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# Giant graviton oscillators

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# Giant Graviton Oscillators

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A Dissertation submitted to the Faculty of Science,
University of the Witwatersrand,
in fulfilment of the requirements for
the degree of Master of Science.

14 May 2013

# Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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**C.R. Mathwin**

14 May 2013

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Love and thanks to my lady, the gravity of her impact on my ability to complete this MSc cannot be understated.
Finally, an infinite mass of appreciation oscillating between my parents, without whom my version of the universe would never have existed.

**Abstract**

We study the action of the dilatation operator on restricted Schur polynomials labeled by Young diagrams with $p$ long columns or $p$ long rows. A new version of Schur-Weyl duality provides a powerful approach to the computation and manipulation of the symmetric group operators appearing in the restricted Schur polynomials. Using this new technology, we are able to evaluate the action of the one loop dilatation operator. The result has a direct and natural connection to the Gauss Law constraint for branes with a compact world volume. Generalzing previous results, we find considerable evidence that the dilatation operator reduces to a decoupled set of harmonic oscillators. This strongly suggests that integrability in $\mathcal{N} = 4$ super Yang-Mills theory is not just a feature of the planar limit, but extends to other large $N$ but non-planar limits.

# Contents


**1 Introduction** **4**

**2 Background** **7**
2.1 The AdS/CFT Correspondence . . . 7
2.1.1 Statement . . . 7
2.1.2 Conformal Field Theory . . . 7
2.1.3 String Theory . . . 8
2.1.4 5-dimensional anti-de Sitter Space ($AdS_5$) . . . 8
2.1.5 Heuristic Motivation for the Conjecture . . . 10
2.1.6 Discussion . . . 11
2.1.7 $D$-branes . . . 11
2.2 Schur-Weyl Duality . . . 12
2.3 Schur Polynomials . . . 13
2.3.1 Definition . . . 13
2.3.2 A Generalization: Restricted Schur Polynomials . . . 13

**3 Giant Gravitons** **16**
3.1 The Stringy Exclusion Principle . . . 16
3.1.1 Development . . . 16
3.1.2 Invasion of the Giant Gravitons . . . 17
3.1.3 Derivation . . . 17
3.1.4 Discussion . . . 22
3.2 Branes in $AdS_5$ . . . 22
3.3 Schur Polynomials as Duals to Giant Graviton Systems . . . 23

**4 Construction of Restricted Schur Polynomials** **26**
4.1 The Construction . . . 26
4.1.1 From $S_{n+m}$ to $S_n \times (S_1)^m$ . . . 26
4.1.2 Young Diagrams with $p$ long rows . . . 27
4.1.3 State Labels - From $S_m$ to $U(p)$ . . . 28
4.1.4 Constructing the Operators Projecting onto $S_n \times S_m$ Carrier Spaces . . . 29
4.1.5 Young Diagrams with $p$ Long Columns . . . 30
4.2 Construction of Projectors: Examples . . . 31
4.2.1 A Three Row Example using $U(3)$ . . . 31
4.2.2 A Four Column Example using $U(4)$ . . . 34

**5 The Dilatation Operator** **37**
5.1 Definition . . . 37
5.2 Why $D$? . . . 37
5.3 Action on Restricted Schur Polynomials . . . 38
5.3.1 Intertwiners . . . 40
5.3.2 $\Gamma_R((1, m+1))$ . . . 42
5.3.3 Dilatation Operator Coefficient . . . 43
5.3.4 Evaluating Traces . . . 44
5.3.5 Long Columns . . . 46

**6 Calculation of the Dilatation Operator** **47**
6.1 Young Diagrams with Two Rows or Columns . . . 47
6.1.1 Diagonal Terms . . . 47
6.1.2 Non-diagonal Terms . . . 50
6.1.3 Result . . . 50
6.2 Young Diagrams with Three Rows or Columns . . . 51
6.2.1 Diagonal Terms . . . 54
6.2.2 Off-Diagonal Terms . . . 57
6.2.3 Some Explicit Examples . . . 58

**7 Diagonalization of the Dilatation Operator** **63**
7.1 The Two-Row Dilatation Operator . . . 63
7.1.1 Continuum Limit . . . 65
7.2 The Three-Row Dilatation Operator . . . 66
7.2.1 Continuum Limit . . . 68
7.2.2 $j^3 = O(1)$ Sector . . . 68
7.3 A Four-Row Example . . . 69

**8 Results & Conclusions** **70**
8.1 Connection to the Gauss Law . . . 70
8.1.1 $\Delta = (1, 1, 1)$ Example . . . 70
8.1.2 $\Delta = (3, 2, 1)$ Example . . . 71
8.1.3 General Action in the Continuum Limit . . . 72
8.2 Holography and Emergent Spacetime . . . 73
8.3 Summary & Outlook . . . 73

**A Elementary Facts from $U(p)$ Representation Theory** **76**
A.1 The Lie Algebra $u(p)$ . . . 76
A.2 Gelfand-Tsetlin Patterns . . . 77
A.3 $\Sigma$ and $\Delta$ Weights . . . 77
A.4 Relation between Gelfand-Tsetlin Patterns and Young Diagrams . . . 78
A.5 Clebsch-Gordon Coefficients . . . 79
A.6 Explicit Association of labelled Young Diagrams and Gelfand-Tsetlin Patterns . . . 80
A.7 Last Remarks . . . 81

**B Elementary Facts from $S_n$ Representation Theory** **82**
B.1 Young-Yamonouchi Basis . . . 82
B.2 Young's Orthogonal Representation . . . 82
B.3 Partially labelled Young diagrams . . . 83
B.4 Simplifying Young's Orthogonal Representation . . . 83

**C Recursion Relations** **85**

**D The Continuum Limit for $p$-row Operators** **86**

# Chapter 1

# Introduction

Over the past 8 years or so, significant progress has been made towards describing the emergence of a higher dimensional quantum gravitational theory from a theory of quantum fields without gravity. Central to the success of such a unification of these two seemingly incompatible theories is Maldacena's conjecture [1], referred to as the AdS-CFT correspondence, which proposes an exact equivalence between Type-IIB String Theory on $AdS_5 \times S^5$ and Conformally Invariant $\mathcal{N} = 4$ Supersymmetric Yang-Mills Field Theory (SYM) on 4-dimensional Minkowski spacetime. It is this correspondence, together with the state-operator correspondence of conformal field theory, which has allowed for the possibility of defining operators in SYM that are dual to states of objects in the string theory, thus enabling one to perform calculations relevant to the string theory while working in the (far more tractable) quantum field theory. This project continues the work of Prof. Robert de Mello Koch and his previous MSc and PhD students, which attempts to explain and utilize the connection between certain quantum field theories and theories of quantum gravity. Part of the novelty of the calculations performed is that the solutions are derived for a system where non-planar diagrams are not suppressed in the large-$N$ limit - following [2], the entire set of diagrams is summed to obtain an answer. The existence of integrability in the non-planar limit is thus examined.

In order to determine whether a given operator in the field theory is an acceptable dual to a string theory state, we must confirm that the physics in both cases is equivalent. Since we measure only conserved charges, and Noether's theorem states an association of symmetries of the action with conserved charges, it suffices to identify an equivalence in the symmetries of the two theories. The isometry group of the $AdS_5 \times S^5$ is $SO(2,4) \times SO(6)$, corresponding to global spacetime symmetries of the two component subspaces of the background. In the gauge theory, we have an $SO(2,4)$ symmetry arising as the conformal group in 4 dimensions (the group of all angle preserving coordinate transformations), and an $SO(6)$ that corresponds to the $\mathcal{R}$-symmetry of the six scalar fields of SYM. The two theories admit the same symmetries, and we can identify charges in the two theories based on which symmetry group they correspond to; for example (and most importantly for the purposes of this dissertation), the $SO(2)$ generator of the maximally compact $SO(2) \times SO(4)$ subgroup of $SO(2,6)$ corresponds to scaling dimension in the gauge theory, and energy in the string theory.

The SYM theory has gauge group $U(N)$ of rank $N$, which we will take to be very large. The effect of taking this limit is usually to suppress the contributions of non-planar diagrams to correlators of operators in the theory, thus significantly simplifying their calculation. However, the operators we consider here have bare dimension proportional to $N$, and thus this simplification does not occur - there are a huge number of ways to form non-planar diagrams when the number of fields in the operator is so large, and the combinatoric factors associated with the non-planar diagrams require their inclusion in the calculation of correlators. This difficulty is circumvented by a change of basis that diagonalizes the two point function, performed for single matrix observables in [2] and generalized to multimatrix observables in [3].

We see then that if one wishes to study some object living in the string theory, an operator can be found in the CFT which is dual to the object (i.e. describes the same physics), and the predictions of calculations performed therewith will be applicable to the state we chose to study. It has been found that the proportionality of the dimension (or $\mathcal{R}$ charge) of the operator with respect to $N$ determines the geometrical object in the string theory to which the operator is associated. Operators with $O(N^2)$

fields have been found to be dual to $\frac{1}{2}$ BPS geometries, with $O(N^{\frac{1}{2}})$ dual to strings etc.. This project continues the study of operators dual to string theoretic states known as giant gravitons. These are D3-branes in the $AdS_5 \times S^5$ which have expanded in either the $AdS_5$ or $S^5$ due to an increase in their angular momentum, by means of a process analogous to the Myers' effect for dielectric branes [4]. This is a result of the increase of the coupling of the brane to the background Ramond-Ramond five form flux with increased angular momentum. The operators proposed to be dual to these objects are *(restricted) Schur Polynomials* containing $O(N)$ fields.

It is these Schur polynomials which form the basis that diagonalizes the two point correlation function for operators containing $O(N)$ fields. Considering the results of [2] in terms of projection operators, an elegant explanation of how and why this occurs is found [5]. One of the basic observations made in [5] is the fact that two point functions of operators of the form

$$\hat{A}_n \equiv A^{i_1\, i_2 \cdots i_n}_{j_1\, j_2 \cdots j_n} Z^{j_1}_{i_1} Z^{j_2}_{i_2} \cdots Z^{j_n}_{i_n} = \mathrm{Tr}(AZ^{\otimes n})$$

are given by

$$\left\langle \hat{A}_n \hat{B}^{\dagger}_n \right\rangle = \sum_{\sigma \in S_n} \mathrm{Tr}(\sigma A \sigma^{-1} B^{\dagger})\,.$$

By choosing $A$ and $B$ to be projection operators projecting onto irreducible representations of the symmetric group,they clearly commute with $\sigma$ (rendering the above sum trivial) and are orthogonal. With this choice for $A$, it turns out that $\hat{A}_n$ is a Schur polynomial.

Since giant gravitons are D3-branes with a compact world volume, Gauss' Law forces the total charge on their surface to vanish. This constraint should be reflected in the description of these objects by Schur polynomials. Compelling evidence that this is indeed the case was provided in [6], where it was shown that the number of operators that can be defined using restricted Schur Polynomials is equal to the number of states of the dual giant graviton systems consistent with the Gauss' Law. In [7] (the article that is the focus of this dissertation), further agreement with the Gauss law constraint of the Schur polynomial description is demonstrated by the study of the action of the one loop dilatation operator on the restricted Schur polynomial operators. Both numerical ([8],[9]) and analytic ([10]) studies of this action on decoupled sectors of the theory containing $p = 2$ giants have previously been performed, with the important and surprising result that the eigenvalues of the dilatation operator, which are dual to the energies in the string theory, are those of a set of decoupled harmonic oscillators. These results are extended to higher numbers of giant gravitons in [7], and the methods utilised to obtain these results are discussed in detail in this dissertation.

The crucial ingredient used in [10] to obtain simple analytic expressions for the action of the dilatation operator is the realization that the symmetric group operators needed to define the restricted Schur polynomials can be computed by mapping the Young diagram labels of the operator onto an auxiliary spin chain. This is essentially an application of Schur-Weyl duality, and it turns out that this is in fact the key to the efficient computation of the projectors in more general cases - the spin chain approach becomes cumbersome upon generalization, but a clarification of the role of the Schur-Weyl duality used in this method lights the way forward. The complexity of the generalization beyond the $p = 2$ sector is seen in the projectors; the subspaces projected to when $p > 2$ may be subduced more than once, and the labelling of states must be organised such that these multiplicities are tracked. Fortunately, it is found that this multiplicity problem is also resolved by the use of Schur-Weyl duality.

The approach developed in [7] provides the following benefits to the further development of the theory:

- It is possible using this approach to construct the restricted Schur polynomials for $p$ giant graviton systems using the representation theory of $U(p)$. For the case of $p$ sphere giant gravitons we obtain an example of Schur-Weyl duality that is, as far as we know, novel.
- The multiplicity of $S_n \times S_m$ irreducible representations subduced from a given $S_{n+m}$ irreducible representation is organised by mapping it into the *inner multiplicity* of the $U(p)$ representation labelled by the $m$-box young diagram. As far as we know, this connection has not been pointed out in the maths literature, although it follows as a rather simple consequence of the Schur-Weyl duality we have found.

- The approach also allows us to evaluate the action of the one loop dilatation operator in terms of known Clebsch-Gordan coefficients of $U(p)$. Although the problem of diagonalizing the dilatation operator has not been solved in general, the cases which have been tested again produce a spectrum of decoupled oscillators.

It has previously been observed, most notably for our purposes in [10], that integrability in the planar limit of $\mathcal{N} = 4$ SYM is a powerful tool for the analysis of that theory. The results of this paper provide additional evidence for the conjecture that the large-$N$ limit of the dilatation operator that we consider, which includes non-planar contributions, is also integrable.

# Chapter 2

# Background

## 2.1 The AdS/CFT Correspondence

### 2.1.1 Statement

The AdS/CFT correspondence was originally submitted as a conjecture by J.M Maldacena in his paper [1]. It proposes an exact equivalence between gauge theories in large-$N$ limits and string theories. The most well-known and tested of this class of equivalences is that between Type-IIB String Theory on spacetimes that are asymptotically $AdS_5 \times S^5$ [1]and Conformally Invariant $\mathcal{N} = 4$ Supersymmetric Yang-Mills Field Theory (SYM) on 4-dimensional Minkowski spacetime.

### 2.1.2 Conformal Field Theory

A conformal transformation is a coordinate transformation $x^\alpha \to \tilde{x}^\alpha(x)$ which causes a rescaling of the metric:

$$g_{\alpha\beta}(x) \to \Omega(x)^2 g_{\alpha\beta}(x)$$

A conformal field theory is a field theory which is invariant under these transformations, and hence is scale invariant - physical predictions of the theory do not depend on lengths, only angles. The interpretation of this metric rescaling depends on the properties of the metric; if the metric is the solution to some equations of motion, i.e. is dynamical, the transformation is a diffeomorphism and the conformal symmetry is a gauge symmetry, if the metric is fixed, we have a global symmetry which has associated conserved currents. A key feature of conformal field theories is that the stress-energy tensor associated to the conserved current is traceless, i.e. $T^\mu_\mu = 0$ - this implies scale invariance. The particular CFT we are considering is $\mathcal{N} = 4$ Super Yang-Mills theory.

$\mathcal{N} = 4$ SYM is special in the sense that its $\beta$ function vanishes so that conformal invariance is not just a feature of the classical limit, but extends into the quantum regime. $\mathcal{N} = 4$ (the theory enoys 4 supersymmetries, that is, there are 4 supersymmetry operators $Q_\alpha$) is the maximal supersymmetry possible for theories describing particles with spin $\leq 1$. This is because each of the $Q_\alpha$ couples with spin $\frac{1}{2}$ to the particles of the theory - if all 4 supersymmetries are applied, it can raise the spin of a particle by at most 2. If there were more supersymmetries, the spin of any particle could be increased to that of a graviton, at which point the theory contains a description of gravity. The particle multiplet of this theory contains 1 vector boson, $A_\mu$, 6 scalar bosons $\phi_i$ which transform under the fundamental representation of the group $SO(6)$ and 4 fermions $\lambda_a$ which transform under the fundamental of $SU(4)$. There are 15 generators of the $SO(2,4)$ conformal algebra: the energy-momentum 4 vector $P^\mu$, associated to translations, angular momentum $\vec{L}$ for rotations, $\vec{K}$ for Lorentz boosts, $\mathcal{K}_\mu$ which generate special conformal transformations and the dilatation operator $D$, which generates scale transformations.

This is not the full symmetry of the theory, as it does not include supersymmetry and hence only describes bosonic symmetries. The full superconformal algebra includes a number of other generators, but we will not be concerned with these - states in the theory are completely specified by the energy

[1]The conjecture was originally stated for $AdS_5 \times S^5$, but it has been found that minor deformations in this spacetime do not affect the validity of the conjecture, so long as the spacetime still has this structure near the boundary.

$E$, total($\vec{L} \cdot \vec{L}$)/$z$-component ($L_z$) angular momentum, a set of $SO(6)$ quantum numbers and scaling dimension $\Delta$, as the set of operators for which these quantities are eigenvalues is the maximal set of mutually commuting operators, even after the generators of the supersymmetry are included. For the purposes of this dissertation, only the dilatation operator is of interest.

### 2.1.3 String Theory

String theory, at its essence, attempts to describe all matter as consisting of tiny, fundamental vibrating strings, i.e. it asserts that electrons and quarks are not in fact point particles, but are extended in one dimension. The entire spectrum of known particles can be associated with the vibrational modes of the strings - each possible excitation of a string corresponds to a different particle type. Originally the strings were considered to be bosonic, such that only bosons were described by the theory. The advent of supersymmetry, which allows for transformations between bosonic and fermionic states, led to the possibility of describing all types of particles within the string theoretical framework. The consequences of such a model are vast and often quite strange. The appearance of massless spin-2 mediating bosons, for which the only consistent mode of interaction is gravity, cannot be eliminated. These are the gravitons, and their natural emergence from the mathematics of string theory provides significant motivation to consider this theory as a canditate for integrating gravity with the strange world of quantum mechanics.

This unification has long been sought after in the physics community, but has been fraught with difficulty. This is a consequence of the mutual incompatibility between the theories of quantum mechanics and general relativity; there exist limiting cases in which both theories should apply, but here their predictions diverge and cannot be reconciled. String theory is thought to be an underlying theory which can accurately describe these limiting cases, and from which both QM and GR can be extracted in the regimes at which each provides a full description. Five distinct consistent string theories have been found, named Type I, Type IIA, Type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. Each one of these requires spacetime supersymmetry on $9 + 1$ dimensional backgrounds for their consistency. Duality transformations between these 5 theories were found to relate the 5 theories to each other, and to a particular 11-dimensional supergravity theory that arises as the strong coupling limit of Type IIA string theory. These dualities led to the postulation of M-theory, for which each of the 5 theories above is a limiting case. The significant duality for our purposes is that between open and closed strings - open string theories reduce to field theories without gravity at low energy, while closed strings are described by theories of gravity in this limit.

The inclusion of supersymmetry in string theory leads to the result that the spacetime on which the theory is defined must be $(9 + 1)$-dimensional. Various theories exist which attempt to explain the appearance of our universe as existing on a $(3 + 1)$-dimensional spacetime by considering that the additional dimensions predicted by string theory are "curled up" on a certain 6-dimensional manifold, and hence not observable. These theories claim that our universe, gravitational interactions included, can be entirely described as existing on the remaining 4-dimensional spacetime. However, as will be discussed, the AdS/CFT correspondence together with the holographic principle may in fact provide a far more elegant and natural explanation, albeit with the emergence of a description of gravity in our universe that is less obvious - the CFT defined in our 4-dimensional spacetime does not account for gravity, but operators defined in this theory can be used to describe gravitational effects in the dual string theory.

### 2.1.4 5-dimensional anti-de Sitter Space ($AdS_5$)

A brief description of the spacetime we consider is now given. The full spacetime is a product space of 5-dimensional anti-de Sitter Space ($AdS_5$) with a 5-dimensional sphere ($S^5$). A 5-sphere of radius $R$ is simply the set of points which are a distance $R$ from a fixed point in 6-dimensional Euclidean space. The $AdS_5$ is a maximally symmetric Lorentzian manifold with constant negative scalar curvature. In plain english, it is a general relativity-like spacetime where time and space are mathematically equivalent in all directions (there are obviously still distinctions between time and space, such as the sign in the metric, but the space possesses the most symmetries possible between them), which is curved such that the curvature is constant across the entire spacetime in the absence of energy (empty spacetime), and is negative. The space essentially describes "gravity in a box"; if one were to stand at the center and throw an object in any direction, the object would always return to the center. A negative curvature

corresponds to an attractive force, and it can be thought that $AdS_5$ has an inherent negative energy associated to it, even when empty.

**Mathematical Description** There exists a broad class of homogeneous spaces that can be described as a quadric surface (a $D$-dimensional hypersurface in $D+1$-dimensional space for which all points on the surface are zeros of a quadratic polynomial) embedded in a flat vector space. $AdS_5$ is such a space, as it can be mathematically defined as the quadric

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 - (t^1)^2 - (t^2)^2 = R^2 \tag{2.1}$$

embedded in 6-dimensional flat Minkowski space with metric

$$dS^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 - (dt^1)^2 - (dt^2)^2$$

For our purposes, it is useful to calculate the induced metric on the submanifold in what are known as global co-ordinates (so named because they cover the entire manifold). This entails using the parametrization:

$$\begin{aligned}
t^1 &= R\text{sinh}\rho\text{cos}\tau \\
t^2 &= R\text{sinh}\rho\text{sin}\tau \\
X^1 &= R\text{cosh}\rho\text{sin}\theta\text{sin}\beta\text{sin}\alpha \\
X^2 &= R\text{cosh}\rho\text{sin}\theta\text{sin}\beta\text{cos}\alpha \\
X^3 &= R\text{cosh}\rho\text{sin}\theta\text{cos}\beta \\
X^4 &= R\text{cosh}\rho\text{cos}\theta
\end{aligned}$$

It is easily verified that this satisfies equation 2.1, and that the induced metric can be written:

$$dS^2 = R^2 d\rho^2 + R^2\text{sinh}^2\rho d\tau^2 - R^2\text{cosh}^2\rho d\theta^2 - R^2\text{cosh}^2\rho\text{sin}^2\theta d\beta^2 - R^2\text{cosh}^2\rho\text{sin}^2\theta\text{sin}^2\beta d\alpha^2$$

One can check that this metric is nondegenerate and has Lorentzian signature.[11]. Another important set of coordinates to consider is the Poincare coordinates. Though with this coordinate choice, the space is divided into two regions that cannot be described simultaneously, it is nonetheless very useful in our study of the AdS/CFT correspondence - the boundary of the $AdS_5$ has a different structure when described by Poincare coordinates, and the correlation functions of the super Yang-Mills theory can be calculated with the interpretation that the CFT lives on this boundary ([12],[13]). The metric of $AdS_5$ can be obtained in Poincare coordinates by first introducing the light cone coordinates ([14]):

$$\begin{aligned}
u &= \frac{t^1 - X^4}{R^2} \\
v &= \frac{t^1 + X^4}{R^2}
\end{aligned}$$

The coordinates not included in these expressions are defined as $x^i = \frac{X^i}{Ru}$ and $t = \frac{t^2}{Ru}$, which yields the following equation upon substitution into 2.1:

$$R^4 uv + R^2 u^2 (t^2 - \bar{x}^2) = R^2$$

where $\bar{x}^2 = \sum_{i=1}^{3} (x^i)^2$. This allows us to write $v$ in terms of $u$, $t$ and $x^i$. After making the substitution $z = \frac{1}{u}$, we obtain the coordinate transformation from which we can calculate the induced metric:

$$\begin{aligned}
t^1 &= \frac{1}{2z}(z^2 + R^2 + \bar{x}^2 - t^2) \\
t^2 &= \frac{Rt}{z} \\
X^i &= \frac{Rx^i}{z} \qquad i = 1..3 \\
X^4 &= \frac{1}{2z}(z^2 - R^2 + \bar{x}^2 - t^2)
\end{aligned}$$

The metric of $AdS_5$ in Poincare coordinates is then:

$$ds^2 = \frac{R^2}{z^2}(-dz^2 - (d\bar{x})^2 + dt^2)$$

We see that the space is divided into two Poincare charts, the first being the region $z > 0$, the second $z < 0$. Usually the $z > 0$ region is used, and the Poincare $AdS$ space is then the region of the full space corresponding to that chart.

$AdS_5$ in Poincare patch coordinates has 4D Minkowski space as its boundary (note this is the spacetime on which our SYM theory is defined). Performing a Wick rotation, this can be transformed to 4D Euclidean spacetime, having metric (note that by Wick rotation, this becomes 4D Minkowski space, the spacetime on which our SYM theory is defined)

$$ds^2 = dt^2 + d\bar{x}^2 \quad ,$$

which can be written in spherical coordinates as

$$ds^2 = dr^2 + r^2 d\Omega_3^2.$$

If we make the substitution $t = \ln r$, we obtain

$$ds^2 = e^{2t}(dt^2 + d\Omega_3^2) \quad ,$$

on which we can apply a conformal transormation to find

$$ds^2 = dt^2 + d\Omega_3^2 \quad ,$$

the metric of $R \times S^3$. This is the boundary of the $AdS_5$ in global coordinates. The existence of this relationship is central to our cause, as it allows for the identification between operators in the SYM theory and states in the string theory (This is discussed further in 5.2). One may wonder how it is that we can make such identifications, given that we have only shown a correspondence with the **boundary** of the string theory spacetime. The central identification necessary to obtain the important results of this paper is between the Hamiltonian in the string theory and the dilatation operator in the Yang-Mills theory; this hinges only on the equivalence of the time dimension of the $AdS_5$ boundary in global and Poincare coordinates, and does not require any equivalence between spatial dimensions. That we present the Schur Polynomials (2.3) in the Yang-Mills theory as dual to Giant graviton states in string theory, and claim that they can be used to describe the full dynamics of these objects, is only possible due to the AdS/CFT correspondence.

### 2.1.5 Heuristic Motivation for the Conjecture

The argument arises by the consideration of the low energy limit of a system of N parallel D3 branes existing in a 10-dimensional spacetime ([1], reviewed in [15]). Energies we consider must be smaller than the string energy scale, $\frac{1}{l_s}$, or alternatively

$$E \ll \frac{1}{\sqrt{\alpha'}}$$

Working in this limit, we need only consider massless excitations of the system of D3 branes. The two possible excitations that are relevant in this limit are closed string excitations which propagate throughout the bulk of the 10 dimensional spacetime, and open string excitations on the branes themselves. The open strings on the branes do not oscillate due to the low energies, and look like particles in this limit. This implies that the theory describing dynamics on the branes should be one of quantum fields which admits the symmetries relevant to a theory living on a D3 brane. It turns out that this is a maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory with $U(N)$ gauge group. The closed strings are massless states of a theory of Type IIB supergravity. The dynamics on the brane is decoupled from the dynamics in the bulk in the low energy limit.

We now consider the same system from a different point of view; one where the D3 branes are considered as solutions to the equations of type IIB supergravity - massive charged objects which deform spacetime and are sources of closed strings. The geometry of these solutions in the space of the dimensions transverse to the brane can be viewed as consisting of an infinite throat at the D3-brane surrounded by flat Minkowski space time. If we consider closed strings in this configuration, as perceived by an observer at infinity [2], two possible excitations are apparent; those far from the throat horizon, in the bulk Minkowski spacetime, and those emanating from near the horizon. The excitations in the bulk will be described by Type IIB supergravity. Those near the horizon are redshifted, such that finite energy excitations of the strings appear to our observer to fall in the low energy limit. These two types of excitations decouple. Studying the metric of the gravitational solution describing the D3 brane system reveals that the near-horizon geometry is in fact $AdS_5 \times S^5$.

So we see that from both viewpoints, the low energy limit results in the D3 brane system splitting into two decoupled subsystems. In both cases, the closed string excitations far from the branes are described by Type IIB supergravity on 10 dimensional Minkowski spacetime. The AdS/CFT conjecture proposes that the other subsystems should exhibit an equivalence too. This is how the conclusion that $\mathcal{N} = 4$ SYM with $U(N)$ gauge group and Type IIB String theory on an $AdS_5 \times S^5$ background should be equivalent was reached. The strong form of the conjecture states in addition that it should hold for all values of the gauge group rank $N$ and 't Hooft coupling $\lambda$.

### 2.1.6 Discussion

The argument presented above is valid for the specific case of the duality that we consider; between Type-IIB String Theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM on 4-d Minkowski spacetime. A more general argument indicating a connection between string theories and large $N$ gauge theories is now briefly explained. The observation that $SU(N)$ gauge theories simplify in the large $N$ limit was first made by Gerard 't Hooft in [16], where it is shown that in this limit an expansion in $\frac{1}{N}$ is admitted. It was also shown that the perturbative expansion for any Feynman diagram of the gauge theory in this limit has precisely the same form as the perturbative expansion over surfaces of increasing genus obtained from oriented closed string theory with string coupling equal to $\frac{1}{N}$. This is compelling evidence that string theories and field theories are related, at least in the perturbative regime.

Another interesting point to note is that the AdS/CFT correspondence is a weak/strong coupling correspondence [17]. The parameters of the two theories are related by:

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4}$$

Perturbation theory in the SYM theory requires $g_{YM}^2 N \ll 1$ for its validity, while in string theory $\frac{R^4}{l_s^4} \gg 1$ is the prerequisite for weakly curved geometries. Due to this, and the given relationship between the parameters of the two theories, studying one of the theories at weak coupling gives insight into the other theory at strong coupling. It is thus possible to make statements about the inaccessible, non-perturbative sector of one theory by performing calculations in the well-understood perturbative sector of the other. This makes the correspondence potentially very powerful, however, it also introduces a difficulty in testing the validity of the correspondence; one must find objects in both theories with some properties which are protected from corrections when extrapolating the perturbative result to the non-perturbative regime. One class of objects having such a protected property are the *D-branes*.

### 2.1.7 *D*-branes

By studying the equations of motion of string theory, one finds that open string endpoints must satisfy one of two types of boundary conditions: Neumann, corresponding to the endpoints being free to move through spacetime, and Dirichlet, where the endpoints are constrained to a fixed submanifold. $D$-branes (introduced in [18]) are objects that have arisen in string theory due to the assignment of Dirichlet boundary conditions to the endpoints of open strings. A number indicating the number of spatial dimensions of

[2] We must make observations relative to an observer at infinity, for reasons due to mathematical concerns. When calculating the variation of the action in a theory of gravity, the only terms that survive are surface terms arising from integration by parts. Hence, invoking Noether's Theorem, any observable quantity must be associated to these term, so that only an observer at the surface (i.e. at infinity) is able to measure it

the brane is often appended to the name, so that a $Dp$-brane has $p$ spatial dimensions. We can imagine that we have open strings propagating in a $(p+q)$-dimensional spacetime, and that we require the open strings to satisfy Dirichlet boundary conditions in $q$ of the coordinates. The strings then satisfy Neumann boundary conditions in the other $p$ co-ordinates, implying that they are free to move on a $p$-dimensional hypersurface - this hypersurface provides a description of a $Dp$-brane.

It was also shown in [18] that the conformal dimension of these branes are in fact protected, because they are BPS states. This is a consequence of supersymmetry and the quantization of $\mathcal{R}$-charge. BPS states are states for which the condition $\Delta = J$ is satisfied - the conformal dimension is equal to the $\mathcal{R}$-charge. The anticommutator relation of the supersymmetry operators is proportional to $(\Delta - J)$, thus the supersymmetry multiplet of BPS states is shortened - the actions of some combinations of the supersymmetry operators are null; for instance, "$\frac{1}{2}$-BPS" refers to the case where half the states in the supersymmetry multiplet vanish. $\mathcal{R}$-charge is known to be quantized and hence cannot vary smoothly, and we must infer that the conformal dimension also has this property. Thus, small variations in the value of the coupling will not in fact change the value of $\Delta$, hence we say that this quantity is protected for BPS states. The conformal dimension of these objects can thus be reliably extrapolated from weak to strong coupling, which makes them ideal candidates for probing the AdS/CFT correspondence.

We hence can see that BPS $D$-branes provide a satisfactory means by which the AdS/CFT conjecture can be tested. This provides some motivation for the study of operators dual to Giant Gravitons, which are $D3$-branes wrapping an $S^3$, for which the relevant protected charge is angular momentum. To summarise, in the Yang-Mills thory conformal dimension is protected when extrapolated from weak to strong coupling because it is equal to $\mathcal{R}$-charge for BPS states, and $\mathcal{R}$-charge is quantized. Thus, on the string theory side, angular momentum on the brane is equal to the energy, which is dual to the conformal dimension. Since renormalized quantities are those for which loop corrections must be applied, we have a case of non-renormalization. For our purposes, we can also apply this same logic to non-BPS states, since we study systems where $N$, and hence $J$, go to infinity - these states are "nearly BPS", since small corrections in this limit dont affect the BPS properties of the state. We are in fact calculating the small corrections to $\Delta$ in this dissertation, with the interpretation that they correspond to excitations of the Giant graviton state.

## 2.2 Schur-Weyl Duality

Schur-Weyl duality ([19],[20]) is a mathematical theorem that provides, for example, a relation between finite dimensional irreducible representations of the general linear group and those of the symmetric group. The rank $n$ unitary group $U(n)$ is the maximal compact subgroup of $GL(n,C)$. Irreducible representations of $U(n)$ have the same dimension as those of $GL(n,C)$, so that Schur-Weyl duality can also be used to relate irreducible symmetric group representations to those of the unitary group ([21]).

Consider the fundamental representation of $U(p)$ (this is the irreducible representation corresponding to the Young diagram containing only a single box; it can be seen as the elementary representation from which all others can be constructed); associated to this representation is a vector space $V_p$ which is known as the carrier space of the representation: it is the vector space in which we can define the group elements to act as matrices. The space will have $p$ states, labelled $\vec{v}(i), i = 1..p$. Now consider taking $n$ copies of this vector space; the result is the space $V_p^{\otimes m}$, which will have $p^m$ states of the form

$$\vec{v}(i_1) \otimes \vec{v}(i_2) \otimes \vec{v}(i_3) \otimes \cdots \otimes \vec{v}(i_{m-1}) \otimes \vec{v}(i_m)$$

The action of the group $U(p)$ on states of this form is to mix the elements of each vector state in the tensor product ($\Gamma(U)$ is the $p \times p$ unitary matrix representing element $U \in U(p)$ in the fundamental representation):

$$U \cdot (\vec{v}(i_1) \otimes \vec{v}(i_2) \otimes \cdots \otimes \vec{v}(i_m)) = \Gamma(U)\vec{v}(i_1) \otimes \Gamma(U)\vec{v}(i_2) \cdots \Gamma(U)\vec{v}(i_m)$$

The action of the symmetric group $S_n$ on these states is:

$$\sigma \cdot (\vec{v}(i_1) \otimes \vec{v}(i_2) \otimes \cdots \otimes \vec{v}(i_m)) = \vec{v}(\sigma(i_1)) \otimes \vec{v}(\sigma(i_2)) \otimes \cdots \otimes \vec{v}(\sigma(i_m))$$

Clearly, the actions of these two groups commute:

$$
\begin{aligned}
U \cdot \Gamma\left(\sigma \cdot \left(\vec{v}(i_1) \otimes \cdots \otimes \vec{v}(i_m)\right)\right) &= U \cdot \left(\vec{v}\left(\sigma(i_1)\right) \otimes \cdots \otimes \vec{v}\left(\sigma(i_m)\right)\right) \\
&= \Gamma(U)\vec{v}\left(\sigma(i_1)\right) \otimes \cdots \otimes \Gamma(U)\vec{v}\left(\sigma(i_m)\right) \\
&= \sigma \cdot \left(\Gamma(U)\vec{v}\left(i_1\right) \otimes \cdots \otimes \Gamma(U)\vec{v}\left(i_m\right)\right) \\
&= \sigma \cdot \left(U \cdot \left(\vec{v}(i_1) \otimes \cdots \otimes \vec{v}(i_m)\right)\right)
\end{aligned}
$$

To be precise, the action of the group algebra of $S_n$ fills out the complete set of operators that commute with $U(p)$, while the commutant of $S_n$ (the set of operators in $V_p^{\otimes n}$ that commute with the group elements of $S_n$) is the enveloping algebra of $u(p)$. The double commutant theorem then allows us to extract the concrete result of Schur-Weyl duality - The vector space $V_p^{\otimes n}$ can be simultaneously decomposed in terms of Young diagrams ($\Lambda$), each of which labels a representation of both the unitary and the symmetric group, into a direct sum of tensor products of carrier spaces of these representations([22],[23]):

$$V_p^{\otimes m} = \oplus_\Lambda V_\Lambda^{U(p)} \otimes V_\Lambda^{S_m}$$

The sum over $\Lambda$ runs over all Young diagrams built from $n$ boxes and having at most $p$ rows. The consequences of this duality are vast, and it will be used extensively in this dissertation. One consequence of this formula is that

$$p^m = \sum_s \mathrm{Dim}(s)\, d_s$$

where $\mathrm{Dim}(s)$ is the dimension of $s$ as an irreducible representation of $U(p)$ and $d_s$ is the dimension of $s$ as an irreducible representation of $S_m$.

## 2.3 Schur Polynomials

### 2.3.1 Definition

These operators are constructed from complex combinations of the six scalar adjoint Higgs fields of the SYM, and labelled by Young diagrams corresponding to irreducible representations of the symmetric group. The reason we must use complex combinations is to prevent mixing between the component scalar fields, thus causing the expectation value of the $Z$'s and $Y$'s with themselves to vanish. If these expectation values did not vanish, we would be required to Wick contract these fields within a single Schur polynomial, resulting in the occurrence of UV divergences that we cannot resolve. Ground state giant graviton systems can be described using Schur Polynomials built using a single complex combination of two of the scalar adjoint Higgs fields of the SYM theory([2]), for example, we define $Z = \phi_1 + i\phi_2$. Then the operator has the form

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}(\sigma Z^{\otimes n})$$

where $R$ is a Young diagram, having $n$ boxes and at most $N$ rows ($N$ is the rank of the gauge group of the Yang-Mills theory), and $\chi_R(\sigma)$ is the character of $\sigma$ in the representation $R$, being the trace over the matrix representation in the carrier space of $R$ of the group element $\sigma$. The Schur polynomials that are useful for the puproses of this dissertation are those corresponding to giant graviton systems - in this case, the number of boxes in the rows (columns) of the Young diagram labelling the state is $O(N)$, with the number in each column (row) being $O(1)$ for AdS (Sphere) giants (See 3.3).

### 2.3.2 A Generalization: Restricted Schur Polynomials

#### Definition

A generalization of these operators, the Restricted Schur Polynomials, have been useful in describing excitations of giant graviton systems. Originally it was thought that this could be achieved by inserting open string words into the operator describing the giant graviton system (corresponding to excitations arising by the attachment of strings to the surface of the graviton)([24]), however, computational studies showed that this results in an instability which causes the string to appear to dissolve into the graviton[25]. The current method is instead to define the restricted Schur polynomial in terms of some number of different complex scalar field combinations (for our purposes, we work with models in which only two of the field combinations participate; we choose $Z = \phi_1 + i\phi_2$ and $Y = \phi_3 + i\phi_4$) , yielding a multi-matrix

model. It is not clear what the exact interpretation of these added "impurities" should be, although sufficient evidence exists to confidently associate them with excitations of the giant graviton. These operators are labelled by a set of 3 Young diagrams:

$$\chi_{R,(r,s)jk}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)jk}(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}Y^{\otimes m})$$

$R$ labels an irreducible representation of $S_{n+m}$, $r$ an irreducible representation of $S_n$ and $s$ an irreducible representation of $S_m$. The latter two Young diagrams together label an irreducible representation of $S_n \times S_m$. $R$ may in general subduce the same irreducible representation of the $S_n \times S_m$ subgroup more than once - the indices $jk$ are multiplicity labels which keep track of these copies, necessary when studying systems containing more than 2 gravitons. Analogous to the character appearing in the Schur polynomial, we define a *restricted character* $\chi_{R,(r,s)jk}(\sigma)$.

