we deal with reduction and integrability of our static Euler-Bernoulli beam equation (1.4) using the results of cases I. and II. above.

4.3 Reduction and integrability of ODE (1.4)

We firstly consider the reduction of the ODE (1.4) when it admits a single symmetry.

For the case I. (a), we have that the zero and first-order invariants corresponding

to the generator X are

$$\begin{aligned} u &= y(\frac{c_3}{c_1} - 2\frac{c_2}{c_1}x + x^2)^{-3/2}, \\ v &= y'(\frac{c_3}{c_1} - 2\frac{c_2}{c_1}x + x^2)^{-1/2} - 3xy(\frac{c_3}{c_1} - 2\frac{c_2}{c_1}x + x^2)^{-3/2}. \end{aligned} \tag{4.19}$$

The invariant representation of the ODE (1.8) then is

$$\frac{dv}{du} = \frac{2v^2 - c_1 + 3uv\frac{c_2}{c_1} - 9u^2\frac{c_3}{c_1}}{3uv + 9u^2\frac{c_2}{c_1}}, c_1 \neq 0. \tag{4.20}$$

The solution of this first-order ODE is complicated. However, we do state that a solution of this equation (4.20) provides a solution of equation (1.8) which in turn can be a candidate for a solution of the beam ODE (1.4).

In case I. (b), we find the invariants

$$\begin{aligned} u &= y(x - \frac{c_3}{2c_2})^{-3/2}, \\ v &= y'(x - \frac{c_3}{2c_2})^{-1/2}. \end{aligned} \tag{4.21}$$

The corresponding first-order ODE is

$$\frac{dv}{du} = \frac{2c_2 + 2v^2 - \frac{3}{2}uv}{3uv - \frac{9}{2}u^2}, \tag{4.22}$$

whose solution is not obvious. Here too we cannot proceed any further in regards to the integrability of our beam equation (1.4). In the third case II. (d) though, we are able to go further as there are two symmetries to take advantage of Lie's double reduction theorem. The use of the X_1 invariants, viz.

$$\begin{aligned} u &= y(2x - \frac{c_1}{c_2}x^2 - \frac{c_2}{c_1})^{-3/2}, \\ v &= y'(2x - \frac{c_1}{c_2}x^2 - \frac{c_2}{c_1})^{-1/2} + 3\frac{c_1}{c_2}(2x - \frac{c_1}{c_2}x^2 - \frac{c_2}{c_1})^{-3/2}, \end{aligned} \tag{4.23}$$

results in the first-order ODE

$$\frac{dv}{du} = \frac{\frac{2}{3}v^2 - uv - 3u^2 + \frac{1}{3}c_2}{uv - 3u^2}, \tag{4.24}$$

which inherits X_2 in (u, v) coordinates as its symmetry which is

$$\tilde{X} = u\frac{\partial}{\partial u} + 3v\frac{\partial}{\partial v}, \tag{4.25}$$

The zero-order invariant corresponding to $\tilde{X}$, $v - 3u = s$, in (4.25) reduces the ODE (4.24) to variables separable form

$$\frac{sds}{3s^2 + c_2} = \frac{du}{3u}, \tag{4.26}$$

which has solution

$$(v - 3u)^2 + \frac{1}{2}c_2 = c_4 u^{4/3}, \tag{4.27}$$

where c_4 ia a constant, which can be written by use of (4.23) as

$$[y'(2x - \frac{c_1}{c_2}x^2 - \frac{c_2}{c_1})^{-1/2} + 3\frac{c_1}{c_2}xu - 3u]^2 + \frac{1}{2}c_2 = c_4 u^{4/3}. \tag{4.28}$$

