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Learners' Participation in the Functions Discourse

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University of the Witwatersrand

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Declaration

I declare that this research report is my own work and no part of it has been copied from any other source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list.

_____

Signature
Date: 30 May 2016

Abstract

This study investigated learners' mathematical discourse on the hyperbola using commognitive theory, with particular focus on the use of words, narratives, routines and visual mediators. Data was collected by means of task-based interviews with nine Grade 10 and 11 learners from a township school in Johannesburg, South Africa. An analytical tool, named the Discourse Profile of the Hyperbola, was adapted from the Arithmetic Discourse Profile of Ben-Yahuda et al (2005) and was used to analyse learners' mathematical discourse. The study focused on three representations of the hyperbola, namely, the formulae (equation); the graph and the table.

Learners' views and definition of the asymptote, in relation to the graph, emerged as a central theme in the analysis. The analysis also focused on the mismatch between what is said and what is done by learners, for example most learners sketched the graph of a hyperbola showing a vertical asymptote yet talked as if there is no vertical asymptote. Most routines were ritualised, for example learners failed to link iconic and symbolic mediators they had used in responding to tasks. However, there were traces of exploratory routines from a few learners, evidenced by links between equations, and identifying the hyperbola from unfamiliar tasks.

While a few learners used literate words, colloquial word use was dominant. The discourse of learners was found to be visual. For example, some reasoned that an equation with a fraction represents a hyperbola while an equation not expressed in standard form does not represent a hyperbola. Some learner narratives are not endorsed by the community of mathematicians, for example, that $y = \frac{3}{x}$ has no asymptotes.

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Chapter 1: Introduction to study

1.1 Introduction

In this study I investigate the mathematical discourse on the hyperbola of nine learners, four in Grade 11 and five in Grade 10, from a secondary school in Johannesburg. The study focuses on the participation of learners in the hyperbola discourse. I have analysed data in chapters five and six using an analytical tool that I adapted from the Arithmetic Discourse Profile by Ben-Yehuda, Lavy, Linchevski and Sfard (2005), and which I have termed the Discourse Profile of the Hyperbola (DPH). I have used the theory of commognition by Sfard (2007) as my theoretic framework and discussed both the general and the school hyperbola, as well as learners' difficulty with functions in chapter two.

Functions is one of the most important topics in the mathematics curriculum, reflected in the number of marks they contribute in the Grade 12 Mathematics paper one. However, the official Department of Basic Education (DBE) Grade 12 reports show that learners struggle with functions. In this study I investigated the manner in which learners engage with Grade 10 and 11 functions, specifically the hyperbola. I used the theory of commognition by Sfard (2007) as my lens. Much of the work on functions is done in Grades 10 and 11; hence I chose these grades for my study.

In this chapter I introduce my study by giving reasons for focusing on functions, with emphasis on the hyperbola. I have also justified the choice of the theory of commognition, and have given a synopsis of this theory. I also discuss the purpose of my study, and its rationale and significance.

1.2 The Problem

In the past two years there has been a steady decline in the Grade 12 Senior certificate examination in Mathematics results (Department of Basic Education, 2015). At the same time, the percentage of learners writing Mathematics has been decreasing (DBE, 2015). The decline in percentage pass rate and the lower percentage in learners that make mathematics an option gives an impression that there is no improvement in the learning of mathematics. Further, statistics show that 68% of the candidates that wrote the examination got marks less than 40% with only 20,3% attained marks that above 50% in the 2015 examination (DBE, 2015). In the same examination 12% of the learners exceeded 60%: of these only 2,9% exceeded 80% (DBE, 2015), as illustrated by table 1.1.

Level	1	2	3	4	5	6	7
Marks	0-29	30-39	40-49	50-59	60-69	70-79	80-100
Percentage of learners	50,9	17,1	11,6	8,3	5,4	3,7	2,9

Table 1.1 Distribution of learners in Grade 12 mathematics in 2015

Functions and their applications take 46% of the questions in paper one of the senior certificate examinations. Therefore, an investigation on functions goes a long way in finding solutions to challenges that learners face, and informs the researcher on the suggestions that may help in the learning of functions, especially the hyperbola.

A closer look at the examination reports for paper one between 2012 and 2015 reveals that learners did not do well in the functions section, and in particular on questions about the hyperbola. Candidates had difficulties with a) the format $y = \frac{a}{x+p} + q$; b) writing an asymptote as an equation; c) failure to translate graphs and d) poor algebraic manipulation (DBE, 2015). In my report some of these challenges will be discussed, as they were prominent with learners in this study. Reports further state that learners found questions on the hyperbola difficult to answer (DBE, 2014). The 2015 mathematics report suggests that more attention should be given to the teaching of functions (DBE, 2015). It was therefore important to investigate the learning of functions and make recommendations based on the findings. In the next section I shall give an overview of the discursive approach to teaching and learning of mathematics by way justifying my choice of the theory of commognition.

1.3 Commognition as a discursive approach

In recent years there has been an emergence of discursive approaches to the teaching and learning of mathematics, and the theory of commognition is one of those.

The theory of commognition brought about by Sfard (2007) considers thinking as communication with one's self. Mathematical thinking is viewed in two ways that are complementary and equivalent, and these are cognition and communication (Sfard, 2007). Learning happens when a discursive conflict is created. This is when communication causes the learner to change the position he/she held. An example is the notion that multiplication makes bigger. When a learner is led through a conversation to realise that in fact multiplication by a fraction does the opposite, then

learning would have happened. The teacher does not merely tell this to the learner, but opens a conversation that will lead the learner to notice that the earlier thinking is correct only for positive integers. The theory of commognition will be discussed more fully in chapter 2.

Given the discursive theories (situated learning; commognition, language in multi-lingual classrooms), I have decided to do a study that is based on mathematical discourse. I shall use commognition to analyse and describe how the Grade 10 and 11 think and justify mathematical concepts in functions.

1.4 Purpose of the study

The purpose of this study is to investigate the words, substitutions, procedures and visual mediators that learners make in completing tasks involving functions, interpreting information from table of values and drawing graphs of functions, in particular in the hyperbola. The study was guided by the following questions:

1. What is the nature of learners' participation in mathematical discourse on the asymptote of the hyperbola in Grade 10 and 11?
2. What difficulties do learners face in working with different representations of the hyperbola?
3. How do learners participate in unfamiliar tasks related to the hyperbola?

1.5 Rationale

In the curriculum functions and their applications account for approximately 45% of Grade 10 to 12 Mathematics (DBE, 2011). The hyperbola is one of those functions. According to previous years' diagnostic reports learners perform poorly on functions, with some questions reflecting an average of less than 40% of the marks in the questions that involve the hyperbola (DBE, 2012; DBE, 2013). One reason suggested for this is that learners do not answer certain parts of the questions. I wanted to find out the mathematical discourse of those learners that do well in functions, and so learners likely to give good practice were the targeted group. I found, as the analysis will show, that their responses to interview questions were not necessarily representative of good practice.

From my experience of teaching Mathematics to learners in different environments, including rural and urban settings, learners struggle with tasks on functions. Investigating the nature of learners' participation in discourse on the hyperbola, with particular attention on the asymptote and unfamiliar tasks, has helped me to discover some of the reasons for their struggles. This has been explained more fully in chapters six and seven.

From my experience, I have noted that learners have some errors they keep on repeating. Some of these errors are made by a number of students. Brodie & Berger (2010) urge that errors are narratives that are endorsed by learners. I hope the substantiations from learners would help those interested in the learning of Mathematics understand where learners are coming from when they have difficulties with the hyperbola.

According to Moalosi (2014) functions are a topic that learners find difficult to learn. The hyperbola is one of the functions that pose difficulties to learners. One of the difficulties comes from the fact that, while other functions are continuous, the hyperbola has two parts, even though the equation does not. For all of the other graphs, points on the table of values can be chosen at random, but for a hyperbola because of the asymptotic behaviour it is not easily revealed without a careful selection of points. Learners should identify the asymptote and where the graph cuts the intercepts. In other functions once the shape and intercepts have been identified sketching is then easy. The calculations in a hyperbola are not as easy. The solution has a clearing of fractions. Working on equations with fractions is not easy, as the diagnostic report for 2014 national senior certificate examinations for 2014 states (DBE, 2015). Therefore, a hyperbola is not an easy function to work with.

Functions are a key component of Mathematics, since they are a meta-discourse of algebra (Sfard, 2012). Mathematics is hierarchical, and as such, learners begin with algebra then go on to functions. Clement (2001) argues that the concept of a function plays a crucial part in the learning of Mathematics throughout the curriculum. Swarthout, Jones, Klespis & Cory (2009) also point out the importance of functions in the Mathematics curriculum, and claim that it is necessary for all learners to understand them. Functions help learners describe the relationships between variables, explain the changes that occur between objects, and interpret and analyse graphs. Carlson, Oehrtman & Engelke (2010) and Carlson & Oehrtman (2005) view functions as a bridge that connects algebra to calculus. It is therefore important to learn functions.

A hyperbola is a function that helps to illustrate the idea of a limit more than other functions studied in the FET phase. From the point of view of learning, a discursive conflict has to be established for successful learning of the hyperbola. My observation has been that learners are used to functions that give a continuous line or curve, and I have observed some learners trying to close the gap. I had hoped to find the reasons for this. Only one learner tried to draw a single continuous graph, but could not give reasons for doing so.

1.6 The significance of the study

Little research has been done on the learning of the hyperbola. My study has produced interesting observations on the difficulties learners have with the asymptote. The concept of the asymptote had not been fully grounded for the learners in my study. They are able to show it in diagrams, but their explanations indicated something else. They were found to do very well with familiar tasks, but struggled with unfamiliar ones. Correct responses to tasks do not necessarily mean that learners have grasped the concept.

I had hoped that, since my study was focused on the top quartile, there would be good practices that would show, but only more difficulties were found. These are learners capable of getting high marks in examinations, and had already done so in the first two terms of the year (2014). The struggles that they faced, in particular with justifications of strategies and generalisations that are not endorsed by the mathematics community, would help the Mathematics community, especially in the Department of Education, with ideas that could be tried in the teaching of functions in general, and particularly the hyperbola.

I also hope that the study of the hyperbola would give practicing teachers a more focused object of learning when teaching functions. The object of learning is what the teacher intends learners to get out of the lesson.

1.7 Conclusion

In this chapter I discussed the purpose of the study, the rationale and how the study is going to benefit teachers, learners and other interested parties. In chapter two, I discuss the mathematical discourse and its special features according to Sfard (2007). I then discuss what literature shows about the learning of functions in chapter two. This will include a discussion on learning of functions, difficulties that learners have with them, and also the study on the general and rectangular hyperbola. Chapter three deals with the methods that were used to collect data, followed by how the data will be analysed. I then go to the ethical considerations and conclude with the issues of reliability and validity. Chapter four carries discussion of the analytical framework tool, the Discourse Profile of the Hyperbola (DPH). Chapter five has analysis and discussion about learners' participation in the functions discourse, with particular attention to asymptotes. Chapter six focuses on learners' responses to unfamiliar tasks on the hyperbola. Chapter seven is the discussion of the findings and the summaries of the study.

Chapter 2: Literature Review and Theoretical Framework

2.1 Introduction

In this chapter I discuss the theory of commognition (Sfard, 2007). I discuss mathematical discourse and how it develops. I will distinguish between object-level learning and meta-level learning, and I will explain the four characteristics of mathematical discourse. It is from these that the analytical tool the Discourse Profile of the Hyperbola (DPH) is formulated. I will then focus on my main area of study, the hyperbola. I will start with the general hyperbola and then go to the rectangular hyperbola. I will then talk of the asymptote, the general definition and the school definition. Thereafter I will discuss the importance of learning of functions, I shall review literature on the learning of functions and discuss the difficulties that learners encounter in their attempts to understand functions.

2.2 Mathematical discourse

Sfard (2012:2) defines a discourse as a "specific type of communication". Mathematical discourse is a subset of discourses, with its own key characteristics. These characteristics are what Berger (2013: 2) refers to as "a range of permissible actions and reactions". A mathematical discourse is characterized by four characteristics of the commognition theory. These characteristics will be discussed later. Mathematical discourse is said to be autopoietic (Nachlieli and Tabach, 2012). This means that it grows from within. As new information is formulated the discourse must expand. For example, some of the rules that learners hold of natural numbers need to be altered to accommodate negative numbers. In natural numbers addition and multiplication are not concerned with signs; but once negative numbers are introduced the mathematical community has to agree on those rules that could remain, and those that must change. The new sub-discourses that develop within the mathematics discourse never existed until they are introduced by mathematicians: hence mathematics develops from *within*, with the mathematics community endorsing or rejecting particular rules. In other discourses, such as, for example, in agriculture, objects like meat, cattle and kraals are known to learners prior to their learning about them in school. Mathematical objects like rational numbers, quadratic equations, calculus, however, are by-products of mathematical conversations, and are met by learners for the first time at school. Learners do not have previous knowledge of mathematical objects, so mathematics grows from within itself. There are two kinds of growth within mathematics: the object-level development and the meta-level development.

In object-level there are new narratives: for example, when learners are introduced to linear functions there are new objects for them to deal with. These objects include axes, intercepts, gradient and the like. The graph is completed by drawing a line that joins all the points. The rules used in the linear functions discourse remain the same as those of plotting points on a Cartesian plane, and the development is cumulative. At the introduction of linear functions learners are already familiar with plotting of points. One major characteristic of object-level development is that it is possible for learners to construct narratives without the interlocutor. For Grade 10 learners that have plotted points in earlier grades it is possible for them to make narratives on the linear function without the assistance of the interlocutor, because the growth is cumulative. There are no contradictions between the narratives of sketching a linear graph and those of plotting points.

Meta-level development is characterized by change in rules, and new objects change in rules of endorsement. In meta-level development there are apparent contradictions between the newly-introduced narratives and previously accepted narratives. Introduction of the hyperbola brings about meta-level development, in that new objects are introduced: a function with two parts, while the other graphs are single and continuous; words and lines like asymptotes are introduced that are absent in the previous endorsed narratives; on the left the graph approaches negative infinity, while on the right hand side it approaches positive infinity; x and y in the function cannot take certain values. When sketching linear and other functions the starting point after axes is plotting the points, but in the hyperbola it would be identifying the asymptotes. In object-level development learning is cumulative, but for meta-level there is a change of rules. The development of mathematical discourse is a result of a team effort. A community of mathematicians agrees on which new narratives to endorse and which to reject, based on the previously-endorsed narratives. Discourses develop vertically when there is a combination of existing discourses to form new meta-discourses at a higher level (Sfard, 2012). Algebra is an example of a vertical development in that it is a combination of arithmetic and numeric patterns. Horizontal development is a result of a combination of separate discourses into a single new discourse. For example, the graphs of motion combine solving equations, rate and calculus. Functions would be classified as vertical development as they mostly emerge from algebra.

Discourses develop in an attempt to get *compression* (Sfard, 2012). Worded and long mathematical statements are expressed in a shortest possible way by expressing the same words or a list of numbers in symbolic form. A cube expressed as x^3 denotes a list of numbers that are being cubed like…-27; -8; -1; 0; 1; 8; 27…. The cube (x^3) may describe a cubic function and may denote the

shape that is produced by that function. The object (x^3) is key to the compression of the mathematical discourse on cubes and cubic functions. Learning as a subset of development is about being able to talk or use new mathematical objects. When new mathematical objects are used there is change in mathematical discourse. The change can happen at meta-level or object-level. The learner has to participate for him/her to benefit from the change in discourse, be it at object-level or meta- level.

The presence of an interlocutor is important in the learning of mathematics for discursive theories. Sfard (2007) says the interlocutor is an experienced discursant in the discourse. The learner comes in to access the formal discourse but enters the scene through the use of his/her informal language. Through a mathematical discussion the interlocutor makes the learner realise that the ideas he held before need some adjustment. For example, a linear function has a constant gradient in a particular line, but when it comes to other functions learners get to know that in a single function the gradient varies. The interlocutor starts the discussion from where the learner is comfortable and moves him/her gradually into noticing that different rules now apply. It is the consciousness that the learners' knowledge is not adequate that motivates him/her to listen to the interlocutor. The learner participates by means of "thoughtful imitation" (Sfard, 2012) of the interlocutor, the knowledgeable other. Thoughtful imitation means that at first the learner does what he/she sees interlocutor doing. When there is success the learner seeks to understand why this is true. The interlocutor scaffolds the learner until he is able to do the work thoughtfully by him/herself. Moschkovich (2002) describes communicating mathematically as participating in mathematical practices such as abstracting, generalising and being precise. In other words, participation in a mathematical discourse means focusing on conceptual understanding rather than just computational fluency. The participation is then individualized when there is talk with others or talk with oneself through thinking that is expressed through discursive action (Sfard, 2012).

Sfard (2008) links communication and cognition. In commognition learners participate in a mathematical discourse through both talking and thinking (Sfard, 2008). "Thinking is regarded as communication with oneself" (Sfard 2012). Thinking is expressed in verbal or written form. So participation can be seen in action. Development is seen in change of discourse (Sfard, 2012). The aim of the discussion is to get learners thinking like mathematicians. If the conversation in the discourse remains the same, then development has not taken place. Learning is a subset of development. The focus is not the change in a learner but the change in the discourse. Development in the discourse is seen when the participants use new rules in the mathematical discourse (Sfard,

2012). In Grade 10 learners are able to plot points, but when they are introduced to a linear function new rules apply. The points plotted are no longer random but follow a straight line. If a point does not lie on a straight line, then there must be something wrong with substituting that resulted in getting that point, or else the point does not belong to that particular line.

The commognitive conflict, as opposed to cognitive conflict from acquisition theories, is distinguished by three differences (Sfard, 2007). Learning is change in how learners participate in a discourse. Firstly, learners experience a need to change how they talk before they can change (*ibid*). The need for new ways of talking is a result of interacting with others (*ibid*). Secondly, the change or development in a mathematical discourse is a consequence of other people's examples (*ibid*). Learners may sketch all functions as if they are linear, but when someone gives an example of a quadratic, the need for change is created. The third way of developing a discourse is that it develops gradually (*ibid*). After learners engage in a quadratic functions sub-discourses they become better participants over time. Initially, they may draw the graphs procedurally, but with time they realise that the sign on the coefficient of the square determines where the graph will face, that the constant is the y-intercept and so on.

Sfard (2012) views algebra as a discourse that is developed from reflection and the processes of arithmetic. In other words, algebra is a meta-discourse of arithmetic. This means that algebra is developed from generalisations of arithmetic. A mathematical discourse is a specific type of communication. Functions are part of algebra, and are composed of the equation, table and graph (Caspi and Sfard, 2012). In the discourse of functions, the discussion rests on words like intercept, co-ordinates, turning point, gradient and the like. The functions discourse is grounded in the understanding of algebra. Learners need the solution of equations for them to get co-ordinates.

Sfard (2008) proposes objectification as replacing the talk about processes and actions to talk about objects. Objectification is illustrated by learners referring to the asymptote rather than to a line that the graph does not pass through or cross. Objectification makes communication in the mathematical discourse more effective. The challenge with mathematical objects is that they are abstract and hidden in the discursive layers and metaphors they are composed of (Caspi & Sfard, 2012). Sfard (2012) suggests that learners can get to objectification if they understand the properties of mathematical concept.

Nachlieli & Tabach (2012:17) discuss four principles of objectification. These are: firstly, the learners' competence in the realization tree. A realization tree, according to Nachlieli & Tabach (2012), is a signifier on which the mathematical object is organised or built. In the functions discourse the realization tree will include the graphs, tables, equations or formulae and verbal expressions of the function. In the hyperbola the realization tree will include the curve, the table of values that clearly show the asymptote and the formula in the form of $y = \frac{a}{x-p} + q$. Thus learners should be able to draw curves, interpret the table of values and work with equations with fractions. Competence in the functions discourse comes after learners have had competence in each of the discourses on graphs, symbolic expressions and tables. Key to objectification in the functions discourse is competence on the realization tree because the introduction of the word hyperbola is new for learners. There is no way learners can know this word prior to it being introduced in the classroom, and therefore objectification is preceded by the competences in the realization tree because mathematics is autopoietic, and its objects do not pre-exist their talk (Nachlieli & Tabach, 2012).

The second principle brought by Nachlieli & Tabach (2012:17) is: "participation in discourse is a precondition for the objectification of functions". Learners start the process by informally, using examples and new words rather than the formal discourse. Participation in the discourse then grows gradually with the informal terms substituted by formal terms because of conjecturing and generalisation. Adler & Ronda (2014) describe this stage as non-mathematical, where the talk is visual, learners describing what they see and routines are highly ritualised. The talk is mainly colloquial (Adler & Venkat, 2014). Learners may not use the terms function or hyperbola at this stage.

The third principle is the gradual and gentle introduction of the formal discourse. The formal discourse includes the covariance of quantities, using the word function as a noun. Learners start using rules and talk about processes (Adler & Ronda, 2014). For example, a hyperbola would be a function with undefined values. Later in the third principle various ways in which the function can be represented and how the different representation of the same is introduced. For example, a point (2;5) on the table of values is the same as (2;5) on the curve, and in the equation, and for all other points. Infinitely many points would have the same relationship.

The final principle is that reflection on the object should happen. Tasks that promote reflection on the objects should be given to the learners. The reflection is promoted by derivation of rules,

equivalent representations (Adler & Ronda, 2014) and being able to explain how they relate to each other. Principles and properties that govern functions are grounded. The examples and tasks given to learners should be such that they lead learners to make conjectures and to generalise. It is important that a function can be expressed by each representation of that function.

2.2.1 Key features of a discourse

Sfard (2008) gives four characteristics of commognition that show the development of the mathematical discourse. These are the words used, the visual mediators, narratives and routines. It is from these characteristics of commognition that the Discourse Profile of the Hyperbola (DPH) was used to analyse the development of the mathematical discourse of learners in this study. They need to be explained:

Key words, or signifiers, are mathematical words or symbols like the equal sign, function, asymptote, intercept, axes, coordinates and so on. It is through these words that learners communicate mathematically. Key words in mathematics are a crucial part of the formal discourse. In the functions discourse the key words are intercepts, domain, coordinates, turning points, minimum/maximum, and gradient and so on. In the hyperbola discourse in general the key words include ***asymptotes***, ***quadrants, axis of symmetry, intercepts, the vertex*** to name just a few. Caspi and Sfard (2012) talk of compression, and it is through these words that we find compression by using a few words to express otherwise wordy statements. For example, instead of saying the point at which the graph or function cuts the axis, the word intercept is used. The DPH has two categories of word use, mathematical and colloquial. The categories are explained fully in chapter five.

The key words are supported by visual mediators, which help the participants to identify the objects of their talk and co-ordinate their talk (Sfard, 2008). Visual mediators include concrete objects, iconic mediators and symbolic mediators. Concrete objects and their images are the usual mediators that are used in everyday life, but in mathematics symbolic artefacts are often used to help communicate. Examples of visual mediators include diagrams, charts, drawings, graphs and symbols. The visual mediators can be drawn on paper or boards in the classroom, and it is possible for them to be done mentally by thinking about them. Most of the mathematics that is written uses the symbolic syntactic mediators: these are the symbols used in mathematical manipulation. The functions notation and symbols used include the following: $y = a(x-p)^2 + q,\ or\ f(x) = \frac{a}{x-p} + q, x \in \mathbb{R}, x \neq p$ for the hyperbola. Iconic mediators are commonly used in the teaching and learning situation. They are the pictorial diagrams that are used either in class discussion or in writing by

learners. This is when an abstract object is represented by a diagram, for example, the graph that represents the motion of a ball. The hyperbola diagrams, (see figure 2.1) helps learners to have an image of the function.

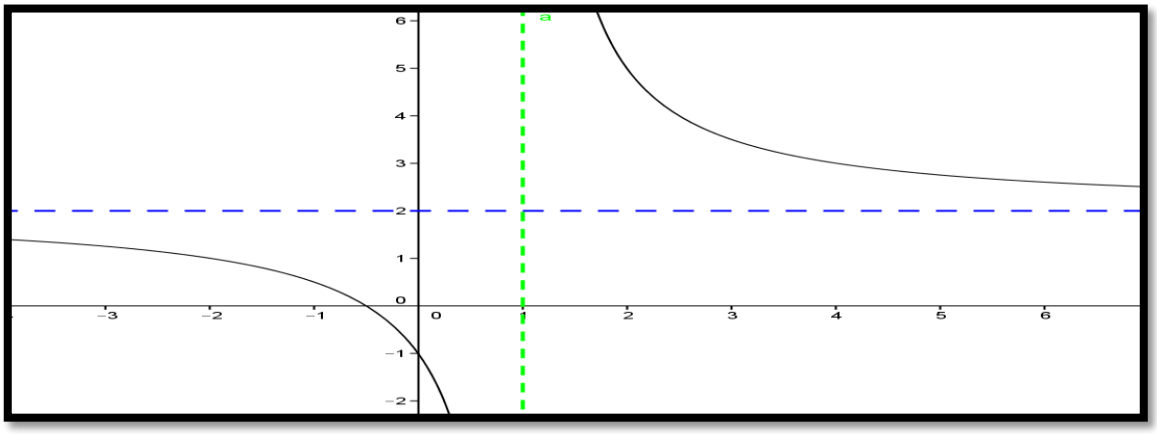


Figure 2.1 Iconic mediator

The iconic mediator illuminates the syntactic mediator by showing the intercepts, asymptotes, axes and the shape of the graph. It helps create an image that learners refer to when faced with the questions that relate to the hyperbola. Concrete mediators are the objects that are used to assist in illuminating mathematical concepts. An example that comes to mind is a rectangular box that is opened in class to help illustrate the number of surfaces that are to be calculated. The hyperbola does not have concrete mediators because it is generally abstract. The DPH has four categories of the visual mediators. The iconic visual mediator has the viewed where a graph or table is provided and the drawn where learners produce a graph. The symbolic mediator has the viewed and the formulated. Chapter five provides a full explanation of the DPH.

The third key feature of mathematical discourse is the narratives. Endorsed narratives are the rules that have been agreed upon by the mathematical community (Sfard, 2008). Narratives include definitions, theorems, proofs and axioms. They are therefore useful in the development of new discourses. Sfard (2012) maintains that once a new discourse has been established, for example, a new function, some basic narratives would have been endorsed. In the development of new discourses mathematicians decide which narratives to keep and which to reject. Sfard (2012) says narratives are a result of either object learning or meta-level learning. Learners engage with the mathematical objects at the object-level. Meta-level is governed by well-defined meta-rules, and learning can only happen through thoughtful imitation and understanding of the narratives that govern the discourse for the hyperbola. The narratives include a graph where there are two separate curves that are mirrors image of each other, and a graph with two asymptotes that are perpendicular to each other. For the rectangular hyperbola, from the equation: $f(x) = \frac{a}{x-p} + q$, we get the

equations of the asymptotes, $y=q$ *and* $x= p$. The asymptotes are a result of a narrative that describes the behaviour of the graph towards a line. In the function $(x) = \frac{1}{x}$, as x approaches zero from the right, y approaches infinity; and as x approaches zero from the left, y approaches negative infinity. The paths close to zero that the function approaches from both the left and the right are the asymptotes. Mathematicians had to endorse this narrative so that the narratives agreed upon earlier would not collapse. A hyperbola has two lines of symmetry that are given by the equation $y - q = \pm(x - p)$. The interlocutors bring these narratives to the fore so that learners can own them. The learner moves from the discourse of others to discourse for one-self (Ben-Zvi & Sfard, 2007).