**The Restricted Character**

A brief overview of what it means to take a restricted trace is now given. The restricted character is defined as the trace over the carrier space of the representation $(r,s)$, a subspace of the vector space associated with $R$. It is here that the multiplicity indices are important; the trace must be performed over the space associated with the correct copy of the subgroup representation. Suppose that upon restricting $R$ to the $S_n \times S_m$ subgroup, we have irreducible representation $(r,s)$ subduced once, and irreducible representation $(t,u)$ is subduced twice. The matrix representation $\Gamma_R(\sigma_{nm})$ of an $S_n \times S_m$ group element $\sigma_{nm}$ can then be written in a suitable basis in block diagonal form, with the diagonal blocks being the matrix representations of the subgroups:

$$\begin{bmatrix} \Gamma_{(r,s)}(\sigma)_{i_1j_1} & 0 & 0 \\ 0 & \Gamma_{(t,u)}(\sigma)_{i_2j_2} & 0 \\ 0 & 0 & \Gamma_{(t,u)}(\sigma)_{i_3j_3} \end{bmatrix}, \qquad \sigma \in S_n \times S_m$$

In this case, the restricted character can be obtained simply by summing over the diagonal elements of the matrix on the diagonal corresponding to the subgroup representation we want. It would seem that only one multiplicity index is required - to specify which copy of $(t,u)$ we want to trace over. However, not all $S_n$ elements that are summed over will be members of the $S_n \times S_m$ subgroup, and it will not generically be possible to block diagonalize the matrix $\Gamma_R(\sigma)$:

$$\begin{bmatrix} A^{(1,1)}_{i_1j_1} & A^{(1,2)}_{i_1j_2} & A^{(1,3)}_{i_1j_3} \\ A^{(2,1)}_{i_2j_1} & A^{(2,2)}_{i_2j_2} & A^{(2,3)}_{i_2j_3} \\ A^{(3,1)}_{i_3j_1} & A^{(3,2)}_{i_3j_2} & A^{(3,3)}_{i_3j_3} \end{bmatrix}, \qquad \sigma \notin S_n \times S_m$$

We now see the need for two multiplicity indices. The restricted trace over the subspace $(t,u)$ can conceivably be performed over any of the 4 lower right entries of the above matrix. To compute the restricted character $\chi_{R,(r,s),jk}(\sigma)$, we trace the row index of $\Gamma_R(\sigma)$ only over the subspace associated to the $j^{\text{th}}$ copy of $(t,u)$ and the column index over the subspace associated to the $k^{\text{th}}$ copy of $(t,u)$. When performing the "trace" over the carrier space of $(t,u)$ the row and column indices can come from different copies of $(t,u)$ so that if $i \neq j$ we are not in fact summing diagonal elements of $\Gamma_R(\sigma)$. Operators constructed by summing these "off diagonal" elements are needed to obtain a complete basis of local operators [**?**]. The restricted character is obtained from the character by the introduction of projection operators, $P_{R\to(r,s)jk}$, which take a matrix representation of the group element in the carrier space of $R$, and project it to the carrier space of the $(r,s)$ subgroup. They obey

$$\begin{gathered} \Gamma_{(r,s)j}(\sigma)P_{R\to(r,s)jk} = P_{R\to(r,s)jk}\Gamma_{(r,s)k}(\sigma) \qquad \sigma \in S_n \times S_m \\ \Gamma_{(r,s)l}(\sigma)P_{R\to(r,s)jk} = 0 = P_{R\to(r,s)jk}\Gamma_{(r,s)q}(\sigma) \quad \sigma \in S_n \times S_m \quad l \neq j, \quad k \neq q \end{gathered}$$

We can write the restricted character in terms of these operators as

$$\chi_{R,(r,s),ji}(\sigma) = \mathrm{Tr}\left(P_{R\to(r,s)ji}\Gamma_R(\sigma)\right)$$

When there are no multiplicities, $P_{R\to(r,s)jk} = P_{R\to(r,s)}$ is an honest projection operator which projects from the carrier space of $R$ to the $(r,s)$ subspace. When there are multiplicities $P_{R\to(r,s)jk}$ is an intertwiner[20] - we are projecting onto one of the copies of an $(r,s)$ subspace, and the non-zero component of the matrix action of the operator will not necessarily be on the diagonal (5.3.1). However,it

is constructed in essentially the same way as a projector and satisfies very similar identities. For these reasons we will sometimes be guilty of an abuse of language and refer to $P_{R\to(r,s)jk}$ simply as a projector even when there are multiplicities. These operators are not easy to construct explicitly, and this is the most significant obstacle when working with the restricted Schur polynomials. The new version of Schur-Weyl duality presented in this dissertation provides an efficient, transparent method by which these operators can be built. This construction is discussed in detail in section 4.1.

**The Multiplicity Problem**

The multiplicity indices $jk$ must be chosen such that they take the correct values to organise the multiplicities arising by the subduction of $S_n \times S_m$ representations for systems containing $p > 2$ gravitons. A proposal to resolve these multiplicities is given in [26], where they are labelled by the eigenvalues of the Cartan subalgebra of elements in the group algebra $CS_{n+m}$ which are invariant under conjugation by $CS_n \times CS_m$. However, we have found that a much simpler method manifests itself by considering our new version of Schur-Weyl duality.

The usual application of Schur-Weyl duality is in the construction of projectors onto good $U(p)$ irreducible representations using the Young symmetrizers i.e. by symmetrizing and antisymmetrizing indices on a tensor. The use of the duality as a means by which we can construct the projection operators appearing in the restricted Schur polynomials, while also resolving this multiplicity problem, turns this argument on its head: by using the irreducible representations of the unitary group, we are able to build symmetric group projectors. The reader should bear in mind that the details of our Schur-Weyl duality are different to the usual construction.

# Chapter 3

# Giant Gravitons

The first description of Giant Gravitons appeared in [27]. They are spherical $D3$ branes which orbit in either of the two component factors of the $AdS_5 \times S^5$ background (although initially only the aptly named sphere Giants were known, $AdS$ counterparts were quick to follow in [28] and [29]), and are the result of the expansion of point gravitons due to the presence of the background Ramond-Ramond five form flux, by a process analogous to the Myers' effect for dielectric branes [4]. This chapter is intended as a brief summary of the present content of the theory of giant gravitons.

## 3.1 The Stringy Exclusion Principle

### 3.1.1 Development

A study of a particular example of the AdS/CFT correspondence relating near-horizon microstates of black holes (obtained as orbifolds of a subset of $AdS_3$) to the states of a conformal field theory [30] led to the discovery that there exists an upper bound on the BPS particle number in the space, following from the unitarity of the superconformal algebra. This is because the requirement of unitarity implies that states in the theory must have positive norm. If one observes the norm of states in our Super Yang-Mills theory as their energy is raised, it is found that the norm remains positive up to a certain point, after which it remains negative - these states of negative norm are not valid for the theory, and hence must be excluded. The existence of this upper bound is commonly referred to as the Stringy Exclusion Principle. The situation is different to another well known exclusion principle (that of Pauli), in that the states are not excluded only by the possibility of their occupancy, but simply do not exist. The physical origins of this principle have never been clear, but they were long thought to be associated with physics at very small distance scales. Although the results were obtained in the context of an $AdS_3$ geometry, the Stringy Exclusion Principle is a property of any theory where operators are built out of matrices. This can be shown by considering the traces of the powers of matrices - suppose we consider a $2 \times 2$ matrix $X$ with eigenvalues $\lambda_1$ and $\lambda_2$, then:

$$\mathrm{Tr}(X) = \lambda_1 + \lambda_2$$
$$\mathrm{Tr}(X^2) = \lambda_1^2 + \lambda_2^2$$

It is useful for us to consider traces, since these are the natural observables of the SYM theory - in order to be gauge invariant, observables in this theory must be invariant under the action of multiplication by unitary matrices, i.e. the gauge transformation is of the form $Z \to UZU^\dagger$. The fact that traces are cyclic implies this invariance for operators built out of traces. It is now a simple algebraic matter to determine that $\mathrm{Tr}(X^3)$ can be written in terms of the first two traces - this implies that the state to which this trace corresponds in not in fact a new state, but rather a bound state of the other two systems. This can be logically extended to the traces of all higher powers of $X$, so that we see the natural emergence of an upper bound on the number of particle states arising due to the fact that we are working with matrices. Upper bounds on possible particle states were previously encountered in other string theoretical contexts, including the case of the duality between IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM.

### 3.1.2 Invasion of the Giant Gravitons

The result of [27] is the acquisition of a new perspective on this matter, one in which the principle emerges as a macroscopic effect, and the physical meaning of the bound is clear. It is considered that the massless single particle states to which the bound is applicable (i.e. the gravitons) in the $S^5$ component of the background expand as their angular momentum is increased. The 5-sphere has a fixed radius, so that the expansion must stop when the radius of the graviton matches that of the sphere. This cut-off in angular momentum was found to agree with the predictions of the stringy exclusion principle, thus validating the theory. Since the Kaluza-Klein (ordinary point-like) graviton is a BPS state, its transformation from point to membrane should not change its energy, and it would be expected that the energy calculated at a given momentum under this interpretation should match the energy of a KK graviton having that same momentum. The fact that these were found to match classically for the case of maximal angular momentum lends further credit to the theory.

### 3.1.3 Derivation

In order to best understand how this process occurs, it is useful to review another case where particles undergo spatial extension proportional to their angular momentum - non-commutative field theories. These are field theories in which the operators corresponding to the spacetime coordinates do not commute with each other. The basics of the theory are embodied in the case of a dipole moving through a magnetic field, which we can define on the surface of a 2-sphere to observe the emergence of an angular momentum bound corresponding to the separation of the ends of the dipole to the antipodes of the sphere. Below is a short summary of the physics of this situation:

#### Dipole Moving in a Magnetic Field

The setup we consider consists of a pair of unit charges of opposite sign that are moving on a plane in a constant magnetic field $B$. The Lagrangian is given as:

$$\mathcal{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{B}{2}\epsilon_{ij}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2}(x_1 - x_2)^2$$

Coulomb and radiation contributions are assumed to be negligible. The terms correspond respectively to kinetic energy, interaction with the background magnetic field and the harmonic potential between the charges. We make the approximation that the identical particle masses are very small, so that the first term vanishes. Note that, since we are assuming negligible mass, we should in fact perform a relativistic analysis. It would also be more correct to perform the analysis, and then set the mass equal to zero. However, this happens not to affect the outcome, and so this section will be presented as it was read in [27]. It is useful to introduce center of mass and relative coordinates, defined as:

$$X = \frac{(x_1 + x_2)}{2}$$
$$\Delta = \frac{(x_1 - x_2)}{2}$$

Applying these changes to the Lagrangian we obtain:

$$\mathcal{L} = B\epsilon_{ij}\dot{X}^i\Delta^j - 2K\Delta^2$$

Using this equation to calculate the commutator $[P, X]$, together with the known fact that this must equal $-i$ (when $\hbar$ is set to one), it is possible to determine that the operators $X$ and $\Delta$ are non-commuting, and satisfy the relation

$$[X^i, \Delta^j] = i\frac{\epsilon_{ij}}{B}$$

The center of mass momentum conjugate to $X$ is given by

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{X}^i} = B\epsilon ij\Delta^j$$

Noting that the coordinate $\Delta$ gives the position of the particles relative to each other, we can rearrange this equation and take absolute values to obtain a formula for the distance between the particles:

$$|\Delta| = \frac{|P|}{B} \tag{3.1}$$

The particles thus separate in the direction perpendicular to the momentum vector by an amount linearly proportional to the momentum of the dipole.

We now imagine that we place the dipole on the surface of a sphere of radius $R$, which has magnetic flux $N$. This is equivalent to the statement that we place a magnetic monopole of strength $2\pi N = \Omega_2 BR^2$ at the center of the sphere. In this arrangement, by symmetry, the center of mass of the dipole will remain on the equator as the components of the dipole separate. By simply glancing at 3.1, we know that the dipole should be as big as the sphere when its momentum is about $2BR$. This corresponds to an angular momentum $L = PR \sim BR^2$. Comparing with the expression for the flux $N$, we see that the angular momentum can be said to be $O(N)$. A more precise analysis of the situation yields the result we want - the maximum angular momentum of the dipole is in fact exactly equal to the flux.

$$|L_{max}| = N$$

This was demonstrated in [27]. It should be noted that this maximum occurs when the dipole ends sit at opposite poles of the sphere, where they are in fact stationary - this may seem odd, but one must recall that the magnetic field carries angular momentum as a result of its coupling with the dipole charges - the angular momentum is in fact associated with the field itself. The fact that the angular momentum of a single field quantum moving in a spherical space in a non-commutative field theory is bounded by $N$ is well known.

**Dielectric Branes**

A brief review of the methods presented in [27] for the case of $AdS_5 \times S^5$ is included following this discussion. The treatment follows a tight analogy with the calculation performed above for the dipole. Before beginning, it is useful to understand how we are able to use such an analogy - in what sense does a $D3$ brane possess a dipole charge? Consider a point particle moving along a worldline described by $x^\mu$ with 1-form gauge potential $A_\mu$; the particle couples to the potential via a term in the action of the form

$$e \int A_\mu dx^\mu.$$

$e$ represents the charge or coupling constant. Note that if the parameterization is changed such that $dx^{\mu\prime} = -dx^\mu$, then the particle will appear to couple to the potential with a negative charge. Appropriate convention choice in this regard allow the charge on a particle to be well defined.

We consider a BPS particle moving on the spherical component of the space, where the background tensor field plays a similar role to the magnetic field considered previously. This background field is the Ramond-Ramond field, and since we know that D-branes carry RR charge [18], we must include a term in the action that accounts for the coupling of brane to background field - this can be referred as the Chern-Simons term, since it resembles the proper Chern-Simons action defined for $(2+1)$ dimensional space. A $Dp$ brane is naturally charged under the $(p+1)$-form RR potential $A_{(4)}$, so that the Chern-Simons action should have the form:

$$S_{CS} = \mu_3 \int_{D3} A_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \mu_3 \int_{D3} A_{(4)}$$

$\mu_3$ is the RR charge of the $D3$ brane. We can define a 5-form field strength for this potential, and it is related to the RR potential by a superposition of derivatives of the potential which is a natural generalisation of the expression for the 2-form field strength in electrodynamics. Denote this as:

$$F_{\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau v} \partial^v A, \quad \text{i.e.} \quad F_{(5)} = dA_{(4)}$$

There are a number of ways to choose $A$ such that this relation is satisfied. The one that is of interest to us is that in which variables that effectively parameterize the $D3$ brane appear - the most convenient parameterization of the brane is in terms of $t$ and $\Omega_3$, and the potential can be written:

$$A_{(4)} = BR\frac{r^4}{4} d\phi d\Omega_3 = BR\dot{\phi}\frac{r^4}{4} dt d\Omega_3$$

Now consider that we begin with a small $D3$ brane and pick a specific point on its surface, and suppose that $d\psi$ is one of the angles parameterizing $d\Omega_3$. If we traverse the surface of the brane in the direction of the infinitesimal vector $d\psi$, we find that by the time we reach the antipode of the original position, $d\psi$ must have changed sign. Since this is a parameterization variable, this change in sign implies that a

negative is picked up by the Chern-Simons term in the action, which can be interpreted as corresponding to a change in the sign of the charge. Thus, the RR charge on the the surface of a $D$-brane appears as being of opposite sign at antipodal points on the brane, and antipodal points will therefore expand in opposite directions as the brane moves through the background. This explains why the dipole analogy is instructive, and why a graviton moving through a background RR potential expands with increasing angular momentum.

**Angular Momentum Bound for BPS Particles on $AdS_5 \times S^5$**

The supergravity equations of motion give the radius of the 5-sphere as

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s$$

where $g_s$ and $l_s$ are the string coupling constant and string length scale respectively. An exact classical analysis of a $D3$ brane wrapping an $S^3$ and moving in the 5-sphere background is performed. The bosonic Lagrangian for the system is given as the sum of the Dirac-Born-Infeld (DBI) Lagrangian corresponding to kinetic energy of the $D3$ brane and the Chern-Simons Lagrangian which is associated with the coupling of the brane to the background field:

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{CS} = -T_{D3}\Omega_3 r^3 \sqrt{1 - (R^2 - r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}$$

**The Dirac-Born-Infeld Action**

The derivation of the DBI term in the brane Lagrangian is now presented. Ignoring the world volume gauge field on the $D3$ brane, the DBI Lagrangian is given by

$$S_{DBI} = -T_{D3} \int_{WV} \sqrt{\det(|g_{ind}|)}$$

The integral is over the worldvolume of the brane, and $g_{ind}$ is the metric induced on the $D3$ brane worldvolume. We begin by embedding the 5-sphere in a $6D$ Euclidean space parameterized by $X_1, ..., X_6$. The usual coordinate transformation to a set of 5 angles $\theta_1, .., \theta_5$ that satisfies the sphere equation $\sum_{i=1}^{6} X_i^2 = R^2$ is used. The first 4 angles range from 0 to $\pi$, while the azimuthal angle $\theta_5$ ranges from 0 to $2\pi$. Since we are interested in the action for a $D3$ brane on the $S^5$, we now embed a 3-sphere in the space. The surface of the spherical membrane can be parameterized by the angles $\theta_3, \theta_4, \theta_5$, so that the brane is free to move in the $X_1 - X_2$ plane. The radius of the brane depends on its position in this plane according to[1]:

$$r = R \sin\theta_1 \sin\theta_2$$

It is useful to note that

$$X_1^2 + X_2^2 = R^2 - r^2,$$

since this tells us that the brane is free to move in circles on the $X_1 - X_2$ plane without its radius changing, and allows us to write:

$$X_1 = \sqrt{R^2 - r^2} \cos\phi$$
$$X_2 = \sqrt{R^2 - r^2} \sin\phi$$

The metric of the $S^5$ is then given by:

$$ds^2_{S_5} = \frac{R^2}{(R^2 - r^2)} dr^2 + (R^2 - r^2) d\phi^2 + r^2 d\Omega_3^2$$

The metric is now in the form where the coordinates can be easily interpreted as corresponding to a 3-sphere moving on a 2-disc with radial coordinate $r$ and angular coordinate $\phi$. All that is left is to calculate $g_{ind}$; this is achieved by embedding a 4-dimensional worldvolume, being parameterized by $t$ and

[1]Note that this limitation on the geometry of the brane is an ansatz introduced by Susskind et. al. in [27]. We are assuming that the Giant gravitons are not subject to major deformations from a spherical shape. The equations of motion should, to be rigorous, be calculated first without this assumption, and then have the ansatz plugged in afterwards. However, as is typical of his work, Susskind's informal treatment happens to produce the correct result.

the 3 angles specifying the brane, into the $AdS_5 \times S^5$. It is assumed that the only coordinate that varies with time is $\phi$, and we obtain:

$$ds^2_{D3} = [(R^2 - r^2)\dot{\phi}^2 - 1]dt^2 + r^2 d\Omega_3^2$$

The DBI Action is thus given by:

$$S_{DBI} = -T_{D3} \int_{WV} \sqrt{1 - (R^2 - r^2)\dot{\phi}^2 dt^2} \sqrt{r^6 d\Omega_3^2} = -T_{D3}\Omega_3 r^3 \int dt \sqrt{1 - (R^2 - r^2)\dot{\phi}^2}$$

In this expression, $\dot{\phi}$ is the angular velocity, $T_{D3}$ is the tension of the brane and $\Omega_3 r^3$ is the volume of a 3-sphere ($= 2\pi^2 r^3$). This means that the coefficient of the integral is in fact the mass of the $D3$ brane, as measured by an observer at a point in the $AdS_5$ (The giant graviton in a 10-dimensional geometry is in fact massless, but an observer that is unaware of the $S^5$ component, as the $AdS_5$ observer is, would measure a mass. This "mass" is in fact the energy of the brane due to its momentum in the additional dimensions). Since the action is the time integral of the Lagrangian, removing this integral from the above expression gives us $\mathcal{L}_{DBI}$. Terms containing derivatives of $r$ have been dropped since we are interested in the case where the radius is constant and close to maximal.

**The Chern-Simons Action**

The term of the Lagrangian referred to in [27] as the Chern-Simons term (it is more commonly associated with Wess and Zumino, although the term bears a resemblance to the action of Chern-Simons theory) is defined as:

$$S_{CS} = \int_{WV} \mathcal{P}[A_4]$$

It is the integral over the $D3$ brane world volume of the pullback of the 4-form potential onto the worldvolume. The pullback is defined as (the $Y$'s are the coordinates on the world volume):

$$\mathcal{P}[A_4] = A_{\mu\nu\rho\tau} \frac{\partial X^\mu}{\partial Y^\alpha} \frac{\partial X^\nu}{\partial Y^\beta} \frac{\partial X^\rho}{\partial Y^\gamma} \frac{\partial X^\tau}{\partial Y^\delta} dY^\alpha \wedge dY^\beta \wedge dY^\gamma \wedge dY^\delta.$$

It is apparent that this is analogous to the definition of the induced metric - in fact, the pullback of a metric onto a certain manifold is exactly the induced metric on that manifold. The relation $F_5 = dA_4$ allows us to implement a form of Stokes theorem, since the action of the curl operator naturally arises from the action of $d$ on a 4-form as a result of the antisymmetricity of wedge products. This antisymmetricity is important for the same reason that any form must be antisymmetric - it ensures that the Jacobian for a coordinate transformation comes out correctly, such that the integral of the form is coordinate independent. The manifold we integrate over must satisfy the condition that its boundary is the world volume of the $D3$:

$$S_{CS} = \int_{WV} \mathcal{P}[A_4] = \int_\Sigma F_5$$

We choose $\Sigma$ to be a 5 dimensional manifold whose boundary is the 4 dimensional surface swept out by the brane as it completes one orbit of the 2-disk, $D_2$. The D3 brane is moving at a constant radius from the center of the $D_2$, this radius is $r$. If we consider the motion of the brane through one orbit ($\phi$ goes from 0 to $2\pi$), it will trace out a 4-dimensional surface that bounds the portion of the $D_2$ with radius less than $r$. $\Sigma$ is the manifold consisting of the 4 dimensional boundary and this region of the disk. We have a constant flux density, so that in analogy with the case of constant magnetic field in electrodynamics, we can define the five-form field strength as $F_5 = Bdvol$, where $dvol$ is the volume form on the 5-sphere. The $S^5$ is a Riemannian manifold, and the volume form is hence given by $dvol = \sqrt{|g|} dr \wedge d\phi \wedge d\Omega_3 = Rr^3 dr \wedge d\phi \wedge d\Omega_3$. The time coordinate that has apparently disappeared is contained in $\phi$. We are now in a position to calculate the action; we simply have to integrate the coordinates given in the volume form over $\Sigma$.

$$\begin{aligned} S_{CS} &= \int_{D3} d\Omega_3 \int_0^{2\pi} d\phi \int_0^r dr' BR(r')^3 \\ &= BR\frac{2\pi}{4}\Omega_3 r^4 \\ &= BR\Omega_5 r^4 \\ &= 2\pi N \frac{r^4}{R^4} \end{aligned}$$

The recursion relation for the hyperarea coefficient of an $n$-sphere ($\Omega_n = \frac{2\pi}{n-1}\Omega_{n-2}$) was used in the second last step, and the last step used the flux quantization condition ($\Omega_5 R^5 B = 2\pi N$). The Chern-Simons Lagrangian can be obtained from the action by dividing by the period of a single orbit (since the time integral which must be performed to obtain the action from this expression contributes a $T$):

$$\mathcal{L}_{CS} = \frac{S_{CS}}{T} = \frac{\dot{\phi}}{2\pi} S_{CS} = \dot{\phi} N \frac{r^4}{R^4}$$

**Angular Momentum and Energy**

The tension of the brane is given in 10 dimensional Planck units by

$$T_{D3} = \frac{1}{(2\pi)^3 l_s^4 g_s}$$

Combining the equations for the brane tension and radius of the sphere we obtain the relation

$$T_{D3}\Omega_3 = \frac{N}{R^4}$$

We can thus obtain the angular momentum from the Lagrangian:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{m\dot{\phi}(R^2 - r^2)}{\sqrt{1 - \dot{\phi}^2(R^2 - r^2)}} + N\frac{r^4}{R^4}$$

We introduce the parameter $m = T_{D3}\Omega_3 r^3 = (\frac{N}{R^4})r^3$, which is the mass of the $D3$ brane. Since a radius $r$ that is greater than $R$ would produce a non-physical result, we see that the radius is bounded by $0 \leq r \leq R$. We must also include the constraint that the linear velocity cannot exceed the speed of light, that is $0 \leq \dot{\phi} r \leq 1$. Plugging the maximal radius into the angular momentum equation, we find the limit of the angular momentum:

$$L_{max} = N$$

This bound is applicable in the regime where $N \gg 1$. We can determine the energy of the brane:

$$E = \sqrt{m^2 + \frac{(L - N\frac{r^4}{R^4})^2}{R^2 - r^2}}$$

Analysing the variation of the energy with respect to $r$ at a constant angular momentum, a stable mimimum is found when

$$r^2 = \frac{L}{N} R^2 \quad , \tag{3.2}$$

indicating the expansion of the brane with increasing angular momentum. The existence of the minimum implies the existence of a stable brane configuration, at least in the classical regime. A quantum mechanical analysis may reveal the possibilty of tunneling, which would compromise the stability of the configuration. Inserting the expression for $r$ at this mimimum into the energy equation, and applying suitable approximations for the $N \gg 1$ limit, we obtain $E = \frac{L}{R}$; this is the energy of the brane for all values of angular momentum - indeed, it is the energy of a Kaluza-Klein graviton at angular momentum $L$. Clearly the brane must exist within the bounds of the 5-sphere, i.e. $r \leq R$, thus we can determine the maximum angular momentum by plugging $r = R$ into 3.2, giving $L_{max} = N$. This is in perfect agreement with the bound enforced by the Stringy Exclusion Principle for a BPS particle. It is interesting to note that, since the angular momentum cannot become infinitely large, Heisenberg's uncertainty principle implies that we can never truly resolve positions to a single point - the geometry is "fuzzy". The implications of this may be interesting to consider, but do not affect the calculations presented in this dissertation. The trivial minimum at $r = 0$ corresponds to the ordinary point-like graviton solution, but this solution is subject to uncontrolled quantum corrections, since it describes a huge energy concentrated at a single point.

Thus we have obtained the angular momentum bound expected from the Stringy Exclusion Principle by envisioning the particle as expanding into a $D3$ brane at high angular momentum. The bound is then a result of the limit on the degree of expansion due to the finite size of the 5-sphere in which it is moving. The argument outlined above provides substantial evidence that for KK gravitons at large angular momenta, a description in terms of expanded $D3$ branes is preferable to one in terms of strings.

### 3.1.4 Discussion

The results obtained by McGreevy, Susskind and Toumbas, while they ingeniously recreate the physics expected by the Stringy Exclusion principle, were in no way rigorously tested nor did they admit much generality. The assumptions made were numerous and restricting - the situation considered takes into account only exactly spherical gravitons of constant radius. In general, the coordinates $\phi$ and $r$ should both be functions of time and the world volume coordinates, and a concrete test of the proposal would be to determine if the giant graviton appears as an exact solution of the resulting equations of motion. However, perhaps by luck or possibly powerful intuition, the results obtained by their minimally technical derivation turn out to be correct. Several more detailed calculations have been performed which show the validity of their interpretation of the graviton as expanding into a brane at large angular momentum. It should also be noted that the Lagrangian derived contains only terms corresponding to bosonic interactions, however, a study of the supersymmetry of the giant graviton allows for the reappearance of the fermionic fields. An analysis of the supersymmetries admitted by giant gravitons in various $AdS \times S$ backgrounds was performed in [29], where it is confirmed that the giant gravitons in fact preserve all the same supersymmetries as the point-like graviton. This is expected, since from the point of view of an observer doing supergravity calculations in the $AdS_5$ the sphere giant graviton is a massive state with charge equal to its mass, and therefore one can determine that they satisfy the appropriate BPS bound (the same bound as the point gravitons).

## 3.2 Branes in $AdS_5$

Soon after Susskind and friends released their paper introducing sphere giants to the world, several authors (notably those of [28] and [29]) discovered that the gravitons could also expand in the $AdS_5$ component of the background. This is seen by following a treatment very similar to that used for the sphere giants; a spherical $D3$ brane wrapping the $\Omega_3$ of the $AdS_5$ background is embedded into the spacetime. The same ansatz of constant radius and time-dependence being limited to the coordinate $\phi$ is substituted into the action, and we consider the same brane configuration of a giant orbiting along the equator of the $S^5$. The Lagrangian of this brane configuration is of the form:

$$\mathcal{L} = -T\Omega_3 R^4 \left( \tan^3\rho \sqrt{\sec^2\rho - \dot{\phi}^2} - \tan^4\rho \right)$$

The energy can then be calculated and is given by:

$$E = N \left( \sec\rho \sqrt{\frac{L^2}{N^2} + \tan^6\rho} - \tan^4\rho \right)$$

The energy has local minima at $\tan\rho = 0$ and $\tan\rho = \sqrt{\frac{L}{N}}$, for which the energy takes value $E = L$. The existence of these minima establishes that there exists a stable configuration of a spherical $D3$ brane embedded in the $AdS_5$. These states are labelled by exactly the same quantum numbers as the sphere giant states and the point graviton states. The $AdS$-branes are often referred to as "dual" to their $S^5$ counterparts, in the sense that the $AdS$-branes couple electrically to the RR field, and can be thought of as dielectric branes, while the $S$-branes couple magnetically and can be called dimagnetic. The authors of [29] also analysed the supersymmetry of the giants expanding in the $AdS_5$ component of the spacetime, and found them to preserve the exact same supersymmetries as the point-like and sphere giant gravitons.

We thus are in a situation where there are three distinct brane configurations (The $AdS$ giant, $S$ giant and point graviton) all of which share the same quantum numbers, and quantum mechanics leads us to expect these three states to mix. It therefore seems prudent to seek instanton solutions describing tunneling between these states. Explicit expressions for the instantons evolving between the $S$-giant and point-like state, as well as between the $AdS$-giant and point-like graviton have been derived in [29], and were found to be $\frac{1}{4}$-BPS states, preserving 8 of the 32 supersymmetries. Instantons involved in direct transitions between the two types of brane gravitons have been sought, but numerical simulations have been performed ([31]) that demonstrate that the direct tunneling solution does not exist - tunnelling between $AdS$- and $S$- giants is a two-instanton transition, with the point graviton acting as an intermediate state. This does not mean, however, that the transition is possible; as stated, the point graviton solution is unstable due to the massive gravitational field resulting from the concentration of energy at a single

point. This transition may only be mathematically possible, and does not necessarily have a physical interpretation. We can visualize the transition by considering a plot of the potential as function of radius, which will be a triple potential well, symmetric about the vertical axis, with one minimum at $r = 0$. The outer minima correspond to the stable giant graviton configurations, one being the sphere giant and the other the $AdS$, while the center minimum is the point graviton.

One may be concerned by the interpretation of gravitons expanding in the $AdS$ space, since this component of the space is not of finite radius - one of the key benefits of the sphere giants was that they provided a natural and intuitive explanation for the bound imposed by the Stringy Exclusion principle. The authors of [29] postulate a resolution to this problem of additional giant graviton states existing above the angular momentum bound by considering supersymmetry breaking. Quantum mechanically, it is expected that a unique supersymmetric ground state representing the true graviton would be the result of a mixing of the three candidate states previously discussed. It was noted that as the angular momentum of the $AdS$ giant is increased above the bound imposed by Stringy Exclusion, the minimum of the potential corresponding to the sphere giant vanishes. In analogy with supersymmetric quantum mechanics in generic models involving a single supersymmetric coordinate, it is postulated that the mixing between the remaining two states would not produce a unique supersymmetric ground state - the short supersymmetry multiplets associated to each of these states combine to give a massive long multiplet, whereas when there are three states the mixing only lifts two short multiplets. This was supported by the work of Lee in [31], where it was shown numerically that there exists no direct instanton transition between the types of giants; at least for instanton effects, the problem is effectively one dimensional, and strong analogy can be made with the one-dimensional quantum mechanical picture, the only difference in calculation being the number of fermionic zero modes. This leads to the conclusion that SUSY is indeed broken for angular momenta above the bound, and there is hence no supersymmetric ground state for the double potential well occurring for these angular momenta. The bound arising by the Stringy Exclusion principle is thus still preserved under the interpretation of gravitons as expanding into branes in the $AdS_5$.

One may also care to note that the two types of branes collapse into the same point graviton state. This is motivated by the matrix description of M-theory in light-cone gauge, where different branes with different geometries can be represented using non-commutative geometry, but when any of the branes are shrunk to zero size, the same state emerges. This state is described by a set of commuting matrices, the entries of which are independent of which geometry it emerges from.

## 3.3 Schur Polynomials as Duals to Giant Graviton Systems

Many reasons to consider the Schur Polynomials described previously as dual to giant gravitons have emerged, and in many cases it occurs that the mathematics associated to the description is completely tractable, even simple in some cases. It was shown in [2] that the space of $\frac{1}{2}$-BPS representations in $\mathcal{N} = 4$ Super Yang-Mills is in one-to-one correspondence with the space of $U(N)$ Young diagrams, and hence with the single matrix Schur Polynomials being labelled by these diagrams. Insights gained by studying the dual quantum gravity have revealed that it is sensible to identify excitations of these $\frac{1}{2}$-BPS states with restricted Schur Polynomials [6]. Most importantly, it was shown that the Neumann and Dirichlet boundary conditions that the $D$-brane excitations (being that they can be described as strings propagating on the brane) must satisfy emerge dynamically in the Yang-Mills theory. This was done in the plane wave limit, using operators dual to giant gravitons that are defined as subdeterminants of one of the complex scalar fields, with the attached string world volume built from another complex scalar and an impurity (corresponding to further oscillator excitations of the string) inserted into this string of scalars, which together represent the excitation ([32]):

$$\hat{O}_k^{Z,Y,X} = \epsilon^{j_1 \dots j_N}_{i_1 \dots i_N} Z^{i_1}_{j_1} \cdots Z^{i_{N-1}}_{j_{N-1}} (Y^k X Y^{J-k})^{i_N}_{j_N}$$
$$\hat{O}_k^{Z,Y,X} = \epsilon^{j_1 \dots j_N}_{i_1 \dots i_N} Z^{i_1}_{j_1} \cdots Z^{i_{N-1}}_{j_{N-1}} (Y^k Z Y^{J-k})^{i_N}_{j_N}$$

The first operator corresponds to a fluctuation parallel to the brane, and hence Neumann boundary conditions emerge, while the second operator corresponds to transverse fluctuations which produce Dirichlet boundary conditions. The matrix of normalized two point correlators of these operators is calculated, where the indices $i, j$ of the matrix $M_{ij}$ correspond to the position of the impurity within the string.

Upon diagonalizing this matrix, one can obtain a basis of energy eigenstates, the form of which reflects the emergence of the boundary conditions desired. The authors of [6] go on to argue that the formalism developed using these operators is equivalent to one arising by the use of operators labelled by Young diagrams corresponding to irreducible representations of the symmetric group - these are basically the Schur Polynomial operators that are the focus of this dissertation.

The two-point function of the single-matrix Schur polynomials was found to be diagonal in [2]. This result was extended to restricted Schurs - their two-point correlator was determined in the free field limit with all Feynman diagrams, non-planar included, summed over. The resulting expression is [3]:

$$\langle \chi_{R,(r,s)jk}(Z,Y)\chi_{T,(t,u)lm}(Z,Y)^{\dagger}\rangle = \delta_{R,(r,s)T,(t,u)}\delta_{kl}\delta_{jm}f_R\frac{\text{hooks}_R}{\text{hooks}_r\text{hooks}_s}$$

$f_R$ is the product of factors in Young diagram $R$, and $\text{hooks}_R$ is the product of the hook lengths. Classically then, we see that there is no mixing between restricted Schurs being labelled by differing Young diagrams. The two point correlators of restricted Schur polynomials were calculated to one-loop level in [33], by associating a Cuntz oscillator chain state to each restricted Schur polynomial, and it was found that mixing of the operators at this level is highly constrained: only restricted Schurs with Young diagram labels that differ by the placement of a single box have non-vanishing two-point functions at this level. It has also been shown [34], by considering the restricted Schurs to have a partonic structure and generalizing the known product rule for single-matrix Schurs, that any higher-point correlator of these operators can be expressed as a sum of two-point correlators.

It is simply shown that the $Z$'s and $Y$'s are each of dimension 1, and contribute one unit of angular momentum in the $1-2$ and $3-4$ plane respectively to the giant graviton system. With this association of the fields to angular momentum, and knowing that each box in $R$ corresponds to a single field, we can infer an association of the Schur polynomial labels with giants expanded in either the $AdS_5$ or the $S^5$. The 5-sphere has a certain radius ($R_{S^5}$), and hence gravitons expanding within it have a maximal size, bounded by $N$ due to the Stringy Exclusion Principle. The relevant relation is

$$R = \sqrt{\frac{J}{N}}R_{S^5}$$

where $R$ is the radius of the giant graviton, and $J$ is angular momentum. It is clear then that it is sensible to label Schur polynomials describing sphere giant gravitons by a Young diagram having long ($O(N)$ boxes) columns, since this implements a natural bound on the angular momentum by the properties of the Young diagram. It also seems natural to associate those Schurs labelled by Young diagrams with long rows as AdS giants, since there is no bound on the radius of these giants and also no bound on the length of a row. This solves an important problem in the theory of giant gravitons as it provides a method by which the three different giant graviton states discussed previously can be distinguished from the point of view of the boundary theory, being the SYM field theory dual to Type IIB String theory on the $AdS_5 \times S^5$ background. The argument for the existence of an upper bound on the angular momentum of AdS giants presented in [29] (and discussed earlier in this section) is based on the interpretation that the true giant graviton state must be a mixture of the three types of gravitons that can occur. This dissertation presents a theory in which the AdS and Sphere giants are described independently. We believe that the results obtained in this framework are powerful enough that we need not be too concerned with the details of how this bound arises.

D-branes are surfaces on which strings, being oriented in terms of the charge they carry, can affix via their endpoints. The interactions between giant gravitons depend on the existence of charge-carrying open strings stretching between their surfaces. Since giant gravitons have a compact worldvolume, we require the total charge localised on their surface to vanish due to Gauss' Law. It was conjectured in [6], and later proven in [35] by the introduction of a double coset ansatz that the restricted Schur Polynomials provide the correct number of states expected from this constraint. Further, one of the results of [7] is the appearance of a direct and natural connection between the action of the dilatation operator on the restricted Schur polynomials and this constraint. Restricted Schurs provide a complete basis for gauge invariant operators built from the Higgs fields due to the fact that any linear combination of multitrace operators can be written as a linear combination of the Schur Polynomials.

Numerous examples demonstrating that the geometry of the dual string theory is encoded in the Young diagram label have been found, one of the more significant being linked to the emergence of locality in our model. The study of string splitting and joining interactions in the context of excited giant graviton systems produces evidence for the interpretation of the number of boxes traversed in moving from one corner of the Young diagram to another as being related to a radial separation distance in the string theory ([24]). This is clearly seen by recalling that the Myers' effect results in the expansion of a giant graviton with increased angular momentum, and that each box on the Young diagram corresponds to a unit of angular momentum. The evidence that the restricted Schur operators are ideal for the description of giant graviton systems is plentiful.

# Chapter 4

# Construction of Restricted Schur Polynomials

In this dissertation, a method for the diagonalization of the dilatation operator within large sectors of decoupled states is presented. Each sector comprises restricted Schur polynomials with a fixed number $p$ of rows or columns. Mixing with restricted Schur polynomials that have $n \neq p$ long rows or columns (or of even more general shape) is suppressed at least by a factor of order $\frac{1}{\sqrt{N}}$. The motivation for this statement is presented in [8] - physically, the giant gravitons are stable semiclassical objects at large $N$, and hence their number is expected to be conserved, implying that any extra rows/columns arising under the action of $D$ would be short. A restricted Schur polynomial labelled by a Young diagram with $p$ long columns/rows and $n-p$ shorter columns would describe a bound state of $p$ giant gravitons with some KK gravitons. The transition amplitude for graviton emission is proportional to the string coupling, which in the 't Hooft limit is related to $N$ by $g_s \sim \frac{1}{N}$. This transition is thus expected to be suppressed - this result is reproduced by studying the action of $D$ on the normalised operators labelled by $n \neq p$ rows/columns. To achieve the diagonalization a key new idea is needed: Schur-Weyl duality is used to construct the restricted Schur polynomials. A connection to this duality was pointed out in [10], but it was only in [7] that the value of the duality was fully realised. In this section we will explain how Schur-Weyl duality arises and how it is exploited.

## 4.1 The Construction

As was previously mentioned, the major difficulty in constructing the restricted Schurs is in constructing the projectors $P_{R\to(r,s)jk}$. The method by which Schur-Weyl duality is used to provide an efficient, transparent construction of these operators is now presented. This construction is not quite completely general, but it does capture many interesting situations and should be a useful tool to explore semi-classical physics dual to the restricted Schur polynomials. We begin with a basis for the carrier space of an $S_{n+m}$ irreducible representation $R$. From this basis, we build another basis that spans the carrier space of an $S_n \times S_m$ irreducible representation $(r,s)j$. It is then a small step to build $P_{R\to(r,s)jk}$. We accomplish the construction in two steps: First we project from $S_{n+m}$ to $S_n \times (S_1)^m$ (this is easy) and second, we assemble the $S_n \times (S_1)^m$ representations into $S_n \times S_m$ representations (this is the trying step). It is this second step that is accomplished using Schur-Weyl duality. As a consequence we learn that the multiplicity index can be organized using $U(p)$ representations, with $p$ the number of rows or columns in $R$. The background material from representation theory needed to understand this section is collected in Appendices A and B.

