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On the Dynamics, Invariance Analysis, Exact Reductions, Series Solutions and Conservation Laws for the Time-Fractional Spherically Symmetric Brain Tumor Model

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Abstract

The Lie symmetry analysis for the time-fractional spherically symmetric brain tumor equation in spherical coordinates, assuming the killing rates are functions of tumor-cells is studied. The classification with regard to the net killing rate is obtained. The underlying equation is transformed into a fractional ordinary differential equation using the Erdélyi-Kober fractional differential operator. The explicit series solutions are obtained with its convergence analysis. The graphs of the solutions are given from which we may extract the behaviour of the solutions. Finally, we construct the conservation laws for the time-fractional brain tumor equation.

Keywords: brain tumor model; invariance analysis; conservation laws.

1 Introduction

The growth of tumor cells in the brain forms a complex field of study and has been taken up in various cases in recent years. The mathematical model was developed with the collaboration of many researchers from fields in biology, chemistry, medicine and mathematics for better explaining the complex process of cancer dynamics and finding appropriate treatments. For example, Jones et al. [12] provided a basic tumor model. A model that describes the formation of a brain tumor while accounting for the mobility and diffusion of the tumor cells has evolved throughout a number of publications [5, 14]. Following this strategy, Tracqui et al. [18] proposed a model that considers the aforementioned factors as well as treatment and the killing rate of brain tumor cells. Other works that may be of benefit to the reader, viz., in [21], the authors discuss the fractional approach in general, Wang [19] looks at the applications of conservation laws and Wang [20] expounds on the symmetry approach. In this instance, the corresponding equation is,

$$\psi_t = \mathcal{D}\Delta^2\psi + \mathcal{P}\psi - k\psi, \tag{1}$$

where $\mathcal{D}$ is the diffusion coefficient, $\mathcal{P}$ is the proliferation rate, k is the killing rate, and ψ is the concentration of tumor cells. Moyo and Leach [14] examined this model under the assumption of complete radial symmetry. Here, $\mathcal{P} - k$ is replaced by $\mathcal{K}(\mathtt{x}, t)$ and is variable. The resultant governing equation takes the form,

$$\psi_t = \psi_{\mathtt{xx}} - \mathcal{K}\psi. \tag{2}$$

They performed a Lie Symmetry Analysis (LSA) and using this method, have provided some exact solutions. Thus, in the case of killing rate $\mathcal{K}(\psi)$ being function of ψ, Bokhari et al. [3] employed Lie symmetry analysis to obtain several invariant reductions and exact solutions. Ali et al. [1] considered diffusivity and the net killing rate as the function of ψ. Therefore, the nonlinear governing equation in spherical coordinates and with radial symmetry assumption becomes,

$$\psi_t = \frac{1}{\mathtt{x}^2}\frac{\partial}{\partial \mathtt{x}}(\mathtt{x}^2\mathcal{D}(\psi)\psi_{\mathtt{x}}) - \mathcal{K}(\psi)\psi, \tag{3}$$

where $\mathcal{K}(\psi)$ is the net killing rate and $\mathcal{D}(\psi)$ is the medium's diffusivity. They have also presented classification of $\mathcal{K}(\psi)$ and $\mathcal{D}(\psi)$ using LSA. Now, after simplifying (3) and assuming $\mathcal{D}(\psi)$ as constant, we get,

$$\psi_t - \frac{2}{\mathtt{x}}\psi_{\mathtt{x}} - \psi_{\mathtt{xx}} + \mathcal{K}(\psi)\psi = 0. \tag{4}$$

A Fractional Differential Equation (FDE) may provide an alternative approach, i.e., if the integer time differential is superseded by a fractional one. An increasing number of researchers are studying this phenomenon with various numerical and analytical techniques - [10] studies expicit approaches to FPDEs, [4] considers numerical approaches, [22] looks at general approaches, [7] studies specific brain tumor models and [8] considers general soliton equations. An LSA study here is proving to be a useful method for dealing with the fractional models. The use of symmetry analysis to convert FPDE into a FODE and the Erdélyi-Kober to obtain an exact solution are covered in [6, 2]. With this in mind, (4) can be recast as in terms of fractional derivatives given by,

$$\mathtt{D}_t^\sigma\psi - \frac{2}{\mathtt{x}}\psi_{\mathtt{x}} - \psi_{\mathtt{xx}} + \mathcal{K}(\psi)\psi = 0, \tag{5}$$

where, $\mathtt{D}_t^\sigma\psi$ denotes the Riemann-Liouville (RL) fractional derivative operator,

$$\mathtt{D}_t^\sigma\psi(\mathtt{x}, t) = \begin{cases} \dfrac{1}{\Gamma(p-\sigma)}\dfrac{\partial^p}{\partial t^p}\displaystyle\int_0^t (t-)^{p-\sigma-1}\psi(\mathtt{x}, \varkappa)d\varkappa, & p-1<\sigma<p, \\ \dfrac{\partial^p\psi}{\partial t^p}, & \sigma = p \in \mathbb{N}. \end{cases} \tag{6}$$

Here, $0 < \sigma \leq 1$ characterizes the order of the fractional derivative, exerting a significant influence on the properties of this equation. A particular version of the LSA of a time-fractional cancer tumor model was performed by Simon et al. [17]. In addition to finding the conservation laws, he presented the series solutions together with their convergence analysis and graphical representation. In what follows, we will present a classification of function $\mathcal{K}(\psi)$ in (5) using LSA.

This paper's work is organized as follows. In Section 2, we provide some basic definitions and a general overview of Time-Fractional Partial Differential Equations (TFPDEs). The infinitesimal group of transformations for the spherically symmetric brain tumor equation is derived in Section 3. The classification of the function $\mathcal{K}(\psi)$ using LSA is covered in Section 4. We perform reductions in Section 5. The series solution with its convergence analysis and graphical illustration are presented in Section 6. Finally, conservation laws are discussed in Section 7 followed by a discussion and conclusions in Section 8.

2 Basic Definitions and General Methodology of Lie Symmetry Method

In this section, we will follow some introductory definitions that will be used throughout our article. Expounding on (6), we get,

Definition 1. *The RL fractional derivative (ordinary) is defined as [16, 9],*

$$\mathtt{D}^{\sigma}h(t) = \begin{cases} \dfrac{d^p}{dt^p} I^{p-\sigma}h(t), & p-1<\sigma<p, \\ \dfrac{d^p h}{dt^p}, & \sigma = p \in \mathbb{N}, \end{cases} \tag{7}$$

where $I^{\sigma}h(t)$ is fractional integration of RL defined by,

$$I^{\sigma}h(t) = \begin{cases} \dfrac{1}{(\sigma-1)!}\displaystyle\int_0^t (t-\varkappa)^{\sigma-1}h(\varkappa)d\varkappa, & \sigma>0, \\ h(t), & \sigma = 0. \end{cases} \tag{8}$$

Definition 2. *The Erdélyi-Kober (EK) fractional differential operator is defined by [9],*

$$\left(\mathcal{P}_{\lambda}^{\tau,\sigma}\digamma\right)(\vartheta) = \prod_{j=0}^{p-1}\left(\tau + j - \frac{1}{\lambda}\vartheta\frac{d}{d\vartheta}\right)\left(\mathcal{K}_{\lambda}^{\tau+\sigma,p-\sigma}\digamma\right)(\vartheta), \tag{9}$$

$$p = \begin{cases} [\sigma]+1, & \sigma \notin \mathbb{N}, \\ \sigma, & \sigma \in \mathbb{N}, \end{cases} \tag{10}$$

such that,

$$\left(\mathcal{K}_{\lambda}^{\tau,\sigma}\digamma\right)(\vartheta) = \begin{cases} \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_1^{\infty}(u-1)^{\sigma-1}u^{-(\tau+\sigma)}\digamma(\vartheta u^{\frac{1}{\lambda}})du, & \sigma>0, \\ G(\vartheta), & \sigma = 0, \end{cases} \tag{11}$$

is the EK fractional integral operator.

Note: EK fractional integral is a generalization and modification of the RL fractional integral and it is more capable of representing the memory property.

