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Exploring heat transfer in extended surfaces through the λ - symmetry technique

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in requirement for the degree Master of Science, January, 2026.

Declaration

I declare that the contents of this thesis are original, except where due references have been made. It is submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

B.E. Ramanyane

Signed at Johannesburg on the 30^{th} day of January 2026.

Abstract

The nonlinear fin equation plays an important role in describing heat transfer through extended surfaces, where both the thermal conductivity and heat generation depend on temperature. This study applies Lie and λ - symmetry analysis to this equation to uncover its underlying properties and obtain analytical solutions for three fin geometries: rectangular, radial, and spherical. Using the invariance criterion, the corresponding determining equations were derived and solved to identify the infinitesimal generators and their admitted symmetries.

The analysis reveals that the nonlinear fin model possesses a symmetry structure that varies with geometry and model parameters. Moreover, the λ - symmetry method successfully extends the analysis, enabling further reductions and exact solutions. To complement the symmetry-based results, the Cheb-Terrab and Roche method was applied to derive first integrals through integrating factors and potential functions, offering a direct and systematic way of identifying conserved quantities.

Together, these approaches provide a systematic framework for understanding the analytical and structural properties of nonlinear fin models. The comparison across geometries highlights how symmetry techniques not only simplify complex differential equations but also reveal meaningful physical relationships that govern heat transfer in extended surfaces.

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Dedication

To the curiosity that first drew me to this field, and the perseverance that kept me here. To the part of me that never stopped wondering, learning, and finding meaning in mathematics,this work is a quiet reflection of that journey.

Contents

List of Publications **6**

List of Abbreviations **7**

List of Figures **8**

List of Tables **9**

1 Introduction **10**

1.1 Outline of Chapters 15

2 Preliminaries **17**

2.1 Lie Symmetries 18

2.2 Reduction of Order 20

2.3 λ - Symmetries 22

2.4 First Integrals 24

2.5 Discussion 27

3 Symmetry Analysis of the Fin Equation **28**

3.1 Application of Lie Symmetry Methods on the Fin Equation 28

3.1.1 Case 1: The rectangular fin $\alpha = 0$ 30

3.1.2 Case 2: The radial fin $\alpha = 1$ 33

3.1.3 Case 3: The spherical fin $\alpha = 2$ 34

3.2 Application of λ - Symmetry Methods on the Fin Equation 37

3.2.1 Case 1: The rectangular fin $\alpha = 0$ 39

3.2.2 Case 2: The radial fin $\alpha = 1$ 42

3.2.3 Case 3: The spherical fin $\alpha = 2$ 44

3.3 Discussion 45

4 Solving the Fin Equation **47**

4.1 Invariant Solutions via λ - Symmetries 47

4.1.1 Case 1: The rectangular fin $\alpha = 0$ 48

4.1.2 Case 2: The radial fin $\alpha = 1$ 54

4.1.3 Case 3: The spherical fin $\alpha = 2$ 54

4.2 Invariant Solutions via Lie Symmetries 56

4.2.1 Case 1: The rectangular fin $\alpha = 0$ 56

4.2.2 Case 2: The radial fin $\alpha = 1$ 58

4.2.3 Case 3: The spherical fin $\alpha = 2$ 59

4.3 Discussion 60

5 First Integrals of the Fin Equation **65**

5.1 A Detailed Calculation 65

5.2 Tabular Results for the Three Geometries 69

5.3 Discussion 72

6 Conclusion **74**

6.1 Lie Symmetry Analysis 75

6.2 λ - Symmetry Analysis 76

6.3 First Integrals 76

6.4 Future Work 77

List of Publications

Details of contributions to publications that form part of the research presented in this dissertation are:

- Ramanyane, B. & Jamal, S. An exploration of heat transfer in extended surfaces through Lie and λ - symmetries, submitted, 2025.

List of Abbreviations

ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations

List of Figures

4.1 Plot of solution (4.13) with varying parameters m and M. 50

4.2 Plot of the solution (4.14), illustrating the effect of varying the thermo-geometric parameter M. 52

4.3 Plot of the solution (4.15), illustrating the effect of varying the thermo-geometric parameter M. 53

4.4 Plot of the solution (4.16), illustrating the effect of varying M. 55

4.5 Plot of the solution (4.17), illustrating the effect of varying M. 61

4.6 Plot of the solution (4.19), illustrating the effect of varying M and m. 62

4.7 Plot of the solution (4.21), illustrating the effect of varying M and m. 63

4.8 Plot of the solution (4.23), illustrating the effect of varying M and m. 64

List of Tables

5.1 Integrating Factors and First Integrals for the Rectangular Fin ($\alpha = 0$) 70

5.2 Integrating Factors and First Integrals for the Radial Fin ($\alpha = 1$) . . 71

5.3 Integrating Factors and First Integrals for the Spherical Fin ($\alpha = 2$) . 72

Chapter 1

Introduction

A differential equation is a mathematical equation that involves an unknown function and its derivatives, which describes how a particular quantity changes with respect to time or space [23]. These equations form the foundation of most models used to explain real-world phenomena in different disciplines, such as economics, biological, physical, and social sciences. When the function depends on a single independent variable, the equation is classified as an ODE, which is central to the analysis of dynamical systems.

Over the years, various techniques and methods have been introduced to obtain exact or approximate solutions to differential equations. Classical methods like separation of variables, integrating factors, and power series have been commonly used to determine exact solutions. [11]. When these are not possible, numerical techniques such as Euler's and Runge-Kutta's methods have been studied to provide practical approximations [23]. In more complex situations, particularly for nonlinear or higher-order differential equations, geometric and qualitative approaches can

be useful in studying the equation without explicitly obtaining the solutions. However, as nonlinear models became increasingly common in science and engineering, researchers recognized the need for more structural approaches that reveal the dynamics of these equations. This recognition led to the development of symmetry analysis [3].

In the late 1860s, the Norwegian mathematician Sophus Lie introduced the theory of continuous transformation group [15], that provides the mathematical foundation for what is known as Lie symmetry analysis. The main idea of this method is to identify transformations that leave a differential equation invariant [4]. These symmetries can then be used to reduce the order of an equation, transform it into a simpler equivalent form or generate exact analytical solutions [3]. Beyond finding solutions, Lie symmetry also plays an important role in identifying and clarifying the underlying structure of an equation, offering a unified framework that connects symmetry, conservation laws, and integrability, see for example, [10, 11, 12].

However, not all differential equations admit nontrivial Lie symmetries. Many nonlinear differential equations possess only trivial or no Lie symmetries at all, yet can still be integrated or have their order reduced [20]. To address these limitations, several extensions of Lie's approach have been introduced to accommodate such equations [19]. Some of the extensions that were made include the concept of hidden symmetries and C^∞ - symmetries also known as λ - symmetries, which was introduced by Muriel and Romeo in 2001, in order to extend the applicability of symmetries to wider class of differential equations [19]. Unlike classical Lie symmetries, which only rely on the invariance of the prolongation of a vector field, X, λ - symmetries allow for a modified vector field prolongation that introduces a function $\lambda = \lambda(x, y)$. This extension allows equations without classical symmetries to be reduced and integrated [19].

A major field where symmetry methods have shown particular promise is heat transfer, especially in the study of fins, extended surfaces designed to enhance heat exchange between a solid and its surrounding fluid [5]. Fins are commonly used in engineering applications such as car radiators, heat exchangers, and electronic cooling systems [13]. The governing equations for heat transfer in fins often become nonlinear when the thermal conductivity or heat transfer coefficient depends on temperature. Because of this nonlinearity, exact solutions are difficult to obtain, and early studies mainly relied on approximate methods such as perturbation techniques and composition methods [2]. Although, these approaches provided useful approximate results, they did not always give a full understanding of the mathematical structure of the problem. In contrast, symmetry-based approaches offer a systematic way of obtaining exact reductions and similarity solutions that preserve the physical properties of the system [20].

Bokhari et al.(2006) made one of the early contributions to symmetry-based fin modeling by performing a group theoretic analysis of nonlinear fin equations with temperature dependent, thermal conductivity, and variable heat transfer coefficients [5]. They applied Lie point symmetry analysis to derive exact and similarity solutions of the fin equation, obtaining both steady-state and traveling-wave temperature profiles. Their results demonstrated how symmetry transformations could reduce nonlinear PDEs to simpler ODEs, thereby producing analytical solutions that describe the physical behaviour of fins under various conditions. The study revealed that for specific conductivity functions and specific heat transfer functions, the equations admit rich symmetry structures that can be exploited to generate new exact solutions.

A decade later, Abdel Kader et al.(2016), used Lie point symmetries methods to derive the general exact solution for fins with variable thermal conductivity [1]. Their

work produced explicit relationships among the fin efficiency, thermal conductivity, and a key thermo-geometric parameter M. They further established that increasing M lowers both the top temperature and the efficiency, a physical consistent trend [1]. Their work established a strong theoretical foundation for applying symmetry methods to nonlinear heat transfer problems.

Further to these developments, R.J. Moitsheki and collaborators have made extensive contributions to the modeling of nonlinear heat transfer in fins and related systems [17]. They studied both steady-state and transient heat conduction in fins with rectangular, exponential, and porous profiles [17]. In early studies, they used Lie symmetry analysis to derive similarity reductions and exact solutions for fins with variable thermal conductivity. Later, they extended this research to include the effects of porosity, radiation, and moving fins, using both analytical and semi analytical approaches such as the differential transform method [18]. These works demonstrated how geometric and material parameters such as porosity, and radiation affect heat distribution and fin efficiency.

In this research, we consider a generalized nonlinear fin equation given by

$$y^m y'' + m y^{m-1} y'^2 - M^2 y^{n+1} + \frac{\alpha\, y^m y'}{x} = 0, \tag{1.1}$$

where $y(x)$ is the dimensionless temperature, x is a dimensionless space variable, m and n are power law parameters associated with temperature dependent thermal conductivity and heat-transfer, respectively, and α represents the geometry of the fin: rectangular ($\alpha = 0$), radial ($\alpha = 1$), or spherical ($\alpha = 2$). As mentioned, $M \neq 0$ is a thermo-geometric fin parameter.

There are many versions of fin equations, each tailored to describe modified heat transfer problems. The best source of heat transfer equations, such as the model above, and their physical discussion can be found in [7, ?].

Although a lot of progress has been made in this field, there are still some open questions. One area that has not been fully explored is how classical Lie symmetries compare with λ - symmetries in fin equations with different geometries. Another gap is in understanding how the geometric parameters α, which distinguishes between rectangular, radial, and spherical fins, interacts with the parameters m and n that are connected to the temperature dependence of thermal conductibility and heat transfer. Exploring these relationships may lead to new analytical solutions and help identify cases where the nonlinear fin equations becomes linearizable.

We seek to apply symmetries to compare fins with rectangular, radial, and spherical profiles. These three categories give rise to interesting subclasses involving the power laws m and n. That is, we shall investigate the relationship between the fin geometry and the exponents linked to the heat transfer and thermal conductivity, from a symmetry perspective. As a byproduct, we find cases whereby the model may be linearizable. In addition, we shall follow this with an exploration of the first integrals admitted by the three categories of fins.

Through this comparative symmetry analysis, this study aims to extend previous research in the literature by combining classical and extended symmetry approaches. The findings are expected to enhance the understanding of nonlinear fin equations and provide new analytical insights into how geometric properties affect heat transfer in fins.

1.1 Outline of Chapters

This dissertation is structured to provide a systematic exploration of the symmetry analysis and first integrals of the nonlinear fin equation (1.1). Each chapter builds on the previous one, gradually connecting the theoretical concepts and computational analysis of the fin equation.

Chapter 2 provides the foundational background on Lie symmetry analysis and its application to differential equations. The chapter introduces the main concepts of infinitesimal generators, prolongations, and the invariance criterion, all of which form the basis for identifying symmetries of a given equation. Furthermore, the chapter presents an overview of λ - symmetries, discussing their origin, importance, and how they extend the classical Lie framework.

Chapter 3 applies these theoretical tools to the fin equation. The chapter begins by determining the Lie symmetries of the fin equation for different geometrical profiles, namely rectangular ($\alpha = 0$), radial ($\alpha = 1$), and spherical ($\alpha = 2$) fins. It then extends the analysis to the λ - symmetry framework, examining how the introduction of a λ - function influences the admitted symmetries. This comparative approach provides a deeper understanding of the symmetry structure of the fin equation under various mathematical conditions.

Chapter 4 focuses on the reduction and solution of the fin equation using Lie and λ - symmetries obtained in the previous chapter. The correspondence invariant functions $I(x, y)$ and solutions $y(x)$ to (1.1) are derived for each case. The chapter also explores graphical plots to visualize the solution trajectories and temperature profiles of the fins, highlighting how geometry affects the thermal distribution.

Chapter 5 shifts attention to the first integrals of the fin equation, derived using the

Cheb-Terrab and Roche method. By introducing an integrating factor and potential function, the method allows the identification of conserved quantities associated with the system. The results for different temperature profiles are presented, showing the integrating factors and first integrals for each geometry.

Finally, Chapter 6 concludes the study with a summary of the main findings, emphasizing the contribution of symmetries and first integral in analyzing nonlinear differential equations. The chapter also discusses possible directions for future research.

Chapter 2

Preliminaries

This chapter presents the mathematical preliminaries that form the foundation for the analytical methods used in this study. In particular, it focuses on the concepts of Lie symmetry analysis and its extension, known as λ - symmetry analysis. The latter provides systematic frameworks for studying nonlinear ODEs enabling equation reduction, the discovery of invariant solutions, and a deeper understanding of the equations.