### 4.1.1 From $S_{n+m}$ to $S_n \times (S_1)^m$

Start from the carrier space for an irreducible representation $R$ of $S_{n+m}$. If we restrict ourselves to an $S_n \times (S_1)^m$ subgroup this space will decompose into a direct sum of invariant subspaces, each of which is the carrier space of a particular irreducible representation of the subgroup. In this subsection it will be explained how to extract a particular $S_n \times (S_1)^m$ invariant subspace from the full carrier space of $R$.

Since $S_1$ has only a single irreducible representation, we need not include it in our labels for the irreducible representation of the subgroup. Consequently, to specify an irreducible representation of the

$S_n \times (S_1)^m$ subgroup, we only need to specify an irreducible representation of $S_n$, that is, a Young diagram $r$ with $n$ boxes. The only representations $r$ that are subduced by $R$ are those with Young diagrams that can be obtained by removing $m$ boxes from $R$. Pulling the same set of $m$ boxes off in different orders leads to different subspaces which all carry the same irreducible representation $r$. To resolve this multiplicity, we only need to specify the order in which the boxes are removed. To specify this order, label the boxes to be removed from $R$ with a label ranging from 1 to $m$, such that box 1 is removed first, then box 2 and so on until box $m$ is removed. Thus, by labeling any given set of boxes in such a way that if we were to remove the boxes in numerical order starting with box 1 we would have a legal Young diagram at each step, we obtain a partially labelled Young diagram with shape $R$, which represents a subspace carrying an irreducible representation of the $S_n \times (S_1)^m$ subgroup. See Appendix B.3 for further discussion.

To build an operator which projects from the carrier space of the $S_{n+m}$ irreducible representation $R$ to the carrier space of an $S_n \times S_m$ irreducible representation $(r, s)j$, we now need to assemble the partially labelled Young diagrams (which already carry a representation $r$ of $S_n$) in such a way that the resulting linear combinations carry an irreducible representation of $S_n \times S_m$. We turn to this task in the next subsection.

### 4.1.2 Young Diagrams with $p$ long rows

We will consider Young diagrams built using $n + m \sim O(N)$ boxes and having $p$ rows. Thus, for the generic diagram, each row has $O(N)$ boxes. We set $m = \alpha N$ with $\alpha \ll 1$, i.e. there are far fewer $Y$ boxes than $Z$'s - this is sensible since the $Z$'s form the bulk of the giant gravitons, while the $Y$'s are impurities corresponding to excitations. After labeling the $m$ boxes, two labelled boxes with labels $i$ and $j$, that are in different rows, will have associated factors $c_i$ and $c_j$ respectively, and $c_i - c_j \sim O(N)$. This approximation implies that we cannot by our methods describe a giant graviton system where any of the giants have similar angular momentum. However, the number of possible operators where the angular momentum of neighbouring giants is similar is tiny in comparison with the number having well-separated angular momenta, so that this approximation is not too restrictive.

Consider the $S_m$ subgroup of $S_{n+m}$ - it acts on the labelled boxes, permuting their positions. We can obtain a matrix representation of this action by thinking about the partially labelled Young diagrams as Young-Yamonouchi states. As discussed in Appendix B.4, the fact that $c_i - c_j \sim O(N)$ for boxes in different rows implies a significant simplification in the representations of $S_m$. When adjacent permutations $(i, i+1)$ act on labelled boxes that belong to the same row, the Young diagram is unchanged and when acting on labelled boxes that belong to the different rows, the labelled boxes are swapped. If we have a Young diagram with $p$ rows and we label $m$ boxes in all possible ways consistent with the rule of the previous subsection, we find a total of $p^m$ possible partially labelled Young diagrams. We associate a particular $p$-dimensional vector to each box that is labelled. This gives a total of $m$ vectors $\vec{v}(i)$ with $i = 1, 2, \cdots, m$. We will denote the components of these vectors as $\vec{v}(i)_n$ where $n = 1, ..., p$. If box $i$ is pulled from the $j^{\text{th}}$ row we have

$$\vec{v}(i)_n = \delta_{nj} \, .$$

For each index $i$ (equivalently, for each labelled box) we have a vector space $V_p$. This vector space is spanned by the set of $p$ vectors $\vec{v}(i)_n$, each of which corresponds to a labelled box sitting in a particular row. Taking the tensor product of these spaces we obtain a set of $p^m$ dimensional vectors (which is the correct dimension, since each vector corresponds to a particular labelling of the Young diagram), of the form

$$\vec{v}(1) \otimes \vec{v}(2) \otimes \vec{v}(3) \otimes \cdots \otimes \vec{v}(m-1) \otimes \vec{v}(m) \, .$$

Call the vector space spanned by these vectors $V_p^{\otimes m}$. When we talk about vectors of the above form we will say that "vector $\vec{v}(i)$ occupies the $i^{\text{th}}$ slot." The matrix action of $S_m$ on the partially labelled Young diagrams described above implies the following action on $V_p^{\otimes m}$

$$\sigma \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = \vec{v}\,(\sigma(1)) \otimes \vec{v}\,(\sigma(2)) \otimes \cdots \otimes \vec{v}\,(\sigma(m)) \, .$$

Thus, $\sigma \in S_m$ will move the vector in the $i^{\text{th}}$ slot to the $\sigma(i)^{\text{th}}$ slot, but does not change its value. We can also define an action of $U(p)$ on $V_p^{\otimes m}$

$$U \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = D(U)\vec{v}\,(1) \otimes D(U)\vec{v}\,(2) \otimes \cdots \otimes D(U)\vec{v}\,(m) \, ,$$

where $D(U)$ is the $p \times p$ unitary matrix representing group element $U \in U(p)$ in the fundamental representation. Thus, $U \in U(p)$ will change the value of the vector in the $i^{\text{th}}$ slot but it will not move it

to a different slot. It acts in exactly the same way on each slot. It is quite clear that these are commuting actions of $U(p)$ and $S_m$ on $V_p^{\otimes m}$

$$\begin{aligned} U\cdot(\sigma\cdot(\vec{v}(1)\otimes\cdots\otimes\vec{v}(m))) &= U\cdot(\vec{v}\,(\sigma(1))\otimes\cdots\otimes\vec{v}\,(\sigma(m))) \\ &= D(U)\vec{v}\,(\sigma(1))\otimes\cdots\otimes D(U)\vec{v}\,(\sigma(m)) \\ &= \sigma\cdot(D(U)\vec{v}\,(1)\otimes\cdots\otimes D(U)\vec{v}\,(m)) \\ &= \sigma\cdot(U\cdot(\vec{v}(1)\otimes\cdots\otimes\vec{v}(m))) \end{aligned}$$

and consequently by Schur-Weyl duality the space can be organized as[1]

$$V_p^{\otimes m} = \oplus_s V_s^{U(p)}\otimes V_s^{S_m}\,, \tag{4.1}$$

where the sum runs over all Young diagrams built from $m$ boxes and each has at most $p$ rows. One consequence of this formula is that

$$p^m = \sum_s \mathrm{Dim}(s)\, d_s$$

where $\mathrm{Dim}(s)$ is the dimension of $s$ as an irreducible representation of $U(p)$ and $d_s$ is the dimension of $s$ as an irreducible representation of $S_m$. The reader is invited to check a few examples herself. Equation 4.1 tells us that states belonging to a definite $U(p)$ irreducible representation can also be represented in the carrier space of a definite $S_m$ irreducible representation, and vice versa. Thus, by identifying states with good $U(p)$ labels we have identified states with good $S_m$ labels. The projectors which are used to define the restricted Schur polynomials are composed of a tensor product of states in the carrier space of an $S_m$ representation. Therefore an important consequence of (4.1) is that it provides an efficient method by which these projectors can be constructed using only the group theory of $U(p)$.[2].

### 4.1.3 State Labels - From $S_m$ to $U(p)$

A necessary step towards building the projectors entails constructing a dictionary between the original symmetric group labels $R,(r,s)jk$ of the restricted Schur polynomial $\chi_{R,(r,s)jk}$ and the new $U(p)$ labels. Exactly the same Young diagram $s$ that originally specifies an $S_m$ irreducible representation, now specifies a $U(p)$ irreducible representation. This is possible as a consequence of equation 4.1. The Young diagram $r$ is included among the new labels and it still specifies an irreducible representation of $S_n$, and is necessary since we must be able to determine the shape of the Young diagram $R$ from the new labels. The final label is the choice of a state from the carrier space of $U(p)$ representation $s$. The $\Delta$ weight of this state (see Appendix A.3) tells us how boxes were removed from $R$ to obtain $r$. This point deserves some explanation. Label the state chosen from the carrier space $s$ by its Gelfand-Tsetlin pattern. This state can be put into one-to-one correspondence with a semi-standard Young tableau and this correspondence plays a central role. Consider for example the $U(3)$ state with Gelfand-Tsetlin pattern

$$\begin{bmatrix} 4 & & 3 & & 2 \\ & 3 & & 2 & \\ & & 2 & & \end{bmatrix}.$$

The uppermost row of the pattern gives the shape of the Young diagram. Each row (starting from the bottom row) tells us how to distribute 1s, then 2s and so on till the semi standard Young tableau is obtained. This connection is reviewed in detail in Appendix A.4. For the Gelfand-Tsetlin pattern shown above the semi-standard Young tableau is

$$\begin{bmatrix} * & & * & & * \\ & * & & * & \\ & & 2 & & \end{bmatrix} \leftrightarrow \begin{array}{|c|c|c|c|}\hline 1&1&*&*\\\hline *&*&*\\\cline{1-3} *&*\\\cline{1-2}\end{array} \qquad \begin{bmatrix} * & & * & & * \\ & 3 & & 2 & \\ & & 2 & & \end{bmatrix} \leftrightarrow \begin{array}{|c|c|c|c|}\hline 1&1&2&*\\\hline 2&2&*\\\cline{1-3} *&*\\\cline{1-2}\end{array} \qquad \begin{bmatrix} 4 & & 3 & & 2 \\ & 3 & & 2 & \\ & & 2 & & \end{bmatrix} \leftrightarrow \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2&3\\\cline{1-3} 3&3\\\cline{1-2}\end{array}\,.$$

[1] Part of what is behind Schur-Weyl duality is simple and familiar: any two operators that commute can be simultaneously diagonalized.

[2] The reader will be familiar with the usual use of Schur-Weyl duality, to construct projectors onto good $U(p)$ irreducible representations using the Young symmetrizers i.e. by symmetrizing and antisymmetrizing indices on a tensor. We are turning this argument on its head by using the irreducible representations of the unitary group to build symmetric group projectors. Bear in mind that the details of our Schur-Weyl duality are different to the usual construction.

Each row in the pattern corresponds to a particular set of boxes carrying the same number in the semi standard tableau. From the definition of the Gelfand-Tsetlin pattern, we also know that each row in the pattern corresponds to a particular subgroup in the chain of subgroups $U(1) \subset U(2) \subset \cdots \subset U(p-1) \subset U(p)$. So, from the point of view of the semi-standard Young tableau or of the Gelfand-Tsetlin pattern, going to the $U(p-1)$ subgroup implies that we consider a subgroup that does not act on one of the numbers appearing in the semi-standard tableau. What does it mean to consider a $U(p-1)$ subgroup of our action of $U(p)$ on the $m$ boxes that have been removed from $R$? Recall that the particular $U(p)$ state that is assigned to each removed box depends on the row it was removed from. Thus going to a $U(p-1)$ subgroup corresponds to considering a subgroup that does not act on the boxes that have been pulled off a particular row. Clearly then, the numbers in the semi-standard tableau can be identified with the row from which the corresponding box has been removed from $R$. Recall that the $\Delta$ weight is a sequence of integers $\Delta(M) = (\delta_n(M), \delta_{n-1}(M), \cdots \delta_1(M))$. The number of boxes labelled $i$ which we have just argued is the number of boxes removed from row $i$ of $R$ to produce $r$, is given by $\delta_i(M)$. Thus, given $r$ and the delta weight we can reconstruct $R$. In our example, the $\Delta$ weight is $(4, 3, 2)$ - the 5 corresponds to the 5 boxes in the semi-standard tableau labelled with a 3 - these boxes were pulled off the third row of $R$.

There is a subtlety that needs to be discussed. Two states that belong to the same $U(p)$ representation and have the same $\Delta$ weight correspond to the same set of labels $R, (r, s)$. Consequently, we find that $(r, s)$ can be subduced more than once in the carrier space of $R$. These multiplicities only arise for $p \geq 3$ and hence were not treated in [10]. Our analysis here shows that this multiplicity index is easily organized using the $U(p)$ representations: The number of states having the same $\Delta$ weight is called the inner multiplicity of the state $I(\Delta(M))$. In this case, we label each state with a multiplicity index which runs from 1 to $I(\Delta(M))$[3]. These multiplicities have been resolved by the $U(p)$ state labels. Finally note that each $U(p)$ representation $s$ will also appear with a particular multiplicity. However, thanks to Schur-Weyl duality, we know that this multiplicity is organized by the $S_m$ representation $s$.

In summary then we trade the labels

| | |
|---|---|
| $R$ | an irreducible representation of $S_{n+m}$ |
| $r$ | an irreducible representation of $S_n$ |
| $s$ | an irreducible representation of $S_m$ |
| $j$ | multiplicity label resolving copies of $(r, s)$ |

for the new labels:

| | |
|---|---|
| $r$ | an irreducible representation of $S_n$ |
| $s$ | an irreducible representation of $U(p)$ |
| $M^i$ | a state in the carrier space of $s$ where $i$ runs over inner multiplicity . |

It has been argued that these labels are interchangeable, and it is possible to recover one set of labels from knowledge of the other - in particular, given $r$ and $M^i$, the delta weight of the $U(p)$ state allows one to reconstruct $R$, while the inner multiplicity provides a means of handling the multiplicities indexed by $jk$ that arise through subduction of multiple identical irreducible representations $(r, s)$ from $R$. At this point we have identified an orthonormal set of states spanning any particular carrier space $(r, s)j$ of the $S_n \times S_m$ subgroup. It is now a trivial task to write down the corresponding projector.

### 4.1.4 Constructing the Operators Projecting onto $S_n \times S_m$ Carrier Spaces

We can now write the symmetric group operator used to define the restricted Schur polynomial as

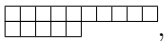
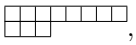
$$P_{R \to (r,s)jk} = \sum_{\alpha=1}^{d_s} |s, M^j, \alpha\rangle\langle s, M^k, \alpha| \otimes \mathbf{I}_r \,,$$

where, by Schur-Weyl duality, the multiplicity label $\alpha$ for the $U(p)$ states is organized by the irreducible representation $s$ of the symmetric group $S_m$ - $\alpha$ indexes the $U(p)$ states having the same Gelfand-Tsetlin pattern, which arise by assembling the removed boxes into the same shape but in different orders. The

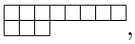
[3] An alternative approach to resolving these multiplicities has been outlined in [36]. The idea is to consider elements in the group algebra $CS_{n+m}$ which are invariant under conjugation by $CS_n \times CS_m$. The Cartan subalgebra of these elements are the natural generalization of the Jucys-Murphy elements which define a Cartan subalgebra for $S_n$[37][38]. The multiplicities will be labelled by the eigenvalues of this Cartan subalgebra[36].

indices $j$ and $k$ pick out states $M$ that have a particular $\Delta$ weight and hence range over $1, 2, ..., I(\Delta(M))$ - these labels index the $U(p)$ states having different Gelfand-Tsetlin patterns but the same $\Delta$ weight. The components $\delta_i$ of the particular $\Delta$ that must be used are equal to the number of boxes removed from row $i$ of $R$ to produce $r$. $\mathbf{I}_r$ is simply the identity matrix in the carrier space of the $S_n$ irreducible representation labelled by $r$.

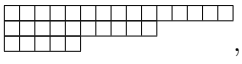
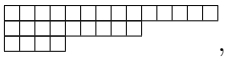
We will end this subsection with a few examples of how to translate between the two sets of labels. The labels

$R =$ , $r =$ , $s =$ 

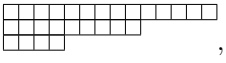
become

$r =$ , $s =$ , $M = \begin{bmatrix} 2 & & 2 \\ & 2 & \end{bmatrix}$

For this example $\Delta = (2, 2)$ because 2 boxes are removed from the first row and two from the second row of $R$ to produce $r$. The first row of $M$ is read off $s$ and the second row is chosen to obtain the correct $\Delta$. The inner multiplicity for this case is 1, so that there is a single possible projection operator. For our second example consider the labels

$R =$ , $r =$ , $s =$ .

The new labels are

$r =$ , $s =$ 

and

$$M_1 = \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} \quad M_2 = \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$

For this example $\Delta = (1, 1, 1)$ because one box is removed from each row. The inner multiplicity is 2. The two possible Gelfand-Tsetlin patterns are shown. Thus, for the $R, (r, s)$ labels given, one can construct a total of four possible restricted Schur polynomials. This second example is discussed in detail in the next section, where the allowed operators $P_{R\to(r,s)jk}$ are explicitly constructed.

### 4.1.5 Young Diagrams with $p$ Long Columns

We will consider Young diagrams with a total of $p$ columns. In this case, boxes that are in different columns, will again have associated factors with $c_i - c_j \sim O(N)$. As discussed in Appendix B.4, the fact that $c_i - c_j \sim O(N)$ for boxes in different rows again implies a significant simplification in the representations of $S_m$. When adjacent permutations $(i, i+1)$ act on states where the labelled boxes being swapped belong to the same column, the state labelled by this Young diagram picks up a negative sign. When acting on labelled boxes that belong to the different columns, the labelled boxes are swapped with no sign change. This is related to the fact that columns of Young diagrams in unitary group theory correspond to antisymmetrized tensor indices. This change in sign for the case that boxes belong to the same column is the only difference to the case of $p$ long rows.

The number of states that can be obtained when $m$ boxes are labelled is again $p^m$ and we again associate a $p$-dimensional vector to each box that is labelled. This again allows us to put partially labelled Young diagrams into one-to-one correspondence with vectors in $V_p^{\otimes m}$. In this case however, we will include some additional phases when we identify vectors in $V_p^{\otimes m}$ with partially labelled Young diagrams. These extra phases occur precisely because adjacent permutations $(i, i+1)$ acting on labelled boxes that belong to the same column flip the sign of the Young diagram. Choose any specific state with a particular set of labels. This state plays the role of a reference state. Any other state with the same boxes labelled but with a different assignment of the labels can be obtained by acting on the reference state with some series of adjacent permutations $(i, i+1)$. Further, the only adjacent permutations $(i, i+1)$ that we are allowed to apply to the reference state to reach any other given state, are those permutations acting on boxes labelled $i$ and $i+1$ that are in different columns when $(i, i+1)$ acts. If we act with $q$ adjacent permutations of this type to get from the reference state to another distinct state, that state is assigned

a phase of $(-1)^q$. See Section 4.2.2 for an explicit example. With this choice for the phases, it is easy to see that the action of $S_m$ on the partially labelled Young diagrams induces the following action on $V_p^{\otimes m}$

$$\sigma \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = \mathrm{sgn}(\sigma)\vec{v}\,(\sigma(1)) \otimes \vec{v}\,(\sigma(2)) \otimes \cdots \otimes \vec{v}\,(\sigma(m)) \ ,$$

where $\mathrm{sgn}(\sigma)$ denotes the signature of permutation $\sigma$: it is $+1$ for even permutations and -1 for odd permutations[4]. Thus, $\sigma \in S_m$ will move the vector in the $i^{\text{th}}$ slot to the $\sigma(i)^{\text{th}}$ slot and may change the overall phase. We can also define an action of $U(p)$ on $V_p^{\otimes m}$

$$U \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = D(U)\vec{v}\,(1) \otimes D(U)\vec{v}\,(2) \otimes \cdots \otimes D(U)\vec{v}\,(m) \ ,$$

where $D(U)$ is the $p \times p$ unitary matrix representing group element $U \in U(p)$. Thus, $U \in U(p)$ will change the value of the vector in the $i^{\text{th}}$ slot but it will not move it to a different slot. It acts in exactly the same way on each slot. It is quite clear that again these are commuting actions of $U(p)$ and $S_m$ on $V_p^{\otimes m}$

$$\begin{aligned} U \cdot (\sigma \cdot (\vec{v}(1) \otimes \cdots \otimes \vec{v}(m))) \quad &= U \cdot \ \mathrm{sgn}(\sigma)\,(\vec{v}\,(\sigma(1)) \otimes \cdots \otimes \vec{v}\,(\sigma(m))) \\ &= \mathrm{sgn}(\sigma) D(U)\vec{v}\,(\sigma(1)) \otimes \cdots \otimes D(U)\vec{v}\,(\sigma(m)) \\ &= \sigma \cdot (D(U)\vec{v}\,(1) \otimes \cdots \otimes D(U)\vec{v}\,(m)) \\ &= \sigma \cdot (U \cdot (\vec{v}(1) \otimes \cdots \otimes \vec{v}(m))) \end{aligned}$$

and consequently by Schur-Weyl duality we can again use $U(p)$ to organize the multiplicity label of the $S_m$ irreducible representations. In this case, the space can be organized as

$$V_p^{\otimes m} = \oplus_s V_{s^T}^{U(p)} \otimes V_s^{S_m} \ , \tag{4.2}$$

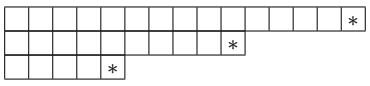
where $s^T$ is obtained by exchanging row and columns in $s$. The discussion from here on is identical to the case of $p$ rows. The reader is invited to consult Section 4.2.2 for a concrete example of a projector constructed using this Schur-Weyl duality.

## 4.2 Construction of Projectors: Examples

In this section we will compute some projectors using the new method outlined in this article. This is done to both check the nuts and bolts of the construction and to make the arguments presented concrete. Indeed, the main technical new result is the understanding that we can use $U(p)$ group theory to construct a basis for the carrier space of an irreducible representation of an $S_n \times S_m$ subgroup from the carrier space of an irreducible representation $R$ of $S_{n+m}$, when $R$ has $p$ long rows or long columns. In this Section we give concrete results illustrating these facts.

### 4.2.1 A Three Row Example using $U(3)$

Consider the following three row Young diagram




.

The starred boxes are to be removed. There are six possible ways to distribute the labels $1, 2, 3$ between these boxes. One possible representation that can be subduced has $r$ as given above but with the starred boxes removed and $s = \begin{array}{|c|c|}\hline \ \ & \ \ \\ \hline \ \ \\ \cline{1-1}\end{array}$. To build the projector $P_{R\to(r,s)jk}$ we need to build the projector onto the $U(3)$ irreducible representation labelled by $s = \begin{array}{|c|c|}\hline \ \ & \ \ \\ \hline \ \ \\ \cline{1-1}\end{array}$. Further, since one box is pulled off each row, the relevant $U(3)$ states have a $\Delta$ weight of $(1, 1, 1)$. The representation $s$ is 8 dimensional, and the Gelfand-Tsetlin patterns that label states in its carrier space are

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$

[4]Recall that a permutation is even (odd) if it can be written as a product of an even (odd) number of two cycles.

$$\begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix}$$

It is easily verified that the fourth and sixth states in the above list have the correct $\Delta$ weight, so that for weight $\Delta = (1,1,1)$ we have inner multiplicity $I(\Delta) = 2$. The fact that there are two states with the correct $\Delta$ weight implies that this particular $S_n \times S_m$ representation, $(r,s)$, is subduced twice from the carrier space of $R$. This in turn implies that there are four possible projection operators, corresponding to the four ways of tensoring two states with each other, and hence four possible restricted Schur polynomials that can be defined.

To build the projector we need to take linear combinations of the subspaces defined by each of the 6 partially labelled Young diagrams in such a way that the resulting combination is an invariant subspace of $S_n \times S_m$ (here $m = 3$) and further that this invariant subspace carries the correct irreducible representation of $S_n \times S_m$. To streamline our notation for the six subspaces we work with, we will set

$|a,b,c\rangle =$ 

.

The $U(3)$ action is defined on the labelled boxes. The box labelled 1 is always in the first slot of the tensor product; its position inside the ket tells you what row (and hence what $U(3)$ state) it is in. Notice that all reference to the carrier space of $S_n$ representation $r$ is omitted. This is perfectly consistent because this subspace is common to all the subspaces we consider and it plays no role in the problem of finding good $S_m$ invariant subspaces. Thus, for example,

$$|1,2,3\rangle = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

and

$$|2,1,3\rangle = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

Using the Clebsch-Gordan coefficients given in section A.5 we easily find that the subspaces considered above break up into subspaces labelled by states from $U(3)$ representations:

$$|1,2,3\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} - \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}^{(2)}$$
$$+\frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 0 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}.$$

$$|2,1,3\rangle = -\frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}^{(2)}$$
$$-\frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 0 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}$$

$$|3,1,2\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} - \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{(1)} - \frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}^{(2)}$$
$$-\frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 0 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}$$

$$|1,3,2\rangle = -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} + \frac{1}{\sqrt{12}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} - \frac{1}{2}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)}$$
$$+\frac{1}{2}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$

$$|2,3,1\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} + \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)}$$
$$-\frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$

$$|3,2,1\rangle = -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)}$$
$$-\frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$

Given these results it is straight forward to write down the two possible sets of states that carry the $S_m$ irreducible representation, $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$, that we are interested in

$$\left|\begin{smallmatrix}\square\square\\\square\end{smallmatrix}, 1\right\rangle^{(1)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} = \tfrac{1}{\sqrt{12}}\left(-|1,2,3\rangle + |2,1,3\rangle - |3,1,2\rangle + |1,3,2\rangle + 2|2,3,1\rangle - 2|3,2,1\rangle\right)$$

$$\left|\begin{smallmatrix}\square\square\\\square\end{smallmatrix}, 2\right\rangle^{(1)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} = \frac{1}{2}\left(|1,2,3\rangle - |2,1,3\rangle - |3,1,2\rangle + |1,3,2\rangle\right)$$

and

$$\left|\begin{smallmatrix}\square\square\\\square\end{smallmatrix}, 1\right\rangle^{(2)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)} = \frac{1}{2}\left(|1,2,3\rangle + |2,1,3\rangle - |3,1,2\rangle - |1,3,2\rangle\right)$$

$$\left|\begin{smallmatrix}\square\square\\\square\end{smallmatrix}, 2\right\rangle^{(2)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} = \tfrac{1}{\sqrt{12}}\left(|1,2,3\rangle + |2,1,3\rangle + |3,1,2\rangle + |1,3,2\rangle - 2|2,3,1\rangle - 2|3,2,1\rangle\right)$$

The superscripts on the kets on the right hand sides of these equations are multiplicity labels, indexing the multiplicity arising by the subduction of two representations labelled by identical $(r,s)$ from $R$ and reflected by the inner multiplicity of the $U(3)$ states. The integer inside each ket indexes states in the carrier space of $s$ as a representation of $S_m$, and hence runs from 1 to $d_s$, the dimension of the symmetric group representation. These formulas have all been obtained using the Clebsch-Gordan coefficients of $U(3)$ - we have not used any symmetric group theory. However, as a consequence of Schur-Weyl duality, we claim that the above states fill out representations of $S_3$. This is easily verified - for example, applying the action of the adjacent group element (12) to the state $\left|\begin{smallmatrix}\square\square\\\square\end{smallmatrix}, 1\right\rangle^{(1)} = A$:

$$(12)A = \frac{1}{\sqrt{12}}\left(-|2,1,3\rangle + |1,2,3\rangle - |3,2,1\rangle + |2,3,1\rangle + 2|1,3,2\rangle - 2|3,1,2\rangle\right)$$

We can see that $A$ and $\left|\begin{array}{|c|c|}\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\cline{1-1}\end{array}, 2\right\rangle^{(1)} = B$ fill out representations of $S_3$ by comparing this result to the linear combination of $A$ and $B$ obtained when acting on $A$ with $\Gamma((12))$ in the Young-Yamonouchi basis:

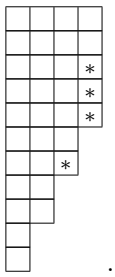
$$\Gamma((12))\left|\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle = \frac{1}{2}\left|\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle + \frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle = \frac{1}{2}A + \frac{\sqrt{3}}{2}B$$

One can check that $(12)A = \frac{1}{2}A + \frac{\sqrt{3}}{2}B$, where $A$ and $B$ are expanded in terms of the $|a,b,c\rangle$ states. This confirms that it is acceptable to use the $U(p)$ state labels for the $S_n \times S_m$ states appearing in our projectors, since the $|a,b,c\rangle$ basis gives good symmetric group reps when we pick a particular $U(p)$ state. The four possible projectors that can be defined are now given by

$$P_{R\to(r,\begin{array}{|c|c|}\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\cline{1-1}\end{array}),ij} = \sum_{k=1}^{2}\left|\begin{array}{|c|c|}\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\cline{1-1}\end{array}, k\right\rangle^{(i)}\quad{}^{(j)}\left\langle\begin{array}{|c|c|}\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\cline{1-1}\end{array}, k\right|.$$

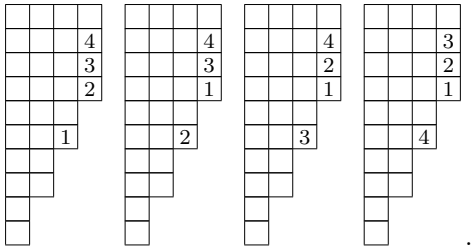
### 4.2.2 A Four Column Example using $U(4)$

Consider the following four column Young diagram



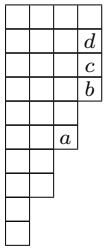
.

The starred boxes are to be removed. There are four possible ways to distribute the labels $1,2,3,4$ between these boxes. One possible $S_n \times S_m$ irreducible representation that can be subduced has $r$ as given above but with the starred boxes removed and $s = \begin{array}{|c|c|}\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}\\\cline{1-1}\end{array}$. To build the corresponding projector we need to build the projector onto the $U(4)$ irreducible representation labelled by $s^T = \begin{array}{|c|c|c|}\hline\multicolumn{1}{|c}{}&\multicolumn{1}{c|}{}&\multicolumn{1}{c|}{}\\\cline{1-1}\end{array}$. Since we pull three boxes off the right most column and one box off the neighboring column, the states we are interested in will have a $\Delta$ weight of $(0,0,1,3)$. For this example, we will need to assign nontrivial phases between the states in $V_p^{\otimes m}$ and the Young diagrams. The four possible ways to distribute the labels are



.

Take the first state shown as the reference state. To get the second state from the first we need to act with (12), so that the second state has a phase of $-1$. The get the third state from the first we need to act with (12) and then with (23), so that it has a phase of 1. Finally, to get the fourth state from the first we need to act with (12) and then (23) and then (34) giving a phase of $-1$. Writing our states as

$|a,b,c,d\rangle =$ 

.

we have

$$|1,2,3,4\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 0 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix}$$

$$= -\frac{\sqrt{3}}{2} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & \\ 4 & 0 & & \\ 3 & & & \end{bmatrix},$$

$$|2,1,3,4\rangle = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 0 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(2)} - \frac{1}{\sqrt{12}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & \\ 4 & 0 & & \\ 3 & & & \end{bmatrix}$$

$$|3,1,2,4\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 0 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix}$$

$$= -\frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(3)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & \\ 4 & 0 & & \\ 3 & & & \end{bmatrix}$$

$$|4,1,2,3\rangle = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 1 & 0 & & \\ 0 & & & \end{bmatrix}$$

$$= -\frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(3)} - \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(2)} - \frac{1}{\sqrt{12}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & \\ 3 & 1 & & \\ 3 & & & \end{bmatrix}^{(1)} - \frac{1}{2} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & \\ 4 & 0 & & \\ 3 & & & \end{bmatrix}$$

Given these results, it is a simple matter to write down the states that carry the $S_m$ irreducible representation $\begin{array}{|cc|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \end{array}$

$$\left| \begin{array}{|cc|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \end{array} , 1 \right\rangle = \begin{bmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 3 & & 1 & & \\ & & & 3 & & & \end{bmatrix}^{(1)} = \frac{1}{\sqrt{12}} \left(-3|1,2,3,4\rangle - |2,1,3,4\rangle + |3,1,2,4\rangle - |4,1,2,3\rangle\right)$$

$$\left| \begin{array}{|cc|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \end{array} , 2 \right\rangle = \begin{bmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 3 & & 1 & & \\ & & & 3 & & & \end{bmatrix}^{(2)} = \frac{1}{\sqrt{6}} \left(2|2,1,3,4\rangle + |3,1,2,4\rangle - |4,1,2,3\rangle\right)$$

$$\left| \begin{array}{|cc|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \end{array} , 3 \right\rangle = \begin{bmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 3 & & 1 & & \\ & & & 3 & & & \end{bmatrix}^{(3)} = -\frac{1}{\sqrt{2}} \left(|3,1,2,4\rangle + |4,1,2,3\rangle\right)$$

These formulas use only the Clebsch-Gordan coefficients of $U(4)$. It is again easy to verify that the above states fill out the representation $\begin{array}{|c|c|}\hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \end{array}$ of $S_4$.

# Chapter 5

# The Dilatation Operator

## 5.1 Definition

Consider the two-point function of a set of conformal fields $O_\alpha$; it has the form

$$\langle O_\alpha(x) O_\beta(y)\rangle = \frac{\delta_{\alpha\beta}}{|x-y|^{2\Delta_\alpha}}$$

The quantity $\Delta_\alpha$ is known as the scaling dimension of the field. There are numerous difficulties associated with the calculation of this quantity by the use of the two point function [39]: calculating the corrections to the two point function perturbatively introduces infinities, where the corrections themselves must be finite, leading to the necessity of complicated renormalization. However, instead of renormalizing two-point functions at each order of the perturbative calculation, one can rather introduce the dilatation operator: its action on a conformal field operator is given by

$$DO_\alpha = \Delta_\alpha O_\alpha$$

The eigenvalue of the dilatation operator when acting on a conformal field give the scaling dimension of the field. When considering an interacting theory, the scaling dimension is found to depend on the coupling constant, and by extension so does the dilatation generator. The dilatation operator can be expanded under perturbation theory in terms of powers of the coupling constant:

$$D = \sum_{k=0}^{\infty} (\frac{g_{YM}^2}{16\pi^2})^k D_{2k}$$

The operator $D_{2k}$ is referred to as the $k$-loop dilatation operator. The 0-loop operator gives the classical scaling dimension, which for fields composed only of scalars (as our Schur operators are) is simply equal to the number of scalars comprising the field, since the engineering dimension of scalars is one. The action of the one-loop dilatation operator was determined in [39] to be

$$D_2 = -g_{YM}^2 \mathrm{Tr}[Y,Z][\partial_Y, \partial_Z] \tag{5.1}$$

## 5.2 Why $D$?

The focus of this project is on the generalization of the study of the action of the dilatation operator on the Schur polynomial operators. The importance of the results obtained can be seen in the context of the AdS/CFT correspondence[1]: by performing a Wick rotation and conformal transformation on the metric of the SYM theory, we obtain an implementation of the gauge theory on $R \times S^3$. States of the theory on this space are in one-to-one correspondence with operators on the 4-D Euclidean space reached by the Wick rotation, by the state-operator correspondence. The boundary of the $AdS_5$ of the string theory can be shown to be $R \times S^3$, thus it is natural to postulate an identification of this boundary with the spacetime on which the SYM is defined. This explains the correspondence of gauge theory operators to string theory states, and allows for a direct comparison between operators of the two theories, where we see that the operator implementing scale transformations in the field theory on Minkowski space (the

[1] See 2.1.4 for further discussion of the points presented in this section

dilatation operator) is dual to the Hamiltonian of the string theory on $R \times S^3$. This identification can also been seen by comparing the charge associated to a symmetry group of the two theories, as mentioned previously.

Hence, if we can determine the eigenvalues of the dilatation operator acting on the restricted Schur polynomials in general, we can associate this with the energy of the giant graviton system in the string theory. We would thus have the energy eigenstates (the restricted Schurs) and the associated eigenvalues for a giant graviton system as described by string theory. This is all the information necessary to determine the time evolution, and hence the dynamics, of any giant graviton system - a problem that has seemed intractable for many years due to the infinities which arise when attempting to sum all possible paths for an oscillating membrane. It was argued in [40] and [41] that the spectrum of anomalous dimensions (obtained by calculating the loop corrections to the dilatation operator) of the operators dual to Giant graviton states reproduces the spectrum of vibrational modes of Giant gravitons calculated in [42]. Numerical study of the one loop dilatation operator acting on restricted Schurs dual to a two giant graviton system with 3 or 4 impurities [9] yields a surprising and powerful result - the dilatation operator was found to be equivalent to a set of decoupled harmonic oscillators, the frequencies of which are determined by the representation organising the Y-fields. The operator has also been successfully analytically diagonalized for a system of two giant gravitons in [10].

## 5.3 Action on Restricted Schur Polynomials

The action of the dilatation operator on restricted Schur polynomials has been studied in [8], [9], [10]. The discussion following uses restricted Schur polynomials having $O(1)$ long rows, however, the discussion for $O(1)$ long columns is very similar, as discussed in Section 5.3.5. Applying the operator defined by 5.1 to the restricted Schur Polynomial, we obtain:

$$D\chi_{R,(r,s)jk} = \frac{g_{\mathrm{YM}}^2}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m}} \mathrm{Tr}_{(r,s)jk} \left(\Gamma_R((1,m+1)\psi - \psi(1,m+1))\right) \times \\ \times \delta^{i_1}_{i_{\psi(1)}} Y^{i_2}_{i_{\psi(2)}} \cdots Y^{i_m}_{i_{\psi(m)}} (YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \cdots Z^{i_{n+m}}_{i_{\psi(n+m)}} \tag{5.2}$$

As a consequence of the $\delta^{i_1}_{i_{\psi(1)}}$ appearing in the summand, the sum over $\psi$ runs only over permutations for which $\psi(1) = 1$. To perform the sum over $\psi$, write the sum over $S_{n+m}$ as a sum over cosets of the $S_{n+m-1}$ subgroup obtained by keeping those permutations that satisfy $\psi(1) = 1$. The result is derived in the same way as the reduction rule for Schur polynomials (see appendix C of [24]). A sum of the form $\sum_{\psi \in S_{n+m}} \chi_R(\psi)$ can be rewritten as $\sum_{\psi \in S_{n+m-1}} \sum_{j=1}^{n+m} \chi_R(\psi \cdot (1,j))$. Using this, together with the knowledge that the operator $\sum_{j=1, j\neq k}^{n+m}(k,j)$ acting on a particular Young-Yamonouchi state has as its eigenvalue the weight of the box labelled $k$ in the diagram, one obtains:

$$D\,\chi_{R,(r,s)jk} = \frac{g_{\mathrm{YM}}^2}{(n-1)!(m-1)!} \sum_{\psi \in S_{n+m-1}} \sum_{R'} c_{RR'}\, \mathrm{Tr}_{(r,s)jk} \Big( \Gamma_R\left((1,m+1)\right) \Gamma_{R'}(\psi) \\ -\Gamma_{R'}(\psi) \Gamma_R\left((1,m+1)\right) \Big) Y^{i_2}_{i_{\psi(2)}} \cdots Y^{i_m}_{i_{\psi(m)}} (YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \cdots Z^{i_{n+m}}_{i_{\psi(n+m)}} .$$

The sum over $R'$ runs over all Young diagrams that can be obtained from $R$ by dropping a single box; $c_{RR'}$ is the factor associated to the box that must be removed from $R$ to obtain $R'$. Since the dilatation operator has derivatives with respect to $Z$ and $Y$ in the same trace, it naturally mixes $Z$s and $Y$s. The appearance of $\Gamma_R\left((1,m+1)\right)$ is thus expected, as the group element of which this is a matrix representation is not an element of the $S_n \times S_m$ subgroup - it mixes indices belonging to $Z$s and indices belonging to $Y$s. To clarify, one can imagine we have an $S_{n+m}$ representation $R$, which is $q$ dimensional, i.e. there are $q$ Young-Yamonouchi labels corresponding to vectors in the basis for its carrier space. We denote these basis vectors by $|YY_i\rangle$, where $i \in [1,q]$ and the $YY_i$ are the Young Tableaux labels. Another set of basis vectors is obtained by numbering only $m$ of the boxes in the Young diagram for $R$. Each possible way of numbering these $m$ boxes provides a partially labelled Young diagram, and the labelled boxes can be removed and assembled in different ways to describe $S_m$ representations. We know that in this way we can obtain $S_n \times S_m$ representations that are subduced by $R$, and that a state in the carrier

space of an $S_n \times S_m$ representation can be decomposed as a linear combination of states in the carrier space of $R$:

$$|R,(r,s)\rangle = \sum_{i=1}^{q} C_{YY_i}^{(r,s)} |YY_i\rangle$$

The carrier space of $R$, instead of being described as a $q$-dimensional vector space with basis vectors carrying Young-Yamonouchi labels, is split up into a number of lower dimensional carrier spaces, each being the carrier space of a representation of some $S_n \times S_m$ subgroup. Acting with $\Gamma_R((1,m+1))$ on one of these states now will produce a linear combination of states in the carrier space of $R$ which may correspond to a linear combination of the states labelled by a different set of $S_n \times S_m$ labels, possibly even including Young Yamonouchi states that do not combine to form states of the subgroup - $\Gamma_R((1,m+1))|R,(r,s)\rangle = \sum_{s'} C_{s'}|R,(r,s')\rangle + \sum_i C_i|YY_i\rangle$. The action of this element of the $S_{n+m}$ group thus mixes states from different representations of the $S_n \times S_m$ subgroup.