Then the invocation of (4.23) in (4.28) gives

$$[2x - \frac{c_1}{c_2}x^2 - \frac{c_2}{c_1}]^2 u'^2 = c_4 u^{4/3} - \frac{1}{2}c_2, \tag{4.29}$$

which easily provides

$$\int \frac{du}{\pm\sqrt{c_4 u^{4/3} - c_2/2}} = \int \frac{dx}{2x - \frac{c_1}{c_2}x^2 - \frac{c_2}{c_1}} + c_5, c_1 \neq 0, c_2 \neq 0, \tag{4.30}$$

where c_5 is a further constant. The substitution of this solution (4.30) into the beam ODE (1.4) results in the additional relation

$$27\delta - 2c_4 c_2 = 0. \tag{4.31}$$

Thus we have two conditions on our solution, viz. $c_2^2 = c_1 c_3$ and (4.31). Therefore we end up with a three-parameter family of solutions, (4.30), to the static Euler-Bernoulli beam equation (1.4). We must re-emphasize that the nonsolvable algebra $sl(2, \mathbb{R})$ admitted by the beam ODE (1.4) via the route of Noether integrals as well as Lie symmetries of the family (1.8) enable us to deduce this new solution. This was not considered in [49]. We can go further as we have two more cases of two-dimensional Lie algebras admitted by equation (1.8). The fourth case II. (e) immediately yields

$$\frac{dv}{du} = \frac{2v^2 - c_3}{3uv}, \tag{4.32}$$

using the X_1 invariants. This ODE is trivially integrable and results in the solution constructed in [49] which is (1.7). In this situation one determines a two-parameter family of exact solutions. The final case II. (f) is new as well. The invariants corresponding to X_1 are

$$u = yx^{-3}, \quad v = y'x^{-1} - 3x^{-2}y. \tag{4.33}$$

The reduced ODE is

$$\frac{dv}{du} = \frac{3v^2 - c_1}{3uv}, \tag{4.34}$$

which can be straightforwardly integrated to give

$$v^2 - \frac{1}{2}c_1 = c_4 u^{4/3}, \tag{4.35}$$

where c_4 is a constant, and hence the first-order ODE in y

$$(y'x^{-1} - 3x^{-2}y) = \frac{1}{2}c_1 + c_4 u^{4/3}. \tag{4.36}$$

The substitution of the invariant (4.33) into (4.36) provides the solution

$$\int \frac{du}{\pm\sqrt{c_1/2 + c_4 u^{4/3}}} = -\frac{1}{x} + c_5, \tag{4.37}$$

where c_5 is a further constant and the constants c_1 and c_4 are related by

$$27\delta + 2c_1c_4 = 0, \tag{4.38}$$

as a consequence of imposing (1.4). In this case too we obtain a two-parameter family of exact solutions for our beam ODE (1.4). However, in this regard we deduce a new family of solutions. In conclusion of this section, we have found new reductions for the two single symmetries. In particular, we have constructed two new families of exact solutions for our static beam ODE (1.4) as given by (4.30) and that by (4.37).

4.4 Concluding remarks

We have performed the complete Lie group classification of the second-order ODE (1.8) that results from the reduction of our static fourth-order Euler-Bernoulli beam ODE (1.4).

Apart from the two cases of single symmetries which enables one reduction of the ODE (1.8), we arrived at two new integrable cases. These were for the ODE (1.8) admitting two symmetries. The integrable cases arose from the negative fractional power law $f = \delta y^{-5/3}, \delta = \pm 1$. The centripetal force distribution for which f is proportional to the normal displacement y gives rise to submaximal Lie point symmetry algebras. It is physical to have negative powers for the normal displacement as a load but it is intriguing that the negative fractional power law $y^{-5/3}$ occurs and yields two new integrable cases. This exceptional case admits the nonsolvable algebra $sl(2, \mathbb{R})$. However it remarkably allows for

a three-parameter family of exact solutions for the beam equation for this case.