2.1 General description of Lie symmetry method for nonlinear TFPDEs

Here, we describe the main concepts of around the Lie symmetry method [9] for the fractional case by introducing the following general (1+1)-dimensional fractional PDEs,

$$\partial_t^\sigma \psi(\mathtt{x},t) = G(t,\mathtt{x},\psi,\psi_{\mathtt{x}},\psi_{\mathtt{xx}},\ldots), \qquad 0<\sigma\le 1. \tag{12}$$

Consider a one parameter Lie (symmetry) group of transformations,

$$\begin{aligned} t^\star &= t+\varepsilon\varrho^{(t)}(\mathtt{x},t,\psi)+O(\varepsilon^2),\\ \mathtt{x}^\star &= \mathtt{x}+\varepsilon\varrho^{(\mathtt{x})}(\mathtt{x},t,\psi)+O(\varepsilon^2),\\ \psi^\star &= \psi+\varepsilon\phi(\mathtt{x},t,\psi)+O(\varepsilon^2), \end{aligned} \tag{13}$$

where $\varepsilon << 1$ is the Lie group parameter and its Lie point symmetry generator has the following form:

$$\mathtt{S} = \varrho^{(t)}\frac{\partial}{\partial t}+\varrho^{(\mathtt{x})}\frac{\partial}{\partial \mathtt{x}}+\phi\frac{\partial}{\partial\psi}. \tag{14}$$

In the Jet space, similarly,

$$\begin{aligned} \frac{\partial^\sigma\psi^\star}{\partial t^{\star\sigma}} &= \frac{\partial^\sigma\psi}{\partial t^\sigma}+\varepsilon\phi_\sigma^0(\mathtt{x},t,\psi)+O(\varepsilon^2),\\ \frac{\partial\psi^\star}{\partial \mathtt{x}^\star} &= \frac{\partial\psi}{\partial\mathtt{x}}+\varepsilon\phi^{\mathtt{x}}(\mathtt{x},t,\psi)+O(\varepsilon^2),\\ \frac{\partial^2\psi^\star}{\partial \mathtt{x}^{\star 2}} &= \frac{\partial^2\psi}{\partial\mathtt{x}^2}+\varepsilon\phi^{\mathtt{xx}}(\mathtt{x},t,\psi)+O(\varepsilon^2),\\ &\vdots \end{aligned} \tag{15}$$

so that the symmetry generator associated to (12) is prolongation of (14) written as,

$$\mathtt{S}^{\sigma,p} = \mathtt{S}+\phi_\sigma^0\frac{\partial}{\partial t^\sigma\psi}+\phi^{\mathtt{x}}\frac{\partial}{\partial\psi_{\mathtt{x}}}+\phi^{\mathtt{xx}}\frac{\partial}{\partial\psi_{\mathtt{xx}}}+\ldots, \tag{16}$$

where p is highest order of (12) and,

$$\begin{aligned} \phi^{\mathtt{x}} &= \mathtt{D}_{\mathtt{x}}(\phi)-\psi_t\mathtt{D}_{\mathtt{x}}(\varrho^{(t)})-\psi_{\mathtt{x}}\mathtt{D}_{\mathtt{x}}(\varrho^{(\mathtt{x})}),\\ \phi^{\mathtt{xx}} &= \mathtt{D}_{\mathtt{x}}(\phi^{\mathtt{x}})-\psi_{t\mathtt{x}}\mathtt{D}_{\mathtt{x}}(\varrho^{(t)})-\psi_{\mathtt{xx}}\mathtt{D}_{\mathtt{x}}(\varrho^{(\mathtt{x})}),\\ \phi^{\mathtt{xxx}} &= \mathtt{D}_{\mathtt{x}}(\phi^{\mathtt{xx}})-\psi_{t\mathtt{xx}}\mathtt{D}_{\mathtt{x}}(\varrho^{(t)})-\psi_{\mathtt{xxx}}\mathtt{D}_{\mathtt{x}}(\varrho^{(\mathtt{x})}), \end{aligned} \tag{17}$$

where the total differential operator $\mathtt{D}_j$ is defined as,

$$\mathtt{D}_j = \frac{\partial}{\partial x^j}+\psi_j\frac{\partial}{\partial\psi}+\psi_{jl}\frac{\partial}{\partial\psi_l}+\ldots, \qquad j,l=1,2,3. \tag{18}$$

Also, generator in (16) is a point symmetry of (12) iff,

$$\mathtt{S}^{\sigma,p}(\Delta)|_{\Delta=0}=0, \tag{19}$$

where

$$\Delta := \partial_t^\sigma\psi(\mathtt{x},t)-G(\mathtt{x},t,\psi,\psi_{\mathtt{x}},,\psi_{\mathtt{xx}},\ldots).$$

The invariance condition,

$$\varrho^{(t)}(\mathtt{x}, t, \psi)|_{t=0} = 0, \tag{20}$$

is necessary to the transformation (13).

Now, extended infinitesimals of fractional order is given as,

$$\begin{aligned}\phi^0_\sigma = \frac{\partial^\sigma \phi}{\partial t^\sigma} + \left(\phi_\psi - \sigma \mathtt{D}_t(\varrho^{(t)})\right)\frac{\partial^\sigma \psi}{\partial t^\sigma} - \psi\frac{\partial^\sigma \phi_\psi}{\partial t^\sigma} + \sum_{p=1}^{\infty}\left[\binom{\sigma}{p}\frac{\partial^\sigma \phi_\psi}{\partial t^\sigma} - \binom{\sigma}{p+1}\mathtt{D}_t^{n+1}(\varrho^{(t)})\right]\mathtt{D}_t^{\sigma-p}(\psi) \\ - \sum_{p=1}^{\infty}\binom{\sigma}{p}\mathtt{D}_t^p(\varrho^{(\mathtt{x})})\mathtt{D}_t^{\sigma-p}(\psi_\mathtt{x}) + \upsilon,\end{aligned} \tag{21}$$

where

$$\binom{\sigma}{p} = \frac{(-1)^{p-1}\sigma\Gamma(p-\sigma)}{\Gamma(1-\sigma)\Gamma(p+1)},$$

and

$$\upsilon = \sum_{p=2}^{\infty}\sum_{q=2}^{p}\sum_{r=2}^{q}\sum_{s=0}^{r-1}\binom{\sigma}{p}\binom{p}{q}\binom{r}{s}\frac{1}{r!}\frac{t^{p-\sigma}}{\Gamma(p+1-\sigma)}(-1)^s\psi^s\frac{\partial^q}{\partial t^q}[\psi^{r-s}]\frac{\partial^{p-q+r}}{\partial t^{p-q}\partial\psi^r}. \tag{22}$$

3 Invariance Analysis of (5)

Suppose that (5) remains invariant under transformations (13) and (15), we get,

$$\mathtt{D}^\sigma_{t^\star}\psi^\star - \frac{2}{\mathtt{x}^\star}\psi^\star_{\mathtt{x}^\star} - \psi^\star_{\mathtt{x}^\star\mathtt{x}^\star} + \mathcal{K}(\psi^\star)\psi^\star = 0. \tag{23}$$

Based on prolongation, we obtain,

$$\phi^0_\sigma - \frac{2}{\mathtt{x}}\phi^\mathtt{x} + \frac{2}{\mathtt{x}^2}\varrho^{(\mathtt{x})}\psi_\mathtt{x} - \phi^{\mathtt{xx}} + \phi\mathcal{K}(\psi) + \phi\psi\mathcal{K}'(\psi) = 0. \tag{24}$$

By putting the values of ϕ^0_σ, $\phi^\mathtt{x}$ and $\phi^{\mathtt{xx}}$ in above equation and comparing the coefficients of terms involving ψ and derivatives of ψ, we obtain the below determining system,

$$\varrho^{(t)}_\psi = \varrho^{(\mathtt{x})}_\psi = \varrho^{(t)}_\mathtt{x} = \varrho^{(\mathtt{x})}_t = \phi_{\psi\psi} = 0, \tag{25a}$$

$$2\varrho^{(\mathtt{x})}_\mathtt{x}\sigma\varrho^{(t)}_t = 0, \tag{25b}$$

$$\frac{2}{\mathtt{x}}(\varrho^{(\mathtt{x})}_\mathtt{x} - \sigma\varrho^{(t)}_t) - 2\phi_{\mathtt{x}\psi} + \varrho^{(\mathtt{x})}_{\mathtt{xx}} + \frac{2}{\mathtt{x}^2}\varrho^{(\mathtt{x})} = 0, \tag{25c}$$

$$\binom{\sigma}{p}\partial^\sigma_t\phi - \binom{\sigma}{p+1}D^{p+1}_t(\varrho^{(t)}) = 0, \qquad p = 1, 2, 3, \dots, \tag{25d}$$

$$\binom{\sigma}{p}\mathtt{D}^p_t(\varrho^{(\mathtt{x})}) = 0, \qquad p = 1, 2, 3, \dots, \tag{25e}$$

$$\partial^\sigma_t\phi - \psi\partial^\sigma_t\phi_u - \frac{2}{\mathtt{x}}\phi_\mathtt{x}\phi_{\mathtt{xx}} + \phi(\mathcal{K}(\psi) + \psi\mathcal{K}'(\psi)) - \mathcal{K}(\psi)(\phi_u - \sigma\varrho^{(t)}_t) = 0. \tag{25f}$$