The methods discussed here are important tools for analyzing the nonlinear fin model presented in the following chapters. Lie symmetry analysis serves as the classical foundation for identifying invariance properties and simplifying differential equations, while λ - symmetries extends this theory. Together, these methods provide a comprehensive analytical framework for investigating nonlinear heat transfer models.

2.1 Lie Symmetries

The study of differential equations forms a central part of mathematics because these equations describe a wide range of natural and physical processes in fields such as physics, engineering, biology, and finance. However, solving nonlinear or higher-order differential equations is often very difficult using traditional analytical methods [11]. Sophus Lie introduced a systematic framework based on the theory of continuous transformation groups, now known as Lie groups - which is a powerful way of studying differential equations through their symmetry properties.

Lie was inspired by the work of Évariste Galois, who had used group theory to study polynomial equations. He realized that similar ideas could be extended to differential equations [8]. The basic concept is that if a differential equation remains unchanged (invariant) under a continuous group of transformations called symmetries, then they can be used to simplify the equation, reduce its order, or generate new solutions from known ones [3]. In this way, Lie symmetry analysis links algebraic structures (groups) with geometric transformations to study and solve differential equations more effectively [8].

Consider a general n^{th} order ODE of the form [3]:

$$\Delta(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{2.1}$$

where x is the independent variable, $y = y(x)$ is the dependent variable, and the y^i for $i \in 1, 2, 3, ...n$ are its derivatives with respect to x. If we consider a space $(x, y) \in \mathbb{R}^2$, a function $y(x)$ that satisfies this equation is called a solution. For nonlinear or higher-order equations, classical methods such as direct integration or separation of variables often fail to produce exact solutions. Lie's approach is based on the idea of invariance under infinitesimal transformations [3, 8].

These infinitesimal transformations are expressed as small perturbations of the independent and dependent variables[3]:

$$\tilde{x}^i = x^i + \epsilon\rho^i(x, y) + O(\epsilon^2), \tag{2.2}$$

$$\tilde{y}^i = y^i + \epsilon\phi^i(x, y) + O(\epsilon^2), \tag{2.3}$$

where ϵ is a small parameter, $\rho(x, y)$ and $\phi(x, y)$ are infinitesimal of the transformation. If this transformation maps any solution $y(x)$ of the differential equation into another solution $\tilde{y}(\tilde{x})$, then the equation is said to be invariant under the transformation. Finding these infinitesimals allows us to determine the symmetry group associated with the equation [8].

To study these transformations systematically, Lie introduced the concept of a vector field, often called the infinitesimal generator, defined by

$$X = \rho(x, y)\frac{\partial}{\partial x} + \phi(x, y)\frac{\partial}{\partial y}. \tag{2.4}$$

Since many differential equations involve derivatives of the dependent variable, the transformation must also extend to these derivatives. This process is called prolongation [20]. The prolonged vector field acts on higher-order derivatives up to order n, and is given by

$$X^n = X + \sum_{k=1}^{n} \phi^{(k)} \frac{\partial}{\partial y^{(k)}}, \tag{2.5}$$

where the coefficients $\phi^{(k)}$ are defined recursively as

$$\phi^{(k)} = D_x(\phi^{(k-1)}) - u^{(k)} D_x(\rho), \tag{2.6}$$

with $\phi^{(0)} = \phi$ and D_x denotes the total derivative operator with respect to x.

The invariance condition determines whether a transformation generated by X is a symmetry of the given equation. This condition is expressed as

$$X^n(\Delta)\big|_{\Delta=0} = 0. \tag{2.7}$$

If this condition holds, the equation is said to admit a Lie symmetry. Solving this invariance condition yields the infinitesimals $\rho(x, y)$ and $\phi(x, y)$, which define the symmetry group of the differential equation [8].

Once the symmetry group is known, several advantages follow. The equation can often be reduced to a lower order, invariant (similarity) variables can be constructed, and new exact solutions can be generated from known ones [3]. In many cases, Lie symmetries also reveals deeper structural properties of the equation, including hidden conservation laws and integrability features that are not apparent through classical methods.

This method forms the foundation for several modern extensions such as λ - symmetry that will follow in the next section.

2.2 Reduction of Order

The reduction of order is one of the most important applications of symmetry analysis. Once the symmetry of a given ODE is known, it can be used to reduce the order of the equation by one degree, which simplifies the process of integration [8].

Consider a general n^{th} - order ODE (2.1) which is invariant under the symmetry generator (2.4).

The first prolongation of the vector field (2.5), can be written by:

$$X^{[1]} = \rho(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y} + \phi^{(1)}(x,y,y')\frac{\partial}{\partial y'}. \tag{2.8}$$

The new variables needed for the reduction are obtained by solving the characteristic equation of the first prolongation, which is written as

$$\frac{dx}{\rho(x,y)} = \frac{dy}{\phi(x,y)} = \frac{dy'}{\phi^{(1)}(x,y,y')}. \tag{2.9}$$

From this relation, we obtain two independent equations, which are then solved to define the new dependent and independent variables that will be used in the reduced equation. These variables are the invariants of the symmetry, and substituting them into the original ODE effectively reduces its order by one. We denote invariant functions as I.

In the next chapters, the concept of invariants will be applied to the nonlinear fin equation to reduce its order and obtain analytical solutions.

2.3 λ - Symmetries

Although the classical Lie symmetry method has proven to be the most powerful technique for identifying symmetries and finding analytical solutions of ODEs, many nonlinear equations do not admit any nontrivial Lie point symmetries. To overcome this challenge, C. Muriel and J.L Romeo (2001), introduced an extension of the Lie symmetry method known as λ - symmetry (also called C^∞ symmetry) [20].

Muriel and Romeo introduced a new function $\lambda = \lambda(x, y)$ that modifies the standard prolongation process in Lie symmetry theory. By doing so, the technique expands the class of differential equations that can be reduced or solved using symmetry based approaches. Even when a differential equation does not admit Lie point symmetries, the introduction of the λ - function enables order reduction and sometimes the derivation of exact solutions [19].

Consider ODE (2.1) again with the associated infinitesimal generator (2.4).

In classical Lie theory, the prolongation of this vector field involves computing higher-order derivatives of y using the total derivative operator. However, in λ - symmetry analysis, this prolongation of the vector field X is expressed as [19]

$$X^{[\lambda,(n)]} = \rho(x, y)\frac{\partial}{\partial x} + \phi^{(0)}(x, y)\frac{\partial}{\partial y} + \sum_{k=1}^{n} \phi^{[\lambda,(k)]}\frac{\partial}{\partial y^{(k)}}, \tag{2.10}$$

where the coefficients $\phi^{[\lambda,(k)]}$ are defined recursively by

$$\phi^{[\lambda,(k)]} = D_x(\phi^{[\lambda,(k-1)]}) - D_x(\rho)y^k + \lambda(\phi^{[\lambda,(k-1)]} - y^k\rho), \tag{2.11}$$

for $k = 1, 2, 3, ..., n$, and where $\phi^{(0)} = \phi$.

For example, the first order $(k = 1)$ λ - prolongation of X, denoted by $\phi^{(1)}$, is

defined as follows:

$$\begin{aligned}\phi^{[\lambda,(1)]} &= D_x(\phi^{(0)}) - D_x(\rho)y' + \lambda(\phi^{(0)} - y'\rho) \\ &= \phi_x^{(0)} + \phi_y^{(0)}y' - y'\rho_x - \rho_y y'^2 + \lambda\phi^{(0)} - \lambda y'\rho \\ &= \phi^{(0)'} - y'\rho' + \Big(\phi^{(0)} - y'\rho\Big)\lambda. \end{aligned} \tag{2.12}$$

Therefore, the first λ - prolonged vector field is given by:

$$X^{[\lambda,(1)]} = \rho\partial_x + \phi^{(0)}\partial_y + \phi^{[\lambda,(1)]}\partial_{y'}. \tag{2.13}$$

Similarly, for $(k = 2)$, the second-order λ - prolongation is defined in a similar way, viz:

$$\begin{aligned}\phi^{[\lambda,(2)]} &= D_x(\phi^{[\lambda,(1)]}) - D_x(\rho)y' + \lambda(\phi^{[\lambda,(1)]} - y'\rho) \\ &= -y'\rho_{xx} - 2\rho_{xy}y'^2 + \lambda_x\phi^{(0)} + 2\lambda\phi_x^{(0)} + \phi_{yy}^{(0)}y'^2 - \rho_{yy}y'^3 \\ &\quad + y''\phi_x^{(0)} - 2y''\rho_x + \lambda^2\phi^{(0)} + 2\phi_{xy}^{(0)}y' - y'\rho - 2\lambda y'\rho_x + y'\lambda_y\phi^{(0)} \\ &\quad - \lambda_y y'^2\rho + 2y'\lambda\phi_y^0 - 2\lambda\rho_y y'^2 - 3y''\rho_y y' - 2y''\lambda\rho - \lambda^2 y'\rho + \phi_{xx}^{(0)} \\ &= \phi^{(0)''} - y'\rho'' - 2y''\rho' + (\phi^{(0)} - y'\rho)\lambda' - 2(y''\rho + y'p' - \phi^{(0)'}) \\ &\quad + (\phi^{(0)} - y'\rho)\lambda^2. \end{aligned} \tag{2.14}$$

Therefore, the second λ - prolonged vector field is expressed as:

$$X^{[\lambda,(2)]} = \rho\partial_x + \phi^{(0)}\partial_y + \phi^{[\lambda,(1)]}\partial_{y'} + \phi^{[\lambda,(2)]}\partial_{y''}. \tag{2.15}$$

This expression shows how the function $\lambda(x, y)$ modifies the standard prolongation process by introducing additional terms that depends both on λ, and shows its interaction with the infinitesimals functions $\rho(x, y)$ and $\phi(x, y)$.

An ODE is said to admit a λ - symmetry generated by the vector field X if it satisfies the following invariance criterion:

$$X^{[\lambda,(n)]}(\Delta)\big|_{\Delta=0} = 0, \tag{2.16}$$

which is equivalent to the classical invariance condition (2.7). This condition ensures that the differential equation remains invariant under the λ - prolonged transformation.

Solving condition (2.16) allows for the determination of the infinitesimal functions $\rho(x, y)$ and $\phi(x, y)$, and the $\lambda(x, y)$ function which defines the λ - symmetry of the equation. In most cases, this leads to a system of PDEs, which can be solved using analytical or computational tools such as Maple to obtain the symmetry functions and corresponding reduced forms of the equation.

The λ - symmetry approach provides significant advantages over the classical Lie point symmetry method. When $\lambda(x, y) = 0$, the λ - prolongation reduces exactly to the standard Lie prolongation, meaning that the classical method is a special case of the λ - symmetry method. However, the introduction of a nonzero λ - function extends the applicability of symmetry methods to equations. The procedure of reduction of the ODE is the same as discussed above.

2.4 First Integrals

A first integral of a differential equation is a function that remains constant along the solution of the equation. If a solution $y(x)$ satisfies the differential equation, then there exists a function $T(x, y, y')$ such that

$$\frac{dT}{dx} = 0. \tag{2.17}$$

This implies that

$$T(x, y, y') = C, \tag{2.18}$$

where C is an arbitrary constant, and $T(x, y, y')$ is conserved along every solution curve. First integrals play an important role in simplifying and analysing differential equations, as they can be used to reduce the order of the equation by one and often correspond to the physical conservation laws [8].

Several methods can be used to determine the first integrals, depending on the structure of the equation. The multiplier method [9, 24] introduces an integrating factor $\mu(x, y, y')$, which transforms the given equation into an exact differential form, making it integrable. When a variational formulation exists, that is, if the equation can be derived from a Lagrangian function, the Noether theorem [21] provides a systematic way to find the first integrals associated with the symmetries of the Lagrangian. This theorem links symmetries to conserved quantities [3].

In this study we apply a method developed by E.S Cheb-Terrab and A.D. Roche in 1996 [6], which is based on the idea that many second-order nonlinear ODEs can be expressed in a quadratic form in the first derivative y', that is,

$$y'' = a(x, y)y'^2 + b(x, y)y + c(x, y), \tag{2.19}$$

where $a(x, y), b(x, y)$, and $c(x, y)$ are functions of x and y.

By introducing an integrating factor $\mu(x, y)$ and a potential function $H(x, y)$, the method allows the construction of a first integral that remains constant along the solution curves of the ODE.

Following [6], these two functions are defined by assuming that the derivative of the first integral $T(x, y, y')$ with respect to y' depends only on x and y, that is,

$$T_{y'} = \mu(x, y). \tag{2.20}$$

Integrating this expression with respect to y' gives

$$T(x, y, y') = \mu(x, y)y' + H(x, y). \tag{2.21}$$

Taking the total derivative of $T(x, y, y')$ with respect to x gives

$$\frac{dT}{dx} = \mu_x y' + \mu_y (y')^2 + \mu y'' + H_x + H_y y'. \tag{2.22}$$

Substituting for y'' from the ODE and grouping the coefficients of $(y')^2, y'$, and the constant term, these yield the following equations:

$$\begin{aligned} y'^3 &: \quad \mu_y - a\mu = 0, \\ y'^2 &: \quad \mu_x + H_y - b\mu = 0, \\ y'^0 &: \quad H_x - c\mu = 0. \end{aligned} \tag{2.23}$$

Rearranging the above deterministic equation gives the following determining equations, which connect the coefficients of the ODE with $\mu(x, y)$ and $H(x, y)$:

$$a(x, y) = \frac{-\mu_y}{\mu}, \tag{2.24}$$

$$b(x, y) = \frac{-H_y + \mu_x}{\mu}, \tag{2.25}$$

$$c(x, y) = \frac{-H_x}{\mu}. \tag{2.26}$$

Hence, the above describes how the coefficients the second-order ODE are related to the integrating factor $\mu(x, y)$ and the potential function $H(x, y)$ [6]. Once these relationships are established, we can determine $\mu(x, y)$ and $H(x, y)$ by solving these determining equations. The solutions of these equations then allow us to construct the first integral associated with the given ODE. The above procedure is built into Maple software.