We will make use of the following notation

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}} .$$

Now, use the identities (bear in mind that $\psi(1)=1$)

$$Y^{i_2}_{i_{\psi(2)}} \cdots Y^{i_m}_{i_{\psi(m)}} (YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}} Z^{i_{m+2}}_{i_{\psi(m+2)}} \cdots Z^{i_{n+m}}_{i_{\psi(n+m)}} = \mathrm{Tr}\left(\Big((1,m+1)\,\psi - \psi\,(1,m+1)\Big) Z^{\otimes n} Y^{\otimes m}\right)$$

and (this identity is proved in [34]; the sum over $T$ runs over all possible irreducible representations of $S_{n+m}$)

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_{T,(t,u)lq} \frac{d_T n!m!}{d_t d_u (n+m)!} \mathrm{Tr}_{(t,u)lq}(\Gamma_T(\sigma^{-1})) \chi_{T,(t,u)ql}(Z,Y)$$

to obtain

$$D\chi_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} M_{R,(r,s)jk;T,(t,u)lq} \chi_{T,(t,u)ql}(Z,Y)\,,$$

where

$$M_{R,(r,s)jk;T,(t,u)lq} = g_{YM}^2 \sum_{\psi \in S_{n+m-1}} \sum_{R'} \frac{c_{RR'} d_T nm}{d_t d_u (n+m)!} \mathrm{Tr}_{(r,s)jk}\Big(\Gamma_R((1,m+1))\Gamma_{R'}(\psi) - \Gamma_{R'}(\psi)\Gamma_R((1,m+1))\Big) \times$$
$$\times\, \mathrm{Tr}_{(t,u)lq}\left(\Gamma_{T'}(\psi^{-1})\Gamma_T((1,m+1)) - \Gamma_T((1,m+1))\Gamma_{T'}(\psi^{-1})\right) .$$

The sum over $\psi$ can be evaluated using the fundamental orthogonality relation

$$M_{R,(r,s)jk;T,(t,u)lq} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \mathrm{Tr}\Big(\Big[\Gamma_R((1,m+1)), P_{R\to(r,s)jk}\Big] I_{R'\,T'} \times \tag{5.3}$$
$$\times \Big[\Gamma_T((1,m+1)), P_{T\to(t,u)ql}\Big] I_{T'\,R'}\Big)\,. \tag{5.4}$$

Sums of this type are discussed in the next subsection and the intertwiners $I_{R'\,T'}$ which arise are discussed in detail. This expression for the one loop dilatation operator is exact in $N$. To obtain the spectrum of anomalous dimensions, we need to consider the action of the dilatation operator on normalized operators. It is only by using the normalized operators that the suppression of the operators labelled by Young diagrams with $n \neq p$ long columns or rows that can arise under the action of the dilatation operator is manifest. The two point function for the restricted Schur polynomials, as we have seen, is not unity. Normalized operators which do have unit two point function can be obtained from

$$\chi_{R,(r,s)jk}(Z,Y) = \sqrt{\frac{f_R\,\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s}} O_{R,(r,s)jk}(Z,Y)\,.$$

In terms of these normalized operators

$$DO_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} N_{R,(r,s)jk;T,(t,u)ql} O_{T,(t,u)ql}(Z,Y)$$

$$N_{R,(r,s)jk;T,(t,u)ql} = -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \, \mathrm{hooks}_T \, \mathrm{hooks}_r \, \mathrm{hooks}_s}{f_R \, \mathrm{hooks}_R \, \mathrm{hooks}_t \, \mathrm{hooks}_u}} \times$$

$$\times \mathrm{Tr}\Big( \Big[ \Gamma_R((1,m+1)), P_{R\to(r,s)jk} \Big] I_{R'\,T'} \Big[ \Gamma_T((1,m+1)), P_{T\to(t,u)lq} \Big] I_{T'\,R'} \Big) .$$

It is this last expression that we evaluate explicitly. The bulk of the work entails evaluating the trace. There are three objects which appear: the symmetric group projection operators $P_{R\to(r,s)jk}$, the intertwiners $I_{T'\,R'}$ and the symmetric group element $\Gamma_R((1,m+1))$. We have already discussed the operators $P_{R\to(r,s)jk}$. The next two subsections are used to discuss $I_{T'\,R'}$ and $\Gamma_R((1,m+1))$.

### 5.3.1 Intertwiners

In this section we will consider the sum over $S_{n+m-1}$ which was performed to obtain (5.3). This will give a very explicit understanding of the intertwiners appearing in the expression for the dilatation operator. When $S^n$ acts on $V^{\otimes n}$, $n > 1$ it furnishes a reducible representation. Imagine that two of the irreducible representations subduced by this representation are $R$ and $S$. Representing the action of $\sigma$ as a matrix $\Gamma(\sigma)$, in a suitable basis we can write

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_R(\sigma) & 0 & \cdots \\ 0 & \Gamma_S(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} .$$

In our construction, we consider the action of $\Gamma(\sigma)$ to be that of a group element of $S_{n+m}$, in a reducible representation of the group. The carrier space of this representation will be a direct sum of the carrier spaces of irreducible representations subduced - the reducible representation acts in the space $V^{\otimes n+m}$, while the subduced representations will act in spaces spanned by various linear combinations of the vectors in $V^{\otimes n+m}$. The linear combinations are constructed such that they span the carrier space of a particular subgroup representation, in such a way that one is not transformed out of that carrier space by the action of the group elements. The representations subduced here are $S_{n+m}$ representations formed by assembling $n+m$ boxes in different ways to form Young diagrams for all the possible irreducible representations of $S_{n+m}$. Continuing with the general discussion above, if we restrict ourselves to an $S_{n-1}$ subgroup of $S_n$, then in general, both $R$ and $S$ will subduce a number of representations. Assume for the sake of this discussion that $R$ subduces $R'_1$ and $R'_2$ and that $S$ subduces $S'_1$ and $S'_2$. This is precisely the situation that arises in the sum performed to obtain (5.3), with the matrix $\Gamma(\sigma)$ being composed of the matrix representations corresponding to the particular $R$ and $T$ for which we are calculating $N_{R,(r,s)jk;T,(t,u)ql}$. Then, for $\sigma \in S_{n-1}$ we have

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & \Gamma_{S'_2}(\sigma) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} .$$

Imagine that as Young diagrams $S'_1 = R'_1$, that is, one of the irreducible representations subduced by $R$ is isomorphic to one of the representations subduced by $S$. If this situation does not occur for the $R$ and $S$ considered, the term being calculated vanishes by the fundamental orthogonality relation. A simple application of this relation for non-zero terms gives

$$\sum_{\sigma\in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ab}$$

$$= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \mathbf{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{aj}$$

$$\equiv \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} (I_{R'_1 S'_1})_{ib} (I_{S'_1 R'_1})_{aj}$$

where the form of the intertwiners has been spelled out. The **1**'s in the first and second matrices are identity matrices of the same dimension as $\Gamma_{R'_1}(\sigma)$ and $\Gamma_{S'_1}(\sigma)$ respectively, and the 0's are zero matrices of appropriate dimension for the intertwiners to act on $\Gamma(\sigma)$. The intertwiners are analogous to the delta functions appearing under usual application of the fundamental orthogonality relation, where the matrices being multiplied do not appear as block diagonal elements of the carrier space of some reducible representation. They appear because we are comparing matrices for which the indices labelling element positions begin at different values. This is the situation that occurs when comparing two matrices corresponding to the same subduced representation, but which reside in different subspaces. The intertwiners can be understood as follows: suppose we use indices $(a, b)$ to reference elements in the matrix $\Gamma(\sigma)$ corresponding to $\Gamma_{R'_1}(\sigma)$, and indices $(i, j)$ for those corresponding to $\Gamma_{S'_1}(\sigma)$. Then $a$ and $b$ will run from $1 \cdots d_{R'_1}$, while $i$ and $j$ will run from $d_R + 1 \cdots d_R + d_{S'_1}$. It is necessary to remain in the vector space that $\Gamma(\sigma)$ lives in, rather than considering the subspaces independently, due to the appearance of both $R$ and $T$ in the same trace in 5.3. The fundamental orthogonality relation in this case is realised as:

$$\sum_{\sigma \in S_{n-1}} [\Gamma(\sigma)]_{ab} [\Gamma(\sigma^{-1})]_{ij} \sim \delta_{a+d_R,j} \delta_{b+d_R,i}$$

Obviously, these are not proper delta functions and cannot be represented as identity matrices - instead we find that they have only one non-zero block as indicated above, and we name them intertwiners. Intertwiners are maps between two isomorphic spaces. For $\sigma \in S_{n-1}$

$$I_{R'T'} \Gamma_{T'}(\sigma) = \Gamma_{R'}(\sigma) I_{R'T'}$$

The box removed to obtain $R'$ and $T'$ can be removed from any corner of the Young diagram.

It is useful to make a few comments on how the intertwiners are realized in our calculation. Since the first box is removed from $R$ or $T$ the intertwiner acts on the first slot of $V_p^{\otimes m}$. Now, look back at formula (5.2). The delta function which appears freezes the 1 index and hence the $S_{n+m-1}$ subgroup of $S_{n+m}$ is obtained by keeping all elements of $S_{n+m}$ that leave index 1 inert. Consequently, with our choice that the intertwiner acts on the first slot of $V_p^{\otimes m}$, we see that the first slot corresponds to index $i_1$. This is important, because we need to specify which indices are acted on by each component of the $S_n \times S_m$ subgroup - the $S_m$ component acts on the first $m$ indices, corresponding to $Y$ boxes. Recall that the particular vector a box corresponds to is determined by the row/column the box belongs to. Thus, the explicit form of the intertwiner is determined once the location of the box removed from $T$ and the box removed from $R$ are specified. The general rule for determining the form of the intertwiner, once given the relevant Young diagram with the location of the removed box specified is known: Consider first the case that $R \neq T$. To obtain $R'$ from $R$ we remove a box from row $i$ and to obtain $T'$ from $T$ we remove a box from row $j$. In this situation we have

$$I_{R'T'} = E_{ij} \otimes \mathbf{1} \otimes \cdots \mathbf{1}\,, \qquad I_{T'R'} = E_{ji} \otimes \mathbf{1} \otimes \cdots \mathbf{1}\,.$$

In the case that $R = T$, the box that must be removed can be removed from any row and we get a contribution to the dilatation operator from each possible removal. Each possible removal must be represented by a different intertwiner and one needs to sum over all possible intertwiners. In this situation, the possible intertwiners are

$$I_{R'T'} = E_{kk} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} = I_{T'R'}, \qquad k = 1, 2, \cdots, p\,.$$

As an example, imagine $R$ and $T$ to be given as Young diagrams with 5 rows, with $R \neq T$. $R'$ is formed from $R$ by removing a box from the first row, and $T'$ obtained from $T$ by removing a box from the 5th row. Note that the Young diagrams $R$ and $T$ must differ only by the placement of a single box in order for the necessary condition $R' = T'$ to hold, so that the sole difference between the isomorphic representations subduced is the location from which this box is removed. The state associated to this box lives in the vector space corresponding to first slot of $V_p^{\otimes m}$, and it is for this reason that the intertwiner need only have an action on this component of the full vector space - if we can transform the vector in this slot for the space associated to $R'$ such that it takes the same value as the vector in the first slot for $T'$, we have a map between the spaces. Using the convention that the basis vector for $V^5$ (one copy of the space spanned by the vectors that can be inserted into a slot of $V_p^{\otimes m}$)) corresponding to a box pulled off row

$j$ is given by $\vec{v}(1)_n = \delta_{nj}$, it is simple to see why the intertwiners have the form seen above. The vectors corresponding to the box pulled off $R'$ and $T'$ will be $\vec{v}_{R'}(1) = [1,0,0,0,0]^T$ and $\vec{v}_{T'}(1) = [0,0,0,0,1]^T$ respectively, so that $I^{(1)}_{R'T'}\vec{v}_{R'}(1) = \vec{v}_{T'}(1)$ and $I^{(1)}_{T'R'}\vec{v}_{T'}(1) = \vec{v}_{R'}(1)$, where $I^{(1)}_{R'T'}$ is the matrix $E_{1,5}$ and $I^{(1)}_{T'R'}$ is the matrix $E_{5,1}$ (the matrices sitting in the first slot of the intertwiners).

This clearly demonstrates that the intertwiners do indeed provide a map between isomorphic spaces. We can imagine that we have a state in the carrier space of representation $T$, which we can denote $|T\rangle$, and consider the inner product with a state from the carrier space of $R$, $|R\rangle$. The inner product between states from the carrier spaces of different representations is zero unless $R = T$. The action of the intertwiner $I_{T'R'}$ on $|T\rangle$ is to project it into the carrier space of $R$, so that while $\langle R|T\rangle = 0$, the inner product $\langle R|I_{T'R'}|T\rangle \neq 0$ for certain states. Since we label states in our formalism by a tensor product of $m$ $p$-dimensional vectors, the intertwiners defined above provide this map. A particular state $|T\rangle$, under the action of the intertwiner, will have non-zero overlap with a particular state $|R\rangle$. The states must have the same vectors in all but the first slot, due to the identity elements in all but this slot in the tensor product comprising the intertwiner. In essence, by mapping the vector occupying the first slot in the tensor product labelling a state in $T$ to that of the state in $R$ which is identical but for the vector in slot 1, we can map between states of $T'$ and states of $R'$, since the box removed from $R$ and $T$ to obtain $R'$ and $T'$ corresponds to the first slot of $V_p^{\otimes m}$.

### 5.3.2 $\Gamma_R((1, m+1))$

This group element acts on one slot from the $Y$s and one slot from the $Z$s. The box removed from $R$ to get $R'$ is the box acted on by the intertwiner and it is a $Y$ box. This is one of the boxes that $\Gamma_R(1, m+1)$ acts on. The second box that $\Gamma_R(1, m+1)$ acts on can be any box associated to the $Z$s. The projectors and intertwiners only have an action on the boxes corresponding to $Y$s (the projectors project onto an $S_n \times S_m$ subgroup $(r, s)$, where for a particular set of removed boxes, the carrier space of $r$ is common to all subspaces involved) and as a result, our discussion has always taken place in the vector space $V_p^{\otimes\, m}$. However, because $\Gamma_R(1, m+1)$ acts on a $Z$ box (the $Y$'s occupy slots 1 to $m$, so $m+1$ corresponds to a $Z$) we must include one more slot and work in $V_p^{\otimes\, m+1}$. The intertwiners and projectors have a trivial action on the $(m+1)^{\text{th}}$ slot and hence the $(m+1)^{\text{th}}$ slot is simply occupied with the identity. For the rest of this Subsection we work in $V_p^{\otimes\, m+1}$ and not in $V_p^{\otimes\, m}$. Acting in $V_p^{\otimes\, m+1}$, $\Gamma_R(1, m+1)$ has a very simple action: it simply swaps the $1^{\text{st}}$ and the $(m+1)^{\text{th}}$ slots. The projectors when acting on $V_p^{\otimes\, m+1}$ are given by

$$\mathcal{P}_{R\to(r,s)ij} = p_{R,(r,s)ij} \otimes \mathbf{1}$$

where the $p \times p$ unit matrix $\mathbf{1}$ acts on the $(m+1)^{\text{th}}$ slot. $p_{R,(r,s)ij}$ acts only in $V_p^{\otimes\, m}$. For comparison, the projectors appearing in the defintion of the restricted Schur polynomial are

$$P_{R\to(r,s)ij} = p_{R,(r,s)ij} \otimes \mathbf{I}_r$$

where $\mathbf{I}_r$ is the identity matrix acting on the carrier space of the $S_n$ irreducible representation $r$. Below we will make use of the obvious formula

$$\mathbf{1} = \sum_{k=1}^{p} E_{kk}\,.$$

In evaluating the dilatation operator, we will need to take products of the intertwiners and $\Gamma(1, m+1)$. These products are easily evaluated

$$\begin{aligned}
\Gamma_R(1, m+1)\, E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \qquad &= \Gamma_R(1, m+1) \sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes E_{kk} \\
&= \sum_{k=1}^{p} E_{kj} \otimes \mathbf{1} \otimes \cdots \otimes E_{ik} \\
E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\, \Gamma_R(1, m+1) \qquad &= \sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes E_{kk}\, \Gamma_R(1, m+1) \\
&= \sum_{k=1}^{p} E_{ik} \otimes \mathbf{1} \otimes \cdots \otimes E_{kj}
\end{aligned}$$

$$\Gamma_R(1,m+1)\, E_{ij}\otimes \mathbf{1}\otimes\cdots\otimes\mathbf{1}\,\Gamma_R(1,m+1) = \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes E_{ij}\,.$$

The reason for the swap occurring only in one of the indices can be seen by writing the matrix $E_{ij}$ as an outer product of two states being labelled by the integers $i$ and $j$, i.e. $E_{ij} = |i\rangle\langle j|$. Each state represents a vector corresponding to the first box (the one the intertwiner acts on) coming from a different row - element $n$ of $|i\rangle = \delta_{ni}$. The action of the matrix $\Gamma_R(1,m+1)$ can only act on either the bra or the ket, so that

$$\Gamma_R(1,m+1)\sum_{k=1}^{p}|i\rangle\langle j|\otimes\mathbf{1}\otimes\cdots\otimes|k\rangle\langle k| = \sum_{k=1}^{p}|k\rangle\langle j|\otimes\mathbf{1}\otimes\cdots\otimes|i\rangle\langle k|$$

From now on we will write the $E_{ij}$ with a superscript, indicating which slot $E_{ij}$ acts on. In this notation we have

$$E_{ik}\otimes\mathbf{1}\otimes\cdots\otimes E_{kj} = E^{(1)}_{ik}E^{(m+1)}_{kj}\,.$$

### 5.3.3 Dilatation Operator Coefficient

In this secton we explain how to evaluate the value of the coefficient

$$g_{YM}^2\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u(n+m)}\sqrt{\frac{f_T\,\mathrm{hooks}_T\,\mathrm{hooks}_r\,\mathrm{hooks}_s}{f_R\,\mathrm{hooks}_R\,\mathrm{hooks}_t\,\mathrm{hooks}_u}} \tag{5.5}$$

in the large $N$ limit. The Young diagrams $R$, $T$, $r$, $t$, $s$ and $u$ each have $p$-rows. We use the symbols $R_i$, $T_i$, $r_i$, $t_i$, $s_i$ and $u_i$ $(i = 1,2,...,p)$ to denote the number of boxes in each row respectively. We assume $p$ is fixed to be $O(1)$. The top row (which is also the longest row) is the value $i = 1$ and the bottom row (shortest row) has $i = p$. It is straight forward to argue that the product of hook lengths, in $r$ for example, is

$$\mathrm{hooks}_R = \frac{\prod_{i=1}^{p}(r_i+p-i)!}{\prod_{j<k}(r_j-r_k+k-j)}\,.$$

For the diagrams $R$ and $T$, the row lengths $R_i$ are of order $N$. Further, $R$ and $T$ differ by at most the placement of a single box. This implies that $R_i = T_i$ for all except two values of $i$, say $i = a,b$. For these values of $i$ we have

$$R_b = T_b + 1, \qquad R_a = T_a - 1\,.$$

This implies that

$$\frac{\mathrm{hooks}_R}{\mathrm{hooks}_T} = \frac{(T_a-1+p-a)!(T_b+1+p-b)!}{((T_a+p-a)!(T_b+p-b)!}\prod_{\substack{k\neq a\\ k\neq b}}\frac{|T_a-T_k|+|k-a|}{|T_a-1-T_k|+|k-a|}\times$$

$$\times\prod_{\substack{k\neq a\\ k\neq b}}\frac{|T_b-T_k|+|k-b|}{|T_b+1-T_k|+|k-b|}\,\frac{|T_b-T_a|+|a-b|}{|T_a-T_b-2|+|a-b|} = \frac{R_b}{R_a}\left(1+O(N^{-1})\right)\,.$$

The most transparent way to see the emergence of the final result is by comparing some example Young diagrams. Suppose $R$ and $T$ are labelled by the following diagrams, where the hook lengths have been filled in:

$R =$

| | | | | | |
|---|---|---|---|---|---|
| 8 | 7 | 5 | 4 | 2 | 1 |
| 5 | 4 | 2 | 1 | | |
| 2 | 1 | | | | |

$T =$

| | | | | |
|---|---|---|---|---|
| 7 | 6 | 5 | 3 | 1 |
| 5 | 4 | 3 | 1 | |
| 3 | 2 | 1 | | |

We see then, that in taking the quotient $\frac{\mathrm{hooks}_R}{\mathrm{hooks}_T}$, we can obviously cancel the 2 and 1 in the bottom rows, the 5, 4 and 1 in the middle row and 7 and 1 in the first row. Notice that for each remaining hook length in $T$, aside from the 8 in row 1 and the 3 in row 3, there is a corresponding hook length in $R$ that differs by 1. Diagrams we consider will have $O(N)$ boxes in each row, so that a difference of 1 in some of the hook lengths can be ignored, and we can safely cancel the remaining hook lengths in $T$ with those differing by 1 in $R$. Thus, the only hook lengths that remain are the 8 from $R$ and the 3 from $T$. Since for the Young diagrams we consider these numbers will be $O(N)$, the number of rows is $O(1)$ and the row lengths in $R$ and $T$ differ by just 1 box, we can approximate them as being the row lengths $R_a$ and $R_b$. Hence we obtain the factor $\frac{R_b}{R_a}$.

Use $R_+$ to denote the row length of the row in $R$ that is longer than the corresponding row in $T$ and let $R_-$ denote the row length of the row in $R$ this is shorter than the corresponding row in $T$. With this notation

$$\frac{\text{hooks}_R}{\text{hooks}_T} = \frac{R_+}{R_-}\left(1+O(N^{-1})\right) .$$

This argument has an obvious generalization to the other hook factors $\frac{\text{hooks}_r}{\text{hooks}_t}$ and $\frac{\text{hooks}_s}{\text{hooks}_u}$. Now consider a Young diagram $R'$ that is obtained by removing a single box from Young diagram $R$. Assuming this box is removed from row $a$, we have the following relation between the lengths of the rows in $R$ and the lengths of the rows in $R'$

$$R_i = R'_i \quad i \neq a, \qquad R_a = R'_a + 1 .$$

Thus, we find

$$\frac{\text{hooks}_R}{\text{hooks}_{R'}} = \frac{(R_a+p-a)!}{(R_a+p-1-a)!}\prod_{j\neq a}\frac{|R_j-R_a-1|+|a-j|}{|R_j-R_a|+|a-j|} = R_a\left(1+O(N^{-1})\right) .$$

The coefficient quoted at the start of this section is multiplied by the trace over an $(r,s)$ subspace. This trace produces a number of order 1 multiplied by $d_{r'}d_s$. The product of the coefficient and the trace now reduces to quantities that we have studied. Thus, we now have all the ingredients needed to estimate the large $N$ values of the combinations of symmetric group dimensions and hook factors that appear in the dilatation operator. Notice that both the product of the hook lengths and the dimensions of symmetric group irreducible representations are invariant under the flip of the Young diagram which exchanges columns and rows. Thus, these conclusions can immediately be recycled when studying the case of $p$ long columns.

Next, recalling that $f_R$ is the product of factors in Young diagram $R$ and $R' = T'$ we learn that

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'}c_{TT'}}$$

where $c_{RR'}$ is the factor associated to the box that must be removed from $R$ to obtain $R'$ and $c_{TT'}$ is the factor associated to the box that must be removed from $T$ to obtain $T'$.

### 5.3.4 Evaluating Traces

In this section we evaluate the trace

$$\mathcal{T} = \text{Tr}\Big(\Big[\Gamma_R((1,m+1)), P_{R\to(r,s)jk}\Big] I_{R'\,T'}\Big[\Gamma_T((1,m+1)), P_{T\to(t,u)lm}\Big] I_{T'\,R'}\Big) .$$

We start by writing this trace as a sum of traces over $m+1$ slots (all the $Y$ slots plus one $Z$ slot) times a trace over $n-1$ slots (the remaining $Z$ slots). This amounts to rewriting the trace as

$$\mathcal{T} = \text{Tr}_{\text{R}^{\text{m+1}}}\Big(\text{Tr}_{\text{V}_\text{p}}\Big(\text{Tr}_{\text{V}_\text{p}^{\otimes\text{m}}}\Big(\Big[\Gamma_\text{R}((1,\text{m}+1)), \text{P}_{\text{R}\to(\text{r,s})\text{jk}}\Big] \text{I}_{\text{R}'\,\text{T}'}\Big[\Gamma_\text{T}((1,\text{m}+1)), \text{P}_{\text{T}\to(\text{t,u})\text{lm}}\Big] \text{I}_{\text{T}'\,\text{R}'}\Big) .$$

The trace over the $n-1$ slots is over the carrier space $R^{m+1}$ which is described by a Young diagram that can be obtained by removing $m+1$ boxes from $R$, or equivalently by removing one box from $r$ or equivalently by removing one box from $t$ - these all give the same Young diagram describing $R^{m+1}$. It should be noted that the trace is not simply over the vector space $V_p^{\otimes n-1}$, which would include all Young diagrams having $n-1$ boxes, since the boxes in the Young diagram $R^{m+1}$ are not rearranged after they are separated from $R$. $R^{m+1}$ has different shapes depending on where the $(m+1)^{\text{th}}$ box is removed. The results from the last subsection clearly imply that the dimension of symmetric group representation $R^{m+1}$, denoted $d_{R^{m+1}}$, depends on the details of this shape. If the $(m+1)^{\text{th}}$ box is removed from row $i$ denote this dimension by $d^i_{R^{m+1}}$ Our general strategy is then to trace over the last $Z$ slot (the $(m+1)^{\text{th}}$ slot) which then leaves a trace over $V_p^{\otimes\, m}$. This trace is then evaluated using elementary $U(p)$ representation theory.

The box removed from $R$ to obtain $R'$ is removed from the $b^{\text{th}}$ row of $R$ and the box removed from $T$ to obtain $T'$ is removed from the $a^{\text{th}}$ row of $T$. First, a trace over the $n-1$ $Z$ slots associated to $R^{m+1}$ is performed - this produces a factor of $d^b_{R^{m+1}}$, since the action of all the operators in the trace on this subspace (which corresponds to slots $m+2$ to $m+n$) is the identity. Next, the symmetric group elements $(1, m+1)$ are multiplied with the intertwiners. After this multiplication is performed, the first of the four terms composing the trace appears as:

$$d^b_{R^{m+1}} \mathrm{Tr}_{V_p^{\otimes(m+1)}} \left( \sum_q E^{(1)}_{aq} E^{(m+1)}_{qb} P_{R\to(r,s)jk} \sum_p E^{(1)}_{bp} E^{(m+1)}_{pa} P_{T\to(t,u)lm} \right)$$

The trace over the $(m+1)^{\text{th}}$ slot is then over the product of the two $E$'s with superscript $m+1$ - the multiplication sets $p$ equal to $b$, while the trace sets $q$ equal to $a$. The cyclicity of the trace can then be used to obtain the form given in the final expression for the trace. The other positive term follows in a very similar manner.

The product between the intertwiners and symmetric group operators in the negative terms appears as $\Gamma_R(1, m+1) I_{R'T'} \Gamma_T(1, m+1)$, which simply evaluates to $E^{(m+1)}_{ba}$. One of the negative terms then appears as

$$-d^b_{R^{m+1}} \mathrm{Tr}_{V_p^{\otimes(m+1)}} \left( P_{R\to(r,s)jk} E^{(m+1)}_{ba} P_{T\to(t,u)lm} E^{(1)}_{ab} \right)$$

The trace over the $(m+1)^{\text{th}}$ slot is a trace over $E_{ba}$, which produces a factor of 1, sets $a=b$ and leaves a trace over $V_p^{\otimes m}$. The cyclicity of the trace can be implemented, and the product of projectors that appears evaluated to obtain a single projector preceded by delta functions between all the projector labels. The multiplicity labels contract as matrix indices, generating a delta function between the second index of the first projector and the first index of the second, and leaving the outer indices as the multiplicity labels of the projector arising from the product i.e. $P_{R\to(r,s)jk} P_{T\to(t,u)lm} = \delta_{RT} \delta_{(r,s)\,(t,u)} \delta_{kl} P_{R\to(r,s)jm}$.

We can now present the trace in its final form:

$$\begin{aligned} \mathcal{T} = &-\delta_{ab}\delta_{RT}\delta_{(r,s)\,(t,u)}\delta_{jm}\delta_{kl} d^b_{R^{m+1}} \left[ \mathrm{Tr}_{V_p^{\otimes m}} \left( P_{R\to(r,s)lk} E^{(1)}_{bb} \right) + \mathrm{Tr}_{V_p^{\otimes m}} \left( P_{R\to(r,s)jm} E^{(1)}_{bb} \right) \right] \\ &+ d^b_{R^{m+1}} \mathrm{Tr}_{V_p^{\otimes m}} \left( P_{R\to(r,s)lk} E^{(1)}_{bb} P_{T\to(t,u)lm} E^{(1)}_{aa} \right) + d^b_{R^{m+1}} \mathrm{Tr}_{V_p^{\otimes m}} \left( P_{R\to(r,s)lk} E^{(1)}_{aa} P_{T\to(t,u)lm} E^{(1)}_{bb} \right). \end{aligned} \tag{5.6}$$

We now need to evaluate the traces over $V_p^{\otimes m}$. Towards this end, write the projector as

$$p_{R\to(r,s)ij} = \sum_{a=1}^{d_s} \left| M^i_s, a \right\rangle \left\langle M^j_s, a \right| .$$

$M^i_s$ and $M^j_s$ label states from $U(p)$ irreducible representation $s$ which have the same $\Delta$ weight. The indices $i, j$ range from $1, ..., I(\Delta(M))$. Index $a$ is a multiplicity index that, as a consequence of Schur-Weyl duality, is organized by representation $s$ of the symmetric group $S_m$. To evaluate the traces over $V_p^{\otimes m}$ we need to allow $E^{(1)}_{kk}$ to act on the state $\left| M^i_s, a \right\rangle$. The state $\left| M^i_s, a \right\rangle$ was obtained by taking a tensor product of $m$ copies (one for each slot) of the fundamental representation of $U(p)$. It is possible and useful to rewrite this state as a linear combination of states which are each the tensor product of the fundamental representation for the first slot with a state obtained by taking the tensor product of states of the remaining $m-1$ slots. This is a useful thing to do because then $E^{(1)}_{kk}$ has a particularly simple action on each state in the linear combination. Towards this end we can write (in the following $\mathbf{0}$ stands for a string of $p-1$ 0s)

$$\left| M^i_s, a \right\rangle = \sum_{M_{s'}, M_{1\mathbf{0}}} C^{M^i_s}_{M_{s'}, M_{1\mathbf{0}}} \left| M_{1\mathbf{0}} \right\rangle \otimes \left| M_{s'}, b \right\rangle$$

where $M_{1\mathbf{0}}$ indexes states in the carrier space of the fundamental representation and $C^{M_s, i}_{M_1, M_{1\mathbf{0}}}$ are the Clebsch Gordan coefficients (discussed in detail in Appendix A.5)

$$C^{M^i_s}_{M_{s'}, M_{1\mathbf{0}}} = \left( \langle M_{1\mathbf{0}}| \otimes \langle M_{s'}, b| \right) \left| M^i_s, a \right\rangle .$$

$s'$ is obtained by removing a single box from $s$. By appealing to the Schur-Weyl duality which organizes the space $V_p^{\otimes\, m-1}$, we know that the multiplicity index $b$ of the state $|M_{s'}, b\rangle$ is organized by the irreducible representation $s'$ of $S_{m-1}$. This allows us to easily evaluate the action of $E^{(1)}_{kk}$: it simply projects onto the state corresponding to box 1 sitting in the $k^{\text{th}}$ row. By this we mean that if we allow $E^{(1)}_{kk}$ to act on a state $\left|M^i_s, a\right\rangle$, we obtain

$$E^{(1)}_{kk}\left|M^i_s, a\right\rangle = \sum_{M_{s'}} C^{M^i_s}_{M_{s'}, M_{\mathbf{1_{0_{(k)}}}}} \left|M_{\mathbf{1_{0_{(k)}}}}\right\rangle \otimes |M_{s'}, b\rangle$$

where the state $\left|M_{\mathbf{1_{0_{(k)}}}}\right\rangle$ is labelled by the Gelfand-Tsetlin pattern of the fundamental $U(p)$ representation that corresponds to a labelled box sitting in row $k$. Evaluating the traces over $V_p^{\otimes\, m}$ is now straight forward.

### 5.3.5 Long Columns

Our computation of the action of the dilatation operator for restricted Schur polynomials labelled by Young diagrams that have a total of $p$ long rows has made extensive use of the fact that we can organize the space of partially labelled Young diagrams into $S_n \times S_m$ irreducible representations $(r, s)$ by appealing to Schur-Weyl duality. We have already argued that it is also possible to perform this organization when considering restricted Schur polynomials labelled by Young diagrams that have a total of $p$ long columns - all that is required is that we fine tune a few phases in our map between partially labelled Young diagrams and vectors in $V_p^{\otimes m}$. The same irreducible representations of $U(p)$ are used for both of these organizations, and further since $d_s = d_{s^T}$, each $U(p)$ representation $s$ appears with the same multiplicity in these two cases[2]. Consequently, the traces computed in the last subsection for labels with $p$ long rows are equal to the values for labels with $p$ long columns. To obtain the action of the dilatation operator all that remains is the computation of the coefficient discussed in 5.3.3. The only quantity appearing in 5.3.3 which is not invariant under exchanging rows and columns is

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'}c_{TT'}}$$

This factor is the only difference between the case of $p$ long rows and $p$ long columns. Consequently, the action of the dilatation operator on restricted Schur polynomials with $p$ long columns is obtained from its action on restricted Schur polynomials with $p$ long rows by making substitutions of the form $N + b \to N - b$. For concrete examples of this substitution see the end of sections 6.1 and 6.2. This generalizes the two row/column relation observed in [10] to an arbitrary number of rows and columns.

This completes the evaluation of the action of the dilatation operator.

[2]Recall that $s^T$ is obtained by exchanging rows and columns in $s$.

# Chapter 6

# Calculation of the Dilatation Operator

In this Chapter the matrix elements $N_{R,(r,s)jk;T,(t,u)lm}$ of the dilatation operator are evaluated explicitly, for the case that the Young diagram labels have either two or three rows or columns.

## 6.1 Young Diagrams with Two Rows or Columns

In this case, we will be using $U(2)$ representation theory. The Gelfand-Tsetlin patterns are extremely useful for understanding the structure of the carrier space of a particular $U(2)$ representation. However, the betweenness conditions make it awkward to work directly with the labels $m_{ij}$ which appear in the pattern. For this reason we will employ a new notation: trade the $m_{ij}$ for $j, j^3$ specified by

$$\begin{bmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{bmatrix} = \begin{bmatrix} m_{22}+2j & & m_{22} \\ & m_{22}+j^3+j & \end{bmatrix}.$$

The new labels are just the familiar angular momenta we usually use for $SU(2)$. It looks as if this trade in labels is not well defined because we have traded three labels $m_{12}, m_{22}, m_{11}$ for two labels $j, j^3$. There is no need for concern: recall that $m$ is fixed. Further,

$$m = 2(m_{22} + j)$$

so that knowing $j, j^3$ and $m$ we can indeed reconstruct $m_{12}, m_{22}, m_{11}$. The benefit of the new labels is that the betweenness conditions are replaced by

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ... \qquad -j \leq j^3 \leq j$$

which are significantly easier to handle. Write our states as kets $|j, j^3\rangle$. The Clebsch-Gordan coefficients we need are (its simple to compute these using Appendix A.5)

$$\left\langle j-\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j+j^3}{2j}}, \quad \left\langle j+\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 \right\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}},$$

$$\left\langle j-\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j-j^3}{2j}}, \quad \left\langle j+\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}}.$$

### 6.1.1 Diagonal Terms

Consider first the case of two rows. To specify $r$ we will specify the number of columns with 2 boxes ($= b_0$) and the number of columns with a single box ($= b_1$). Thus, our operators are labelled as $O(b_0, b_1, j, j^3)$. We will evaluate the diagonal terms (that is, the terms that don't change the value of $j$, these are terms for which $s = u$) in detail and simply gloss over the other necessary terms. To compute the diagonal term in the dilatation operator we need to evaluate the traces appearing in (5.6). Using our new $U(p)$ state labels, we can write the projector as $P_{R\to(r,s)jk} = \sum_a |j, j^3; a\rangle\langle j, j^3; a|$, where $a$ is a multiplicity

label indexing the possible $S_m$ states to which each $j, j^3$ label can correspond. Starting with the first term, which is only non-zero when $R = T$, we can rewrite the trace as simply the inner product of the operator $E^{(1)}_{bb}$ with the states composing the projector and expand one of the states as described in 5.3.4. This allows us to write the trace as:

$$\sum_p \langle j, j^3; p|E^{(1)}_{bb}|j, j^3; p\rangle = \sum_{p,q} \langle j, j^3; p|E^{(1)}_{bb}[\alpha|j + \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; q\rangle + \beta|j + \frac{1}{2}, j^3 + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; q\rangle +$$
$$\gamma|j - \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; q\rangle + \delta|j - \frac{1}{2}, j^3 + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; q\rangle]$$

The action of $E^{(1)}_{bb}$ on the expansion is to remove all terms except those corresponding to a box being pulled from row $b$ - this means only terms with a particular value of $j^3$ remain. The greek letters stand in place of the appropriate Clebsch-Gordon coefficients. As an example, take $b = 1$; only terms 1 and 3 remain. Expanding the bra in the same way we obtain:

$$\begin{aligned}\sum_p (\langle j, j^3; p|E^{(1)}_{bb}|j, j^3; p\rangle) &= \sum_q \alpha^2 \langle j + \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; q|j + \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; q\rangle \\ &+ \sum_q \gamma^2 \langle j - \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; q|j - \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; q\rangle \\ &= ds'_2 (C^{jj^3}_{j+\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}})^2 + ds'_1 (C^{jj^3}_{j-\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}})^2\end{aligned}$$

This can be written as:

$$\sum_p (\langle j, j^3; p|E^{(1)}_{bb}|j, j^3; p\rangle) = \sum_i ds'_i C^{jj^3}_{(j+(-1)^i \frac{1}{2}, j^3+(-1)^b \frac{1}{2}), (\frac{1}{2}, (-1)^{(b+1)} \frac{1}{2})} \tag{6.1}$$

where $ds'_i$ is the symmetric group dimension of the representation obtained by removing box $i$ from $s$ - $i$ is reflected by the sign following $j$. The dimension appears because the sum over $q$ is over all possible $S_{m-1}$ states corresponding to representation $s'_i$. Since there are no multiplicities for the two-row case, the first two terms in 5.6 are equal. We now move on to the determination of an expression for the last two terms in the trace; those that do not require $R = T$ to be non-vanishing. Using the $j, j^3$ labels and rewriting the trace again, the first of these terms can be written:

$$d^b_{R^{m+1}} \sum_p \langle j, j^3; p|E^1_{bb}|j, j^3; p\rangle \langle j, j^3; p|E^1_{aa}|j, j^3; p\rangle$$

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**Matrix Structure and the bra/ket Notation**

It is important to note at this point that, while using bra/ket notation to express the trace as the multiplication of inner products is the most transparent way to express the problem, it is easy to lose track of the matrix structure of the calculation by using it. Looking at the calculation purely in this way, one may be inclined to think that if the inner products each produce a sum of two terms, each term corresponding to one way of pulling a box off $s$ to obtain $s'$, the four numbers should all be multiplied together since the outcome of an inner product is simply a number. This is not the case - only coefficients from each inner product that correspond to the same $s'$ are multiplied to obtain the answer. The easiest way to see this is to revert back to the trace notation, :

$$\mathrm{Tr}\left(\sum_p |j, j^3; p\rangle \langle j, j^3; p|E^1_{bb}|j, j^3; p\rangle \langle j, j^3; p|E^1_{aa}\right)$$

Consider the projector $\sum_p |j, j^3; p\rangle \langle j, j^3; p|$, and suppose that upon expanding in terms of tensor products of states corresponding to the 2 possible $s'$ ($|1\rangle$ and $|2\rangle$) it becomes

$$\sum_p |j, j^3; p\rangle \langle j, j^3; p| = \alpha^2 |1\rangle\langle 1| \oplus \beta^2 |2\rangle\langle 2|$$

Representing the projector as a matrix, this corresponds to:

$$\sum_p |j, j^3; p\rangle \langle j, j^3; p| = \begin{bmatrix} X & X \\ X & X \end{bmatrix} = \begin{bmatrix} \alpha^2|1\rangle\langle 1| & 0 \\ 0 & \beta^2|2\rangle\langle 2| \end{bmatrix}$$

The projectors $|1\rangle\langle 1|$ and $|2\rangle\langle 2|$ form square block diagonal elements of the matrix, with as many rows as the dimension of the representation $s'$ to which they correspond. Since the matrices $E^1_{bb}$ and $E^1_{aa}$ are of the same dimension as the projectors, the same expansion is also possible for $E^1_{bb}|j,j^3;p\rangle\langle j,j^3;p|E^1_{aa}$. So what we are in fact calculating is of the form:

$$\mathrm{Tr}\left(\begin{bmatrix} \alpha^2|1\rangle\langle 1| & 0 \\ 0 & \beta^2|2\rangle\langle 2| \end{bmatrix}\begin{bmatrix} \alpha^2 E^1_{bb}|1\rangle\langle 1|E^1_{aa} & 0 \\ 0 & \beta^2 E^1_{bb}|2\rangle\langle 2|E^1_{aa} \end{bmatrix}\right)$$

It should now be clear why only coefficients corresponding to the same $s'$ are to be multiplied - it is due to the matrix nature of the projectors, and the manner in which the matrix representations of $s'$ compose the matrix of $s$.