Brodie & Berger (2010) talk of errors as narratives that are endorsed by learners but of which mathematicians and experienced participants would not approve. Narratives should be universally approved by the community of mathematicians. In teaching and learning narratives can help in identifying errors learners make. Substantiations from learners make sense to them, but are not in agreement with what is universally accepted. One example of narratives from learners is when they ignore the asymptote and join all the points they see in the graph. When erroneously endorsed narratives have been identified, corrective measures can be taken. In the DPH I have explained two categories the substantiations and the recall narratives.

Routines are well-defined patterns repeated over time, and are characteristic of a given discourse (Sfard, 2007). Routines include mathematical procedures used to perform mathematical tasks. Methods of proving, comparing graphs, looking at key words in questions, generalisations and many more are part of the routines. Sfard (2008) argues that it is within the routines that words, mediators and narratives meet.

The question comes to mind: "How will learners start with sketching of graphs of the hyperbola?" Routines for the hyperbola include completion of tables, calculating intercepts, identifying the equations of the asymptotes, drawing lines that show the asymptotes and drawing of the curves. Sfard (2008) urges that the key is in knowing ***how*** and ***when*** to use these routines. Focusing on ***how*** to use the routines results in the discourse of rituals rather than explorations (Sfard, 2008). Ritualised routines or focusing on the how is about following the rules. A learner may plot points for a hyperbola and draw the asymptotes but still be doing ritualised routines because of not understanding why those steps are done. Exploratory routines are all about *when* a certain step should be taken. This is about the appropriate circumstances for taking particular actions. When sketching a hyperbola, understanding how the graph relates to the asymptote is an exploratory

routine, but just following the interlocutor's instructions is not. An action that may be an exploratory routine when done by an interlocutor may be a ritualised routine when done by a student. Exploratory routines are about making an individual choice about mathematical decisions, and this would show that individualization has taken place. It is important to note that learning usually starts from ritualised followed by exploratory routines. Learning will have occurred when learners know ***when*** to take certain actions in their doing of mathematics

.

I have termed the ritualised and exploratory routines as kinds of routines in the DPH. Flexibility which is the use of more than a single routine so as to arrive at the same narrative and is mostly used to prove endorsed narratives (Ben-Yehuda et al, 2005). Flexibility is seen when learners use the table of values to show values approaching an asymptote and at the same time sketching a graph that shows points getting closer to the asymptote. Applicability routines are about the likelihood of a routine procedure to be used or applied in mathematics tasks (*ibid*). In corrigibility, routines are used to check the correctness of answers arrived at. An example of corrigibility routine is to use intercepts of a graph solve an equation of a function equated to zero and algebra to check the correctness of the solution from the graph. In DPH, applicability, flexibility and corrigibility routines are termed *the use of routines* because they explain how routines can be used in enhancing the learning of mathematics.

2.2.3 Learning as Participation

The learning of mathematics is important in that it enables learners to have tools through which they can be able to participate in everyday life. These tools include interpreting, analysing, describing, making predictions, or even solving problems they may face in their school or adult life. Unfortunately, not all learners are able to access this knowledge or skills at the same time. Educational theories have been put forward as a way of trying to find a solution or a way of maximising learning. These include cognitive (Piaget, 1972), social constructivist (Vygotsky, 1978, situated (Lave & Wenger, 1991), and social learning theories (Bandura, 1986).

One such theory that seeks to find a solution to the learning problem is the situated learning theory. In this theory there is no separation between learning and participation (Lave & Wenger, 1991); hence Rogoff (1995) argues that learning is a process of becoming rather than acquisition. For a situated learning theorist, learning is participation. Newcomers participate from the periphery in the community of practice (Lave & Wenger, 1991). The 'old-timer' is responsible for helping the newcomer participate in the community.

Commognition is one of the partipationist theories. Learning is the process of individualising the mathematical discourse (Sfard, 2007). Learning becomes the means through which one communicates mathematical concepts with others and oneself in an objectified manner. For Sfard (2012) learning is becoming a competent participant in a discourse or practice. This participation is evident in communication by a learner. Commognition looks at talking and thinking as part of communication (Sfard, 2008). Thinking is seen as an individualized form of activity of communication (Sfard, 2007). There are two main ways in which thinking can be expressed, one through talking and the other through writing. So learners participate when they talk and bring forth their reasoning in class or group discussion or otherwise participation can be seen in solving problems in their writing. Thinking is a discourse with oneself (Sfard, 2008). Rogoff (1995) urges that individuals change through being involved in activities. I would put it that participation of learners in the mathematical discourse is measured by learners' talk, gestures and writing in the particular sub-discourse they are engaged in.

2.3 The hyperbola

I will discuss the hyperbola from two fronts, the general hyperbola and the rectangular or school hyperbola. The general hyperbola is disciplinary mathematics while the school mathematics has been contextualised from disciplinary mathematics through particular selections and emphases. The general hyperbola is disciplinary mathematics, and is beyond the school curriculum. The school hyperbola is, of course, the one found in the curriculum. I shall first discuss the general hyperbola because it is from the general hyperbola that the school hyperbola is derived.

2.3.1 The general hyperbola

The definition for mathematicians is: *A hyperbola is the set of all points P in a plane such that the difference of the distances from point P to two other points in the plane is a positive constant* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Narasimhan, 2009). I shall discuss this definition of the hyperbola.

A hyperbola is a function that has two separate curved parts. The two curves are a mirror image of each other about a given line. A hyperbola is an open curve - in other words, the curve does not have an end, but approaches a certain line from the left and the right or from the top and the bottom - that is, in four directions. In the curve of the hyperbola the ratio of the focus (a fixed point) and the directrix (a fixed straight line) are always the same. In all hyperbolas this ratio is always greater than one. Eccentricity is the name given to this ratio. Another characteristic of the hyperbola is that it has an axis of symmetry that goes through each focus, at right angles to the directrix. The point where the curve makes its sharpest turn is known as the vertex, and there are always two of these.

Two asymptotes, perpendicular to each other, are not part of the diagram, but denote the path that the curve approaches as the graph tends to infinity or negative infinity. The asymptotes of the general hyperbola are not axes as shown in figure 2.1. In the school hyperbola the asymptotes can be axes and are confused with axes.

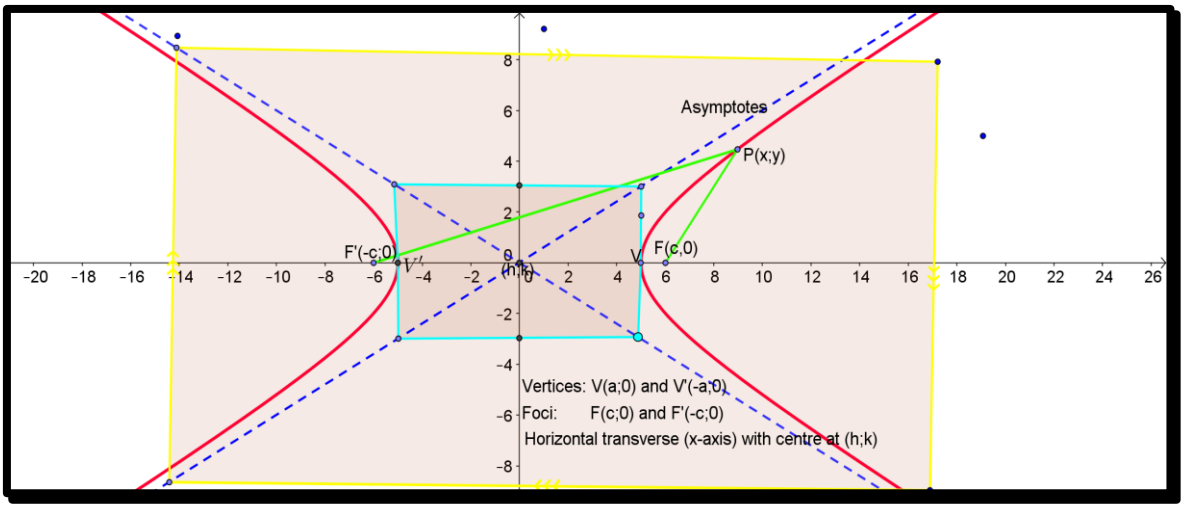


Figure 2.2 The general hyperbola

Points F and F' in figure 3.1 are called the ***foci*** of the hyperbola, and the midpoint of the segment FF' is the centre. V and V' are the vertices. The segment VV' is called the transverse axis. On a Cartesian plane the conic section has an equation of: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with vertices at (a; 0) and (-a;0) for a horizontal vertex and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ for a vertical vertex hyperbola. The two equations of the asymptotes are the straight lines: $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ for a horizontal vertex hyperbola and $x = \pm\frac{b}{a}y$ for a vertical vertex hyperbola. The eccentricity shows how the hyperbola's curve is different from that of a circle. An ellipse has eccentricity between zero and one, for a parabola it is exactly one and a circle has an eccentricity of zero. The eccentricity shows how "un-circular" the curve is. The bigger the eccentricity, the less curved it is. In other words, the hyperbola is the least curved of the conic sections followed by the parabola, and the ellipse is most curved. There are two formulas for eccentricity: one is the ratio $\frac{PF}{PN}$, where P is a point on the curve, F is the focus and N is the point on the directrix such that PN is perpendicular to the directrix. This ratio for the hyperbola is always greater than one. The other way of determining the eccentricity is by the use of the formula: $e = \frac{\sqrt{a^2+b^2}}{a}$, where a and b are the same values in the equation of the hyperbola (Narasimhan, 2009).

The Mathematical objects presented above present the discourse of the mathematicians on the hyperbola. In the paragraphs that follow, I discuss the rectangular, or school hyperbola

2.3.2 The rectangular hyperbola

A rectangular hyperbola is a one where the asymptotes are perpendicular to each other. The South African textbooks do not define the hyperbola; they simply express the formula. For example, one textbook defines the hyperbola as a graph with an equation $xy = k$ .This definition will have an impact on what a hyperbola is for school learners. The other name for a rectangular hyperbola is an equilateral, or right hyperbola. A hyperbola becomes rectangular when the semi-major and the semi-minor axes are equal. The semi-major axis is a line segment from the centre to the vertex of the hyperbola. The length of the semi-major is denoted by the letter a. A semi-minor axis is a line perpendicular to the semi-major axis and has a length of b. Therefore, in a rectangular hyperbola a=b, and the eccentricity is equal to $\sqrt{2}$. The asymptotes of a rectangular hyperbola are generally given by the equation $y = \pm x$ (if there are no translations) because a=b will give a one. The asymptotes make an angle of $\pm 45^o$ to the axis. The equation $x^2 - y^2 = a^2$ gives a rectangular hyperbola (Narasimhan, 2009).

The rectangular hyperbola has a simple form where the asymptotes coincide with axes, and this study focuses is on this hyperbola. The formula is reduced to $xy = \frac{a^2}{2}$, and is mostly written in the form of $xy = k$. For a positive *k*, the graph is in the first and third quadrants, while for the negative it will be on the second and fourth. The asymptotes of any kind of rectangular hyperbola can be translated around the Cartesian plane, but the asymptotes remain parallel to the axes (Narasimhan, 2009).

As stated in section 3.3 above, the school mathematical discourse on the hyperbola focuses on the simple equations where the coefficients of x and y are one. I will now explain the hyperbola as it is given in the Curriculum and Assessment Policy Statement (DBE, 2011).

The curriculum does not expect Grade 10 and 11 learners to know the definition of a function. It expects them to convert flexibly between four representations of a function: tables, graphs, words and formulae. In this study I have investigated the use three of these representations, that is, tables, graphs and formulae.

Grade 10 learners are expected to first plot points of graphs and then make generalisations by sketching the graphs, showing the key points, like the intersection with the axes, the shape of the graph and the line which the function can approach but not reach. The curriculum refers to hyperbola functions as some rational functions. In Grade 10 learners do a vertical shift on all

functions and in Grade 11 a horizontal shift. Content clarification gives the rational functions as those that are in the form of $y = \frac{a}{x} + q$ in Grade 10, and $y = \frac{a}{x-p} + q$ for Grade 11. The curriculum expects learners to first plot the functions and then make generalisations so that they can sketch the graphs. The parameter q has the effect of moving the graph vertically. A positive q moves the graph upwards, while the negative moves it down. The parameter p moves the graph horizontally, either to the left or right. A positive p moves the graph to the right, while a negative moves it to the left. This movement, resulting from these parameters moves the whole graph. (This means all points on the graph move by either q or p units.)

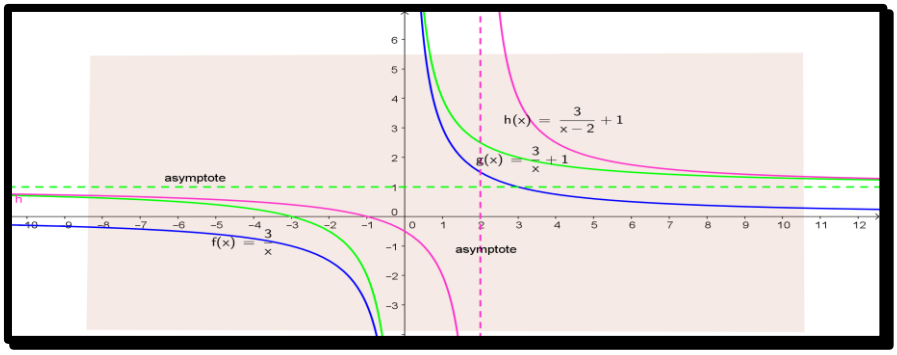


Figure 2.3 Transformation of the hyperbola

The curriculum expects the equations given above to be mastered by learners such that they can sketch the graphs from given formulae, given ordered points or a table. Learners are also expected to find the equation of functions given the graphs and some points on the graph.

The learning of the hyperbola can be traced back to the inverse proportion relationship. This relationship is done in arithmetic only in the lower grades in South African curriculum (DBE, 2011). Learners move from worded questions to arithmetic, and it is never expressed in algebraic form. The expression "y varies inversely to x" or "y is inversely proportional to x", written in symbolic form as $y = \frac{1}{x}$ means that $y = \frac{k}{x}$, where k is the constant of the variation. The difference is that in inverse variation, there are only positive values, since variation deals with real objects, for example, the number of hours it takes to build a tower is inversely proportional to the number of workers. The asymptotes and the graph on the positive side of the axis are the same, but there are no negative values for this graph. Yet the same function in algebra will have the values (ordered pairs) in the first and third quadrants for positive k, because algebra deals with abstract objects (Sfard, 2012). In my opinion this link is important in getting the relevance of the hyperbola in the curriculum and in helping learners understand the path (shape) of the hyperbola.

In most South African school textbooks there are no application questions on the hyperbola. Real-life examples of the hyperbola include the intersection points where waves interfere and cancel each other. The shapes of the curve of interference follow that of a hyperbola. Other examples that follow a hyperbola in shape can be used for mathematical questions so that the question: "Where will the learning of the hyperbola help us?" can be answered. There are real-life objects like some of the curved roofs, most potato chips, cooling towers or the arch. The designers of questions need to think of these when choosing items for their textbooks. Teachers tend to use examples from textbooks, and would rarely come up with their own. In this study some of the findings were about the understanding of the asymptote by learners. The following paragraphs discuss the asymptote.

2.3.3 The asymptote of the hyperbola

The word "asymptote" is derived from a Greek word that literally means "not falling together" (Smith, 1958). Information about a curve is conveyed in an asymptote, and in sketching functions one of the important steps is to determine it. The mathematical definition of the asymptote, used by the community of mathematicians, differs from the school definition. The school definition is a contextualisation of the disciplinary one. I shall discuss the difference between the definitions of the asymptotes by the two communities, and then show the difference between the asymptote and the removable discontinuity.

Mathematicians define an asymptote of a curve as a line such that the distance between the line and the curve tends zero as they approach infinity (Kuptsov, 2001). As the values of the curve increase, the curve and the asymptote approach each other. In this definition a curve may cross the asymptote. While some functions like $f(x) = \frac{x^2-3x}{x-1}$ will have curves not crossing the asymptote, there are functions like $g(x) = \frac{sinx}{x}$ where the asymptote is crossed infinitely many times as the graph tends to infinity. The diagram below illustrates these curves and their asymptotes.

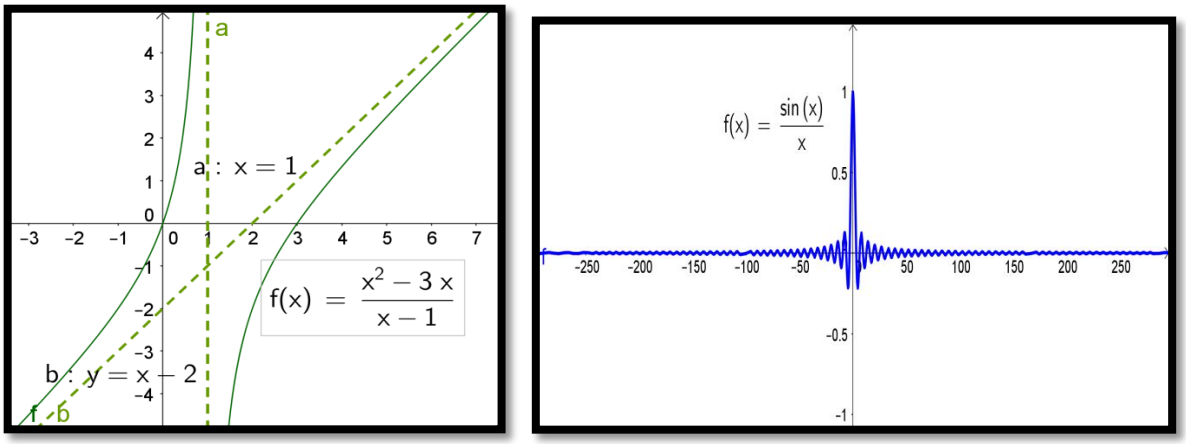


Figure 2.4 Illustration of curves and asymptotes

The asymptotes of the function $f(x) = \frac{x^2-3x}{x-1}$ are $x = 1$ and $y = x - 2$. The curve of the function f does not cross the asymptote. The asymptote for g is the x-axis, and the curve crosses the asymptote several times.

For the school definition the asymptote would be a line whose distance from the curve tends to zero as they approach infinite. Grade 10 and 11 learners are not expected to do limits, so the definition with limits would not be suitable for them. At the same time, to include that the curve should not cross the asymptote is not mathematically correct, and should therefore be avoided in the teaching and learning situation. This does not mean that learners should be exposed to questions that show the asymptote crossing many times. What is important is for the community of mathematics not to have contradictions. As discussed earlier, endorsed narratives are agreed upon by the community of mathematicians, and the purpose is to have the same narratives applied in the community. Thus there is a dilemma on how to define the asymptote at school.

There are generally three kinds of linear asymptotes: first a vertical asymptote, where the gradient of the asymptote is undefined and the asymptote is presented in the form of $x = a$. The vertical asymptote is parallel to, or coincides with, the y-axis. The second type of an asymptote is horizontal, and is expressed in the form of an equation $y = b$. This asymptote is parallel to, or coincides with, the x-axis. An oblique asymptote is expressed in the form of $y = mx + c, where\ m \neq 0$. The diagram below gives a diagrammatic representation of the three kinds of asymptotes.

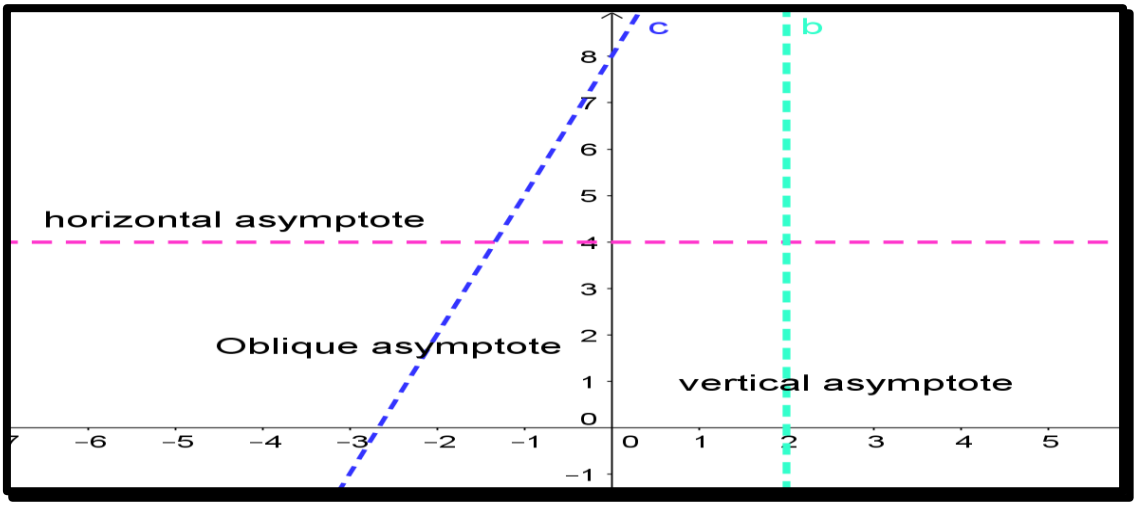


Figure 2.5 Graphical representations of kinds of asymptotes

The rectangular hyperbola, as stated earlier, will always have the vertical and horizontal asymptotes.

Asymptotes are generally linear in nature, but there are instances where the distance between two curves tends to zero as they tend to infinity. This is known as a curvilinear asymptote. The curve $h(x) = \frac{x^3+2x^2+3x+4}{x}$ has an asymptote of $y = x^2 + 2x + 3$. Figure 2.5 illustrates the curvilinear asymptote, a curve in a green dotted line and the function *h* represented by the purple curve.

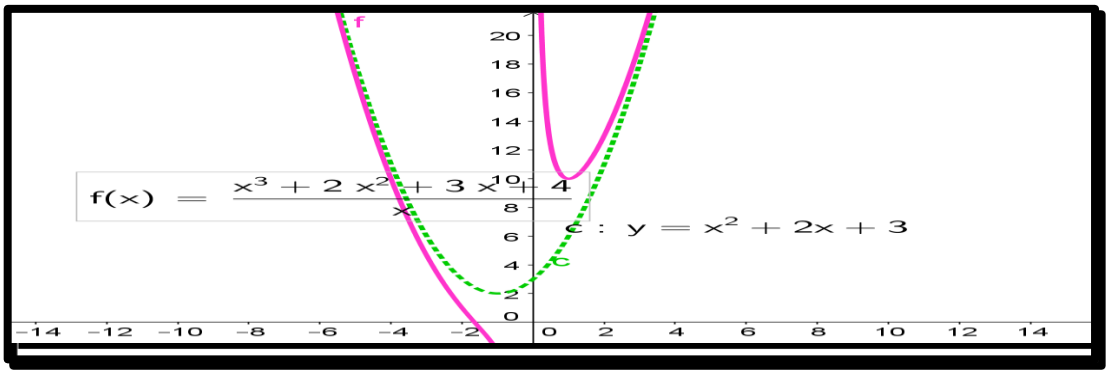


Figure 2.6 An example of a curvilinear asymptote

Thus there are four types of asymptotes, three of which are linear and one that is a curve. Asymptotes are not lines that are part of the graph. The dotted line shows where the asymptote for a curve is and is drawn for that purpose. A graph that does not show an asymptote is mathematically correct, provided all the other requirements for that function are clearly represented.

An asymptote and a removable discontinuity occur when the denominator is zero. The difference between the two is that an asymptote cannot be redefined to make the function continuous at that point. On the other hand, there is a way of defining a function in removable discontinuities such that they are continuous. If a denominator of a rational function is made zero at a certain value of x, but the numerator is not zero, then at that point there is a vertical asymptote; but when both the numerator and the denominator are zero there is a removable discontinuity (Kuptsov, 2001). For example, $f(x) = \frac{4}{x-2}$ at x=2 would give $f(2) = \frac{4}{0}$ and $h(x) = \frac{4(x-4)^2}{x-4}$ at x=4 would result in $h(4) = \frac{0}{0}$. The function *f* has a vertical asymptote while the function *h* has a removable discontinuity. The curriculum at Grade 10 and 11 does not require the knowledge of this difference.

In the coming section I define the function, the South African curriculum with regards to functions, the importance of learning of functions and learner difficulties with functions

2.4 Functions

A function is a relation where a set of inputs have exactly one output corresponding to it. A function is defined by a formula or algorithm that gives instructions on how to get an output from the given input. Sometimes a function is represented as a graph or alternatively as a table with inputs and

corresponding outputs. A function has a well-defined set of inputs, called the domain, and the set of outputs known as the codomain, or the range.

2.5 Functions in the South African curriculum

A hyperbola is one of the functions that learners in South Africa start sketching and learning in Grade 10. Although the hyperbola is introduced in Grade 10 is also learnt in Grade 11 and examined in Grade 12. Grade 10's do the hyperbola with a vertical shift only, and the horizontal shift is introduced in Grade 11. The work on the hyperbola starts by plotting points. The importance of this process lies in choosing the values of the input that clearly show the output approaching the asymptote as the input gets both greater and smaller. Once they have understood the concept, learners are expected to generalise by sketching the graph showing only key points, like the intercepts and the asymptotes. The CAPS document expects learners to be able to move flexibly between the four representations of a hyperbola (DBE, 2011). Recommendations of CAPS go further in asking learners to make conjectures, and prove them, so that they make generalisations, especially with the asymptote, intercepts and translations. In the next paragraph I shall discuss the learning of functions.

2.6 Learning of functions

The hyperbola best explains what happens in limits as x approaches an asymptote. Understanding of functions at school level creates good ground for further studies. However, there is agreement within the research fraternity that functions are not an easy concept for students (Carlson et al, 2010; Clement, 2001, and Yavuz, 2010 and Moalosi, 2014). It is therefore not surprising that functions are poorly done in schools. At the same time, when the reasons for that difficulty are laid bare, solutions can be found.

