Section 2C : Szegö Asymptotics

We can now state and prove formally the theorem concerning strong Szegö asymptotics off τ.

Theorem 2.17

Let $d\mu(e^{i\theta}) = w(\theta)d\theta$, $w > 0$ a.e.(meas), $\log w \in L^1[0, 2\pi]$. Then given $R > 1$,

$$\frac{\varphi_n(z)}{z^n\overline{\left(D\left(w, (\bar{z})^{-1}\right)\right)^{-1}}} \to 1 \tag{2.56}$$

uniformly for $|z| \geq R$ as $n \to \infty$. In particular,

$$\lim_{n\to\infty} \gamma_n(d\mu) = D(w, 0)^{-1} = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta)d\theta\right). \tag{2.57}$$

(This is a Strong Asymptotic off τ.) We also include two mean Asymptotics on τ:

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|\varphi_n(z) - z^n\overline{D\left(w, (\bar{z})^{-1}\right)}\right|^2 w(\theta)d\theta = 0, \ (z = e^{i\theta}); \tag{2.58}$$

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|D(w, z)\varphi_n^*(z) - 1\right|^2 d\theta = 0, \ (z = e^{i\theta}); \tag{2.59}$$

$$\left[D(w, 0)D(w, z)\right]^{-1} = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(0)}; \tag{2.60}$$

$$\pi(z) = \left(D(w, z)\right)^{-1}, \ (\pi \text{ as in Theorem 2.10}). \tag{2.61}$$

Proof:

We notice first that (2.57) has been already shown in Theorem 2.5. Now write $\varphi_n(z) = \gamma_n z^n + \cdots$. Then this implies $\varphi_n^*(z) = \gamma_n + \cdots$, where $\varphi_n^*(0) = \gamma_n$. So $D(w, z)\varphi_n^*(z)$ is analytic in $|z| < 1$. So we can write

$$D(w, z)\varphi_n^*(z) - 1 = \sum_{j=0}^{\infty} d_{nj}z^j, \ |z| < 1$$

where

$$d_{n0} = D(w,0)\varphi_n^*(0) - 1 = D(w,0)\gamma_n - 1 \to 0 \tag{2.62}$$

as $n \to \infty$ by (2.57). Now given $0 < r < 1$, we write

$$\begin{aligned}
\Delta_n(r) &= \frac{1}{2\pi}\int_0^{2\pi} \left| D(w,re^{i\theta})\varphi_n^*(re^{i\theta}) - 1\right|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |D(w,re^{i\theta})|^2 |\varphi_n^*(re^{i\theta})|^2 d\theta - \\
&\quad - 2Re\left[\frac{1}{2\pi}\int_0^{2\pi} D(w,re^{i\theta})(\varphi_n^*(re^{i\theta}))d\theta\right] + 1 \\
&\quad \text{(by orthonormality)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} |D(w,re^{i\theta})|^2 |\varphi_n^*(re^{i\theta})|^2 d\theta - \\
&\quad - 2D(w,0)\varphi_n^*(0) + 1
\end{aligned}$$

(by the mean value theorem for analytic functions at least if $r < 1$). Thus by Theorem 2.14

$$\begin{aligned}
&\lim_{r\to 1-} \Delta_n(r) \\
&= \frac{1}{2\pi}\int_0^{2\pi} w(\theta)|\varphi_n^*(e^{i\theta})|^2 d\theta - 2D(w,0)\gamma_n + 1 \\
&= \frac{1}{2\pi}\int_0^{2\pi} w(\theta)|\varphi_n(e^{i\theta})|^2 d\theta - 2D(w,0)\gamma_n + 1 \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\varphi_n(z)|^2 d\mu(z) - 2D(w,0)\gamma_n + 1 (z = e^{i\theta}) \\
&= 2(1 - D(w,0)\gamma_n)
\end{aligned}$$

(by orthonormality). Then

$$\begin{aligned}
\sum_{j=0}^{\infty} |d_{n_j}|^2 &= \lim_{r\to 1-} \sum_{j=0}^{\infty} |d_{n_j}|^2 r^{2j} \\
&= \lim_{r\to 1-} \Delta_n(r) = 2(1 - D(w,0)\gamma_n) \to 0
\end{aligned}$$

by (2.62). So given $r < 1$, by the Cauchy Schwarz inequality we have

$$\begin{aligned}
\max_{|z|\le r}\left|D(w,z)\varphi_n^*(z)-1\right| &= \max_{|z|\le r}\left|\sum_{j=0}^{\infty} d_{nj}z^j\right| \\
&\le \max_{|z|\le r}\left(\sum_{j=0}^{\infty}|d_{nj}|^2\right)^{1/2}\left(\sum_{j=0}^{\infty}|z|^{2j}\right) \\
&= \left(\sum_{j=0}^{\infty}|d_{nj}|^2\right)^{1/2}\left(\frac{1}{1-r^2}\right) \to 0
\end{aligned}$$

as $n \to \infty$. Thus we have

$$\max_{|z|\le r}\left|D(w,z)z^n\overline{\varphi_n\big((\bar z)^{-1}\big)}-1\right| \to 0 \text{ as } n\to\infty. \tag{2.63}$$

Setting $R = \frac{1}{r}$, $u = (\bar z)^{-1}$ and substituting into (2.63) yields (2.56). Next noting the definition of $D(w, e^{i\theta})$ as in Lemma 2.15,

$$\Delta_n(1) = \frac{1}{2\pi}\int_0^{2\pi}\left|D(w,e^{i\theta})\varphi_n^*(e^{i\theta})-1\right|^2 d\theta = \sum_{j=0}^{\infty}|d_{nj}|^2 \to 0$$

as $n \to \infty$. So (2.59) is proved.

Next, write (2.59) as

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|z^n\overline{\varphi_n\big((\bar z)^{-1}\big)}-\big(D(w,z)\big)^{-1}\right|^2 w(\theta)d\theta = 0. \tag{2.64}$$

Then use (2.36) for $z = e^{i\theta}$, and $z = (\bar z)^{-1}$ so that (2.64) becomes

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|(\bar z)^n\varphi_n(z) - z^n(\bar z)^n\overline{D\big(w,(\bar z)^{-1}\big)}\right| w(\theta)d\theta = 0.$$

So (2.58) is proved. Now compare (2.13) with (2.58) and we deduce that

$$\overline{\big(D\big(w,(\bar z)^{-1}\big)\big)^{-1}} = K\big((\bar z)^{-1}\big)\cdot\big(K(0)\big)^{-1/2} \ a.e.\,(meas)\ \theta\in[0,2\pi],\ (z=e^{i\theta}).$$

That is,

$$\big(D(w,u)\big)^{-1} = K(u)\big(K(0)\big)^{-1/2} \ a.e.\,(meas)\ u = e^{it},\ t\in[0,2\pi]. \tag{2.65}$$

Now for $|z| < 1$,

$$|K(z)| \le \left(\sum_{j=0}^{\infty}|\varphi_j(z)|^2\right)^{1/2}\left(\sum_{j=0}^{\infty}|\varphi_j(0)|^2\right)^{1/2}$$
$$\le |D(w,z)|^{-1}(1-|z|)^{-1/2}\left(\sum_{j=0}^{\infty}|\varphi_j(0)|^2\right)^{1/2}.$$

So the series defining K converges in compact subsets of $|z| < 1$ uniformly as D is analytic and $\lim \gamma_n$ is finite. So K is analytic in $|z| < 1$. Also

$$(D(w,0))^{-1} = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi}\log w(\theta)d\theta\right) = \left(\sum_{j=0}^{\infty}|\varphi_j(0)|^2\right)^{1/2}$$
$$= K(0)K(0)^{-1/2}$$

by Theorem 2.6. So both functions on both sides of (2.65) are analytic in $\{z\colon |z| < 1\}$, have the same value at 0 and their boundary values satisfy

$$\frac{1}{2\pi}\int_0^{2\pi}\left|(D(w,e^{it}))^{-1} - K(e^{it})K(0)^{-1/2}\right|^2 dt = 0.$$

Thus all the coefficients of the Maclaurin Series of the difference of these two functions $= 0$. Thus

$$(D(w,u))^{-1} = K(u)(K(0))^{-1/2} = K(u)\exp\left(\frac{1}{4\pi}\int_0^{2\pi}\log w(\theta)d\theta\right)$$
$$= K(u)D(w,0),\ |u| < 1.$$

Now (2.61) follows from the definition of $\pi(z)$ and (2.60). ∎

Remark

We note that (2.58) and (2.59) will have more general analogues in Theorem 5.13.