2.5 Discussion

This chapter introduced the main mathematical ideas and background needed for the analysis that follows. The discussion covered the basic concepts of differential equations, the idea of symmetry transformations, and how infinitesimal generators and their prolongations are used to study the invariance of an equation.

We also discussed both the classical Lie symmetries and their extension through the λ - symmetry approach. The λ - symmetry method extends the classical framework by introducing a modifying function $\lambda(x, y)$. We included a procedure of how symmetries are used to reduce the order of an ODE and finally, we detailed how integrating factors and first integrals can be found for a second-order ODE.

Overall, this chapter sets the theoretical groundwork for the next stage of the study. The tools and definitions presented here will be directly applied in Chapter 3, 4 and 5.

Chapter 3

Symmetry Analysis of the Fin Equation

In this chapter, we delve into the symmetry analysis of the nonlinear fin equation (1.1). We consider the well known Lie symmetry method and the newer method of λ - symmetries.

3.1 Application of Lie Symmetry Methods on the Fin Equation

Recall the Lie point symmetry generator (2.4), then the corresponding invariance condition is:

$$X^{[2]}\Big(y^m y'' + m y^{m-1} y^2 - M^2 y^{n+1} + \frac{\alpha\, y^m y'}{x}\Big)\Big|_{(1.1)} = 0, \tag{3.1}$$

where

$$X^{[2]} = X + \phi^{(1)}\partial_{y'} + \phi^{(2)}\partial_{y''}, \tag{3.2}$$

denotes the second prolongation of the vector field X. The prolonged infinitesimals $\phi^{(1)}$ and $\phi^{(2)}$ are given by:

$$\begin{aligned} \phi^{(1)} &= D_x(\phi) - y' D_x(\rho), \\ \phi^{(2)} &= D_x(\phi^{(1)}) - y'' D_x(\rho). \end{aligned} \tag{3.3}$$

Substituting the above expressions into the invariance criterion condition (3.1) for the governing equation (1.1), yields the determining equations involving the infinitesimals $\rho(x, y)$ and $\phi(x, y)$.

However, for arbitrary values of the parameters α, m and n, the determining equations becomes algebraically complex, making it impractical to obtain a closed-form general solution.

Therefore, in this study, several cases are considered by evaluating different cases of the fin profiles corresponding to the parameter α.

To illustrate the process, we display the case $\alpha = 0$ (rectangular fin) with $n = -1$, in detail. For the remaining geometries, namely the radial fin ($\alpha = 1$) and the spherical fin ($\alpha = 2$), as well as other subcases for parameters m and n, the same analytical procedure was followed.

3.1.1 Case 1: The rectangular fin $\alpha = 0$

The system admits several subcases based on the parameter values of m and n.

Subcase 1: $n = -1$, $m \neq -1$

Substituting these parameters and expanding the expression with respect to the powers of the highest order of y' yield the following determining equations for the infinitesimals $\rho(x, y)$ and $\phi(x, y)$, namely,

$$\begin{aligned} y'^3 &: m\rho_y - y\rho_{yy} = 0, \\ y'^2 &: -m\phi + y(m\phi_y + y(\phi_{yy} - 2\rho_{xy})) = 0, \\ y' &: mM^2\phi + y(M^2\phi_y - 2M^2\rho_x + y^m\phi_{xx}) = 0, \\ y'^0 &: -3M^2y\rho_y + y^m(2m\rho_{xy} - y\rho_{xx}) = 0. \end{aligned} \tag{3.4}$$

This resulting system is solvable manually but can also be solved using a computational algebra system to obtain the corresponding infinitesimals.

Hence, the admitted Lie point symmetries for this subcase are given by:

$$\begin{gathered} X_1 = \partial_x, \\ X_2 = y^{-m}\partial_y, \\ X_3 = xy^{-m}\partial_y \\ X_4 = x\partial_x + M^2x^2y^{-m}\partial_y, \\ X_5 = \Big(y - \tfrac{M^2x^2(m+1)y^{-m}}{2}\Big)\partial_y, \\ X_6 = \tfrac{x^2}{2}\partial_x + \tfrac{x\left(M^2x^2(m+1)y^{-m}+2y\right)}{4m+4}\partial_y, \\ X_7 = y^{m+1}\partial_x - \tfrac{x\,M^2\left(M^2x^2(m+1)y^{-m}-6y\right)}{4}\partial_y, \\ X_8 = -\tfrac{x\left(-2y^{m+1}+M^2x^2(m+1)\right)}{2}\partial_x + \tfrac{-M^4(m+1)^2x^4y^{-m}+4y^{m+2}}{4m+4}\partial_y. \end{gathered} \tag{3.5}$$

The above is a maximal 8-dimensional Lie algebra and its presence indicates that (1.1) is linearizable in this case. In particular, the such an equation is equivalent to $y'' = 0$ [16] and transformations exist to linearize it. We find more linearizable cases below.

Subcase 2: $n = m$

For this subcase, two distinct configurations arise, each yielding a different set of admitted symmetries. These are discussed separately below.

i) $n = m = -1$

In this configuration, the symmetries are given by:

$$\begin{gathered} X_1 = \partial_x, X_2 = y\partial_y, X_3 = xy\partial_y, \\ X_4 = x\partial_x + (M^2x^2y)\,\partial_y,\ \ X_5 = (2y\ln(y) - M^2x^2y)\,\partial_y, \\ X_6 = (2x^2)\,\partial_x + (M^2x^3y + 2xy\ln(y))\,\partial_y, \\ X_7 = (2\ln(y) - 3M^2x^2)\,\partial_x - (2M^4x^3y)\,\partial_y, \\ X_8 = (4x\ln(y) - 2M^2x^3)\,\partial_x + \left(4y\ln^2(y) - M^4x^4y\right)\partial_y. \end{gathered} \tag{3.6}$$

ii) $n = m \neq -1$

Similarly, in the second configuration, the symmetries are as follows:

$$\begin{gathered}
X_1 = \partial_x, X_2 = y\partial_y, X_3 = \left(y^{-m}e^{\sqrt{m+1}Mx}\right)\partial_y, \\
X_4 = \left(y^{-m}e^{-\sqrt{m+1}Mx}\right)\partial_y, \\
X_5 = \frac{\left(y^{m+1}e^{\sqrt{m+1}Mx}\right)}{m+1}\partial_x + \frac{\left(My^{m+2}e^{\sqrt{m+1}Mx}\right)}{(m+1)^{3/2}}\partial_y, \\
X_6 = \frac{\left(y^{m+1}e^{-\sqrt{m+1}Mx}\right)}{m+1}\partial_x - \frac{\left(My^{m+2}e^{-\sqrt{m+1}Mx}\right)}{(m+1)^{3/2}}\partial_y, \\
X_7 = \left(\frac{M^2ye^{2x\sqrt{(m+1)M^2}}}{((m+1)M^2)^{3/2}} + \frac{mM^2ye^{2x\sqrt{(m+1)M^2}}}{((m+1)M^2)^{3/2}}\right)\partial_x + \left(\frac{me^{2x\sqrt{(m+1)M^2}}}{(m+1)M^2} + \frac{e^{2x\sqrt{(m+1)M^2}}}{(m+1)M^2}\right)\partial_y, \\
X_8 = \left(-\frac{M^2ye^{-2x\sqrt{(m+1)M^2}}}{((m+1)M^2)^{3/2}} - \frac{mM^2ye^{-2x\sqrt{(m+1)M^2}}}{((m+1)M^2)^{3/2}}\right)\partial_x + \left(\frac{me^{-2x\sqrt{(m+1)M^2}}}{(m+1)M^2} + \frac{e^{-2x\sqrt{(m+1)M^2}}}{(m+1)M^2}\right)\partial_y.
\end{gathered} \tag{3.7}$$

We find that the rectangular fin is not linearizable for $n = \pm k,\ k = 2, 3, \ldots$. An example of symmetries of such a case is given below.

Subcase 3: $n = 2$ & $m = -2$

For this subcase, the admitted symmetries are as follows:

$$\begin{aligned}
X_1 &= \partial_x, \\
X_2 &= x\partial_x + \tfrac{y}{2}\partial_y, \\
X_3 &= \tfrac{x^2}{2}\partial_x - \tfrac{xy}{2}\partial_y.
\end{aligned} \tag{3.8}$$

3.1.2 Case 2: The radial fin $\alpha = 1$

Let $n = -1,\ m \neq -1$. The admitted symmetries for this are obtained as follows:

$$\begin{gathered}
X_1 = y^{-m}\partial_y, \\
X_2 = \ln(x)y^{-m}\partial_y, \\
X_3 = x\partial_x + \tfrac{y^{-m}M^2x^2}{2}\partial_y, \\
X_4 = \left(y - \tfrac{M^2x^2(m+1)y^{-m}}{4}\right)\partial_y, \\
X_5 = x\ln(x)\partial_x + \tfrac{M^2x^2\ln(x)y^{-m}}{2}\partial_y, \\
X_6 = x\ln(x)^2 + \tfrac{2\left(M^2(m+1)x^2\left(\ln(x)-\frac{1}{2}\right)y^{-m}+2y\right)\ln(x)}{4m+4}\partial_y, \\
X_7 = -\tfrac{\left(-4y^{m+1}+M^2x^2(m+1)\right)x}{4}\partial_x - \tfrac{M^2\left(M^2x^2(m+1)y^{-m}-4y\right)x^2}{8}\partial_y, \\
X_8 = -\tfrac{x\ln(x)\left(-4y^{m+1}+M^2x^2(m+1)\right)}{4}\partial_x \\
+ \tfrac{-2M^4(m+1)^2x^4\left(\ln(x)-\frac{1}{2}\right)y^{-m}+8y(m+1)x^2M^2\ln(x)-8M^2x^2(m+1)y+16y^{m+2}}{16m+16}\partial_y.
\end{gathered} \tag{3.9}$$

Note that the case $n = -1$ & $m = 1$ can be recovered from the above by setting $m = 1$, as well as other values for m, $m \neq -1$.

For the case $n = m$, symmetries exist for $n = m = -1$, which cannot be recovered

from the above case. They are

$$
\begin{gathered}
X_1 = y\partial_y, \\
X_2 = (y\ln(x))\partial_y, \\
X_3 = (4y\ln(y) - M^2x^2y)\,\partial_y, \\
X_4 = (2x)\partial_x + (M^2x^2y)\,\partial_y, \\
X_5 = (2x\ln(x))\partial_x + (M^2x^2y\ln(x))\,\partial_y, \\
X_6 = (8x\ln(y) - 2M^2x^3)\,\partial_x + (4M^2x^2y\ln(y) - M^4x^4y)\,\partial_y, \\
X_7 = \left(4x\ln^2(x)\right)\partial_x + \left(2M^2x^2y\ln^2(x) - M^2x^2y\ln(x) + 4y\ln(x)\ln(y)\right)\partial_y, \\
X_8 = (16x\ln(x)\ln(y) - 4M^2x^3\ln(x))\,\partial_x \\
+ \left(M^4x^4y - 2M^4x^4y\ln(x) - 8M^2x^2y\ln(y) + 8M^2x^2y\ln(x)\ln(y) + 16y\ln^2(y)\right)\partial_y.
\end{gathered}
\tag{3.10}
$$

3.1.3 Case 3: The spherical fin $\alpha = 2$

We now examine this profile following the same procedure, the admitted inifinitesimals for each subcase are determined below.

Subcase 1: $n = m$

For this subcase, two distinct configurations arise, as in the rectangular case.

i) $n = m = -1$

The symmetries for this subcase are given by:

$$X_1 = y\partial_y, X_2 = \tfrac{y}{x}\partial_y,$$
$$X_3 = \left(6y\ln(y) - M^2x^2y\right)\partial_y, X_4 = 3x\partial_x + \left(M^2x^2y\right)\partial_y,$$
$$X_5 = \left(3x^2\right)\partial_x + \left(M^2x^3y\right)\partial_y, X_6 = 2\partial_x + \left(M^2xy - \tfrac{2y\ln(y)}{x}\right)\partial_y,$$
$$X_7 = \left(18x^2\ln(y) - 3M^2x^4\right)\partial_x + \left(6M^2x^3y\ln(y) - M^4x^5y\right)\partial_y,$$
$$X_8 = \left(12x\ln(y) - 2M^2x^3\right)\partial_x + \left(-M^4x^4y + 8M^2x^2y\ln(y) - 12y\ln^2(y)\right)\partial_y. \tag{3.11}$$

ii) $n = m \neq -1$

Similarly, for the second configuration, the symmetries take the form:

$$X_1 = \frac{\left(y^{-m}e^{x\left(-\sqrt{(m+1)M^2}\right)}\right)\partial_y}{x}, X_2 = \frac{\left(y^{-m}e^{x\sqrt{(m+1)M^2}}\right)\partial_y}{x\sqrt{(m+1)M^2}},$$
$$X_3 = \frac{\left(xy^{m+1}e^{\sqrt{m+1}Mx}\right)\partial_x}{m+1} + \left(\frac{Mxy^{m+2}e^{\sqrt{m+1}Mx}}{(m+1)^{5/2}} + \frac{mMxy^{m+2}e^{\sqrt{m+1}Mx}}{(m+1)^{5/2}} - \frac{y^{m+2}e^{\sqrt{m+1}Mx}}{(m+1)^2}\right)\partial_y,$$
$$X_4 = \frac{\left(xy^{m+1}e^{-\sqrt{m+1}Mx}\right)\partial_x}{m+1} + \left(-\frac{y^{m+2}e^{-\sqrt{m+1}Mx}}{(m+1)^2} - \frac{Mxy^{m+2}e^{-\sqrt{m+1}Mx}}{(m+1)^{5/2}} - \frac{mMxy^{m+2}e^{-\sqrt{m+1}Mx}}{(m+1)^{5/2}}\right)\partial_y,$$
$$X_5 = yF_2(x)\partial_y, X_6 = F_1(x)\partial_x, \tag{3.12}$$

where $F_1(x)$ and $F_2(x)$ satisfy the conditions

$$2mx^2F_2' + 2x^2F_2' - x^2F_1'' + 2xF_1' - 2F_1 = 0,$$

and

$$2M^2xF_1' - 2F_2' - xF_2'' = 0.$$

This case does not appear to possess an 8 dimensional Lie algebra, so may not be linearizable.