After solving above equations, we initially get the following infinitesimals:

$$\varrho^{(\mathtt{x})} = b_1\mathtt{x} + b_2, \qquad \varrho^{(t)} = \frac{2}{\sigma}b_1 t + b_3, \qquad \phi = \left(-\frac{b_2}{\mathtt{x}} + b_4\right)\psi + B(\mathtt{x}, t),$$

where b_1, b_2, b_3 and b_4 are constants. Putting $\varrho^{(\mathtt{x})}$, ϱ^t and ϕ in (25f), we get,

$$\partial_t^\sigma B - \frac{2}{\mathtt{x}}B_{\mathtt{x}} - B_{\mathtt{xx}} + \mathcal{K}(\psi)(B + 2b_1\psi) + \mathcal{K}^{'}(\psi)\left(\left(\frac{-b_2}{\mathtt{x}} + b_4\right)\psi^2 + B\psi\right) = 0. \tag{26}$$

After comparing coefficients in the above equation and solving the system, we get the following infinitesimals:

$$\varrho^{(\mathtt{x})} = 0, \qquad \varrho^{(t)} = b_3, \qquad \phi = 0.$$

Therefore, we obtain the translation symmetry,

$$\mathtt{S}_1 = \frac{\partial}{\partial t}. \tag{27}$$

However, due to (20), time translation does not feature as a symmetry.

4 Classification

We now perform a further classification to find the possible for of $B(\mathtt{x}, t)$ in ϕ. For this purpose, we differentiate (26) with respect to ψ and get,

$$\frac{\mathcal{K}^{'}(\psi)}{\mathcal{K}(\psi)}\left[2\left(\frac{-b_2}{\mathtt{x}} + b_4\right)\psi + 2B + 2b_1\psi\right] + \frac{\mathcal{K}^{''}(\psi)}{\mathcal{K}(\psi)}\left[\left(\frac{-b_2}{\mathtt{x}} + b_4\right)\psi^2 + B\psi\right] = -2b_1. \tag{28}$$

Case 1: If $B(\mathtt{x}, t) = 0$ in (26), then we get,

$$\left[\frac{\mathcal{K}^{'}(\psi)}{\mathcal{K}(\psi)}\psi\right]_\psi \left(\frac{-b_2}{\mathtt{x}} + b_4\right) = 0. \tag{29}$$

Subcase 1.1: If,

$$\frac{-b_2}{\mathtt{x}} + b_4 = 0,$$

this implies,

$$\varrho^{(\mathtt{x})} = b_1\mathtt{x}, \qquad \varrho^{(t)} = \frac{2}{\sigma}b_1 t + b_3, \qquad \phi = 0.$$

Therefore, we get the scaling symmetry,

$$\mathtt{S}_1 = \mathtt{x}\frac{\partial}{\partial \mathtt{x}} + \frac{2}{\sigma}t\frac{\partial}{\partial t}. \tag{30}$$

Subcase 1.2: If,

$$\left[\frac{\mathcal{K}^{'}(\psi)}{\mathcal{K}(\psi)}\psi\right]_\psi = 0,$$

after solving above equation, we get,

$$\mathcal{K}(\psi) = c_2\psi^{c_1}. \tag{31}$$

Now, putting $B(\mathtt{x}, t) = 0$ and $\mathcal{K}(\psi) = c_2\psi^{c_1}$ in (26), we have,

$$c_2\left[\left(\frac{-b_2}{\mathtt{x}} + b_4\right)c_1 + 2b_1\right] = 0.$$

Finally, we get,

$$\varrho^{(\mathtt{x})} = b_1\mathtt{x}, \qquad \varrho^{(t)} = \frac{2}{\sigma}b_1 t + b_3, \qquad \phi = \frac{-2b_1}{c_1}\psi,$$

and the corresponding symmetry is,

$$\mathtt{S}_1 = \mathtt{x}\frac{\partial}{\partial \mathtt{x}} + \frac{2}{\sigma}t\frac{\partial}{\partial t} - \frac{2}{c_1}\psi\frac{\partial}{\partial \psi}. \tag{32}$$

Case 2: For arbitrary $B(\mathtt{x}, t)$, we consider (28). Putting $\mathcal{K}(\psi) = c\psi + d$ in (28), we get,

$$\begin{aligned} &\frac{c}{c\psi + d}\left[2\left(\frac{-b_2}{\mathtt{x}} + b_4\right)\psi + 2B + 2b_1\psi\right] = -2b_1, \\ &2c\psi\left[\left(\frac{-b_2}{\mathtt{x}} + b_4\right) + 2b_1\right] + 2Bc + 2b_1 d = 0. \end{aligned} \tag{33}$$

After solving the above equations, we get the below infinitesimals,

$$\varrho^{(\mathtt{x})} = b_1\mathtt{x}, \qquad \varrho^{(t)} = \frac{2}{\sigma}b_1 t + b_3, \qquad \phi = -2b_1\psi - \frac{b_1 d}{c},$$

with

$$\mathtt{S}_1 = \mathtt{x}\frac{\partial}{\partial \mathtt{x}} + \frac{2}{\sigma}t\frac{\partial}{\partial t} - \left(2\psi + \frac{d}{c}\right)\frac{\partial}{\partial \psi}. \tag{34}$$

Case 3: In this case, if we substitute $\mathcal{K}(\psi) = \psi^m$ in (28), we get,

$$\frac{m\psi^{m-1}}{\psi^m}\left[2\left(\frac{-b_2}{\mathtt{x}} + b_4\right)\psi + 2B + 2b_1\psi\right] + \frac{m(m-1)\psi^{m-2}}{\psi^m}\left[\left(\frac{-b_2}{\mathtt{x}} + b_4\right)\psi^2 + B\psi\right] = -2b_1. \tag{35}$$

So, after simplification and comparing coefficients, we get these equations,

$$2m\left(\frac{-b_2}{\mathtt{x}} + b_4\right) + m(m-1)\left(\frac{-b_2}{\mathtt{x}} + b_4\right) + 2mb_1 + 2b_1 = 0, \tag{36a}$$

$$m(m+1)B = 0. \tag{36b}$$

From (36b), we have three situations:

i. $m = 0$,
ii. $m = -1$,
iii. $B(\mathtt{x}, t) = 0$.

Subcase 3.1: If $m = 0$, then $\mathcal{K}(\psi) = \psi^0 = 1$. And putting $m = 0$ in (36a) gives $b_1 = 0$. Now, substituting $b_1 = 0$ and $\mathcal{K}(\psi) = \psi^0 = 1$ in (26), we have,

$$\partial_t^\sigma B - \frac{2}{\mathrm{x}} B_{\mathrm{x}} - B_{\mathrm{xx}} + B = 0, \tag{37}$$

so that an 'infinite dimenesional' is of the form $B(x,t)\dfrac{\partial}{\partial u}$ where $B(x,t)$ satisfies (37). For example,

(a) If $B(\mathrm{x}, t) = B(t)$, then (37) becomes,

$$\partial_t^\sigma B + B = 0.$$

If $\sigma = \dfrac{1}{2}$, we have,

$$\begin{aligned} \partial_t^{\frac{1}{2}} B &= -B, \\ \partial_t^{\frac{1}{2}} (\partial_t^{\frac{1}{2}} B) &= -\partial_t^{\frac{1}{2}} (B). \end{aligned} \tag{38}$$

After solving the above equations, we get $B(t) = b_5 \exp t$. Thus, we have the following infinitesimal:

$$\varrho^{(\mathrm{x})} = b_2, \qquad \varrho^{(t)} = b_3, \qquad \phi = \left(-\frac{b_2}{\mathrm{x}} + b_4 \right) \psi + b_5 \exp t,$$

with corresponding symmetries,

$$\mathtt{S}_1 = \frac{\partial}{\partial \mathrm{x}} - \frac{\psi}{\mathrm{x}} \frac{\partial}{\partial \psi}, \qquad \mathtt{S}_2 = \psi \frac{\partial}{\partial \psi}, \qquad \mathtt{S}_3 = \exp t \frac{\partial}{\partial \psi}. \tag{39}$$