Theorem 2.18

Let $d\mu$ be a finite positive measure on τ satisfying a Szegö condition there. Then

(2.66) $$\lim_{n\to\infty}\gamma_n(d\mu) = \gamma < \infty;$$

(2.67) $$\lim_{n\to\infty} \gamma_{n+1}(d\mu)\cdot\left(\gamma_n(d\mu)\right)^{-1} = 1;$$

(2.68) $$\lim_{n\to\infty} \gamma_n^{1/n}(d\mu) = 1;$$

(2.69) $$\lim_{n\to\infty} \Phi_{n+1}(d\mu, 0) = 0;$$

(2.70) $$\lim_{n\to\infty} \Phi_{n+1}(z)\cdot\left(\Phi_n(z)\right)^{-1} = \lim_{n\to\infty} \varphi_{n+1}(z)\left(\varphi_n(z)\right)^{-1} = z,\ |z| \geq 1;$$

(2.71) $$\lim_{n\to\infty} \Phi^*_{n+1}(z)\left(\Phi^*_n(z)\right)^{-1} = \lim_{n\to\infty} \varphi^*_{n+1}(z)\left(\varphi^*_n(z)\right)^{-1} = 1,\ |z| \leq 1;$$

(2.72) $$\lim_{n\to\infty} \Phi^*_n(z)\left(\Phi_{n+1}(z)\right)^{-1} = \lim_{n\to\infty} \varphi_n(z)\left(\varphi_{n+1}(z)\right)^{-1} = 0,\ |z| > 1,$$

where the convergence in (2.68), (2.69) and (2.70) is uniform in compact subsets of the prescribed regions.

Proof:

(2.66) is just (2.28). For the rest see the much more general Theorem 3.8 ■

Remarks:

(a) We note that Theorem 3.8 will prove (2.66) — (2.72) for much more general measures satisfying an Erdös–Turan condition, but of course because of what we have seen we cannot hope to prove (2.64).

(b) A famous theorem of Erdös–Turan states: Let E be compact with $cap(E) > 0$ and equilibrium measure μ_E. Suppose μ is a Borel measure with $supp\,\mu$ in E such that $\frac{d\mu}{d\mu_E} > 0$ *a.e.* (μ_E) then $\lim_{n\to\infty}(\gamma_n)^{1/n} =$ (capacity of $E)^{-1}$. For $E = \tau$, this reduces to $\lim_{n\to\infty}(\gamma_n)^{1/n} = 1$. So Theorem 3.8 is far more general than this.

(c) We end this chapter with a remark concerning Strong Asymptotics on τ for Szegő weights. This is often referred to as Tauber's problem, that is, given the existence of the radial boundary value $D(w, e^{i\theta})$, deduce the convergence properties of the sequence $\left(\varphi_n(e^{i\theta})\right)_{n=0}^{\infty}$ (see [5] p.75). As the purpose of this dissertation is primarily concerned with non Szegő Asymptotics, we do not state or prove all the results in this area but refer the reader to the relevant references ([4] pp.221–245, [5], Chapter 5 and tables 3,4,5 pp.200–203). Two results are however worth stating as they have ratio analogues in Chapter 5. They will constitute the final theorem in this chapter.

Theorem 2.19

(a) Let $d\mu = f d\theta$, f continuous, $|f(t) - f(\theta)| \le C_3|t - \theta|$, $t, \theta \in [0, 2\pi]$. Then Tauber's problem is solvable. In fact (a) holds under the weaker conditions $w(f, \delta) \le \left(\log(\delta)^{-1}\right)^{-\gamma}$, $\gamma > 1$, $C_4 \le f(t) \le C_5$.

(b) A famous weaker version of (a) due to Freud: Let $\log \mu'(\theta) \in L^1[0, 2\pi]$ and suppose in a neighbourhood $[\gamma - \delta, \gamma + \delta]$ of γ the function $\mu(\theta)$ is absolutely continuous. Furthermore, suppose that for $\theta \in [\gamma - \delta, \gamma + \delta]$, $0 < m \le \mu'(\theta) \le M$, holds and let

$$\int_{-\pi}^{\pi} \left| \frac{\mu'(\theta) - \mu'(\gamma)}{\theta - \gamma} \right|^2 d\theta < \infty.$$

Then Tauber's problem is solvable.

(c) In 1979 Paul Nevai established asymptotics for all the derivatives of the orthogonal polynomials under the hypothesis of (b) that is: Let K be a fixed positive integer. Then under the hypotheses of (b) we have for $z = e^{it}$,

$$\lim_{n \to \infty} \varphi_n^{(k)}(z)(z^{n-k} n^k)^{-1} = \overline{(D(f, z))^{-1}}.$$

An Introduction to non Szegö Asymptotics

For weights that satisfy $\log w \in L^1[0, 2\pi]$, asymptotic properties of the sequence $\left(\varphi_n(z)\right)_{n=0}^{\infty}$ have been studied in Chapter 2. However, it is well known that there exist many classes of weights that fail to satisfy Szegö's condition. Consider for example the weight $w(z) = \exp\left(-|z-1|\right)^{-\beta}$, $\beta \geq 0$, $|z| = 1$. Then $\log w \notin L[0, 2\pi]$, $\beta \geq 1$ and $\log w \in L[0, 2\pi]$, $\beta < 1$. Another example would be the Pollaczek polynomials described in ([27], pp.388–390) with weight function

$$w(\cos\theta, a, b) = 2\exp(a+b)\left(1 - \frac{\pi}{\theta}\right)$$

as $\theta \to 0^+$. Now writing $x = \cos\theta$, we see that when $x = \pm 1$, $\log w(\cos\theta)$ becomes non integrable. We now study asymptotics of $(\varphi_n)_{n=0}^{\infty}$ for these more general weights.

Chapter 3

The ratio Asymptotic $\lim_{n\to\infty} \varphi_{n+1}(z)(\varphi_n(z))^{-1} = z$ and strong and weak convergence of orthogonal polynomials on τ

The main results in this chapter are the proofs of Theorem 3.8 for non Szegõ weights and some other strong and weak convergence results. The proofs of the above are taken from [10] although the author wishes to point out that the ideas are taken from the original proofs of Nevai et al in [14] and [18]. We also point out that the proof of Theorem 3.8 was proved first by Rakhmanov.