Leshota (2015), citing Leinhardt, Zaslavsky and Stein (1990), states that in the learning of functions tasks given to learners should enhance understanding. The tasks that enhance the learning of functions include, but are not limited to, interpretation and construction of functions. In interpretation the mathematical activities include identification of coordinates from a function, generalisation on the behaviour of the graph for certain equations, and many more. The movement of the graph in relation to the asymptote can be one of the foci on interpreting the graph. The construction activities include drawing the graph using ordered pairs and sketching them. The design of the tasks should be such that objectification is achieved.

Development of the functions discourse is the second perspective that Leshota (2015) mentions. Understanding of functions depends on the selection of examples and tasks that learners are exposed to. The success of the development of the functions discourse depends on what is made available for learners to learn (Adler & Ronda, 2014). Functions discourse can be developed using the four principles of objectification explained in chapter two above. The development of the functions discourse is not easy to achieve, as learners have difficulties.

Moalosi (2014) suggests that the learning of functions should focus on relationship rather than the process. The formula or the equation is regarded by some learners as a process. The reason is that they see the formula as a machine for producing the output. Leshota (2015) refers to the same scenario as the procedural action. The learning of functions focuses on the procedure. Learning functions concentrates on computation and plotting of points without relating the ordered pair to each other. Leshota (2015) then emphasises that learning should aim at covariation rather than correspondence.

Learning of functions should focus on multi-representations of functions and terminology (Moalosi, 2014). A function can be represented in more than one way, and the different ways in which the function is represented is a *translation* of the same function. A function expressed as a graph should be matched to an equation or table. The challenge with the table is that sometimes learners choose values for a hyperbola that are the same as those of a linear function. Correct terminology is important in the learning of functions. The definition of an asymptote, for example, has a bearing in how learners understand the relationship between the asymptote and the graph.

2.7 Learners' difficulties with functions

Clement (2001) mentions some of the limited ideas on functions that learners have. These ideas include that:

- a function is given by a single rule
- A piece-wise relation would not be a function
- The graph of a function should be continuous
- Once there is a gap between the graph then it ceases to be a function, e.g. the hyperbola is not regarded as a function because it has two parts.

A function should be a one-on-one; $f(x) = 4$ is not regarded as a function. At the same time, the same statement [f(x) =4] drawn on paper is regarded as a function. While learners are correct that an

equation written as x is equal to a constant is not a function, the equivalent of the same drawn on a Cartesian plane is regarded as a function. It is also difficult for learners to distinguish between y is equal to a constant and x is equal to a constant ($y = a$ and $x = a$) (*ibid*).

There is a belief among learners that a function should be continuous (Wilson, 1994). Wilson (1994) goes on to argue that learners would not classify as functions those graphs that are not continuous. The test that distinguishes a relation from a function is the vertical test. For a function each input value must have a distinct output. The only time a relation is regarded as a non-function is when an input has at least two output values, for example in $y = \sqrt{x}$: for every value of x there are two distinct y values for all values of x (except for x is equal to zero). All disjoint graphs are regarded by some learners as non-functions. A hyperbola would fall under this category.

Some of the challenges that learners face are given by Moalosi (2014) as definition of the function, the x and y intercepts, the effects of the parameters on the graph, interpreting the graph, translating between the representation of the graph and the ability to distinguish between the functions. Learners in this study had difficulties with the effects of the parameters on the function, interpreting information on the table or graph and the ability to translate between the representations of the functions.

2.8 Conclusion

In this chapter I discussed the theory of commognition, detailing what the theory entails. I explained the difference between the object-level and meta-level development. I also discussed the four characteristics of commognition, that is, the use of words, the visual mediators that are used for communication, the routines and endorsed narratives. I also discussed learning as participation because the focus of this study is on learning of the hyperbola. I chose participation as the theory of learning as I believe that participation theory best explain how learning occurs. The next chapter will be a discussion on literature that I reviewed on the importance of learning of functions in general and in particular the hyperbola, and the difficulties that learners face in learning it.

I also discussed the requirement of the South African mathematics curriculum on the functions has been explained and how my study fits in. I then looked at the learning of functions and difficulties that learners face in learning functions. I have looked at the general and the rectangular hyperbola. I have explained how the rectangular hyperbola is arrived at when the semi-major axis and the semi-minor axes are equal to each other. The asymptote has been defined, and various types described.

I used is based on the literature discussed above. The four key features of the mathematical discourse as presented by Sfard (2008) form the lenses through which the study was viewed. The four features are: word use, visual mediators, narratives and routines. I investigated how learners used the four features of a mathematical discourse in the three representations of functions, which are the expression of a function as a table, a graph and an equation. My aim was to find out the success and difficulties that learners experience as they work on functions from different representations.

Learners participate differently in mathematical practices. Moschkovich (2002) talks of unequal participation of learners. I propose that full participation occurs when the learner is able to show that the mathematical discourse has been mastered. The learner is able to explain, verbally or written, the concepts learned. There is some sense of generalisations and justification of the mathematical concepts used. In my study I investigated how far learners are from objectified talk (Sfard, 2012). In the next section I discuss the general and school hyperbola.

In the next chapter I shall discuss the methods used to collect data and steps that were taken to ensure that the findings were valid and credible. In Chapter four the analytical tool (DPH) will be discussed.

Chapter 3: Methodology

3.1 Introduction

This study is qualitative because I wanted to understand learners' participation in the functions discourse. The research is working from the interpretivist position (Denzin & Lincoln, 1994; Ely Anzul, Friedman Garner and Steinmetz, 1991). I made sense of the learners' meanings of the hyperbola especially with regards to the asymptote, unfamiliar questions and representations of the functions. I conducted task-based interviews with nine learners. As in most qualitative research, both verbal and visual data was analysed (Flick, 1998). Discussed below are where the study was conducted; the sample size, the instruments used for collection data. I also discussed ethical considerations, validity and reliability.

3.2 Research approach

In this section I outline the methods for investigating how learners communicate in the task on the hyperbola. The approach to the research is qualitative (Hatch, 2002). A qualitative study is one in which techniques used are flexible and suitable for discovering underlying patterns of learners' thinking with regard to use of words, visual mediators, routines and narratives. The qualitative research approach enables me to understand the nature of learners' participation in the sub-discourse of the hyperbola of the mathematical discourse.

The focus of the study is the learning of the hyperbola. I looked at the discourse of learners on the asymptote: how learners use words about the asymptote of the hyperbola, how they interact and interpret of the visual mediators, both iconic and symbolic. The kinds of routines and how they were used during the interview. Learner narratives will be discussed in terms of whether they were endorsed by the community of mathematicians. I also examined learner difficulties with unfamiliar tasks.

3.3 Setting

The study was conducted in one of the Wits Maths Connect Secondary Project schools in one of the townships in Johannesburg. I chose the school because of the relationship I have with it through the professional development programme conducted by the project. The collection of data was done in the afternoon at the school – during the morning they were having classes, and also the level of noise is less at this time of the day. An office was used for the task and the interviews. The interviews were done at the school as it is convenient for the learners.

3.4 Sample

I purposely sampled four learners in Grade 11 and five in Grade 10 (2015). Marshall (1996) says that in purposive sampling the researcher actively selects the most productive participants to answer the research questions. The most productive sample for my study is one where learners are able to talk and justify their answers. For this reason, high-performing learners were the target of my study. These are learners that were in the upper quartile in their first term marks for Mathematics. The head of the Mathematics department and the FET Mathematics teachers at the school played a big role in identifying learners with top marks in term 1 in each Grade. I asked for at least 16 learners so that I had several options if anything happened to one of the learners or they chose not to participate. (In my invitation letter I stated that learners were free to choose not to participate or to withdraw at any time.) The learners that participated chose to do so. I put the four cards with labels (two As and two Bs) in a hat and asked them to pick a single card. This is how I randomly placed them into two pairs for each Grade. One of the groups had three learners, because a third learner wanted to participate after the random selection, and neither of the others were willing to withdraw. The three pairs and the group with three make a total of nine learners. The initial plan was to have pairs in the same grade so that they could discuss the mathematical discourse of the asymptote of the hyperbola, on the table of values and their response to unfamiliar tasks.

3.5 Task-based Interview

Data was collected using task-based interviews. The task-based interview allows learners to work and simultaneously explain their solutions to the mathematical tasks (Goldin, 2000). In task-based interviews the participants do not only interact with the researcher but have an opportunity of working with a task (Goldin, 2000). In task-based interviews it is possible to gain access to learners' participation and the strategies they use to find solutions to the task. Mathematical discourse is not just about conversations but is also expressed in the form of written work and learners' non-verbal communication. Participation in the mathematical discourse included what learners said and did during the interview. This helped me to have an in-depth understanding of the words, visual mediators, routines and narratives used by learners.

Data obtained from an interview is affected by the interviewer's questions and non-verbal prose like facial expressions. Denscombe (2007) says the researcher gets better information from observing people performing than when they tell the stories of what they do. As stated earlier, mathematical discourse is more than just conversations it was important for me to both observe and listen as

learners worked on the tasks. I allowed the learners to talk freely and interjected where I felt there was a need for more explanation, for example, when the learners said “my asymptote is undefined”. The focus of the interview was to find out how learners participate in the hyperbola discourse.

I used a semi-structured interview, so not all questions were included in the interview protocol. Most of the questions depended on learner utterances. The questions asked during the interview had the aim of getting learners to explain the methods, reasoning and choice of words and procedures. I put many follow-up questions during the interview as I sought to have an understanding of their learning of hyperbola function.

A task-based interview is one where students are involved in explaining solutions to a task (Goldin, 2000; Maher, Powell & Uptegrove, 2011). The researcher was able to probe and prompt the learners to say more about their solutions. This kind of an interview enables the researcher to the word use, interpretation and use of visual mediators, narratives endorsed by learners and types of routine used.

I have chosen this type of interview because in it, it is possible to investigate words, visual mediators, narratives and routines and ask for explanations on their mathematical discourse. Learners were asked to explain how they got their solutions. In the process of explaining their reasoning would be expected to emerge. The other good thing about using a task-based interview is that it is possible to identify the strategies and reasoning that learners used. The intention I had in the study was to the word use, interpretation and use of visual mediators, narratives endorsed by learners and types of routine used. I also intended to find out what narratives on the hyperbola shaped their understanding of the hyperbola. Learner understanding is shown when learners are able to work with visual mediators, the graph, equation and table of values. I was also interested in the words they used. I wanted to find out the depth of this understanding. The task-based interview was done in sets of pairs (and one group of three). A discussion between a particular pair was encouraged, though with little success because of their reluctance to talk. I therefore had to deliberately ask them to respond individually to the task to ensure that they explained their reasoning. The interviews lasted less than an hour to reduce the chances of fatigue and restlessness. There was a single interview per day to avoid fatigue for the researcher and the participants. I spent the rest of day after the interview reviewing the appropriateness of my questioning and the quality of the answers learners had given so that the next interview was improved in quality. The result was that the first interviews for each grade were longer than the second. I did this by reviewing the day’s work through watching the videos and making notes of interesting words, narratives or routines

learners had used. I also reflected on the questions or comments I had omitted to raise so as to prepare for the next day. I found during analysis of the data that I would have benefited from more follow-up questions.

I designed two tasks: one each for Grade 10 and Grade11. Each one had four main questions, with each question having sub-questions. The first two questions were the same as those that can be found in textbooks and examination questions. I used these type of questions because I wanted learners to participate through talk or action. Problem-solving questions would have hindered my objective, to have learners participate in the discourse. These would be classified as routine procedure in the CAPS (2011). All learners in the study were able to attempt them. The third and fourth questions were not normal classroom questions, but are within the curriculum. The fourth question was added after the piloting, once I realised that learners participating were comfortable with questions from textbooks.

Each of the questions had a purpose. The first question was on the hyperbola in algebraic form. In this question I wanted to find out about the routines and narratives that learners use on algebraic formulae. On the second question I wanted to see how learners work with iconic mediators. The third question was on how learners make use of data in a table. I wanted to find out using the questions if learners understood that a function can be represented using the three representations. The fourth question was on unfamiliar tasks, where an equation was presented in product form. Learners were given a task to answer, after which they explained their solutions in the interview. I gave students four questions, two on the equation and one for each of the other representations of a hyperbola. The three representations are the hyperbola expressed in algebraic form, as a graph and in tabular form.

Learners first answered these questions individually and then explained their solutions in a group or in pairs. The change from the original plan of having the task questions done in pairs was that during the piloting of the study one of the learners in the pair did everything, so I decided to give each learner an opportunity to do the task and then explain their views. The reasoning of each learner was easy to determine, as all had some written work and were given a chance of explaining their reasoning.

3.6 Data collection method

Video recordings provided the main mode of data collection. Video has the advantage of capturing the voices and actions of learners, and work can always be referred to time and again (Opie, 2004). The recording was visual and audio. I had a back-up to the video recorder in the form of an audio recorder.

Video recordings afford the observer to notice even the non-verbal communication of the participants do, besides writing and talking. However, there was a challenge to doing both recording and asking questions. Some of the issues that I could have followed were left hanging, partly because of doing both recording and interviewing. For example, Sifiso said that a question: $x(y-4)=3$ was wrongly written, and this thought was not followed-up sufficiently.

3.7 Piloting

Piloting is a mini-version of what would happen in the full study. The purpose of piloting is to check, among other things, the appropriateness of the instruments that are to be used in the study. I did a pilot study with two Grade 11 learners, and found that the questions in the task allowed learners to discuss and express their views on their understanding of the hyperbola as a function. I found that the interview lasted about 30 minutes, meaning that a maximum of 40 minutes would be sufficient. I found that an empty office a little removed from the classrooms was the most appropriate place to carry out the interviews. Teachers kept on coming in to the HOD's office. The appropriate time was after other learners had left because of the amount of noise when learners were at school. The learners that took part in the piloting helped me to improve the quality of the questions for the task and the interview. For example, in question 3 I asked learners to name the graph represented by the table before drawing it. The question sought to understand if learners can identify the graph from the table. I realised that learners would plot the points, identify the shape, but then derive the name from the graph instead from the table. I also needed to video record the discussion when learners were completing the task, as most of what happened during this discussion was lost.

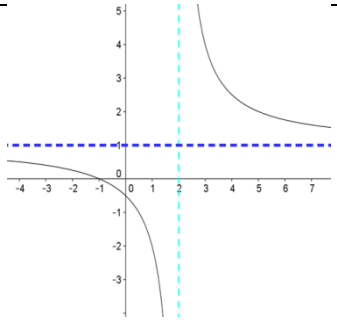
3.8 Data Analysis

Since the study is a qualitative one, data was analysed from the moment of interviews. The analysis came in the form of understanding learners' responses so as to ask good follow-up questions. It was my responsibility to listen to the responses of learners, make sense of them and probe further. The researcher has the responsibility of making sense of the data before him/her (Hatch, 2002).

The task and interview generated the data on learner participation. This is in terms of the words used, the visual mediators and how they are used to communicate in the functions discourse. The data from the task and interview will be analysed using typological and inductive analysis (Hatch, 2002).

3.8.1 Typological Analysis

Typological analysis is generally used for qualitative data analysis with the aim of developing sets of categories that are different from each other. Hatch (2002) argues that typologies are generated from theories, common sense or research objectives. These categories place data in cells that are mutually exclusive from each other. Typological analysis seeks to reduce ambiguity when classifying data. In this research I describe and define the categories that emerged from the data. Secondly, I assigned data to the appropriate groups or matrix. Raw data would be classified under one given typology. The typologies in this study come from the research questions. Examples of the categories are: word use, visual mediators, routines and narratives. Table 4.1 below represents the categories on which the data was classified for the question: Find the equation for the graph given below. What is said or done is used to populate the table. For example, consider the key word "quadrant". The iconic visual mediator indicates to the learner that there is a mistake in his working. The learner uses symbolic and iconic to correct his mistake, the evidence of corrigibility routine. The endorsed narrative is evoked and used to check the correctness of the working. This is how the categories were used in the analysis of data.

What is said or done	Word use	Visual mediators	Routines	Narratives
Interviewer: I see that you had written -4, then you changed it to 4. Can you explain why you did that? **Learner 1**: I looked at the graph and realised it was in the first quadrant and third quadrant	Quadrant		Solving a linear equation with a fraction	For a Positive "a" the graph is on the first and third quadrants. If "a" was negative graph could have been on the second and fourth quadrants

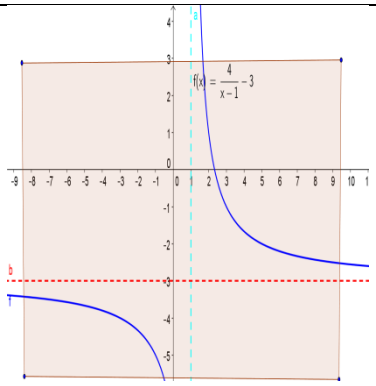
Interviewer: Show me the x-intercept **Learner 2**: points at the asymptote then changes **Interviewer:** where did you want to point and why? **Learner 2**: I wanted to point at the asymptote What about the y-axis? Learner points at the vertical asymptote	Intercept; asymptote (learner does not distinguish between an asymptote and an intercept)			

Table 3.1 An example of typological analysis

The four characteristics of commognition given in this table 3.1 were used to classify the transcription of learner utterances. The data was further analysed using the DPH.

3.8.2 Inductive analysis

The inductive approach is a way of analysing qualitative data that allows findings to emerge from the frequent, dominant or significant themes in raw data (Thomas, 2003). Thomas (2003) describes inductive data analysis as a way of condensing large sums of raw data into a summarized format. He goes on to say that inductive analysis establishes clear links between data summaries and the research objectives. In this study inductive analysis is used to condense raw data into a brief, summary format and to establish clear links between the research objectives and the summary of findings derived from the raw data.

Data was read and analysis themes were identified (Hatch, 2002). The asymptote of a hyperbola is one such. Analysis was then done using the DPH where differences in data that seemed similar were exposed. More on the DPH is explained in chapter five.

3.9 Ethical Considerations

3.9.1 Ethics of the study

The study was conducted with the ethical considerations that guide research at the University of the Witwatersrand. I applied to the Ethics Committee for approval of my research and obtained clearance with protocol number: **2015ECE011M**. The research is located within the Wits Maths Connect Secondary Project with Protocol number: **2010ECE60C**. I explained the purpose of my research to all the participants that were involved in this study. I respected and protected the participants' rights in terms ensuring that they were not inconvenienced by participating in this study in any way. All the consent forms to the parents, school and learners with their covering letters were completed and signed. Letters spell out the purpose of the study, how long it will take, how learners' confidentiality would be protected and the activities in which the school and learners were to take part in. All the parties were invited to be part of the study.

3.9.2 Anonymity and Confidentiality

Neither learners' names nor their school will be used in any form in reporting this study. Pseudonyms have been used for this study. These names were used during transcription and analysis of the data. All information that might allow anyone to guess learners' identity was removed during analysis, for example, their names, written or spoken. The school will not be identified. Anonymity will be guaranteed throughout this study.

Only the learners being interviewed were in the room. No one else in the school or class will have access to the specifics of which learner made what contribution to the study.

3.10 Validity and Reliability

Reliability is the consistency in results over time if the research could be done under same conditions (Joppe, 2000). It is important, as it establishes the truthfulness of the research. Joppe (2000) states that validity determines that the research has done what it purposed to find out at the inception of the study. Since my study is qualitative, it is difficult to measure validity and reliability because the results of the study cannot be generalised, but its trustworthiness will be examined (Seale, 1999). The measure is on the truthfulness of the results gleaned from a research. Creswell & Miller (2002) note that the researcher's perception affects validity and reliability in qualitative

research. According to Brink (2004) the researcher can reduce bias by being aware of that he/she may introduce bias at various stages of the research. On the participants' information being biased, I gave them a task and asked them to take me through it in an interview, giving me the reasons for the decisions they took. I confirmed with them the understanding I had of their information by asking questions, to ensure I correctly interpret their statements. On the appendix section a sample of the transcripts has been added. A full transcription of this data has been added onto a CD, submitted, to ensure credibility.

3.11 Limitations

The results of this study cannot be generalised because they do not represent all learners in South Africa. However, the findings of the study will provide insights into the ways in which some of the learners regarded as high performers in their school participate in the functions discourse, and this will inform further work related to all learners of all abilities.

The focus of this study is on learning rather than teaching. Nothing will be said about teaching and teachers. In chapter seven I discuss limitations further.

3.12 Conclusion

In this chapter I discussed the methods of collecting data that were employed in this study. I have explained that the task-based interviews were used to collect data from nine learners. I captured data using video recording. I also explained how data was analysed and the ethical considerations that were taken. In the next chapter I talk about the Discourse Profile of the Hyperbola.

Chapter 4: The Discourse Profile of the Hyperbola

4.1 Introduction

The discourse profile of the hyperbola (DPH) is the framework I adapted to analyse data. It has been adapted from the Arithmetic Discourse Profile that was used to distinguish between the discourses of two learners that seem to be inseparable (Ben-Yehuda, Lavy, Linchevski and Sfard, 2005). As this framework was used for arithmetic, I have adapted it and called it "The discourse profile of the hyperbola" (DPH). The Functions Discourse Profile by Gcasamba (2014) played a big role in the design of the DPH. I chose to use this profile because the discourse of learners in this study on the hyperbola seemed to be the same, and the profile helped expose the differences. I have used this profile because they discuss and explain the analysis of data using the four characteristics of the mathematical discourse - words and how they are used, the visual mediators, routines and narratives. Although the arithmetic profile was not organised into a table, I have summarised the profile into a table in this chapter. There are some aspects of the arithmetic profile that I have not included in my profile for two reasons; firstly, some of the analysis tools are never used in the theory of commognition such as the subject dimension that includes self-assessment and self-report and routines like closure conditions. The second reason is that some of the aspects used in the arithmetic discourse profile did not emerge from my data and I excluded them and these include but not limited to the deeds routines, derivations narratives and concrete visual mediators.

4.2 The Discourse Profile of the Hyperbola

The DPH was used to analyse the task-based interviews for this study. In the adaption of the DPH some of the categories of the ADP were left out for two reasons: firstly, some of the terms found in the ADP are not found in later works of Sfard - 'proficiency' 'routine', 'closure' 'conditions'. Secondly, some of the categories, like concrete mediators and derivative narratives, although they are still being used in Sfard's work, were not investigated in this study.

4.3 Explanation of the DPH

The words and their use are divided into two categories colloquial and literate. Words used in a colloquial way include all words that have a combination of mathematical and non-mathematical language, for example, when a learner says "make your y to be is equal to zero then you work it out solving for x". Words that are considered literate are those that are mathematically correct, like "an intercept is where the graph intersects with an axis".

I have classified visual mediators into two groups, the iconic and symbolic mediators. Iconic mediators are pictorial. In this category are graphs and tables. The symbolic mediators are equations. I have further sub-divided these into the viewed iconic mediator and those sketched or drawn for tables and graphs. A visual mediator is classified as viewed if the action taken is based on a presented visual mediator. For example, a drawn hyperbola graph and the act of drawing the graph are both classified as drawn. When a symbolic mediator is generated from a presented iconic mediator I have classified it as formulated symbolic mediator. As stated above, I shall not include concrete visual mediators, as none are present in this study.

Routines have been divided into kinds and properties. Kinds of routines include ritualised deeds and exploratory routines. An example of a deed routine is when one can tell the number of pieces that can be got from cutting two oranges into four each, but cannot divide four into two. The ritualised routines are those where learners are able to do required procedures such that they may get full marks if it were a test, but cannot explain mathematically, how or why he/she arrived at that answer. A ritualised routine is where a learner's routine is close to that of the teacher. An exploratory routine is used for verifying endorsed narratives. The exploratory routine includes knowing why certain steps are carried out. Properties of routines include applicability, flexibility and corrigibility. Applicability routine is about when certain routine procedures are likely to be produced (Ben-Yehuda et al, 2005). In this study the applicability routine is seen in the following circumstances i) solving problems to find the intercepts and points. ii) sketching or drawing graphs and a table of values; iii) Use of a table to identify key features of a hyperbola; iv) Use of the key features (intercepts, asymptotes) to sketch a graph; v) Using a visual trigger, for example an asymptote signifying a vertical or horizontal translation. The flexibility routine refers to the capability of using more than one routine to solve problems and come up with the same endorsed narrative. Flexibility is also seen in working with unfamiliar tasks. In this framework use of multiple strategies and being able to transform any equation to standard form is an example of a flexibility routine. Flexibility is seen in being able to move from one visual mediator to the other and giving mathematically valid reasons for such. Corrigibility is a routine that deals with self-evaluation and leads to self-correction. Corrigibility is achieved sometimes through mediational change. This is moving from one mode of visual mediator to another and checking that the endorsed narrative has not changed, or, if it has changed, why the change has happened.

Lastly, I shall use the narratives to analyse the discourse of learners on the hyperbola. Narratives that result in establishment new theorems, definitions, axioms etc. are called derivations. In this

study no new narratives were derived as the learners involved in the study were not taught by the researcher. I therefore did not include the derivations narratives in this analysis. The analysis will use substantiations, the actions through which we decide to endorse previously constructed narratives. The substantiations that govern school mathematics are not as rigorous as those that govern the community of mathematicians. The justifications and reasons for particular actions are classified as substantiations. Some of these substantiations are endorsed by the community of mathematicians, while others are not. The other narrative that was used to analyse the learners' discourse is the memorisation narrative. Memorisation narrative is about remembering previously endorsed narratives. For example, $y = q$ is an equation of a horizontal asymptote. It is a memorisation narrative, as it had been endorsed for the Grade 10 and 11 learners at the time of the study.

The table 4.1 is the DPH:

4 Key characteristics of mathematical discourse		***Classification***	***Description***	***Examples***
Words/words use		Colloquial	Combination of literate and colloquial	an asymptote is an imaginary line that a graph can't pass
		Literate	Mathematical	A line whose distance to a given curve tends to zero. An asymptote may or may not intersect its associated curve.
Visual mediators		Iconic	Viewed	Use of a hyperbola function to identify asymptotes
			Drawn	Sketching a graph; table of values
		Symbolic	Viewed	Identification of the asymptotes from the equation
			Formulated/generated	Equations from a graph or table
Routines	Kinds of routines	Ritualised	Correct procedure wrong and/no justification	Sketching a graph showing that the y-axis is an asymptote but talking as if there is no vertical asymptote
		Exploratory	Verification of narratives; Working with unfamiliar tasks	Choice of numbers on a table that show values moving towards a limit
	Use of routines	Applicability	Solving equations	Solving to find intercepts; asymptotes;
			Drawn	Hyperbola graph from an equation or table
			Use of a table of values	Identification of key features from a table of values
			Use of key features	Sketch a curve using intercepts, asymptotes and the equation
			Using visual trigger	An asymptote signifying a vertical or horizontal translation
		Corrigibility	Correction	Self-evaluate and correct
		Flexibility	Use of multiple routines	Using key features and/or table of values to sketch graphs
			translating	Being able to transform an equation to standard form
Narratives		Substantiation	Justifications and reasons	Justifications for actions, e.g. it is an asymptote because…
		Memorisation	Formula/rule	$y = \frac{a}{x-p} + q$ is a hyperbola
			Visual (justification based on what learners can see).	y = q is an asymptote

Table 4.1 Discourse profile of the hyperbola

4.4 An example of how the DPH was used to analyse Mathematical discourse of learners

In this section I will illustrate how the analytical tool (DPH) is used to analyse learners' mathematical discourse on the asymptote. I chose Palesa's response to the task:

> Given the function: $f(x) = \frac{3}{x} - 4$
> *1.1 Sketch the graph of* $f(x)$
> *1.2 Explain the transformation from g(x)* $= \frac{3}{x}$ to f(x)

On the word use, I classified Palesa' mathematical discourse as colloquial because she used two different mathematical terms as if they had the same meaning. The excerpt below illustrates this point.