(b) If $B(\mathrm{x}, t) = B(\mathrm{x}) = B$, then (37) becomes,

$$B_{\mathrm{xx}} + \frac{2}{\mathrm{x}} B_{\mathrm{x}} - B = 0.$$

By the first integral,

$$\mathcal{I} = -\mathrm{x} \sinh \mathrm{x} B + B_{\mathrm{x}} \mathrm{x} \cosh \mathrm{x} + B \cosh \mathrm{x},$$

the reduced first order ordinary differential equation is,

$$\begin{aligned} -\mathrm{x} \sinh \mathrm{x} B + B_{\mathrm{x}} \mathrm{x} \cosh \mathrm{x} + B \cosh \mathrm{x} &= k_1, \\ \text{or} \quad B_{\mathrm{x}} + \left(\frac{1}{\mathrm{x}} - \tan \mathrm{x} \right) B &= \frac{k_1}{\mathrm{x} \cosh \mathrm{x}}. \end{aligned} \tag{40}$$

After solving the above equation, we have,

$$B = B(\mathrm{x}) = \frac{k_1}{\cosh \mathrm{x}} + \frac{k_2}{\mathrm{x} \cosh \mathrm{x}}.$$

We get these corresponding infinitesimals,

$$\varrho^{(\mathrm{x})} = b_2, \qquad \varrho^{(t)} = b_3, \qquad \phi = \left(-\frac{b_2}{\mathrm{x}} + b_4 \right) \psi + \frac{k_1}{\cosh \mathrm{x}} + \frac{k_2}{\mathrm{x} \cosh \mathrm{x}},$$

and symmetries,

$$\mathtt{S}_1 = \frac{\partial}{\partial \mathrm{x}} - \frac{\psi}{\mathrm{x}} \frac{\partial}{\partial \psi}, \quad \mathtt{S}_2 = \psi \frac{\partial}{\partial \psi}, \quad \mathtt{S}_3 = \frac{1}{\cosh \mathrm{x}} \frac{\partial}{\partial \psi}, \quad \mathtt{S}_4 = \frac{1}{\mathrm{x} \cosh \mathrm{x}} \frac{\partial}{\partial \psi}. \tag{41}$$

5 Symmetry Reductions

In this section, we perform reductions of the model for some cases.

Reduction by symmetry $\mathtt{S}_1$ in (32):
For the symmetry $\mathtt{S}_1$, we have following characteristic equation:

$$\frac{\sigma dt}{2t} = \frac{dx}{x} = \frac{c_1}{2}\frac{d\psi}{-\psi},$$

and we get following similarity transformations:

$$\vartheta = \mathtt{x}t^{-\frac{\sigma}{3}}, \qquad \psi(\mathtt{x}, t) = t^{-\frac{\sigma}{c_1}} F(\vartheta), \tag{42}$$

where F is a function of ϑ. After putting (42) in (5), we get non-linear fractional ODE. Following this, we have,

Theorem 1. *Equation (5) reduces to the form,*

$$\left(\mathcal{P}_{\frac{2}{\sigma}}^{1-(\frac{c_1\sigma+\sigma}{c_1}),\sigma} F\right)(\vartheta) - \frac{2}{\vartheta}F_{\vartheta} - F_{\vartheta\vartheta} - c_2 F^{c_1} = 0, \tag{43}$$

where the EK fractional differential operator defined in (2).

Proof 1. *Suppose that $p-1<\sigma<p$, $p = 1, 2, 3, \ldots$ and under the similarity transformation (42), Definition 1 turns into,*

$$\frac{\partial^\sigma \psi}{\partial t^\sigma} = \frac{\partial^p}{\partial t^p}\left[\frac{1}{\Gamma(p-\sigma)}\int_0^t (t-\varkappa)^{p-\sigma-1}\varkappa^{\frac{-\sigma}{c_1}} F\left(x\varkappa^{\frac{-\sigma}{2}}\right) d\varkappa\right]. \tag{44}$$

Substituting $\mathtt{w} = t/\varkappa \Rightarrow d\varkappa = -(t/\mathtt{w}^2)d\mathtt{w}$, then the above equation becomes,

$$\frac{\partial^\sigma \psi}{\partial t^\sigma} = \frac{\partial^p}{\partial t^p}\left[t^{p-\frac{4\sigma}{3}}\frac{1}{\Gamma(p-\sigma)}\int_1^\infty (\mathtt{w}-1)^{p-\sigma-1}\mathtt{w}^{-(1-\frac{\sigma}{c_1}+p-\sigma)} F\left(\vartheta \mathtt{w}^{\frac{\sigma}{2}}\right) d\mathtt{w}\right], \tag{45}$$

keeping (11) in mind, we have,

$$\frac{\partial^\sigma \psi}{\partial t^\sigma} = \frac{\partial^p}{\partial t^p}\left[t^{p-(\frac{c_1\sigma+\sigma}{c_1})}\left(\mathcal{K}_{\frac{2}{\sigma}}^{1-\frac{\sigma}{c_1},p-\sigma} F\right)(\vartheta)\right]. \tag{46}$$

Now, simplifying the right hand side of above equation, we get,

$$\begin{aligned}
\frac{\partial^\sigma \psi}{\partial t^\sigma} &= \frac{\partial^{p-1}}{\partial t^{p-1}}\left[\frac{\partial}{\partial t}\left(t^{p-(\frac{c_1\sigma+\sigma}{c_1})}\left(\mathcal{K}_{\frac{2}{\sigma}}^{1-\frac{\sigma}{c_1},p-\sigma} F\right)(\vartheta)\right)\right],\\
&= \frac{\partial^{p-1}}{\partial t^{p-1}}\left[t^{p-1-(\frac{c_1\sigma+\sigma}{c_1})}\left(p-(\frac{c_1\sigma+\sigma}{c_1})-\frac{\sigma}{2}\vartheta\frac{d}{d\vartheta}\right)\left(\mathcal{K}_{\frac{2}{\sigma}}^{1-\frac{\sigma}{c_1},p-\sigma} F\right)(\vartheta)\right],\\
&\vdots\\
&= t^{-(\frac{c_1\sigma+\sigma}{c_1})}\prod_{j=0}^{p-1}\left[\left(1-(\frac{c_1\sigma+\sigma}{c_1})+j-\frac{\sigma}{2}\vartheta\frac{d}{d\vartheta}\right)\left(\mathcal{K}_{\frac{2}{\sigma}}^{1-\frac{\sigma}{c_1},p-\sigma} F\right)(\vartheta)\right].
\end{aligned} \tag{47}$$

Applying (9) *in above equations, we get,*

$$\frac{\partial^p}{\partial t^p}\left[t^{p-(\frac{c_1\sigma+\sigma}{c_1})}\left(\mathcal{K}_{\frac{2}{\sigma}}^{1-\frac{\sigma}{c_1},p-\sigma}\digamma\right)(\vartheta)\right]=t^{-(\frac{c_1\sigma+\sigma}{c_1})}\left(\mathcal{P}_{\frac{2}{\sigma}}^{1-(\frac{c_1\sigma+\sigma}{c_1}),\sigma}\digamma\right)(\varkappa). \tag{48}$$

Inserting (48) *into* (46), *we obtained,*

$$\frac{\partial^\sigma\psi}{\partial t^\sigma}=t^{-(\frac{c_1\sigma+\sigma}{c_1})}\left(\mathcal{P}_{\frac{2}{\sigma}}^{1-(\frac{c_1\sigma+\sigma}{c_1}),\sigma}\digamma\right)(\vartheta). \tag{49}$$

Hence, (5) *is converted into the following fractional ODE:*

$$\left(\mathcal{P}_{\frac{2}{\sigma}}^{1-(\frac{c_1\sigma+\sigma}{c_1}),\sigma}\digamma\right)(\vartheta)-\frac{2}{\vartheta}\digamma_{\vartheta}-\digamma_{\vartheta\vartheta}-c_2\digamma^{c_1}=0. \tag{50}$$

Equation (43) will be dealt with in Section 6.