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The expression thus evaluates to

$$\sum_p \langle j,j^3;p|E^1_{bb}|j,j^3;p\rangle\langle j,j^3;p|E^1_{aa}|j,j^3;p\rangle = ds'_2 (C^{jj^3}_{j+\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2 (C^{jj^3}_{j+\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2 + ds'_1 (C^{jj^3}_{j-\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2 (C^{jj^3}_{j-\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2 \tag{6.2}$$

Note that when $R = T$, $a$ will equal $b$ and fourth powers of the Clebsch-Gordon coefficients with the correct $j^3$ for the row from which the box is removed will appear. Clearly, the second term of this type will evaluate exactly as the first, since swapping $a$ with $b$ does not affect the outcome of the calculation. Using these calculations, we see that in order to compute the diagonal terms in the dilatation operator when $R = T$, one must evaluate (the Clebsch-Gordon coefficients are labelled by the original $U(p)$ state labels, since the new labels are very clumsy to express when summed over):

$$-\frac{2g^2_{YM} c_{RR'} r_k m}{R_k d_s} \sum_{s'} d_{s'} \left[ (C^{M_s}_{M_{s'},M^k_{10}})^2 - (C^{M_s}_{M_{s'},M^k_{10}})^4 \right] \delta_{jl}\delta_{iq}\,. \tag{6.3}$$

For the case of two rows, there are no multiplicity labels and further for each $s'$ only a single state contributes, so that there is no sum over $M_{s'}$. Consider the contribution obtained when $R'$ is related to $R$ by removing a box from the first row of $R$. In this case

$$c_{RR'} = (N + b_0 + b_1)\left(1 + O\left(\frac{n_1}{N+b_0+b_1}\right)\right), \qquad \frac{r_1}{R_1} = 1 + O\left(\frac{n_1}{b_0+b_1}\right)$$

and

$$M^1_{10} \leftrightarrow \left|\frac{1}{2},\frac{1}{2}\right\rangle, \qquad M_s \leftrightarrow |j,j^3\rangle\,.$$

When we pull a box from the first row of $s$ to obtain $s'$ we have

$$m\frac{d_{s'}}{d_s} = \frac{\mathrm{hooks}_s}{\mathrm{hooks}_{s'}} = \frac{2j}{2j+1}\frac{m+2j+2}{2}, \qquad M_{s'} = \left|j-\frac{1}{2},j^3-\frac{1}{2}\right\rangle\,.$$

When we pull a box from the second row of $s$ to obtain $s'$ we have

$$m\frac{d_{s'}}{d_s} = \frac{\mathrm{hooks}_s}{\mathrm{hooks}_{s'}} = \frac{2j+2}{2j+1}\frac{m-2j}{2}, \qquad M_{s'} = \left|j+\frac{1}{2},j^3-\frac{1}{2}\right\rangle\,.$$

It is now a simple matter to show that (6.3) evaluates to

$$-\frac{g^2_{YM}}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) \tag{6.4}$$

This will be the coefficient of $(2N + 2b_0 + b_1)O(b_0, b_1, j, j^3)$ (the contribution arising from factors like $c_{RR'}$ is the same for all operators corresponding to the same $r$ and $t$ - varying $j$ will not affect it since the coefficient does not depend on the shapes of $s$ and $u$) in the dilatation operator expression, since it was obtained using the Schur Polynomial labels $R, (r, s) = T(t, u)$.

The second contribution to the diagonal terms is obtained when $R \neq T$, in which case we need to evaluate

$$\frac{2g^2_{YM}\sqrt{c_{RR'}c_{TT'}}\sqrt{r_w t_x}m}{\sqrt{R_w T_x}d_u} \sum_{s'} d_{s'} (C^{M_s}_{M_{s'},M^2_{10}})^2 (C^{M_s}_{M_{s'},M^1_{10}})^2\,. \tag{6.5}$$

When $s'$ is obtained by removing a box from the first row of $s$ we computed $m\frac{d_{s'}}{d_s}$ above and we have

$$(C^{M_s}_{M_{s'},M^2_{10}})^2(C^{M_s}_{M_{s'},M^1_{10}})^2 = \left\langle j-\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle^2\left\langle j-\frac{1}{2},j^3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}|j,j^3\right\rangle^2 .$$

When $s'$ is obtained by removing a box from the second row of $s$ we computed $m\frac{d_{s'}}{d_s}$ above and we have

$$(C^{M_s}_{M_{s'},M^2_{10}})^2(C^{M_s}_{M_{s'},M^1_{10}})^2 = \left\langle j+\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle^2\left\langle j+\frac{1}{2},j^3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}|j,j^3\right\rangle^2 .$$

It is now easy to see that (6.5) evaluates to

$$\frac{g^2_{YM}}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right). \tag{6.6}$$

This will be the coefficient of the term $\sqrt{(N+b_0)(N+b_0+b_1)}O(b_0-1,b_1+2,j,j^3)$ and of the term $\sqrt{(N+b_0)(N+b_0+b_1)}O(b_0+1,b_1-2,j,j^3)$, since it was obtained by considering the case where $R\neq T$, and hence $r\neq t$ - the differences in labels correspond to the the way in which the labels $T,(t,u)$ differ from $R,(r,s)$. In this case, $t$ differs from $r$ by the placement of a single box, reflected by the differences in $b_0$ and $b_1$. Notice that although they were computed in completely different ways (6.4) and (6.6) are identical up to a sign. It is due to "accidents" like this that the final dilatation operator has such a simple dependence on operators of varying $b_0$ and $b_1$.

### 6.1.2 Non-diagonal Terms

Calculating the other terms is now straightforward. The calculations are performed in a manner very similar to the diagonal case. The value of $j$ can be shifted to either $j+1$ or $j-1$ - these are the only possible changes to the value of $j$, since at one loop level terms corresponding to the Young diagrams $s$ and $u$ differing by more than the placement of a single box vanish. Consider 6.2 in the case where $u$ has one more box in one of its rows (and one less in the other) than $s$; now the projector $P_{T\to(t,u)lm}=\sum_a|j\pm1,j^3;a\rangle\langle j\pm1,j^3;a|$, so that we have:

$$\begin{aligned}\sum_a\langle j,j^3;a|E^1_{bb}|j,j^3;a\rangle\langle j\pm1,j^3;a|E^1_{aa}|j\pm1,j^3;a\rangle = {}& ds'_2(C^{jj^3}_{j+\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2(C^{j\pm1,j^3}_{j+\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2\\ &+ds'_1(C^{jj^3}_{j-\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2(C^{j\pm1,j^3}_{j-\frac{1}{2},j^3\mp\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}})^2\end{aligned} \tag{6.7}$$

The Clebsch-Gordons are simply calculated by replacing $j$ by $j\pm1$ in the expressions given at the beginning of the section [1]. If the extra box is in the first row of $u$ insert $j+1$, or $j-1$ if it is in the second row. Using this method one can easily obtain the coefficients of the terms $(2N+2b_0+b_1)O(b_0,b_1,j\pm1,j^3)$, $O(b_0-1,b_1+2,j\pm1,j^3)$ and $\sqrt{(N+b_0)(N+b_0+b_1)}O(b_0+1,b_1-2,j\pm1,j^3)$ by plugging different values of $a$ and $b$ into (6.7).

### 6.1.3 Result

The result of these calculations is a surprisingly simple formula for the action of the dilatation operator on Schur Polynomial operators corresponding to systems of giant gravitons with $p=2$ rows. It is convenient to define the operators:

$$\begin{aligned}\Delta O(b_0,b_1,j,j^3) \quad &= \sqrt{(N+b_0)(N+b_0+b_1)}(O(b_0+1,b_1-2,j,j^3)+O(b_0-1,b_1+2,j,j^3))\\ &\quad -(2N+2b_0+b_1)O(b_0,b_1,j,j^3)\end{aligned} \tag{6.8}$$

[1] To illustrate this point, consider the Clebsch-Gordon coefficient $\langle j+\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j+1,j^3\rangle$; this combination of labels can be obtained by replacing $j$ by $j+1$ in the coefficient $\langle j-\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\rangle$. Thus the value of the coefficient labelled with a $j+1$ can be found by making this replacement in the value of the coefficient labelled by a $j$:

$$\sqrt{\frac{j+j^3}{2j}}\to(j\to(j+1))\to\sqrt{\frac{j+1+j^3}{2(j+1)}}$$

since the factor in the coefficient coming from $c_{RR'}$ and $c_{TT'}$ does not depend on $j$, and the factor from the product of the Clebsch-Gordons with the symmetric group dimensions does not depend on $b_0$ and $b_1$. In terms of these operators, the dilatation operator action is:

$$\begin{aligned} DO(b_0,b_1,j,j^3) &= g_{YM}^2\left[-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O(b_0,b_1,j,j^3)\right. \\ &+\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\Delta O(b_0,b_1,j+1,j^3) \\ &\left.+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\Delta O(b_0,b_1,j-1,j^3)\right] \end{aligned} \tag{6.9}$$

This reproduces the result of [10]. The fact that the dilatation operator does not change the $j^3$ label of the operator it acts on is a consequence of the fact that the $\Gamma(1,m+1)$ factor in $D$ ensures that the block removed comes from the same row of $R$ and $r$ to produce $T$ and $t$ (in the term $\chi_{T,(t,s)}$ produced by the action of $D$ on $\chi_{R,(r,s)}$). This conclusion only follows in the approximation outlined in section B.4 of Appendix B. If we study the limit in which $j^3 \ll j$ we obtain the significantly simpler result

$$\begin{aligned} DO(b_0,b_1,j,j^3) &= g_{YM}^2\left[-\frac{m}{2}\Delta O(b_0,b_1,j,j^3)\right. \\ &\left.+\frac{\sqrt{(m+2j)(m-2j)}}{4}\left(\Delta O(b_0,b_1,j+1,j^3)+\Delta O(b_0,b_1,j-1,j^3)\right)\right] \end{aligned} \tag{6.10}$$

The system (6.10) retains the essential feature that it is again equivalent to a set of decoupled oscillators. When generalizing to $p > 2$ rows, it is straight forward to compute the analog of (6.9). The resulting expressions are quite lengthy and difficult to interpret. For that reason, we will focus on simplified expressions which are the analog of (6.10). This completes our evaluation of the dilatation operator for two rows.

Using the results of section 5.3.5 we can immediately obtain the action of the dilatation operator on restricted Schur polynomials with $p$ long columns. Transpose the Young diagram labels. In this case, for example, the number of rows in $r$ with 2 boxes is $b_0$ and the number of rows with 1 box is $b_1$, while the number of rows in $s$ with 2 boxes is given by $\frac{m-2j}{2}$ and the number of rows with one box is $2j$. Denote the corresponding normalized operators by $Q(b_0,b_1,j,j^3)$. The action of the dilatation operator in this case is given by

$$\begin{aligned} DQ(b_0,b_1,j,j^3) &= g_{YM}^2\left[-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta Q(b_0,b_1,j,j^3)\right. \\ &+\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\Delta Q(b_0,b_1,j+1,j^3) \\ &\left.+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\Delta Q(b_0,b_1,j-1,j^3)\right] \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} \Delta Q(b_0,b_1,j,j^3) \quad &= \sqrt{(N-b_0)(N-b_0-b_1)}(Q(b_0+1,b_1-2,j,j^3)+Q(b_0-1,b_1+2,j,j^3)) \\ &-(2N-2b_0-b_1)Q(b_0,b_1,j,j^3). \end{aligned} \tag{6.12}$$

Notice that the sphere giant and AdS giant cases are related by replacing expressions like $N+b_0$ with $N-b_0$.

## 6.2 Young Diagrams with Three Rows or Columns

In this case, we will be using $U(3)$ representation theory. It is again useful to trade the $m_{ij}$ appearing in the Gelfand-Tsetlin patterns for a new set of labels $j,k,j^3,k^3,l^3$ specified by

$$\begin{bmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{bmatrix} = \begin{bmatrix} j+k+m_{33} & & k+m_{33} & & m_{33} \\ & j^3+k+m_{33} & & k^3+m_{33} & \\ & & l^3+k^3+m_{33} & & \end{bmatrix}.$$

It again looks like we are trading 5 variables for 6. However, we can again recover the value of $m_{33}$ from the value of $m$ using

$$m = 3m_{33} + 2k + j\,.$$

The variables satisfy

$$j \geq 0, \quad k \geq 0, \quad j \geq j^3 \geq 0, \quad k \geq k^3 \geq 0, \quad k + j^3 - k^3 \geq l^3 \geq 0,$$

which are again much easier to handle than the betweenness conditions. We will write our states as kets $|j,k,j^3,k^3,l^3\rangle$. The $U(3)$ state $|j,k,j^3,k^3,l^3\rangle$ can also be viewed as an $S_m$ state, and like the $U(2)$ example, can be decomposed into a superposition of states each corresponding to the tensor product of an $S_{m-1}$ state with an $S_1$ state - a superposition of states each corresponding to a different way of pulling a box off the Young diagram of $s$. There will be a fairly vast number of states to consider in the three row case. The expansion is:

$$\begin{aligned}
|j,k,j^3,k^3,l^3\rangle = &\sum\sum_{i=1}^{3} \langle j-1,k,j^3\pm1,k^3\pm1,l^3\pm1;m_i|j,k,j^3,k^3,l^3\rangle\,|j-1,k,j^3\pm1,k^3\pm1,l^3\pm1;m_i\rangle \\
&+\sum\sum_{i=1}^{3} \langle j+1,k-1,j^3\pm1,k^3\pm1,l^3\pm1;m_i|j,k,j^3,k^3,l^3\rangle\,|j+1,k-1,j^3\pm1,k^3\pm1,l^3\pm1;m_i\rangle \\
&+\sum\sum_{i=1}^{3} \langle j,k-1,j^3\pm1,k^3\pm1,l^3\pm1;m_i|j,k,j^3,k^3,l^3\rangle\,|j,k-1,j^3\pm1,k^3\pm1,l^3\pm1;m_i\rangle
\end{aligned}$$

where the unlabelled sum is over all possible combinations of variations of the labels $j^3,k^3,l^3$ such that the betweenness conditions are satisfied. Since $j$ and $k$ are the only labels that affect the first row of the Gelfand-Tsetlin pattern, they indicate from which row of $s$ (or $u$) the box is removed; clearly the first sum is over states where the box is removed from row 1, the second from row 2 and the third from row 3. The label $m_i$ indicates that the first box is removed from row $i$ of $R$. Thankfully, not all the Clebsch-Gordan coefficients appearing in this sum contribute; the ones we will need are (its simple to compute these using Appendix A.5)

$$\langle j-1,k,j^3,k^3,l^3;m_1|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j-j^3)(j+k-k^3+1)}{j(j+k+1)}},$$

$$\langle j+1,k-1,j^3+1,k^3,l^3;m_1|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j^3+1)(k-k^3)}{k(j+2)}},$$

$$\langle j,k+1,j^3,k^3+1,l^3;m_1|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(k^3+1)(k+j^3+2)}{(j+k+3)(k+2)}},$$

$$\langle j-1,k,j^3-1,k^3,l^3;m_2|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j+k-k^3+1)j^3(k+j^3+1)(j^3-k^3-l^3+k)}{j(j+k+1)(k+j^3-k^3+1)(j^3+k-k^3)}},$$

$$\langle j-1,k,j^3,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j-j^3)(k-k^3+1)k^3(k+j^3-k^3-l^3+1)}{j(j+k+1)(k+j^3-k^3+1)(k+j^3-k^3+2)}},$$

$$\langle j+1,k-1,j^3,k^3,l^3;m_2|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(k-k^3)(j-j^3+1)(k+j^3+1)(k+j^3-k^3-l^3)}{(j+2)k(j^3+k-k^3+1)(k+j^3-k^3)}},$$

$$\langle j+1,k-1,j^3+1,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j^3+1)(j+k-k^3+2)k^3(k+j^3-k^3-l^3+1)}{(j+2)k(k+j^3-k^3+1)(k+j^3-k^3+2)}},$$

$$\langle j,k+1,j^3-1,k^3+1,l^3;m_2|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(k^3+1)(j-j^3+1)j^3(k+j^3-k^3-l^3)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3)}},$$

$$\langle j,k+1,j^3,k^3,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle =$$
$$-\sqrt{\frac{(k+j^3+2)(j+k-k^3+2)(k-k^3+1)(k+j^3-k^3-l^3+1)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3+2)}},$$

$$\langle j-1,k,j^3-1,k^3,l^3-1;m_3|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j+k-k^3+1)j^3(k+j^3+1)l^3}{j(j+k+1)(k+j^3-k^3+1)(j^3+k-k^3)}},$$

$$\langle j-1,k,j^3,k^3-1,l^3;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(j-j^3)(k-k^3+1)k^3(l^3+1)}{j(j+k+1)(k+j^3-k^3+1)(k+j^3-k^3+2)}},$$

$$\langle j+1,k-1,j^3,k^3,l^3-1;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(k-k^3)(j-j^3+1)(k+j^3+1)l^3}{(j+2)k(j^3+k-k^3+1)(k+j^3-k^3)}},$$

$$\langle j+1,k-1,j^3+1,k^3-1,l^3;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(j^3+1)(j+k-k^3+2)k^3(l^3+1)}{(j+2)k(k+j^3-k^3+1)(k+j^3-k^3+2)}},$$

$$\langle j, k+1, j^3-1, k^3+1, l^3-1; m_3 | j, k, j^3, k^3, l^3\rangle = -\sqrt{\frac{(k^3+1)(j-j^3+1)j^3l^3}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3)}},$$

$$\langle j, k+1, j^3, k^3, l^3; m_3 | j, k, j^3, k^3, l^3\rangle = \sqrt{\frac{(k+j^3+2)(j+k-k^3+2)(k-k^3+1)(l^3+1)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3+2)}},$$

where

$$m_1 = 1,0,0,0,0, \quad m_2 = 1,0,1,0,0, \quad m_3 = 1,0,1,0,1\,.$$

Consider first the case of three rows. To specify $r$ we specify the number of columns with three boxes ($b_0$), the number of columns with two boxes ($b_1$) and the number of columns with a single box ($b_2$). Thus, our operators $O(b_1, b_2, j, k, j^3, k^3, l^3)$ carry seven labels. To simplify the notation a little we do not explicitly display $b_0$ since it is fixed once $b_1$ and $b_2$ are chosen by $b_0 = (n - b_2 - 2b_1)/3$. To obtain $r$ from $R$ we remove $n_i$ boxes from each row where

$$n_1 = \frac{m + 2j + k - 3k^3 - 3j^3}{3}, \qquad n_2 = \frac{m + k - j + 3j^3 - 3l^3}{3},$$

$$n_3 = \frac{m - j - 2k + 3l^3 + 3k^3}{3}\,.$$

We can read $j$, $k$ and $m$ directly from the Young diagram label $s$. One might have thought that by employing the above expressions for the $n_i$ one could obtain a formula for $j^3, k^3, l^3$ in terms of the $n_i$. This is not possible. Indeed, this conclusion follows immediately upon noting that

$$n_1 + n_2 + n_3 = m\,.$$

The reason why it is not possible to express $j^3, k^3, l^3$ in terms of the $n_i$, is simply that in all situations where the inner multiplicity is greater than 1, there is no unique $j^3, k^3, l^3$ given the $n_i$. Recall that the dilatation operator, when acting on restricted Schur polynomials labelled by Young diagrams with two rows, preserved the $j^3$ label of the operator. What is the corresponding statement that would be valid for any number of rows? In general, the dilatation operator preserves the $\Delta$ weight of the operator it acts on. In the two row case, preserving $j^3$ is equivalent to preserving the $\Delta$ weight. Further, the reason why the $\Delta$ weight is preserved can again be traced back to the factors of $\Gamma(1, m+1)$ appearing in the dilatation operator and again this conclusion only follows in the approximation outlined in section B.4 of Appendix B. For the case of three rows it is simple to give this inner multiplicity a nice characterization: States that belong to the same inner multiplicity multiplet

- Have the same first row in their Gelfand-Tsetlin pattern because they belong to the same $U(3)$ irreducible representation.
- Have the same last row because the $\Delta$ weight is conserved.
- Have the same sum of numbers in the second row of the Gelfand-Tsetlin pattern again because the $\Delta$ weight is conserved.

This implies that states in the same inner multiplet can be written as

$$\begin{bmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12}-i & & m_{22}+i & \\ & & m_{11} & & \end{bmatrix}$$

with different values of $i$ giving the different states, and that the number of states in the inner multiplet is

$$N = \max(m_{12} - m_{11}, m_{12} - m_{23}, \frac{m_{12} - m_{22}}{2}) + \min(m_{13} - m_{12}, m_{22} - m_{33}) + 1\,,$$

where $\max(a, b, c)$ means take the largest of $a, b, c$ and $\min(a, b)$ means take the smallest of $a, b$.

In the case of two rows, we saw that the action of the dilatation operator could be expressed entirely in terms of the combination $\Delta O(b_0, b_1, j, j^3)$. There is a generalization of this result for $p = 3$ rows: after applying the dilatation operator we only obtain the linear combinations:[2]

$$\Delta_{12} O(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N + 2b_0 + 2b_1 + b_2) O(b_1, b_2, j, k, j^3, k^3, l^3) + \tag{6.13}$$
$$\sqrt{(N + b_0 + b_1)(N + b_0 + b_1 + b_2)}\left(O(b_1 - 1, b_2 + 2, j, k, j^3, k^3, l^3) + O(b_1 + 1, b_2 - 2, j, k, j^3, k^3, l^3)\right),$$

[2]The combination $\Delta_{ij}$ is relevant for terms in the dilatation operator which allow a box to move between rows $i$ and $j$.

$$\Delta_{13}O(b_1,b_2,j,k,j^3,k^3,l^3) = -(2N+2b_0+b_1+b_2)O(b_1,b_2,j,k,j^3,k^3,l^3) \tag{6.14}$$
$$\sqrt{(N+b_0)(N+b_0+b_1+b_2)}\left(O(b_1-1,b_2-1,j,k,j^3,k^3,l^3)+O(b_1+1,b_2+1,j,k,j^3,k^3,l^3)\right),$$

$$\Delta_{23}O(b_1,b_2,j,k,j^3,k^3,l^3) = -(2N+2b_0+b_1)O(b_1,b_2,j,k,j^3,k^3,l^3)+ \tag{6.15}$$
$$\sqrt{(N+b_0)(N+b_0+b_1)}\left(O(b_1-2,b_2+1,j,k,j^3,k^3,l^3)+O(b_1+2,b_2-1,j,k,j^3,k^3,l^3)\right).$$

To obtain this generalization, we start by evaluating one of the diagonal terms in the dilatation operator.

### 6.2.1 Diagonal Terms

To compute the diagonal terms in the dilatation operator, the trace we need to evaluate is

$$\begin{aligned}\mathcal{T} =& -\delta_{ab}\delta_{RT}\delta_{(r,s)\,(t,u)}\delta_{j^3j'^3}\delta_{k^3k'^3}\delta_{l^3l'^3}d^b_{R^{m+1}}\Big[\mathrm{Tr}_{V_p^{\otimes\, m}}\left(|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}\right)\\
&+\mathrm{Tr}_{V_p^{\otimes\, m}}\left(|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}\right)\Big]\\
&+d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes\, m}}\left(|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{aa}\right)\\
&+d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes\, m}}\left(|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{aa}|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}\right).\end{aligned}$$

If the projector in question is not formed from two different states from the same inner multiplet, then $\{j^3,k^3,l^3\}=\{j'^3,k'^3,l'^3\}$. In the same way as for the 2 row case, and using the fact that for the diagonal elements, the last two terms are are equal, this is equivalent to evaluating

$$\begin{aligned}\mathcal{T} =& -2\delta_{ab}\delta_{RT}\delta_{(r,s)\,(t,u)}\delta_{j^3j'^3}\delta_{k^3k'^3}\delta_{l^3l'^3}d^b_{R^{m+1}}\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}|j,k,j^3,k^3,l^3\rangle\\
&+2d^b_{R^{m+1}}\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{aa}|j,k,j^3,k^3,l^3\rangle\end{aligned}$$

Notice that we are using a bra/ket notation. Remembering that a particular set of labels $j,k,j^3,k^3,l^3$ corresponds to a particular Gelfand-Tsetlin pattern $M^i_s$, and expanding the states in the above expression into a superposition of tensor products of states as discussed for the two row case, we see that for terms where $R,(r,s)=T,(t,u)$ we must evaluate:

$$-\frac{2g^2_{YM}c_{RR'}r_km}{R_kd_s}\sum_{s'}d_{s'}\left[C^{M^i_s}_{M_{s'},M^k_{100}}C^{M^j_s}_{M_{s'},M^k_{100}}-(C^{M^i_s}_{M_{s'},M^k_{100}}C^{M^j_s}_{M_{s'},M^k_{100}})^2\right]\delta_{jl}\delta_{iq}\,. \tag{6.16}$$

When dealing with the case that $i=j$, this becomes:

$$-\frac{2g^2_{YM}c_{RR'}r_km}{R_kd_s}\sum_{s'}d_{s'}\left[(C^{M^i_s}_{M_{s'},M^k_{100}})^2-(C^{M^i_s}_{M_{s'},M^k_{100}})^4\right]\delta_{jl}\delta_{iq}\,. \tag{6.17}$$

In the three row case we do need to worry about multiplicities, which is why the labels $i,j$ appear on $M_s$. For each $s'$, again only a single state contributes so that there is no sum over $M_{s'}$. Consider the contribution obtained when $R'$ is obtained from $R$ by removing a box from the first row of $R$. In this case

$$c_{RR'}=(N+b_0+b_1+b_2)\left(1+O\left(\frac{n_1}{N+b_0+b_1+b_2}\right)\right),\qquad \frac{r_1}{R_1}=1+O\left(\frac{n_1}{b_0+b_1+b_2}\right)$$

and

$$M^1_{100}\leftrightarrow|1,0,0,0,0\rangle\,,\qquad M^i_s\leftrightarrow\left|j,k,j^3,k^3,l^3\right\rangle\,.$$

Changing the value of $i$ in $M^i_s$ corresponds to changing the value of $j^3,k^3,l^3$ while keeping the $\Delta$ weight of the Gelfand-Tsetlin pattern fixed. We can now explicitly perform some calculations; we will do this for $i=j$. The calculations for $i\neq j$ are discussed afterwards. To obtain $s'$ from $s$ we can remove a box from any one of three rows. When we pull a box from the first row of $s$ we have

$$m\frac{d_{s'}}{d_s}=\left(\frac{m+2j+k+6}{3}\right)\left(\frac{j+k+1}{j+k+2}\right)\left(\frac{j}{j+1}\right),\qquad M_{s'}=\left|j-1,k,j^3,k^3,l^3\right\rangle\,.$$

When we pull a box from the second row of $s$ we have

$$m\frac{d_{s'}}{d_s}=\left(\frac{m-j+k+3}{3}\right)\left(\frac{j+2}{j+1}\right)\left(\frac{k}{k+1}\right),\qquad M_{s'}=\left|j+1,k-1,j^3+1,k^3,l^3\right\rangle\,.$$

Finally, when we pull a box from the third row of $s$ we have

$$m\frac{d_{s'}}{d_s}=\left(\frac{m-2k-j}{3}\right)\left(\frac{j+k+3}{j+k+2}\right)\left(\frac{k+2}{k+1}\right),\qquad M_{s'}=\left|j,k+1,j^3,k^3+1,l^3\right\rangle\,.$$

**$\mathbf{R},(\mathbf{r},\mathbf{s})\mathbf{jk}=\mathbf{T},(\mathbf{t},\mathbf{u})\mathbf{lm}$ :**

Using these results, we easily obtain the contribution obtained when $R'$ is obtained from $R$ by removing a box from the first row

$$-2g_{YM}^2(N+b_0+b_1+b_2)A\equiv -2g_{YM}^2(N+b_0+b_1+b_2)\times$$
$$\times\left[\frac{(m+2j+k+6)(j+k+1)j}{3(j+k+2)(j+1)}\left(\frac{(j-j^3)(j+k-k^3+1)}{j(j+k+1)}-\frac{(j-j^3)^2(j+k-k^3+1)^2}{j^2(j+k+1)^2}\right)\right.$$
$$+\frac{(m-j+k+3)(j+2)k}{3(j+1)(k+1)}\left(\frac{(j^3+1)(k-k^3)}{k(j+2)}-\frac{(j^3+1)^2(k-k^3)^2}{k^2(j+2)^2}\right)$$
$$\left.+\frac{(m-2k-j)(j+k+3)(k+2)}{3(j+k+2)(k+1)}\left(\frac{(k^3+1)(k+j^3+2)}{(j+k+3)(k+2)}-\frac{(k^3+1)^2(k+j^3+2)^2}{(j+k+3)^2(k+2)^2}\right)\right]\,.$$

There are two other diagonal terms, one proportional to $N+b_0+b_1$ and one proportional to $N+b_0$, evaluated in exactly the same way.

**$\mathbf{R}\neq\mathbf{T},\mathbf{r}\neq\mathbf{t},\mathbf{s}=\mathbf{u},\mathbf{i}=\mathbf{j}$**

Notice that the coefficient of the operator $O(b_1,b_2,j,k,j^3,k^3,l^3)$ in $\Delta_{ij}O$ is equal to $(c^{(i)}_{RR'}+c^{(j)}_{RR'})$, i.e. the sum of the factors for a box pulled off row $i$ and $j$ of $R$. This factorization of the factors is impossible to obtain in some cases, so that it is not useful to calculate these terms, as their exact contribution to each of the $\Delta_{ij}O$'s is not obvious. Instead, we can calculate the term obtained when $s=u$ but $R\neq T$, and use identities relating to these coefficients and the fact that the matrix elements values do not depend on the relation between $R$ and $T$ to argue that this will also be the element associated to the operators for which $R=T$. The terms for which this condition holds are evaluated by considering terms of the form:

$$\frac{2g_{YM}^2\sqrt{c_{RR'}c_{TT'}}\sqrt{r_wt_x}m}{\sqrt{R_wT_x}d_u}\sum_{s'}d_{s'}\sum_{M_{s'}}\sum_{\tilde{M}_{s'}}C^{M_s^i}_{\tilde{M}_{s'},M^x_{\mathbf{1_0}}}C^{M_u^l}_{\tilde{M}_{s'},M^x_{\mathbf{1_0}}}C^{M_s^i}_{M_{s'},M^w_{\mathbf{1_0}}}C^{M_u^l}_{M_{s'},M^w_{\mathbf{1_0}}}\,.\tag{6.18}$$

Referring to (6.16), this corresponds to setting $a=x$ and $b=w$. In a bra-ket notation, we are evaluating

$$\langle j,k,j^3,k^3,l^3|E^{(1)}_{xx}|j,k,j'^3,k'^3,l'^3\rangle\langle j,k,j^3,k^3,l^3|E^{(1)}_{ww}|j,k,j'^3,k'^3,l'^3\rangle\quad.$$

When the box to be removed from $R$ to obtain $R'$ must be removed from the first row ($w=1$) and the box to be removed from $T$ to obtain $T'$ must be removed from the second row ($x=2$), we obtain

$$c_{TT'}=(N+b_0+b_1)\left(1+O\left(\frac{n_2}{N+b_0+b_1}\right)\right),\qquad\frac{r_2}{R_2}=1+O\left(\frac{n_2}{b_0+b_1}\right),$$

$$M^2_{100}\leftrightarrow|1,0,1,0,0\rangle$$

and hence

$$2g_{YM}^2\sqrt{(N+b_0+b_1+b_2)(N+b_0+b_1)}B\equiv 2g_{YM}^2\sqrt{(N+b_0+b_1+b_2)(N+b_0+b_1)}\times$$
$$\times\left[\frac{(m+2j+k+6)(j-j^3)(j+k-k^3+1)}{3(j+k+2)(j+1)}\left(\frac{(j+k-k^3+1)j^3(k+j^3+1)(j^3-k^3-l^3+k)}{j(j+k+1)(k+j^3-k^3+1)(j^3+k-k^3)}\right.\right.$$
$$\left.+\frac{(j-j^3)(k-k^3+1)k^3(k+j^3-k^3-l^3+1)}{j(j+k+1)(k+j^3-k^3+1)(k+j^3-k^3+2)}\right)$$
$$+\frac{(m-j+k+3)(j^3+1)(k-k^3)}{3(j+1)(k+1)}\left(\frac{(k-k^3)(j-j^3+1)(k+j^3+1)(k+j^3-k^3-l^3)}{(j+2)k(j^3+k-k^3+1)(k+j^3-k^3)}\right.$$

$$+\frac{(j^3+1)(j+k-k^3+2)k^3(k+j^3-k^3-l^3+1)}{(j+2)k(k+j^3-k^3+1)(k+j^3-k^3+2)}\Big)$$
$$+\frac{(m-2k-j)(k^3+1)(k+j^3+2)}{3(j+k+2)(k+1)}\left(\frac{(k^3+1)(j-j^3+1)j^3(k+j^3-k^3-l^3)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3)}\right.$$
$$\left.+\frac{(k+j^3+2)(j+k-k^3+2)(k-k^3+1)(k+j^3-k^3-l^3+1)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3+2)}\Big)\right].$$

This will be a diagonal element associated with the operator $\Delta_{12}O$. Notice, we multiply the factor for a box being removed from row $p$ of $s$ with the Clebsch-Gordon associated to removing a box from row 1 of $R$ and row $p$ of $s$, and then multiply this product with the sum of the Clebsch-Gordons associated to removing a box from row 2 of $R$ and row $p$ of $s$. When the box to be removed from $R$ to obtain $R'$ must be removed from the first row ($w=1$) and the box to be removed from $T$ to obtain $T'$ must be removed from the third row ($x=3$), we obtain

$$c_{TT'}=(N+b_0)\left(1+O\left(\frac{n_3}{N+b_0}\right)\right),\qquad \frac{r_3}{R_3}=1+O\left(\frac{n_3}{b_0}\right),$$

$$M^3_{100}\leftrightarrow|1,0,1,0,1\rangle$$

and hence

$$2g^2_{YM}\sqrt{(N+b_0+b_1+b_2)(N+b_0)}C\equiv 2g^2_{YM}\sqrt{(N+b_0+b_1+b_2)(N+b_0)}\times$$
$$\times\left[\frac{(m+2j+k+6)(j-j^3)(j+k-k^3+1)}{3(j+k+2)(j+1)}\left(\frac{(j+k-k^3+1)j^3(k+j^3+1)l^3}{j(j+k+1)(k+j^3-k^3+1)(j^3+k-k^3)}\right.\right.$$
$$\left.+\frac{(j-j^3)(k-k^3+1)k^3(l^3+1)}{j(j+k+1)(k+j^3-k^3+1)(k+j^3-k^3+2)}\right)$$
$$+\frac{(m-j+k+3)(j^3+1)(k-k^3)}{3(j+1)(k+1)}\left(\frac{(k-k^3)(j-j^3+1)(k+j^3+1)l^3}{(j+2)k(j^3+k-k^3+1)(k+j^3-k^3)}\right.$$
$$\left.+\frac{(j^3+1)(j+k-k^3+2)k^3(l^3+1)}{(j+2)k(k+j^3-k^3+1)(k+j^3-k^3+2)}\right)$$
$$+\frac{(m-2k-j)(k^3+1)(k+j^3+2)}{3(j+k+2)(k+1)}\left(\frac{(k^3+1)(j-j^3+1)j^3l^3}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3)}\right.$$
$$\left.\left.+\frac{(k+j^3+2)(j+k-k^3+2)(k-k^3+1)(l^3+1)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3+2)}\right)\right].$$

This will be a diagonal element associated with the operator $\Delta_{13}O$. It is easy to verify that $A=B+C$. Thanks to this identity and others like it, we see that the dilatation operator only depends on the combinations of operators contained in the $\Delta_{ij}$'s. This is the generalization of the fact that for two rows the dilatation operator is expressed entirely in terms of the combination $\Delta O(b_0,b_1,j,j^3)$.

**$\mathbf{i\neq j}$**

When calculating the diagonal matrix elements where $i\neq j$, one would expect to calculate:

$$\mathrm{Tr}\left(|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{aa}|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{bb}\right)$$

However, examining the equation (5.3), one notices that the multiplicity labels $ql$ on the LHS are swapped on the RHS - these are the multiplicity labels associated to $T,(t,u)$, so that when we calcuate terms involving inner multiplet operators, the projector $|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|$ in the expression above should have the states swapped, i.e. we are in fact calculating:

$$\mathrm{Tr}(|j,k,j^3,k^3,l^3\rangle\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{aa}|j,k,j'^3,k'^3,l'^3\rangle\langle j,k,j^3,k^3,l^3|E^{(1)}_{bb})$$
$$=\langle j,k,j'^3,k'^3,l'^3|E^{(1)}_{aa}|j,k,j'^3,k'^3,l'^3\rangle\langle j,k,j^3,k^3,l^3|E^{(1)}_{bb}|j,k,j^3,k^3,l^3\rangle$$

This has the same form as the diagonal elements not involving inner multiplet operators, and is calculated in the same way.

### 6.2.2 Off-Diagonal Terms

To further clarify how the dependence on only the $\Delta_{ij}$'s comes about, we can consider the off diagonal terms in the dilatation operator. To make the argument transparent, we use the bra/ket notation, so that, for example

$$\sum_{\tilde{M}_{s'}} C^{M_s^i}_{\tilde{M}_{s'},M^x_{1\mathbf{0}}} C^{M_u^l}_{\tilde{M}_{s'},M^x_{1\mathbf{0}}} = \left\langle M_s^i | E_{xx} | M_u^l \right\rangle .$$

The terms multiplying (as an example) $(N + b_0 + b_1 + b_2)$ come multiplied by

$$\left\langle M_1^j | E_{11} | M_2^i \right\rangle \left\langle M_2^j | E_{11} | M_1^i \right\rangle ,$$

the terms multiplying $\sqrt{(N + b_0 + b_1 + b_2)(N + b_0 + b_1)}$ come multiplied by

$$\left\langle M_1^j | E_{11} | M_2^i \right\rangle \left\langle M_2^j | E_{22} | M_1^i \right\rangle ,$$

and finally the terms multiplying $\sqrt{(N + b_0 + b_1 + b_2)(N + b_0)}$ come multiplied by

$$\left\langle M_1^j | E_{11} | M_2^i \right\rangle \left\langle M_2^j | E_{33} | M_1^i \right\rangle .$$

If we are to have a dependence only on the $\Delta_{ij} O(b_1, b_2, j, k, j^3, k^3, l^3)$s we need the first number above to be minus the sum of the second two (plus some additional conditions which follow in the same way). Using the identity $\mathbf{1} = E_{11} + E_{22} + E_{33}$ and $\langle M_1 | M_2 \rangle = 0$ (for the off diagonal terms in the dilatation operator $M_1$ and $M_2$ are by definition different states) we easily find that this is indeed the case. Note also that this argument generalizes trivially to $p > 3$ rows.