Definition 3.1

Let $d\mu$ be a finite positive Borel measure on the unit circle with $supp(d\mu)$ infinite. Let $(W_n)_{n=1}$ be a sequence of polynomials of degree n with zeros $(w_{n,i})$, $1 \le i \le n$ in $|z| < 1$. We assume that the indices are taken so that if $w = 0$ is a zero of W_n of order m then $w_{n,1} = \ldots w_{n,m} = 0$. Now fix $n \in \mathbb{N}$. Let $d\mu_n(\theta) = d\mu(\theta) \cdot |W_n(z)|^{-2}$, $(z = e^{i\theta})$ and let $\varphi_{n,m}(d\mu_n, z) = \varphi_{n,m}(z)$ be the orthonormal polynomial of degree m with respect to $d\mu_n$ uniquely determined by the conditions

(3.1) $$\frac{1}{2\pi}\int_0^{2\pi} \varphi_{n,m}(z)\overline{\varphi_{n,\ell}(z)}\, d\mu_n(\theta) = \delta_{m,\ell}$$

degree $\varphi_{n,m} = m$, $\gamma_{n,m}(d\mu_n) > 0$.

Notation

Throughout, $d\mu_n$ will denote the varied measure of $d\mu$ with respect to $(W_n)_{n=1}$ and $\varphi_{n,m}(d\mu_n, z) = \varphi_{n,m}(z)$ will be the orthonormal polynomial of degree m with respect to $d\mu_n$, $\varphi_n(d\mu, z) = \varphi_n(z)$ will be the orthonormal polynomial of degree n with respect to $d\mu$. If a_n and a are positive Borel measures on $[0, 2\pi]$ then $a_n \xrightarrow{*} a$ read as "a_n converges weak star to a" iff $\lim_{n\to\infty} \int_0^{2\pi} f da_n(\theta) = \int_0^{2\pi} f da(\theta)$ for all continuous functions f. $C_1, C_2 \ldots$ will denote constants independent of n, ξ, z, w.

Definition 3.2

Let K be a fixed integer. We shall say that $(\mu, (W_n), K)$ is admissible on $[0, 2\pi]$ if:

$$\mu' > 0 \; a.e. \, (meas); \tag{3.2}$$

$$\|d\mu_n\| = \int_0^{2\pi} d\mu_n(\theta) < \infty; \tag{3.3}$$

$$\int_0^{2\pi} \prod_{i=1}^{-K} |z - w_{n,i}|^{-2} d\mu(\theta) \leq m < \infty, \tag{3.4}$$

$z = e^{i\theta}$, $K = -1, -2, \ldots,$ m a finite constant

$$\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = \infty. \tag{3.5}$$

Statement 3.3

We now state the main results to be proved in this chapter. We remark that Lemmas 3.3, 3.4 and so on, which are used to prove these theorems, will be proved below.

Theorem 3.6

Let $(\mu, (W_n), K)$ be admissible on $[0, 2\pi]$ then,

$$\lim_{n\to\infty} \int_0^{2\pi} \left| \left(|\varphi_{n,n+k}(z)| |\varphi_{n,n+k+m}(z)|^{-1} \right)^2 - 1 \right| d\theta = 0, \; (z = e^{i\theta})$$

uniformly in $m \in \mathbb{N}$.

Theorem 3.8

Let $(\mu, (W_n), K)$ be admissible on $[0, 2\pi]$. Then, the following are true:

$$\lim_{n\to\infty} \Phi_{n,n+k+1}(0) = 0; \tag{3.6}$$

$$\lim_{n\to\infty} \gamma_{n,n+k+1}(\gamma_{n,n+k})^{-1} = 1; \tag{3.7}$$

$$\begin{aligned} &\lim_{n\to\infty} \Phi_{n,n+k+1}(w)\left(\Phi_{n,n+k}(w)\right)^{-1} \\ &= \lim_{n\to\infty} \varphi_{n,n+k+1}(w)\left(\varphi_{n,n+k}(w)\right)^{-1} = w, \ |w| \geq 1; \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\lim_{n\to\infty} \Phi^*_{n,n+k+1}(w)\left(\Phi^*_{n,n+k}(w)\right)^{-1} \\ &= \lim_{n\to\infty} \varphi^*_{n,n+k+1}(w)\left(\varphi^*_{n,n+k}(w)\right)^{-1} = 1, \ |w| \leq 1; \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\lim_{n\to\infty} \Phi^*_{n,n+k}(w)\left(\Phi_{n,n+k}(w)\right)^{-1} \\ &= \lim_{n\to\infty} \varphi^*_{n,n+k}(w)\left(\varphi_{n,n+k}(w)\right)^{-1} = 0, \ |w| > 1, \end{aligned} \tag{3.10}$$

where the convergence in (3.8), (3.9), (3.10) is uniform in compact subsets of the prescribed regions.

Theorem 3.9

Let $(\mu, (W_n), K)$ be admissible on $[0, 2\pi]$. Let $z = e^{i\theta}$ then,

$$\lim_{n\to\infty} \int_0^{2\pi} \left(\left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right| \left(\mu'(\theta)\right)^{1/2} - 1 \right)^2 d\theta = 0 \tag{3.11}$$

$$\lim_{n\to\infty} \int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}\, z^{\ell} \mu'(\theta)}{\left|W_n(z)\right|^2} - 1 \right| d\theta = 0. \tag{3.12}$$

Theorem 3.10

Let $(\mu, (W_n), K)$ be admissible on $[0, 2\pi]$, then if $z = e^{i\theta}$ for every bounded measurable f on $[0, 2\pi]$ and $\ell \in \mathbb{N}$ we have,

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{f(\theta)\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}z^{\ell}\mu'(\theta)}{|W_n(z)|^2} d\theta = \int_0^{2\pi} f(\theta)d\theta; \tag{3.13}$$

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{f(\theta)\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}z^{\ell}d\mu(\theta)}{|W_n(z)|^2} = \int_0^{2\pi} f(\theta)d\theta. \tag{3.14}$$

Remarks:

(a) We note that Theorem 3.8 is more general than Theorem 2.8 as it works for non Szegõ weights.

(b) When $W_n \equiv 1$, Theorems 3.6 — 3.10 reduce to those results proved in [13], [14], [15] and [18] although as already pointed out, the method of proof comes from the latter papers.

(c) From now on the weaker $\lim_{n\to\infty} \gamma_{n,n+k+1}(\gamma_{n,n+k})^{-1} = 1$ will replace the stronger $\lim_{n\to\infty} \gamma_{n,n+k+1} = \gamma$ and the more general Erdõs Turan condition $\mu' > 0$ *a.e.* (*meas*) and $\lim_{n\to\infty} \Phi_{n,n+k+1}(0) = 0$ will replace the less general $\log \mu' \in L^1[0, 2\pi]$.