Scalar higher-order equations that admit $sl(2,\mathbb{R})$ symmetry are of immense importance in diverse applications. The equation $y^{(4)} = \delta y^{-5/3}$ is the simplest fourth-order equation that admits the nonsolvable algebra $sl(2,\mathbb{R})$ and allows for a three-parameter family of exact solutions. More importantly it arises in the study of a physical problem, viz. the Euler-Bernoulli beam equation. It should also be stated that it is in the hierarchy of familiar equations which have $sL(2,\mathbb{R})$ symmetry such as the Ermakov equation.

Chapter 5

Four-Dimensional Symmetry Algebras and Fourth-Order ODEs

5.1 Introduction

The aim of this chapter is to provide a supplementation of the results on the canonical forms for scalar fourth-order ordinary differential equations which admit four-dimensional Lie algebras obtained recently. Together with these new canonical forms, a complete list of scalar fourth-order ODEs that admit four-dimensional Lie algebras is provided. This sets the scene for future work on conditional symmetries of such equations subject to lower order ODEs. We have

seen an interesting example of a static Euler-Bernoulli beam equation which is conditionally integrable in the previous chapter. Moreover, conditional linearization of fourth-order ODEs subject to Lie linearizable lower order ODEs were considered in [42].

5.2 Comparison of the results of [58] and [55]

We show here that the results on realizations of four-dimensional algebras in the plane given in [55] is a special case of the corresponding set of realizations given in [58]. We make a comparison of the lists of realizations given in [55] and [58]. It should be remarked that in general a result of classification of realizations may contain errors of two types, viz.

1. missing of some inequivalent cases
2. mutually equivalent cases.

In the following comparison, some first type of errors exist in [55]. Five cases are missing. There are some other cases which can be combined in a compact form and also some arbitrary parameters and functions need modification according to the results of [58] related to realizations in the plane. Below we keep the notations of both; on the left hand side the notations of [55] and on right hand side that of [58]. However, for the final results and further utilization, we keep the notations of [58].

Four-dimensional algebras

We use the nomenclature of Patera and Winternitz [56] in the naming of the algebras such as $4A_1$ etc. Thus we do not provide a table of the abstract algebras of dimension four as this is easily available. Here the Nu refers to the realizations given in the work [55] and R to that of [58].

(i) $Nu(4A_1) \sim (4A_1)(\bar{e_1} = e_2, \bar{e_2} = e_3, \bar{e_3} = e_1, \bar{e_4} = e_4)$
$Nu(4A_1) \sim R(4A_1, 11)$ $(\bar{t} = y,\ \ \bar{x} = x)$.

(ii) $Nu(A_2 \oplus 2A_1) \sim (A_{2.1} \oplus 2A_1)(\bar{e_1} = e_2, \bar{e_2} = -e_1, \bar{e_3} = e_3, \bar{e_4} = e_4)$.
No realization exists in (1+1)-dimension.

(iii) $Nu(2A_2) \sim (2A_{2.1})(\bar{e_1} = e_2, \bar{e_2} = -e_1, \bar{e_3} = e_4, \bar{e_4} = -e_3)$.
$Nu(2A_2) \sim R(2A_{2.1}, 5)(\bar{t} = x, \bar{x} = y)$ whereas $R(2A_{2.1}, 7)$ is missing in [55].

(iv) $Nu(A_{3,1} \oplus A_1) \sim (A_{3.1} \oplus A_1)$. No realization exists in (1+1)-dimension.

(v) $Nu(A_{3,2} \oplus A_1, f(x) = 0) \sim (A_{3.2} \oplus A_1)$
$Nu(A_{3,2} \oplus A_1, f(x) = 0) \sim R(A_{3.2} \oplus A_1, 9)(\bar{t} = y, \bar{x} = -x)$.

(vi) $Nu(A_{3,3} \oplus A_1) \sim (A_{3.3} \oplus A_1)$. No realization exists in (1+1)-dimension.

(vii) $Nu(A_{3,4} \oplus A_1,\ f(x) = 0) \sim (A^a_{3.4} \oplus A_1, a = -1)$ $Nu(A_{3,4} \oplus A_1, f(x) = 0) \sim$ $R(A^a_{3.4} \oplus A_1, 9, a = -1)(\bar{t} = y, \bar{x} = x^2)$.