Subcase 2: $n = -1$, $m \neq -1$

$$
\begin{aligned}
X_1 &= y^{-m}\partial_y, \\
X_2 &= \tfrac{y^{-m}}{x}\partial_y, \\
X_3 &= (3x)\partial_x + (M^2x^2y^{-m})\,\partial_y, \\
X_4 &= (3x^2)\,\partial_x + (M^2x^3y^{-m})\,\partial_y, \\
X_5 &= (-mM^2x^2y^{-m} - M^2x^2y^{-m} + 6y)\,\partial_y, \\
X_6 &= \left(-\tfrac{3mM^2x^4}{m+1} - \tfrac{3M^2x^4}{m+1} + \tfrac{18x^2y^{m+1}}{m+1}\right)\partial_x + \left(-\tfrac{mM^4x^5y^{-m}}{m+1} - \tfrac{M^4x^5y^{-m}}{m+1} + \tfrac{6M^2x^3y}{m+1}\right)\partial_y, \\
X_7 &= (2m+2)\partial_x + \left(mM^2xy^{-m} + M^2xy^{-m} - \tfrac{2y}{x}\right)\partial_y, \\
X_8 &= \left(-\tfrac{2m^2M^2x^3}{(m+1)^2} - \tfrac{4mM^2x^3}{(m+1)^2} - \tfrac{2M^2x^3}{(m+1)^2} + \tfrac{12mxy^{m+1}}{(m+1)^2} + \tfrac{12xy^{m+1}}{(m+1)^2}\right)\partial_x \\
&+ \left(-\tfrac{m^2M^4x^4y^{-m}}{(m+1)^2} - \tfrac{2mM^4x^4y^{-m}}{(m+1)^2} - \tfrac{M^4x^4y^{-m}}{(m+1)^2} + \tfrac{8mM^2x^2y}{(m+1)^2} + \tfrac{8M^2x^2y}{(m+1)^2} - \tfrac{12y^{m+2}}{(m+1)^2}\right)\partial_y.
\end{aligned}
\tag{3.13}
$$

3.2 Application of λ - Symmetry Methods on the Fin Equation

Since we are dealing with a second-order nonlinear ODE, we begin by considering its second-order λ - prolongation. Taking the second-order prolongation (2.15) with formula (2.12) and (2.14), yields the simplified form of the second-order prolongation, viz.

$$\begin{aligned} X^{[\lambda,(2)]} &= \rho\partial_x + \phi^{(0)}\partial_y + \big[\phi^{(0)'} - y'\rho' + \lambda(\phi^{(0)} - y'\rho)\big]\partial_{y'} \\ &\quad + \big[\phi^{(0)''} - y'\rho'' - 2y''\rho' + \lambda'(\phi^{(0)} - y'\rho) - 2(y''\rho + y'p' - \phi^{(0)'}) \\ &\quad + \lambda^2(\phi^{(0)} - y'\rho)\big]\partial_{y''}. \end{aligned} \tag{3.14}$$

We now apply the invariance criterion (2.16) and this leads to the following relation:

$$\begin{aligned} &\frac{2my^m y'\lambda\phi}{y} - \frac{2my^m y'^2\lambda\rho}{y} + \frac{\phi y^m my''}{y} + \frac{\phi m^2 y^m y'^2}{y^2} - \frac{\phi my^m y'^2}{y^2} + \frac{2my^m y'\phi_x}{y} \\ &+\frac{2my^m y'^2\phi_y}{y} - \frac{2my^m y'^2\rho_x}{y} - \frac{2my^m y'^3\rho_y}{y} - y^m\lambda_x y'\rho - 2y^m\lambda y'\rho_x + y^m y'\lambda_y\phi \\ &-y^m\lambda_y y'^2\rho + 2y^m y'\lambda\phi_y - 2y^m\lambda\rho_y y'^2 - 3y^m y''\rho_y y' - 2y^m y''\lambda\rho - y^m\lambda^2 y'\rho \\ &+y^m\phi_{y,y}y_1^2 - 2y^m\rho_{x,y}y'^2 + y^m\lambda_x\phi + 2y^m\lambda\phi_x - y^m y'\rho_{x,x} + 2y^m\phi_{x,y}y' \\ &-y^m\rho_{y,y}y'^3 + y^m y''\phi_y - 2y^m y''\rho_x + y^m\lambda^2\phi + y^m\phi_{x,x} = 0. \end{aligned} \tag{3.15}$$

By replacing the highest-order derivative of y from (1.1) and expanding the expression, we obtain

$$\begin{aligned}
&\frac{2my^m y'\lambda\phi}{y} - \frac{\phi m y^m y'^2}{y^2} + \frac{2my^m y'\phi_x}{y} + \frac{my^m y'^2\phi_y}{y} + \frac{my^m y'^3\rho_y}{y} + \frac{\phi m\, M^2}{y} \\
&-y^m\lambda_x y'\rho - 2y^m\lambda y'\rho_x + y^m y'\lambda_y\phi - y^m\lambda_y y'^2\rho + 2y^m y'\lambda\phi_y - 2y^m\lambda\rho_y y'^2 \\
&-y^m\lambda^2 y'\rho - 3\rho_y y' M^2 - 2\lambda\rho M^2 - 2\rho_x M^2 + \phi_y M^2 + y^m\phi_{y,y}y'^2 - 2y^m\rho_{x,y}y'^2 \\
&+y^m\lambda_x\phi + 2y^m\lambda\phi_x - y^m y'\rho_{x,x} + 2y^m\phi_{x,y}y' - y^m\rho_{y,y}y'^3 + y^m\lambda^2\phi + y^m\phi_{x,x} = 0.
\end{aligned} \tag{3.16}$$

Since our objective is to determine the unknown functions $\phi(x,y), \rho(x,y)$, and $\lambda(x,y)$, we separate the above relation according to the powers of derivatives of the dependent variable y. This procedure yields the following overdetermined system of equations:

$$\begin{aligned}
\dot{y}^3 \;&:\; \frac{\rho_y m y^m}{y} - y^m\rho_{yy} = 0, \\
\dot{y}^2 \;&:\; -y^m\lambda_y\rho - 2y^m\lambda\rho_y - 2y^m\rho_{xy} + y^m\phi_{yy} - \frac{\phi m y^m}{y^2} + \frac{my^m\phi_y}{y} = 0, \\
\dot{y} \;&:\; \frac{2my^m\lambda\phi}{y} - y^m\lambda_x\rho - 2y^m\lambda\rho_x + y^m\lambda_y\phi + 2y^m\lambda\phi_y - y^m\lambda^2\rho - y^m\rho_{xx} \\
&\quad +2y^m\phi_{xy} - 3\rho_y M^2 y^n y + \frac{2my^m\phi_x}{y} = 0, \\
\dot{y}^0 \;&:\; -2\lambda\rho M^2 y^n y + \phi m\, M^2 y^n - \phi M^2 y^n n - 2\rho_x M^2 y^n y + \phi_y M^2 y^n y \\
&\quad -\phi M^2 y^n + y^m\lambda^2\phi + 2y^m\lambda\phi_x + y^m\lambda_x\phi + y^m\phi_{xx} = 0.
\end{aligned} \tag{3.17}$$

The above system of determining equations is nonlinear and cannot be solved directly by classical methods due to its complexity, we employ an ansatz approach in order to obtain explicit solutions. To this end, we assume:

$$\rho(x,y) = 0, \tag{3.18}$$

which serves to simplify the system considerably.

With this assumption, we proceed to solve for λ and ϕ. In doing so, we consider the temperature profiles corresponding to fins of different geometries, namely with rectangular fin, radial fin and spherical fin. This classification leads to different solutions in each of the cases of the system (3.17).

3.2.1 Case 1: The rectangular fin $\alpha = 0$

The overdetermined system (3.17) is analyzed for different values of parameters m and n, which gives rise to the distinct subcases below.

Subcase 1: $n = -1$

Substitution into the system gives the functions

$$\begin{aligned} \lambda(x, y) &= -\tfrac{f_{1x}(x)}{f_1(x)} + \tfrac{1}{x - c_1}, \\ \phi(x, y) &= f_1(x)\, y^{-m}, \end{aligned} \tag{3.19}$$

where $f_1(x)$ is an arbitrary function and c_1 is an arbitrary constant, hereunder.

This corresponds to the following symmetry:

$$X = f_1(x)\, y^{-m} \partial_y. \tag{3.20}$$

In addition, the system admits an alternative form of a solution, given by:

$$\lambda(x,y) = -\tfrac{f_{1_x}(x)}{f_1(x)},$$
$$\phi(x,y) = -\frac{f_1(x)\left(\left(x^2M^2(m+1)-2c_1x-2c_2\right)y^{-m}-2y\right)}{2}, \tag{3.21}$$

where c_2 is an arbitrary constant, hereunder. This corresponds to the following symmetry:

$$X = -\frac{f_1(x)\left(\left(x^2M^2(m+1)-2c_1x-2c_2\right)y^{-m}-2y\right)}{2}\partial_y. \tag{3.22}$$

Subcase 2: $n = m$

For $n = m$, we obtain the following functions:

$$\lambda(x,y) = \tfrac{M\sqrt{m+1}\,f_1(x)-f_{1_x}(x)}{f_1(x)} - \tfrac{\mathrm{e}^{-2M\sqrt{m+1}\,x}}{c_1+\frac{\mathrm{e}^{-2M\sqrt{m+1}\,x}}{2\sqrt{m+1}\,M}},$$
$$\phi(x,y) = f_1(x)\,y^{-m}. \tag{3.23}$$

This corresponds to the following symmetry:

$$X = f_1(x)\,y^{-m}\partial_y. \tag{3.24}$$

In addition, the system admits an alternative form of solution, given by:

$$\lambda(x,y) = -\tfrac{f_{1_x}(x)}{f_1(x)},$$
$$\phi(x,y) = f_1(x)\left(\sinh\left(\sqrt{(m+1)\,M^2}\,x\right)y^{-m}c_1 + \cosh\left(\sqrt{(m+1)\,M^2}\,x\right)y^{-m}c_2 + y\right).$$

Similarly, this corresponds to the symmetry:

$$X = f_1(x)\left(\sinh\left(\sqrt{(m+1)\,M^2}\,x\right)y^{-m}c_1 + \cosh\left(\sqrt{(m+1)\,M^2}\,x\right)y^{-m}c_2 + y\right)\partial_y.$$

Note that the case $n = -1,\ m = 1$ can be recovered from Subcase 1 by setting $m = 1$.

Subcase 3: $n = 2$**,** $m = -2$

Here, we obtain the following solution:

$$\begin{aligned} \lambda(x, y) &= \tfrac{2Mf_1(x)y^2 - f_{1_x}(x)}{f_1(x)}, \\ \phi(x, y) &= f_1(x)\, y. \end{aligned} \tag{3.25}$$

This corresponds to the following symmetry:

$$X = f_1(x)\, y\partial_y. \tag{3.26}$$

In addition, the system admits an alternative form of solution, given by:

$$\begin{aligned} \lambda(x, y) &= \tfrac{-2Mf_1(x)y^2 - f_{1_x}(x)}{f_1(x)}, \\ \phi(x, y) &= f_1(x)\, y. \end{aligned} \tag{3.27}$$

This corresponds to the following symmetry:

$$X = f_1(x)\, y\partial_y. \tag{3.28}$$

3.2.2 Case 2: The radial fin $\alpha = 1$

For the radial fin, we proceed in the same way.

Subcase 1: $n = -1$

Substituting $n = -1$ gives the following functions:

$$\lambda(x,y) = \frac{-x\Big(c_1+\ln(x)\Big)\Big(f_{1_x}(x)\Big)+f_1(x)}{f_1(x)x\Big(c_1+\ln(x)\Big)},$$
$$\phi(x,y) = f_1(x)\, y^{-m}, \tag{3.29}$$

which corresponds to the symmetry:

$$X = f_1(x)\, y^{-m}\partial_y. \tag{3.30}$$

In addition, the system admits an alternative form of a solution, given by:

$$\lambda(x,y) = -\tfrac{f_{1_x}(x)}{f_1(x)},$$
$$\phi(x,y) = -\frac{\left(\left(-4c_1\ln(x)+x^2M^2(m+1)-4c_2\right)y^{-m}-4y\right)f_1(x)}{4}, \tag{3.31}$$

which corresponds to the symmetry:

$$X = -\frac{\left(\left(-4c_1\ln(x)+x^2M^2(m+1)-4c_2\right)y^{-m}-4y\right)f_1(x)}{4}\partial_y. \tag{3.32}$$

Subcase 2: $n = -1,\ m = 1$

Similarly, here we obtain the following solution:

$$\lambda(x,y) = \frac{-x\Big(c_1+\ln(x)\Big)\Big(f_{1_x}(x)\Big)+f_1(x)}{f_1(x)x\Big(c_1+\ln(x)\Big)},$$
$$\phi(x,y) = \frac{f_1(x)}{y}, \tag{3.33}$$

which corresponds to the following symmetry:

$$X = \frac{f_1(x)}{y}\partial_y. \tag{3.34}$$

In addition, the system admits a second solution, given by:

$$\lambda(x,y) = \frac{-x\Big(M^2x^2-2c_2\ln(x)+2c_1\Big)\Big(f_{1_x}(x)\Big)+2f_1(x)\Big(M^2x^2-c_2\Big)}{\Big(M^2x^2-2c_2\ln(x)+2c_1\Big)xf_1(x)},$$
$$\phi(x,y) = \frac{f_1(x)}{y} - \frac{2f_1(x)y}{M^2x^2-2c_2\ln(x)+2c_1}, \tag{3.35}$$

which corresponds to the following symmetry:

$$X = \left(\frac{f_1(x)}{y} - \frac{2f_1(x)y}{M^2x^2-2c_2\ln(x)+2c_1}\right)\partial_y. \tag{3.36}$$

Note that the case $n = m$ produces a complicated solution that is not useful.