Who	**What is said**
Interviewer	can you take me through your first question 1.1 how did you plot the graph
Palesa	ok I used the table method and my cube my asymptote is negative 4 and I drew the symmetrical line on the y axis on the negative side which is -4 then I plotted my graph using the table method

Palesa expresses an asymptote as if it is a number. She says "my asymptote is -4". Negative 4 does not represent a line on the Cartesian plane. Further the words *asymptote* and symmetrical are used as if they have the same meaning. I therefore classified her use of words on this part of the question is a colloquial for two reasons. Firstly she does not express the asymptote as an equation and she uses two different terms as if they mean the same thing.

Palesa shows the ability to interpret the symbolic mediator. From the above excerpt, she is able to identify the asymptote from the symbolic visual mediator $f(x) = \frac{3}{x} - 4$, although she does not express it as an equation. I classify her use of the symbolic mediator as being viewed.

The figure 4.1 below illustrates Palesa's response to the question to sketch the function.

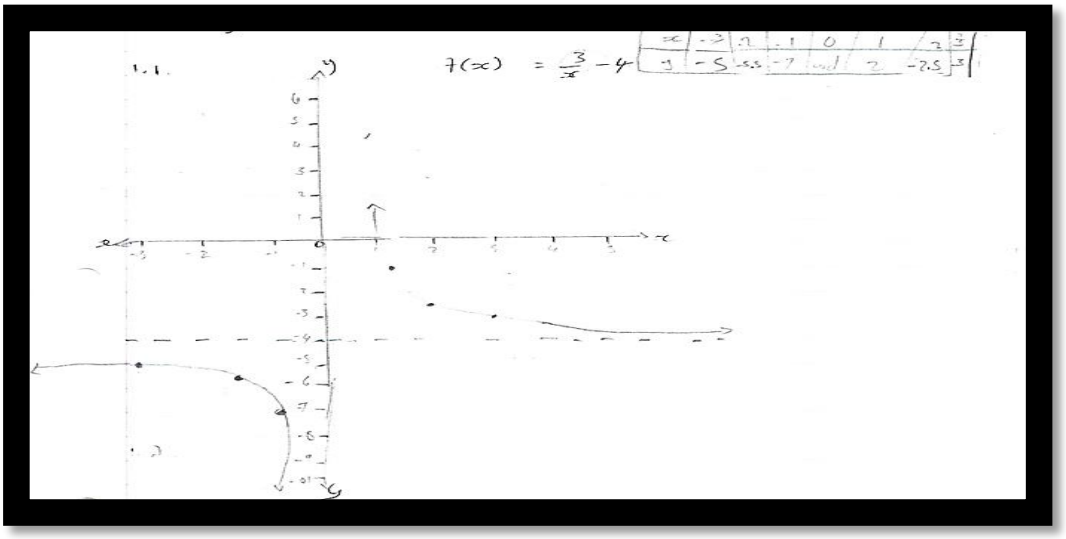


Figure 4.1 Palesa's response to Sketch the graph of f(x)= $= \frac{3}{x} - 4$

From the above diagram Palesa shows ability to sketch the hyperbola, showing all the key features like intercepts, asymptotes and the general shape. Her iconic visual mediator is drawn according to the DPH.

Palesa's routines are ritualised because although she does not use literate mathematical language, she is able to sketch a hyperbola. She does not adequately explain the relationship between the two functions $f(x) = \frac{3}{x} - 4$ and $g(x) = \frac{3}{x}$.

Who	What is said
interviewer	and what do you think?
Palesa	ok the transformation from g(x) which is 3 over x at first there was no asymptote and then when it was formed into f(x) we had 3 over x minus 4 whereby we have our asymptote. Ahh

Palesa said at first there were no asymptotes meaning although she is able to sketch a hyperbola with a vertical shift she does not relate it to the parent function. I then classified her routines as ritualised rather than exploratory because of that.

On the use of routines I have classified her mathematical discourse as applicability because she is able to sketch the graph using a table of values, she also shows the key features of the hyperbola in her graph, she calculates the x-intercept but she fails to explain that from the asymptote we can see the vertical and/or horizontal translation. Therefore her routines cannot be considered flexible.

I classified Palesa's narratives as memorisation based on visuals. In the above excerpt she says $g(x) = \frac{3}{x}$ has no asymptote because she does not see the parameters "p" and "q".

4.4 Conclusion

In this chapter I have discussed the DPH, a framework I used to analyse the data. I have explained where I got my analysis profile from. In the next two chapters I shall use the above profile to analyse the data. Chapter five focuses on the learners' mathematical discourse on the asymptote of the hyperbola and chapter six deals with learners' discourse on unfamiliar tasks of the hyperbola.

Chapter 5: Learners' mathematical discourse on the asymptote

5.1 Introduction

In this chapter I shall use the Discourse Profile on the Hyperbola (DPH) discussed in chapter five to analyse the mathematical discourse of learners on the hyperbola with particular attention on the asymptote. The analysis will be described in two chapters. In this chapter I shall analyse the mathematical discourse of learners on the asymptote. Chapter seven will focus on the analysis of the mathematical discourse of learners in unfamiliar tasks of the hyperbola.

The features that emerged from the data most prominently are the links between equations of the hyperbola; naming of the asymptote as if it is a number; asymptotes said to be demarcating quadrants; and the y-axis not seen as an asymptote mostly by Grade 10 learners in this study. There were also mismatches between what some learners said and what they did. I shall argue that the learners' mathematical discourse is affected by the dominance of the ritualised routines. In the following paragraphs I shall refresh the key features of the mathematical discourse again.

5.2 Key features of the mathematical discourse on the hyperbola

Key words in mathematics are a crucial part of the formal discourse. It is through these words that learners communicate mathematically. I have classified the use of words as 'literate' when the sentence or phrase used contained only mathematical language as described in the analytical tool DPH. An example of words that are used in a literate manner is "the asymptote is: x is equal to four (x = 4)". 'Colloquial' words represent a mixture of non-mathematical and mathematical words in a phrase or sentence. An example of what I have classified as colloquial is a phrase "my asymptote is undefined". The words used are mathematical, but cannot be regarded as literate because an asymptote cannot be undefined. Only certain points cannot be defined. Using the DPH such words would be classified as colloquial. The analysis will also look at the visual mediators.

In the functions discourse visual mediators include the graphs, tables and equations that represent various functions. The symbolic mediator used for the hyperbola is usually presented as $f(x) = \frac{a}{x-p} + q$, and the iconic mediator would be a table of values or a graph with two parts that are a reflection of each other, moving in 4 directions approaching the asymptotes as seen in figure 5.1

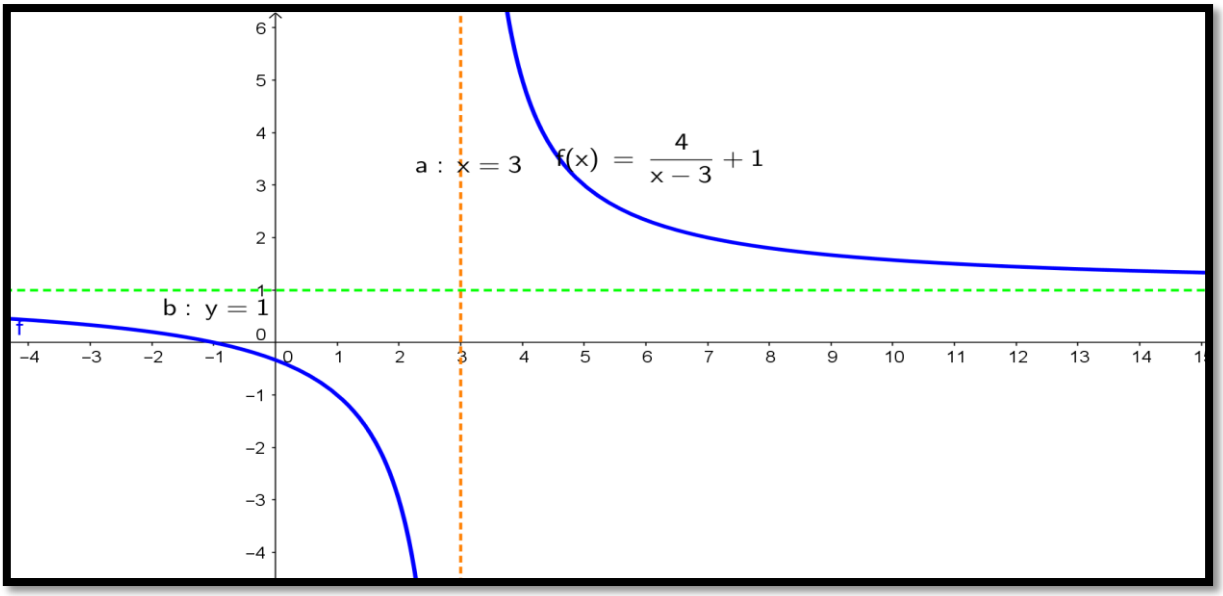


Figure 5.1 Example of an iconic mediator

Visual mediators help participants in the mathematical discourse to coordinate their talk and to identify their objects. What is seen in a visual mediator and the interpretation thereof should have the same meaning for all the participants of the mathematical discourse. In the DPH I have four categories for analysis of the visual mediators. The iconic visual mediators can either be viewed or drawn and the symbolic visual mediators can be viewed or formulated. Next I shall discuss the routines.

In learning mathematics all learners begin with ritualised routines: the learner would do what he sees the interlocutor do (Sfard, 2012). The aim of learning mathematics is to achieve exploratory routines where learners are able reflect and find meaning from what they see their interlocutor do. Reflective imitation brings the ritualised routines to exploratory routines, by understanding the reasons behind the actions of the interlocutor (ibid). Learners often struggle to achieve this transition.

One of the key issues with the routines – whether it is drawing of tables of values, point plotting, curve sketching etc., is that learners already have some knowledge of the shape of the hyperbola from the procedures rather than reasons for action. A shape can therefore be sketched without necessarily exploratory routines for that particular function. As stated earlier, a ritualised routine is one where the learner does what he/she sees the teacher doing, mostly without understanding why those steps should be taken. The work would be correct, but the reasons behind doing particular things may not be clear to the learner. The learning that is made available to learners, whether in textbooks or other sources on functions, tends to emphasise the procedures on key features of a

particular function. The DPH has classified routines as kinds of routines where we find the ritualised and exploratory routines and use of routines has applicability, flexibility and corrigibility. The focus on how mathematics is done then supersedes why and when it is done, and the routines then affect the narratives that learners substantiate.

Narratives include mathematical theories and definitions. Some narratives are accepted or endorsed by the community of mathematicians, while others are not. It is not uncommon for narratives that are not endorsed by the community of mathematicians to be uttered by learners, especially if their routines are ritualised. In this study a number of narratives that are not endorsed by the mathematicians were said, for example, there is no asymptote for $f(x) = \frac{3}{x}$, or that the asymptote is undefined when the equations of the asymptotes are x = 0 or y = 0. In the `analytical tool the DPH I would classify such narratives as memorisation based on visuals. Learner narratives provide a window of viewing their thinking on particular subjects; in this case it is the asymptote and related aspects.

5.3 Learners' discourse about asymptotes

One of the key words often used is "asymptote". An asymptote is a line whose distance to a given curve tends to zero. An asymptote may or may not intersect its associative curve. As explained earlier in chapter three, a rectangular hyperbola has two asymptotes that are perpendicular to each other and parallel to each of the axes. I shall discuss below some of the ways in which the word "asymptote" was used.

5.3.1 Naming the asymptotes

In this study most of the learners (6 out of 9) expressed the equation of the asymptote as if it were a number. Secondly, for most of the Grade 10 learners there is only one asymptote, the horizontal. The way they have used their words has an impact on how they understand and apply the knowledge of the asymptote.

Three learners named the asymptotes mathematically in the form of $x = a$ and $y = b$, where a and b are real numbers. An equation has an equal sign. The equation clears ambiguity, in that one understands which asymptote is being referred to, and standardisation is achieved. I will show that learners spoke about the asymptote as if it were a number. I provide evidence from transcripts later in this section.

While most of the learners in this study spoke of an asymptote as if it was a number, Connie, (Grade 10) and Sean and Malusi (Grade 11) expressed the asymptote mathematically, in terms of DPH, their use of words was literate. This was a marked difference from the other learners. Sean not only gave equations during the interviews, but also labelled the equations on his graph. The utterances of Connie is representative of how asymptotes were named mathematically. I chose this learner because she displays hesitance on her answer.

Connie, working on the equation $x(y-4)=3$, states that there are two asymptotes: y =4 and the y-axis.

Who	**What is said**
Question: x(y-4) =3	
Interviewer	No, [unclear] how many asymptotes do you have there? In a, in a hyperbola how many asymptotes do you have? Yes? How many?
Connie	Not shoo, maybe two.
Interviewer	Maybe two? What do you think?
Connie	I think, I'm not sure. But I think it could be y equals to four, and then the other one which is the y axis.

One of the reasons I think Connie seems unsure of the second asymptote is that there is only a vertical translation for Grade 10 graphs in the curriculum. The vertical shift makes the horizontal asymptote more visible, as the asymptote and the x-axis will not coincide.

The community of mathematicians understands, and can identify, the objects named by Connie. The equation has been named in a literate manner. It is possible to tell the difference between the horizontal and the vertical asymptote from her statement. Her use of words is classified as literate according to the DPH. The symbolic mediator $x(y-4)=3$ is not written in a way learners are familiar with, but Connie, after translating the equation to standard form, is able to name the asymptotes from it. This according to the DPH shows exploratory and flexibility routines in her answering of this question. A memorisation narrative was used as the rule for expressing the asymptote. This is in contrast to how Pat (Grade 11) named the asymptotes.

I shall discuss Pat and Palesa's responses to a question that asked them to name asymptote from a symbolic mediator $f(x)=\frac{3}{x-2}+1$. The two learners are examples of how six out of 9 learners named the asymptotes. Their discourse on naming the asymptote is colloquial, as they name the asymptote as if it were a number. Furthermore, Palesa uses the words "asymptote" and "symmetric"

as if they were interchangeable. I classify their use of words on naming the asymptote as colloquial and their routines are ritualised.

In the excerpt below Pat gives the equation of the asymptote as if it were a number. It is difficult to distinguish between the horizontal and the vertical asymptotes from such a statement. It is not possible for any other participant in the mathematical discourse to locate the asymptote from such an utterance and for DPH such talk is colloquial.

Who	What is said
	Question Sketch the function: $f(x) = \frac{3}{x-2} + 1$
Interviewer	Can you define them? Suppose I am blind I am only hearing. Can you tell me which ones are the asymptotes?
Pat	Ja my, my asymptote is, is at negative 9 and at negative 2.
Interviewer	Asymptote negative 9?

When she says "my asymptote is, is at negative 9 and at negative 2" it is not possible to distinguish the vertical asymptote from the horizontal. Both are named as if they are just numbers on a number line or intercepts with axes. In spite of the way she named asymptotes, she was able to sketch a graph that would have got her full marks (see figure 5. 5) in an examination.

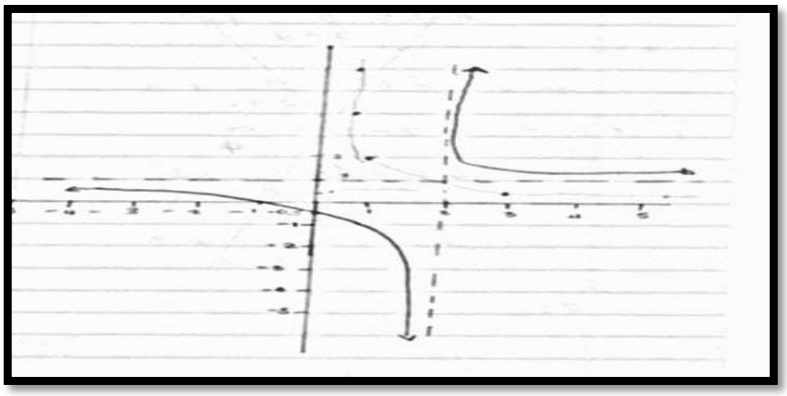


Figure 5.2 Pat's diagram of a hyperbola

The asymptotes are clearly shown as straight lines. Pat's routines are ritualised because she produces a graph with proper asymptotes, yet names them like numbers. DPH classifies her narratives on naming of the asymptotes as recall and are rejected by the community of mathematicians.

Palesa, (Grade 10), in the excerpt below, says: "I found my asymptote negative four, then I drew my symmetrical line negative four". The equation of the asymptote is given as if it is a number. Palesa says she found the asymptote negative four. She is probably referring to the horizontal asymptote, although this is not clear from her utterances. Apart from naming an equation as if it was

a number, she does not name the vertical asymptote. It is as if there is only one asymptote on her hyperbola. DPH classifies her talk as colloquial, her memorisation narratives are viewed as they are influenced by what she sees.

This is what she says on the asymptote of the equation $y = \frac{3}{x} - 4$.

Who	What was said
Interviewer	Given the function: $f(x) = \frac{3}{x} - 4$ *Sketch* the graph of *f(x)* Take me through the steps you took to draw your graph.
Palesa	I used a table method I started from negative three… my x was from negative to three. I substituted x with my x-values in the table to find my y values Then I plotted the graph Then I found my asymptote…. negative four Then I drew my symmetrical line, negative four… below the y-axis

Palesa also sketched a diagram similar to Pat's in figure 6.2, showing the two asymptotes, although she did not recognise the y-axis as an asymptote. She spoke much about processes rather than objects. The dominant routine is applicability, because she is able to draw the graph but could not self-correct nor use multiple routines.

Naming the equation of the asymptote as if it were a number was common with learners during the interview. Some knew the mathematical way of naming the equation, but opted to use the colloquial way of expressing the equation of the asymptote. By colloquial way I mean the use of words in an informal way. It would seem that giving the equation of the asymptote as negative four (-4) is acceptable to them. Their naming of objects is visual. They name the equations according to what they see from the equations. The naming of the asymptotes by the learners in this study shows a lack of disambiguation. The only person that understands the "lines" referred to is the one speaking; the rest of the people in the practice may not have a common understanding. Stating the asymptotes as equations increases the likelihood that members of the community of mathematics have the same understanding of the objects referred to.

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In the excerpt Palesa uses the words "symmetrical line" for asymptote. She uses the two words interchangeably as if they mean the same thing. When asked to explain why she uses them in that manner she says that is how she remembers it. In other words, she does not know the difference between the words. These two words are used in functions, and would apply in the hyperbola meaning different things. A symmetrical line is a line that divides shapes such that they are mirror images of each other, while an asymptote is as defined in earlier chapters. She does not have a full

understanding of the two terms. The terms have been used in a colloquial way by Palesa according to the DPH.

5.3.2 Asymptotes as demarcating quadrants

Quadrants are four regions formed by the x and y-axes on the coordinate plane. The parent rectangular hyperbola would lie in two of these regions, either the first and third or second and fourth. In this study the word "quadrant" is used a little differently by learners from the two grades. When learners refer to the quadrants they do not necessarily mean the quadrants as the mathematical community understands, but "quadrants" determined by the asymptotes. Asymptotes in a rectangular hyperbola are always parallel to the axes, and hence form what look like quadrants. The hyperbola is drawn within the regions demarcated by the asymptotes; learners then refer to these regions as "quadrants". Teachers have often done this as well, evidenced by anecdotal data in the Wits Maths Connect Secondary Project.

In explaining the steps, they took in sketching the graph learners were heard saying that the value of "a" determined the quadrants on which graph would lie. If the hyperbola is translated vertically and/or horizontally it would lie in at least 3 quadrants. For example, $f(x) = \frac{1}{x-3} + 2$ lies in quadrants I, II and III, as the figure 6.3 shows.

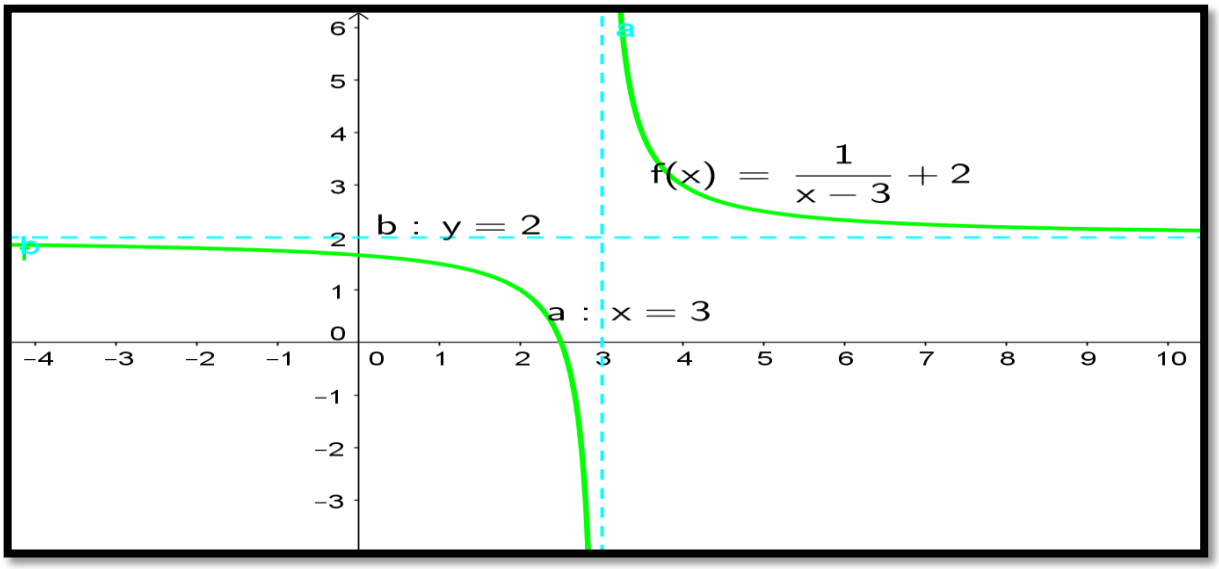


Figure 5.3 A hyperbola in 3 quadrants

The probable reason that they refer to these regions as quadrants is that the two asymptotes are perpendicular to the each other and parallel to the axes, as seen in the diagram above. All the learners referred to the regions formed by drawing the asymptotes as quadrants. I will use what Malusi said to illustrate how they used the word "quadrant".

In the excerpt below Malusi explains how he sketched the graph of $f(x) = \frac{3}{x-2} + 1$. He says the value of "a" determines the quadrants in which the graph would lie. He says a positive value of "a" will have a graph in the first and third quadrants, and for a negative in the alternate quadrants.

Who	What is said
	Explain how you sketched $f(x) = \frac{3}{x-2} + 1$
Malusi	I started finding the shape too and the asymptote, then I worked out the y intercept and the x intercept that made me to draw the, sketching the draft to find the x asymtote. I said x is supposed $x -$ just goes to zero then I took two to the other side, then y asymptote was already there and then to know that my graph is on the first and the third quadrant the x should be positive. Then if the, the a was negative my, my shape was going to be at the second and fourth quadrant.

According to this statement we have two sets of Cartesian plane with eight regions that can be referred to as quadrants. This substantiation narrative cannot be accepted in the practice of mathematics because it does not distinguish between two objects that look similar but actually are different. Quadrants are only the regions formed by the axes. To say that asymptotes demarcate quadrants is a visual cue that is meant to have learners understand the function more easily, but is a colloquial way of expressing the regions. Using the DPH, the substantiation is a result of application of the visual trigger routine, where a set of asymptotes are regarded as the axes. Learners tend to be stuck with this visual cue.

5.3.3 The y-axis is not seen as an asymptote

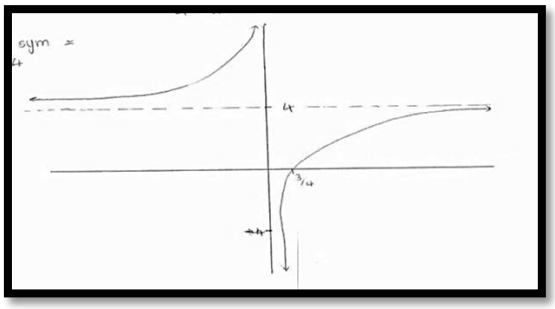
One of the distinguishing features of the hyperbola graph is the asymptotes. For the Grade 11 hyperbola the two asymptotes are clearly distinguished from the axes because of the vertical and horizontal shifts. In Grade 10, I found out learners have difficulty in recognising the vertical asymptote. Some of the reasons for this include that the vertical asymptote coincides with the y-axis, and therefore not "visible" as an asymptote. My observation has been when sketching the hyperbola with a vertical shift that the general practice is to start with the axes and then draw the horizontal asymptote. The vertical asymptote is then ignored. From there the graph is sketched. So the routines in the classroom may be the reason learners define a hyperbola as a single-asymptote graph. Secondly, graphs that are drawn at Grade 10 level do not clearly indicate the y-axis as a vertical asymptote. The asymptote is usually denoted by a broken vertical line. There is nothing to show the difference between the asymptote and the y-axis. An asymptote is not a "visible" line that is part of the graph. A dotted line is accepted as a representation of the asymptote, but a function does not necessarily have to show the dotted line. The purpose of the dotted line aids in showing the asymptote easily. If the axes are asymptotes the dotted line is not usually shown because the axes are part of the graph, but the dotted line is not. Thirdly, in the equation and graph the horizontal

asymptote is easily identified, because (for Grade 10s) of the vertical shift, yet the horizontal remains static. The vertical asymptote is then not emphasised as much as the horizontal.