(viii) $Nu(A^a_{3,5} \oplus A_1, 0 < |a| < 1, f(x) = 0) \sim (A^a_{3.4} \oplus A_1, |a| \leq 1, a \neq 0, \pm 1)$
$Nu(A^a_{3,5} \oplus A_1, 0 < |a| < 1,\ f(x) = 0) \sim R(A^a_{3.4} \oplus A_1, 9)$ $(\bar{t} = y,\ \ \bar{x} = x^{1-a})$.

(ix) $Nu(A_{3,6} \oplus A_1, f(x) = 0) \sim (A^b_{3.5} \oplus A_1, b = 0)$ $Nu(A_{3,6} \oplus A_1, f(x) = 0) \sim R(A^b_{3.5} \oplus A_1, 8, b = 0)(\bar{t} = y, \bar{x} = (x^2 - 1)^{\frac{1}{2}})$.

(x) $Nu(A^b_{3,7} \oplus A_1, b > 0, f(x) = 0) \sim (A^b_{3.5} \oplus A_1, b > 0)$ $Nu(A^b_{3,7} \oplus A_1, b > 0, f(x) = 0) \sim R(A^b_{3.5} \oplus A_1, 8, b > 0)(\bar{t} = y, \bar{x} = x)$.

(xi)$Nu(A_{3,8} \oplus A_1,\ f(x) = 1) \sim (Sl(2, \mathbb{R}) \oplus A_1)$ $(\bar{e_1} = e_1,\ \bar{e_2} = e_2,\ \bar{e_3} = -e_3,\ \bar{e_4} = e_4)$ $Nu(A_{3,8} \oplus A_1, b > 0,\ f(x) = 1) \sim R(Sl(2, \mathbb{R}) \oplus A_1, 9)$ $(\bar{t} = y,\ \bar{x} = x)$whereas $R(Sl(2, \mathbb{R}) \oplus A_1, 8))$ is missing in [55].

(xii) $Nu(A_{3.9} \oplus A_1) \sim (So(3) \oplus A_1)$. No realization exists in (1+1)-dimension.

(xiii) $Nu(A_{4,1}, f(x) = 0) \sim (A_{4.1})$. $Nu(A_{4,1}, f(x) = 0) \sim R(A_{4.1}, 8)$ $(\bar{t} = y,\ \bar{x} = x)$.

(xiv) $Nu(A^b_{4,2}, b \neq 0, 1) \sim (A^b_{4.2}, b \neq 0, 1), (\bar{e_1} = e_1, \bar{e_2} = -e_2,\ \bar{e_3} = -e_3,\ \bar{e_4} = e_4)$ $Nu(A^b_{4,2}) \sim R(A^b_{4.2}, 8)(\bar{t} = e^{(b-1)x}y, \bar{x} = e^{(b-1)x})$.

(xv) $Nu(A^1_{4.2}) \sim (A^b_{4.2},\ b = 1)$. No realization exists in (1+1)-dimension.

(xvi) $Nu(A_{4,3}, f(x) = 0) \sim (A_{4.3})$ $Nu(A_{4,3}, f(x) = 0) \sim R(A_{4.3}, 8)$ $(\bar{t} = y, \bar{x} = x)$.

(xvii) $Nu(A_{4,4}, f(x) = 0) \sim (A_{4.4})$ $Nu(A_{4,4}, f(x) = 0) \sim R(A_{4.4}, 7)$ $(\bar{t} = y, \bar{x} = x)$.