Reduction by symmetry $\mathtt{S}_1$ in (39):
For this symmetry, the corresponding characteristic equation is,

$$\frac{dt}{0}=\frac{dx}{1}=\frac{\mathtt{x}d\psi}{-\psi},$$

so that the similarity transformation and variables are,

$$\zeta=t, \tag{51}$$

$$\psi(\mathtt{x},t)=\frac{\mathcal{G}(\zeta)}{\mathtt{x}}, \tag{52}$$

where $\mathcal{G}$ is a function of ζ. After putting (51) in (5), we get this form of fractional ODE.

$$\partial_t^\sigma\mathcal{G}(\zeta)+\mathcal{G}(\zeta)=0.$$

As $\zeta=t$, so the above equation can be written as,

$$\partial_t^\sigma\mathcal{G}(t)+\mathcal{G}(t)=0.$$

For the special case $\sigma=\frac{1}{2}$,

$$\partial_t^{\frac{1}{2}}\mathcal{G}(t)=-\mathcal{G}(t),\quad \text{so that}\quad \partial_t^{\frac{1}{2}}(\partial_t^{\frac{1}{2}}\mathcal{G}(t))=-\partial_t^{\frac{1}{2}}(\mathcal{G}(t)), \tag{53}$$

from which we get $\mathcal{G}(t)=b_6\exp t$. Hence,

$$\psi(\mathtt{x},t)=\frac{b_6\exp t}{\mathtt{x}}, \tag{54}$$

see Figure 1, i.e.,

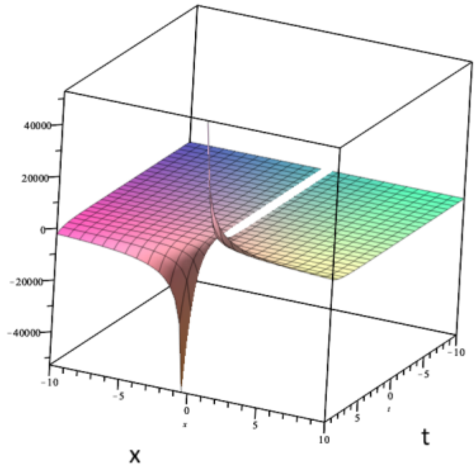

Figure 1: The profile of (54).

Note: Reduction by symmetry $\mathtt{S}_1$ in (41) lead to the same solution.

6 Series Solutions for (5)

In this section, we find the explicit series solutions of (5) via its reduced form (43). Suppose the solution of (43) is,

$$\mathcal{N}(\vartheta) = \sum_{k=0}^{\infty} d_k \vartheta^{k+r}, \tag{55}$$

where d_k are coefficients of the above series. Now, putting (55) into (43), we get,

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)} d_k \vartheta^{k+r} - \sum_{k=0}^{\infty}(k+r)d_k\vartheta^{k+r-2} - \sum_{k=0}^{\infty}(k+r)(k+r-1)d_k\vartheta^{k+r-2} \\ - \sum_{k=0}^{\infty}\sum_{l=0}^{k} d_l d_{k-l}\vartheta^{k+r} = 0.\end{aligned} \tag{56}$$

This implies,

$$-\sum_{k=0}^{\infty}(k+r)^2 d_k \vartheta^{k+r-2} + \sum_{k=0}^{\infty}\left[\frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)} d_k - \sum_{l=0}^{k} d_l d_{k-l}\right]\vartheta^{k+r} = 0, \tag{57}$$

expanding the first term for $k = 0, 1$,

$$-r^2 d_0 \vartheta^{r-2} - (1+r)^2 d_1 \vartheta^{r-1} - \sum_{k=2}^{\infty} (k+r)^2 d_k \vartheta^{k+r-2} + \sum_{k=0}^{\infty} \left[\frac{\Gamma\left(2 - 2\sigma - \frac{k\sigma}{2}\right)}{\Gamma\left(2 - \sigma - \frac{k\sigma}{2}\right)} d_k - \sum_{l=0}^{k} d_l d_{k-l} \right] \vartheta^{k+r} = 0,$$

$$\vartheta^r \left[-r^2 d_0 \vartheta^{-2} - (1 + 2r + r^2) d_1 \vartheta^{-1} - \sum_{k=0}^{\infty} \left[(k+2+r)^2 d_{k+2} + \frac{\Gamma\left(2 - 2\sigma - \frac{k\sigma}{2}\right)}{\Gamma\left(2 - \sigma - \frac{k\sigma}{2}\right)} d_k - \sum_{l=0}^{k} d_l d_{k-l} \right] \vartheta^k \right] = 0, \tag{58}$$

where d_0 and d_1 are arbitrary constants. Now, comparing coefficients,

$$r^2 = 0, \tag{59a}$$

$$1 + 2r + r^2 = 0, \tag{59b}$$

$$-(k+2+r)^2 d_{k+2} + \frac{\Gamma\left(2 - 2\sigma - \frac{k\sigma}{2}\right)}{\Gamma\left(2 - \sigma - \frac{k\sigma}{2}\right)} d_k - \sum_{l=0}^{k} d_l d_{k-l} = 0, \tag{59c}$$

putting $k = 0$ in (59c),

$$-(2+r)^2 d_2 + \frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)} d_0 - d_0^2 = 0, \tag{60}$$

putting $r = 0$ from (59a) in (60), we get,

$$-4d_2 + \frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)} d_0 - d_0^2 = 0,$$

above equation implies,

$$d_2 = \frac{1}{4}\left(\frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)} d_0 - d_0^2 \right). \tag{61}$$

So, for $k \geq 1$, (59c) gives,

$$d_{k+2} = \frac{1}{k+2} \left(\frac{\Gamma\left(2 - 2\sigma - \frac{k\sigma}{2}\right)}{\Gamma\left(2 - \sigma - \frac{k\sigma}{2}\right)} d_k - \sum_{l=0}^{k} d_l d_{k-l} \right). \tag{62}$$

Now, with $r = -1$ from (59b) in (60), we have,

$$-d_2 + \frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)} d_0 - d_0^2 = 0,$$

which gives,

$$d_2 = \frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)} d_0 - d_0^2. \tag{63}$$

Therefore, for $k \geq 1$, (59c) implies,

$$d_{k+2} = \frac{1}{k-1}\left(\frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)}d_k - \sum_{l=0}^{k} d_l d_{k-l}\right). \tag{64}$$

So, we have two explicit solutions given below:

(i) For $r = 0$,

$$\mathcal{W}_1(\vartheta) = d_0 + d_1\vartheta + \frac{1}{4}\left(\frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)}d_0 - d_0^2\right)\vartheta^2 + \sum_{k=3}^{\infty}\left[\frac{1}{k+2}\left(\frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)}d_k - \sum_{l=0}^{k} d_l d_{k-l}\right)\right]\vartheta^{k+2}. \tag{65}$$

Therefore, we acquire the following explicit power series of (5) corresponding to above solution,

$$\psi_1(\mathtt{x},t) = d_0 t^{-\sigma} + d_1 \mathtt{x} t^{\frac{-3\sigma}{2}} + \frac{1}{4}\left(\frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)}d_0 - d_0^2\right)\mathtt{x}^2 t^{-2\sigma} + \sum_{k=3}^{\infty}\left[\frac{1}{k+2}\left(\frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)}d_k - \sum_{l=0}^{k} d_l d_{k-l}\right)\right]\mathtt{x}^{k+2} t^{\frac{-(k+4)\sigma}{2}}. \tag{66}$$

(ii) Similarly, for $r = -1$,

$$\mathcal{W}_2(\vartheta) = d_0\vartheta^{-1} + d_1\vartheta^0 + \left(\frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)}d_0 - d_0^2\right)\vartheta + \sum_{k=3}^{\infty}\left[\frac{1}{k-1}\left(\frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)}d_k - \sum_{l=0}^{k} d_l d_{k-l}\right)\right]\vartheta^{k+1}. \tag{67}$$

Hence, we get the below explicit power series corresponding to (67),

$$\psi_2(\mathtt{x},t) = d_0 \mathtt{x}^{-1} t^{\frac{-\sigma}{2}} + d_1 t^{-\sigma} + \left(\frac{\Gamma(2-2\sigma)}{\Gamma(2-\sigma)}d_0 - d_0^2\right)\mathtt{x} t^{\frac{-3\sigma}{2}} + \sum_{k=3}^{\infty}\left[\frac{1}{k-1}\left(\frac{\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)}{\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)}d_k - \sum_{l=0}^{k} d_l d_{k-l}\right)\right]\mathtt{x}^{k+1} t^{\frac{-(k+3)\sigma}{2}}. \tag{68}$$