Section 3A : Ratio Asymptotics on τ

We begin with two technical lemmas:

Lemma 3.4

Let $(\mu, (W_n), K)$ be admissible on $[0, 2\pi]$. Then for $z = e^{i\theta}$ as $n \to \infty$ we have

$$d\beta_n(\theta) = \left(|W_n(z)|\,|\varphi_{n,n+k}(z)|^{-1}\right)^2 d\theta \xrightarrow{*} d\mu(\theta).$$

Proof:

Let $z = e^{i\theta}$, by Lemma 1.8,

$$\frac{1}{2\pi}\int_0^{2\pi} z^j |\varphi_{n,m}(z)|^{-2} d\theta = \frac{1}{2\pi}\int_0^{2\pi} z^j d\mu_n(\theta),\ j = 0, \pm 1, \ldots, \pm n.$$

Thus for all trigonometric polynomials T_m of degree $\le m$ we have,

$$\frac{1}{2\pi}\int_0^{2\pi} T_m(\theta)|\varphi_{n,m}(z)|^{-2} d\theta = \frac{1}{2\pi}\int_0^{2\pi} T_m(\theta) d\mu_n(\theta). \tag{3.15}$$

Now let

$$A_n(z) = \prod_{i=1}^{-\ell}(z - w_{n,i}),\ \ell = -1, -2, \ldots$$

and take $A_n \equiv 1$, $\ell = 0, 1, 2, \ldots$. Then firstly for $\ell = 0, 1, 2 \ldots$ we have.

$$\int_0^{2\pi} d\beta_n(\theta) = \int_0^{2\pi} \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} d\theta = \int_0^{2\pi} |W_n(z)|^2 d\mu_n(\theta) = \int_0^{2\pi} d\mu(\theta)$$

(by Lemma 1.8). Also for $\ell = -1, -2, \ldots$ we see that we may apply (Lemma 1.8) again to yield

$$\int_0^{2\pi} |A_n(z)|^{-2} d\beta_n(\theta) = \int_0^{2\pi} |A_n(z)|^{-2} d\mu(\theta) \le C_1 < \infty \tag{3.16}$$

for all ℓ (using Definition 3.2). Also as

$$\|d\mu_n(\theta)\| = \int_0^{2\pi} d\mu_n(\theta) < \infty$$

the sequences $\left(|A_n(z)|^{-2} d\beta_n(\theta)\right)_{n=1}$ and $\left(|A_n(z)|^{-2} d\mu(\theta)\right)_{n=1}$ are bounded in norm. We must show that,

$$dh_n(\theta) = |A_n(z)|^{-2} d(\beta_n - \mu)(\theta) \xrightarrow{*} 0 \text{ as } n \to \infty.$$

It is sufficient to show that the subsequences (dh_n), $n \in \Omega \subseteq \mathbb{N}$ converge weak star to zero. Thus let $\Omega \subseteq \mathbb{N}$ be chosen and consider (dh_n), $n \in \Omega \subseteq \mathbb{N}$. Also let

$$e_n(w) = \int_0^{2\pi} \left(\frac{z}{z - w}\right) dh_n(\theta),\ z = e^{i\theta},\ |w| < 1.$$

Now, $\left|\frac{z}{z-w}\right| \le C_2$ as $z = e^{i\theta}$, $|w| < 1$ and by (3.16) we have that $e_n(w)$ is analytic in $|w| < 1$. Furthermore we claim that $e_n(w)$ is in Nevalinna's class N (see Remarks after Lemma 2.15). To this end let

$$\hat{\beta}_n(w) = \int_0^{2\pi} |A_n(z)|^{-2}\left[\frac{z}{z-w}\right] d\beta_n(\theta)$$

and

$$\hat{\mu}_n(w) = \int_0^{2\pi} |A_n(z)|^{-2}\left[\frac{z}{z-w}\right] d\mu(\theta), \ |w| < 1.$$

Then writing

$$\frac{z}{z-w} = \frac{z(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-z\bar{w}}{|z-w|^2} = \frac{1-Re(z\bar{w})+iIm(z\bar{w})}{|z-w|^2}$$

we have that

$$\hat{\beta}_n(w) = \int_0^{2\pi} \frac{1-Re(e^{i\theta}\bar{w})}{|z-w|^2|A_n(z)|^2} d\beta_n(\theta) + i\int_0^{2\pi} \frac{Im(e^{i\theta}\bar{w})d\beta_n(\theta)}{|z-w|^2|A_n(z)|^2}.$$

Also

$$Re(\hat{\beta}_n(w)) \ge (1-|w|)\int_0^{2\pi} \frac{d\beta_n(\theta)}{|z-w|^2|A_n(z)|^2}$$

and

$$Im(\hat{\beta}_n(w)) \le |w|\int_0^{2\pi} \frac{d\beta_n(\theta)}{|z-w|^2|A_n(z)|^2}.$$

Thus we have,

$$\frac{Im(\hat{\beta}_n(w))}{Re(\hat{\beta}_n(w))} \le \frac{|w|}{1-|w|}.$$

Now taking the arc tangent on each side of the above and repeating the same argument for $(\hat{\mu}_n)$ yields

$$-\frac{\pi}{2} < -\frac{\delta\pi}{2} \le \arg\hat{\beta}_n(w) < \frac{\delta\pi}{2} < \frac{\pi}{2} \tag{3.17}$$

and

$$-\frac{\pi}{2} < -\frac{\delta\pi}{2} \le \arg\hat{\mu}_n(w) < \frac{\delta\pi}{2} < \frac{\pi}{2}$$

for some $0 < \delta < 1$.

Next using the inequality $\log x \le P^{-1}(x-1)^P$, $x \ge 1$, $0 < P < 1$ we obtain

$$\log^+|a| \le \delta^{-1}|a|^\delta \le (Re(a))^\delta \left(\delta\cos\left(\frac{\pi\delta}{2}\right)\right)^{-1} \tag{3.18}$$

when

$$\frac{-\delta\pi}{2} < \arg a < \frac{\delta\pi}{2},\ a \in \mathbb{C}.$$

Thus using the definition of $\hat{\beta}_n$, the mean value theorem for analytic functions, (3.17) and (3.18) we get for $\theta' \in [0, 2\pi]$,

$$\begin{aligned}
&\int_0^{2\pi} \log^+|\hat{\beta}_n(re^{i\theta'})|d\theta' \\
&\le \int_0^{2\pi} Re(\hat{\beta}_n(re^{i\theta'}))^\delta \left(\delta\cos\left(\frac{\delta\pi}{2}\right)\right)^{-1} d\theta \\
&= Re(\hat{\beta}_n(0))^\delta \left(\delta\cos\left(\frac{\pi\delta}{2}\right)\right)^{-1} \\
&= \left(\int_0^{2\pi} |A_n(z)|^{-2} d\beta_n(\theta)\right)^\delta \left(\delta\cos\left(\frac{\pi\delta}{2}\right)\right)^{-1} \\
&\le (C_1)^\delta \left(\delta\cos\left(\frac{\pi\delta}{2}\right)\right)^{-1},\ 0 < \delta < 1.
\end{aligned}$$

Thus

$$\int_0^{2\pi} \log^+|\hat{\beta}_n(re^{i\theta'})|d\theta' \le (C_1)^\delta \left[\delta\cos\left(\frac{\pi\delta}{2}\right)\right]^{-1},\ 0 < \delta < 1 \tag{3.19}$$

and similarly

$$\int_0^{2\pi} \log^+|\hat{\mu}_n(re^{i\theta'})|d\theta' \le (C_1)^\delta \left[\delta\cos\left(\frac{\pi\delta}{2}\right)\right]^{-1},\ 0 < \delta < 1 \tag{3.20}$$

Thus we have using (3.19) and (3.20)

(3.21)

$$\sup_r \lim_{r\to 1-} \int_0^{2\pi} \log^+\left|(\hat{\mu}_n - \hat{\beta}_n)(re^{i\theta'})\right|d\theta' \le 2(C_1)^\delta \left[\delta\cos\left(\frac{\pi\delta}{2}\right)\right]^{-1} < \infty.$$

Thus from (2.51) and (3.21) we deduce that $e_n(w) \in N$ and hence we may write $e_n(w) = B_n(w)(C_n(w))^{-1}$, $n \in \mathbb{N}$ where, B_n, C_n are analytic in

$|w| < 1$, $\lim_{r\to 1-} \max_{|w|=r} |B_n(w)| \le 1$, $\lim_{r\to 1-} \max_{|w|=r} |C_n(w)| \le 1$, $C_n(w) \neq 0$, $|w| < 1$ and

(3.22)

$$C_n(w) = \lim_{r\to 1-} \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|(\hat{\mu}_n - \hat{\beta}_n)(re^{i\theta'})\right|\left(\frac{re^{i\theta'}+w}{re^{i\theta'}-w}\right)d\theta\right).$$

(for details of above see [3] pp.16–17).

