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LIE GROUP APPROACH TO CAUCHY'S PROBLEMS: SOLUTION OF AN INITIAL VALUE PROBLEM FOR THE BLACK-SCHOLES MODEL

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ABSTRACT

A Lie group assisted method is used to solve explicitly an arbitrary initial value problem for the Black-Scholes equation. This equation plays a crucial role in the mathematics of finance. It was first solved by its inventors for a special initial data . Our solution generalises the well-known Black-Scholes formula.

DECLARATION

I declare that the contents of this research report are original except where due references have been made. It has not been submitted before for any degree to any institution.



C. Wafo Soh

DEDICATION

To my Family and Friends.

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Preface

In the Cauchy problem for linear differential operators, the notion of a fundamental solution plays a central role. For equations with constant coefficients, fundamental solutions are usually constructed by using the Fourier transform method and the solution to the Cauchy problem is given in terms of the convolution of a fundamental solution with the initial data. When it comes to equations with variable coefficients, two problems occur: How to construct fundamental solutions explicitly? How to represent the solution of the initial value problem? N. H. Ibragimov showed in 1989 that the first issue can be addressed by Lie group analysis (see Ibragimov (1989)). In this research report, we continue the group approach and show that the invariance principle together with a method due to Hadamard (see Hadamard (1923)) provide a systematic approach to the second problem as well. This approach is applied

to parabolic equations with variable coefficients. As an illustration, we give an integral representation of the solution to an arbitrary Cauchy problem for the Black-Scholes equation widely used in the mathematics of finance. This equation was first solved by its inventors for specific initial data (see Black and Scholes (1973)). The method they used was rather an *ad hoc* method and cannot be extended to the general case.

In more detail, the outline of this work is as follows:

Chapter 1 provides a brief introduction to Lie group analysis of differential equations. The topics covered in this preliminary chapter are biased by the fact that we will used Lie group methods to solve a specific problem: The Cauchy problem for the Black-Scholes equation.

Since generalised functions are widely used in the theory of differential equations, it was necessary to extend Lie group analysis concepts to the space of distributions. This step was done in Ibragimov (1989). The second chapter exposes mainly Ibragimov's idea. Namely, we show in this chapter how symmetry analysis can be used to construct the fundamental solution of Cauchy's problem explicitly.

In the third chapter, it is shown after Hadamard that the solution to the Cauchy problem for linear operators can be represented by means of the fun-

damental solution and t[illegible] data.

The fourth chapter is devoted to the application of concepts developed in the previous chapters. In this last chapter, we solve an arbitrary initial value problem for the Black-Scholes equation explicitly (see Ibragimov and Wafo Soh (1997)). The solution obtained generalises the well-known Black-Scholes formula.

Chapter 1

Differential Equations from Group Standpoint

1.1 Introduction

Systematic investigation of continuous transformation groups was initiated by the Norwegian mathematician Sophus Lie (1842-1899). He was the first to notice that classical devices for the integration of differential equations can be explained from a single standpoint—that of (infinitesimal) transformations leaving the equation unchanged (called symmetries of the equation). What is still more striking is that Lie gave an algorithm to calculate symme-

tries of a given equation and showed how these symmetries can be utilised to investigate or integrate differential equations.

In this preliminary chapter, we aim at discussing the simplest methods of Lie group analysis of differential equations: Continuous one-parameter group of transformations, calculation of symmetries and invariant solutions. Throughout this chapter, we shall assume that functions are sufficiently smooth.

1.2 One-parameter group of transformations

The object of this section is to define and illustrate the concept of a group of transformations whose equations involve a single parameter. Each such group is generated by an *infinitesimal operator*. The latter will prove to be more convenient in applications than the group itself.

1.2.1 Continuous one-parameter groups of transformations

Definition 1.2.1 A transformation of $\mathcal{R}^n$ is a one-to-one mapping from $\mathcal{R}^n$ onto itself.

Example 1.2.1 The identity mapping

$$I : \mathcal{R}^n \longrightarrow \mathcal{R}^n$$

$$x \longmapsto x$$

is a transformation of $\mathcal{R}^n$.