3.2.3 Case 3: The spherical fin $\alpha = 2$

Finally, for the geometry of the spherical fin, the system again admits multiple solutions for different values of the parameters m and n.

Subcase 1: $n = m$

For $n = m$, we obtain the solution:

$$\lambda(x,y) = \frac{M\sqrt{n+1}\, f_1(x)x - (f_{1_x}(x))x - f_1(x)}{f_1(x)x} - \frac{\mathrm{e}^{-2M\sqrt{n+1}\,x}}{c_1 + \frac{\mathrm{e}^{-2M\sqrt{n+1}\,x}}{2\sqrt{n+1}\,M}}, \tag{3.37}$$

$$\phi(x,y) = f_1(x)\, y^{-n}, \tag{3.38}$$

which corresponds to the following symmetry:

$$X = f_1(x)\, y^{-n}\partial_y. \tag{3.39}$$

In addition, the system admits a second solution, given by:

$$\begin{gathered}\lambda(x,y) = -\frac{f_{1_x}(x)}{f_1(x)}, \\ \phi(x,y) = \frac{f_1(x)\Big(\sinh\Big(\sqrt{(n+1)M^2}\,x\Big)y^{-n}c_1 + \cosh\Big(\sqrt{(n+1)M^2}\,x\Big)y^{-n}c_2 + xy\Big)}{x},\end{gathered} \tag{3.40}$$

which corresponds to the following symmetry:

$$X = \frac{f_1(x)\Big(\sinh\Big(\sqrt{(n+1)M^2}\,x\Big)y^{-n}c_1 + \cosh\Big(\sqrt{(n+1)M^2}\,x\Big)y^{-n}c_2 + xy\Big)}{x}\partial_y. \tag{3.41}$$

Subcase 2: $n = -1, m = 1$

Similarly, for $n = -1, m = 1$, we obtain the following solution:

$$\lambda(x, y) = \frac{-(f_{1_x}(x))x - f_1(x)}{x f_1(x)} + \frac{1}{-c_1 + x}, \tag{3.42}$$

$$\phi(x, y) = \frac{f_1(x)}{y}\partial_y, \tag{3.43}$$

which corresponds to the symmetry:

$$X = \frac{f_1(x)}{y}\partial_y. \tag{3.44}$$

In addition, the system admits an alternative form of a solution, given by:

$$\lambda(x, y) = -\frac{f_{1_x}(x)}{f_1(x)}, \tag{3.45}$$

$$\phi(x, y) = -\frac{f_1(x)\left(M^2x^3 + \left(-3y^2 - 3c_2\right)x - 3c_1\right)}{3xy}, \tag{3.46}$$

which corresponds to the symmetry:

$$X = -\frac{f_1(x)\left(M^2x^3 + \left(-3y^2 - 3c_2\right)x - 3c_1\right)}{3xy}\partial_y. \tag{3.47}$$

To illustrate, for the nonlinear fin equation corresponding to the geometry cases considered, selecting $f_1(x) = x$ allows us to observe explicit λ-symmetries; this particular case is further analysed in detail in Chapter 4

3.3 Discussion

In this chapter, we applied both Lie and λ - symmetry analysis to the nonlinear fin equation to identify its admitted symmetries and explore how they differ across

different temperature profiles. The analysis was carried out for the rectangular, radial and spherical fin geometries, where the determining equations were derived and solved to obtain the corresponding symmetry structures.

The results showed that each geometry admits distinct sets of symmetry generators, reflecting the influence of the fin profile on the underlying invariance properties of the equation. In particular, the arbitrary nature of the function $f_1(x)$, which appears in every case above, indicates that infinitely many λ - symmetries exist for the nonlinear fin equation for certain parameter values. Moreover, by identifying the maximal symmetry cases, we have established the parameter values of the power laws for which the fin equation can be linearized.

Overall, this chapter established a symmetry classification of the fin equation. The obtained symmetries form the basis for the reductions of order and invariant solutions presented in the next chapter.

Chapter 4

Solving the Fin Equation

In this chapter, we discuss the exact solutions of the nonlinear fin equation based on the symmetries obtained earlier. Rather than giving a complete list of solutions, which would be too long, we present only a selection of those that are particularly interesting.

4.1 Invariant Solutions via λ - Symmetries

In this section, we will show how the λ - symmetries in selected cases can be used to reduce the order of the fin equation and thereby find exact solutions. For simplicity, we let the arbitrary function in the λ - symmetry be $f_1(x) = x$.

4.1.1 Case 1: The rectangular fin $\alpha = 0$

Consider the subcase $n = -1$. The first λ - prolongation of symmetry (3.20), given by (2.13). Explicitly, this is expressed as

$$X^{[\lambda,(1)]} = 0\partial_x + xy^{-m}\partial_y + \left(y^{-m} - \frac{xy^{-m}my'}{y} + \left(-\frac{1}{x} + \frac{1}{-c_1+x}\right)xy^{-m}\right)\partial_{y'}. \tag{4.1}$$

The above symmetry yields the following characteristic equation:

$$\frac{dx}{0} = \frac{dy}{xy^{-m}} = \frac{dy'}{y^{-m} - \frac{xy^{-m}my'}{y} + \left(-\frac{1}{x} + \frac{1}{-c_1+x}\right)xy^{-m}}. \tag{4.2}$$

Hence, the first invariant is given by:

$$I_1 := v(x) = x. \tag{4.3}$$

Solving the characteristic equation,

$$\frac{dy}{xy^{-m}} = \frac{dy'}{y^{-m} - \frac{xy^{-m}my'}{y} + \left(-\frac{1}{x} + \frac{1}{-c_1+x}\right)xy^{-m}} \tag{4.4}$$

$$\frac{dy}{x} = \frac{dy'}{1 - \frac{xy^{-m}my'}{y} + \left(-\frac{1}{x} + \frac{1}{-c_1+x}\right)x}$$

$$\frac{dy'}{dy} = \frac{1 - \frac{xy^{-m}my'}{y} + \left(-\frac{1}{x} + \frac{1}{-c_1+x}\right)x}{x}, \tag{4.5}$$

yields a solution

$$y' = -\frac{y}{(m+1)(c_1-x)} + y^{-m}c_0, \tag{4.6}$$

which leads to the second invariant,

$$I_2 := w(x) = y'y^m + \frac{y^{m+1}}{(c_1-x)(m+1)}. \tag{4.7}$$

To obtain the reduced equation, it is necessary to determine $\frac{dw}{dv}$, viz

$$\frac{dw}{dx} = y^m\left(y'' + \frac{y''m}{y} + \frac{y'}{c_1 - x} + \frac{y}{(c_1 - x)^2(m+1)}\right), \tag{4.8}$$

and substituting y'' from (1.1) into the above equation then we get the following result

$$\frac{dw}{dv} = M^2 + \frac{y^m y'}{c_1 - x} + \frac{y^{m+1}}{(c_1 - x)^2(m+1)}, \tag{4.9}$$

which can be written as a first order ODE by invoking (4.7):

$$\frac{dw}{dv} = M^2 + \frac{w}{c_1 - x}, \tag{4.10}$$

This reduced equation yields a solution

$$w(x) = \frac{M^2\left(c_1 x - \frac{1}{2}x^2\right) + c_2}{c_1 - x}, \tag{4.11}$$

where c_2 is a constant. Thereafter, a substitution of the above equation (4.11) into equation (4.7) provides

$$y' = -\frac{y}{(m+1)(c_1 - x)} + \frac{y^{-m}\left(M^2\left(c_1 x - \frac{1}{2}x^2\right) + c_2\right)}{c_1 - x}. \tag{4.12}$$

We integrate the above equation to obtain a solution for $y(x)$, that is (c_3 is a constant)

$$\begin{aligned} y(x) = \frac{1}{2^{\frac{1}{m+1}}}\Bigg(& M^2c_1^2m - M^2c_1mx + M^2m\,x^2 + M^2c_1^2 - M^2c_1x + x^2M^2 + 2c_3c_1 \\ & + 2c_2m - 2c_3x + 2c_2\Bigg)^{\frac{1}{m+1}}. \end{aligned} \tag{4.13}$$

Overall from (4.13), increasing M amplifies the magnitude of the solution $y(x)$, while variations in m alter the curvature of the profile, smaller m values flatten the

curve, while large m values make it steeper. The case $m = -1$ is excluded due to a singularity. Overall, M influences the scale of the solution, whereas m determine its shape.

Finally, a plot of the solution for the above equation is generated by varying the parameters m and M, while we let $c_1 = c_2 = c_3 = 1$ in Figure 4.1. This allows for an analysis of how changes in these parameters affect the overall behaviour of the solution.

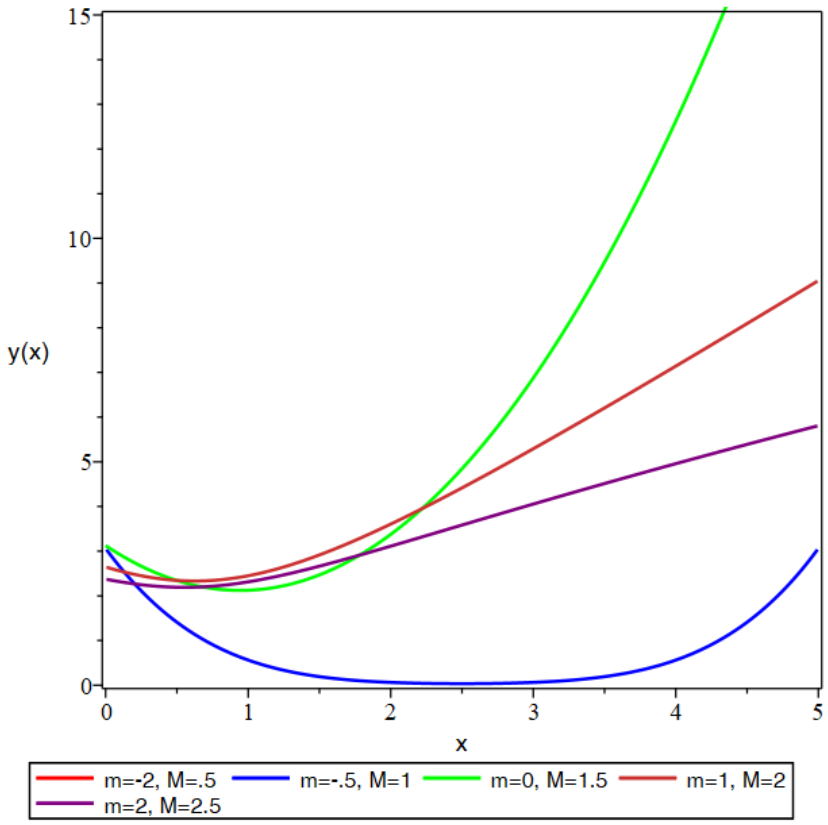


Figure 4.1: Plot of solution (4.13) with varying parameters m and M.

As in subcase 4.1.1, we apply the standard λ - symmetry reduction method below. The calculations are analogous and thus are omitted here for brevity. Note that due

to the ansatz of (3.18), the first invariant (4.3) will appear in all cases below. The resulting invariant and solution are as follows:

Let $n = -1$ and $m = 1$. In this case, the admitted symmetry (3.20) leads to the following second invariant:

$$I = y'y + \tfrac{y^2}{2(c_1 - x)}.$$

Using these invariants, we reduce the ODE (1.1) to find the corresponding solution:

$$y(x) = \pm\sqrt{M^2c_1^2 - M^2c_1x + M^2x^2 + c_3c_1 - c_3x + 2c_2}. \tag{4.14}$$

The plot of this solution is shown in Figure 4.2 below, where the parameter M is varied and $c_1 = c_2 = c_3 = 1$.