Now as B_n, C_n are analytic in $|w| < 1$ and by uniform boundedness we may apply the Vitali–Montel theorem for analytic functions to obtain $\Omega' \subseteq \Omega \subseteq \mathbb{N}$ such that $\lim_{n\in\Omega'} B_n(w) = B(w)$ and $\lim_{n\in\Omega'} C_n(w) = C(w)$ uniformly on each compact subset of $|w| < 1$, C, B analytic in $|w| < 1$ and $|C(w)|$, $|B(w)| \le 1$, $|w| < 1$. Now $C_n(w) \neq 0$ in $|w| < 1$, so by Hurwitz's theorem on the zeros of analytic functions $C(w) \equiv 0$ or $C(w) \neq 0$ in $|w| < 1$. Now using (3.22) we have

$$\inf_{n\in\mathbb{N}} |C_n(0)| = \exp\left(-\frac{1}{2\pi}\sup_{n\in\mathbb{N}}\lim_{r\to 1}\int_0^{2\pi}\log^+(\hat{\mu}_n - \hat{\beta}_n)(re^{i\theta'})d\theta'\right)$$

$$\geq \exp\left(-(C_1)^{v}\left[\pi\delta\cos\left(\frac{\pi\delta}{2}\right)\right]^{-1}\right) > 0 \tag{3.23}$$

(by 3.19 and 3.20). Now if $C \equiv 0$, $C(0) = 0$, thus by Hurwitz in a close enough neighbourhood $C_n(0) = 0$ a contradiction to (3.23). Thus $C \neq 0$ anywhere. We show $B(w) = 0$, $|w| < 1$. Suppose first that w' is an arbitrary zero of $W_n(A_n)^{-1}$ inside $|w| < 1$ of order $m > 0$. Assume first that $w' \neq 0$. Then using the definition of $d\beta_n$ and Lemma 1.8 we get that

$$\int_0^{2\pi} |A_n(z)|^{-2}\frac{z}{(z-w')^j}d\beta_n(\theta)$$

$$= \int_0^{2\pi} |A_n(z)|^{-2}\frac{z}{(z-w')^j}d\mu(\theta),\ 1 \le j \le m. \tag{3.24}$$

Now since

$$(z-w)^{-1} = \sum_{j=1}^{m}\frac{(w-w')^{j-1}}{(z-w')^j} + \frac{(w-w')^m}{(z-w')^m}(z-w)^{-1}$$

it follows using the definition of $e_n(w)$ that $e_n(w)$ has a zero of order m at w'. Next if $w' = 0$ is a zero of $W_n(A_n)^{-1}$ inside $|w| < 1$ of order $m > 0$ using the definition of $d\beta_n(\theta)$ and Lemma 1.8 we get that

$$(3.25) \quad \int_0^{2\pi} (z)^{-j} |A_n(z)|^{-2} d\beta_n(\theta) = \int_0^{2\pi} (z)^{-j} |A_n(z)|^{-2} d\mu(\theta), \; 1 \le j \le m.$$

Now since

$$(z-w)^{-1} = \sum_{j=1}^{m} \frac{w^{j-1}}{z^j} + \frac{w^{m+1}}{z^{m+1}}(z-w)^{-1},$$

it follows by the definition of $e_n(w)$ that $e_n(w)$ has a zero of order $m+1$ at w'. Now consider the sequence of functions,

$$(3.26) \quad H_n(w) = B_n(w) \prod{}' (w - w_{n,i})^{-1}(1 - \bar{w}_{n,i} w), \; n \in \Omega' \le \Omega \le \mathbb{N}$$

where $\prod'$ is taken over all i such that $w_{n,i}$ is a zero of $W_n(A_n)^{-1}$ and of modulus < 1. Now as $C_n(w) \ne 0$ in $|w| < 1$ and by the previous discussion on the zeros of $e_n(w)$ we see that $H_n(w)$ is analytic in $|w| < 1$. Now as $\lim_{r \to 1} \max_{|w|=r} |B_n(w)| \le 1$ and $|w_{m,i}| < 1$ we have by the maximum principle of analytic functions that $|H_n(w)| \le 1$, $n \in N'$, $|w| < 1$. Thus by (3.26)

$$(3.27) \quad |B_n(w)| \le \prod{}' \left| (w - w_{n,i})(1 - \overline{w_{n,i}} w)^{-1} \right|; \; n \in \Omega', \; |w| < 1.$$

Now by definition 3.2 we see that the right hand side of (3.27) $\to 0$ as $n \to \infty$. Thus we must have that $B(w) \equiv 0$ and $C(w) \ne 0$ in $|w| < 1$. Thus uniformly in compact subsets of $|w| < 1$ we have $\lim_{\substack{n \to \infty \\ n \in \Omega'}} e_n(w) = 0$. Now

$$e_n(w) = \int_0^{2\pi} z[z-w]^{-1} dh_n(\theta) \text{ and } z \cdot [z-w]^{-1} = \sum_{j=0}^{\infty} \left| \frac{w}{z} \right|^j \text{ as } \left| \frac{w}{z} \right| < 1.$$

Thus

$$e_n(w) = \sum_{i=0}^{\infty} \left(\int_0^{2\pi} (z)^{-i} dh_n(\theta) \right) w^i$$

and

(3.28) $$\lim_{n\in\Omega'}\int_0^{2\pi}(z)^{-i}dh_n(\theta)=0.$$

Thus by Weierstrass' approximation theorem (3.28) implies that $dh_n \xrightarrow{*} 0$, $n \in \Omega' \subseteq \Omega \subseteq \mathbb{N}$ As Ω is arbitrary we have shown that

(3.29) $$dh_n(\theta) \xrightarrow{*} 0, \ n \in \mathbb{N}.$$

If $\ell \in \mathbb{N}$, then $A_n \equiv 1$ and (3.29) implies the result. Suppose $\ell = -1, -2, \ldots$. Then since the zeros of $A_n(z)$ are in $\{w\colon |w| \le 1\}$ and A_n is monic, the coefficients of the trigonometric polynomial

$$|A_n(z)|^2 = \sum_{i=\ell}^{-\ell} c_{n,i}z^i$$

are uniformly bounded and hence,

$$|c_{n,i}| \le C_3, \quad i = -|\ell|, \ldots, |\ell|, \ n \in \mathbb{N}.$$

Thus for every integer m we have

$$\int_0^{2\pi} z^m |A_n(z)|^2 dh_n(\theta) = \sum_{i=\ell}^{-\ell} \ell_{n,i}\int_0^{2\pi} z^{m+i}dh_n(\theta) \to 0 \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty}\int_0^{2\pi} z^m d\beta_n(\theta) = \int_0^{2\pi} z^m d\mu(\theta)$$

which implies

$$\beta_n \xrightarrow{*} \mu(\theta), \ n \in \mathbb{N}. \ \blacksquare$$

Lemma 3.5

Let V be a finite positive Borel measure that is singular with respect to "meas". Then there exists a sequence $(h_n)_{n=1}^{\infty}$ of continuous functions defined on the real line, $0 \le h_n(x) \le 1$ for every x, $\lim_{n\to\infty} h_n(x) = 1$ a.e.

(meas) and $\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(x)dV(x) = 0$. Further if V is confined to a finite interval and $T > 0$, then each h_n can be made to be periodic with period T.