(xviii) $Nu(A^{a,b}_{4,5}, -1 \leq a < b < 1, ab \neq 0) \sim (A^{a,b}_{4,5}, -1 \leq a < b < c = 1, abc \neq 0)(\bar{e_1} = e_2, \bar{e_2} = e_3, \bar{e_3} = e_1, \bar{e_4} = e_4)$

$Nu(A^{a,b}_{4,5}, -1 \leq a < b < 1, ab \neq 0) \sim R(A^{a,b}_{4,5}, 7, -1 \leq a < b < c = 1, abc \neq 0,\ b > 0\ if\ a = -1)$ $(\bar{t} = x^{a-1}y + x^a \ln \frac{1}{|x|}, \bar{x} = \ln|x|)$

(xix) $(A^{a,a}_{4,5},\ -1 \leq a < 1, a \neq 0) \sim (A^{1,1,a^{-1}}_{4.5}, -1 \leq a < 1, a \neq 0), (\bar{e_1} = e_3, \bar{e_2} = e_2, \bar{e_3} = e_1, \bar{e_4} = e_4)$. No realization exists in (1+1)-dimension.

(xx) $Nu(A^{a,1}_{4,5}, -1 \leq a < 1, a \neq 0) \sim (A^{1,1,a}_{4.5}, -1 \leq a < 1, a \neq 0), (\bar{e_1} = e_1, \bar{e_2} = e_3, \bar{e_3} = e_2, \bar{e_4} = e_4)$. No realization exists in (1+1)-dimension.

(xxi) $Nu(A^{1,1}_{4,5}) \sim (A^{a,b}_{4,5}, a = b = c = 1)(\bar{e_1} = e_2, \bar{e_2} = e_3, \bar{e_3} = e_1, \bar{e_4} = e_4)$ $Nu(A^{1,1}_{4,5}) \sim R(A^{a,b}_{4,5}, 10, a = b = c = 1)(\bar{t} = y, \bar{x} = x)$

(xxii) $Nu(A^{a,b}_{4,6}, a \neq 0, b \geq 0) \sim (A^{a,b}_{4.6}, a > 0), (\bar{e_1} = e_1, \bar{e_2} = e_3, \bar{e_3} = -e_2, \bar{e_4} = e_4)$. $Nu(A^{a,b}_{4,6}, a \neq 0, b \geq 0) \sim R(A^{a,b}_{4.6}, 6, a > 0)(\bar{t} = y(1 + x^2)^{-\frac{1}{2}} e^{(a-b)tan^{-1}x}, \bar{x} = tan^{-1}x)$.

(xxiii)$Nu(A_{4,7}) \sim (A_{4.7})$. $Nu(A_{4,7}) \sim R(A_{4.7}, 5)(\bar{t} = y + \frac{x^2}{4}(1 - 2\log|x|), \bar{x} = x)$.

(xxiv) $Nu(A_{4,8}) \sim (A^b_{4.8}, |b| \leq 1, b = -1)$. $Nu(A_{4,8}) \sim R(A^b_{4.8}, |b| \leq 1, 5, b = -1)$ $(\bar{t} = y,\ \ \bar{x} = x)$.

(xxv) $Nu(A^b_{4,9}, 0 < |b| < 1) \sim (A^b_{4.8}, |b| \leq 1, b \neq \pm 1, 0)$. $Nu(A^b_{4,9}, 0 < |b| < 1) \sim (A^b_{4.8}, |b| \leq 1, b \neq \pm 1, 0)$. $Nu(A^b_{4,9}) \sim R(A^b_{4.8}, 5)(\bar{t} = y, \bar{x} = x)$ whereas $R(A^b_{4.8}, 7, b \neq \pm 1, 0)$ is missing in [55].

(xxvi) $Nu(A^1_{4,9}) \sim (A^b_{4.8}, b = 1)$. $Nu(A^1_{4,9}) \sim R(A^1_{4.8}, 5)(\bar{t} = y, \bar{x} = x)$.

(xxvii) $Nu(A^0_{4,9}) \sim (A^b_{4.8}, b = 0).Nu(A^0_{4,9}) \sim R(A^0_{4.8}, 5)(\bar{t} = y, \bar{x} = x)$ whereas

$R(A^b_{4.8}, 7, b = 0)$ is missing in [55].