Four out of the five learners in Grade 10 spoke as if there was one asymptote, the horizontal one. Below is an excerpt where Leo and Connie state the number of asymptotes from an equation. Leo names the horizontal asymptote, but says nothing of the vertical. When pressed to give the number of asymptotes on a hyperbola, he did not respond

Who	What is said
	In the equation x(y-4) =3. What are the asymptotes in this equation?
Leo	Asymptote is four (4)
Interviewer	In a hyperbola how many asymptotes do we have?
Leo	(no response)
Connie	Uhmmm…. not sure maybe two (2) y is equal to four ($y = 4$) and the y-axis
Interviewer	Maybe two (2)? What do you think?
Interviewer	Can there be more than 2?
Connie	(no response)

Again, the diagram below represents Leo's answer to question 1.1. In his answer he clearly marks the horizontal asymptote, but does not do the same for the vertical because it is the y-axis. He only mentions the horizontal asymptote, and says nothing of the vertical. When asked to for the number of asymptotes, he does not answer. This signifies ritualised routines, where the answer is based on what the learner already knows, rather than the mathematics behind action. The graph as drawn does show the y-axis as an asymptote, yet he only mentions the horizontal asymptote.



5.4 Leo's response to $f(x) = \frac{3}{x} - 4$

The narratives of learners are affected by the interpretation they attach to the equation of the hyperbola and iconic mediator, the graph, for Grade 10 learners. So, the y-axis is not seen as an asymptote.

When drawing or sketching graphs learners would have correct diagrams that are not informed by correct mathematical understanding. Not even one of the Grade 10 graphs cut the y-axis, yet it was

never referred to as an asymptote. One of the task questions for Grade 10 was: Explain why $f(x) = \frac{3}{x} - 4$ has no y-intercept.

Who	What is said
interviewer	Okay, alright. Okay, thank you very much. Let's, one point three, explain why F of X has no Y intercept, what's the reason? What do you think?
Leo	The graph does not touch the Y axis.
Interviewer	Yes, it does not touch, but what's the reason, why doesn't it touch it? From the graph, if you look at your equation, maybe it can give you a cue.
Connie	I think it's because x is in the denominator so if we divide… so if we divide three by zero it would be then undefined.
Interviewer	Undefined, okay. Do you agree, you're nodding your head?
Leo	Yes.
Interviewer	Do you agree?
Leo	Yes, I do.
Interviewer	Do you agree, why do you agree?
Leo	Because when I calculated the y-intercept I found that it was undefined.

In their talk of the asymptotes, learners were silent on the vertical asymptote. The learner narratives are affected by the equation of the hyperbola. On the hyperbola, the emphasis is put on the value of the constant as the asymptote rather than the asymptote being identified as an equation. The evidence of ritualised routines, where learners are able to sketch the graph, yet fail to explain how they sketched it.

The horizontal asymptote is "visible" from both the equation and the graph for Grade 10 hyperbola. Learners would normally show an asymptote by drawing a dotted line. This does not happen with the vertical asymptote because it coincides with the y-axis and is not "seen" on the equation since its value is zero. The y-axis is already drawn as a solid line, so any broken line on it would not be visible. So the y-axis is not seen as an asymptote, but as a line that the graph should not pass through. Figure 5.5 illustrates this. The asymptote is viewed as a wall that blocks the graph from passing through.

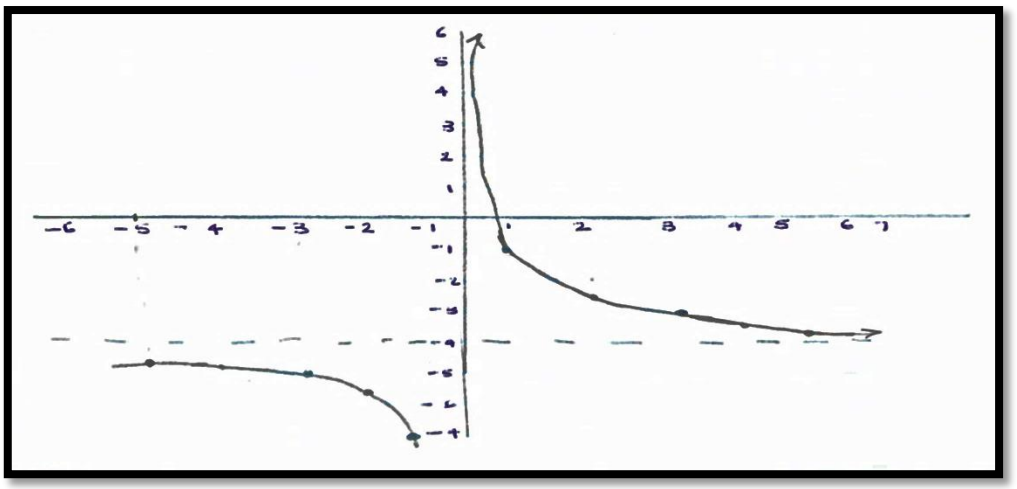


Figure 5.5 A typical Grade 10 Hyperbola

The vertical asymptote, as seen in the above diagram (figure 5.5), is visible in all their diagrams as x tends to zero. Learners do not see the y-axis as an asymptote. There is a difference between what is available on the diagram to be seen and what is perceived. Their diagrams and what learners say are different, meaning that although their diagrams show the y-axis as an asymptote, learners do not perceive it as such, but as a line that the graph does not have to pass through. Therefore, what is made available for learners to see through a symbolic mediator is not interpreted that way.

To conclude this section, it appears that learners in Grade 10 know that a hyperbola has a horizontal asymptote, and can identify it from the both the iconic and symbolic visual mediators, but do not "see" the vertical asymptote from either. It is a generalisation they hold about the hyperbola that is not endorsed by the community of mathematicians. The reason why learners have difficulties with the asymptotes is that their justifications and generalisations are mostly visual, and their routines are ritualised. If the asymptote is not clearly seen, then it does not exist. For example, for Grade 10 learners the vertical asymptote is x is equal to zero (x = 0), and therefore not written on either the iconic or symbolic mediator for them - so it is, for them, not there. According to the DPH, their use of words is colloquial, iconic and symbolic mediators are interpreted from the visual point of view, the routines are ritualised while their substantiations are also visually based.

5.4 Mismatch: What is said versus what is done

In this section I discuss Palesa and Sifiso's discourses on the asymptote of the hyperbola. The two learners' graphs are correct, but their justification of their actions is not. There is a mismatch between what they say and what they wrote.

As explained above, Grade 10 learners in this study talk as if the vertical asymptote does not exist, but they draw the graph as if they recognise the y-axis as an asymptote. Also, they choose points when making a table of values that do not reveal either asymptote. They just plot some general points as they would do for a linear function. They get the curve correct (or partially correct) because of what they already know. In figure 5.6, Sifiso chose integral values of *x* for the table from negative three (-3) to positive three (+3).

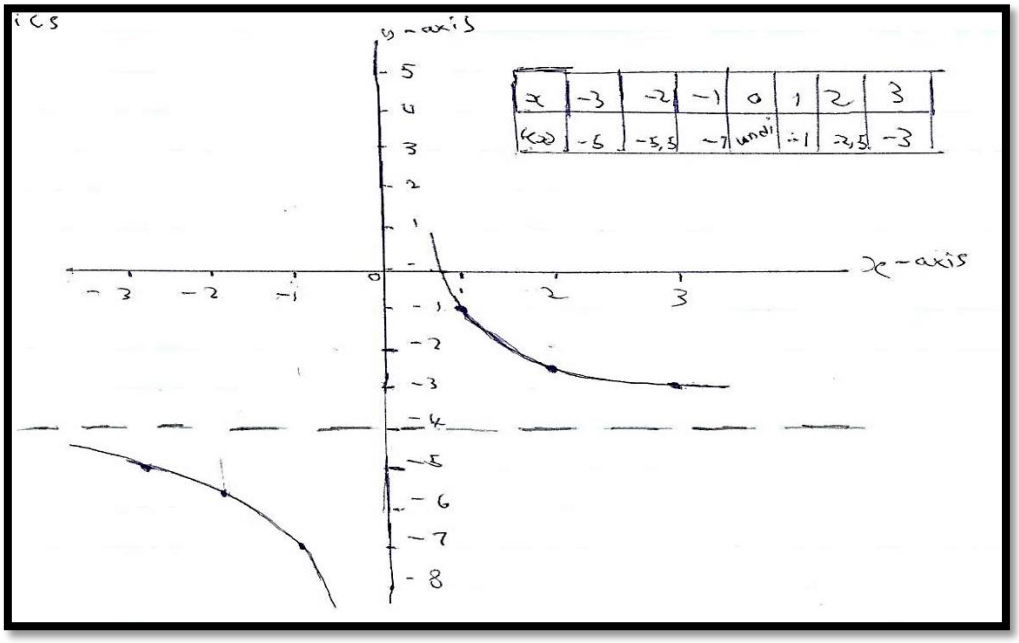


Figure 5.6 An illustration of the table of values and graph

The points on the table do not show the asymptote, yet the graph shows the asymptotes. This is evidence of applicability routines. A visual trigger has been used to identify the asymptotes. In this case the symbolic mediator $f(x) = \frac{3}{x} - 4$ has been used to sketch the graph. The table of values does not show either the vertical or the horizontal asymptotes, yet the graph shows both.

If learners in this study are compared with learners who are encountering the hyperbola for the first time; often first timers will try to join the two pieces of the graph and so they end up with a line that cuts the y-axis (see figure 5.7).

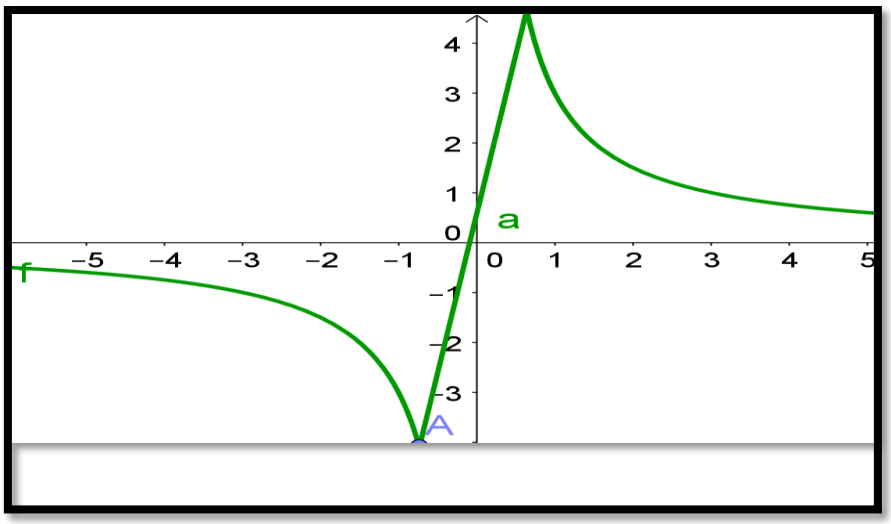


Figure 5.7 An illustration of what first timers would do

Here they do not actually choose x-values that show the asymptotic behaviour of the hyperbola. So they just plot their general points and then complete the rest of the shape of the graph (see figure 5.8).

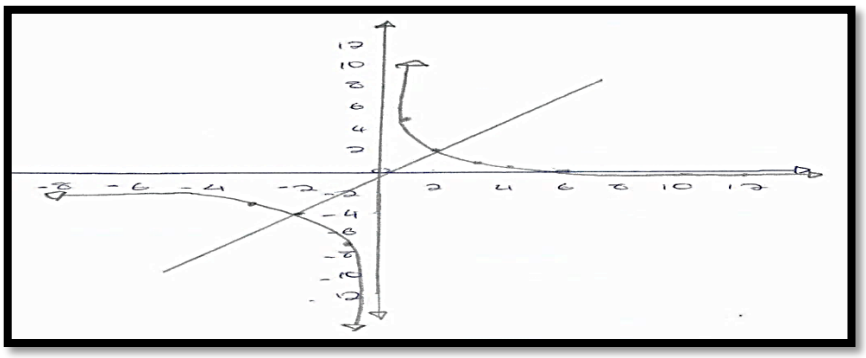


Figure 5.8 An example of iconic mediator drawn from a table

The excerpt below is Palesa's answer to the question "Explain the transformation from g(x) = $\frac{3}{x}$ to f(x) = $\frac{3}{x} - 4$". She says the graph would cut the y-axis. It is not clear which of the two functions she is referring to, but what is clear is that for someone that drew the diagram below (figure 5.9) it is surprising that she talks of the graph cutting the y-axis.

Who	**What is said**
Question	Explain the transformation from g(x)=$\frac{3}{x}$ to f(x)=$\frac{3}{x} - 4$
Interviewer	right if you are saying its zero so if your asymptote is x is equal to zero how does that affect the graph that you drew? What would … can you show me on your graph how this will affect your graph.
Palesa	ok ahh the graph would have maybe cut pass on the y axis and it would have to lie it would have to lie below the asymptote as it will not have an asymptote so it will pass through the y-axis
Interviewer	ok it will pass through the…

Palesa	the y axis
Interviewer	why would it pass through the y axis??
Palesa	because there is no att... asymptote

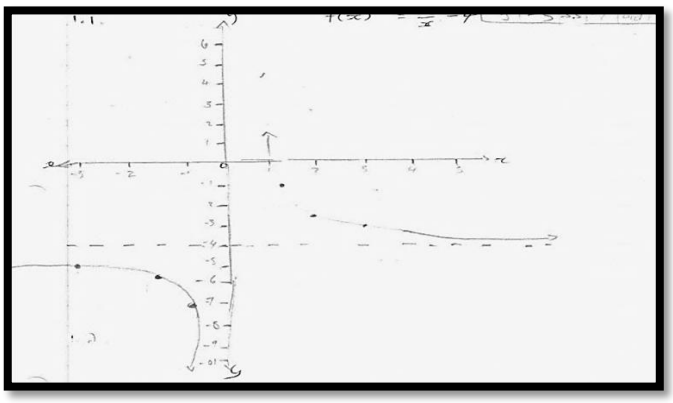


Figure 5.9 Palesa's response to $f(x) = \frac{3}{x} - 4$

She goes on to say "there will be no asymptote". In comparing the two functions, $g(x) = \frac{3}{x}$ and $f(x) = \frac{3}{x} - 4$, she says in $g(x)$ there is no asymptote but $f(x)$ has an asymptote. The asymptote, according to Palesa prevents the graph from cutting the axis. She talks of the y-axis and at the same time of the graph being below the asymptote. The asymptote in this case is a horizontal one and the y-axis. The absence of the asymptote, according to her, allows the graph to cut the y-axis meaning the asymptote prevents the graph from cutting the y-axis. (Note that she does not regard the y-axis as an asymptote). The y axis is not an asymptote, the "absence" of the parameter "q" (in this case $q = 0$) to her means there is no asymptote. So the parameter "q" is the asymptote. She "sees" the asymptote from the symbolic mediator rather than from the iconic. Words used are mainly colloquial as the use of the word asymptote has to do with the symbolic mediator rather than the iconic. There is no link between the iconic and the symbolic mediators. This strengthens my assertion that her routines are ritualised and that she uses the visual trigger (applicability routines). A hyperbola must have asymptotes. For Grade 10s there is no horizontal shift, so the talk of a graph cutting the y-axis shows some lack of understanding. She gives her reason as "because there is no asymptote". The y-axis is not regarded as an asymptote, yet the diagram shows that it is one. Her recall narratives are visual, and not endorsed by mathematicians, since all hyperbolas have an asymptote. Again using the DPH, words used are colloquial, iconic and symbolic visual mediators are visually interpreted, routines are ritualised and visual trigger enables them to sketch correct diagrams and their memorisation narratives are visually based.

The graph below that Palesa had sketched would probably earn her full marks for the diagram, yet her explanations show that the diagram is a result of being familiar with the procedure for sketching the graph using ritualised routines.

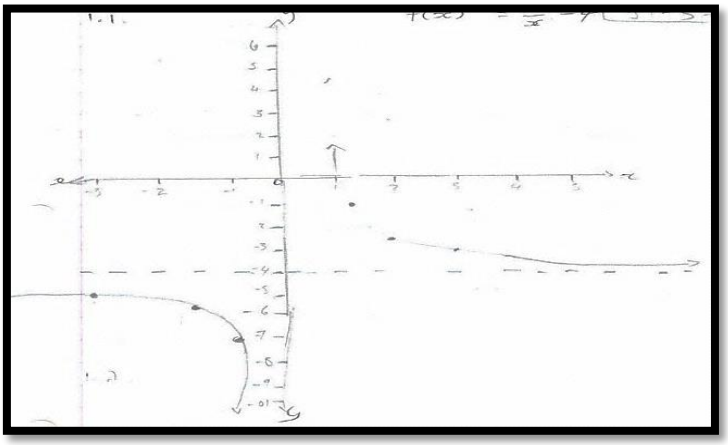


Figure 5.10 Palesa's response to $f(x) = \frac{3}{x} - 4$

Palesa was further asked to give a reason for her graph not having a y-intercept. She could not give a reason. The graph above clearly shows the y-axis as an asymptote, hence showing a mismatch between what is said and what is done. This clearly indicates ritualised applicability routines.

In the next paragraphs I will discuss Sifiso's understanding of the effect of the asymptote on the graph.

For Sifiso the asymptote determines the quadrant on which the two parts of the hyperbola graph lies. On explaining the steps, he took to draw the function $f(x) = \frac{3}{x} - 4$, he said the sign of the asymptote determines where the graph would lie, the positive means the graph is on the first and third quadrants and a negative sign would make the graph to be on the second and fourth quadrants. Below is what Sifiso had to say:

Who	What is said
Sifiso	I started by determining x-intercept and the y-intercepts and asymptotes. In order to determine the fact of where my graph will lie on. To see at the end if I am right and wrong. As my graph is the asymptote is negative four, which means the asymptote is negative… it is easier for me to determine the quadrants where the graph would lie…
Interviewer	So then from there what did you do?
Sifiso	From there I started making a table to determine the values of x and y…then I sketched it.

If Sifiso had said this only here, I might have regarded it as a slip, but he mentions the same thing again. On answering the question "What would happen to the graph of h if the value of q is -3? $[h(x) = \frac{2}{x} + 3]$ He talks again about the sign on the asymptote affecting the quadrants on which the graph lies.

Who	What was said
Interviewer	What would happen to the graph of h if the value of q was negative three?
Sifiso	What would happen to the graph of h?
Interviewer	Yes
Sifiso	The thing that would happen as our asymptote is changed to negative three…the graph would no longer lie on the same quadrants…
Interviewer	Why?
Sifiso	For example, now it lies on the first and the third quadrant
Interviewer	So where would it lie?
Sifiso	It would lie on the second and fourth quadrant
Interviewer	Why would that happen?
Sifiso	Because the asymptote it would change to negative three
Interviewer	So the asymptote changes the quadrants, am I hearing right?
Sifiso	Eh…yes…the way I understand it

Sifiso's substantiation narrative is not endorsed by the community of mathematicians. While it is true that the asymptotes demarcate the "four regions", they do not determine where the graph would lie. On question one, where the asymptote is negative, Sifiso's graph was on the first and the third quadrants. The reason for this is that he already knows the shape that comes out of such an equation. There is a mismatch between what he understands to be mathematically true and what he does, meaning that the routines are ritualised. The misunderstanding shows the conflict between his narratives and what he believes to be mathematically true.

5.5 Link between equations

In this section I shall discuss learners' talk on the link between the symbolic mediator $f(x) = \frac{a}{x-p} + q$ and the parent function $g(x) = \frac{a}{x}$. In the form in which function f has been given it represents a vertical and/or horizontal translation. Learners immediately associate the symbolic visual mediator $f(x) = \frac{a}{x-p} + q$ with the hyperbola, and can most probably draw a sketch from such equations. My intention was to investigate if learners could link the above equation ($f(x) = \frac{a}{x-p} + q$) with $g(x) = \frac{a}{x}$. It was difficult for most of the learners to explain that function f is a translation of function g. Moreover, learners could identify the asymptotes of function f and fail to identify the asymptotes of function g.

5.5.1 Translation of a hyperbola

Learners struggled to explain the relationship between symbolic mediators $f(x) = \frac{a}{x-p} + q$ and $g(x) = \frac{a}{x}$. Two learners of the nine managed to link the two visual mediators. In this section I shall

discuss how Sean and Connie explained the relationship between the equations. These two learners could make links between the two equations.

Sean (Grade 11) explains that the function $f(x)$ is a result of a translation both vertical and horizontal. The transformation is explained from the symbolic visual mediator without a diagram to back it. Each of the parameters is clearly distinguished, and their effect on the graph is explained.

Who	What is said
Explain how $f(x) = \frac{3}{x-2} + 1$ is transformed from $g(x) = \frac{3}{x}$	
Sean	Oh from, from graph $g\ of\ x\ to\ f\ of\ x$ the graph was shifted one unit upward to form, to form $g\ of\ x = \frac{3}{x} + 1$ and also it was shifted two units to the right to create the new graph which is $f\ of\ x$ which then the equation of $f\ of\ x$ would be $\frac{3}{x-2} + 1$

His communication shows some disambiguation, as all in the community of mathematics will have the same interpretation of the equation. Therefore, his talk is standardised and literate, as the rules of communication are the same as others in the community. He also shows that the symbolic mediator is a compressed version of what could have been said in many words. The routines are exploratory, and show some flexibility, as he is able to explain how a hyperbola is translated by horizontal and vertical shifts.

In the equation $f(x) = \frac{3}{x} - 4$, Connie explains that the graph has been shifted or translated 4 units down from an equation $g(x) = \frac{3}{x}$. She goes further to explain that her communication is determined by the visual mediator when she says she knew there was a vertical shift of negative 4 from equation. This explanation can be seen in her graph (see figure 5.11 below).

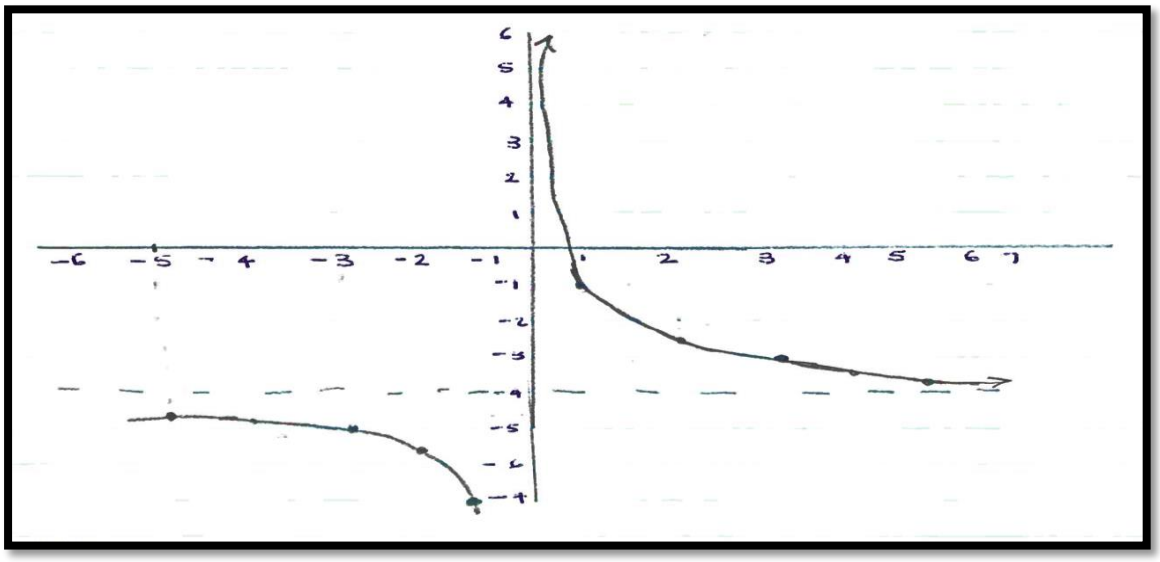


Figure 5.11 Connie's graph $f(x) = \frac{3}{x} - 4$

She could also explain equations that were asked verbally. For example, "What if it was, f(x) is equal to three over x plus eight [$f(x) = \frac{3}{x} + 8$], what would have happened to the graph of three over x?" Her answer is that it would move 8 units up. Connie can link the equation, the graph and the transformation. Her participation in the mathematical discourse would count as competent as far as translation, equation and graph are concerned. She was the only one among the Grade 10s that managed to display such competence.

Who	**What is said**
Question: Explain the transformation from g(x)=$\frac{3}{x}$ to f(x)= $\frac{3}{x} - 4$	
Connie	It's that the graph has been shifted four units downwards.
Interviewer	Four units downwards. Where do you see that in the equation?
Connie	By the minus four.
Interviewer	The minus four?
Connie	Yes.
Interviewer	So the minus four tells you the graph has been?
Connie	Shifted four units downwards.
Interviewer	Right. What if it was, f of x was three over x plus eight, what would have happened to the graph of three of x?
Connie	It would have shifted upwards, eight units upwards.

Connie's routines show exploration, and uses applicability's visual trigger. She uses negative four (–4) from the symbolic mediator $f(x) = \frac{3}{x} - 4$. Her narratives are based on a recall where a rule has been applied from a visual symbolic mediator.

Sean and Connie were the only learners in his study that managed to show some competence with transformation of the hyperbola. They managed to interpret the symbolic visual mediator in the same way as other competent participants in the practice would do. In the DPH their use of words is classified as literate, they are able to link the viewed and the drawn iconic and viewed and formulated symbolic visual mediators, there is some form of flexibility as they are able to link translated equations to the parent equations without an iconic visual mediator and their substantiations are accepted by the community of mathematicians. The next section discusses the focus of learners that could not link the two equations.

5.5.2 Failure to link equations

In this section I focus on the transformation from the function $g(x) = \frac{a}{x}$ to $f(x) = \frac{a}{x-p} + q$. Learners were asked to explain the transformation from g(x)=$\frac{3}{x}$ to f(x)= $\frac{3}{x} - 4$.

Three learners focused on the asymptotes. These were two Grade 11s and one Grade 10. I shall discuss the response of Palesa, a Grade 10 learner. The reason is that Palesa's response is clearly focused on the vertical asymptote, unlike the Grade 11s that spoke of the "undefined intercepts" and "undefined asymptotes".

Palesa associates the two equations with asymptotes. She says "at first there is no asymptote" referring to g(x). She goes on to say f(x) has an asymptote.

Who	What was said
Question: Explain the transformation from g(x)=$\frac{3}{x}$ to f(x)= $\frac{3}{x}-4$	
interviewer	How did you do one point two (1.2)
Palesa	The transformation from g of x [g(x)] which is three over x ($\frac{3}{x}$) …at first there was no asymptote…and when we transformed into f of x is equal to three over x minus four [f(x) =$\frac{3}{x}=-4$] whereby we have our asymptote…
interviewer	Let's start again…you say at first there was no asymptote?
Palesa	It was …it was…it was one then …
interviewer	It was one, Why?
Palesa	It was not written which means it was one

The equation $f(x)=\frac{a}{x}+q$ has an asymptote, but for $f(x)=\frac{a}{x}$ there is no asymptote for Palesa. This suggests that "q" is the asymptote, not y = q. Earlier in this chapter I mentioned that the asymptote is given as if it were a number. This suggests that when they say the asymptote is -4, for example, they actually mean the -4 on the equation rather than the line on the graph. Consequently, if the parameter is not written for a parent graph then there is no asymptote. Since there is no mention of the vertical asymptote I can infer that the reason is because of the "absence" of the parameter p.