Proof:

As V is a singular Borel measure with respect to "meas" we can find a Borel set E with meas $E = 0$ and $V(E^c) = \int_{E^c} dV = 0$. Now choose a decreasing sequence $(E_n)_{n=1}^{\infty}$ of open sets with $E \subset E_n$, $E = \bigcap_{n=1}^{\infty} E_n$ and meas $E_n \to 0$ as $n \to \infty$. Now define for each $n \in \mathbb{N}$

$$g_n(x) = \inf\Big\{|x-y| : y \notin E_n\Big\}.$$

Then g_n is continuous, $g_n(x) > 0$, $x \in E_n$ and $g_n(x) = 0$, $x \notin E_n$. Also as g_n is defined as an inf, by specifying in advance that meas $E_n < 2$ say we can get $g_n(x) < 1$ for every x. Now set $h_{n,k}(x) = \big(1 - g_n(x)\big)^k$. Then as $g_n(x) < 1$ we have $1 - g_n(x) < 1$ which implies $\lim_{k\to\infty} h_{n,k}(x) = 0$ for every $x \in E_n$ and hence for all $x \in E$. Thus as $g_n(x) < 1$ for every x we have that $0 \le h_{m,k} < 1$. Also V is finite so by dominated convergence there exists an integer K_n such that

$$0 \le \int h_{n,K_n} dV < \frac{1}{n}. \tag{3.30}$$

Set $h_n = h_{n,K_n}$. Then if $x \notin E_n$, $g_n = 0$ which implies $\lim_{n\to\infty} h_n(x) = 1$. Also if $x \in \bigcap_1^n E_n$ then as meas $E_n \to 0$,

$$meas\left(\bigcap_1^n E_n\right) \le (meas\ E_n) \to 0$$

which implies $\lim_{n\to\infty} h_n(x) = 1$ a.e. (meas). Also by (3.30)

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(x)dV(x) = 0.$$

Now let V be confined to the interval $(0, MT)$, some positive m. Let h_n be as above and $E_n \subset (0, MT)$. As $h_n(x) = 1$, $x \notin (0, MT)$ we have $h_n(0) = h_n(MT) = 1$. Now define

$$\eta_n(x) = \prod_{K=0}^{M} h_n(x - KT), \quad x \in [0, T].$$

Then as h_n is continuous, η_η is. Also as $h_n(0) = h_n(MT) = 1$ we have $\eta_n(0) = \eta_n(T)$. Thus η_n can be extended to be continuous in $(-\infty, \infty)$. Also for $x \notin [0, T]$, $\eta_n = 1$ and meas $\left(\bigcap_1^n E_n\right) = 0$. Thus $\lim_{n\to\infty} \eta_n(x) = 1$ a.e. (meas). Also as

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n dV = 0$$

this implies

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \eta_n dV = 0,$$

also η_n is periodic of period T. Thus the result is proved. ■

We can now give the **Proof of Theorem 3.6:**

<u>Proof:</u>

Let f be a 2π periodic, non negative continuous function and let m be a non negative integer. By Holder's inequality we have that,

$$\frac{1}{2\pi}\int_0^{2\pi} (\mu' f)^{1/4} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\varphi_{n,n+k}}{\varphi_{n,n+k+m}}\right|^{1/2} \left(\left|\frac{\varphi_{n,n+k+m}}{W_n}\right|^{1/2} (\mu')^{1/4}\right) \left(\left|\frac{W_n}{\varphi_{n,n+k}}\right|^{1/2} f^{1/4}\right) d\theta$$

$$\le I_1 \cdot I_2 \cdot I_3$$

where

$$I_1 = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| d\theta\right)^{1/2},$$

$$I_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\varphi_{n,n+k+m}}{W_n}\right|^2 \mu'(\theta) d\theta\right)^{1/4},$$

$$I_3 = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{W_n}{\varphi_{n,n+k}}\right|^2 f d\theta\right)^{1/4}.$$

Consider first I_2:

$$I_2^4 \le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k+m}}{W_n}\right|^2 d\mu(\theta) \quad \text{(by (1.5))}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} |\varphi_{n,n+k+m}|^2 d\mu_n(\theta) = 1.$$

Consider next I_3: By Lemma 3.4,

$$I_3^4 \to \frac{1}{2\pi}\int_0^{2\pi} f\,d\mu(\theta) \quad \text{as } n \to \infty.$$

Thus we have as $n \to \infty$

$$\left(\frac{1}{2\pi}\int_0^{2\pi}(\mu' f(\theta))^{1/4}d\theta\right)^4$$
$$\le \left[\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi_{n,n+k}(z)|}{|\varphi_{n,n+k+m}(z)|}d\theta\right]^2 \times$$
$$\times\left(\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\mu(\theta)\right),\ (z = e^{i\theta}). \tag{3.31}$$

Now by Lemma 3.5 choose a sequence of continuous 2π periodic functions $(h_n)_{n=1}^{\infty}$ such that $0 < h_m(\theta) \le 1$, $\lim_{m\to\infty} h_m(\theta) = 1$ a.e. (meas), $\lim_{m\to\infty}\int_{-\infty}^{\infty} h_m d\mu_s = 0$. Also for a fixed $\epsilon > 0$, choose a sequence of 2π periodic functions $(g_k)^{\infty}$ such that

$$0 < g_k(\theta) \le (\epsilon)^{-1},\ \lim_{k\to\infty} g_k(\theta) = (\mu' + \epsilon)^{-1}\ a.e.\,(meas). \tag{3.32}$$

Let $f = h_m g_k$ in (3.31). Then we obtain

$$\left[\frac{1}{2\pi}\int_0^{2\pi}(\mu' h_m g_k)^{1/4}d\theta\right]^4$$
$$\le \left[\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}}{\varphi_{n,n+k+m}}(z)\right|d\theta\right]^2\left(\frac{1}{2\pi}\int_0^{2\pi} h_m g_k d\mu(\theta)\right).$$

Now let $m \to \infty$, use dominated convergence, (3.32), and Lemma (3.5) in (3.33) to get,

$$\left[\frac{1}{2\pi}\int_0^{2\pi}(\mu' g_k)^{1/4}d\theta\right]^4$$
$$\le \left[\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}}{\varphi_{n,n+k+m}}\right|d\theta\right]^2\left(\frac{1}{2\pi}\int_0^{2\pi} g_k\mu'(\theta)d\theta\right).$$

Now let $k \to \infty$. Use again (3.32) and Lemma 3.5 to get

$$\left[\frac{1}{2\pi}\int_0^{2\pi}(\mu'(\mu'+\epsilon)^{-1})^{1/2}d\theta\right]^4$$
$$\leq \left[\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}}{\varphi_{n,n+k+m}}\right|d\theta\right]^2\left(\frac{1}{2\pi}\int_0^{2\pi}(\mu'+\epsilon)^{-1}\mu' d\theta\right).$$

Finally, let $\epsilon \to 0$. Then we get

$$1 \leq \liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}}{\varphi_{n,n+k+m}}\right|d\theta, \ (z = e^{i\theta}) \tag{3.34}$$

uniformly in m as I_3 does not depend on m. Also, using Lemma 1.8 again,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right|^2 d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}|\varphi_{n,n+k}(z)|^2 d\mu_n(\theta) = 1, \ (z = e^{i\theta}). \tag{3.35}$$

Now by (3.34) and (3.35) we deduce that

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\varphi_{n,n_k}(z)|}{|\varphi_{n,n+k+m}(z)|}-1\right)^2 d\theta = 0, \ (z = e^{i\theta}). \tag{3.36}$$