(xxviii) $Nu(A_{4.10}) \sim (A^a_{4.9}, a = 0)$. No realization exists in (1+1)-dimension.
(xxix) $Nu(A^a_{4.11}, a > 0) \sim (A^a_{4.9}, a > 0)$. No realization exists in (1+1)-dimension.

(xxx) $Nu(A_{4,12}) \sim (A_{4.10})$. $Nu(A_{4,12}) \sim R(A_{4.10}, 7)(\bar{t} = y, \bar{x} = x)$ whereas $R(A_{4.10}, 6)$ is missing in [55].

*Table*1

Lie Algebra	N	*Realizations (Generators)*
$4A_1$	11	$\partial_t,\ x\partial_t,\ k(x)\partial_t,\ h(x)\partial_t$
$2A_{2.1}$	5	$\partial_t,\ t\partial_t,\ \partial_x,\ x\partial_x$
$2A_{2.1}$	^{*7}	$\partial_t,\ t\partial_t + x\partial_x,\ x\partial_t,\ -x\partial_x$
$A_{3.2} \oplus A_1$	9	$\partial_t,\ x\partial_t,\ t\partial_t - \partial_x,\ e^{-x}\partial_t$
$A^a_{3.4} \oplus A_1, \lvert a\rvert \leq 1, a \neq 0, 1$	9	$\partial_t,\ x\partial_t,\ t\partial_t + (1-a)x\partial_x,\ \lvert x\rvert^{\frac{1}{1-a}}\partial_t$
$A^b_{3.5} \oplus A_1, b \geq 0$	8	$\partial_t,\ x\partial_t,\ (b-x)t\partial_t - (1+x^2)\partial_x,\ \sqrt{1+x^2}e^{-b\arctan x}\partial_t$
$sl(2,\mathbb{R}) \oplus A_1$	^{*8}	$\partial_t,\ t\partial_t + x\partial_x,\ t^2\partial_t + 2tx\partial_x,\ x\partial_x$
$sl(2,\mathbb{R}) \oplus A_1$	9	$\partial_t,\ t\partial_t,\ t^2\partial_t,\ \partial_x$
$A_{4.1}$	8	$\partial_t,\ x\partial_t,\ \frac{1}{2}x^2\partial_t,\ -\partial_x$
$A^b_{4.2}, b \neq$	8	$\partial_t,\ x\partial_t,\ \frac{1}{1-b}x\ln\lvert x\rvert\partial_t,\ bt\partial_t + (b-1)x\partial_x, b \neq 1$
$A_{4.3}$	8	$\partial_t,\ x\partial_t,\ -x\ln\lvert x\rvert\partial_t,\ t\partial_t + x\partial_x$
$A_{4.4}$	7	$\partial_t,\ x\partial_t,\ \frac{1}{2}x^2\partial_t,\ t\partial_t - \partial_x$
$A^{a,b,c}_{4.5}, -1 \leq a < b < c = 1, abc \neq 0$	7	$\partial_t,\ e^{(a-b)x}\partial_t, e^{(a-1)x}\partial_t, at\partial_t + \partial_x, b > 0 if a = -1$
$A^{a,b,c}_{4.5}, -1 \leq a < b < c = 1, abc \neq 0$	10	$\partial_t, x\partial_t, \phi(x)\partial_t,\ t\partial_t, \phi''(x) \neq 0, a = b = c = 1$
$A^{a,b}_{4.6}, a > 0$	6	$\partial_t, e^{(a-b)x}\cos x\partial_t, -e^{(a-b)x}\sin x\partial_t, at\partial_t + \partial_x$
$A_{4.7}$	5	$\partial_t,\ x\partial_t,\ -\partial_x,\ (2t - \frac{1}{2}x^2)\partial_t + x\partial_x$
$A^b_{4.8}, \lvert b\rvert \leq 1$	5	$\partial_t, \partial_x, x\partial_t, (1+b)t\partial_t + x\partial_x$
$A^b_{4.8}, \lvert b\rvert \leq 1$	7	$\partial_t, x\partial_t, -\partial_x, (1+b)t\partial_t + bx\partial_x, b \neq \pm 1$
$A_{4.10}$	^{*6}	$\partial_t,\ \partial_x,\ t\partial_t + x\partial_x,\ x\partial_t - t\partial_x$
$A_{4.10}$	7	$\partial_t, x\partial_t, t\partial_t, -tx\partial_t - (1+x^2)\partial_x$