Secondly, Palesa uses the discourse on algebraic terms in the functions discourses. When she is asked to explain the meaning of "there is no asymptote" she changes to that it is 1. This may suggest that she is using the discourse on algebraic terms where the coefficient of 1 is not written. I could not push it further, as she went back to that it was zero.

Who	What is said
	Explain the transformation from g(x)=$\frac{3}{x}$ to f(x)= $\frac{3}{x}-4$
Interviewer	in g(x) what are your asymptotes
Palesa	I think its plus one
Interviewer	Is it y is equal to one?
Palesa	no its zero
Interviewer	Its zero? (Repeating what she had said)
Palesa	Yes

This shows that she does not understand the relationship between the two functions. Her focus is on parts of the equation in which "the fraction would mean a hyperbola" and that the number is an "asymptote". She has not yet reached a level where her talk about objects has the same interpretation as that of mathematicians. Although she uses mathematical words like asymptote, her communication is not coordinated by the meaning of the visual mediator presented. Her routines are ritualised, having sketched the graph of a hyperbola, and are based on visual triggers like the symbolic visual mediators $f(x) = \frac{3}{x} - 4$ and $g(x) = \frac{3}{x}$. The justification (Recall narratives) are based on what she sees at the time. I shall explore how her interpretation of the two equations is different or mismatched in the coming paragraphs below.

The reason for this may be that their routines are close to what the teacher taught them (ritualized). In their talk they do not mention the vertical asymptote, nor do they explain how they got the horizontal asymptote. This may suggest that the routine for some learners in this study is mostly ritualised. They do what they see the teacher do in class, perhaps, without necessarily understanding the reasons for the steps. Learning mathematics is about change in practices and justifications for the changes effected. Part of that change is seeing new routines and giving justifications for choosing them. Learning of mathematics is about reasons behind the actions. Finding reasons for actions is aim of learning mathematics.

5.6 Conclusion

The use and interpretation of the word *asymptote* by learners has been prominent in this study. While the graphs drawn by learners indicated the asymptote, their explanation of how they used this word shows that their routines are ritualised for most of them. Asymptotes are named as if they are numbers, but the graphs are drawn with asymptotes in their proper positions. Some of the Grade 10s did not realise that the y-axis was an asymptote, yet the diagrams they drew show the y-axis as an asymptote. This led to mismatches between what they wrote and what they said. The x-values chosen for the table of values are the same as that of any other function, yet for a hyperbola because of the asymptotic behaviour it is not easily revealed without a careful selection of points. The graphs sketched show the asymptotic behaviour. While some learners could sketch a function translated vertically and/or horizontally, they could not explain the link between a parent function and the transformed function. This suggests that learners' routines are ritualised, and that the applicability routine was more dominant. The flexibility and corrigibility routines that could be have shown some explorations were not evident for most of the learners in the study. In fact, there were two learners who exhibited some explorations and used the flexibility routines. There was no

evidence of corrigibility. Answers were rarely checked and corrected. There was a rare use of multiple routines to a solution so that they could check their answers.

As a result, most of the learners had visually-based narratives. The narratives were not those that are endorsed by the community of mathematicians, especially with regard to the asymptotes. Asymptotes were said to be demarcating quadrants, or not present in some hyperbola functions, or said to be undefined. Most of the words were mathematical words but learners in most instances could not explain the use of those words in the same way as those in the community of mathematicians.

The naming of objects affects the way learners view the objects. They say the asymptote is "three" and the number referred to would be the parameter, and this explains why in the graph it would be a line and the utterance is about a number. The asymptote is viewed as a number. When learners look at the equation they see numbers as an asymptote. Therefore, absence of a number signifies absence of the asymptote. I would say this is a processual description of objects. In the processual stage objects are described according to what they do rather than what they are.

Learners did not relate the output to input that produces the curve and hence the asymptote. The table of values did not show values approaching the asymptote as either x or y increased. Instead, integral values of x only were chosen, but the graph showed an asymptote.

Learners' mathematical discourse is affected by routines that are ritualised. Most of the words used by learners were colloquial. Although the words were mathematical, their application was misplaced in most cases. Mathematical words were uttered in most instances, but their use or explanation was colloquial, mainly because of the ritualised routines. The narratives and symbolic mediators where mostly visual, and the narratives were not those accepted by the community of mathematicians.

Chapter 6: Learners' discourse on unfamiliar tasks of the hyperbola

6.1 Introduction

There are two types of learning in commognition: object-level and meta-level learning. In this study most of the focus was on object-level learning because my focus was on expanding the existing discourse, since I did not teach the learners or influence their learning prior to this study. The discourse expands through enhanced vocabulary producing new endorsed narratives or constructing new routines. I shall discuss how learners fared when faced with new situations that needed new routines.

In chapter six I discussed the discourse of learners on the asymptote of the hyperbola. The focus was mainly on questions that they would meet in classroom exercises and examinations. I use the DPH again in this chapter as I examine learners' discourse on the hyperbola in working with unfamiliar tasks. In this study "unfamiliar task" refers to tasks not normally found in South African classrooms and textbooks. My focus shall be on how learners dealt with the questions that were expressed in an unfamiliar form as far as their learning of the hyperbola is concerned and the linear equations with a fraction coefficient in the form of $y = \frac{x}{2}$. The task for the first part of this discussion is:

For Grade 10

Given an equation: x(y-4)=3

1.1 Name the graph represented by the above equation

1.2 Identify any special features of the graph from the given equation

1.3 Explain how you identified the special features in 1.2 above

1.4 If the graph is moved three units to the left, will the equations of the asymptotes change? Write them down. Will this movement change the position of the intercepts? Why or why not

For Grade 11

Given an equation: (x-2)(y-4)=3

1.1 Name the graph represented by the above equation

1.2 Identify any special features of the graph from the given equation

1.3 Explain how you identified the special features in 1.2 above

1.4 If the graph is moved three units to the left, will the equations of the asymptotes change? Write them down. Will this movement change the position of the intercepts? Why or why not

The questions are similar for the two grades, except that the Grade 10 question has a vertical shift only. This question gave me an opportunity to examine the routines of the learners. I looked at the flexibility routine, (the ability to work with unfamiliar routine procedures) and corrigibility (the ability to correct one's discursive procedure). In the previous chapter it is evident that learners were comfortable with the ritualised routine. I shall therefore examine the discourse of learners in the question given above because in unfamiliar tasks flexibility routines are usually brought to the fore.

6.2 Unfamiliar task

Eight of nine learners could satisfactorily work with an equation given in the form of $f(x) = \frac{a}{x+p} - q$ and a sketched graph. The first question was in the form of an equation, and learners were required to sketch a graph. The second was given in a graph and learners were asked to give the equation to the graph. These were typical of questions in school textbooks or even in the examinations. Even though learners had struggles with giving reasons for their actions, there was evidence that they could do familiar procedures. There were some difficulties with interpreting data from a table, but my focus is on the questions given above. In a normal learning situation questions are rarely given in this form. I have classified learner responses into three categories, those that thought the equation represented any other function, those who correctly identified it as a hyperbola, but could not give substantiations that are endorsed by the community of mathematics as a narrative for their answers and finally those who realised it was a hyperbola and gave a mathematical justification for their answer.

6.2.1 Competence to link equations

Two Grade 10s, Leo and Connie, realised that the equation could be transformed to a standard form they are familiar with. They could work independently from the classroom examples and tasks. Their discourse shows traces of exploration. Although there is still a long way before these two learners can fluently express themselves mathematically, there is a sense that says their routines are flexible. They are not only able to do familiar tasks but can go further and transform equations. The aim of learning mathematics is such that they are able to apply what they have learned in new situations.

Who	What was said
Interviewer	What is the function represented by the equation in 1.1?
Connie	It's a hyperbola
Interviewer	Why do you say it's a hyperbola?
Connie	When we divide by x both side the fraction would be a fraction which would be three over x, so a fraction then a hyperbola

Interviewer	A fraction therefore hyperbola and then
Leo	It's a hyperbola
Interviewer	Why?
Leo	When I simplify the original equation there is a fraction.
Interviewer	Can you give me the equation that you got
Leo	Y is equal to three plus four x over x.

Although Connie and Leo show exploratory (as defined in the DPH), or flexibility routine, they still talk of the fraction as signifying a hyperbola. In their expressions the notion of 'fraction therefore a hyperbola' keeps cropping up. The idea of a fraction signifying hyperbola will be discussed later in this chapter.

6.2.2 Non-mathematical reasons

The equation was identified as hyperbola, but no sufficient reasons were given. The hyperbola is associated with the asymptote. The reasons given arise from the presence of the word *asymptote,* so therefore the equation is for a hyperbola. Sometimes the reason is given that there is an intercept, but the learners failed to give further reasons for their assertion. Using the analytical tool, DPH, they arrive at their conclusion because of the visual trigger, in this case the word *asymptote.* This indicates applicability, the use of ritualised routines. Their narratives are therefore recall based on what they see.

Malusi identified the equation as that of a hyperbola, but gave a reason that would fit for any function. His reasons did not distinguish a hyperbola from other functions. Below is what was captured during an interview:

Speaker	**What was said**
Interviewer	What graph is that?
Malusi	At first I thought it was a linear graph, then I cancelled and said hyperbola
Interviewer	Why a hyperbola?
Malusi	Because I had x-intercept and y-intercept

These reasons apply to any function. While it is true that this equation is for a hyperbola, the reasons for arriving at that conclusion are not enough. From the figure 7.4 below he solves the equation as if it were a quadratic equation. He then takes his solution to the equation as the intercepts. It is likely, (although not stated) that the presence of the word asymptote in the questions that followed may have influenced him to change his mind from a linear function to a hyperbola. While he could realise that his classification was incorrect, he did not use a different method or change his working. This disqualifies his use of routines to corrigibility, because no alternative

working was used to change his mind. Malusi was able to work comfortably with equations or graphs that are expressed in standard form, and his routine is classified by the DPH as ritualised.

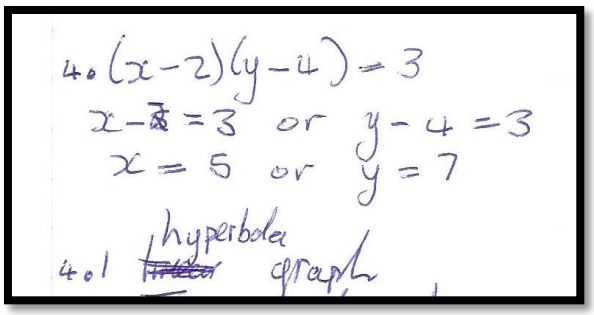


Figure 6.1 Malusi's response to: Name the graph represented by $(x-2)(y-4)=3$

Palesa (Grade 10) and Dolly (Grade 11) also classified the equation as hyperbola. They gave the same reason, namely, that they saw the word asymptote in one of the questions, and concluded that it was an equation of a hyperbola. For Palesa a hyperbola is denoted by an asymptote, signifying a visual trigger. While this question would have gone as far as getting to the asymptote, the intention was to test the flexibility of learners with mathematical procedures on the hyperbola. Inasmuch as learners have done the exponential function, where one would find an asymptote, the asymptote is emphasised with the hyperbola only. Using the DPH, I classify the narratives of the two learners are recall-based on the visual, and are not endorsed by the community of mathematicians. Thus Palesa says the only function with an asymptote is a hyperbola.

Who	**What is said**
Palesa	To be honest it's my first time to see an equation like this but as I read through the questions I realised because of asymptote then I said it is a hyperbola
Interviewer	Let me ask you; which graphs have got asymptotes?
Palesa	It's a hyperbola
Interviewer	And?
Palesa	It's the only graph

Dolly gives the same reasons as Palesa, associating the asymptote with the hyperbola. For Dolly, when there is an asymptote or shift in the graph, then the conclusion is that it is a hyperbola. This gives the impression that only the hyperbola has an asymptote. This substantiation narrative of Dolly is not endorsed by the community of mathematicians.

Who	**What is said**
Interviewer	What function does the equation represent?
Dolly	It's a hyperbola
Interviewer	Why?
Dolly	I classified it as hyperbola graph because what I did is I worked out the equation that I have been given, to find the x and the y value, and I couldn't get the clarification of the kind of graph. So I skipped the question and somewhere in the questions some other questions where talking of the asymptotes the shift in the graph and I then concluded that it was a hyperbola.

In general, learners are used to working with equations in standard form. Sifiso said that the equation was "wrongly written". During the interview this is what he said:

Who	What is said
Interviewer	What function is represented by the equation in 1.1
Sifiso	I said it's a hyperbolic function
Interviewer	Why do you say it's a hyperbolic function?
Sifiso	Other functions doesn't have the asymptote
Interviewer	So how do you see the asymptote?
Sifiso	The asymptote? From this equation? I see by asymptote by its equation it's going to be three.
Interviewer	The asymptote is three? How do you see that from the equation?
Sifiso	The asymptote is in the equation where q is equal to the asymptote. So q is equal to three. It's like solving for x and y… The way they place it's like it's incorrect

What comes out of this discourse is that Sifiso views the equation $y = \frac{a}{x} + q$ as the only way of expressing the equation of the hyperbola; he says **x(y-4) =3** is incorrectly written. The only way of writing an equation is the standard form. Secondly, he says three (3) is the asymptote. In this case it appears that the constant in the equation is three, and because of that he says that the asymptote is three. In this case the asymptote is denoted by its being a constant. "q" is a constant in a standard form, and therefore in $x(y-4) = 3$ the constant 3 is also an asymptote according to Sifiso. His operation on functions is visual, and there is no indication he goes beyond what he sees.

6.2.3 Equation not recognised

Two Grade 11 learners and 1 Grade 10 did not realise that the equation was for a hyperbola. Sean, (Grade 11), initially said the equation was for a parabola, but after doing the algebraic manipulation he changed his mind to exponential function. He chose to expand, factorise and then divide to make y the subject of the formula. He then introduced an exponent. I could not understand where this exponent came from or what rules are used. The equation $y = \frac{-5+4x}{x-2}$ that he should have got had he not slipped and written -8 instead of +8, is the same as equation $y = \frac{3}{x-2} + 4$. Sean does not realise the algebraic mistake he made, and he classifies the equation as that of an exponent. The realization tree of the functions, in this case the algebraic manipulation and the exponents sub-discourse of algebra, are the ones that prevented him from working flexibly with the equation. He does not use other routines to establish the correctness of his answer. The DPH classifies the routines as in lacking flexibility and corrigibility. His routines are, therefore, ritualised.

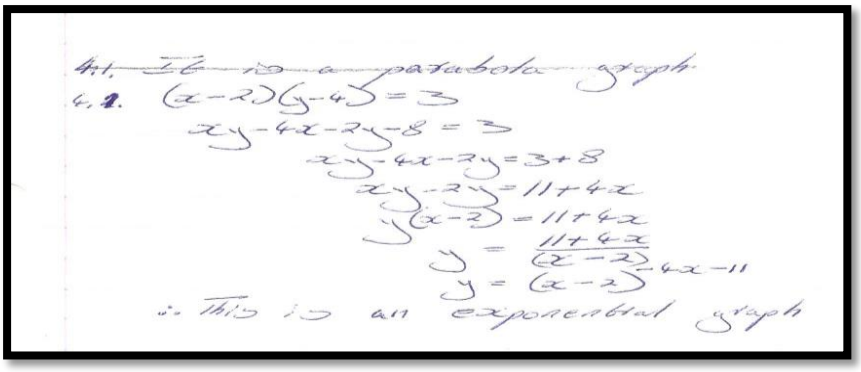


Figure 6.2 Sean's working for name the graph represented by $(x-2)(y-4)=3$

In his explanation Sean does not mention the algebraic manipulation he did, but goes straight to talk about exponents. He recognises the equation as a hyperbola when it is written in standard form only. He is operating only at ritualised and applicability routine. In the equation $y=\frac{2+2x}{x}$ he thinks that the equation is not for a hyperbola until he is given instructions that simplify the equation to the hyperbola form. There is no evidence of flexibility in his discourse, because if questions are given in a way different from what he has done in class he seems to be lost. He also fails to do self-correction (Corrigibility). he looked comfortable with familiar questions, but had difficulty with those that are expressed in a different form. This is what he said:

Who	**What is said**
Given an equation: (x-2)(y-4)=3 Name the graph represented by the above equation	
Sean	I said it's an exponential graph. I worked out the equation then made y the subject of the formula. I think thus where she got … a and said it's a hyperbola. I remembered the exponential rules… of which it can say if something is a numerator an exponent with a negative sign. I concluded that it's an exponential graph
Interviewer	What if I say that you divide both sides by x-2. What will you get?
Sean and Dolly	Oh it's a hyperbola

Sean and Dolly explain that the equation is that of a hyperbola when the researcher asked them to divide both sides by x-2. Their justification is based on visual trigger. An equation expressed in standard form is easily identified as a hyperbola. Using the DPH, I classify their narrativyes as recall based on visuals.

Andrew also said the equation was for an exponential function. Below is what he wrote, and it is difficult to understand. The reasons he gives, for example, include: "By looking and finding the answers from the x-value and minus y to get it and double it will the x-value number".

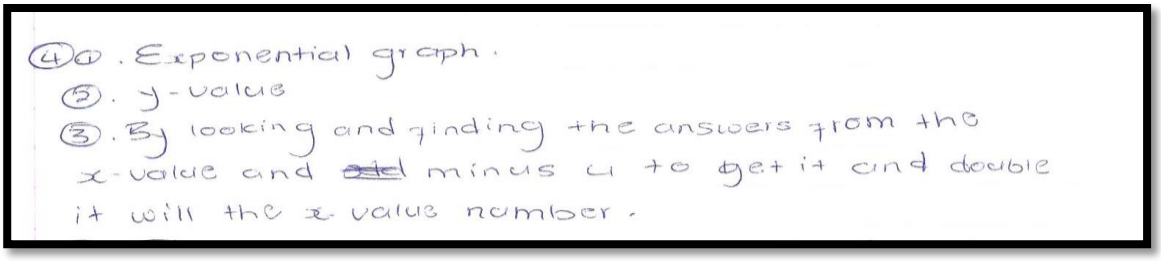


Figure 6.3 Andrew's answer to $x(y-4)=3$

Andrew has not yet reached even the ritualised routine. He is still in the periphery. While he recalls the names of the functions, for example, hyperbola, exponential and some of the key features, he has not yet managed to do what he sees the teacher and other learners doing. While he could draw the horizontal asymptote he failed to draw the graphs correctly. When the learner has not yet reached the ritualised routine it is impossible for him to attain higher routines like corrigibility and flexibility.

The other learner that failed to identify the equation as that of a hyperbola is Pat. Initially she had identified the function as a parabola, probably because of the brackets. She changed her mind when she realised that she could not get the turning points, so opts for a linear equation. The figure below shows her working and reason for saying the equation is for a linear function. She looks at the term on the right as separate single terms by ignoring the presence of the brackets, and says "there is x and y and there is an equal sign"; therefore, it must be an equation for a straight line.

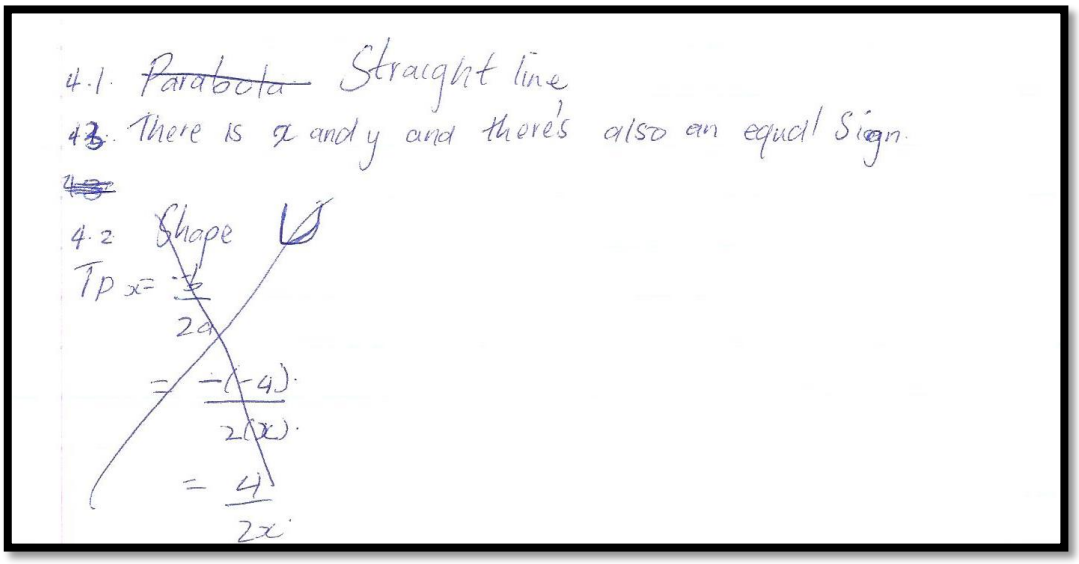


Figure 6.4 Pat's answer to $(x-2)(y-4)=3$

One thing common to Andrew and Sean is that they both associated the equation of the hyperbola with other functions. That gives an impression that they are not sure of the equations of those other

functions either. For example, for Andrew to say the equation is that of an exponential function means he may not distinguish the equation of an exponential function from other equations. In the next section I shall examine the response of learners that named the equation correctly, but did not give mathematically sound reasons.

Considering that learners have successfully done other questions that were given in the standard form and failed to give reasons for the equation that has been expressed in a different form suggests ritualised routine. They do not go beyond examples and tasks from the classroom situation.

Learners in this study failed to give adequate justifications for their mathematical procedures. Most of their justifications appear to be based on visual cues, for example, 'constant is an asymptote', and 'fraction means hyperbola'. These reasons form justifications for their actions.

Only 2 learners showed some flexibility on the equation of the hyperbola. They managed to bring in their knowledge of solving the equations. In the next section I shall discuss learners' response to a linear function whose coefficient is a fraction.

6.3 Equation with a fraction

While in general a hyperbola function is expressed in the form of a fraction, this does not mean this is the only way of expressing it, for example $xy = 2.$, Neither are all equations with fractions for a hyperbola, for example $y = \frac{x}{2} + 4$. In general, learners in my study referred to a fraction as a hyperbola, and would say an equation with a fraction was a hyperbola. In this section I discuss the learners' response to a question on naming the function $= \frac{x}{2}$. I have grouped their responses as, first, those that said the equation is for a hyperbola because of the presence of a fraction; the second that could not recognise that the coefficient of the equation is half; and lastly those that classified the equation as a linear equation

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In the interview the learners were presented with the equation $y = \frac{x}{2}$, and they gave a name other than a linear function for the equation. Four out of the 9 said it was a hyperbola. What is surprising is that 3 of those were in Grade 11. In fact, none of the Grade 11s realised that the equation is for a linear function. This is not the only occasion where learners said an equation with a fraction would represent a hyperbola.

Leo (Grade 10), said more than once that when he sees an equation with a fraction he labels it as a hyperbola. Leo's answers were brief; in most cases as a single word without much elaboration. Below is an excerpt:

Who	What was said
Leo	I think it's a hyperbola,
Interviewer	Why do you say it's a hyperbola?'
Leo	Because there is a fraction.
Interviewer	So whenever there's a fraction it's a hyperbola?
Leo	Yes.

Leo affirms that whenever there is a fraction he thinks the equation is that of a hyperbola. He is not alone in this: Pat, Malusi and Dolly in Grade 11 said the same. Of all the functions that learners do the hyperbola is the only one that is expressed in the form of a fraction. Disambiguation has not been yet been achieved: the learners are not able to distinguish between different mathematical objects that look similar, like $y = \frac{x}{2}$ and $y = \frac{2}{x}$.

Sean (Grade 11) said the equation $y = \frac{x}{2}$ was for an exponential function. In order to try and understand their reasons (Sean and also Dolly) I asked them for the coefficient of the equation. Both said the coefficient of x is 1. This means that they do not only fail to distinguish between different fractions, but also to determine coefficients in a term with a fraction. When I suggested that the coefficient was $\frac{1}{2}$, Sean said that the coefficient could be both 1 and $\frac{1}{2}$, probably because he does not view $\frac{x}{2}$ as a single term. The difficulties that learners have as with learning of algebraic terms.

This is how they responded:

Who	What is said
Interviewer	Because now if we, if we take that, what is the coefficient of x there?
Sean	It is one.
Dolly	Yes it is one.
Sean	Positive one.
Dolly	1 is the coefficient of x.
Interviewer	If I say it is half, am I wrong?
Dolly	I am not sure …[laughing].
Sean	I think you are both correct.
Interviewer	…[laughing]
Sean	I think we are both correct.
Interviewer	Can it, can it be one, can it also be, can it also be half, can it also be 1, what is it, is it 1 or half? Which is which?

Andrew noticed that the equation was not that of a hyperbola. Inasmuch as he is correct, he goes on to label the equation as of that of a parabola. He does not talk of objects, but rather describes the

process. He talks of 2 as being below and explains the division process. Andrew's talk is not yet objectified. To say that the equation is not of a hyperbola does not necessarily mean that he actually recognises the equation. For the equation $y = \frac{x}{2}$ learners needed to recognise that x is a numerator and that the exponent is 1, making it a linear function. This is how he responded to the question:

Who	What is said
Interviewer	Do you think it's a hyperbola or not?
Andrew	It's not
Interviewer	Why
Andrew	Because the 2 is below and the x is above you can't take the 2 and divide by x
Interviewer	So what function do you think it is?
Andrew	It's a parabola

Andrew did not give an explanation for saying that the equation $y = \frac{x}{2}$ is for a parabola. What I can infer is that he realises that for a hyperbola the equation should have a denominator of x if it is given in the form of $y = \frac{a}{b}$. His mathematical discourse seems to be at processural stage (Caspi & Sfard, 2012), where he struggles to explain that *x* should be a denominator. I did not probe him further on why he decided it was a parabola. His word use is mostly colloquial. His essential difficulty is that he is not competent in the realization tree of the functions. He struggles to describe an equation. The DPH classifies his narratives are recall, based on what he sees.

Palesa and Connie, on the other hand, recognised that the equation was not of a hyperbola. Connie goes further, to naming the function as linear, though with some doubt. She is not confident of her answer as she says "I think maybe it's a straight line or something". What is important is that she is able to talk about objects instead of describing the processes. She uses words like 'numerator' and 'straight line' that show that objectification has been achieved. She gives substantiations that are endorsed by the community of mathematicians.