When calculating the off-diagonal terms, we are required to calcuate terms like:

$$\langle j', k', j'^3, k'^3, l'^3 | E^{(1)}_{aa} | j, k, j^3, k^3, l^3 \rangle$$

Since the ket does not carry the same labels as the bra, we need to determine which of the states in the expansion of the ket have non-zero Clebsch-Gordon coefficients and have a non-zero overlap with a state in the expansion of the bra. Lets consider the case where $a = 2$; then we have:

$$\begin{aligned}
E^{(1)}_{22} | j, k, j^3, k^3, l^3 \rangle = & \langle j-1, k, j^3-1, k^3, l^3; m_2 | j, k, j^3, k^3, l^3 \rangle \, | j-1, k, j^3-1, k^3, l^3; m_2 \rangle \\
& + \langle j-1, k, j^3, k^3-1, l^3+1; m_2 | j, k, j^3, k^3, l^3 \rangle \, | j-1, k, j^3, k^3-1, l^3+1; m_2 \rangle \\
& + \langle j+1, k-1, j^3, k^3, l^3; m_2 | j, k, j^3, k^3, l^3 \rangle \, | j+1, k-1, j^3, k^3, l^3; m_2 \rangle \\
& + \langle j+1, k-1, j^3+1, k^3-1, l^3+1; m_2 | j, k, j^3, k^3, l^3 \rangle \, | j+1, k-1, j^3+1, k^3-1, l^3+1; m_2 \rangle \\
& + \langle j, k+1, j^3-1, k^3+1, l^3; m_2 | j, k, j^3, k^3, l^3 \rangle \, | j, k+1, j^3-1, k^3+1, l^3; m_2 \rangle \\
& + \langle j, k+1, j^3, k^3, l^3+1; m_2 | j, k, j^3, k^3, l^3 \rangle \, | j, k+1, j^3, k^3, l^3+1; m_2 \rangle
\end{aligned}$$

Expanding the bra and considering the states in the expansion carrying the label $m_2$, we may see that one or more of the states carry the same labels as states in the expansion of the ket. Say, for example, that $(j+1, k-1, j^3, k^3, l^3) = (j'+1, k'-1, j'^3+1, k'^3-1, l'^3+1)$ and $(j-1, k, j^3-1, k^3, l^3) = (j'-1, k', j'^3, k'^3-1, l'^3+1)$. In this case, this term evaluates to:

$$\begin{aligned}
& \left\langle j-1, k, j^3-1, k^3, l^3; m_2 | j, k, j^3, k^3, l^3 \right\rangle \left\langle j'-1, k', j'^3, k'^3-1, l'^3+1; m_2 | j', k', j'^3, k'^3, l'^3 \right\rangle \\
& + \left\langle j+1, k-1, j^3, k^3, l^3; m_2 | j, k, j^3, k^3, l^3 \right\rangle \left\langle j'+1, k'-1, j'^3+1, k'^3-1, l'^3+1; m_2 | j', k', j'^3, k'^3, l'^3 \right\rangle
\end{aligned}$$

Note that in terms of Gelfand-Tsetlin pattern labels, this is equivalent to:

$$\sum_{s'} C^{M_s^i}_{M_{s'},M^2_{1\mathbf{0}}} C^{M_s^j}_{M_{s'},M^2_{1\mathbf{0}}}$$

By thinking about the Young diagrams associated to the states under consideration, it is easy to determine which of the states in each of the expansions could possibly coincide - this will become obvious when concrete examples are presented in the following subsection. Also, since $s \neq u$ for the off-diagonal terms, instead of having to multiply a factor of $m\frac{ds'}{ds}$, we find by examining the dilatation operator coefficient 5.5 that this is replaced by a factor of

$$\frac{ds'}{du} m \frac{\sqrt{\text{hooks}_s \text{hooks}_u}}{\text{hooks}_u} = \frac{\sqrt{\text{hooks}_s \text{hooks}_u}}{\text{hooks}_{s'}}$$

The calculation of the off-diagonal elements is now straightforward. General expressions for the off diagonal terms can be obtained using these methods, but will require the specification of relations between the labels, and hence are not all that useful. For this reason, the calculation will rather be discussed in the context of a specific example in the following subsection.

### 6.2.3 Some Explicit Examples

$\Delta = (1,1,1)$ **States of the** $m = 3$ **Sector:** By applying the above results, it is straight forward to evaluate the action of the dilatation operator for the case that we have 3 $Y$ fields and we set $\Delta = (1,1,1)$. There are four possible $U(3)$ states

$$|3,2,0,0,1\rangle \leftrightarrow \begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \qquad |0,0,0,0,0\rangle \leftrightarrow \begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}$$

$$|1,1,1,0,1\rangle \leftrightarrow \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \qquad |1,1,0,1,0\rangle \leftrightarrow \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}$$

This example was chosen because it is the simplest case in which we have a nontrivial inner multiplicity: indeed, the last two states belong to an inner multiplicity multiplet. This implies that there are a total of 6 symmetric group operators

$$P_1 = |3,2,0,0,1\rangle\langle 3,2,0,0,1| \qquad P_2 = |0,0,0,0,0\rangle\langle 0,0,0,0,0|$$

$$P_3^{(1,1)} = |1,1,1,0,1\rangle\langle 1,1,1,0,1| \qquad P_3^{(1,2)} = |1,1,1,0,1\rangle\langle 1,1,0,1,0|$$

$$P_3^{(2,1)} = |1,1,0,1,0\rangle\langle 1,1,1,0,1| \qquad P_3^{(2,2)} = |1,1,0,1,0\rangle\langle 1,1,0,1,0|$$

which define 6 restricted Schur polynomials. The corresponding normalized operators will be denoted $O_1(b_1,b_2)$, $O_2(b_1,b_2)$, $O_3(b_1,b_2)$, $O_4(b_1,b_2)$, $O_5(b_1,b_2)$ and $O_6(b_1,b_2)$. In terms of the linear combinations of these operators,(6.13), (6.14) and (6.15), the action of the dilatation operator is given by

$$DO_i(b_1,b_2) = -g_{YM}^2 \left( M_{ij}^{(12)} \Delta_{12} O_j(b_1,b_2) + M_{ij}^{(13)} \Delta_{13} O_j(b_1,b_2) + M_{ij}^{(23)} \Delta_{23} O_j(b_1,b_2) \right) \tag{6.19}$$

where

$$M^{(12)} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{3\sqrt{2}} \\ -\frac{2}{3\sqrt{2}} & 0 & \frac{1}{3} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & 1 & 0 & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & 0 & 1 & \frac{1}{2\sqrt{3}} \\ 0 & -\frac{2}{3\sqrt{2}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{3} \end{bmatrix} \quad M^{(13)} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{3\sqrt{2}} \\ -\frac{2}{3\sqrt{2}} & 0 & \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & 1 & 0 & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & 0 & 1 & -\frac{1}{2\sqrt{3}} \\ 0 & -\frac{2}{3\sqrt{2}} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

$$M^{(23)} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{2}{3} & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{5}{6} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{2} & 0 & 0 & \frac{5}{6} \end{bmatrix}.$$

To obtain this result we have used the exact expressions for the Clebsch-Gordan coefficients given at the beginning of this section. Examples of each of the types of matrix elements that are calculated are now given.

**Diagonal Term,** $i = j$

Lets consider the diagonal matrix element associated to operator $O_3$, where $R$ is related to $T$ by swapping a box from row 1 to row 2; in bra/ket nototation we are calculating:

$$\langle 11101|E_{11}|11101\rangle\langle 11101|E_{22}|11101\rangle$$

The factor $m\frac{ds'}{ds} = \frac{3}{2}$ in the case of a box pulled off the first or second row of $s$, and 0 in the case of a box pulled off the 3rd row - this is expected, since there is no third row of the Young diagram associated to this state. In terms of the Clebsch-Gordon coefficients listed in the beginning of the section, the second of the factors in the product is:

$$\begin{aligned}\langle 11101|E_{22}|11101\rangle &= (\langle j-1,k,j^3-1,k^3,l^3;m_2|j,k,j^3,k^3,l^3\rangle)^2 + (\langle j-1,k,j^3,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle)^2 \\ &+ (\langle j+1,k-1,j^3,k^3,l^3;m_2|j,k,j^3,k^3,l^3\rangle)^2 + (\langle j+1,k-1,j^3+1,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle)^2 \\ &= \frac{1}{2}^{(1)} + 0^{(1)} + \frac{1}{6}^{(2)} + 0^{(2)}\end{aligned}$$

The bracketed superscripts indicate from what row of $s$ a box was removed to obtain the state to which the coefficient corresponds, i.e. they indicate to which $s'$ the coefficient is associated; this is necessary because we only take products of coefficients that are associated to the same $s'$, as discussed for the two row case. The first factor in the product is:

$$\begin{aligned}\langle 11101|E_{11}|11101\rangle &= (\langle j-1,k,j^3,k^3,l^3;m_1|j,k,j^3,k^3,l^3\rangle)^2 + (\langle j+1,k-1,j^3+1,k^3,l^3;m_1|j,k,j^3,k^3,l^3\rangle)^2 \\ &= 0^{(1)} + \frac{2}{3}^{(2)}\end{aligned}$$

So, including the coefficient $m\frac{ds'}{ds}$, we are multiplying:

$$\frac{3}{2}(\frac{2}{3}^{(2)})(\frac{1}{2}^{(1)} + \frac{1}{6}^{(2)}) = \frac{1}{6}$$

We now multiply by the factor of 2 arising because the last two terms in the trace expression are related by swapping $E_{11}$ with $E_{22}$, since in this case this swap does not affect the answer, to obtain the final answer of $\frac{1}{3}$. This appears in row 3, column 3 of $M^{(12)}$, since it is the coefficient of $\sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2)}\left(O_3(b_1-1,b_2+2,j,k,j^3,k^3,l^3)+O_3(b_1+1,b_2-2,j,k,j^3,k^3,l^3)\right)$ in $DO_3(b_1,b_2,j,k,j^3,k^3,l^3)$. It is also the coefficient of $-(2N+2b_0+2b_1+b_2)O_3(b_1,b_2,j,k,j^3,k^3,l^3)$, by arguments outlined previously.

**Diagonal Term,** $i \neq j$ Lets consider the diagonal matrix element associated to operator $O_4$, where $R$ is related to $T$ by swapping a box from row 1 to row 2; remembering to swap the bra and ket in the projector associated to $T$, in bra/ket notation we are calculating:

$$\langle 11010|E_{11}|11010\rangle\langle 11101|E_{22}|11101\rangle$$

We can calculate each of these factors individually in the same way for the terms with $i = j$; multiply the Clebsch-Gordon coefficients corresponding to the same $s'$ together, then include the factor $m\frac{ds'}{ds}$ and the factor of 2 (since one can check that in this case swapping the $E$'s does not alter the outcome), to obtain the desired answer of 1.

**Off Diagonal Term,** $i = j$

Now we consider an element associated to mixing between two different operators, where neither belong to the inner multiplet and so are both outer products of two of the same states. For this example, we will consider mixing between operators $O_1$ and $O_3$, where $R$ is related to $T$ by swapping a box between rows 2 and 3. We must calculate:

$$\mathrm{Tr}(|30201\rangle\langle 30201|E_{22}|11101\rangle\langle 11101|E_{33}) = \langle 30201|E_{22}|11101\rangle\langle 11101|E_{33}|30201\rangle$$

We must now determine which states in the expansion of $E_{33}|30201\rangle$ have non-zero overlap with states in the expansion of $\langle 11101|$, and which states overlap between $E_{11}|11101\rangle$ and $\langle 30201|$. We can consider the Young diagrams to determine which states could possibly have an overlap; the state $\langle 30201|$ corresponds to the Young diagram , and the state $\langle 11101|$ corresponds to the diagram . This means that we only need to look at Clebsch-Gordons of $\langle 11101|$ corresponding to a box being pulled off row 2 (those that have $j+1, k-1$ in the label of the tensor-product-state), and those of $|30201\rangle$ where a box is pulled off the first row ($j-1, k$ in the tensor-product-state labels), since it is by pulling these boxes off that we obtain the same $s'$ from both states in terms of Young diagrams - it remains to be checked that there exists a state in both expansions that corresponds to the same Gelfand-Tsetlin pattern, and that have non-zero Clebsch-Gordon coefficients. The relevant states with their coeefficients in the expansion of $E_{22}|11101\rangle$ are:

$$\left\langle j+1,k-1,j^3,k^3,l^3;m_2|j,k,j^3,k^3,l^3\right\rangle|j+1,k-1,j^3,k^3,l^3;m_2\rangle = \langle 20101|11101\rangle|20101\rangle$$

and

$$\left\langle j+1,k-1,j^3+1,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\right\rangle |j+1,k-1,j^3+1,k^3-1,l^3+1;m_2\rangle = \mathbf{DNW}$$

The relevant states in the expansion of $\langle 30201|$ are:

$$\left\langle j-1,k,j^3-1,k^3,l^3;m_2|j,k,j^3,k^3,l^3\right\rangle \langle j-1,k,j^3-1,k^3,l^3;m_2| = \langle 20101|30201\rangle\langle 20101|$$

and

$$\left\langle j-1,k,j^3,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\right\rangle \langle j-1,k,j^3,k^3-1,l^3+1;m_2| = \mathbf{DNW}$$

Some of the states turn out not to be valid as they produce negative numbers for some of the labels, hence the **DNW**. The product $\langle 20101|20101\rangle = \sum_\alpha \langle 20101;\alpha|20101;\alpha\rangle = ds'$, the dimension of the symmetric group representation labelled by the Young diagram to which the labels 20101 correspond - this is where the $ds'$ in the factor that simplifies to $\frac{\sqrt{\text{hooks}_s\text{hooks}_u}}{\text{hooks}_{s'}}$ comes from (the sum over $\alpha$ has previously been omitted for simplicity). Multiplying the coefficients that are attached to the same state in both expansions, we end up with:

$$(\frac{1}{\sqrt{3}})(-\frac{1}{\sqrt{6}}) = -\frac{1}{3\sqrt{2}}$$

In exactly the same way, but using Clebsch-Gordons with the label $m_3$, we obtain for the second inner product:

$$(\frac{1}{\sqrt{3}})(-\frac{1}{\sqrt{6}}) = -\frac{1}{3\sqrt{2}}$$

Multiplying these together with the factor $\frac{\sqrt{\text{hooks}_s\text{hooks}_u}}{\text{hooks}_{s'}}$, we obtain:

$$3\frac{\sqrt{2}}{2}\frac{1}{18} = \frac{1}{6\sqrt{2}}$$

We are dealing with the case where $i=j$, and so we can be certain that swapping $E_{22}$ and $E_{33}$ will not affect the answer, since $\langle 30201|E_{22}|11101\rangle = \langle 11101|E_{22}|30201\rangle$. We can thus multiply by a factor of 2, giving the desired final answer of $\frac{1}{3\sqrt{2}}$ appearing in row 1, column 3 of $M^{(23)}$. It is the coefficient of $\sqrt{(N+b_0)(N+b_0+b_1)}(O_3(b_1-2,b_2+1,j,k,j^3,k^3,l^3)+O_3(b_1+2,b_2-1,j,k,j^3,k^3,l^3)$ and $-(2N+2b_0+b_1)O_3(b_1,b_2,j,k,j^3,k^3,l^3)$ in $DO_1(b_1,b_2)$.

**Off-Diagonal Term,** $i \neq j$

We now consider mixing between operators $O_1$ and $O_5$ when $R$ is related to $T$ by swapping a box between rows 1 and 2. This is performed in the exact same way as for the case where $i=j$, except that we must perform the calculation explicitly for both of the possible positions of $E_{11}$ and $E_{22}$. We are calculating (it looks as if this should be the expression for $O_4$, since $O_4$ appears as the second projector; but remember, the multiplicity indices are swapped - it just so happens that $O_4$ is what we get when we swap the states in $O_5$):

$$\text{Tr})(|30201\rangle\langle 30201|E_{11}|11101\rangle\langle 11010|E_{22}) = \langle 30201|E_{11}|11101\rangle\langle 11010|E_{22}|30201\rangle$$

We proceed with the calculation exactly as before, to find that after including the factor $\frac{\sqrt{\text{hooks}_s\text{hooks}_u}}{\text{hooks}_{s'}}$ we obtain:

$$3\frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{3}\frac{1}{\sqrt{6}}) = \frac{1}{\sqrt{6}}$$

We now swap the $E$'s, so we must calculate:

$$\langle 30201|E_{22}|11101\rangle\langle 11010|E_{11}|30201\rangle$$

Comparing the expansion of $E_{11}|30201\rangle$ with that of $\langle 11010|$, it is seen that there are no common states - this implies that this factor, and hence the entire term, vanishes. The only contribution is from the first product calculated, and so the final answer is $\frac{1}{\sqrt{6}}$, as desired. It appears in row 1, column 4 of $M^{(12)}$, and is the coefficient of

$$\sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2)}\left(O_4(b_1-1,b_2+2,j,k,j^3,k^3,l^3)+O_4(b_1+1,b_2-2,j,k,j^3,k^3,l^3)\right)$$

and $-(2N+2b_0+2b_1+b_2)O_4(b_1,b_2,j,k,j^3,k^3,l^3)$ in $DO_1(b_1,b_2,j,k,j^3,k^3,l^3)$.

$j^3 = O(1)$ **Sector:**

We assume that the remaining quantum numbers ($j, k, k^3, l^3$ and $m$) are all order $N$. The Clebsch-Gordan coefficients simplify considerably in this limit. The non-zero Clebsch-Gordan coefficients are

$$\langle j-1,k,j^3,k^3,l^3;m_1|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{j+k-k^3}{j+k}},$$

$$\langle j,k+1,j^3,k^3+1,l^3;m_1|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{k^3}{j+k}},$$

$$\langle j-1,k,j^3,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{k^3(k-k^3-l^3)}{(j+k)(k-k^3)}},$$

$$\langle j+1,k-1,j^3,k^3,l^3;m_2|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{k-k^3-l^3}{k-k^3}},$$

$$\langle j,k+1,j^3,k^3,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(j+k-k^3)(k-k^3-l^3)}{(j+k)(k-k^3)}},$$

$$\langle j-1,k,j^3,k^3-1,l^3;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{k^3l^3}{(j+k)(k-k^3)}},$$

$$\langle j+1,k-1,j^3,k^3,l^3-1;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{l^3}{k-k^3}},$$

$$\langle j,k+1,j^3,k^3,l^3;m_3|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j+k-k^3)l^3}{(j+k)(k-k^3)}}.$$

Looking at the non-zero Clebsch-Gordan coefficients, the reason for the simplification of this limit is clear. Indeed, notice that in the limit that we are considering the $j^3$ quantum number is fixed. This in turn implies that a single state from each inner multiplicity multiplet participates - a considerable simplification. Indeed, if $j, k, m$ and the $\Delta$ weight $\Delta = (n_1, n_2, n_3)$ are given, then we know

$$k^3 = \frac{m-3n_1-3j^3+2j+k}{3}, \qquad l^3 = \frac{m-3n_2+3j^3+k-j}{3}.$$

Thus, after specifying $\Delta$ and $j^3$ the $k^3, l^3$ labels are not needed. For this reason we can now simplify the notation for our operators to $O(b_1, b_2, j, k)$ for a given problem which is specified by $j^3$ and $\Delta^3$. The action of the dilatation operator is

$$\begin{aligned}
DO(b_1,b_2,j,k) = -g_{YM}^2 \Bigg[ & \frac{k^3(j+k-k^3)(k-k^3-l^3)(2m+j-k)}{3(j+k)^2(k-k^3)}\Delta_{12}O(b_1,b_2,j,k) \\
& + \frac{l^3k^3(j+k-k^3)(2m+j-k)}{3(j+k)^2(k-k^3)}\Delta_{13}O(b_1,b_2,j,k) - \frac{l^3k^3(k-k^3-l^3)(j+k-k^3)(2m+j-k)}{3(j+k)^2(k-k^3)^2}\Delta_{23}O(b_1,b_2,j,k) \\
& + \frac{l^3(k-k^3-l^3)(j+k-k^3)(2m-2j-k)}{3(j+k)(k-k^3)^2}\Delta_{23}O(b_1,b_2,j,k) + \frac{k^3l^3(k-k^3-l^3)(2m+j+2k)}{3(j+k)(k-k^3)^2}\Delta_{23}O(b_1,b_2,j,k) \\
& - \frac{(j+k-k^3)k^3(k-k^3-l^3)\sqrt{(m+2j+k)(m-j-2k)}}{3(j+k)^2(k-k^3)}\left(\Delta_{12}O(b_1,b_2,j-1,k-1)+\Delta_{12}O(b_1,b_2,j+1,k+1)\right) \\
& - \frac{l^3k^3(j+k-k^3)\sqrt{(m+2j+k)(m-j-2k)}}{3(j+k)^2(k-k^3)}\left(\Delta_{13}O(b_1,b_2,j-1,k-1)+\Delta_{13}O(b_1,b_2,j+1,k+1)\right) \\
& + \frac{l^3k^3(k-k^3-l^3)(j+k-k^3)\sqrt{(m+2j+k)(m-j-2k)}}{3(j+k)^2(k-k^3)^2}\left(\Delta_{23}O(b_1,b_2,j-1,k-1)+\Delta_{23}O(b_1,b_2,j+1,k+1)\right) \\
& - \frac{l^3(k-k^3-l^3)(j+k-k^3)\sqrt{(m-j-2k)(m-j+k)}}{3(j+k)(k-k^3)^2}\left(\Delta_{23}O(b_1,b_2,j+1,k-2)+\Delta_{23}O(b_1,b_2,j-1,k+2)\right) \\
& - \frac{l^3k^3(k-k^3-l^3)\sqrt{(m+2j+k)(m-j+k)}}{3(j+k)(k-k^3)^2}\left(\Delta_{23}O(b_1,b_2,j-2,k+1)+\Delta_{23}O(b_1,b_2,j+2,k-1)\right)\Bigg]
\end{aligned}$$

This rather lengthy expression can be simplified by the introduction of the operators $\Delta^{(a)}, \Delta^{(b)}$ and $\Delta^{(c)}$, defined as follows:

$$\begin{aligned}
\Delta^{(a)}O(b_1,b_2,j,k) &= (2m+j-k)O(b_1,b_2,j,k) \\
&-\sqrt{(m+2j+k)(m-j-2k)}\left(O(b_1,b_2,j-1,k-1)+O(b_1,b_2,j+1,k+1)\right)
\end{aligned}$$

$$\Delta^{(b)}O(b_1,b_2,j,k) = (2m-2j-k)O(b_1,b_2,j,k)$$

[3]The symmetric group operators used to define the restricted Schur polynomials are $P = \sum |j,k,j^3,k^3,l^3\rangle\langle j,k,j^{3\prime},k^{3\prime},l^{3\prime}|$ where we could have $j^3 \neq j^{3\prime}$, $k^3 \neq k^{3\prime}$, $l^3 \neq l^{3\prime}$. For simplicity we consider only the $j^3 = j^{3\prime}$ case. It is a simple extension of our analysis to consider the general case.

$$-\sqrt{(m-j-2k)(m-j+k)}\,(O(b_1,b_2,j+1,k-2)+O(b_1,b_2,j-1,k+2))$$

$$\Delta^{(c)}O(b_1,b_2,j,k)=(2m+j+2k)O(b_1,b_2,j,k)$$
$$-\sqrt{(m+2j+k)(m-j+k)}\,(O(b_1,b_2,j-2,k+1)+O(b_1,b_2,j+2,k-1))$$

The action of the dilatation operator becomes

$$\begin{aligned}DO(b_1,b_2,j,k) \quad &= -g_{YM}^2\left[\frac{k^3(j+k-k^3)(k-k^3-l^3)}{3(j+k)^2(k-k^3)}\Delta^{(a)}\Delta_{12}O(b_1,b_2,j,k)\right. \\ &+\frac{l^3k^3(j+k-k^3)}{3(j+k)^2(k-k^3)}\Delta^{(a)}\Delta_{13}O(b_1,b_2,j,k)-\frac{l^3k^3(k-k^3-l^3)(j+k-k^3)}{3(j+k)^2(k-k^3)^2}\Delta^{(a)}\Delta_{23}O(b_1,b_2,j,k) \\ &\left.+\frac{l^3(k-k^3-l^3)(j+k-k^3)}{3(j+k)(k-k^3)^2}\Delta^{(b)}\Delta_{23}O(b_1,b_2,j,k)+\frac{k^3l^3(k-k^3-l^3)}{3(j+k)(k-k^3)^2}\Delta^{(c)}\Delta_{23}O(b_1,b_2,j,k)\right]\end{aligned} \tag{6.20}$$

# Chapter 7

# Diagonalization of the Dilatation Operator

## 7.1 The Two-Row Dilatation Operator

The dilatation operator when acting on two giant systems has already been diagonalized in [10]. We start with a quick review of this material because it is relevant for the multiple giant systems we consider next. Make the following ansatz for the operators of good scaling dimension[1]

$$O(p,n) = \sum_{b_1} f(b_0,b_1)\, O_{p,j^3}(b_0,b_1) \; = \sum_{j,b_1} C_{p,j^3}(j)\, f(b_0,b_1)\, O_{j,j^3}(b_0,b_1)\,.$$

We are required to solve the eigenproblem

$$DO(p,n) = \kappa O(p,n)$$

where $\kappa$ is the one loop anomalous dimension. Plugging the expression for $O(p,n)$ given in the ansatz into this equation, we apply the effect of $D$ on the basis operators $O_{j,j^3}(b_0,b_1)$ given in (6.9) to obtain

$$\begin{aligned} g_{YM}^2 \sum_{b_1,j}[ \quad & h_1(j) C_{p,j^3}(j) f(b_0,b_1) \Delta O_{j,j^3}(b_0,b_1) + \\ & h_2(j) C_{p,j^3}(j) f(b_0,b_1) \Delta O_{j+1,j^3}(b_0,b_1) + \\ & h_3(j) C_{p,j^3}(j) f(b_0,b_1) \Delta O_{j-1,j^3}(b_0,b_1) \quad ] = \kappa \sum_{j,b_1} C_{p,j^3}(j) f(b_0,b_1) O_{j,j^3}(b_0,b_1) \end{aligned}$$

where for brevity we have defined:

$$\begin{aligned} h_1(j) &= -\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) \\ h_2(j) &= \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} \\ h_3(j) &= \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\frac{(j+j^3)(j-j^3)}{2j} \end{aligned}$$

We can then relabel the indices in the sum to obtain this equation in the form

$$\begin{aligned} g_{YM}^2 \sum_{b_1,j}[ \quad & h_1(j) C_{p,j^3}(j) f(b_0,b_1) \Delta O_{j,j^3}(b_0,b_1) + \\ & h_2(j-1) C_{p,j^3}(j-1) f(b_0,b_1) \Delta O_{j,j^3}(b_0,b_1) + \\ & h_3(j+1) C_{p,j^3}(j+1) f(b_0,b_1) \Delta O_{j,j^3}(b_0,b_1) \quad ] = \kappa \sum_{j,b_1} C_{p,j^3}(j) f(b_0,b_1) O_{j,j^3}(b_0,b_1) \end{aligned}$$

[1] $f(b_0,b_1)$ is not a function of $b_0$ and $b_1$ separately because $2b_0 + b_1$ is fixed equal to the number of $Z$s.

It is easy to check that $h_2(j-1) = h_3(j)$ and $h_3(j+1) = h_2(j)$. We thus obtain the equation in the form

$$g_{YM}^2 \sum_{b_1,j} \left[h_1(j)C_{p,j^3}(j) + h_2(j)C_{p,j^3}(j+1) + h_3(j)C_{p,j^3}(j-1)\right] f(b_0,b_1)\Delta O_{j,j^3}(b_0,b_1) \\ = \kappa \sum_{j,b_1} C_{p,j^3}(j) f(b_0,b_1) O_{j,j^3}(b_0,b_1) \tag{7.1}$$

Studying this equation, we see by comparing terms on each side having a definite value of the eigenvalue $p$ that the linear combination of $C_{p,j^3}$'s on the left hand side should be proportional to the $C_{p,j^3}(j)$ on the right hand side. In order to obtain the most general equation encoding this fact, we introduce a constant of proportionality that we will call $\alpha_{p,j^3}$, since each value of these two labels will produce a different equation in terms of $j$. This gives us the first recursion relation that we must solve in order to diagonalise the dilatation operator:

$$-\alpha_{p,j^3} C_{p,j^3}(j) = \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}} \frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} C_{p,j^3}(j+1) \\ \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}} \frac{(j+j^3)(j-j^3)}{2j} C_{p,j^3}(j-1) - \frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) C_{p,j^3}(j)\,. \tag{7.2}$$

If we now combine this equation with (7.1), we find

$$-\alpha_{p,j^3} g_{YM}^2 \sum_{b_1,j} C_{p,j^3}(j) f(b_0,b_1) \Delta O_{j,j^3}(b_0,b_1) = \kappa \sum_{j,b_1} C_{p,j^3}(j) f(b_0,b_1) O_{j,j^3}(b_0,b_1)$$

We define $O_q(b_0,b_1) = \sum_j C_{p,j^3} O_{j,j^3}(b_0,b_1)$, and rewrite to obtain

$$-\alpha_{p,j^3} g_{YM}^2 \sum_{b_1} f(b_0,b_1) \Delta O_q(b_0,b_1) = \kappa \sum_{b_1} f(b_0,b_1) O_q(b_0,b_1)$$

Applying the effect of $\Delta$ to the operator $O_q$, we have

$$-\alpha_{p,j^3} g_{YM}^2 \sum_{b_1} f(b_0,b_1)[\quad (2N+2b_0+b_1)O_q(b_0,b_1) - \sqrt{(N+b_0+b_1)(N+b_0)}(O_q(b_0+1,b_1-2) \\ + O_q(b_0-1,b_1+2))\quad ] = \kappa \sum_{b_1} f(b_0,b_1) O_q(b_0,b_1)$$

, which becomes

$$-\alpha_{p,j^3} g_{YM}^2 \sum_{b_1} [\quad f(b_0,b_1)(2N+2b_0+b_1) - (f(b_0-1,b_1+2) + f(b_0+1,b_1-2)) \times \\ \times \sqrt{(N+b_0+b_1)(N+b_0)}\quad ] O_q(b_0,b_1) = \kappa \sum_{b_1} f(b_0,b_1) O_q(b_0,b_1)$$

It is now a simple matter of equating the coefficients of $O_q(b_0,b_1)$ to obtain the second recursion relation:

$$-\alpha_{p,j^3} g_{YM}^2 \quad [\sqrt{(N+b_0)(N+b_0+b_1)}(f(b_0-1,b_1+2) + f(b_0+1,b_1-2)) \\ -(2N+2b_0+b_1) f(b_0,b_1)] = \kappa f(b_0,b_1)\,. \tag{7.3}$$

So, once the possible values of $\alpha_{p,j^3}$ have been determined from the first relation, each value can be plugged into the second relation in order to obtain a particular value of $\kappa$. The recursion relations are solved by

$$C_{p,j^3}(j) = (-1)^{\frac{m}{2}-p} \left(\frac{m}{2}\right)! \sqrt{\frac{(2j+1)}{(\frac{m}{2}-j)!\,(\frac{m}{2}+j+1)!}}\, {}_3F_2\left(\begin{matrix}|j^3|-j, j+|j^3|+1, -p \\ |j^3|-\frac{m}{2}, 1\end{matrix}; 1\right) \tag{7.4}$$

and

$$f(b_0,b_1) = (-1)^n (\frac{1}{2})^{N+b_0+\frac{b_1}{2}} \sqrt{\binom{2N+2b_0+b_1}{N+b_0+b_1}\binom{2N+2b_0+b_1}{n}}\, {}_2F_1\left(\begin{matrix}-(N+b_0+b_1), -n \\ -(2N+2b_0+b_1)\end{matrix}; 2\right) \tag{7.5}$$

The range of $j$ and $p$ are $|j^3| \leq j \leq \frac{m}{2}$(from the properties of angular momentum, and the fact that the number of rows with one box in $s$ is equal to $2j$), $0 \leq p \leq \frac{m}{2} - |j^3|$(by the definition of the Hahn polynomial, to which the hypergeometric function in our solution is dual), and the associated eigenvalues are

$$-\alpha_{p,j^3} = -2p = 0, -2, -4, ..., -(m - 2|j^3|)$$

and

$$\kappa = 4n\alpha_{p,j^3} g_{YM}^2 = 8png_{YM}^2 \qquad n = 0, 1, 2, \ldots .$$

We can check that these solutions are indeed valid by plugging them into the recursion relations. Substituting (7.4) into (7.2), we obtain by trivial algebra:

$$-\alpha_{p,j^3} {}_3F_2\left(\begin{matrix}|j^3|-j, j+1+|j^3|, -p \\ |j^3|-\frac{m}{2}, 1\end{matrix}\middle| 1\right) = \frac{(j+j^3+1)(j-j^3+1)(m-2j)}{2(j+1)(2j+1)} {}_3F_2\left(\begin{matrix}|j^3|-j-1, j+1+|j^3|, -p \\ |j^3|-\frac{m}{2}, 1\end{matrix}\middle| 1\right)$$

$$-\left(\frac{m}{2} - \frac{(m+2)(j^3)^2}{2j(j+1)}\right) {}_3F_2\left(\begin{matrix}|j^3|-j, j+1+|j^3|, -p \\ |j^3|-\frac{m}{2}, 1\end{matrix}\middle| 1\right) + \frac{(j+j^3)(j-j^3)(m+2j+2)}{2j(2j+1)} {}_3F_2\left(\begin{matrix}|j^3|-j+1, j+1+|j^3|, -p \\ |j^3|-\frac{m}{2}, 1\end{matrix}\middle| 1\right)$$

This has the form of the second recursion relation given in Appendix C, which is the defining relation for the hypergeometric function ${}_3F_2$ (which is also dual to the Hahn Polynomial) - this clearly implies that our chosen solution is valid. It is clear by comparing to the relation in the appendix that $-\alpha_{p,j^3} = -2p$. We can then proceed to perform the same check for the other solution; plugging (7.5) into (7.3), and using the large $N$ approximation, we obtain:

$$-\kappa_2 F_1\left(\begin{matrix}-(N+b_0+b_1), -n \\ -(2N+2b_0+b_1)\end{matrix}\middle| 2\right) = \alpha_{p,j^3} g_{YM}^2 (N+b_0) {}_2F_1\left(\begin{matrix}-(N+b_0+b_1)-1, -n \\ -(2N+2b_0+b_1)\end{matrix}\middle| 2\right) - [2N+2b_0+b_1] {}_2F_1\left(\begin{matrix}-(N+b_0+b_1), -n \\ -(2N+2b_0+b_1)\end{matrix}\middle| 2\right)$$

$$+(N+b_0+b_1) {}_2F_1\left(\begin{matrix}-(N+b_0+b_1)+1, -n \\ -(2N+2b_0+b_1)\end{matrix}\middle| 2\right)$$

Substituting the values $x = n$, $p = \frac{1}{2}$, $n = (N + b_0 + b_1)$ and $N = (2N + 2b_0 + b_1)$ into the first recursion relation given in Appendix C, we see that if $\kappa = 4ng_{YM}^2\alpha_{p,j^3}$ then the above relation is the same as the defining relation for the hypergeometric function ${}_2F_1$. This again clearly implies that our solution is valid.

### 7.1.1 Continuum Limit

Since our quantum numbers are very large, one might also consider examining the above recursion relations in a continuum limit where one would expect them to become differential equations. This is indeed the case[10]. Consider first (7.3). Introduce the continuous variable $\rho = \frac{b_1}{2\sqrt{N+b_0}}$ and replace $f(b_0, b_1)$ with $f(\rho)$. Now, expand

$$\sqrt{(N + b_0 + b_1)(N + b_0)} = (N + b_0)\left(1 + \frac{1}{2}\frac{b_1}{N + b_0} - \frac{1}{8}\frac{b_1^2}{(N + b_0)^2} + ....\right)$$

and

$$f\left(\rho - \frac{1}{\sqrt{N + b_0}}\right) = f(\rho) - \frac{1}{\sqrt{N + b_0}}\frac{\partial f}{\partial \rho} + \frac{1}{2(N + b_0)}\frac{\partial^2 f}{\partial \rho^2} + ...$$

These expansions are only valid if $b_1 \ll N + b_0$, which is certainly not always the case. However, for eigenfunctions with all of their support in the small $\rho$ region (which corresponds to operators having large differences between their row lengths, an assumption we have made all along) the continuum limit of the recursion relation will give accurate answers. The recursion relation becomes

$$(\alpha_{p,j^3} g_{YM}^2)\left[-\frac{\partial^2}{\partial \rho^2} + \rho^2\right] f(\rho) = \kappa f(\rho) \tag{7.6}$$

which is a harmonic oscillator with frequency $2\alpha_{p,j^3} g_{YM}^2$. This can be clearly seen in a quantum mechanical sense if we define creation and annihilation operators as $a = i\frac{\partial}{\partial \rho} + i\rho$ and $a^\dagger = i\frac{\partial}{\partial \rho} - i\rho$ - after normalization, these have a commutator equal to one, as expected for a harmonic oscillator. We should only keep half of the oscillator states because the lengths of the rows (or columns) of the Young diagram are non-increasing, which implies that $b_1 \geq 0$ and hence that $\rho \geq 0$. The necessity of this has also been seen in numerical calculations of the wavefunction, which was found to have a value of zero at the origin, implying that we should only keep oscillator states having the property that their wavefunction passes

through the origin; only alternate excited states have this property. In short, only wave functions that vanish at $\rho = 0$ are allowed solutions. By considering the definition of $\rho$, we see that it can be viewed as the relative coordinate between $b_0$ and $b_1$ - it is a measure of the difference in length between the two rows, and hence a measure of the radial separation of the giant gravitons. With our interpretation of the excitations as being the excited states of strings connecting the gravitons, it is therefore sensible that we should see a vanishing of the wavefunction at the origin, since this is the point where the giant gravitons have exactly the same size, and therefore the strings between them should have zero energy. Thus, the energy spacing of the half oscillator states is $4\alpha_{p,j^3} g_{YM}^2$. Clearly the description of the coefficients $f(b_0, b_1)$ obtained by solving (7.6) will be accurate for the low lying oscillator eigenstates. Any operators corresponding to a finite energy state are accurately described.

A few comments are in order. The solutions of the discrete recursion relations can be compared to the solution of the continuum differential equations. The agreement is perfect[10]. Although the solution of our discrete recursion relation is in complete agreement with the solution of the corresponding differential equation obtained by taking a continuum limit, notice that the solution of the recursion relation does not make any additional assumptions. To obtain our differential equation we assumed that $b_1 \ll N + b_0$. Thus, although solving the differential equation is easier, the solution is not as general.