Let $x = (x^1, \ldots, x^n) \in \mathcal{R}^n$, $n \in \mathcal{N}^* = \mathcal{N} - \{0\}$. Consider a one-parameter set of transformations T_a:

$$T_a \equiv \bar{x} = f(x, a), \tag{1.1}$$

or in coordinates

$$\bar{x}^i = f^i(x, a); \quad i = 1, \ldots, n. \tag{1.2}$$

Let U be a neighbourhood of $a = 0$ in $\mathcal{R}$. Given $a \in U$, the transformation T_a carries the point $x \in \mathcal{R}^n$ to the point $\bar{x} \in \mathcal{R}^n$. We suppose that (1.1) is the identity transformation if and only if $a = 0$ i.e.

$$f(x, a) = x \quad \forall x \in \mathcal{R}^n \Leftrightarrow a = 0. \tag{1.3}$$

Definition 1.2.2 A set of transformations (1.1) is called a one-parameter (local) group of transformations in $\mathcal{R}^n$ if the functions $f^i(x, a)$

satisfy the condition (1.3) and the group property

$$f^i(f(x,a),b) = f^i(x,c); \quad i = 1,\ldots,n; \tag{1.4}$$

for a, $b \in U$, where $c \in U$ is a certain (smooth) function of a and b

$$c = \gamma(a,b), \tag{1.5}$$

such that

$$\gamma(a,b) = 0 \tag{1.6}$$

has a unique solution $b \in U$ for any $a \in U$. Given a, the solution b of (1.6) is denoted a^{-1}. The function γ is termed the *group composition law.*

According to this definition, a continuous one-parameter group G of transformations contains the (unique) identity transformation $I = I_0$. Further, the group property (1.4) means that any two transformations T_a, $T_b \in G$ carried out one after another result in a transformation which belongs to G for any a, $b \in U$. The solvability of the equation (1.6) together with the group property (1.4), provide the inverse transformation $T_a^{-1} = T_{a^{-1}} \in G$ to $T_a \in G$: $T_{a^{-1}}T_a = T_aT_{a^{-1}} = I$ for any $a \in U$.

Example 1.2.2 (Group of translations)

$$G = \{T_a, \quad a \in \mathcal{R}\},$$

where T_a is defined by

$$T_a: \quad \bar{x} = f(x, a) = x + a.$$

(i) $f(x, a) = x \ \forall x \in \mathcal{R}$ if and only if $a = 0$.

(ii) $f(f(x, a), b) = f(x + a, b) = x + a + b = f(x, c)$, where

$$= a + b = \gamma(a, b).$$

(iii) $\gamma(a, b) = 0$ if and only if $b = -a$. For this $a^{-1} = -a$ and $T_{a^{-1}} = T_{-a}$.

Note that in this example we can take $U = \mathcal{R}$. But this is an exception: In most of the cases occuring in applications we have only $U \subset \mathcal{R}$. Whence the terminology *local group.*

In the subsequent sections, any reference to group means continuous one-parameter (local) group of transformations. The next discussion shows how to construct continuous one-parameter groups as we please.

1.2.2 Group found by integrating a differential equation

We consider the system of differential equations

$$\frac{d\bar{x}}{dt} = \xi(\bar{x}), \tag{1.7}$$

under the initial condition

$$\bar{x} \mid_{t=0} = x \tag{1.8}$$

and where

$$x = (x^i), \quad \xi = (\xi^i); \quad i = 1, \ldots, n; \quad \xi \neq 0.$$

The system (1. plies the equations

$$\frac{d\bar{x}^1}{\xi^1(\bar{x})} = \cdots = \frac{d\bar{x}^n}{\xi^n(\bar{x})}. \tag{1.9}$$

We suppose that the functions ξ^i are sufficiently smooth and without loss of generality we assume also that $\xi^1 \neq 0$. The equations (1.9) are known to possess $(n-1)$ first integrals $v^j(\bar{x}) = c^j$; $j = 2, \ldots, n$; passing through x and defined for all $\bar{x}$ of a certain neighbourhood of x. Obviously $c^j = v^j(x)$; $j = 2, \ldots, n$. In order to solve (1.7), we solve [1] $v^j(\bar{x}) = c^j$; $j = 2, \ldots, n$ for $\bar{x}^2, \ldots, \bar{x}^n$ in terms of $\bar{x}^1$ and c^j; $j = 2, \ldots, n$. Susbtitute the result in the first equation of (1.7) and integrate. Let the result be $\omega(\bar{x}^1, c^2, \ldots, c^n) - t = c^1$, where c^1 is an arbitrary constant of integration. Replacing c^j by its value $v^j(\bar{x})$; $j = 2, \ldots, n$; we obtain a result of the form $u(\bar{x}) - t = c^1$. In view of

[1] This is possible because $v^j(\bar{x}) = c^j$ implies after differenting with respect to t that $\xi^i(\bar{x}) \frac{\partial v^j}{\partial \bar{x}^i} = 0$. Setting $t = 0$ in the last equation, we obtain $\xi^i(x) \frac{\partial v^j}{\partial x^i} \mid_{t=0} = 0$. Since $\xi \neq 0$, this implies rank$\| \frac{\partial v^j}{\partial \bar{x}^i} \mid_{t=0} \| = n - 1$.

the initial condition, we have

$$u(\bar{x}) = u(x) + t, \quad v^j(\bar{x}) = v^j(x), \quad j = 2, \ldots, n. \tag{1.10}$$

Let the solved form of these equations (1.10) be

$$\bar{x} = \Phi(x, t), \tag{1.11}$$

in which the functions Φ^i; $i = 1, \ldots, n$; are defined for values t which are sufficiently small. We can also solve (1.10) for x. Thus (1.11) defines a transformation of $\mathcal{R}^n$ provided t is sufficiently small.