6.1 Convergence analysis of the explicit series solution

We examine the convergence of the explicit solution (55) in this analysis, using coefficients from (61) and (62). It is possible to expand (62) as,

$$|d_{k+2}| \leq \frac{\left|\Gamma\left(2-2\sigma-\dfrac{k\sigma}{2}\right)\right|}{\left|\Gamma\left(2-\sigma-\dfrac{k\sigma}{2}\right)\right|}|d_k| + \sum_{l=0}^{k}|d_l||d_{k-l}|, \tag{69}$$

but, $\dfrac{\Gamma(k)}{\Gamma(j)<1}$ for any $k(j)$ numbers. Therefore, (69) convert into,

$$|d_{k+2}| \leq |d_k| + \sum_{l=0}^{k}|d_l||d_{k-l}|. \tag{70}$$

To proceed further, we consider another power series suggestion as,

$$\mathcal{M}(\vartheta) = \sum_{k=0}^{\infty} m_k \vartheta^{k+r}, \tag{71}$$

with $m_i = |d_i|$ with $i = 0$. So, one can write,

$$m_{k+2} = m_k + \sum_{l=0}^{k} m_l m_{k-l}, \tag{72}$$

where $k = 0, 1, 2, \ldots$ and it is easily noted that,

$$|d_{k+2}| \leq m_{k+2} \quad \Rightarrow \quad |d_k| \leq m_k.$$

Corresponding to above result, we may conclude that the series provided by (71) is a majorant series of (55). Then, by computations, (71) can be written as,

$$\begin{aligned}\mathcal{M}(\vartheta) &= m_0 + m_1\vartheta + \sum_{k=2}^{\infty} m_k\vartheta^{k+r},\\ &= m_0 + m_1\vartheta + \left[\sum_{k=0}^{\infty}\left(m_k + \sum_{l=0}^{k} m_l m_{k-l}\right)\vartheta^k\right]\vartheta^r,\\ &= m_0 + m_1\vartheta + [\mathcal{M}\vartheta + \mathcal{M}^2\vartheta^2]\vartheta^r.\end{aligned} \tag{73}$$

Next, we will show that series $\mathcal{M} = \mathcal{M}(\vartheta)$ has positive radius of convergence. For this purpose, we will consider an implicit functional theorem regarding the variable ϑ by the form of,

$$\mathcal{R}(\vartheta, \mathcal{M}) = \mathcal{M} - m_0 - m_1\vartheta - [(\mathcal{M} - m_0)\vartheta^2 - (\mathcal{M}^2 - m_0^2)\vartheta^2]\vartheta^r, \tag{74}$$

since $\mathcal{R}$ is an analytic function in the neighborhood of $(0, m_0)$ where $\mathcal{R}(0, m_0) = 0$ and $\dfrac{\partial \mathcal{R}}{\partial \mathcal{M}}(0, m_0) \neq 0$. Hence, by implicit functional theorem, we reach convergence.

6.2 The physical explanation of the explicit solution for brain-tumor (5)

We plot various dimensional graphics of solutions (66) and (68) displayed in Figures 2-5 using sample values for parameters in order to analyze the properties of power series solutions.

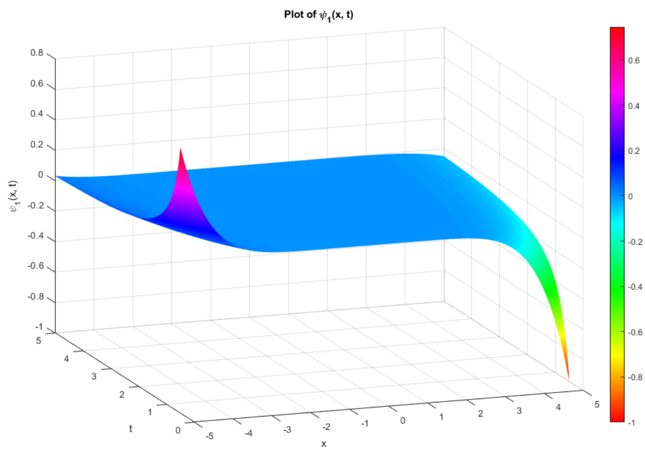

Figure 2: The visualization of profile (66) of (5) by setting $k = 5$ with constants $d_0 = 1$, $d_1 = 1$ and $\sigma = 0.3$.

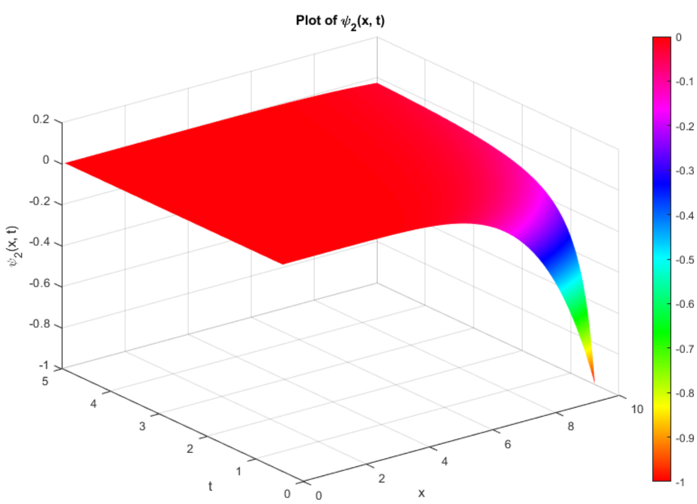

Figure 3: The visualization of solution (68) of (5) by fixing $k = 5$, $d_0 = 1$, $d_1 = 0.5$ and $\sigma = 0.4$.

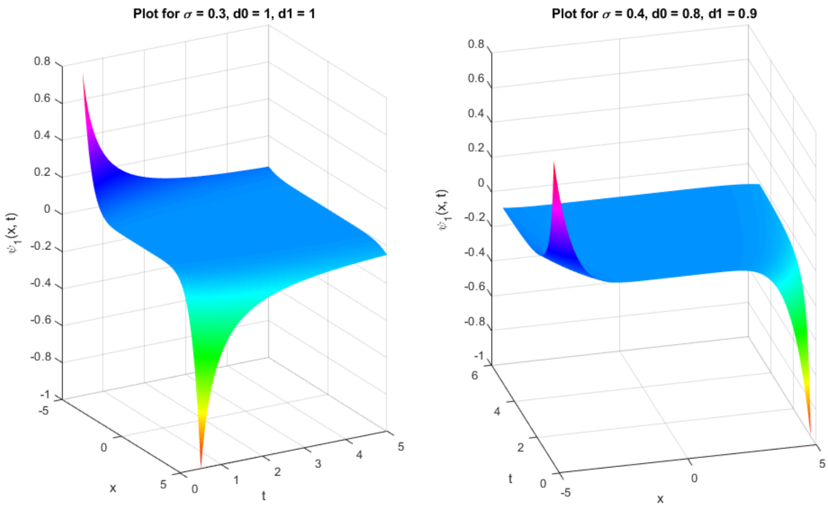

Figure 4: The visualization of profile (66) of (5) by setting $k = 5$, $d_0 = 0.8,\ 1$, $d_1 = 0.9,\ 1$ and $\sigma = 0.3,\ 0.4$.

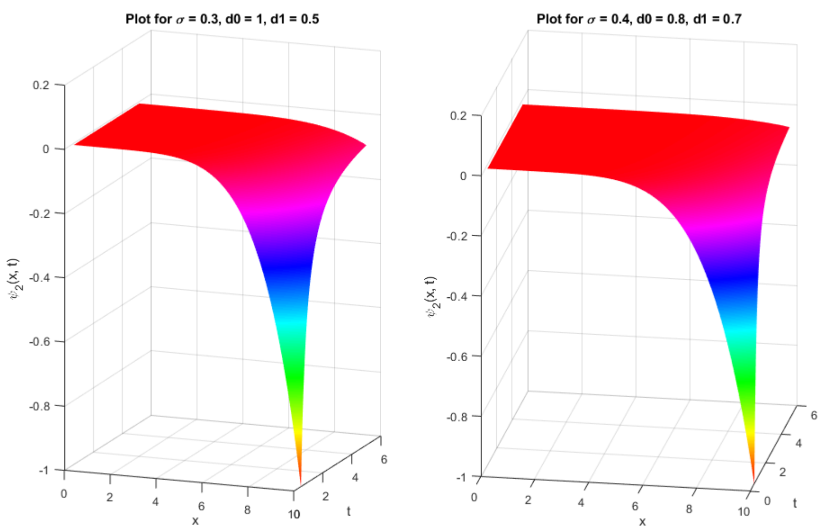

Figure 5: The visualization of solution (68) of (5) by fixing $k = 5$ with constants $d_0 = 0.8,\ 1$, $d_1 = 0.5,\ 0.7$ and $\sigma = 0.3,\ 0.4$.