## 7.2 The Three-Row Dilatation Operator

Consider now the action of the dilatation operator when acting on three giant systems. We study the $\Delta = (1, 1, 1)$ example first. It is a simple matter to check that the matrices $M^{(12)}$, $M^{(13)}$ and $M^{(23)}$ appearing in (6.19) commute and hence can be simultaneously diagonalized. All three matrices have eigenvalues $(2, 1, 1, 0, 0, 0)$. It is convenient to use the eigenvectors corresponding to the non-degenerate eigenvalue 2 in each matrix - it turns out that these eigenvectors have eigenvalue 0 when acted on by either of the other two matrices. The 2 eigenvectors in each matrix corresponding to eigenvalue 1 are common to all three matrices, and are thus an obvious choice, since they have the same eigenvalue of 1 when acted on by any of the 3 matrices. We thus have 5 common eigenvectors; the last one can obviously be the null vector, since we have zero eigenvalues in all three matrices, thus providing us with a complete set of 6 common eigenvectors that simultaneously diagonalise the set of matrices. We define eigenoperators as the linear combination of operators $O_i = \sum_j v_{ij} O_j$, where $v_{ij}$ denotes the $j$th element of eigenvector $i$. The result of the diagonalization is thus the following 6 decoupled equations (the label $i$ is counted in Roman numerals, to clarify the distinction from the operators $O_j$)

$$\begin{aligned} DO_I(b_1, b_2) &= -2g_{YM}^2 \Delta_{23} O_I(b_1, b_2), \qquad DO_{II} = -2g_{YM}^2 \Delta_{12} O_{II}(b_1, b_2), \\ DO_{III}(b_1, b_2) &= -2g_{YM}^2 \Delta_{13} O_{III}(b_1, b_2), \qquad DO_{IV}(b_1, b_2) = -g_{YM}^2 (\Delta_{23} + \Delta_{12} + \Delta_{13}) O_{IV}(b_1, b_2), \\ DO_V(b_1, b_2) &= -g_{YM}^2 (\Delta_{23} + \Delta_{12} + \Delta_{13}) O_V(b_1, b_2), \qquad DO_{VI}(b_1, b_2) = 0. \end{aligned} \tag{7.7}$$

where:

$$\begin{aligned} O_I &= \frac{-1}{\sqrt{2}} O_1 + \frac{1}{\sqrt{2}} O_2 - O_3 + O_6 \\ O_{II} &= \sqrt{2} O_1 - \sqrt{2} O_2 - O_3 + \sqrt{3} O_4 + \sqrt{3} O_5 + O_6 \\ O_{III} &= \sqrt{2} O_1 - \sqrt{2} O_2 - O_3 - \sqrt{3} O_4 - \sqrt{3} O_5 + O_6 \\ O_{IV} &= -\sqrt{2} O_1 - \sqrt{2} O_2 + O_3 + O_6 \\ O_V &= -O_4 + O_5 \\ O_{VI} &= 0 \end{aligned}$$

In a matrix notation, this is given by:

$$DO_i(b_1, b_2) = -g_{YM}^2 \left( M_{ik}^{(12)} \Delta_{12} O_k(b_1, b_2) + M_{ik}^{(13)} \Delta_{13} O_k(b_1, b_2) + M_{ik}^{(23)} \Delta_{12} O_k(b_1, b_2) \right) \qquad i = I, \cdots, VI$$

with

$$\begin{aligned} M^{(12)} &= \mathbf{Diag}(0, 2, 0, 0, 1, 1) \\ M^{(13)} &= \mathbf{Diag}(0, 0, 2, 0, 1, 1) \\ M^{(23)} &= \mathbf{Diag}(2, 0, 0, 0, 1, 1) \end{aligned}$$

It is interesting to ask if we can diagonalize (7.7) directly without taking a continuum limit, since the resulting spectrum is not computed with any assumptions about the relative lengths of the rows, and will hence be completely general. Consider first the equation for $O_{II}(b_1, b_2)$; we proceed in the same manner as for the two row case, once we have determined what a reasonable ansatz to apply to the operators is. It is clear that $\Delta_{12}$ does not change the value of $b_0$. In addition, the dilatation operator does not change the number of $Z$s in our operator, so that $n_Z = 3b_0 + 2b_2 + b_1$ is fixed. This motivates the ansatz

$$O = \sum_{b_1} f(b_1, b_2) O_{II}(b_1, b_2)\Big|_{b_2 = n_Z - 3b_0 - 2b_1}$$

Requiring that $DO = 2g_{YM}^2 \alpha_n O$ ($\alpha_n$ is analogous to $\kappa$ used in the two row case; the factor $2g_{YM}^2$ is inserted for convenience) we obtain the recursion relation[2]

$$-(2N + 2b_0 + 2b_1 + b_2) f_n(b_1, b_2) + \sqrt{(N + b_0 + b_1)(N + b_0 + b_1 + b_2 + 1)} f_n(b_1 - 1, b_2 + 2)$$
$$+\sqrt{(N + b_0 + b_1 + 1)(N + b_0 + b_1 + b_2)} f_n(b_1 + 1, b_2 - 2) = 2g_{YM}^2 \alpha_n f_n(b_1, b_2)$$

where in the above equation $b_2 = n_Z - 3b_0 - 2b_1$. Using the results of Appendix C, it is a simple matter to verify that this recursion relation is solved by

$$f_n = (-1)^n \left(\frac{1}{2}\right)^{N + b_0 + b_1 + \frac{b_2}{2}} \sqrt{\binom{2N + 2b_0 + 2b_1 + b_2}{N + b_0 + b_1 + b_2} \binom{2N + 2b_0 + 2b_1 + b_2}{n}} {}_2F_1 \left(\begin{matrix} -(N + b_0 + b_1 + b_2), -n \\ -(2N + 2b_0 + 2b_1 + b_2) \end{matrix}; 2\right)$$

$$2g_{YM}^2 \alpha_n = 4n g_{YM}^2, \qquad n = 0, 1, 2, \cdots, \text{int}\left(\frac{n_Z - 3b_0}{2}\right)$$

where $n_Z$ is the number of $Z$s in the restricted Schur polynomial, $b_0$ is fixed, $b_2 = n_Z - 3b_0 - 2b_1$ and int$(\cdot)$ is the integer part of the number in braces. Again, only half the states are retained because $b_1, b_2 > 0$ so that we finally obtain a spacing of $8ng_{YM}^2$ - we will see that this is in perfect agreement with what is found in the continuum limit. Notice that we obtain a set of eigenfunctions for each value of $b_0$, so that at infinite $N$ we have an infinite degeneracy at each level.

The equation for $O_{III}(b_1, b_2)$ can be solved in the same way, by introducing the same ansatz but obviously with $O_{II}$ replaced by $O_{III}$ . The recursion relation obtained is then clearly very similar, and we can show that it is solved by

$$f_n(b_0, b_1) = (-1)^n \left(\frac{1}{2}\right)^{N + b_0 + \frac{b_1 + b_2}{2}} \sqrt{\binom{2N + 2b_0 + b_1 + b_2}{N + b_0 + b_1 + b_2} \binom{2N + 2b_0 + b_1 + b_2}{n}} {}_2F_1 \left(\begin{matrix} -(N + b_0 + b_1 + b_2), -n \\ -(2N + 2b_0 + b_1 + b_2) \end{matrix}; 2\right)$$

$$n = 0, 1, ...., \min(J, n_Z - 2J)$$

where $J = b_0 + b_1$ is fixed, $b_2 = n_Z - 3b_0 - 2b_1$ and $\min(a, b)$ is the smallest of the two integers $a$ and $b$. Only half the states are retained because $b_1, b_2 > 0$ and we again obtain a spacing of $8ng_{YM}^2$. Notice that we obtain a set of eigenfunctions for each value of $J$, so that at infinite $N$ we again have an infinite degeneracy at each level. In the same way, we find for $O_I(b_1, b_2)$ that

$$f_n(b_0, b_1) = (-1)^n \left(\frac{1}{2}\right)^{N + b_0 + \frac{b_1}{2}} \sqrt{\binom{2N + 2b_0 + b_1}{N + b_0 + b_1} \binom{2N + 2b_0 + b_1}{n}} {}_2F_1 \left(\begin{matrix} -(N + b_0 + b_1), -n \\ -(2N + 2b_0 + b_1) \end{matrix}; 2\right)$$

$$n = 0, 1, ...., \text{int}\left(\frac{n_Z - J}{2}\right)$$

where $J = b_0 + b_1 + b_2$ is fixed and $b_2 = n_Z - 3b_0 - 2b_1$. Only half the states are retained because $b_1, b_2 > 0$ and we again obtain a spacing of $8ng_{YM}^2$. Notice that we obtain a set of eigenfunctions for each value of $J$, so that at infinite $N$ we again have an infinite degeneracy at each level. The solution for the null eigenvector $O_{VI}$ is, trivial. It would be interesting to solve the recursion relations arising from $O_V(b_1, b_2)$ and $O_{VI}(b_1, b_2)$, and it is possible to obtain specific solutions for the resulting equations, but a general solution will not be presented here. Instead, we will continue onto the more tractable continuum limit solutions.

[2]Notice that we have replaced $N + b_0 + b_1 + b_2 \to N + b_0 + b_1 + b_2 + 1$ under the square root in the second term on the left hand side and we have replaced $N + b_0 + b_1 \to N + b_0 + b_1 + 1$ under the square root in the third term on the left hand side. This may seem odd, since we have already dropped finite additions to these terms arising from the changes in the $b_i$. The purpose is purely the simplification of the solution, and we can do this with negligible error in the large $N$ limit.

### 7.2.1 Continuum Limit

Taking a continuum limit implies assuming that $b_1, b_2 \ll N + b_0$. The square roots can be expanded in a Taylor series, and we can define continuous variables $x = b_1/\sqrt{N+b_0}$ and $y = b_2/\sqrt{N+b_0}$ in which the functions $f(b_1, b_2)$ can be Taylor expanded also. We find:

$$\Delta_{12} O(b_1, b_2) \to \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)^2 O(x,y) - \frac{y^2}{4} O(x,y)$$

$$\Delta_{13} O(b_1, b_2) \to \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 O(x,y) - \frac{(x+y)^2}{4} O(x,y)$$

$$\Delta_{23} O(b_1, b_2) \to \left(2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 O(x,y) - \frac{x^2}{4} O(x,y)$$

These all correspond to oscillators with an energy level spacing of[3] 2. However, again because $b_1, b_2 > 0$ we keep only half the states and hence obtain oscillators with a level spacing of 4. The corresponding eigenvalues of the dilatation operator are $8ng_{YM}^2$ with $n$ an integer. This is remarkably consistent with what we found for the anomalous dimensions for the two giant system. Of course, a very important difference is that since these oscillators live in a two dimensional space, there will be an infinite discrete degeneracy in each level. Finally, it is also straight forward to show that

$$\Delta_{23} + \Delta_{12} + \Delta_{13} = 3\frac{\partial^2}{\partial x^{+2}} - \frac{3}{4}(x^+)^2 + 9\frac{\partial^2}{\partial x^{-2}} - \frac{1}{4}(x^-)^2$$

where

$$x^+ = \frac{x+y}{\sqrt{2}}, \qquad x^- = \frac{x-y}{\sqrt{2}}\,.$$

After rescaling the $x^- \to \sqrt{3}x^-$ we obtain a rotation invariant 2d harmonic oscillator with an energy level spacing of 3. Again because $b_1, b_2 > 0$ we keep only half the states and hence obtain oscillators with a level spacing of 6. The corresponding eigenvalues of the dilatation operator are $6ng_{YM}^2$ with $n$ an integer. This is the solution we expect to obtain by solving the discrete recursion relations for $O_V(b_1, b_2)$ and $O_{VI}(b_1, b_2)$ in general. Equivalent equations can be found using the results of Appendix D.

### 7.2.2 $j^3 = O(1)$ Sector

We now turn to the $j^3 = O(1)$ example. We have already studied the continuum limit of the operators $\Delta_{12}$, $\Delta_{13}$, and $\Delta_{23}$. In addition to these three operators, we will also need the continuum limit of $\Delta^{(a)}$, $\Delta^{(b)}$ and $\Delta^{(c)}$. Taking $j, k \ll m$ and defining the continuum variables $w = k/\sqrt{m}$, $z = j/\sqrt{m}$ it is straight forward to obtain

$$\Delta^{(a)} O(j,k) \to \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 - \frac{9}{4}(z+w)^2$$

$$\Delta^{(b)} O(j,k) \to \left(\frac{\partial}{\partial z} - 2\frac{\partial}{\partial w}\right)^2 - \frac{9}{4}w^2$$

$$\Delta^{(c)} O(j,k) \to \left(\frac{\partial}{\partial w} - 2\frac{\partial}{\partial z}\right)^2 - \frac{9}{4}z^2\,.$$

These all correspond to oscillators with an energy level spacing of 3. Once again, because $j, k > 0$, only half the states are valid solutions implying a final level spacing of 6. Finally, we need to consider the continuum limit of the coefficients appearing in (6.20). Things simplify very nicely if we focus on those operators for which $\Delta = (n, n, n_3)$ and $n_3 \gg n$. In this case, we find

$$k^3 = l^3 = \frac{m}{3} - n$$

so that after taking the continuum limit (6.20) becomes

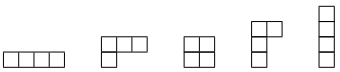
$$DO(w,x,y,z) = g_{YM}^2 \frac{(k^3)^2}{3(j+k)^2}\left[9\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 - \frac{(x-y)^2}{4}\right]\left[\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 - 9\frac{(z+w)^2}{4}\right] O(w,x,y,z)$$

which is a direct product of harmonic oscillators! Although many interesting questions could be pursued at this point, we will not do so here.

[3] For example, for the oscillator corresponding to $\Delta_{12}$ we have $H = \frac{1}{2}(aa^\dagger + a^\dagger a)$, $[a, a^\dagger] = 2$, $a = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} + \frac{y}{2}$ and $a^\dagger = -\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + \frac{y}{2}$.

## 7.3 A Four-Row Example

Finally, we have studied the action of the dilatation operator when acting on four giant systems. We will report the result for a four giant system with four impurities and $\Delta = (1,1,1,1)$. The $S_4$ representations that can be subduced are



The second and fourth representations are each subduced 3 times from $R$, and each form inner multiplicity multiplets for which 9 projectors can be defined. The third representation is subduced twice, so that 4 operators can be defined from this Young diagram. The other two are subduced only once, and hence there are a total of 24 operators that can be defined. The action of the dilatation operator when acting on these 24 operators can be written in terms of (only the labels of the Young diagram for the $Z$s is shown; the $b_i$ are again the difference in the lengths of the rows)

$$\Delta_{12}O(b_1,b_2,b_3) = -(2N+2b_0+2b_1+2b_2+b_3)O(b_1,b_2,b_3) + \sqrt{(N+b_0+b_1+b_2)(N+b_0+b_1+b_2+b_3)}\,(O(b_1,b_2+1,b_3-2)+O(b_1,b_2-1,b_3+2))\,, \tag{7.8}$$

$$\Delta_{13}O(b_1,b_2) = -(2N+2b_0+2b_1+b_2+b_3)O(b_1,b_2,b_3) + \sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2+b_3)}\,(O(b_1+1,b_2-1,b_3-1)+O(b_1-1,b_2+1,b_3+1))\,, \tag{7.9}$$

$$\Delta_{14}O(b_1,b_2,b_3) = -(2N+2b_0+b_1+b_2+b_3)O(b_1,b_2,b_3) + \sqrt{(N+b_0)(N+b_0+b_1+b_2+b_3)}\,(O(b_1-1,b_2,b_3-1)+O(b_1+1,b_2,b_3+1))\,. \tag{7.10}$$

$$\Delta_{23}O(b_1,b_2,b_3) = -(2N+2b_0+2b_1+b_2)O(b_1,b_2,b_3) + \sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2)}\,(O(b_1+1,b_2-2,b_3+1)+O(b_1-1,b_2+2,b_3-1))\,. \tag{7.11}$$

$$\Delta_{24}O(b_1,b_2,b_3) = -(2N+2b_0+b_1+b_2)O(b_1,b_2,b_3) + \sqrt{(N+b_0)(N+b_0+b_1+b_2)}\,(O(b_1-1,b_2-1,b_3+1)+O(b_1+1,b_2+1,b_3-1))\,. \tag{7.12}$$

$$\Delta_{34}O(b_1,b_2,b_3) = -(2N+2b_0+b_1)O(b_1,b_2,b_3) + \sqrt{(N+b_0)(N+b_0+b_1)}\,(O(b_1-2,b_2+1,b_3)+O(b_1+2,b_2-1,b_3))\,. \tag{7.13}$$

The action of the dilatation operator can then be expressed in terms of six $24\times 24$ matrices multiplying these operators, and these six matrices can be simultaneously diagonalized. After diagonalizing on the impurity labels we obtain the following decoupled problems: One BPS state

$$DO(b_1,b_2,b_3) = 0\,, \tag{7.14}$$

six operators with two rows participating

$$DO(b_1,b_2,b_3) = -2g_{YM}^2\Delta_{ij}O(b_1,b_2,b_3), \quad (ij) = \{(12),(13),(14),(23),(24),(34)\}\,, \tag{7.15}$$

four doubly degenerate operators with three rows participating (so each equation appears twice) giving eight more operators

$$DO(b_1,b_2,b_3) = -g_{YM}^2(\Delta_{12}+\Delta_{13}+\Delta_{23})O(b_1,b_2,b_3), \quad \text{plus 3 more}\,, \tag{7.16}$$

six operators of the type

$$DO(b_1,b_2,b_3) = -g_{YM}^2(\Delta_{12}+\Delta_{23}+\Delta_{34}+\Delta_{14})O(b_1,b_2,b_3), \quad \text{plus 5 more}\,, \tag{7.17}$$

and finally three operators of the type

$$DO(b_1,b_2,b_3) = -2g_{YM}^2(\Delta_{12}+\Delta_{34})O(b_1,b_2,b_3), \quad \text{plus 2 more}\,. \tag{7.18}$$

The equations (7.14), (7.15) and (7.16) can be solved with a very simple extension of what was done for the three giant system.

# Chapter 8

# Results & Conclusions

Technology for working with restricted Schur polynomials has been developed [6, 24, 25, 33, 34, 3, 8, 9, 10] and is now at the stage where it is becoming useful. In this dissertation we have further added to this technology by describing a new version of Schur-Weyl duality that provides a powerful approach to the computation and manipulation of the symmetric group operators appearing in the restricted Schur polynomials. Using this new technology we have shown that it is straight forward to evaluate the action of the one loop dilatation operator on restricted Schur polynomials. We studied the spectrum of one loop anomalous dimensions on restricted Schur polynomials that have $p$ long columns or rows. For $p = 3, 4$ we have obtained the spectrum explicitly in a number of examples, and have shown that it is identical to the spectrum of decoupled harmonic oscillators. This generalizes results obtained in [8, 9, 10]. The articles [8, 9, 10] provided very strong evidence that the one loop dilatation operator acting on restricted Schur polynomials with two long rows or columns is integrable. In this article we have found evidence that the dilatation operator when acting on restricted Schur polynomials with $p$ long rows or columns is an integrable system. To obtain this action we had to sum much more than just the planar diagrams so that *integrability in $\mathcal{N} = 4$ super Yang-Mills theory is not just a feature of the planar limit, but extends to other large $N$ but non-planar limits.*

The operators we have studied are dual to giant gravitons in the $AdS_5 \times S^5$ background. These giant gravitons have a world volume whose spatial component is topologically an $S^3$. The excitations of the giant graviton will correspond to vibrational excitations of this $S^3$. At the quantum level, the energy in any particular vibrational mode will be quantized and consequently, the free theory of giant gravitons should be a collection of decoupled oscillators, which provides a rather natural interpretation of the oscillators we have found.

## 8.1 Connection to the Gauss Law

Giant gravitons are D-branes. Attaching open strings to a D-brane provides a concrete way to describe excitations. Are these open strings visible in our work? Recall that, since the giant graviton has a compact world volume, the Gauss Law implies that the total charge on the giant's world volume must vanish. When enumerating the possible stringy excitation states of a system of giant gravitons, only those states consistent with the Gauss Law should be retained. In [6], restricted Schur polynomials corresponding to giants with "string words" attached were constructed and, remarkably, the number of possible operators that could be defined in the gauge theory matches the number of stringy excitation states of the system of giant gravitons. In this study we have replaced open strings words with impurities $Y$, which does not modify the counting argument of [6].

### 8.1.1 $\Delta = (1, 1, 1)$ Example

Our results add something new and significant to this story: not only does the counting of states match with that expected from the Gauss Law, but, as we now explain, the structure of the action of the dilatation on restricted Schur polynomials itself is closely related to the Gauss Law. Consider the three giant system with $\Delta = (1, 1, 1)$. For this $\Delta$ we have three impurities and hence we consider open string configurations with 3 open strings participating. There are three rows in the Young diagrams, corresponding to three giant gravitons. Draw each giant graviton as a solid dot as shown in figure 8.1.

The Gauss Law constraint then becomes the condition that there are an equal number of open strings coming to each particular dot as there are leaving the particular dot. We find six possible open string configurations consistent with the Gauss Law as shown in figure 8.1. Our results suggest that the action of the one loop dilatation operator is also coded into these diagrams. For each figure associate a factor of $\Delta_{ij}$ for a string stretching between dots $i$ and $j$[1]. Since $\Delta_{ij} = \Delta_{ji}$, the last two figures shown translate into the same equation, but because the string orientations are different they do represent different states. A string starting and ending on the same dot does not contribute a $\Delta$. Once the complete set of $\Delta_{ij}$ are read off the diagram, the action of the dilatation operator is given by summing them and multiplying by $-g_{YM}^2$. Thus, the first diagram shown translates into

$$DO(b_1, b_2) = 0\,.$$

The last two diagrams both give

$$DO(b_1, b_2) = -g_{YM}^2(\Delta_{23} + \Delta_{12} + \Delta_{13})O(b_1, b_2)\,.$$

Finally, the remaining three diagrams give

$$DO(b_1, b_2) = -2g_{YM}^2\Delta_{12}O(b_1, b_2), \quad DO(b_1, b_2) = -2g_{YM}^2\Delta_{13}O(b_1, b_2),$$
$$DO(b_1, b_2) = -2g_{YM}^2\Delta_{23}O(b_1, b_2)\,.$$

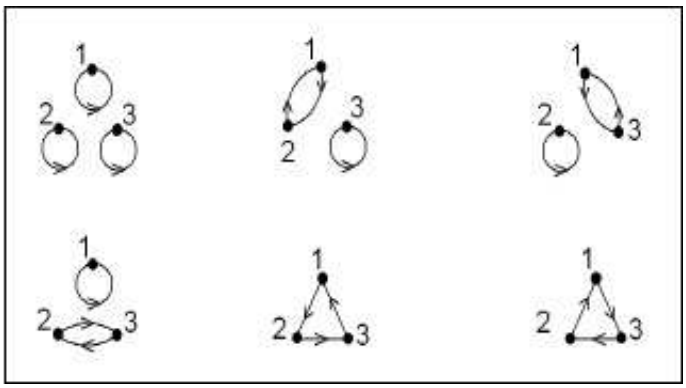
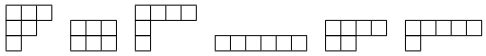



Figure 8.1: A schematic representation of the possible excitations of a three giant system that are consistent with the Gauss Law. Each giant graviton is represented by a labelled point. Lines represent open strings.

This is exactly the action we finally obtained in (7.7)! The reader is invited to check that this matching between the possible open string configurations and the action of the dilatation operator continues for the four giant system with $\Delta = (1, 1, 1, 1)$. These two examples remove exactly one box from each row. However, the connection to the Gauss Law is general. It is easy to check that it is consistent with the exact two row results obtained in [8, 9, 10].

### 8.1.2 $\Delta = (3, 2, 1)$ Example

We now report the result of the computation of the action of the dilatation operator for restricted Schur polynomials with three rows and $\Delta = (3, 2, 1)$. There are a total of 60 states (i.e. 60 Young-Yamonouchi labels corresponding to states in the carrier space of representations of $S_6$) that can be obtained by removing 6 boxes as specified by the $\Delta$ weight. The 6 $S_6$ irreducible representations that can be suduced are



with the last two irreducible representations being suduced twice. Thus, there are a total of 12 operators that can be defined - 4 from the first four representations, and 4 from each of the last two, since each

[1] $\Delta_{ij}$ in general is the natural generalization of the operators we defined in section 3, with boxes moving between rows $i$ and $j$.

of these corresponds to two distinct Gelfandt-Tsetlin patterns forming an inner multiplicity multiplet. The action of the dilatation operator can be determined in terms of functions $\Delta_{ij}$ and expressed using matrices $M^{(ij)}$ as for the $\Delta = (1,1,1)$ example. After diagonalizing the action of the dilatation operator (by simultaneously diagonalizing the $M^{(ij)}$'s), we find

$$DO = 0 \tag{8.1}$$

$$DO = -2g_{YM}^2 \Delta_{12} O \tag{8.2}$$

$$DO = -2g_{YM}^2 \Delta_{23} O \tag{8.3}$$

$$DO = -2g_{YM}^2 \Delta_{13} O \tag{8.4}$$

$$DO = -2g_{YM}^2 (\Delta_{12} + \Delta_{13}) O \tag{8.5}$$

$$DO = -2g_{YM}^2 (2\Delta_{12} + \Delta_{13}) O \tag{8.6}$$

$$DO = -2g_{YM}^2 (\Delta_{12} + \Delta_{23}) O \tag{8.7}$$

$$DO = -4g_{YM}^2 \Delta_{12} O \tag{8.8}$$

$$DO = -g_{YM}^2 (\Delta_{12} + \Delta_{13} + \Delta_{23}) O \tag{8.9}$$

$$DO = -g_{YM}^2 (\Delta_{13} + 3\Delta_{12} + \Delta_{23}) O \tag{8.10}$$

The last two equations each appear twice. The corresponding diagrams are shown in figure 8.2.

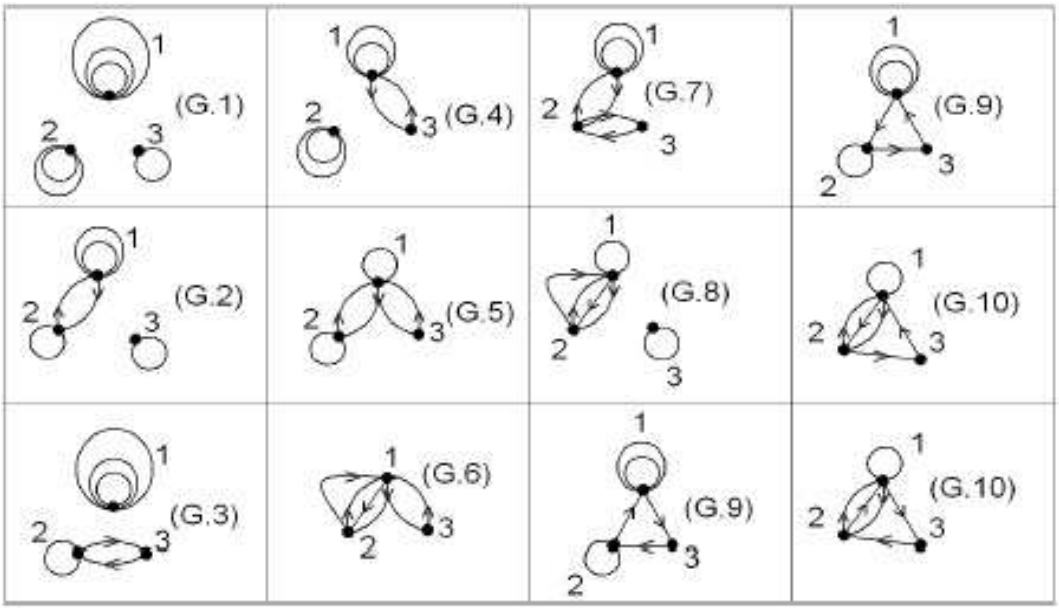



Figure 8.2: The open string configurations consistent with the Gauss Law for a three giant system with $\Delta$ weight $\Delta = (3,2,1)$. The figure labels match the corresponding equation.

### 8.1.3 General Action in the Continuum Limit

This connection provides a remarkably simple and general way of describing the action of the one loop dilatation operator in the large $N$ but non-planar limit. For example, we learn that the action of the dilatation operator is given by summing a collection of operators $\Delta_{ij}$, each appearing some integer $n_{ij}$ number of times.

$$DO(b_1, b_2) = -g_{YM}^2 \sum_{ij} n_{ij} \Delta_{ij} \, O(b_1, b_2) \, .$$

In Appendix D the action of this operator in a natural continuum limit is studied and found to take the form

$$-g_{YM}^2 \sum_{ij} n_{ij} \Delta_{ij} \to g_{YM}^2 \sum_I D_I \left[ -\frac{\partial^2}{\partial x_I^2} + \frac{x_I^2}{4} \right] .$$

Thus, at one loop and in this continuum limit, the dilatation operator reduces to an infinite set of decoupled oscillators. The open string excitations of the $p$ giant graviton system are, at low energy, described by a Yang-Mills theory with $U(p)$ gauge group. It seems natural to identify the $U(p)$ which played a central role in our new Schur-Weyl duality with this gauge group - this connection is expanded upon in the next subsection.

## 8.2 Holography and Emergent Spacetime

Clearly, since the two theories to which the AdS/CFT correspondence applies are defined on spacetimes of different dimensions, we must give some explanation as to how the extra dimensions of the string theory are reconstructed in the SYM theory. Relevant to the description of the emergence of these extra dimensions is the content of the Holographic principle, an idea originally put forth by Gerard 't Hooft [43], which states that all the information required to describe the dynamics of the volume of a $d$ dimensional space can be encoded on the $d-1$ dimensional boundary of the space. As has been explained, the spacetime of the SYM theory can be conformally mapped to the spacetime of the boundary of $AdS$ space. We can say that the extra dimensions of the string theory are *holographically reconstructed* in the SYM theory. A discussion of how details of the geometry and locality of the $(9+1)$-dimensional string theory spacetime can be extracted from Schur polynomial operators in the $(3+1)$ dimensional Yang-Mills theory can be found in [44]. The content of this article is briefly discussed hereforth.

$\frac{1}{2}$-BPS (LLM) geometries of the string theory are dual to Schur Polynomials containing $O(N^2)$ fields in the Yang-Mills theory; it is by considering the Young diagram labels of these operators that details of the dual geometry are emergent [45]. The Schur polynomial description can be mapped into a description as an energy eigenstate of $N$ free fermions. The energy levels that are occupied are found by identifying the energy of a level with $r_i + i - 1$, where $r_i$ is the number of boxes in row $i$ of the Young diagram, and the label $i$ increases from the bottom to the top of the Young diagram. These energies map to rings in the fermion phase space, which is identified with the LLM plane - the two dimensional plane on which the boundary condition for the LLM geometry is specified. If the rings corresponding to occupied energy levels are colored black, and the unoccupied levels white, we obtain from any given Young diagram a pattern of concentric rings which specifies a boundary condition for the dual LLM geometry. We can see locality on the dual geometry by the relation between the radius on the LLM plane and the position on the Young diagram, or equivalently the energy of the excitation. An excitation is implemented by attaching additional boxes ($O(1), O(\sqrt{N}), O(N)$ for graviton, string and giant graviton respectively) representing the excitation to a certain position on the Young diagram labelling the geometry, which corresponds to a certain radius on the LLM plane. Thus we have the natural emergence of both the geometry of the extra dimensions as well as locality of interactions from the Young diagram labels of the Schur polynomial.

As is well known, the low energy dynamics of open strings on a D-brane are described by field theory. Another aspect of emergent spacetime to consider then is the emergence of a new gauge theory from the low energy interactions of strings on the giant graviton (a D3-brane). This has been investigated in [6], where the conclusion was reached that these string dynamics are described by a theory with emergent gauge symmetry on a $3+1$ dimensional space that arises from the matrix degrees of freedom of the original SYM theory - i.e. due to the $Z$'s and $X$'s. This is the emergent theory for which it was suggested at the end of the previous section that the gauge group should be the $U(p)$ (where $p$ is the giant graviton number) that was used in the construction of the projectors appearing in our Schur polynomials.

## 8.3 Summary & Outlook

This disseration has focused on the material presented in [7], wherein the energies of excitations of giant gravitons in an $AdS_5 \times S^5$ background are calculated. There has been much debate as to whether or not such a system is integrable in the non-planar limit; it has now been shown that the system is indeed integrable, not by a conceptual argument, but by performing the calculation which produces the solutions. The solutions themselves are remarkable - by following a lengthy and involved procedure, to which it can confidently be said that no unreasonable simplifications are applied, it is found that the spectrum of energies of the system are identical to those of a set of decoupled harmonic oscillators. In addition, a strong connection to the Gauss law is also observed - in fact, the equations for the diagonalized dilatation operator, from which the oscillator energies are determined, can be extracted by the invention of simple

diagrams that illustrate this law for the Giant Graviton system (8.1). The authors of [7] have thus created a powerful and intuitive method by which the energy eigensystem for a number of examples of weakly interacting Giant Graviton systems in the given background can be computed.

The most important prerequisite for the theory is the AdS/CFT conjecture of J.M. Maldacena ([1], Section 2.1), or more specifically, the proposed equivalence between $\mathcal{N} = 4$ Super Yang-Mills Theory, a conformal field theory, and Type-IIB String Theory on $AdS_5 \times S^5$. This allows us to define operators in field theory which are dual to string theory states - the operators we define are the Schur polynomials as duals to ground state Giant gravitons ([46]), and restricted Schur polynomials for the excited states (Section 2.3), that is, Giant graviton states having stringy excitations attached to their surface. Many of the properties of Giant gravitons (Chapter 3) can be conveniently replicated using the Schur Polynomial operators, and they admit numerous identities relevant to our calculation. The Schur polynomial operators carry labels in the form of Young diagrams, associated with representations of the symmetric group, the shape of which indicates the configuration of the Giant graviton system. In previous attempts at this calculation (relevant articles listed at the beginning of this Chapter), it was found that the primary difficulty preventing a generalized solution was in the complexity of the calculations necessary to construct the projectors appearing in the restricted Schur polynomials - a remedy born of the Schur-Weyl duality (Section 2.2) was realised in [7].

In order to construct the operators $P_{R\to(r,s)jk}$ appearing in the Schur polynomials, which project from the carrier space of the $S_{n+m}$ group associated to $R$ to the $S_n \times S_m$ subspace labelled by the Young diagrams $(r, s)$ and being subduced with multiplicity labelled by $jk$, Schur-Weyl duality is applied in an unconventional manner. As projectors, these operators can be written as the outer product of two states in the space to which they are projecting. By exploiting the duality between symmetric and unitary group representations, it is possible to trade the symmetric group state labels $R, (r, s)jk$ of the restricted Schur polynomial for equivalent $U(p)$ state labels, where $p$ is the number of rows in $R$ (the number of giant gravitons in the system). Employing this labelling allows one to use the Clebsch-Gordon coefficients of unitary group theory to decompose the $S_n \times (S_1)^m$ states, obtained by removing $m$ boxes (corresponding to the impurities) from $R$ in different orders, into linear combinations of $U(p)$ states. It is then possible to obtain the $U(p)$ states as linear combinations of the $S_n \times (S_1)^m$ states. Checking that the $U(p)$ states fill out representations of $S_m$ is simply executed by examining the action of the symmetric group elements in this basis - this confirms that it is correct to label the $S_n \times S_m$ states from which the projectors are constructed by picking a particular $U(p)$ state. Another major advantage of this method is that, because states in the same $U(p)$ representation having the same $\Delta$ weight correspond to the same symmetric group labels, the multiplicity indices $jk$ are easily organised by the inner multiplicity of the $U(p)$ representation. Schur-Weyl duality has thus provided an efficient, transparent method by which the projectors can be constructed in terms of excited Giant graviton states represented by partially labelled Young diagrams.

The goal of this dissertation is to present a method by which the energy spectrum of the Giant graviton excitations can be computed using the restricted Schur Polynomials as duals to these objects. Hence, we require an operator in the CFT in which the Schurs are defined which is dual to the Hamiltonian of the string theory - it has been discovered that this operator is $D$, the dilatation operator ([39], Section 5.2). An eigenvalue of this operator, which tells us the scaling dimension of the state on which it acts, is equal to the energy of a state in the dual string theory. There is sufficient evidence that the loop corrections to the eigenvalues of this operator when acting on our restricted Schurs are equivalent to the energies of the stringy excitations of the Giant gravitons [41],[40]. Obtaining the eigenvalues is a three step process: first, an equation for the action of the dilatation operator [39] on our Schur polynomial operators must be obtained in a useful form (Section 5.3). Elements of this action must then be explicitly calculated for the particular giant graviton system we consider (Section 6) - the terms in the equation factorize such that the action reduces to a set of matrices acting on operators formed of linear combinations of Giant graviton configurations. Finally, the dilatation operator must be diagonalized by introducing operators with good scaling dimension, decomposing them into linear combinations of the operators in terms of which the dilatation operator action was calculated, and then solving the resulting recursion relations to determine the spectrum of eigenvalues (Section 7).

In all examples presented in this dissertation, the result of this calcuation was that the set of eigenvalues was equivalent to the energy spectrum of a set of decoupled harmonic oscillators (Section 7). This has a

natural physical interpretation: the Giant gravitons are essentially 3-spheres, so that excitations on them are expected to be simply the vibrational modes of the $S^3$ - as the energy in any mode is quantized, a set of decoupled oscillators is exactly the description one would expect for these excitations. Another important and, frankly, incredible physical outcome of this calculation is seen when one considers the set of equations for the action of $D$ on its eigenstates, which arise after the diagonalization. It has been postulated (and shown for a number of examples) in [7] (Section 8.1 of this dissertation) that these equations reflect the restrictions imposed by Gauss' Law on the possible configurations of the Giant graviton system. This can be easily visualized using simple diagrams, and in turn the diagrams can be used to extract the equations, in a manner reminiscent of the diagrams introduced by Feynman for correlation functions in QFT. This is not only an extremely useful, physically relevant nuance of the calculation, but also lends credence to the interpretation of the excitatations as reflecting the presence of energetic strings attached to the giant gravitons.

Although we have written most of our formulas for Young diagrams with $p$ long rows, there is a straight forward relation to the case with $p$ long columns - see section 5.3.5. Further, although we have focused on the $SU(2)$ sector of the theory, it is not difficult to add another impurity flavor. Indeed, a remarkable and surprising result of [47] which studied the $p = 2$ case, is the fact that projectors from $S_{n+m+p}$ to $S_n \times S_m \times S_p$ can be constructed by taking a direct product of two $SU(2)$ projectors. We have checked that this extends to the general case of projectors from $S_{n_1+n_2+\ldots+n_k}$ to $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$, and for general $p$. This is presumably closely related to the math result [48]. The Gauss Law constraint is an exact statement about the worldvolume physics of giant gravitons. For this reason we are optimistic that the connection we have found between the Gauss Law constraint and the action of the one loop dilatation operator persists to higher loops. Clearly despite the enormous number of diagrams that need to be summed to construct this large $N$ but non-planar limit, we are finding evidence that a simple integrable system emerges in the end.

I believe that the work presented in this dissertation is an important step in a promising field of research. For the first time, a method by which the energy eigensystem, and hence the dynamics, of a system of quantum membranes has been developed and tested. In addition, this method exploits and hopes to provide further insight into the Gauge-Gravity duality which has garnered so much attention in recent years. The systems studied here (Giant gravitons in $AdS_5 \times S^5$) were chosen out of necessity, since the space in which their associated string theory is defined is one of the very few for which a gauge theory dual is known, and because it is believed that the giants are the only membranes which arise on this space. The calculation performed is thus a "toy model", being probably the simplest of all systems that could be studied using these methods; however, the convenience and intuitiveness afforded by the use of group theory as a means to label and organise configurations hints that this approach (or another like it) may eventually prove its worth in a more realistic setting. There is an enormous amount to be done before this is realised - as a species, we have only a very limited understanding of the gauge-gravity duality which is so central to our cause, not to mention our entirely incomplete description of gravity itself. The authors of [7] and the articles preceding it believe that we have presented a theory in its infancy, which we hope one day will mature and unlock some of the deeper secrets of the universe.

# Appendix A

# Elementary Facts from $U(p)$ Representation Theory

In this appendix we collect the background $U(p)$ representation theory needed to understand our construction and diagonalization of the dilatation operator. There are many excellent references for this material. We have found [49, 50] useful. See also [51] for an extremely useful Clebsch-Gordan calculator.