Thus writing

$$\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\varphi_{n,n_k}(z)|^2}{|\varphi_{n,n+k+m}(z)|^2}-1\right|d\theta\right]^2$$
$$\leq \left(\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\varphi_{n,n_k}(z)|}{|\varphi_{n,n+k+m}(z)|}+1\right)^2 d\theta\right) \times$$
$$\times \left(\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\varphi_{n,n_k}(z)|}{|\varphi_{n,n+k+m}(z)|}-1\right)^2 d\theta\right), \ (z = e^{i\theta}) \leq 4(o(1))$$

as $n \to \infty$ by (3.35) and (3.36). The result follows and is uniform as we have uniform convergence in (3.34) ■

<u>Lemma 3.7</u>

$$Re\left[\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}-1\right] = \frac{1}{2}\left[\frac{|\varphi_{n,m}(z)|^2}{|\varphi_{n,m+1}(z)|^2}-1\right] \ (z = e^{i\theta}) \tag{3.37}$$

and there exists a constant C_7 such that

$$(3.38)\qquad \left|\Phi_{n,m+1}(0)\right| \le C_7 \int_0^{2\pi} \left| \frac{\left|\varphi_{n,m}(z)\right|^2}{\left|\varphi_{n,m+1}(z)\right|^2} - 1 \right| d\theta, \ (z = e^{i\theta}).$$

Proof:

By Lemma 1.5 we have,

$$(3.39)\qquad \Phi^*_{n,m+1}(z) = \Phi^*_{n,m}(z) + z\overline{\Phi_{n,m+1}(0)}\Phi_{n,m}(z).$$

Hence,

$$(3.40)\qquad \left|\Phi_{n,m+1}(0)\right| = \left|\frac{\Phi^*_{n,m+1}(z)}{\Phi^*_{n,m}(z)} - 1\right|, \ |z| = 1.$$

Now also the zeros of $\Phi_{n,m}$ lie in $|z| < 1$ (by Lemma 1.8). So we may write

$$\Phi_{n,m+1}(z) = \prod_{i=1}^{m+1} (z - z_j), \ |z_j| < 1$$

which implies $\left|\Phi_{n,m+1}(0)\right| \le 1$. Also by (3.39) and (1.6),

$$\left|\Phi^*_{n,m+1}(z)\right| \le 2\left|\Phi^*_{n,m}(z)\right|, \ |z| = 1.$$

Thus by (3.40),

$$(3.41)\qquad \left|\Phi_{n,m+1}(0)\right| \le 2\left|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)} - 1\right|, \ |z| = 1.$$

Now divide both sides of (3.39) by $\Phi^*_{n,m+1}(z) \neq 0$ to get

$$(3.42)\qquad \frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)} - 1 = \frac{-z\overline{\Phi_{n,m+1}(0)}\Phi_{n,m}(z)}{\Phi^*_{n,m+1}(z)}.$$

Taking $|\ |^2$ on both sides of (3.42) yields for $|z|=1$,

$$\begin{aligned}\Big|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big|^2 &- 2Re\Big[\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big]+1\\ &=|\Phi_{n,m+1}(0)|^2\Big|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big|^2\\ &=\Big(\frac{\gamma^2_{n+1}-\gamma^2_n}{\gamma^2_{n+1}}\Big)\Big|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big|^2\\ &\text{(by (1.18))}\\ &=\Big(1-\frac{\gamma^2_n}{\gamma^2_{n+1}}\Big)\Big|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big|^2\\ &=\Big|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big|^2-\Big|\frac{\varphi^*_{n,m}(z)}{\varphi^*_{n,m+1}(z)}\Big|^2.\end{aligned}$$

So we obtain after cancelling,

$$-2Re\Big[\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big]+1=-\Big|\frac{\varphi^*_{n,m}(z)}{\varphi^*_{n,m+1}(z)}\Big|^2$$

which implies

$$Re\Big[\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}\Big]-\frac{1}{2}=\frac{1}{2}\Big|\frac{\varphi^*_{n,m}(z)}{\varphi^*_{n,m+1}(z)}\Big|^2$$

which implies after adding $-\frac{1}{2}$ to each side that (3.37) follows. Next we prove (3.38). To this end we make use of an identity found in ([28] p.254) that states that if F is analytic in $|z|<1$,

$$F(z)=u(z)+iv(z),\ v(0)=0,$$

then there exists C_4 such that

$$\Big(\int_{|z|=1}|F(z)|^{1/2}d\theta\Big)^2\le C_4\Big(\int_{|z|=1}|u(z)|^{1/2}d\theta\Big)^2.$$

So let

$$F(z)=\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)}-1=-z\overline{\Phi_{n,m+1}(0)}\,\frac{\Phi_{n,m}(z)}{\Phi^*_{n,m+1}(z)}.$$

Then $F(0) = 0$, F is analytic in $|z| < 1$ and $I_m F(0) = 0$. Thus,

$$\left(\int_0^{2\pi} \left|\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)} - 1\right|^{1/2} d\theta\right)^2$$

$$\le C_5\left(\int_0^{2\pi} Re\left(\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)} - 1\right)^{1/2} d\theta\right)^2$$

$$\le C_6 \int_0^{2\pi} \left|Re\left[\frac{\Phi^*_{n,m}(z)}{\Phi^*_{n,m+1}(z)} - 1\right]\right| d\theta.$$

Now using (3.37) and (3.41) the result holds. ∎

We now give **The proof of Theorem 3.8:**

(3.6) follows from Theorem 3.6 and (3.38). (3.7) follows by (3.6) and Lemma 1.5. Now (3.8) for Φ follows by Lemma 1.5 and for φ by (3.6) and (3.7). (3.9) follows for Φ by Lemma 1.5 and by (3.6) and (3.7), we now prove (3.10). Let $\lambda_{n,n+k}(z) = \Phi_{n,n+k}(z)\left(\Phi^*_{n,n+k}(z)\right)^{-1}$ analytic in $|z| < 1$. We show $\lim_{n\to\infty} \lambda_{n,n+k}(z) = 0$ uniformly for $|z| \le R < 1$. Note as $\left|\lambda_{n,m}(z)\right| \le 1$, $|z| \le 1$, it is sufficient to show convergence for $|z| \le \frac{1}{4}$. Thus for $|z| \le \frac{1}{4}$, using Lemma 1.5,

$$\text{(3.43)} \qquad |\lambda_{n,m}| = \left|\frac{z\lambda_{n,m-1} + \Phi_{n,m}(0)}{1 + z\overline{\Phi_{n,m}(0)}\lambda_{n,m-1}}\right| \le \frac{1}{2}|\lambda_{n,m-1}| + 2\left|\Phi_{n,m}(0)\right|.$$

Fix $\epsilon > 0$ and choose m such that $(2)^{-m} \le \frac{\epsilon}{2}$. Now by (3.6) there exists N such that for $n \ge N$ and $i = 0, \ldots, m$, $2\left|\Phi_{n,m+k-i}(0)\right| \le \frac{\epsilon}{4}$. Then for $n \ge N$ using (3.43) we have that

$$\text{(3.44)} \qquad \left|\lambda_{n,n+k-i}(z)\right| \le \frac{1}{2}\left|\lambda_{n,n+k-i-1}(z)\right| + \frac{\epsilon}{4},\ i = 0, \ldots, m,\ |z| \le \frac{1}{4}.$$

Now apply (3.44) for $i = 0, \ldots, m$. Then,

$$\left|\lambda_{n,n+k}(z)\right| \le 2^{-m-1}\left|\lambda_{n,n+k-m-1}(z)\right| + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots \frac{\epsilon}{2^{m+2}}$$

$$< 2^{-m-1} + \frac{\epsilon}{2} \le \epsilon,\ |z| \le \frac{1}{4},\ n \ge N.$$

As $\epsilon > 0$ is arbitary $\lim_{n\to\infty} \lambda_{n,n+k}(z) = 0$ uniformly for $|z| \leq R < 1$. Finally as

$$\frac{\Phi^*_{n,m}(z)}{\Phi_{n,m}(z)} = \left(\bar{\Phi}_{n,m}(\bar{z})^{-1}\right)\left(\bar{\Phi}_{n,m}(\bar{z})^{-1}\right)^{-1},\ |z| > 1$$

we are done. ■

Section 3B : Strong and weak convergence of orthogonal polynomials on $|z| = 1$

We now proceed to prove some important results concerning strong and weak convergence of orthonormal polynomials on τ. To this end we recall that we obtained weak results for example in Theorem 2.6 which involved Szegõ weights. As the Szegõ function is no longer defined, the results use its analogue in this setting $\mu'(\theta)$ instead of $\left|D(w,z)\right|^2 = \mu'(\theta)$ *a.e.* (*meas*).