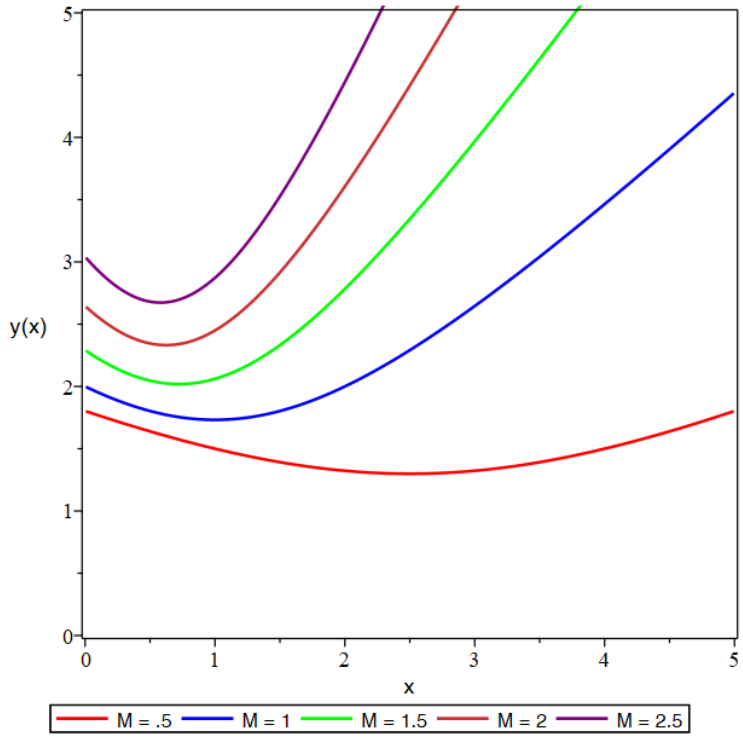


Figure 4.2: Plot of the solution (4.14), illustrating the effect of varying the thermo-geometric parameter M.

The graph of $y(x)$ shows a smooth, parabolic behaviour for all positive values of M. As M increases, it is confirmed that M acts as a scaling parameter influencing the amplitude of the solution.

Next, let $n = 2$ and $m = -2$. The admitted symmetry (3.26) yields the following invariant:

$$I = \frac{y'}{y} - My^2.$$

From this, the corresponding solution is obtained as

$$y(x) = \pm \frac{1}{\sqrt{c_1^2 c_2 - 2c_1 c_2 x + c_2 x^2 - 2Mc_1 + 2Mx}}. \tag{4.15}$$

The plot of the solution for this alternative case is shown in Figure 4.3 below, obtained by varying the parameter M while $c_1 = c_2 = 1$.

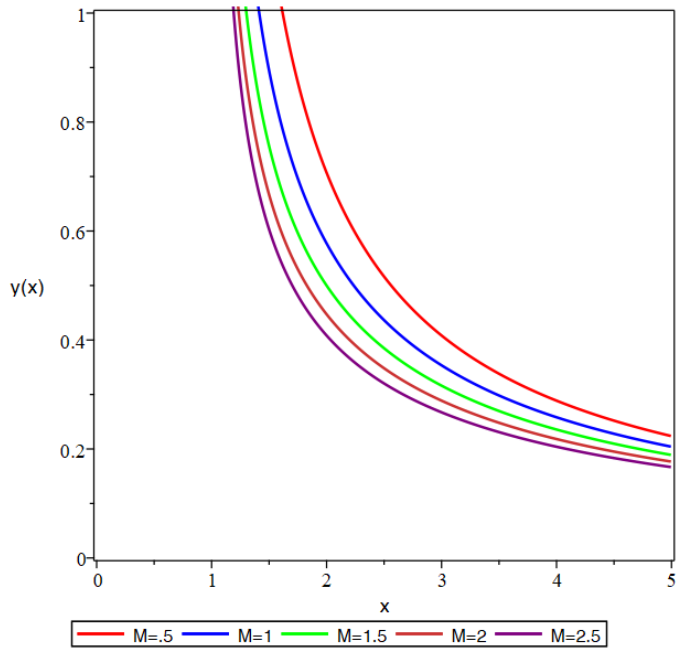


Figure 4.3: Plot of the solution (4.15), illustrating the effect of varying the thermo-geometric parameter M.

The function is only defined where the term under the square root is positive, giving each M a limited range. As M increases, the domain narrows and the amplitude of $y(x)$ decreases. The curve remains smooth and continuous.

4.1.2 Case 2: The radial fin $\alpha = 1$

Under the radial fin geometry, we derive several explicit solutions. Let $n = -1$ and $m = 1$. In this case, the admitted symmetry (3.34) leads to the following second invariant:

$$I = y'y - \frac{y^2}{2x(c_1 + \ln(x))},$$

which leads to the solution:

$$y(x) = \pm \frac{\sqrt{2x^2M^2 + 4c_3 \ln(x) + 4c_3c_1 - 8c_2}}{2}. \tag{4.16}$$

The plot of the solution (4.16) for this case is shown in Figure 4.4 below, obtained by varying M while keeping all other constants equal to 1.

4.1.3 Case 3: The spherical fin $\alpha = 2$

Here, consider the case $n = -1$ and $m = 1$. In this case, the admitted symmetry (3.44) leads to the following second invariant:

$$I = y'y + \frac{c_1y^2}{2x(c_1 - x)},$$

and the corresponding solution is:

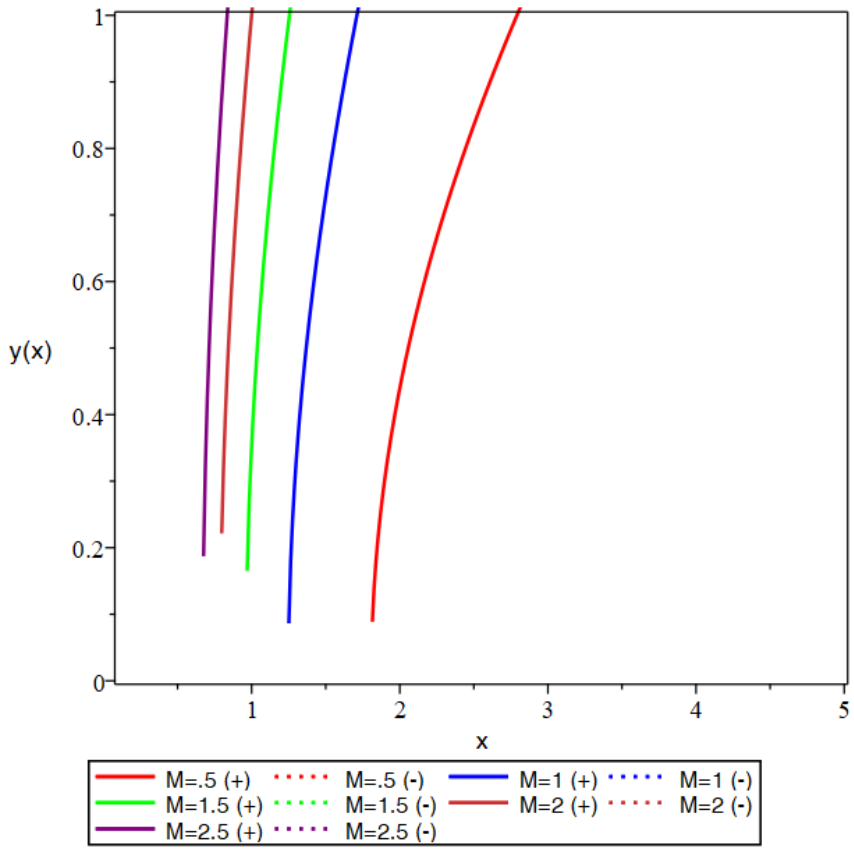


Figure 4.4: Plot of the solution (4.16), illustrating the effect of varying M.

$$
\begin{aligned}
y(x) \;=\; & \pm\frac{1}{3xc_1}\Bigg(xc_1\Big(6M^2\ln(-c_1+x)\,c_1^4-6M^2\ln(-c_1+x)\,c_1^3x-6M^2\ln(-c_1+x)\,c_1x^3 \\
& +6M^2\ln(-c_1+x)\,x^4+6M^2\ln(x)\,c_1x^3-6M^2\ln(x)\,x^4-11M^2c_1^4+17M^2c_1^3x \\
& -3M^2c_1^2x^2-3M^2c_1x^3-6c_2c_1^2x^3+6c_2c_1x^4-9c_3c_1^2+9c_3c_1x\Big)\Bigg)^{\frac{1}{2}}. \qquad (4.17)
\end{aligned}
$$

The plot of the solution (4.17) for this case is shown in Figure 4.5 below, obtained by varying M while keeping all other constants equal to 1.

The graph shows the positive trajectories of the solution for different M values. The function is only defined where the term under the root is positive, giving each M a limited range. As M increases, the domain narrows and the amplitude of $y(x)$ decreases. The curve remains smooth and continuous, while the negative branch would be the reflection about the x-axis.

4.2 Invariant Solutions via Lie Symmetries

In this section, we determine the invariant and corresponding analytical solutions for the nonlinear fin equation using the Lie symmetry approach. The procedure follows the same framework as that applied for the λ - symmetries, but with the (2.5) and (2.6) prolongation. Hence only the final results are presented here for brevity.

We consider the same three geometrical cases of the fin model. For each case, one admitted symmetry is selected to compute the corresponding invariant and to obtain the reduced or explicit solution of the fin equation. The calculations for the remaining subcases are analogous and therefore omitted for conciseness.

4.2.1 Case 1: The rectangular fin $\alpha = 0$

Let $n = -1$. In this case, the admitted symmetry X_4 from (3.5) leads to the following invariants:

$$
\begin{aligned}
I_1 &= 2^{-\frac{m}{1+m}} \left(2^{\frac{1}{1+m}} y\right)^m y - \frac{M^2 m\, x^2}{2} - \frac{M^2 x^2}{2}, \\
I_2 &= x \left(M^2 x - \left(\left(2\, 2^{-\frac{m}{1+m}} \left(y 2^{\frac{1}{1+m}}\right)^m y\right)^{\frac{1}{1+m}}\right)^m 2^{-\frac{m}{1+m}} y'\right),
\end{aligned} \tag{4.18}
$$

which yield the solution

$$
y(x) = 4^{-\frac{1}{1+m}} \left((1+m)\left(2^{\frac{1}{1+m}} M^2 \left(2^{\frac{1}{1+m}}\right)^m x^2 + 2^{\frac{2+m}{1+m}} x c_1 + 2^{\frac{2+m}{1+m}} c_2\right)\right)^{\frac{1}{1+m}}. \tag{4.19}
$$

A plot of the solution for the above equation is produced by varying the parameters M and m simultaneously for selected values while letting $c_1 = c_2 = 1$. This plot, shown in Figure 4.6, provides a clear visual representation of how the parameters M and m influence the real trajectories of the solution and reveal the corresponding changes in amplitude and curvature.

The plot shows that M mainly affects the height of the curves, i.e. as M increases, the overall values of $y(x)$ become larger. The parameter m changes the shape of the curves; smaller or negative m values make the curves flatter, while larger m values make them steeper. This means that m controls how quickly $y(x)$ grows with x. Together, M and m determine both the size and the steepness of the solutions. The fixed y-axis scale makes it easy to compare the curves and clearly shows that as both M and m increase, the solution becomes more nonlinear and grows faster.

4.2.2 Case 2: The radial fin $\alpha = 1$

Let $n = -1$. In this case, the admitted symmetry X_1 of (3.9) leads to the following invariants:

$$\begin{aligned} I_1 &= x, \\ I_2 &= y' y^m, \end{aligned} \tag{4.20}$$

which yield the solution

$$y(x) = \frac{\left((m+1)\left(2^{\frac{1}{m+1}} M^2 \left(2^{\frac{1}{m+1}}\right)^m x^2 + 2^{\frac{m+2}{m+1}} c_1 x + 2^{\frac{m+2}{m+1}} c_2\right)\right)^{\frac{1}{m+1}} 2^{-\frac{1}{m+1}}}{2^{\frac{1}{m+1}}}. \tag{4.21}$$

The plot of the above solution is produced by varying the parameters M and m for selected values while letting $c_1 = c_2 = 1$ in Figure 4.7. The graph illustrates how changes in M and m affects the solution and shows the differences in amplitude and curvature. The plot shows how the solution $y(x)$ behaves for different values of M and m. The curves are smooth and increase as x increases, showing that the solution remains real and behaves well for positive values of x. The parameter M mainly controls the height of the curves. When M is larger, the solutions rise higher, meaning the solution's overall size increases. The parameter m changes the shape of the curves; when m is small or negative, the curves are flatter and grow slowly; when m is larger, the curves rise faster and become steeper. This shows that m affects how strongly the solution grows with x.

4.2.3 Case 3: The spherical fin $\alpha = 2$

Let $n = -1$. In this case, the admitted symmetry X_3 of (3.13) leads to the following invariants:

$$\begin{aligned} I_1 &= 6^{-\frac{m}{1+m}}\left(6^{\frac{1}{1+m}}y\right)^m y - \frac{M^2 m\, x^2}{6} - \frac{M^2 x^2}{6}, \\ I_2 &= x\left(M^2 x - 3\left(\left(6y6^{-\frac{m}{1+m}}\left(6^{\frac{1}{1+m}}y\right)^m\right)^{\frac{1}{1+m}}\right)^m 6^{-\frac{m}{1+m}}y'\right), \end{aligned} \tag{4.22}$$

which yield the solution

$$y(x) = 9^{-\frac{1}{1+m}}4^{-\frac{1}{1+m}}\left(\frac{(1+m)\left(6^{\frac{1}{1+m}}x^3\left(6^{\frac{1}{1+m}}\right)^m M^2 + 6^{\frac{2+m}{1+m}}xc_2 - 6^{\frac{2+m}{1+m}}c_1\right)}{x}\right)^{\frac{1}{1+m}}. \tag{4.23}$$

Finally, a plot of the above solution is generated by varying the parameters M and m for selected values while letting $c_1 = c_2 = 1$. This plot, shown in Figure 4.8, helps visualize how changes in M and m affect the real trajectories of the solution and reveal the differences in shape and amplitude.

The curves are real, smooth, and defined only for positive x. The parameter M mainly controls how high the curves rise: larger M values make the trajectories higher, showing a stronger overall magnitude of the solution. The parameter m changes the shape and steepness of the curves. When m is small or negative, the curves are flatter and increase slowly, but as m becomes larger, they rise faster and become more curved. Together, M and m determine how quickly $y(x)$ grows with x and how nonlinear the solution becomes.