## A.1 The Lie Algebra $u(p)$

It is simpler to study the Lie algebra $u(p)$ instead of the group $U(p)$ itself. Most results obtained for representations of $u(p)$ carry over to $U(p)$. In particular, the Clebsch-Gordan coefficients (which play a central role in our construction) of their representations are identical. The structure of the $u(p)$ algebra is easily illustrated using a specific basis. Let $E_{ij}$ with $1 \leq i, j \leq p$ be the matrix

$$(E_{ij})_{rs} = \delta_{ir}\delta_{js}\,,$$

so that it has only one non-zero matrix element. A convenient basis for the Lie algebra is generated by the matrices

$$iE_{kk}, \qquad 1 \leq k \leq p,$$
$$i(E_{k,k-1} + E_{k-1,k}), \qquad E_{k,k-1} - E_{k-1,k}, \qquad 1 < k \leq p\,.$$

$u(p)$ is spanned by real linear combinations of these matrices. The restriction of any irreducible representation of $GL(p,C)$ onto the subgroup $U(p)$ is also irreducible. Thus the carrier space of the irreducible representations of $U(p)$ share the same basis as the irreducible representations of $GL(p,C)$ and consequently, a labeling for $gl(p,C)$ irreducible representations is also a labeling for $u(p)$ irreducible representations. For this reason, it is correct and convenient to rather work with a basis for $gl(p,C)$ given by the complex linear combinations

$$J_z^{(l)} = E_{ll}, \qquad 1 \leq l \leq p$$
$$J_+^{(l)} = E_{l,l+1}, \qquad J_-^{(l)} = E_{l+1,l}, \qquad 1 \leq l \leq p-1$$

which satisfy, for each $l$, the Lie algebra

$$\left[J_z^{(l)}, J_\pm^{(l)}\right] = \pm J_\pm^{(l)},$$
$$\left[J_+^{(l)}, J_-^{(l)}\right] = 2J_z^{(l)}.$$

The $p$ matrices $J_z^{(l)}$ form a maximal set of mutually commuting matrices, i.e. they span the Cartan subalgebra of $u(p)$. We have used a notation which stresses the connection to the familiar $su(2)$ Lie algebra. To generate an anti-Hermitian basis from $J_\pm^{(l)}$ simply use

$$E_{k,q} = [J_-^{(k-1)}, [J_-^{(k-2)}, \ldots [J_-^{(q+1)}, J_-^{(q)}]] \ldots] \quad \text{for} \quad k > q\,,$$
$$E_{k,q} = [J_+^{(k)}, [J_+^{(k+1)}, \ldots [J_+^{(q-2)}, J_+^{(q-1)}]] \ldots] \quad \text{for} \quad k < q\,.$$

Thus, once we know the representations for all $J_\pm^{(l)}$ on a given carrier space, the representations of all other elements of the algebras $gl(p,C)$ and $u(p)$ are also known.

## A.2 Gelfand-Tsetlin Patterns

Gelfand and Tsetlin have introduced a powerful labeling for $u(p)$ irreducible representations and the basis states of their carrier spaces[52]. This labeling chooses basis states that are simultaneous eigenstates of all the matrices $J_z^{(l)}$, and further, explicit formulas are known for the matrix elements of the $J_\pm^{(l)}$ with respect to these basis states. An inequivalent irreducible representation for $GL(p,C)$ is uniquely given by specifying the sequence of $p$ integers

$$\mathbf{m} = (m_{1p}, m_{2p}, \ldots, m_{pp}), \tag{A.1}$$

satisfying $m_{kp} \geq m_{k+1,p}$ for $1 \leq k \leq p-1$. Throughout this dissertation we call this sequence the weight of the irreducible representation. The restriction of this irreducible representation onto the subgroup $GL(p-1,C)$ is reducible. It decomposes into a direct sum of $GL(p-1,C)$ irreducible representations with highest weights

$$\mathbf{m}' = (m_{1,p-1}, m_{2,p-1}, \ldots, m_{p-1,p-1}), \tag{A.2}$$

for which the "betweenness" conditions

$$m_{kp} \geq m_{k,p-1} \geq m_{k+1,p} \qquad \text{for} \qquad 1 \leq k \leq p-1$$

hold. The carrier spaces of the $GL(p,C)$ irreducible representations now give rise to (after restricting to the $GL(p-1,C)$ subgroup) $GL(p-1,C)$ irreducible representations. We can keep repeating this procedure until we get to $GL(1,C)$ which has one-dimensional carrier spaces. The Gelfand-Tsetlin labeling exploits this sequence of subgroups to label the basis states using what are called *Gelfand-Tsetlin patterns*. These are triangular arrangements of integers, denoted by $M$, with the structure

$$M = \begin{bmatrix} m_{1p} & & m_{2p} & \cdots & m_{p-1,p} & & m_{pp} \\ & m_{1,p-1} & & m_{2,p-1} & \cdots & m_{p-1,p-1} & \\ & & \cdots & \cdots & \cdots & & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{bmatrix}$$

The top row contains the weight that specifies the irreducible representation of the state and the entries of lower rows are subject to the betweenness condition. Thus, the lower rows give the sequence of irreducible representations our state belongs to as we pass through successive restrictions from $GL(p,C)$ to $GL(p-1,C)$ to ... to $GL(1,C)$. The dimension of an irreducible representation with weight $\mathbf{m}$ is equal to the number of valid Gelfand-Testlin patterns having $\mathbf{m}$ as their top row.

## A.3 $\Sigma$ and $\Delta$ Weights

We make extensive use of two weights in our construction: $\Sigma$-weights and $\Delta$ weights. Define the row sum

$$\sigma_l(M) = \sum_{k=1}^{l} m_{k,l}\,.$$

The sequence of row sums defines the sigma weight

$$\Sigma(M) = (\sigma_p(M), \sigma_{p-1}(M), \cdots, \sigma_1(M))\,.$$

The sigma weights do not provide a unique label for the states in the carrier space. Indeed, it is possible that $\Sigma(M) = \Sigma(M')$ but $M \neq M'$. The number of states $\vec{v}(M)$ in the carrier space that have the same $\Sigma$ weight $\Sigma = \Sigma(M)$ is called the inner multiplicity $I(\Sigma)$ of the state. The inner multiplicity plays an important role in determining how many restricted Schur polynomials can be defined. The $\Delta$ weights are defined in terms of differences between row sums

$$\begin{aligned}\Delta(M) &= (\sigma_p(M) - \sigma_{p-1}(M), \sigma_{p-1}(M) - \sigma_{p-2}(M), \cdots, \sigma_1(M) - \sigma_0(M)) \\ &\equiv (\delta_p(M), \delta_{p-1}(M), \cdots \delta_1(M))\end{aligned}$$

where $\sigma_0 \equiv 0$. We could also ask how many states in the carrier space have the same $\Delta$, denoted $I(\Delta)$. It is clear that $I(\Delta) = I(\Sigma)$.

The $\Delta$ weights play an important role in determining how the three Young diagram labels $R, (r,s)$ of the restricted Schur polynomials $\chi_{R,(r,s)jk}$ translate into a set of $U(p)$ labels. It tells us how boxes were removed from $R$ to obtain $r$. Further, the multiplicity labels $jk$ of the restricted Schur polynomial each run over the inner multiplicity.

## A.4 Relation between Gelfand-Tsetlin Patterns and Young Diagrams

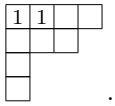
There is a one-to-one correspondence between $\Sigma$ weights and Young diagrams, and between Gelfand-Tsetlin patterns and semi-standard Young tableaux. The language of semi-standard Young tableau is a key ingredient in understanding how the three Young diagram labels $R, (r, s)$ of the restricted Schur polynomials $\chi_{R,(r,s)jk}$ translate into the $U(p)$ language, so we will review this connection here. Recall that a Young diagram is an arrangement of boxes in rows and columns in a single, contiguous cluster of boxes such that the left borders of all rows are aligned and each row is not longer than the one above. The empty Young diagram consisting of no boxes is a valid Young diagram. For a $u(p)$ irreducible representation there are at most $p$ rows. Every Young diagram uniquely labels a $u(p)$ irreducible representation.

A (semi-standard) Young tableau is a Young diagram, with labelled boxes. The rules for labeling are that each box contains a single integer between 1 and $p$ inclusive, the numbers in each row of boxes weakly increase from left to right (each number is equal to or larger than the one to its left) and the numbers in each column strictly increase from top to bottom (each number is strictly larger than the one above it). The basis states of a $u(p)$ representation identified by a given Young diagram $D$ can be uniquely labelled by the set of all semi-standard Young tableaux. The dimension of a carrier space labelled by a Young diagram is equal to the number of valid Young tableaux with the same shape as the Young diagram.

Each Gelfand-Tsetlin pattern $M$ corresponds to a unique Young tableau. We will now explain how to construct the Young tableau given a Gelfand-Tsetlin pattern. Each step in the procedure is illustrated with a concrete example given by the following Gelfand-Tsetlin pattern

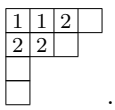
$$\begin{bmatrix} 4 & & 3 & & 1 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 3 & & 2 & & \\ & & & 2 & & & \end{bmatrix}.$$

Start with an empty Young diagram (no labels). The first line of the Gelfand-Tsetlin pattern tells you the shape of the Young diagram - $m_{in}$ is the number of boxes in row $i$. Thus, the information specifying the irreducible representation resides in the topmost row of the pattern. The last row of the Gelfand-Tsetlin pattern tells us which boxes are labelled with a 1. Imagine superposing the smaller Young diagram defined by the last row of the pattern onto the full Young diagram, so that the topmost and leftmost boxes of the two are identified. Label all boxes of this smaller Young diagram with a 1. For the example we consider



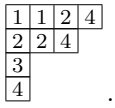

.

The second last row of the pattern tells us which boxes are labelled with a 2. Again superpose the smaller Young diagram defined by the second last row of the pattern onto the full Young diagram and again identify the topmost and leftmost boxes of the two. Label all empty boxes of this smaller Young diagram with a 2. For the example we consider




.

Keep repeating this procedure until you have used the first row to identify the boxes labelled $p$. The result is a semi-standard Young tableau. The semi standard Young tableau for the example we consider is




.

The number of boxes containing the number $l$ in tableau row $k$ is given by $m_{kl} - m_{k,l-1}$ and we set $m_{kl} \equiv 0$ if $k > l$. The converse process of transcribing a semi-standard Young tableau to a Gelfand-Tsetlin pattern is now obvious.

The components $\delta_l(M)$ of the $\Delta$ weight of a Gelfand-Tsetlin pattern $M$, is the number of boxes containing $l$ in the tableau corresponding to $M$. Thus, the tableau corresponding to two patterns with the same $\Delta$ weight contain the same set of entries (i.e. the same number of $l$-boxes) but arranged in different ways. One interpretation for the inner multiplicity is that it simply counts the number of ways to arrange the relevant fixed set of entries in the tableau.

## A.5 Clebsch-Gordon Coefficients

Let $R$ and $S$ be two irreducible unitary representations of the group $U(p)$. The tensor product of these representations decomposes into a direct sum of irreducible components

$$R \otimes S = \sum_T \oplus \nu(T) T \,. \tag{A.3}$$

In general a particular irreducible representation $T$ can appear more than once in the product $R \otimes S$. The integer $\nu(T)$ indicates the multiplicity of $T$ in this decomposition. For the applications we have in mind, we will need the direct product of an arbitrary representation with weight $\mathbf{m}_n$ with the defining representation which has weight $(1, \mathbf{0})$. In this case all multiplicities are equal to 1 and we need not worry about tracking multiplicities. Use the notation $\mathbf{m}_R$ to denote the weight of irreducible representation $R$ and $M_R$ to denote the Gelfand-Tsetlin pattern for a particular state in the carrier space of this irreducible representation. There are two natural bases for $R \otimes S$. The first is simply obtained by taking the direct product of the states spanning the carrier spaces of $R$ and $S$. The states in this basis are labelled, using a bra/ket notation, as[1]

$$|\mathbf{m}_R, M_R; \mathbf{m}_S, M_S\rangle \,.$$

The second natural basis is given as a direct sum over the bases of the carrier spaces for the irreducible representations $T$ appearing in the sum on the right hand side of (A.3). The states in this basis are labelled as[2]

$$|\mathbf{m}_T, M_T\rangle$$

where $T$ runs over all irreducible representations appearing in the sum on the right hand side of (A.3). The Clebsch-Gordan coefficients supply the transformation matrix which takes us between the two bases. They are written as the overlap

$$\langle \mathbf{m}_R, M_R; \mathbf{m}_S, M_S | \mathbf{m}_T, M_T\rangle \,.$$

From now on we will drop the $R, S, T$ labels which are actually redundant since the particular irreducible representations we consider are uniquely labelled by the weight which is recorded in the first row of the corresponding Gelfand-Tsetlin patterns. It is known that we can write the Clebsch-Gordan coefficients of $U(p)$ in terms of the Clebsch-Gordan coefficients of $U(p-1)$ as[3]

$$\langle \mathbf{m}_p, M; \mathbf{m}'_p, M' | \mathbf{m}''_p, M''\rangle = \left(\begin{array}{ccc} \mathbf{m}_p & \mathbf{m}'_p & \mathbf{m}''_p \\ \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \end{array}\right) \langle \mathbf{m}_{p-1}, M_1; \mathbf{m}'_{p-1}, M'_1 | \mathbf{m}''_{p-1}, M''_1\rangle \,.$$

On the right hand side we have the Clebsch-Gordan coefficients of the group $U(p-1)$ and on the left hand side we have the Clebsch-Gordan coefficients of the group $U(p)$. The weights $\mathbf{m}_p, \mathbf{m}'_p, \mathbf{m}''_p$ label irreducible representations of $U(p)$, while weights $\mathbf{m}_{p-1}, \mathbf{m}'_{p-1}, \mathbf{m}''_{p-1}$ label irreducible representations of $U(p-1)$. The Gelfand-Tsetlin patterns $M_1, M'_1$ and $M''_1$ are obtained from $M, M'$ and $M''$ respectively by removing the first row. Thus, the weights $\mathbf{m}_{p-1}, \mathbf{m}'_{p-1}, \mathbf{m}''_{p-1}$ correspond with the second rows in $M, M'$ and $M''$. The coefficients $\left(\begin{array}{ccc} \mathbf{m}_p & \mathbf{m}'_p & \mathbf{m}''_p \\ \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \end{array}\right)$ are called the scalar factors of the Clebsch-Gordan coefficients $\langle \mathbf{m}_p, M; \mathbf{m}'_p, M' | \mathbf{m}''_p, M''\rangle$. Applying the above factorization to the chain of subgroups referenced by the Gelfand-Tsetlin pattern, we obtain

$$\langle \mathbf{m}_p, M; \mathbf{m}'_p, M' | \mathbf{m}''_p, M''\rangle = \left(\begin{array}{ccc} \mathbf{m}_p & \mathbf{m}'_p & \mathbf{m}''_p \\ \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \end{array}\right) \left(\begin{array}{ccc} \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \\ \mathbf{m}_{p-2} & \mathbf{m}'_{p-2} & \mathbf{m}''_{p-2} \end{array}\right) \left(\begin{array}{ccc} \mathbf{m}_{p-2} & \mathbf{m}'_{p-2} & \mathbf{m}''_{p-2} \\ \mathbf{m}_{p-3} & \mathbf{m}'_{p-3} & \mathbf{m}''_{p-3} \end{array}\right) \cdots$$

[1] When discussing and using the Clebsch-Gordan coefficients, we prefer to use a bra/ket notation. In our previous notation we could write this basis vector as $\vec{v}(M_R) \otimes \vec{v}(M_S)$.

[2] In general one would also need to include a multiplicity label among the labels for these states.

[3] Again, we are using the fact that for our applications multiple copies of the same representation are absent. In general one needs to worry about multiplicities.

Thus, the Clebsch-Gordan coefficients can be written as a product of scalar factors. There is a selection rule for the Clebsch-Gordan coefficients. The Clebsch-Gordan coefficients vanish unless

$$\sum_{i=1}^{j} m_{ij} + \sum_{i=1}^{j} m'_{ij} = \sum_{i=1}^{j} m''_{ij} \qquad j = 1, 2, ..., p\,.$$

The only Clebsch-Gordan coefficient that we will need for our applications come from taking the product of some general representation $\mathbf{m}_p$ with the fundamental representation. The weight of the fundamental representation is $(1, 0, ..., 0)$ with $p-1$ 0s appearing. The product we consider has been studied and the following result is known

$$\mathbf{m}_p \otimes (1, \mathbf{0}) = \sum_{i=1}^{m} \mathbf{m}_p^{+i}\,. \tag{A.4}$$

where $\mathbf{m}_p^{+i}$ is obtained from $\mathbf{m}_p$ by replacing $m_{ip}$ by $m_{ip}+1$. Of course, if this replacement does not lead to a valid Gelfand-Tsetlin pattern there is no corresponding representation. The term with the illegal pattern should be dropped from the right hand side of (A.4). From (A.4) we see that multiple copies of the same irreducible representation are absent on the right hand side. We have made use of this repeatedly in this subsection. These Clebsch-Gordan coefficients factor into products of scalar factors of the form

$$\left(\begin{array}{ccc} \mathbf{m}_p & (1,\mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (1,\mathbf{0})_{p-1} & \mathbf{m}_{p-1}^{+j} \end{array}\right) \quad \text{or} \quad \left(\begin{array}{ccc} \mathbf{m}_p & (1,\mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (0,\mathbf{0})_{p-1} & \mathbf{m}_{p-1} \end{array}\right).$$

Explicit formulas for these scalar factors are known

$$\left(\begin{array}{ccc} \mathbf{m}_p & (1,\mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (1,\mathbf{0})_{p-1} & \mathbf{m}_{p-1}^{+j} \end{array}\right) = S(i,j)\left|\frac{\prod_{k\neq j}^{p-1}(l_{k,p-1}-l_{ip}-1)\prod_{k\neq i}^{p}(l_{kp}-l_{j,p-1})}{\prod_{k\neq i}^{p}(l_{kp}-l_{ip})\prod_{k\neq j}^{p-1}(l_{k,p-1}-l_{j,p-1}-1)}\right|^{\frac{1}{2}}$$

$$\left(\begin{array}{ccc} \mathbf{m}_p & (1,\mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (0,\mathbf{0})_{p-1} & \mathbf{m}_{p-1} \end{array}\right) = \left|\frac{\prod_{j=1}^{p-1}(l_{j,p-1}-l_{ip}-1)}{\prod_{j\neq i}^{p}(l_{jp}-l_{ip})}\right|^{\frac{1}{2}}$$

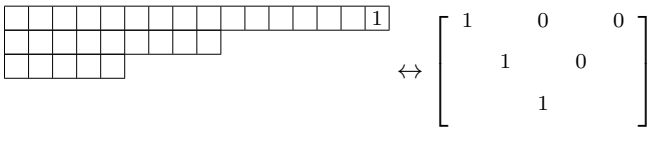
where $l_{sk} = m_{sk} - s$, $S(i,j) = 1$ if $i \leq j$ and $S(i,j) = -1$ if $i > j$.

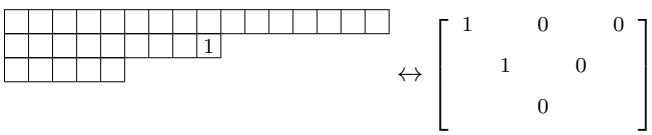
## A.6 Explicit Association of labelled Young Diagrams and Gelfand-Tsetlin Patterns

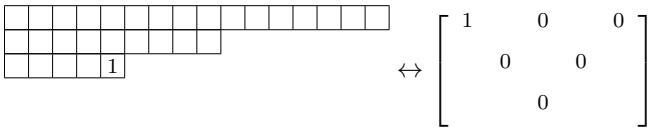
The association we spell out in this section is at the heart of our new Schur-Weyl duality and it demonstrates how we associate an action of $U(p)$ to a Young diagram with $p$ rows or columns. First consider the case of a Young diagram with $O(1)$ rows and $O(N)$ columns. This situation is relevant for the description of AdS giant gravitons. We consider Young diagrams in which a certain number of boxes are labelled. To keep the argument general assume that the Young diagram has $p$ rows. These labelled boxes are put into a one-to-one correspondence with $p$-dimensional vectors. If box $i$ appears in the $q^{\text{th}}$ row it is associated to vector with components

$$\vec{v}(i)_k = \delta_{kq}\,.$$

These states live in the carrier space of the fundamental representation of $U(p)$. In this subsection we would like to clearly spell out the Gelfand-Testlin pattern labeling of these vectors. We will spell out our conventions for $U(3)$. The generalization to any $p$ is trivial. Our conventions are

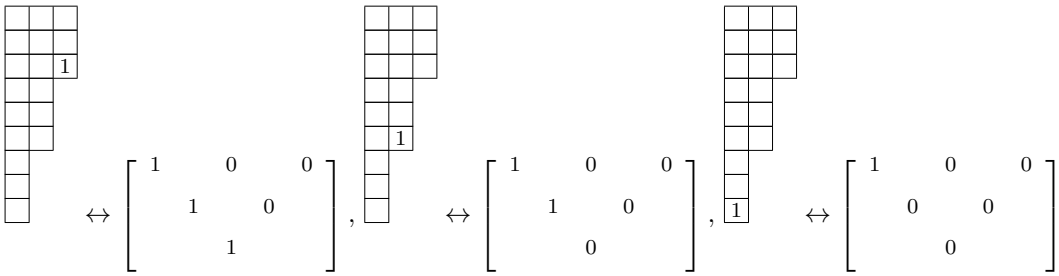




The particular label (the 1 in this case) is irrelevant - its the row the label appears in that determines the pattern.

For the case of Young diagrams with $O(N)$ rows and $O(1)$ columns we have



This situation is relevant for the description of sphere giant gravitons. Note that in addition to specifying the above correspondence between Gelfand-Tsetlin patterns and labelled Young diagrams, one also needs to assign the phases of the different states carefully. For a discussion see section 5.3.5.

## A.7 Last Remarks

A box in row $i$ and column $j$ has a factor equal to $N - i + j$. To obtain the hook length associated to a given box, draw a line starting from the given box towards the bottom of the page until you exit the Young diagram, and another line starting from the same box towards the right until you again exit the diagram. These two lines form an elbow - the hook. The hook length for the given box is obtained by counting the number of boxes the elbow belonging to the box passes through. Here is a Young diagram with the hook lengths filled in

| | | |
|---|---|---|
| 5 | 3 | 1 |
| 3 | 1 | |
| 1 | | |

For Young diagram $R$ we denote the product of the hook lengths by $\text{hooks}_R$.

# Appendix B

# Elementary Facts from $S_n$ Representation Theory

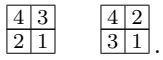
The complete set of irreducible representations of $S_n$ are uniquely labelled by Young diagrams with $n$ boxes. From this Young diagram we can construct both a basis for the carrier space of the representation as well as the matrices representing the group elements. We will review these constructions in this Appendix. A useful reference for this material is [53].

## B.1 Young-Yamonouchi Basis

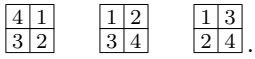
A particularly convenient basis for the carrier space of an irreducible representation of the symmetric group is provided by the Young-Yamonouchi basis. The elements of this basis are labelled by numbered Young diagrams - a Young tableau. For a Young diagram with $n$ boxes, each box in the tableau is labelled with a unique integer $i$ with $1 \leq i \leq n$. In our conventions this numbering is done in such a way that if all boxes with labels less than $k$ with $k < n$ are dropped, a valid Young diagram remains. As an example, if we consider the irreducible representation of $S_4$ corresponding to



then the allowed labels are



Examples of labels that are not allowed include



For any given Young diagram the number of valid labels is equal to the dimension of the irreducible representation and each label corresponds to a vector in the basis for the carrier space. This basis is orthonormal so that, for example

$$\left\langle \begin{array}{|c|c|}\hline 4&3\\\hline 2&1\\\hline\end{array} \middle| \begin{array}{|c|c|}\hline 4&3\\\hline 2&1\\\hline\end{array} \right\rangle = 1, \qquad \left\langle \begin{array}{|c|c|}\hline 4&3\\\hline 2&1\\\hline\end{array} \middle| \begin{array}{|c|c|}\hline 4&2\\\hline 3&1\\\hline\end{array} \right\rangle = 0\,.$$

## B.2 Young's Orthogonal Representation

A rule for constructing the matrices representing the elements of the symmetric group is easily given by specifying the action of the group elements on the Young-Yamonouchi basis. The rule is only stated for "adjacent permutations" which correspond to cycles of the form $(i, i+1)$. This is enough because these adjacent permutations generate the complete group. To state the rule it is helpful to associate to each box a factor[1]. The factor of a box in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column is given by $K - i + j$. Here $K$ is an

[1]This number is also commonly called the "weight" of the box. Here we will refer to it as the factor since we do not want to confuse it with the weight of the Gelfand-Tsetlin pattern.

arbitrary integer that will not appear in any final results. We will denote the factor of the box labelled $l$ by $c_l$. Let $\hat{T}$ denote a Young tableau corresponding to Young diagram $T$ and let $\hat{T}_{ij}$ denote exactly the same tableau, but with boxes $i$ and $j$ swapped. The rule for the action of the group elements on the basis vectors of the carrier space is

$$\Gamma_T\left((i,i+1)\right)\left|\hat{T}\right\rangle = \frac{1}{c_i - c_{i+1}}\left|\hat{T}\right\rangle + \sqrt{1-\frac{1}{(c_i-c_{i+1})^2}}\left|\hat{T}_{i,i+1}\right\rangle \ .$$

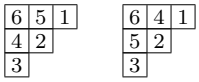
## B.3 Partially labelled Young diagrams

Consider a Young diagram containing $n+m$ boxes so that it labels an irreducible representations of $S_{n+m}$. We will often consider "partially labelled" Young diagrams, which are obtained by labeling $m$ boxes. The remaining $n$ boxes are not labelled. We only consider labelings which have the property that if all boxes with labels $\leq i$ are dropped, the remaining boxes are still arranged in a legal Young diagram. We refer to this as a "sensible labeling". What is the interpretation of these partially labelled Young diagrams? To make the discussion concrete, we will develop the discussion using an explicit example. For the example we consider take $n = m = 3$ and use the following partially labelled Young diagram




. (B.1)

If the labeling is completed, this partially labelled diagram will give rise to a number of Young tableau. For our present example two tableau are obtained




.

Each of these represents a vector in the carrier space of the $S_6$ irreducible representation labelled by the Young diagram  . Thus, a partially labelled Young diagram stands for a collection of states. Next, note that the subspace formed by this collection of states is invariant (you don't get transformed out of the subspace) under the action of the $S_3$ subgroup which acts on the boxes labelled 4,5 and 6. Thus, this subspace is a representation of $S_3$. In fact, it is easy to see that it is the irreducible representation labelled by . This Young diagram can be obtained by dropping all the labelled boxes in (B.1). From this example we can now extract the general rule:

**Key Idea:** *A partially labelled Young diagram that has $n+m$ boxes, $m$ of which are labelled, stands for a collection of states which furnish the basis for an irreducible representation of $S_n \times (S_1)^m$. The Young diagram that labels the representation of the $S_n$ subgroup is given by dropping all labelled boxes.*

Finally, note that the only representations $r$ that are subduced by $R$ are those with Young diagrams that can be obtained by pulling boxes off $R$. This follows immediately from the well known subduction rule for the symmetric group which states that an irreducible representation of $S_n$ labelled by Young diagram $R$ with $n$ boxes will subduce all possible representations $R'_i$ of $S_{n-1}$, where $R'_i$ is obtained by removing any box of $R$ that can be removed such the we are left with a valid Young diagram after removal. Each such irreducible representation of the subgroup is subduced once.

## B.4 Simplifying Young's Orthogonal Representation

In this section we would like to consider a collection of partially labelled Young diagrams. A total of $m$ boxes are labelled, with a unique integer $i$ ($1 \leq i \leq m$) appearing in each box. The set of boxes to be removed are the same for every partially labelled Young diagram. The set of partially labelled Young diagrams we consider is given by including all possible ways in which the $m$ boxes in the Young diagrams can sensibly be labelled. We can consider the action of the $S_m$ subgroup which acts on the labelled boxes. This action will mix these partially labelled Young diagrams.

We will consider Young diagrams with $p$ rows built out of $O(N)$ boxes. For the generic operator we consider, the difference in the length between any two rows will be $O(N)$. If we consider the case $m = \gamma N$

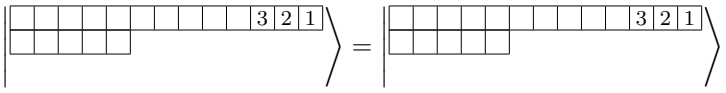
with $\gamma \sim O(N^0) \ll 1$, any two labelled boxes ($i$ and $j$ say) that are not in the same row will have factors that obey $|c_i - c_j| \sim O(N)$. Young's orthogonal representation is particularly useful because it simplifies dramatically in this situation. Indeed, if the boxes $i$ and $i+1$ are in the same row, $i+1$ must sit in the next box to the left of $i$ so that

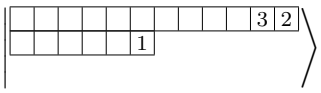
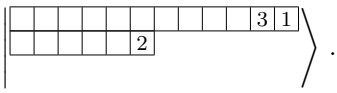
$$\Gamma_R\left((i,i+1)\right)|\text{same row state}\rangle = |\text{same row state}\rangle\,. \tag{B.2}$$

The same state appears on both sides of this last equation. If $i$ and $i+1$ are in different rows, then $c_i - c_{i+1}$ must itself be $O(N)$. In this case, at large $N$ replace $\frac{1}{c_i - c_{i+1}} = O(b_1^{-1})$ by 0 and $\sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1})$ by 1 so that

$$\Gamma_R\left((i,i+1)\right)|\text{different row state}\rangle = |\text{swapped different row state}\rangle\,. \tag{B.3}$$

The notation in this last equation is indicating two things: $i$ and $i+1$ are in different rows and the states on the two sides of the equation differ by swapping the $i$ and $i+1$ labels. An example illustrating these rules is

$\Gamma_R\left((1,2)\right)$  $=$ 

$\Gamma_R\left((1,2)\right)$  $=$  .

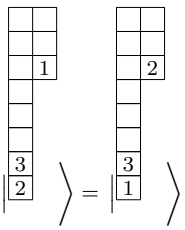
We will also consider Young diagrams with $p$ columns built out of $O(N)$ boxes. For the generic operator we consider, the difference in the length between any two columns will be $O(N)$. Since we consider the case $m = \gamma N$ with $\gamma \sim O(N^0) \ll 1$, any two labelled boxes ($i$ and $j$ say) that are not in the same column will again have factors that obey $|c_i - c_j| \sim O(N)$. If the boxes $i$ and $i+1$ are in the same column, $i+1$ must sit above $i$ so that

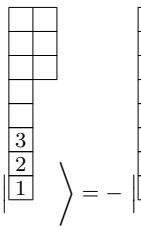
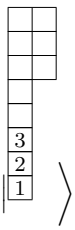
$$\Gamma_R\left((i,i+1)\right)|\text{same column state}\rangle = -|\text{same column state}\rangle\,. \tag{B.4}$$

The same state appears on both sides of this last equation. If $i$ and $i+1$ are in different columns, then $c_i - c_{i+1}$ must itself be $O(N)$. In this case, at large $N$ again replace $\frac{1}{c_i - c_{i+1}} = O(b_1^{-1})$ by 0 and $\sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1})$ by 1 so that

$$\Gamma_R\left((i,i+1)\right)|\text{different column state}\rangle = |\text{swapped different column state}\rangle\,. \tag{B.5}$$

An example illustrating these rules is:

$\Gamma_R\left((1,2)\right)$  $=$ 

$\Gamma_R\left((1,2)\right)$  $= -$ 

Thus, the representations of the symmetric group simplify dramatically in this limit.

# Appendix C

# Recursion Relations

The recursion relations needed in the diagonalization of the dilatation operator acting on restricted Schur polynomials labelled with two rows/columns are

$$-2x\,{}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix}\Big|\frac{1}{p}\right)=p(N-n)\,{}_2F_1\left(\begin{matrix}-n-1,-x\\-N\end{matrix}\Big|\frac{1}{p}\right)-[p(N-n)+n(1-p)]\,{}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix}\Big|\frac{1}{p}\right)$$

$$+n(1-p)\,{}_2F_1\left(\begin{matrix}-n+1,-x\\-N\end{matrix}\Big|\frac{1}{p}\right)$$

and

$$-p\,{}_3F_2\left(\begin{matrix}j^3-j,j+1+j^3,-p\\1,j^3-\frac{m}{2}\end{matrix}\Big|1\right)=\frac{(j+j^3+1)(j-j^3+1)(m-2j)}{4(j+1)(2j+1)}\,{}_3F_2\left(\begin{matrix}-1+j^3-j,j+2+j^3,-p\\1,j^3-\frac{m}{2}\end{matrix}\Big|1\right)$$

$$-\left(\frac{m}{4}-\frac{(m+2)(j^3)^2}{4j(j+1)}\right)\,{}_3F_2\left(\begin{matrix}j^3-j,j+1+j^3,-p\\1,j^3-\frac{m}{2}\end{matrix}\Big|1\right)+\frac{(j+j^3)(j-j^3)(m+2j+2)}{4j(2j+1)}\,{}_3F_2\left(\begin{matrix}1+j^3-j,j+j^3,-p\\1,j^3-\frac{m}{2}\end{matrix}\Big|1\right)$$

The first relation is equation (1.10.3) in [54] and is used to obtain the $f(b_0,b_1)$. The second relation is equivalent to equation (1.5.3) in [54] and is used to obtain the $C_{p,j^3}(j)$.

# Appendix D

# The Continuum Limit for $p$-row Operators

In this Appendix, we obtain the continuum limit equation for the dilatation operator when acting on $p$-row Schur polynomials. We will begin by studying the action of $\Delta_{ij}$ on a Young diagram with $p$ rows. The row closest to the top of the page is row 1 and the row closest to the bottom of the page is row $p$. The number of boxes in row $i$ minus the number of boxes in row $i+1$ is given by $b_{p-i}$. $\Delta_{ij}$ exchanges boxes between rows $i$ and $j$; we always have $i \neq j$. If $|i-j| > 1$ we have

$$\Delta_{ij} O(b_0, ..., b_{p-1}) = -(2N + \sum_{k=0}^{p-j} b_k + \sum_{q=0}^{p-i} b_q) O(b_0, ..., b_{p-1})$$

$$+\sqrt{(N + \sum_{k=0}^{p-j} b_k)(N + \sum_{q=0}^{p-i} b_q)}\,[O(b_0, ..., b_{p-j} - 1, b_{p-j+1} + 1, ..., b_{p-i} + 1, b_{p-i+1} - 1, ..., b_{p-1})$$

$$+O(b_0, ..., b_{p-j} + 1, b_{p-j+1} - 1, ..., b_{p-i} - 1, b_{p-i+1} + 1, ..., b_{p-1})]$$

It proves convenient to introduce the variables

$$l_i = \sum_{k=1}^{p-i} b_k \qquad i = 1, 2, ..., p-1.$$

Making the ansatz

$$O = \sum_{b_0, l_i, ..., l_{p-1}} f(b_0, l_1, ..., l_{p-1}) O(b_0, l_1, ..., l_{p-1})$$

for operators of a good scaling dimension, we find

$$\Delta_{ij} O = \sum_{b_0, l_i, ..., l_{p-1}} f(b_0, l_1, ..., l_{p-1}) \Delta_{ij} O(b_0, l_1, ..., l_{p-1}) = \sum_{b_0, l_i, ..., l_{p-1}} \tilde{\Delta}_{ij} f(b_0, l_1, ..., l_{p-1}) O(b_0, l_1, ..., l_{p-1})$$

where[1]

$$\tilde{\Delta}_{ij} f(b_0, l_1, ..., l_{p-1}) = -(2N + 2b_0 + l_i + l_j) f(b_0, l_1, ..., l_{p-1})$$

$$-\sqrt{(N + b_0 + l_i)(N + b_0 + l_j)}\,[f(b_0, ..., l_i - 1, ..., l_j + 1, ..., l_{p-1}) + f(b_0, ..., l_i + 1, ..., l_j - 1, ..., l_{p-1})]\ .$$

The operator $\tilde{\Delta}_{ij}$ has the same action as $\Delta_{ij}$, but it acts on $f$ rather than $O$. The continuum limit we consider takes $N + b_0 \to \infty$ holding the variables[2]

$$x_i = \frac{l_i}{\sqrt{N + b_0}}$$

[1] As the reader can easily check, this formula is also true when $|i-j| = 1$ i.e. its completely general.

[2] Note that this differs from the variables introduced when studying the continuum limit for $p = 3$ in Section 7.2.1; comparing the variables used, we find $x_1 = x + y$ and $x_2 = x$. It is simple to check that plugging in this change of variables for $x$ and $y$ gives the same resulting equation for $\tilde{\Delta}_{ij}$

fixed. Using the expansions

$$\sqrt{(N+b_0+l_i)(N+b_0+l_j)} = N+b_0+\frac{x_i+x_j}{2}\sqrt{N+b_0}-\frac{(x_i-l_x)^2}{8}+...$$

and

$$f(b_0,...,l_i-1,...,l_j+1,...) \to f(b_0,...,x_i-\frac{1}{\sqrt{N+b_0}},...,x_j-\frac{1}{\sqrt{N+b_0}},...)$$
$$= f(b_0,...,l_i,...,l_j,...)-\frac{1}{\sqrt{N+b_0}}\frac{\partial f}{\partial x_i}+\frac{1}{\sqrt{N+b_0}}\frac{\partial f}{\partial x_j}+\frac{1}{2(N+b_0)}\frac{\partial^2 f}{\partial x_i^2}$$
$$+\frac{1}{2(N+b_0)}\frac{\partial^2 f}{\partial x_j^2}-\frac{1}{N+b_0}\frac{\partial^2 f}{\partial x_i \partial x_j}+...$$

we find that in the continuum limit we have

$$\tilde{\Delta}_{ij} f = \left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)^2 f - \frac{(x_i-x_j)^2}{4} f = m_{ab}\left(\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b}-\frac{x_a x_b}{4}\right) f\,,$$

where

$$m_{ab} = \delta_{ai}\delta_{bi}+\delta_{aj}\delta_{bj}-\delta_{ai}\delta_{bj}-\delta_{aj}\delta_{bi}\,.$$

In general, the action of the dilatation operator is given by summing a collection of operators $\Delta_{ij}$, each appearing some integer $n_{ij}$ number of times

$$DO(b_1,b_2) = -g_{YM}^2 \sum_{ij} n_{ij}\Delta_{ij}\, O(b_1,b_2)\,.$$

The result that we obtained above implies that in the continuum limit we have

$$\sum_{ij} n_{ij}\Delta_{ij} \to M_{ab}\left(\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b}-\frac{x_a x_b}{4}\right)\,,$$

where the explicit formula for $M_{ab}$ depends on the $n_{ij}$. We can construct an orthogonal matrix $V$ that diagonalizes $M$

$$V_{ik} M_{ij} V_{jl} = D_k \delta_{kl}$$

$V$ has as its columns a set of orthogonal eigenvectors of $M$ - we label these eigenvectors $v_k$. We then define the new variable $y_k = Norm(v_k)_i x_i$. Written in terms of the new $y$ variables we have

$$\sum_{ij} n_{ij}\Delta_{ij} \to \sum_a D_a\left(\frac{\partial^2}{\partial y_a^2}-\frac{y_a^2}{4}\right)\,,$$

which is (minus) the Hamiltonian of a set of decoupled oscillators. The $D_a$'s, which are the eigenvalues of $M$, set the frequencies of the oscillators. For

$$\sum_{ij} n_{ij}\Delta_{ij} = 2\Delta_{12}\,,$$

we have

$$M = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

The matrix whose columns are the eigenvectors of $M$ diagonalizes $M$, so that we can use this matrix as $V$. However, in order to obtain the correct factors attached to the terms in the differential equation, we must choose the correct multiple of $V$. In this case $V$ must be multiplied by a constant factor of $\frac{1}{\sqrt{2}}$ to obtain the desired form of the equation for $\Delta_{12}$:

$$y_1 = \frac{1}{\sqrt{2}}(-x_1+x_2) \quad , \quad y_2 = \frac{1}{\sqrt{2}}(x_1+x_2), \qquad D_1 = 4, \quad D_2 = 0\,.$$

For

$$\sum_{ij} n_{ij}\Delta_{ij} = \Delta_{12}+\Delta_{23}+\Delta_{13}\,,$$

we have

$$M = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

To diagonalize, we must construct the diagonalizing matrix $V$ with its columns being orthogonal eigenvectors of $M$.. The relation between the $y$'s and $x$'s is then determined by normalization of these eigenvectors, so that we have:

$$y_1 = \frac{1}{\sqrt{2}}(-x_1 + x_2), \quad , y_2 = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3), \quad , y_3 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \qquad D_1 = 3 = D_2, \quad D_3 = 0\,.$$

These are consistent with the results given in Section 7.2.1. One might wonder if the $D_i$ are always integers. This is not the case. Indeed, for

$$\sum_{ij} n_{ij}\Delta_{ij} = \Delta_{12} + \Delta_{23} + \Delta_{34} + ... + \Delta_{1d}\,,$$

we have

$$M = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

In this case it is rather simple to see that the eigenvalues are

$$D_n = 2 - 2\cos\left(\frac{n\pi}{d}\right), \qquad n = 0, 1, ..., d\,.$$

These are not, in general, integer.

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