Table2

Lie Algebra	N	*Realizations (Generators)*
$2A_{2.1}$	^{*7}	$\partial_t,\ t\partial_t + x\partial_x,\ x\partial_t,\ -x\partial_x$
$2A_{2.1}$	^{*7}	$x^{\cdot\cdot\cdot\cdot} = \frac{10x^{\cdot}x^{\cdot\cdot\cdot}}{x} - \frac{15x^{\cdot\cdot}{}^3}{x^{\cdot}{}^2} + \frac{x^{\cdot}{}^2x^{\cdot\cdot}}{x^2} + f(\frac{x^{\cdot\cdot}x^{\cdot\cdot\cdot}}{x^{\cdot}x^{\cdot\cdot}} - \frac{3xx^{\cdot\cdot}}{x^{\cdot}{}^2})$
$sl(2,\mathbb{R}) \oplus A_1$	^{*8}	$\partial_t,\ t\partial_t + x\partial_x,\ t^2\partial_t + 2tx\partial_x,\ x\partial_x$
$sl(2,\mathbb{R}) \oplus A_1$	^{*8}	$x^{\cdot\cdot\cdot\cdot} = -\frac{2x^{\cdot}x^{\cdot\cdot\cdot}}{x} + \frac{1}{x^3}(\frac{x^{\cdot}{}^2}{2} - xx^{\cdot\cdot})^2 + f(\frac{x^2x^{\cdot\cdot\cdot}}{(\frac{x^{\cdot}{}^2}{2} - xx^{\cdot\cdot})^{\frac{3}{2}}})$
$A_{4.8}^b, \lvert b\rvert \leq 1$	^{*7}	$\partial_t,\ x\partial_t,\ -\partial_x,\ (1+b)t\partial_t + bx\partial_x, b \neq \pm 1$
$A_{4.8}^b, \lvert b\rvert \leq 1$	^{*7}	$x^{\cdot\cdot\cdot\cdot} = \frac{10x^{\cdot}x^{\cdot\cdot\cdot}}{x} - \frac{15x^{\cdot\cdot}{}^3}{x^{\cdot}{}^2} + x^{\cdot\cdot}{}^{\frac{1-3b}{1-b}}x^{\cdot}{}^{\frac{4-2b}{1-b}}f((\frac{x^{\cdot\cdot}}{x^{\cdot}{}^4} - \frac{3x^{\cdot\cdot}{}^2}{x^{\cdot}{}^5})(\frac{x^{\cdot\cdot}}{x^{\cdot}}{}^3)^{\frac{1-3b}{1-b}})$
$A_{4.10}$	^{*6}	$\partial_t,\ \partial_x,\ t\partial_t + x\partial_x,\ x\partial_t - t\partial_x$
$A_{4.10}$	^{*6}	$x^{\cdot\cdot\cdot\cdot} = \frac{10x^{\cdot}x^{\cdot\cdot}x^{\cdot\cdot\cdot}}{1+x^{\cdot}{}^2} - \frac{15x^{\cdot}{}^2x^{\cdot\cdot}{}^3}{(1+x^{\cdot}{}^2)^2}f(\frac{(1+x^{\cdot}{}^2)x^{\cdot\cdot\cdot}}{x^{\cdot\cdot}{}^2} - 3x^{\cdot})$

Remarks for Table1 •$4A_1$:1, x, $h(x)$ and $k(x)$ form a linearly independent set.