The behaviour of the solutions are fairly consistent over the range of parameters considered; there is no blow up and the convergence is obviously apparent.

7 Conservation Laws For (5)

In this section, we will find the conservation laws for brain-tumor (5) based on Ibragimov's formal Lagrangian and Lie symmetries. A conservation law satisfies the following condition:

$$\Big[\mathtt{D}_t(\hat{\mathtt{T}}^t) + \mathtt{D}_{\mathtt{x}}(\hat{\mathtt{T}}^{\mathtt{x}})\Big]_{Eq(5)} = 0, \tag{75}$$

where $\hat{\mathtt{T}}^t$ and $\hat{\mathtt{T}}^{\mathtt{x}}$ are conserved quantities. Noether in [15] describes the conservation laws constructed for variational systems, Ibragimov [11] extends this to general systems and the authors in [9] applies these methods to FDEs, so the formal Lagrangian of (5) is given by,

$$\mathcal{L} = \nu(\mathtt{x}, t)\Big[\mathtt{D}_t^{\sigma}\psi - \frac{2}{\mathtt{x}}\psi_{\mathtt{x}} - \psi_{\mathtt{xx}} + \mathcal{K}(\psi)\psi\Big], \tag{76}$$

where $\nu(\mathtt{x}, t)$ is a pseudo dependent variable. By using (76), the action integral takes the below form,

$$\int_0^t \int_{\Omega} \mathcal{L}(\mathtt{x}, t, \psi, \nu, \mathtt{D}_t^{\sigma}\psi, \psi_{\mathtt{x}}, \psi_{\mathtt{xx}})dxdt. \tag{77}$$

Now, the Euler-Lagrange (EL) operator is defined by,

$$\mathcal{E} = \frac{\partial}{\partial\psi} - (\mathtt{D}_t^{\sigma})^* \frac{\partial}{\partial \mathtt{D}_t^{\sigma}\psi} - \mathtt{D}_{\mathtt{x}}\frac{\partial}{\partial\psi_{\mathtt{x}}} + \mathtt{D}_{\mathtt{x}}^2\frac{\partial}{\partial\psi_{\mathtt{xx}}}, \tag{78}$$

where $(\mathtt{D}_t^{\sigma})^*$ represents adjoint operator of $(\mathtt{D}_t^{\sigma})$ and the adjoint equation can be defined as EL equation,

$$\mathcal{E}\mathcal{L} = 0. \tag{79}$$

Adjoint operator $(\mathtt{D}_t^{\sigma})^*$ for RL is defined as,

$$(\mathtt{D}_t^{\sigma})^* = (-1)^p I_T^{p-\sigma}(\mathtt{D}_t^p) = {}_t^C\mathtt{D}_T^{\sigma}, \tag{80}$$

where $I_T^{p-\sigma}$ has following form,

$$I_T^{p-\sigma} h(\mathtt{x}, t) = \frac{1}{\Gamma(p-\sigma)}\int_t^{\tau} \frac{h(\mathtt{x}, \tau)}{(\tau - t)^{\sigma - m + 1}}d\tau. \tag{81}$$

Next, symmetry condition is given below,

$$\hat{\mathtt{S}} + \mathtt{D}_t(\varrho^t)\hat{I} + \mathtt{D}_{\mathtt{x}}(\varrho^{\mathtt{x}})\hat{I} = W\mathcal{E} + \mathtt{D}_t(\hat{\mathtt{T}}^t) + \mathtt{D}_{\mathtt{x}}(\hat{\mathtt{T}}^{\mathtt{x}}), \tag{82}$$

where $\hat{I}$ is an identity operator. So, $\hat{\mathtt{S}}$ is given by,

$$\hat{\mathtt{S}} = \varrho^t\frac{\partial}{\partial t} + \varrho^{\mathtt{x}}\frac{\partial}{\partial x} + \psi_{\sigma}^0\frac{\partial}{\partial D_t^{\sigma}\psi} + \psi^{\mathtt{x}}\frac{\partial}{\partial\psi_{\mathtt{x}}} + \psi^{\mathtt{xx}}\frac{\partial}{\partial\psi_{\mathtt{xx}}}, \tag{83}$$

and W is the characteristic function and it is written as $W = \psi - \varrho^t\psi_t - \varrho^{\mathtt{x}}\psi_{\mathtt{x}}$. By using RL-fractional derivative, conserved component $\hat{\mathtt{T}}^t$ is given as,

$$\hat{\mathtt{T}}^t = \varrho^t\mathcal{L} + \sum_{u=0}^{m-1}(-1)_0^u\mathtt{D}t^{\sigma-1-u}(W_{\iota})\mathtt{D}t^u\frac{\partial\mathcal{L}}{\partial_0 D_t^{\sigma}\psi} - (-1)^m J\Big(W_{\iota}, D_t^m\frac{\partial\mathcal{L}}{\partial_0 D_t^{\sigma}\psi}\Big), \tag{84}$$

where $J(.)$ is defined by,

$$J(f,g)=\frac{1}{\Gamma(m-\sigma)}\int_0^t\int_t^\tau \frac{f(\tau,x)g(\kappa,x)}{(\kappa-\tau)^{\sigma-m+1}}d\kappa d\tau, \tag{85}$$

and the flux components $\hat{\mathtt{T}}^{\mathtt{x}}$, for independent variables x defined as,

$$\begin{aligned}\hat{\mathtt{T}}^{\mathtt{x}}=\varrho^{\mathtt{x}}\mathcal{L}+W_\iota\left[\frac{\partial\mathcal{L}}{\partial\psi_j^\iota}-\mathtt{D}_j(\frac{\partial\mathcal{L}}{\partial\psi_{jk}^\iota})+\mathtt{D}_j\mathtt{D}_k(\frac{\partial\mathcal{L}}{\partial\psi_{jkl}^\iota})-\dots\right]+\mathtt{D}_j(W_\iota)\left[\frac{\partial\mathcal{L}}{\partial\psi_{jk}^\iota}-\mathtt{D}_k(\frac{\partial\mathcal{L}}{\partial\psi_{jkl}^\iota})+\dots\right]\\ +\mathtt{D}_j\mathtt{D}_k(W_\iota)\left[\frac{\partial\mathcal{L}}{\partial\psi_{jkl}^\iota}-\dots\right]+\dots,\end{aligned} \tag{86}$$

where $j,k,l=1,2$ and $\iota=1,2,3$.

Now, we will consider all the symmetries (30, 32, 34, 39, 41) to find their conserved vectors.

(a) In this phase, we will find the conserved vectors for symmetries (30) and (32).

i. Here, for symmetries (30), (76) will not change. For symmetry $\mathtt{S}_1$, $W_1=-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t$, $\varrho^{\mathtt{x}}=\mathtt{x}$, $\varrho^t=\frac{2}{\sigma}t$ and

$$\hat{\mathtt{T}}^t=\frac{2}{\sigma}t\mathcal{L}+\nu\mathtt{D}_t^{\sigma-1}\left(-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t\right)+J\left(-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t,\nu_t\right),$$

$$\hat{\mathtt{T}}^{\mathtt{x}}=\nu\left(\mathtt{x}\mathtt{D}_t^\sigma\psi+\mathtt{x}\mathcal{K}(\psi)\psi+\frac{4}{\sigma\mathtt{x}}t\psi_t+\psi_{\mathtt{x}}+\frac{2}{\sigma}t\psi_{\mathtt{xt}}\right)+\nu_{\mathtt{x}}\left(-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t\right).$$

ii. Here, for symmetries (32), (76) will convert into the below form,

$$\mathcal{L}=\nu(\mathtt{x},t)\left[\mathtt{D}_t^\sigma\psi-\frac{2}{\mathtt{x}}\psi_{\mathtt{x}}-\psi_{\mathtt{xx}}+c_2\psi^{c_1+1}\right], \tag{87}$$