Who	What is said
Interviewer	The first question. Is $y = \frac{x}{2}$ an equation for the hyperbola?
Connie	No,
Interviewer	Why not?
Connie	Because *x* is in the numerator
Interviewer	*x* is the numerator, so what, what function would that be?
Connie	Sir?
Interviewer	What function would that be? If *x* is the numerator.
Connie	I think maybe a straight line or something

Most of the learners (7 out of 9) have routines that are based on what they see. All equations with fractions are regarded as equations for a hyperbola. Their routines are mostly ritualised and based on visuals. Their narratives are visual and often not endorsed by the community of mathematicians.

6.4 Conclusion

Most of the learners in this study struggled with unfamiliar tasks related to the hyperbola. Although these equations could be transformed to familiar standard forms using equivalence, an opportunity for "new endorsed narratives" is lost. Some learners lost an opportunity to establish that a hyperbola does not necessarily have to be expressed in the form of $y = \frac{a}{x-p} + q$ but can be expressed as $xy = a$, or $y = \frac{a+q(x-p)}{x-p}$ and many other equivalents of the hyperbola.

Most of the justifications were based on what could be viewed. The narratives are mostly based on visual recall. Their narratives are most based on visual recall. When some learners see a fraction in an equation, they conclude that the equation is for a hyperbola because of the fraction without a thorough examination of that fraction. They would then say all equations with fractions represent a hyperbola.

Construction of new routines is difficult in situations where there is no flexibility. In flexibility more than one routine response is explored before a conclusion is arrived at. For most of the learners in this study they did not explore new routines even in situations where new routines where required, for example, $(x-a)(y-b) = c$, presented learners an opportunity to try different routines in producing conjectures and determining that the equation was for a hyperbola. I classify their routines as ritualised as they used only applicability routines according to the DPH.

The majority of the learners are operating on ritualised routines, as described in chapter six. They are competent in the mathematical discourse on the hyperbola as far as it is presented in the classroom. In my opinion there is no difference in terms of routines between those that identified the equation as that of a hyperbola without reasons, and those that could not identify the equation at all. I say this because the reasons put forward by learners as justification for their choice where either general for any function for example, when a learner says "It is a hyperbola because there is x-intercept and y-intercept". All 7 learners demonstrated in other questions that they are still on ritualised routines. When they made mistakes there was very little evidence of self-correction.

Mathematics is autopoietic, being built on previously established mathematically-endorsed narratives. In this case the discourse on algebraic terms is not yet fully developed, and it would be difficult for learners to develop the functions discourse. Discourse on algebra is a prerequisite for the functions discourse. Competence in algebra, or the lack of it, is a step towards or hindrance to, explorations in the discourse of the hyperbola function.

The discourse of learners in the hyperbola discourse is mostly visual. The symbolic mediators are interpreted from the visual trigger perspective. For some learners an equation with a fraction is for a hyperbola. An equation not expressed as a fraction is wrongly written or is not for a hyperbola for some learners. The naming of fractions, compared with seeing the parameters "q" and "p" in $y = \frac{a}{x-p} + q$ as asymptotes, and when the parameters are zero and therefore not written, the function has no asymptotes. This makes most of their narratives to be recall based on visual justifications.

The routines are ritualised; they would do well on the work normally done in the classroom, but struggle with questions that are not necessarily found in their textbooks. Most of the learners struggled with unfamiliar tasks on the hyperbola. One of the reasons for this is, in my opinion, that their routines are ritualised, and they do not check the correctness of their answers using multiple methods.

In the next chapter I conclude the study by presenting the findings, limitations to the study and recommendations and implications. I also discuss how the study impacts learning of functions, with the hyperbola in mind.

Chapter 7: Conclusion

7.1 Introduction

My study was conducted in the functions field of mathematics, with particular attention on the hyperbola. I chose this study because of the struggles with functions learners in my classes experienced during my teaching days, and because functions is a key topic in mathematics both at school level and beyond. My interest was enhanced by the work that has been done at Wits Maths Connect Secondary Project on the hyperbola. The purpose of this study was to investigate the participation in mathematical discourse that learners make in solving problems on the hyperbola, their choice of information for the table of values, and drawing graphs of the hyperbola.

I used the theory of commognition enunciated by Sfard (2007) to investigate learners' reasons and justifications of their actions in solving problems on the hyperbola. Thinking in the theory of commognition is communicating with oneself, as has been explained fully in chapter 2. The four key characteristics of the mathematical discourse were used to analyse the communication of the Grade 10 and 11 learners on the hyperbola. The four characteristics of commognition are word use, the routines, visual mediators and endorsed narratives. I then adapted the Arithmetic Discourse Profile by Ben-Yahuda et al (2005) to an analytical tool I named the Discourse Profile of the Hyperbola. I used this framework for all three of my research questions given below. The categories on the profile were influenced by the data I collected and what I made available to learners through questions presented during the interviews.

The three questions that guided the study are as follows:
What is the nature of learners' mathematical discourse on the asymptote of the hyperbola in Grade 11 and 10?
What difficulties do learners face in working with different representations of the hyperbola?
How do learners participate in unfamiliar tasks on the hyperbola (or something like this)?

Data was collected by means of a task-based interview with nine learners from a secondary school in Johannesburg; five in Grade10 and four in Grade 11. The interviews were conducted, in pairs except for one group of three learners, over a period of four days. The interviews were held in the afternoon for two hours, and were video-recorded and transcribed. Data was examined and analysed in order to answer the three research questions for this study using the analytical framework (DPH) described in chapter 5.

7.2 Findings

Functions are a key topic for learners doing mathematics. As mathematics is an auto-poietic subject, what learners endorse as mathematics affects the way they reason. What they communicate reveals their thinking on mathematics. Their justifications (or lack thereof) reflect the mathematics they have learned. So the narratives and routines of learners affect their present and future understanding of mathematics. I shall therefore detail how the above questions have been answered.

What is the nature of learners' mathematical discourse on the asymptote of the hyperbola in Grades 10 and 11?

I found that the mathematical discourse of the participants generally comprised ritualised and applicability routines. They would sketch a graph that shows two asymptotes, but only talk of one asymptote, the horizontal. I shall discuss each of the two findings in the paragraphs that follow.

All participants in the community of mathematics need to pass through this stage of ritualised routines. However, there is a need for exploratory routines, as the aim of learning mathematics is for learners to solve complex and new problems they may face. Reflective imitation helps learners reach exploratory routines. In exploratory routines it is expected that learners use the applicability, flexibility and corrigibility routines. Applicability routines are about the likelihood of certain procedures to be produced. In flexibility routines more than one routine is used to do a task. Self-evaluation leads to self-correction and this use of a routine is known as corrigibility. Flexibility routines were noticed only rarely in this study: Connie exhibited them when she linked the parent equation ($y = \frac{3}{x}$) to the translated function ($y = \frac{3}{x} - 4$). There was a general lack of use of multiple routines. These enable learners to check their answers and derive new narratives. Although there were opportunities for corrigibility, there was no evidence that learners ever did a self-evaluation of their work using different routines so as to self-correct. As a result, their mathematical discourse was generally full of ritualised routines, but with the quality of their explanations to support their answers less mathematically acceptable.

Operating at ritualised routines is the reason they could not tackle unfamiliar tasks, or link the different representations of the hyperbola. For most of the learners their explanation was based on what they *saw*, meaning that their justifications were mostly visually based, for example, "an equation with a parameter "p" and/or "q" has asymptotes but when "p" and/or "q" are zero then there is no asymptote". For $y = \frac{a}{x-p} + q$ there are asymptotes and $y = \frac{a}{x}$ there if no asymptote.

There is a connection between ritualised routines and how some learners used their words. Words used were mostly colloquial, with statements like “the asymptote is undefined”, or “a hyperbola is a graph that does not cross the y-axis”. The asymptotes were often named as if they were numbers. Instead of mentioning equations, numbers were given, for example, learners would say “the asymptote is negative four” (-4). The parameters “p” and “q” were often talked about as if they were asymptotes in their own right, not signifying how the input or output values had moved from the parent function. Ritualised routines, plus failure to exercise reflective imitation are the main reason for words that are classified as colloquial. The classification is given according to the DPH where colloquial is a mixture of mathematical and non-mathematical word use. Ritualised routines are when learners do what they see the interlocutor do without understanding. When learners are asked to make explanations they mix mathematical and non-mathematical words because they are explaining something they do not actually understand. Although they cannot explain what and why they do it, they are able to write it correctly or partially correct.

Narratives were based on what they see, for example, the asymptote was defined as a line that the graph must not cut. Most of their narratives be it substantiations or recall were mostly visually justified and often not endorsed by the community of mathematicians. For example, learners would say “there is no asymptote” for y=3/x because they do not “see” the parameters “p” and “q” in the equation.

The visual mediators were also used in a “ritualised” manner, for example, there would a mismatch between what was given in the diagram and what was said about it. There was a general inability to link the relationship between the parent equation and the translated one. For example, learners’ talk would shift to discuss the presence or absence of asymptotes in these two equations. There was also a lack of recognition of the y-axis as a vertical asymptote by most of the Grade 10 learners, yet their graphs show the y-axis as one. The asymptote was mostly associated with parameters only, but not related to the graph. Learners would say the graph of $y = \frac{x}{3}$ has no asymptotes and name the asymptotes for $y = \frac{3}{x-2} + 1$. Learners easily identify the asymptote from the translated equation than the parent function.

While the dominance of ritualised routines, colloquial talk, visualised narratives was the pattern of seven of the learners, there were two of the learners moved between the ritualised routines and the exploratory, using literate language in their explanations and justifications. They recognised an asymptote in symbolic form as being represented by an equation, and they could link the

transformed equation to the parent one. They could make use of applicability and flexibility routines, e.g., their narratives included both substantiations and recall, and were mostly endorsed by the community of mathematicians.

What difficulties do learners face in working with different representations of the hyperbola?

Learners were comfortable working with an iconic representation of the hyperbola, though, as stated above, had difficulty explaining the relationship between parent and translated function. The difficulty was reflected in a tendency to focus on the parameters "p" and "q" in the equation $y = \frac{a}{x-p} + q$ as entities disconnected from the whole equation. For learners the presence of these parameters represents the presence of asymptotes, while a parent function is regarded as having no asymptotes. This gives credence to the view that the parameters are viewed as asymptotes. This partly explains why asymptotes would be named as if they are numbers. Asymptotes were given as numbers; for example, a learner would say "the asymptotes are negative four (-4) and two (2)".

In this paragraph, I talk about how learners work with tables for the hyperbola. The x-values for the table were similar to those of a linear or quadratic function. Values chosen do not show the asymptotic behaviour of the hyperbola. The asymptotes is then drawn from their previous knowledge of how the hyperbola should look like rather than ordered pairs coming from the graph. These values did not show or illustrate the asymptote although the resulting graph would did so, as shown in figure 7.1

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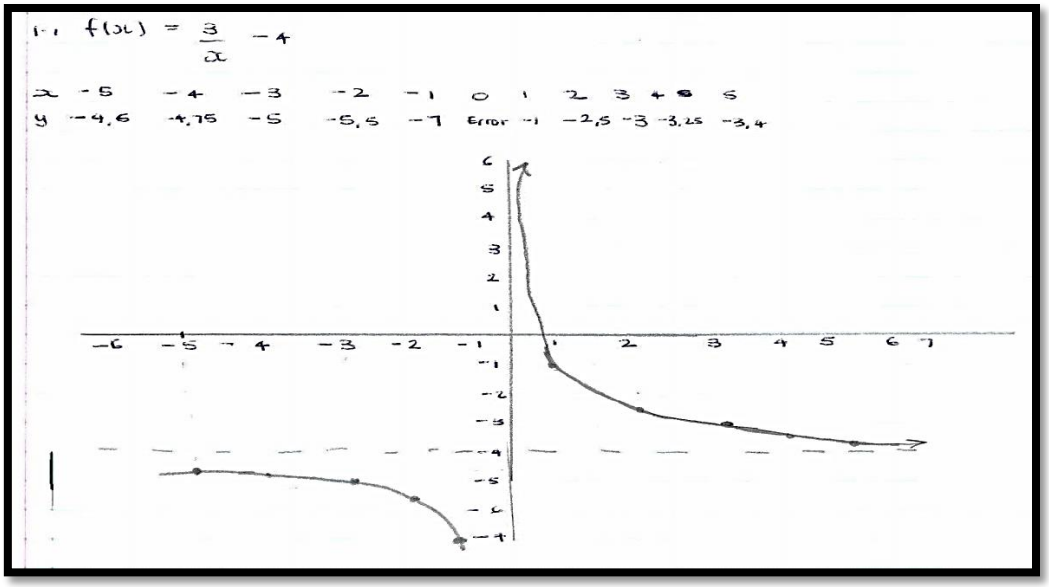


Figure 7.1 an illustration of a table of values and the graph

This begs the question of how learners reach the generalisations on the representations of a hyperbola. The table is just used to provide ordered pairs so that that particular hyperbola can be

written for plotting purposes - and beyond that there was no more use for it as a representation. It was difficult to distinguish the table as an iconic representation for a hyperbola from the tables of other functions. Iconic mediators are pictures that distinguish or communicate mathematics to those that view them. In the tables that were presented one could not immediately see that they represented a hyperbola, because values chosen did not tend to show that they approach the asymptote.

How do learners participate in unfamiliar tasks relating to the hyperbola?

This question sought to test flexibility and corrigibility routines by requiring learners to use multiple routines, and self-correct if necessary. I chose to base my analyses on two questions: $(x-2)(y-4)=3$ for Grade 11 and $x(y-4)=3$ for Grade 10. The Grade 10s do a vertical shift only and the Grade 11s do both a vertical and horizontal shifts and the two equations accommodate the difference. Six out of nine learners identified the equation as that of a hyperbola, while only two of them gave mathematical reasons for their assertion. They had to use equivalence and transform the equation to the standard form. Flexibility was seen only in these two learners. Four of the learners gave answers that had no mathematical base. Some of the answers included that they saw the word 'asymptote' in follow-up questions, and concluded that it must therefore be an equation of a hyperbola. Some even reasoned that they saw an equal sign. Seven of the learners had difficulty working with these equations specifically, but they could work comfortably with equations given in a more familiar form. Learners could only work with familiar tasks, signifying ritualised routines. There was no evidence of corrigibility, as there was no attempt to use multiple routines for self-correction or verification of answers. Learners are to be encouraged to justify their mathematical decisions, as these should be the basis for their actions.

The second interview task that helped to answer the question is a linear equation with a fraction coefficient ($y=\frac{x}{2}$). Two learners classified the equation as a non-hyperbola function while the rest classified it as a hyperbola. The common reason for classifying the equation as a hyperbola was the presence of a fraction in the equation. The justification is visual. Unfamiliar tasks were not explained with the competence that was apparent in familiar tasks.

7.3 Recommendations and Implications of this study for teaching

Several areas that affect teaching and learning of mathematics can be gleaned from this study. The curriculum emphasises that teaching should not focus solely on "how" mathematics is done, but should also address the "why" and "when" questions (DBE, 2011). The study shows that the 'how'

is done well, but that the reason for doing certain procedures and when they should be done seemed to be lacking. Therefore, for teachers it is important to challenge learners with tasks that address reasons and justification for actions. Objectification can only be achieved through exploratory routines, where flexibility and corrigibility are dominant routines, in which learners are able to use multiple tasks and self-correct. This can be achieved through, among other things, exposing learners to unfamiliar tasks.

Advice for teachers is that the language used in a teaching and learning situation is often taken 'as is' by learners. It should be stated and emphasised that the asymptote in a rectangular hyperbola is a straight line, and it should always be presented as an equation, whether verbally or in writing. The parameters "p" and "q" represent vertical and horizontal shifts on the function, and should not to be referred to as asymptotes.

Teachers should pay particular attention to the connections between the four representations of the functions. CAPS recommends that learners should move flexibly between four representations: the table, formulae (equation), graph and words. The table for a hyperbola should be different from that of other functions. As an iconic mediator, the table of values should show values getting closer to the asymptotes.

Emphasis should be placed on generalisations and justifications. All mathematical actions are done for a reason, and that mathematical reason is the key to learning mathematics. Generalisations should be arrived at through discovery by learners. The relationship between the parent and translated hyperbolas should be the starting point of reference and the translated hyperbolas should be viewed in the light of the parent hyperbola. The translation of the parent hyperbola results in a hyperbola in the form of $y = \frac{a}{x-p} + q$.

7.4 Recommendations for research

This study was focused on learners in the top quartile (best) in a township school whose first language is not English. It could be repeated in a former model C school or a well-resourced government or private school, with the best-performing learner or with students of mixed ability whose first language is English.

Subsequent research may add word representation of a hyperbola, as I have limited my study to the hyperbola represented as formula, graph and table. Word representation

Future research studies could focus on the use of technology in the introduction of the concept of the asymptote of the hyperbola in Grades 10 and 11. Technology can be used to support the learners' mathematical discourse on the hyperbola in relation to transformations, limit or the behaviour of the graph towards the asymptote.

7.5 Limitations to the study

The findings of this study are limited to the nine participants of this study from a particular township school in Johannesburg, and as such cannot be generalised. The learners in the top quartile of term one mathematics results were selected. The selection of top students was an advantage as they were able to talk and give as much information. Another set of learners would probably give a different outcome. The trends and patterns of this study are only relevant to this group of participants, and could not be applied to any other group without further research.

All learners were interviewed in English, the language of teaching and learning, and in which mathematics textbooks are written, but not the first language of communication for learners. This may have impacted their communication, and so the analytical framework was also affected, as some of the issues could not have been expressed to the best of their ability. The instruments that are used for research should be carefully thought. the tabling question failed to test what I had intended to gain from the research because of the integral values of x that I chose. Learners could not tell the kind of graph they represented without plotting it. Some of the follow-up questions were giving away answers of the previous questions. An example is the question where learners claim to have realised that it was one of a hyperbola because of word "asymptote" used in a later question.

7.6 Reflections on the study

It has not been easy to conduct task-based interviews as I did not have enough time to look at what learners had written before the interviews and some of the questions I could have asked I realised when I was looking at the data. Another challenge was recording and conducting the interviews simultaneously, I discovered some of the issues I did not follow them enough. For example, when a learner said Leo said for a graph to be identified as a hyperbola there should be a fraction and the same learner later on identified an equation expressed in the form of $(x-a)(y-b)=c$ as representing a hyperbola. I did not probe further what he meant by a hyperbola is seen by a fraction.

This study has helped me to recognise the following:

My research helped me to redefine my understanding of the hyperbola as a function, and the definition of the asymptote.

The instruments used for research should be carefully thought through. The tabling question failed to test what I had intended to gain from the research because of the values of x that I chose. Learners could not tell the kind of graph they represented without plotting it. Some of the follow-up questions were giving away the answers to the previous questions. An example is the question where learners claimed to have realised that the function was one of a hyperbola because of the word 'asymptote' used in a later question.

Piloting helped me refine questions for the task-based interview and the way I conducted the interview. I had to change the order of questions and add further questions because I noticed that some of what I had intended to investigate was not fully covered by the tasks I had prepared. I had to add an unfamiliar task. Furthermore, during the pilot study, one learner was dominant in everything. He answered all the task questions and did the explanation during the interview, leaving the other learner uninvolved. I realised that in group work each participant needs a specific role for all to benefit from the exercise. I then asked them to take turns in answering questions. Where one learner seemed to dominate I was careful to ask the other learner to respond to the next task. In this way I managed to get all their opinion on all tasks.

It was not easy to conduct task-based interviews, as I did not have enough time to look at what learners had written before the interviews. Other questions I could have asked I realised only when I was looking at the data. Another challenge was recording and conducting the interviews simultaneously. I discovered a number of issues which I would have done well to pursue further. For example, when the learner Leo said that for a graph to be identified as a hyperbola there should be a fraction, and then later on identified an equation expressed in the form of $(x-a)(y-b)=c$ as representing a hyperbola, I did not probe further what he had meant by a hyperbola being identified by a fraction.

It would have helped had I observed one or two lessons on the hyperbola, in the context of the learners with their teacher, to give an understanding of what to expect during the interview. The last time I taught learners on the hyperbola was three years ago, and observing even one such lesson, not necessarily the one with learners I interviewed, would have helped me focus on the actual issues

that affect learners. For example, I took it for granted that learners in Grade 10 would notice the y-axis as an asymptote.

It was to my advantage that I did not know the participants of the study personally, as that might have placed undue pressure on the participants. As the participants were unknown to me I did not expect them to behave in a certain way, nor interpret their response according to how I knew them.

7.7 Conclusion

As I conclude this study there are two points that I would like to emphasise based on what I have learned from this study. Firstly, ritualised routines and reasoning from visuals shaped the narratives and the word use of the learners in this study. There were a few explorations from the study. Reflective imitation moves learners from ritualised routines to explorations. Reflective imitation does not just happen, but is a result of exposure to different kinds of questions in terms of cognitive level and level of difficulty. If learners remain at ritualised routines, they do not develop the ability to tackle challenging questions, and the mathematics results will remain low.

Secondly, emphasis should be put on generalisations based on learners discovering mathematical facts, rather than merely following what the teacher does. Reasons for mathematical actions and can be justified are necessary for learners to move into explorations. Learning mathematics should not only focus on ability to answer basic questions, but should also focus on answering the questions ‘why’ and ‘when’. A variety of unfamiliar tasks may help learners to participate fully in the hyperbola discourse. Learning mathematics should focus on justification of mathematics steps, and connections between what is being learned and its realization tree.

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Appendix A

Task 1 Grade 10

Attempt all questions

Write all your answers on the answer sheet provided

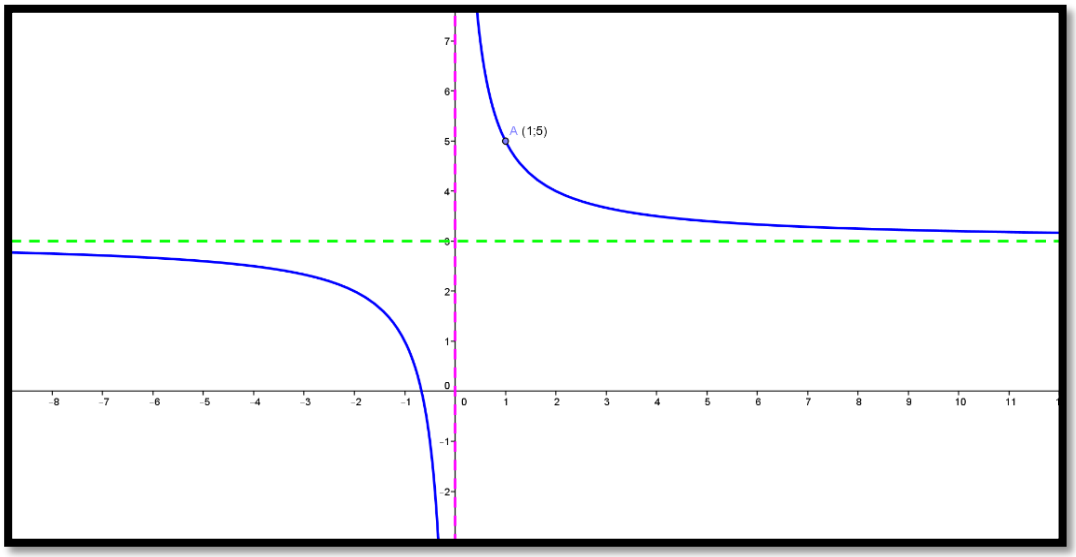
1 Given the function: $f(x) = \frac{3}{x} - 4$

1.1 Sketch the graph of *f(x)*

1.2 Explain the transformation from g(x) = $\frac{3}{x}$ to f(x)

1.3 Explain why the there is no y-intercept in *f*

2. In the diagram below, $h(x) = \frac{a}{x} + q$. The graph passes through the point A(1;5).



2.1 Find the equations of the function *h*

2.2 What do you notice on the graph as x (i) gets bigger (ii) moves towards zero from both sides (iii) gets smaller. Explain the possible reason(s) for the behaviour graph for each of your answers

3.Given a table below, answer the questions that follow:

X	-3	-2	-1	0	1	2	3	4	6	12
Y	-3	-4	-7	Undefined	5	2	1	$\frac{1}{2}$	0	$\frac{-1}{2}$

a. Is this a linear graph, parabola, exponential or hyperbola? (Name before plotting)

b. Draw this graph

2. Given an equation: x(y-4) = 3
   a. Name the graph represented by the above equation
   b. Identify any special features of the graph from the given equation
   c. Explain how you identified the special features in 1.2 above
   d. If the graph is moved three units to the left, will the equations of the asymptotes change? Write them down. Will this movement change the position of the intercepts? Why or why not

Appendix B

Task 2 Grade 11

Attempt all questions

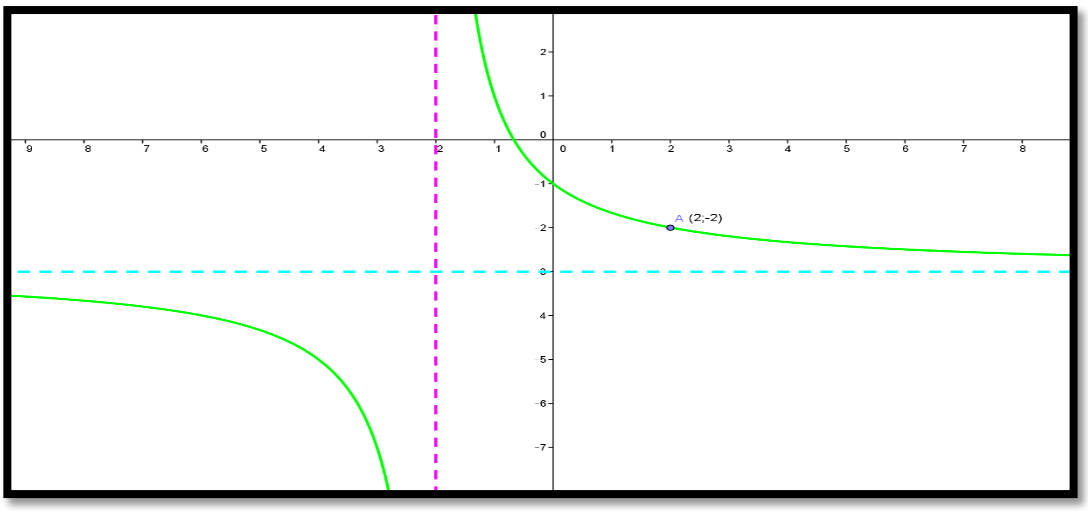
Write all your answers on the answer sheet provided