4.3 Discussion

This chapter focused on solving the nonlinear fin equation using the symmetry results obtained earlier. By applying the admitted Lie and λ - symmetries, the equation was reduced to simpler forms that could be integrated to obtain explicit analytical solutions. These solutions were then analysed and plotted for different geometric profiles, namely rectangular, radial, and spherical fins.

The graphical analysis showed that the parameters M and m and play a significant role in shaping the temperature profiles along the fin. Larger values of M generally increase the amplitude of the solution, while variations in m affect the curvature and rate of temperature decay. The results confirmed that the obtained solutions are physically valid, real, and continuous for $x \geq 0$.

Overall, the chapter demonstrated how symmetry-based reduction can simplify complex nonlinear equations and reveal the physical behaviour of the system.

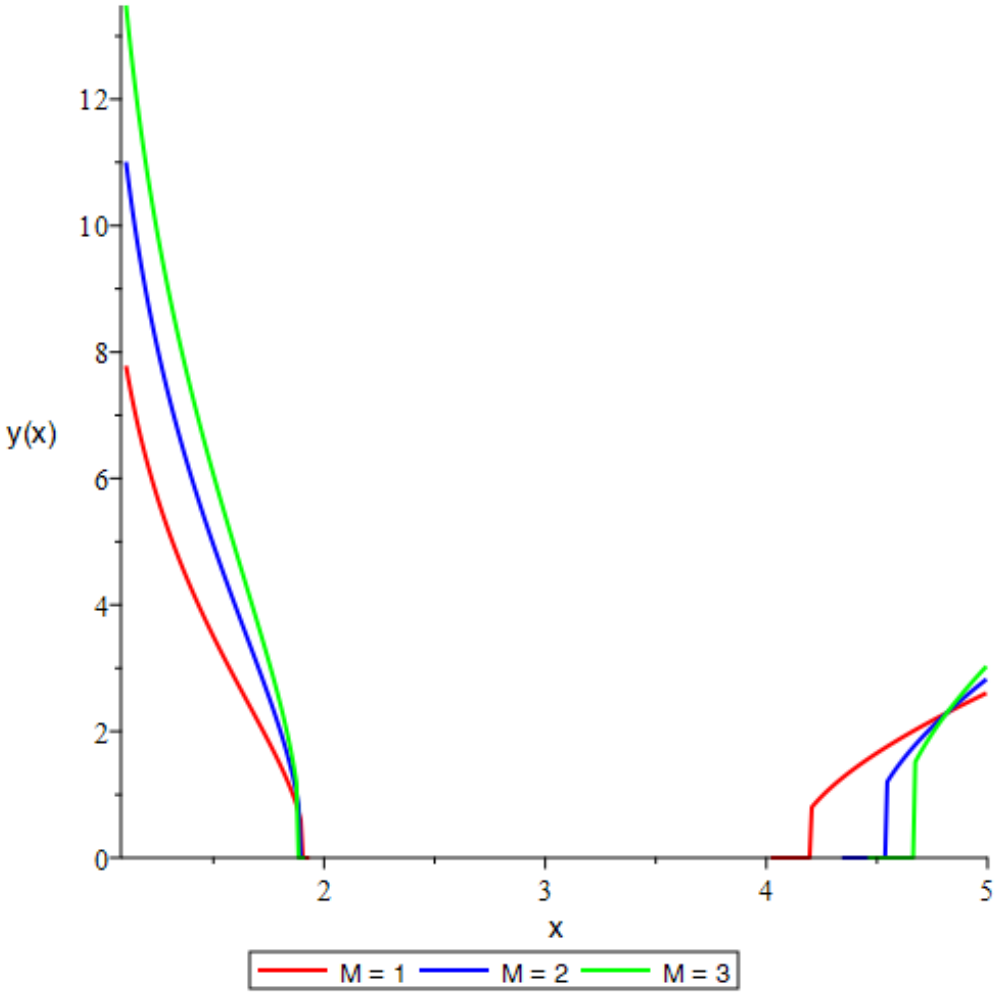


Figure 4.5: Plot of the solution (4.17), illustrating the effect of varying M.

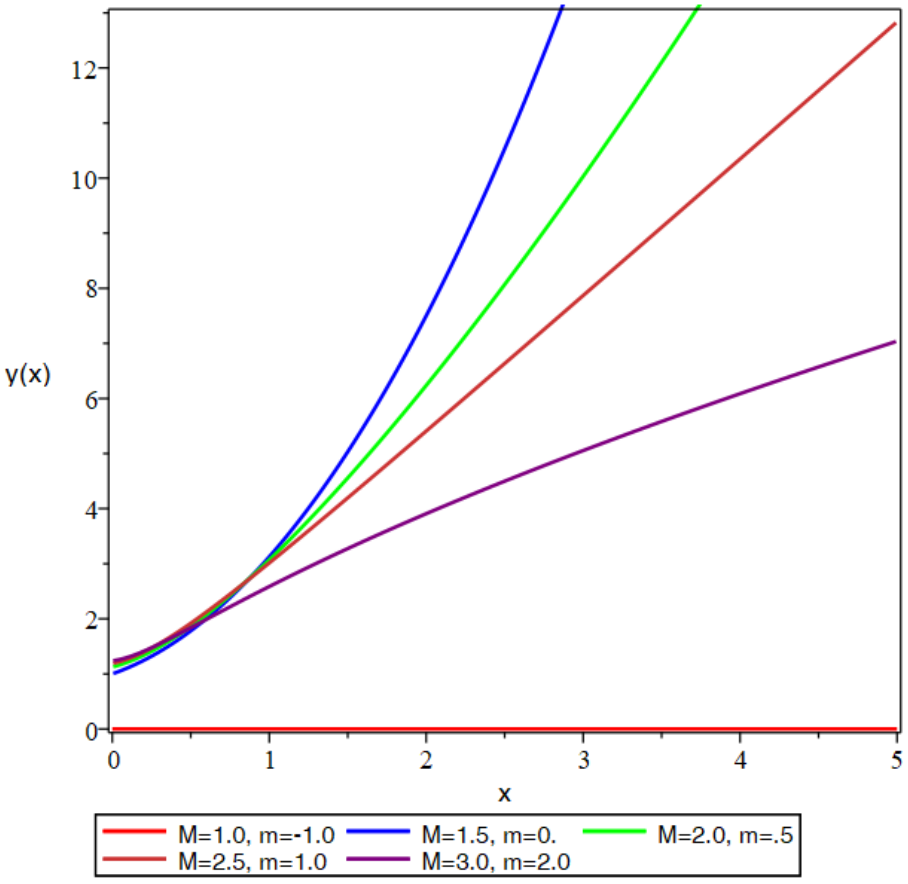


Figure 4.6: Plot of the solution (4.19), illustrating the effect of varying M and m.

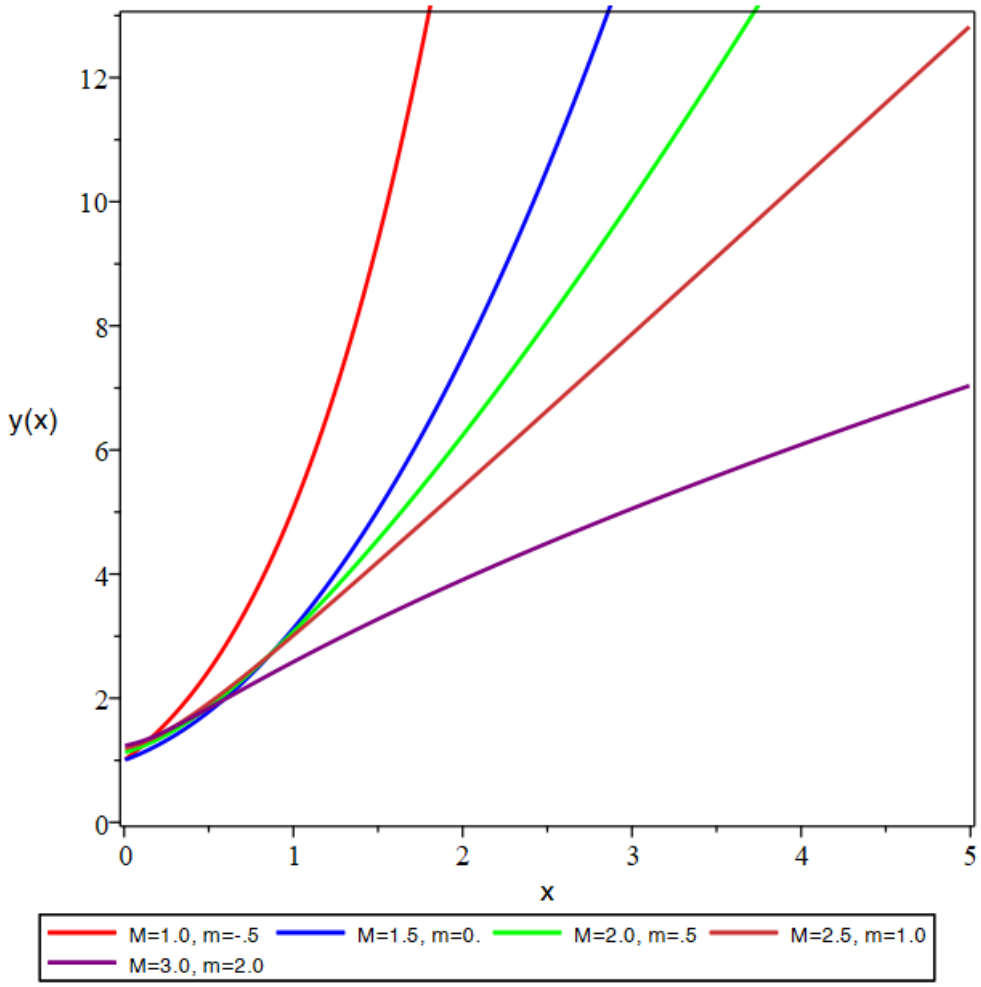


Figure 4.7: Plot of the solution (4.21), illustrating the effect of varying M and m.

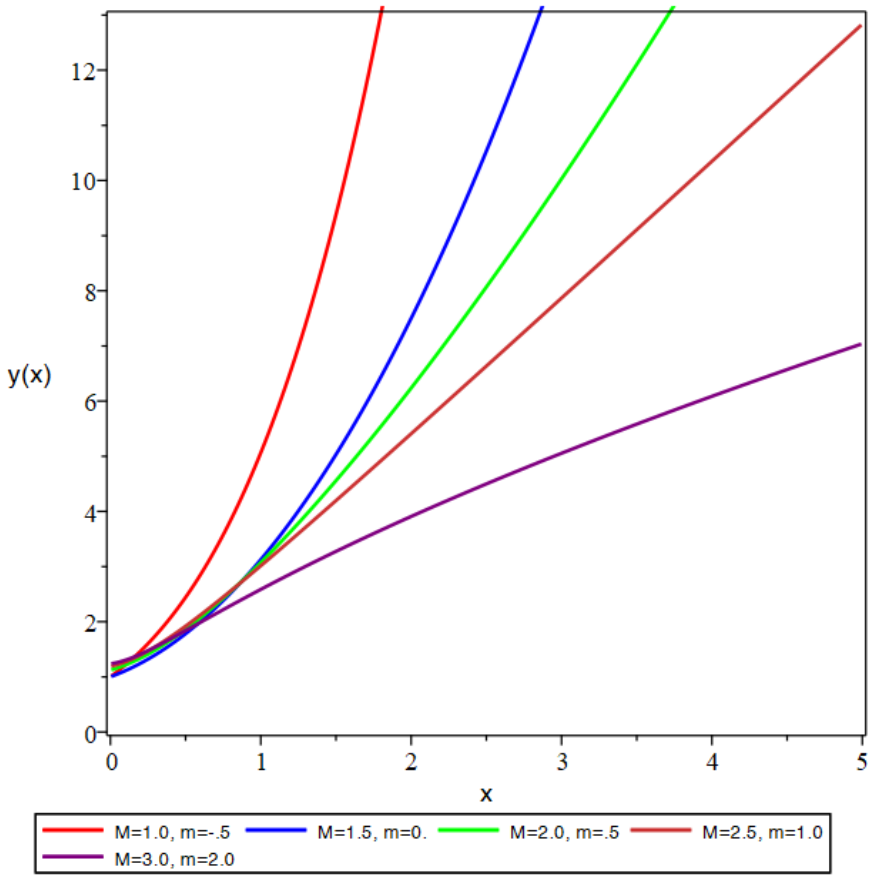


Figure 4.8: Plot of the solution (4.23), illustrating the effect of varying M and m.

Chapter 5

First Integrals of the Fin Equation

In this chapter, we focus on obtaining integrating factors and first integrals of the Fin equation (1.1) using the Cheb-Terrab & Roche method [6], which provides a systematic way of identifying conserved quantities for second-order ODEs. We start with the rectangular fin case, followed by the same procedure for the radial and spherical fins.

5.1 A Detailed Calculation

As before, the nonlinear fin equation is analyzed for different parameter values of m and n, giving rise to several distinct subcases. Each subcase is examined individually to determine the corresponding integrating factor and first integral. The first subcase for $n = -1$ in a rectangular fin is done in detail.