Remarks for Tables1 & 2 In both Tables

• h, k,and ϕ are arbitrary functions with specified conditions mentioned in the corresponding realizations.

• a, b and c are parameters and arbitrary constants, whose range and values are mentioned in each of the realizations.

•$*$: these are the cases of realizations which are missing in [55].

Remarks for Table 2

•$*$: these are the canonical forms of fourth-order ODEs which are missing in [55]. Only these are given here with their corresponding algebras and realizations.

5.3 Concluding remarks

In this contribution, we have supplemented the work [55] for the canonical forms of scalar fourth-order ODEs and have obtained four new forms as listed in Table 2. The integrability of these equations have the same route as the others which are discussed at length in [55]. Moreover, the notion of conditional symmetries as shown in the previous chapter becomes amenable to such equations. The conditional linearization subject to root ODEs have already been looked at in [42].

Chapter 6

Conclusion

In this thesis, we have provided the algebraic structure of ordinary differential equations (ODEs) when they have insufficient number of Lie point symmetries. These lost symmetries are called conditional symmetries and are very useful for integration as well. An algorithm has been given for the calculation of conditional symmetries of ODEs subject to lower order ODEs. An effective criteria as to when a derived ODE system inherits the symmetries of the root ODE is proved. We have considered scalar linear and linear system of ODEs for their conditional symmetries by giving many simple examples. Moreover, we have given four propositions in which we have shown when derived linear ODEs inherit the symmetries of the root linear ODEs. We have proved that a system of ODEs admit a conditional symmetry subject to the first-order ODEs if it has exact solutions related to the invariant curve conditions arising from the known solution curve. We have extended our investigations to third-order non-linear ODEs which ad-

mit conditional symmetries subject to nonlinear second-order ODEs with three symmetries. We have given applications of conditional symmetries to ODEs that arise in mechanics and have obtained new solutions.

We have extended our idea and take the base as scalar second-order nonlinear ODEs which have seven equivalence classes with each possessing three Lie point symmetries. We have stated the algorithm for the calculation of conditional symmetries of nonlinear third-order ODEs subject to nonlinear second-order ODEs admitting symmetries. Some examples were given. Further, we have considered first integrals corresponding to base ODEs for their conditional symmetries. We have also shown that derived nonlinear ODEs have in general less inherited symmetries. We have embarked on the conditional criteria for $sl(2,\mathbb{R})$ of scalar nonlinear third-order ODE obtained after differentiation of the scalar second-order ODE with $sl(2,\mathbb{R})$ symmetry algebra. This has been carried out by using the Tressé invariants.

For the first time we have analyzed (1.8) for symmetries and showed that equation (1.4) remarkably allows for a three parameters family of exact solutions. Thus the admittance of $sl(2,\mathbb{R})$ symmetry algebra for the static Euler-Bernoulli fourth-order ODE (1.4) gives rise to a three-parameter family of exact solutions. This is the simplest in the hierarchy of fourth-order ODEs that has this property and joins the well-known Ermakov second-order ODE as well as third-order ODEs having $sl(2,\mathbb{R})$ symmetry algebra.

Finally, we have provided a comparison of results between [55] and [58] related to realization of four-dimensional Lie algebra as vector fields in the plane. The pa-

per [55] contains two types of the errors, namely (i) missing of some inequivalent cases, (ii) mutually equivalent cases. Some first type of errors exist in [55]. Five cases are missing. There are some other cases which can be combined in a compact form and also some arbitrary parameters and functions need modifications according to the results of [58] related to realizations in the plane.

The idea of conditional symmetries becomes amenable to fourth-order ODEs as studied in Chapter 4. The conditional linearization subject to root ODEs have already been looked at in [42]. It will be of interest in future works to investigate the conditional symmetry classification of scalar fourth-order ODEs subject to nonlinear root ODEs of say second-order.

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