Now, for $\mathtt{S}_1$, we have $W_1=-\frac{2}{c_1}\psi-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t$, $\varrho^{\mathtt{x}}=\mathtt{x}$ and $\varrho^t=\frac{2}{\sigma}t$ with the following conserved quantities:

$$\hat{\mathtt{T}}^t=\frac{2}{\sigma}t\mathcal{L}+\nu\mathtt{D}_t^{\sigma-1}\left(-\frac{2}{c_1}\psi-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t\right)+J\left(-\frac{2}{c_1}\psi-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t,\nu_t\right),$$

$$\begin{aligned}\hat{\mathtt{T}}^{\mathtt{x}}&=\nu\left(\mathtt{x}\mathtt{D}_t^\sigma\psi+\mathtt{x}c_2\psi^{c_1+1}+\frac{4}{c_1\mathtt{x}}\psi+\frac{4}{\sigma\mathtt{x}}t\psi_t+\frac{2}{c_1}\psi_{\mathtt{x}}+\psi_{\mathtt{x}}+\frac{2}{\sigma}t\psi_{\mathtt{xt}}\right)\\&\quad+\nu_{\mathtt{x}}\left(-\frac{2}{c_1}\psi-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t\right).\end{aligned}$$

(b) In this phase, for symmetries (34), (76) will change into the following form:

$$\mathcal{L}=\nu(\mathtt{x},t)\left[\mathtt{D}_t^\sigma\psi-\frac{2}{\mathtt{x}}\psi_{\mathtt{x}}-\psi_{\mathtt{xx}}+(c\psi+d)\psi\right], \tag{88}$$

But for $\mathtt{S}_1$, we have $W_1=-\left(2\psi+\frac{d}{c}\right)-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t$, $\varrho^{\mathtt{x}}=\mathtt{x}$ and $\varrho^t=\frac{2}{\sigma}t$ with following conserved components:

$$\hat{\mathtt{T}}^t=\frac{2}{\sigma}t\mathcal{L}+\nu\mathtt{D}_t^{\sigma-1}\left(-\left(2\psi+\frac{d}{c}\right)-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t\right)+J\left(-\left(2\psi+\frac{d}{c}\right)-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t,\nu_t\right),$$

$$\begin{aligned}\hat{\mathtt{T}}^{\mathtt{x}}&=\nu\left(\mathtt{x}\mathtt{D}_t^\sigma\psi+\mathtt{x}(c\psi^2+d\psi)+\frac{4}{\mathtt{x}}\psi+\frac{4}{\sigma\mathtt{x}}t\psi_t+3\psi_{\mathtt{x}}+\frac{2}{\sigma}t\psi_{\mathtt{xt}}\right)\\&\quad+\nu_{\mathtt{x}}\left(-\left(2\psi+\frac{d}{c}\right)-\mathtt{x}\psi_{\mathtt{x}}-\frac{2}{\sigma}t\psi_t\right).\end{aligned}$$

(c) In this phase, we will find conserved components for symmetries (39). Here, (76) will change into this form,

$$\mathcal{L} = \nu(\mathtt{x}, t)\Big[\mathtt{D}_t^\sigma \psi - \frac{2}{\mathtt{x}}\psi_{\mathtt{x}} - \psi_{\mathtt{xx}} + \psi\Big]. \tag{89}$$

Now, $(\hat{\mathtt{T}}^t, \hat{\mathtt{T}}^{\mathtt{x}})$ for the symmetries is given below:

i. For $\mathtt{S}_1$, we get $W_1 = -\frac{\psi}{\mathtt{x}} - \psi_{\mathtt{x}}$, $\varrho^{\mathtt{x}} = 1$ and $\phi = \frac{\psi}{\mathtt{x}}$ with below conserved components,

$$\hat{\mathtt{T}}^t = \nu\mathtt{D}_t^{\sigma-1}\Big(-\frac{\psi}{\mathtt{x}} - \psi_{\mathtt{x}}\Big) + J\Big(-\frac{\psi}{\mathtt{x}} - \psi_{\mathtt{x}}, \nu_t\Big),$$
$$\hat{\mathtt{T}}^{\mathtt{x}} = \nu\Big(\mathtt{D}_t^\sigma\psi + \psi + \frac{\psi}{\mathtt{x}^2} + \frac{2}{\mathtt{x}^2}\psi\Big) + \nu_{\mathtt{x}}\Big(-\frac{\psi}{\mathtt{x}} - \psi_{\mathtt{x}}\Big).$$

ii. For $\mathtt{S}_2$, we have $W_2 = \psi$, $\phi = \psi$ and,

$$\hat{\mathtt{T}}^t = \nu\mathtt{D}_t^{\sigma-1}(\psi) + J(\psi, \nu_t),$$
$$\hat{\mathtt{T}}^{\mathtt{x}} = \nu\Big(-\psi_{\mathtt{x}} - \frac{2}{\mathtt{x}}\psi\Big) + \nu_{\mathtt{x}}\psi.$$

iii. For $\mathtt{S}_3$, we have $W_3 = \exp t$, $\phi = \exp t$ and,

$$\hat{\mathtt{T}}^t = \nu\mathtt{D}_t^{\sigma-1}(\exp t) + J(\exp t, \nu_t),$$
$$\hat{\mathtt{T}}^{\mathtt{x}} = -\frac{2}{\mathtt{x}}\exp t\nu + \nu_{\mathtt{x}}\exp t.$$

(d) For symmetries (41) Lagrangian is same as (89) and noticeably first two symmetries in (41) are similar to the first two symmetries of (39). So, their corresponding conserved components will be similar to the components which we discussed in (c) above. Now, for remaining symmetries we have following conserved quantities:

i. For $\mathtt{S}_3$ in (41), we acquire $W_3 = \frac{1}{\cosh\mathtt{x}}$, $\phi = \frac{1}{\cosh\mathtt{x}}$ and,

$$\hat{\mathtt{T}}^t = \nu\mathtt{D}_t^{\sigma-1}\Big(\frac{1}{\cosh\mathtt{x}}\Big) + J\Big(\frac{1}{\cosh\mathtt{x}}, \nu_t\Big),$$
$$\hat{\mathtt{T}}^{\mathtt{x}} = -\Big(\frac{2}{\mathtt{x}\cosh\mathtt{x}} + \mathtt{D}_{\mathtt{x}}\Big(\frac{1}{\cosh\mathtt{x}}\Big)\Big)\nu + \nu_{\mathtt{x}}\frac{1}{\cosh\mathtt{x}}.$$

ii. For $\mathtt{S}_4$ in (41), we have $W_4 = \frac{1}{\mathtt{x}\cosh\mathtt{x}}$, $\phi = \frac{1}{\mathtt{x}\cosh\mathtt{x}}$ and,

$$\hat{\mathtt{T}}^t = \nu\mathtt{D}_t^{\sigma-1}\Big(\frac{1}{\mathtt{x}\cosh\mathtt{x}}\Big) + J\Big(\frac{1}{\mathtt{x}\cosh\mathtt{x}}, \nu_t\Big),$$
$$\hat{\mathtt{T}}^{\mathtt{x}} = -\Big(\frac{2}{\mathtt{x}^2\cosh\mathtt{x}} + \mathtt{D}_{\mathtt{x}}\Big(\frac{1}{\mathtt{x}\cosh\mathtt{x}}\Big)\Big)\nu + \nu_{\mathtt{x}}\frac{1}{\mathtt{x}\cosh\mathtt{x}}.$$

Notes: In (c)iii, (d)i and (d)ii, the components of the conserved vectors lead to trivial conservation laws. The role of trivial conservation laws has been established for partial differential equations [13], but it has yet to be extended to fractional differential equations.

8 Conclusions

8.1 Discussion

We have, in detail, studied the time-fractional spherically symmetric brain tumor equation in spherical coordinates, assuming the killing rates are functions of tumor-cells. The classification was obtained with respect to the net killing rate. We attained this by transforming the equation into a fractional ordinary differential equation using the Erdélyi-Kober fractional differential operator. The explicit series solutions were obtained with its convergence behaviour. The nature of the solutions were graphically represented. We also constructed the conservation laws for the equation.

The results obtained serve as benchmarks for accuracy testing and comparison with numerical results elsewhere. The exploration of symmetry properties in FRDEs is in its early stages, warranting further investigation. Nevertheless, we have shown that the method is useful in dealing with the model studied here, especially when combined with other approaches studying convergence, for instance. Our analysis currently involves dependent variable, ψ and independent variables (x, t). Extending this analysis to time FDEs with more independent and dependent variables raises questions about the derivation of non-local results. Addressing these issues requires further research to advance our understanding of the symmetry properties of FDEs.

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Conflicts of Interest The authors declare there is none conflicts of interest.

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