Proof of Theorem 3.9

We write

$$\begin{aligned}
0 &\leq \int_0^{2\pi} \left(\left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right| (\mu'(\theta))^{1/2} - 1 \right)^2 d\theta \\
&= \int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right|^2 \mu'(\theta) d\theta - 2 \int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right| (\mu'(\theta))^{1/2} d\theta + \\
&\quad + \int_0^{2\pi} d\theta = I_1 + I_2 + I_3.
\end{aligned}$$

Firstly,

$$I_1 \leq \int_0^{2\pi} \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right|^2 d\mu(\theta) = \int_0^{2\pi} \left|\varphi_{n,n+k}(z)\right|^2 d\mu_n(\theta) = 2\pi.$$

Also $I_3 = 2\pi$. Thus $I_1 + I_3 \leq 4\pi$. We must show that

$$\liminf_{n\to\infty} \int_0^{2\pi} (\mu'(\theta))^{1/2} \left| \frac{\varphi_{n,n+k}}{W_n}(z) \right| d\theta \geq 2\pi$$

for then

$$\lim_{n\to\infty}\int_0^{2\pi}\left(\mu'(\theta)\right)^{1/2}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|d\theta \geq 2\pi$$

and (3.11) holds.

Let f be a 2π periodic, non negative continuous function then we have as in the proof of Theorem 3.6,

$$\left(\int_0^{2\pi}(f\mu')^{1/4}d\theta\right)^4 \leq 2\pi I_1 I_2$$

where

$$I_1 = \int_0^{2\pi} f(\theta)\left|\frac{W_n(z)}{\varphi_{n,n+k}(z)}\right|^2 d\theta \to \int_0^{2\pi} f\,d\mu \quad \text{as } n\to\infty$$

and

$$I_2 = \left(\int_0^{2\pi}\left(\mu'(\theta)\right)^{1/2}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\theta\right).$$

Now choose an appropriate sequence of functions as in the proof of Theorem 3.6. We obtain $\liminf_{n\to\infty}(I_2) \geq 2\pi$. So (3.11) holds. Now to prove (3.12) we proceed as follows.

$$\begin{aligned}
&\left(\int_0^{2\pi}\left|\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\mu'(\theta)-1\right|d\theta\right)^2\\
&\leq \left(\int_0^{2\pi}\left(\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\left(\mu'(\theta)\right)^{1/2}-1\right)^2 d\theta\right)\times\\
&\quad\times\left(\int_0^{2\pi}\left(\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\left(\mu'(\theta)\right)^{1/2}+1\right)^2 d\theta\right)\\
&=: I_4 I_5.
\end{aligned}$$

Now $I_4 \to 0$ by (3.11). Also $I_5 \leq C_8$ using (3.44) thus (3.12) holds with $\ell = 0$. But recalling (3.8) we see it is sufficient to prove (3.12) for $\ell = 0$. So we have (3.12) ∎

The proof of Theorem 3.10

(3.13) follows immediately from (3.12) by writing

$$\left|\frac{\int_0^{2\pi} f(\theta)\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}z^\ell\mu'(\theta)d\theta}{|W_n(z)|^2} - \int_0^{2\pi} f(\theta)d\theta\right|$$
$$\leq \left|\int_0^{2\pi} f(\theta)\left(\frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}z^\ell\mu'(\theta)}{|W_n(z)|^2} - 1\right)d\theta\right|$$

using the Schwarz inequality and the boundedness of f. To get (3.14) we show that

$$\text{(3.45)} \qquad \lim_{n\to\infty}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}z^\ell}{|W_n(z)|^2}\right|^2 d\mu_s(\theta) = 0.$$

We note that from (3.13) with $f = 1$ we get

$$2\pi = \lim_{n\to\infty}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\mu'(\theta)d\theta$$
$$\leq \lim_{n\to\infty}\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\mu(\theta) = 2\pi.$$

Thus

$$\lim_{n\to\infty}\int_0^{2\pi}\left|(\varphi_{n,n+k}(z))(W_n(z))^{-1}\right|^2 d\mu_s(\theta) = 0.$$

So by the Schwarz inequality,

$$\int_0^{2\pi}\left|(\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+\ell}(z)}z^\ell)(W_n^2(z))^{-1}\right|d\mu_s(\theta)$$
$$\leq \left[\int_0^{2\pi}\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 d\mu_s(\theta)\right]\left[\int_0^{2\pi}\left|\frac{\varphi_{n,n+k+\ell}(z)z^\ell}{W_n(z)}\right|^2 d\mu_s(\theta)\right] \to 0.$$

Thus (3.45) holds. Therefore (3.45) and (3.13) give (3.14). ∎

To end this chapter we give some strong convergence results of orthonormal polynomials in this setting. We need:

Definition 3.11

Let $w = re^{i\alpha}$, $r < 1$ or $r > 1$ and $z = e^{i\theta}$. Define

$$g_{n,n+k}(w) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\overline{\varphi_{n,n+k}(z)}}{w-z}\right]d\mu_n(\theta)$$

called the function of the second kind of order $n+k$ with respect to $d\mu_n$.

Remark

We note that these functions will be defined more extensively in Chapter 6.

Lemma 3.12

$$(3.46) \qquad \frac{1}{2\pi}\int_0^{2\pi}(w-z)^{-1}d\theta = \begin{cases} 0, & |w| < 1, \\ (w)^{-1}, & |w| > 1, \end{cases} \qquad (z = e^{i\theta}).$$

Proof:

Write

$$\frac{1}{2\pi}\int_0^{2\pi}(w-z)^{-1}d\theta = \frac{1}{2\pi}\int_{|z|=1}(w-e^{i\theta})^{-1}(ie^{i\theta})^{-1}dz$$
$$= \frac{1}{2\pi i}\int_{|z|=1}(wz-z^2)^{-1}dz = (2\pi iw)^{-1}\int_{|z|=1}(z)^{-1}dz +$$
$$+ (2\pi iw)^{-1}\int_{|z|=1}(w-z)^{-1}dz = \begin{cases} (w)^{-1}, & |w| > 1, \\ 0, & |w| < 1, \end{cases}$$

(by the residue theorem).

We can now state and prove our last theorem of this chapter.

Theorem 3.13

Let $(\mu, \{W_n\}, K)$ be admissible on $[0, 2\pi]$. Then the following limits hold uniformly on each compact subset of the corresponding region:

$$(3.47) \qquad \lim_{n\to\infty}(\varphi_{n,n+k}g_{n,n+k})(w) = 0, \ |w| < 1;$$