The nonlinear fin equation (1.1) can be written as:

$$y'' = \frac{-m}{y}y'^2 + \frac{M^2}{y^m}, \tag{5.1}$$

which is a quadratic of y'. Thus, in the notation from Section 2.4,

$$a(x,y) = \frac{-m}{y}, \tag{5.2}$$

$$b(x,y) = 0, \tag{5.3}$$

$$c(x,y) = \frac{M^2}{y^m}. \tag{5.4}$$

From equation (2.24) and equation (5.2) we have

$$\frac{\mu_y}{\mu} = \frac{m}{y}. \tag{5.5}$$

Since the above expression is a first order ODE, we can integrate both sides with respect to y:

$$\begin{aligned}
\int \frac{\mu_y}{\mu} dy &= \int \frac{m}{y} dy, \\
\ln(\mu) &= m\ln(y) + A(x), \\
e^{\ln(\mu)} &= e^{\ln(y)^m + A(x)}, \\
\mu &= y^m e^{A(x)}.
\end{aligned}$$

Hence the integrating factor is

$$\mu(x,y) = y^m \nu(x), \tag{5.6}$$

where $\nu(x) = e^{A(x)}$. From equation (2.25) and (5.3), we have

$$\frac{-(H_y + \mu_x)}{\mu} = 0. \tag{5.7}$$

This implies that

$$H_y = -\mu_x. \tag{5.8}$$

But from equation (5.6), we have

$$\mu_x = y^m \nu'(x), \tag{5.9}$$

so that

$$H_y = -y^m \nu(x). \tag{5.10}$$

Integrating the above equation, we obtain:

$$\int H_y dy = \int -y^m \nu'(x) dy,$$

which yields

$$H(x,y) = \frac{-y^{m+1}\nu'(x)}{m+1} + B(x). \tag{5.11}$$

From equation (2.26) and equation (5.4), we have

$$\frac{-H_x}{\mu} = \frac{M^2}{y^m}. \tag{5.12}$$

so that by (5.11), we have

$$H_x = \frac{-y^{m+1}\nu''(x)}{m+1} + B'(x).$$

Then it follows that:

$$\begin{aligned}
\frac{\left(\frac{y^{m+1}\nu''(x)}{m+1} + B'(x)\right)}{y^m \nu(x)} &= \frac{M^2}{y^m}, \\
\frac{y^{m+1}\nu''(x)}{(m+1)y^m\nu(x)} + \frac{B'(x)}{y^m\nu(x)} &= \frac{M^2}{y^m}, \\
\frac{y}{(m+1)}\frac{\nu''(x)}{\nu(x)} + \frac{B'(x)}{y^m\nu(x)} &= \frac{M^2}{y^m}.
\end{aligned} \tag{5.13}$$

Since the above equation must hold for all values of y, the following expression must therefore be satisfied:

$$\frac{\nu''(x)}{(m+1)\nu(x)} = 0.$$

This implies that

$$\nu''(x) = 0. \tag{5.14}$$

Hence, we obtain

$$\nu(x) = Ax + F. \tag{5.15}$$

Where A and F are arbitrary constants. From equation (5.14) and (5.13) we have

$$\begin{aligned} \frac{B'(x)}{\nu(x)} &= -M^2, \\ B'(x) &= -M^2\nu(x), \\ \int B'(x)dx &= -\int M^2(Ax+F)dx, \\ B(x) &= -M^2\Big(\frac{Ax^2}{2} + Fx\Big) + c_1, \end{aligned} \tag{5.16}$$

where c_1 is an constant of integration. For simplicity, we let $A = 0$ and $F = 1$. Then

$$B(x) = -M^2x + c_1, \tag{5.17}$$

and

$$\nu(x) = 1,$$

by (5.15), so that

$$\mu(x, y) = y^m,$$

by (5.6).

However, according to definition (2.21) of the first integral, and by using formula (5.17) to find (5.11), we obtain the first integral

$$T(x, y, y') \;=\; y^m y' - M^2 x + c_1. \tag{5.18}$$

Similarly, a second first integral,

$$T(x, y, y') = -\frac{y^{m+1}}{m+1} + x\, y^m y' - \frac{M^2 x^2}{2} + c_1,$$

corresponding to the integrating factor $\mu(x, y) = x\, y^m$ can be found.

5.2 Tabular Results for the Three Geometries

Having obtained the first integral for the rectangular fin geometry corresponding to $\alpha = 0$ and $n = -1$, we now summarize the results for the other selected subcases. Since the method follows the same procedure as those outline for the first case, only the final results are presented below.

Table 5.1: Integrating Factors and First Integrals for the Rectangular Fin ($\alpha = 0$)

Integrating Factor	**First Integral**
Case: $n = m$	
$\mu = \sin\big(M\sqrt{-m-1}\,x\big)\,y^m$	$T = \frac{M\,y^{m+1}\cos\left(M\sqrt{-m-1}\,x\right)}{\sqrt{-m-1}} + \sin\big(M\sqrt{-m-1}\,x\big)\,y^m y' + c_1$
$\mu = \cos\big(M\sqrt{-m-1}\,x\big)\,y^m$	$T = -\frac{M\,y^{m+1}\sin\left(M\sqrt{-m-1}\,x\right)}{\sqrt{-m-1}} + \cos\big(M\sqrt{-m-1}\,x\big)\,y^m y' + c_1$
Case: $n = -1$**,** $m = 1$	
$\mu = y$	$T = -M^2 x + y\,y' + c_1$
$\mu = x\,y$	$T = -\frac{y^2}{2} + x\,y\,y' - \frac{M^2x^2}{2} + c_1$
$\mu = \frac{y'}{(M-y')(M+y')}$	$T = 2\ln(y) + \ln\Big((M-y')(M+y')\Big) + c_1$

Table 5.2: Integrating Factors and First Integrals for the Radial Fin ($\alpha = 1$)

Integrating Factor	**First Integral**
Case: $n = -1$	
$\mu = x\, y^m$	$T = x\, y^m y' - \dfrac{M^2 x^2}{2} + c_1$
$\mu = x \ln(x)\, y^m$	$T = -\dfrac{y^{m+1}}{m+1} + x \ln(x)\, y^m y' - \dfrac{M^2 \ln(x)\, x^2}{2} + \dfrac{M^2 x^2}{4} + c_1$
Case: $n = -1$**,** $m = 1$	
$\mu = y$	$T = -M^2 x + y\, y' + c_1$
$\mu = x\, y$	$T = -\dfrac{y^2}{2} + x\, y\, y' - \dfrac{M^2 x^2}{2} + c_1$
$\mu = \dfrac{y'}{(M - y')(M + y')}$	$T = 2 \ln(y) + \ln\big((M - y')(M + y')\big) + c_1$
Case: $n = m$	
$\mu = x\, Y_0\big(M\sqrt{-m-1}\, x\big)\, y^m$	$T = -\, y^{m+1} x M\, Y_1\big(M\sqrt{-m-1}\, x\big)\, \dfrac{1}{\sqrt{-m-1}}$ $+ x\, Y_0\big(M\sqrt{-m-1}\, x\big)\, y^m y' + c_1$
$\mu = x\, J_0\big(M\sqrt{-m-1}\, x\big)\, y^m$	$T = -\, y^{m+1} x M\, J_1\big(M\sqrt{-m-1}\, x\big)\, \dfrac{1}{\sqrt{-m-1}}$ $+\, x\, J_0\big(M\sqrt{-m-1}\, x\big)\, y^m y' + c_1$

J and Y are the Bessel functions of the first and second kind, respectively. The subscript index refers to the order of the Bessel functions.

Table 5.3: Integrating Factors and First Integrals for the Spherical Fin ($\alpha = 2$)

Integrating Factor	**First Integral**
Case: $n = -1$	
$\mu = x\,y^m$	$T = \dfrac{y^{m+1}}{m+1} + x\,y^m y' - \dfrac{M^2x^2}{2} + c_1$
$\mu = x^2y^m$	$T = x^2y^m y' - \dfrac{M^2x^3}{3} + c_1$
Case: $n = -1$**,** $m = 1$	
$\mu = x\,y$	$T = \dfrac{y^2}{2} + x\,y\,y' - \dfrac{M^2x^2}{2} + c_1$
$\mu = x^2y$	$T = x^2y\,y' - \dfrac{M^2x^3}{3} + c_1$
Case: $n = m$	
$\mu = \sinh(\gamma)\,xy^m,\quad \gamma = \sqrt{M^2(m+1)}\,x$	$T = -\dfrac{y^{m+1}}{m+1}\Big[\gamma\,\cosh(\gamma) - \sinh(\gamma)\Big] + x\,\sinh(\gamma)\,y^m y' + c_1$
$\mu = x\,\cosh(\gamma)\,y^m$	$T = -\dfrac{y^{m+1}}{m+1}\Big[\gamma\,\sinh(\gamma) - \cosh(\gamma)\Big] + x\,\cosh(\gamma)\,y^m y' + c_1$

5.3 Discussion

In this chapter, we obtained first integrals of the nonlinear fin equation. We calculated explicit expressions for the integrating factors and first integrals for each geometric configuration, namely the rectangular, radial, and spherical fin profiles.

First integrals complement the Lie and λ - symmetry analyses by providing an-

other pathway to construct invariant quantities and provide solutions. Although, we have not provided solutions from the first integrals, it can easily be done by using the Tables presented above. Furthermore, the obtained expressions reveal how the geometric and thermal parameters influence the first integrals along the fin.

We note that the case $n = -1$ and m arbitrary provides two first integrals, but for the special case $n = -1$ and $m = 1$, a third integrating factor and first integral can be found for the rectangular and radial case.

Overall, this chapter demonstrated the usefulness of integrating factors in providing first integrals of nonlinear differential equations that model physical systems. The first integrals derived here form an important link to the symmetry-based results obtained earlier.

Chapter 6

Conclusion

The main aim of this study was to explore and analyze the nonlinear fin equation using analytical techniques based on symmetry analysis to uncover its underlying structure, invariant properties, and possible first integrals. The nonlinear fin model, which describes heat transfer in extended surfaces with temperature-dependent thermal conductivity and heat generation, was studied under three different geometries; rectangular ($\alpha = 0$), radial ($\alpha = 1$), and spherical ($\alpha = 2$).

The research began with a review of the fundamental concepts of differential equations and symmetry theory, highlighting how continuous transformations can be used to reduce the order of nonlinear ODEs.

Lie symmetry analysis was first introduced as a classical tool for identifying transformations that leave the differential equation invariant. The study then extended this framework by considering λ - symmetries, which generalize Lie symmetries through the introduction of a modifying function $\lambda(x, y)$, allowing for further solutions of equations. Finally, the method developed by Cheb-Terrab and Roche, was pre-

sented as a systematic way to compute first integrals through integrating factors and potential functions, complementing the symmetry-based approach.

For each geometry, the analysis focused on identifying the admitted symmetries, determining the corresponding invariants, and obtaining analytical solutions. The behaviour of the solutions was examined graphically, providing an illustration of how parameters such as M (thermo-geometric fin parameter) and m (power law parameter) affect the temperature distribution along the fin.

6.1 Lie Symmetry Analysis

The Lie symmetry analysis provided the foundation for exploring the invariance properties of the nonlinear fin equation. By applying the classical invariance criterion to the fin model for different geometrical profiles, we defined the conditions under which analytical linearization is possible.

We have shown that the nonlinear fin equation admits a maximal 8 dimensional Lie algebra and is therefore linearizable in certain cases. That is, the rectangular fin is linearizable whenever $n = m,\ m \neq -1$, and for $n = -1$ and $m \neq -1$. The radial fin is linearizable whenever $n = m = -1$, and for $n = -1$ and $m \neq -1$. Finally, the spherical fin is linearizable whenever $n = m = -1$, and for $n = -1$ and $m \neq -1$. The nonlinear fins may not be linearizable for $n = \pm j,\ j = 2, 3, \ldots$, under any α profile. This classification for linearizability, provides valuable theoretical insight into the geometric and algebraic structure of the nonlinear fin equation.

From the symmetries, the corresponding invariants were obtained by solving the associated characteristic equations. Each invariant was then used to explicitly obtain

analytical solutions $y(x)$. Plots of $y(x)$ for different parameters confirmed that increasing M amplifies the solution magnitude, while increasing m sharpens the temperature gradient.

6.2 λ - Symmetry Analysis

Building on the classical results, the λ - symmetry method introduced by Muriel and Romero was applied to overcome the limitations observed in the Lie analysis. This approach modifies the prolongation process through the inclusion of a function $\lambda(x, y)$. This method is advantageous for equations without classical Lie symmetries, but also systematically provides symmetries and invariants for other equations. In the cases considered, we showed that infinite λ - symmetries exist.

Using λ - symmetries, the nonlinear fin equation was analysed for different parameter combinations and geometries. Explicit analytical solutions $y(x)$ were obtained for each case, and the resulting temperature profiles were plotted to illustrate the physical behaviour of the fin.

This method proved to be a more powerful and flexible tool than the classical Lie framework. It extended the scope of solvable equations and confirmed the consistency of the derived results with the symmetries identified earlier.

6.3 First Integrals

In addition to the symmetry-based approaches, we determined first integrals and integrating factors for the fin equation. The method used, assumes that the nonlinear

ODE can be written in a quadratic form in y' and introduces an integrating factor $\mu(x, y)$ to find corresponding first integrals $T(x, y, y')$, which remains constant along solution curves.

The results were summarised for different cases of m and n across all three geometries. The method proved to be a highly reliable algebraic approach for identifying conserved quantities. The first integrals may also be used to construct solutions to the fin equation.

6.4 Future Work

Future research can extend this study in several meaningful directions. Possible extensions include examining fin equations with variable cross-sectional areas. A symmetry approach could also be applied to fractional-order or higher-dimensional models to capture nonlocal behaviours.

Moreover, future investigations can explore how generalized symmetry approaches can be used to simplify and solve similar equations and reveal new invariants and integrable structures.

In addition, future studies may explore hidden symmetries or nonlocal symmetries to complement the analytical results. Such extensions would further uncover the mathematical structure of the nonlinear heat conduction models and related thermal systems.

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