1. Given the function: $f(x) = \frac{3}{x-2} + 1$

1.1 Sketch the graph of $f(x)$

1.1 *Explain the transformation from g(x)* $= \frac{3}{x}$ to f(x)

2. In the diagram below, $h(x) = \frac{a}{x-p} + q$. The graph of g passes through the point A(2;-2).



2.1 Find the equation of the function h

2.2 What do you notice on the graph as x (i) gets bigger (ii) moves towards -2 from both sides (iii) gets smaller. Explain the possible reason(s) for the behaviour graph for each of your answers

3. Given a table below, answer the questions that follow:

X	-4	-2	-1	0	1	2	3	4	5	8
Y	-2	$\frac{-5}{2}$	-3	-4	-7	undefined	5	2	1	0

3.1 Is this a linear graph; parabola; exponential or hyperbola? (Name before plotting)

3.2 Draw this graph

4. *Given an equation: (x-2)(y-4)=3*

4.1 Name the graph represented by the above equation

4.2 Identify any special features of the graph from the given equation

4.3 Explain how *you identified the special features in 4.2 above*

4.4 If the graph is moved three units to the left, will the equations of the asymptotes change? Write them down. Will this movement change the position of the intercepts? Why or why not

Appendix C

Interview Questions Grade 10

1.1 Briefly explain the steps you took to sketch the graph of: $f(x) = \frac{3}{x} - 4$

1.2 What challenges did you face in doing this question?

1.3 How do you distinguish a hyperbola from other functions you have done?

2.1 Tell me how you determined the equation of this graph

2.2 Show me on the graph the x-intercept

2.3 How did you calculate the coordinates of the x-intercept?

3.1 How did you identify the type of graph represented by the table in question 3?

3.2 Which co-ordinate represents the x-intercept?

Now try this one

4. State which of following functions are hyperbola. Give reasons why it is a hyperbola and do the same for those that are not hyperbolas

$(i) y = \frac{x}{2} \quad (ii)\ y = \frac{2}{x} \quad (iii)\ y = \frac{2}{x^2} \quad (iv)\ y = \frac{2}{x+2} \quad (v)\ y = \frac{2}{x} + 2 \quad (vi) y = \frac{x}{2} + 2$

$(vii)\ y = \frac{2 + 2x}{x}$

Appendix D

Interview questions Grade 11

1.1 Briefly explain the steps you took to sketch the graph of: $f(x) = \frac{3}{x-2} + 1$

1.2 What challenges did you face in doing this question?

1.3 How do you distinguish a hyperbola from other functions you have done?

2.1 Tell me how you determined the equation of this graph?

2.2 Show me on the graph the x-intercept and the y-intercept

2.3 How would you calculate the co-ordinates of the intercepts?

3.1 How did you identify the type of graph represented by the table in question 3?

3.2 From the table, identify the intercepts?

Now try this one

4. State which of following functions are hyperbola. Give reasons why it is a hyperbola and do the same for those that are not hyperbolas

$(i) y = \frac{x}{2}$ $(ii)\ y = \frac{2}{x}$ $(iii)\ y = \frac{2}{x^2}$ $(iv)\ y = \frac{2}{x+2}$ $(v)\ y = \frac{2}{x} + 2$ $(vi) y = \frac{x}{2} + 2$

$(vii)\ y = \frac{2 + 2x}{x}$

Appendix E

Ethics Clearance

Wits School of Education27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

11 May 2015

Student number: 440617

Protocol Number: 2015ECE011M

Dear Sihlobosenkosi Mpofu

Application for ethics clearance: Master of Science

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Learner participation in Functions Discourse

The committee recently met and I am pleased to inform you that clearance was granted.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

MMaseko

Wits School of Education

011 717-3416

cc Supervisor: Dr Craig Pournara

Appendix F

GDEClearance

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

For administrative use:
Reference no: D2016 / 006

GDE RESEARCH APPROVAL LETTER

Date:	**2 April 2015**
Validity of Research Approval:	**2 April 2015 to 2 October 2015**
Name of Researcher:	**Mpofu S.**
Address of Researcher:	**23 Ebony Street; Primrose; 1401**
Telephone / Fax Number/s:	**011 822 9274; 079 236 4875**
Email address:	**sihlobosekosi@gmail.com**
Research Topic:	**Learners' participation in the functions discourse**
Number and type of schools:	**ONE Secondary School**
District/s/HO	**Johannesburg East**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved. A separate copy of this letter must be presented to the Principal, SGB and the relevant District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted. However participation is VOLUNTARY.

The following conditions apply to GDE research. The researcher has agreed to and may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

CONDITIONS FOR CONDUCTING RESEARCH IN GDE

1. The District/Head Office Senior Manager/s concerned must be presented with a copy of this letter;
2. A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB);

Office of the Director: Knowledge Management and Research
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

Appendix G

Letters Participants

Learner Consent Form

Please fill in the reply slip below if you agree to participate in my study called: **Learners' participation in the functions discourse**

My name is: _____

Permission for my written work to be used for the study	**Circle one**
I agree that answers to a task during an interview can be used in this study and in the WMCS project study only.	YES/NO
Permission for task	
I agree to write a task during an interview for this study.	YES/NO
Permission to be interviewed	
I would like to be interviewed for this study.	*YES/NO*
I know that I can stop the interview at any time and don't have to answer all the questions asked.	YES/NO
Permission to be audiotaped	
I agree to be audiotaped during the interview	YES/NO
I know that the audiotapes will be used in this study and in the WMCS project study only.	YES/NO
Permission to be photographed	
I know that I can stop this permission at any time.	YES/NO
I know that the photos will be used in this study and in the WMCS project only.	YES/NO
Permission to be videotaped	
I agree to be videotaped during the interview.	YES/NO
I know that the videotapes will be used in this study and in the WMCS project only.	YES/NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed 3 years after the completion of the WMCS project.

Sign_____ Date_____

Appendix H

INFORMATION SHEET PARENTS DATE: 15 April 2015

Dear Parent

My name is Sihlobosenkosi Mpofu and I am a Masters student in the School of Education at the University of the Witwatersrand.

I am doing research on learning of functions. My study forms part of the Wits Maths Connect Secondary Project (WMCS) which has been running since 2010.

My investigation involves giving a task to learners in pairs and then asking them to discuss the answers. I will also ask them questions based on what they say.

The reason why I have chosen your child is because he/she is keen student of mathematics.

I was wondering whether you would mind allowing your child to take part in the research by writing the task and taking part in the discussion that would follow. I shall video record all the proceedings.

I shall also audio record the proceedings as a way of backing up. I shall also take photos of the work they do. None of the faces of learners shall appear in these videos or photos. The task shall take about 30 minutes and the discussion thereafter about 40 minutes.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

I am inviting you to be part of this research by allowing your child to participate.

Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study. Apart from me, the only other person who will know what your child said in the interview, will be his/her interview partner.

All research data will be destroyed 3 years after the completion of the WMCS project.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

SIGNATURE

NAME: Sihlobosenkosi Mpofu

ADDRESS: 23 Ebony Street, Primrose

EMAIL: sihlobosenkosi@gmail.com

TELEPHONE NUMBERS: 0792364875

Appendix I

INFORMATION SHEET LEARNER

DATE: 15 April 2015

Dear Learner

My name is Sihlobosenkosi Mpofu and I am a Masters Student in the School of Education at the University of the Witwatersrand.

I am doing research on learning of functions. My study forms part of the Wits Maths Connect Secondary Project (WMCS) which has been running since 2010.

My investigation involves giving a task on functions to eight learners working in pairs. The task will take about 30 minutes. All material needed for the project shall be provided.

I was wondering whether you would mind if I asked you to be part of this research.

I need your help with writing a task and participating in an interview. During the interview you will work with a partner and discuss your answers. I will ask you a few questions. The discussion shall take about 40 minutes. I will do a video recording but none of your faces shall be video recorded as my interest would be what you say and do on paper. I shall have an audio recorder as a backup in case something goes wrong to the video recorder. Photos will be taken of the working on functions you shall do. None of your faces shall appear on these photos.

Remember, this is not a test, it is not for marks and it is voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way.

I am inviting you to take part in this research.

I will not be using your own name but I will make one up so no one can identify you. All information about you will be kept confidential in all my writing about the study. The only person who will know what you have said, is your partner in the interview Also, all collected information will be stored safely and destroyed after 3 years of the completion of the WMCS project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you

SIGNATURE

NAME: Sihlobosenkosi Mpofu

ADDRESS: 23 Ebony Street, Primrose

EMAIL: sihlobosenkosi@gmail.com

TELEPHONE NUMBERS: 0792364875

Appendix J

LETTER TO THE PRINCIPAL, SGB Chair, etc. DATE: 15 April 2015

Dear Mr. ……

My name is Sihlobosenkosi Mpofu I am a Masters student in the School of Education at the University of the Witwatersrand.

I am doing research on learning of functions. My study forms part of the Wits Maths Connect Secondary Project (WMCS) which has been running since 2010.

My research involves giving a task to learners in grade 10 and 11. This task shall be written in pairs for about 30 minutes. After the task has been written, learners, in pairs, will discuss their solutions writing their supposed solution on a flip chart. I will ask a few questions. This interview is expected to take about 40 minutes. I would prefer a room where there would not be any disturbance. This is in terms of noise and people coming in because in my interview there shall be a video recording and I would like the learners to feel comfortable. I shall use a video recording because there is a need to capture learners' voices and what they do, for example, what they write. The video camera shall be placed in a manner that it takes what learners write and the movement of their hands. No faces shall be captured. I shall also have an audio recording as a back-up to the video. The audio recorder shall be placed at the corner of a desk where learners would be working from. Only the work on functions shall be captured. Photos may be taken of work that learners shall do. No faces of learners shall be photographed. Photos are necessary to provide clearer images than images from a video camera. I am requesting the mathematics HOD to assist me in the selection process of the learners.

The reason why I have chosen your school is because among the schools in the Wits Maths Connect Secondary project your results for matric have been good.

I am inviting your school to participate in this research through the learners selected.

The research participants will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in participating in this study. The participants will not be paid for their taking part in the research.

The names of the research participants and identity of the school will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed 3 years after the completion of the Wits Maths Connect Secondary Project.

Please let me know if you require any further information. I look forward to your response as soon as is convenient.

Yours sincerely,

SIGNATURE

NAME: Sihlobosenkosi Mpofu

ADDRESS: 23 Ebony Street, Primrose

EMAIL: sihlobosenkosi@gmail.com

TELEPHONE NUMBERS: 0792364875

Appendix K

Parent's Consent Form

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called: Learners' participation in the functions discourse

I, _____ the parent of _____

Permission for written work to be used for this study Circle one

I agree that my child's task can be used in this study and in the

WMCS project only. YES/NO

Permission for task

I agree that my child may write a task for this study. YES/NO

Permission to be interviewed

I agree that my child may be interviewed for this study. YES/NO

I know that he/she can stop the interview at any time and doesn't have to

answer all the questions asked. YES/NO

Permission to be audiotaped

I agree that my child may be audiotaped during interview. YES/NO

I know that the audiotapes will be used in this study and in the

WMCS project only YES/NO

Permission to be photographed

I agree that my child's work may be photographed during the study. YES/NO

I know that I can stop this permission at any time. YES/NO

I know that the photos will be used in this study and in the

WMCS project only. YES/NO

Permission to be videotaped

I agree my child may be videotaped during the interview. YES/NO

I know that the videotapes will be used in this study and in the WMCS project only. YES/NO

Informed Consent

I understand that:

- my child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- he/she does not have to answer every question and can withdraw from the study at any time.
- he/she can ask not to be audiotaped, photographed and/or videotaped.
- all the data collected during this study will be destroyed 3 years after completion of the WMCS project.

Sign_____ Date_____

Appendices L:

Transcripts

Transcript of a videoed lesson - HDV-0036
TRANSCRIBER: JULIA MARTINELLI

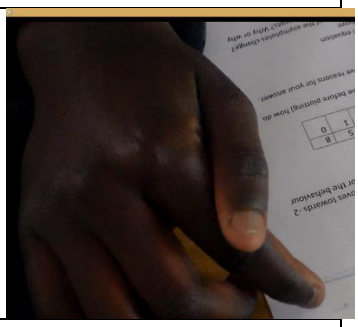
T: teacher; M: Malusi; P: Pat

Time	What is said?	What is done?	Board work
The lesson takes place in a classroom with a Maths teacher and a class full of learners *The learners are sitting at their desks and the teacher walks around the classroom looking at the learners' work. The teacher then resumes the lesson by standing in front of the class at the board.*			
1	Okay, right this is interesting, maybe firstly I would like to see your papers, how you have written your papers. Let me just have your things? It's not like a lot of questions to ask you. …inaudible…Alright. Okay I will start, so you, you also got this ...[inaudible]	Teacher collects learners' work.	
2	Mmmmm	Teacher looks at learners' work.	
3	Okay, alright let us start by asking you to explain how you have done, the steps you took to sketch the graph in question 1.1?	Teacher speaks and hands work back.	
4	Okay I started by finding the shape and then the asymptotes and the I did the y and the x intercept and then that led me to draw, how to draw the graph.	Learner answers.	
5	Okay by saying the shape, what do you mean?	Teacher asks question.	
6	I mean like there is, the equations says y is equal to $y = \frac{a}{x} - p + q$ so the shape I already know, the, the function they gave me it says $fx = \frac{3}{x} - 2 + 1$ so I already know from my head that my a it's positive. When I see the equation, my a it's positive so the shape will then be a smiling face because of my a is positive	Learner answers.	
7	What do you mean by smiling face?	Teacher asks question.	
8	Oh, I am confusing pythagora and hyperbola. Oh I know that is, my a it is positive then, it will start from the first quadrant and the quadrant.	Learner answers	
9	Okay it will be the first quadrant?		
10	Ja.	Learner answers.	
11	Is that what you meant by the shape?		
12	Yes.	Learner answers	
13	Okay and?	Teacher asks question.	
14	I started finding the shape too and the asymptote, then I worked out the y intercept and the x intercept that made me to draw the, sketching the draft to find the x asymtote. I said x is supposed $x -$ just goes to zero then I took two to the other side, then y asymptote was already there and then to know that my graph is on the first and the third quadrant the x should be positive. Then if the, the a was negative my, my ship was going to be at the second and fourth quadrant.	Learner answers.	
15	Okay right, I was glad you said that, I wanted to ask you one, anyway but let's continue, lets continue, let's go on. So what challenges did you face in this question? Any challenges you faced? It was okay? What about you?	Teacher speaks.	
16	Ja question 1 was not challenging.	Learner answers.	
17	At all? Okay what about question 2?	Teacher asks question.	
18	Question.	Learner speaks	
19	Question 2.	Learner speaks	
20	Question 1.2	Teacher speaks	
21	Oh question 1.2 it was a bit trickish because I	Learner answers	

		could not find, my x, my y intercept and my x intercept were both undefined.		
22		Your x intercept?	Teacher asks question.	
23		Your x intercept and y intercept they were both undefined. I could not find a value for y intercept and x intercept.	Learner answers	
24		For question 1.2?	Teacher asks question.	
25		1.2 yes.	Learner answers	
26		1.2.	Learner speaks	
27		What does 1.2 say?	Teacher asks question.	
28		Explain the transformation, show how we can see the transformation on the graph.	Learner speaks	
29		So what do you mean by that? I don't understand what you are saying.	Teacher asks question.	
30		Okay.	Learner speaks	
31		You are saying there is no intercept?	Teacher asks question.	
32		No.	Learner speaks	
33		The y intercept is undefined.	Learner answers	
34		Undefined. And the x intercept is undefined.	Learner speaks	
35		So if it is like that then what is the problem?	Teacher asks question.	
36		My problem became like, my asymptotes $y = 0$ and $x = 0$ and I did not have any y intercept and x intercept, they were both undefined. So I could not draw the graph because I had no value for y and no value for x. Actually I did it is zero, zero but the graph was in origin I think.	Learner answers	
37		Okay what are you saying about to it?	Teacher asks question.	
38		We are not going to be able to draw the graph because y and x are both zeros so to, to draw the graph you will need to find y Intercept and the x intercept. So ...[intervened]	Learner answers	
39		So you cannot, if there is no x and y intercept you cannot draw the graph? Is that what you are saying?	Teacher asks question.	
40		No you can. The thing is our y and x intercept they are, they are both undefined.	Learner answers.	
41		Yes.	Teacher speaks.	
42		And our asymptotes they y is zero and the x is zero so ...[intervened]	Learner speaks.	
43		y is equal to zero and x is equal to zero?	Teacher speaks.	
44		Yes.	Learner speaks	
45		Right so how does that stop you from drawing the graph? That's my question?	Teacher asks question.	
46		How, like drawing the graph it was like putting a dot in the origin and then that was my graph.	Learner speaks	
47		Why?	Teacher asks question.	
48		Because I couldn't, I couldn't , I couldn't, I didn't know where to go because I don't know what is my y and I don't know what's my x.	Learner answers	
49		Okay suppose okay, you, you have told me that the x axis, the x intercept rather is not there. And you have said that the ...[intervened]	Teacher asks question.	
50		The y intercept.	Learner speaks	
51		The y intercept is also not there and you have also told me that the y axis and the x axis, they are asymptotes.	Teacher asks question.	
52		Yes.	Learner speaks	
53		That's correct right. I heard you right. Now okay your equation is, is y is equal to? What it is given as?	Teacher asks question.	
54		...[inaudible] by x.	Learner speaks	
55		...[inaudible] of the x is equal to $\frac{3}{x}$.	Teacher speaks.	
56		Ja.	Learner speaks	

57		Yes.	Learner speaks	
58		Okay. Now let us suppose you get x is equal to, to 1.	Teacher asks question.	
59		x is equals to 1.	Learner speaks	
60		What will be y?	Teacher asks question.	
61		y is zero.	Learner speaks	
62		No on that graph?	Teacher asks question.	
63		Oh here?	Learner speaks.	
64		Yes.	Teacher speaks.	
65		I replace this x with 1.	Learner answers	
66		Yes.	Teacher speaks.	
67		Now y is going to be 3.	Learner speaks	
68		It is going to be 3?	Teacher asks question.	
69		Yes.	Learner speaks	
70		Then that 3 can you, can you not plot that 3?	Teacher asks question.	
71		I can.	Learner speaks. Learner plots graph.	
72		I can.	Learner speaks	
73		Right let's suppose x is ½ what will be y?	Teacher asks question.	
74		x is half.	Learner answers	
75		Yes.	Teacher speaks.	
76		y is 6.	Learner answers	
77		y is 6? Can you not plot that point?	Teacher asks question.	
78		I can.	Learner answers	
79		I can.	Learner answers	
80		Let's suppose that y is a third what is y?	Teacher asks question.	
81		It's 1.	Learner answers	
82		Uhuh x is a third what is y?	Teacher asks question.	
83		x is a third?	Learner repeats question.	
84		Yes. If x is a third what would be y.	Teacher asks question.	
85		You replace this x with ...[inaudible] here?	Learner answers	
86		With a third, a third, 1/3.	Teacher asks question.	
87		Oh. 1/3.	Learner speaks	
88		It is going to be 9.	Learner answers	
89		To be 9?	Teacher asks question.	
90		Yes.	Learner answers	
91		So can't we plot those points?	Teacher asks question.	
92		We can.	Learner answers	
93		We can.	Learner answers	
94		So if we can why can we, why are you saying that we cannot have a graph then?	Teacher asks question.	
95		We can't, we can't plot zero.	Learner answers	
96		Why do you want to plot zero?	Teacher asks question.	
97		Because it's in the origin.	Learner answers	
98		Okay maybe you can tell me, why do you want to put zero? Let me understand that.	Teacher asks question.	
99		Sir?	Learner asks question.	
100		Yes.	Teacher speaks	
101		If, look sir, our y intercept and x intercept are undefined, both of them, then our ...[intervened]	Learner asks question.	
102		I am fine with that one.	Teacher speaks	
103		Then our asymptotes, you can't plot zero because when we draw our graph, the graph will not like, how can I put it.	Learner speaks	

104	T	Okay, can we, can we try it with the graph?	Teacher asks question.	
105	P	Yes.	Learner answers	
106	P	Yes.	Learner answers	
107	P	Let's do it.	Learner answers	
108	T	Can you try and draw the graph? Use pens hey, on the dialogic ...[inaudible] this one.	Teacher asks question.	
109	P	Pens. This one?	Learner ask question.	
110	T	Yes, use a pen not a pencil. So at least you can see this is what you have drawn together with me.	Teacher answers.	
111	P	Okay.	Learner speaks.	
112	T	Right when x is 1 you said y is 3?	Teacher asks question.	
113	P	Yes.	Learner answers	
114	T	Can you put there on the graph your x is 1 y is 3.? Right then when x is a ½ y is 6.	Teacher asks question.	
115	P	x is a ½ .	Learner answers	
116	T	Yes. y is 6. When x is a 1/3 y is 9. A 1/3 , a 1/3 .	Teacher asks question.	
117	P	1/3?	Learner speaks.	
118	T	Ja 1/3 that side close to zero.	Teacher speaks	
119	P	y is 9?	Learner asks question	
120	T	Yes y is 9. 9 will be somewhere on top. Can you see that when x is 1 y is 3 right? When x is 1 y is 3. Can you put that one as well? When x 1.	Teacher asks question.	
121	P	Yes x is 1.	Learner answers	
122	T	Okay when x is 3 y is 1.	Teacher asks question.	
123	P	When x is 3 y is 1.	Learner answers	
124	T	Yes. Then when x is 6 y is ½. Can you draw ...[inaudible]	Teacher asks question.	
125	P	I can.	Learner answers	
126	T	Can you ...[inaudible], I don't understand what you meant by you cannot draw it? Can you, do you see that? And then even on the other side, this ...[inaudible] they can be drawn on what, on the other side because it is a reflection, you just changing the what, the signs of, of your, of your values and you put them on the other side. Right?	Teacher asks question.	
127	P	Maybe we didn't understand the question.	Learner speaks	
128	T	No but now the question was asking about transformation. What has happened to that equation for, to *g* for you to get *f*. That was the question. So what, how, how do you go from *g* to *f*?	Teacher asks question.	
129	P	How do I get to ...[intervened]	Learner speaks	
130	T	Yes.	Teacher speaks	
131	P	Oh.	Learner speaks	
132	T	How did you do that?	Teacher asks question.	
133	P	You move the graph to one step to the left.	Learner answers	
134	T	One step to the left?	Teacher asks question.	
135	P	Ja.	Learner answers	
136	T	Are you sure? One step to the left?	Teacher asks question.	
137	P	To the right sir.	Learner answers	
138	T	Are you sure one step to the right? Are you sure it is one step?	Teacher asks question.	
139	P	There is not one step.	Learner answers	
140	T	How many steps is it?	Teacher asks question.	
141	P	Oh, oh, there, two, two, two.	Learner speaks	
142	T	Two steps to the what? To the left.	Teacher asks question.	
143	P	To the left.	Learner answers	
144	P	Left ja.	Learner answers	

145		And?	Teacher asks question.	
146		One step upwards to ...[intervened]	Learner answers	
147		Up, one step up. Okay I think from, from the definition, from your asymptotes, you should have seen that now my asymptote was x is equal to 2 right?	Teacher asks question.	
148		Ja.	Learner answers	
149		So what was it before? It was x is equal to zero. So we have actually move two units to the what?	Teacher asks question.	
150		To the ...[intervened]	Learner answers	
151		To the right …[laughing]	Teacher asks question. Teacher laughs.	
152		Oh to the right is positive.	Learner answers	
153		Yes, yes, yes. Okay so how do you distinguish a hyperbola from other graphs?	Teacher asks question.	
154		The value of x is 10 okay.	Learner answers	
155		What do you expect the value of y to be?	Teacher asks question.	
156		The value of y will be somewhere at	Learner answers	

NAME OF AUDIO : **HDV_0035**
DATEOF AUDIO :
LENGTH OF AUDIO : **16:33**
TRANSCRIBER NAME : **JULIA MARTINELLI**
TRANSCRIPTION LEGEND : **TEACHER** **T**
: **LEARNERS** **L: Connie & M: Leo**

L Because it never, it never touched the *y* axis.
T It never crossed the y axis? But also if we have a, a look at the graph did you seen it, have you seen it. What about exponential graph? It also has undefined somewhere.
L Mmm,
T Right?
L Yes,
T It also has an asymptote, let me just put it like that. Okay write from the table and identify the *x* intercept, which one is the *x* intercept in the table there?
L [unclear] six
T *x* intercept is?
L Six.
T Six, how did you see it?
L Because *y* is zero.
T *y* is zero and,
M I agree.
T You agree?
M Yes,
T Why do you say so? Why do you agree?
M Because to, to find the *x* intercept you say let *y* equal zero. So that zero because [unclear]
T Okay. Okay let us look at our graph again. Can you see the values of, of y?
L This one?
T Yes before zero on the graph, on the table, sorry on the table.
L No.
T Can you see the values of *y*?
L Yes,
T What's happening to them, are the increasing or decreasing?
L They are increasing…
T They are?
L Increasing and then the last one is decreasing.
T Uh-uh. From *x* is equal negative three up to *x* is equal to zero let's just look at that point.
L Mmm.
T Are they increasing or decreasing?
L Increasing.
T Which number is bigger, negative three and negative four?
M Negative three.
T Negative three.
L Three yes.
T So are the numbers there increasing or decreasing?
L Increasing, no decreasing,
M Mmm
T They are decreasing?
L Yes.
T Okay, so which means that we have a second point and they are what?,
M Mmm
T They are decreasing?
L Yes.
T Okay, so which means that we have a second point and they are what?
L Decreasing,
T They are decreasing right?
M Yes.
T Towards what? The asymptote. Can you see that?
L & M Yes.
T And then if we, if we go to the second part from one up to twelve, can we look at that part and see what's happening there?
M They are decreasing,
L They are decreasing.
T They are also what?
L & M Decreasing.
T They are decreasing from the
L Right,
T Right. So that's one way we could have seen that is a what?
L A hyperbola.
T An hyperbola right. The, the values that decreased towards the what?
L The asymptote,

T	The asymptote, and the other part also starts with decreasing values okay. Okay that way you can see but it starts from the values that are above what? The first value that you had negative three right? Okay let's go to question four. I think we are about to finish here. We are about to finish here. Question four, name the graph represented by the above equation.
L	It's a hyperbola.
T	Why do you say it's a hyperbola?
L	Because when you divide by *x* to the other side it will turn into a, into a fraction it will be three over *x*, so the fraction makes a hyperbola.
T	Okay, if it's a fraction then it's a hyperbola. Okay that's very good, that's very good. And then?
M	It's a hyperbola,
T	Why?
M	Because when I simplifies the whole general equation

why do you say so?

because there is an x there is an x that is, that is squared.