Let us show that the product of (1.11) by a second transformation

$$\bar{\bar{x}} = \Phi(\bar{x}, t') \tag{1.12}$$

of the same set is the following transformation

$$\bar{\bar{x}} = \Phi(x, t + t'). \tag{1.13}$$

We return from (1.11) to the equivalent equations (1.10) and likewise from (1.12) to the equivalent equations

$$u(\bar{\bar{x}}) = u(\bar{x}) + t', \quad v^j(\bar{\bar{x}}) = v^j(\bar{x}), \quad j = 2, \ldots, n. \tag{1.14}$$

whose solved form is (1.12) for the same reason that (1.11) is the solved form of (1.10). Eliminating $\bar{x}$ between (1.10) and (1.14), we get

$$u(\bar{\bar{x}}) = u(x) + t + t', \quad v^j(\bar{\bar{x}}) = v^j(x) \quad j = 2, \ldots, n; \tag{1.15}$$

whose solved form is (1.13). Hence the product of two transformations (1.11) whose parameters t and t' are sufficiently small is a transformation (1.10) with $t + t'$.

Since (1.15) is the identity if $t + t' = 0$, the inverse of (1.11) is derived from (1.15) by replacing t by $-t$. Finally we summarize the results obtained in the following theorem.

Theorem 1.2.1 When the parameter t is restricted to values sufficiently small, the solution of (1.7) under the initial condition (1.8) forms a continuous one-parameter (local) group of tranformations.

Example 1.2.3 (Group of rotations)

$$n = 2, \quad \xi^1 = -y, \quad \xi^2 = x.$$

In this case, (1.7) becomes

$$\frac{d\bar{x}}{dt} = -\bar{y}, \qquad \frac{d\bar{y}}{dt} = \bar{x}.$$

Differentiate the second equation with respect to t:

$$\frac{d^2\bar{y}}{dt^2} = \frac{d\bar{x}}{dt} = -\bar{y}.$$

Thus

$$\bar{y} = A\cos t + B\sin t$$

and

$$\bar{x} = \frac{d\bar{y}}{dt} = -A\sin t + B\cos t.$$

But $\bar{x}\,|_{t=0} = x$ and $\bar{y}\,|_{t=0} = y$. Hence $A = y$ and $B = x$. Therefore

$$\bar{x} = x\cos t + y\sin t,$$

$$\bar{y} = x\sin t - y\cos t.$$

This transformation describes a rotation of angle t about the origin. The set of such transformations is called the *group of rotations.*

1.2.3 Infinitesimal transformations: Symbol of a continuous one-parameter group of transformations

Consider a continuous one-parameter group G of transformations (1.1). The expansion of the functions f^i into Taylor series in the parameter $a = 0$ (i.e. the value of the parameter which gives the identity), taking into account the initial condition ($f(x, 0) = x$), yields the *infinitesimal transformation* (to the first order of precision)

$$\bar{x} = x + \xi(x)a, \tag{1.16}$$

where

$$\xi^i = \frac{\partial f^i}{\partial a}\Big|_{a=0}; \quad i = 1, \ldots, n. \tag{1.17}$$

The vector $\xi = (\xi^i)$ with components defined by (1.17) is the tangent vector at $x = (x^i)$ to the curve described by $\bar{x} = (\bar{x}^i)$ and therefore called the *tangent vector field* of the group. This tangent vector field is sometimes written as the first order differential operator

$$X = \xi^i(x)\frac{\partial}{\partial x^i}. \tag{1.18}$$

Lie called the operator (1.18), the *symbol* of the infinitesimal transformation. A natural question arises at this point of our discussion: Does there exist a one-to-one correspondence between the transformation (1.1) and the corresponding infinitesimal transformation (1.16)? This question is motivated by what we have seen in the previous section : For a group obtained by the integration of an ordinary differential equation, we can check easily that the knowledge of the infinitesimal transformation yields the transformation and vice versa. Therefore, we can replace our first question by the following one: Does there exist a one-to-one correspondence between continuous one-parameter groups and one-parameter groups obtained by the integration of an ordinary differential equation? Before giving an answer to this question,

we shall first treat the typical case of the group of similarity transformations

$$S_a \quad : \bar{x} = ax. \tag{1.19}$$

Since $S_a S_b = S_{ab}$, the parameter of the product is the product of the parameters of the component transformations, whereas it was their sum in the case of the group of transformations (1.11). Hence we cannot put the transformations S_a into one-to-one correspondence with the transformations of any group of the type (1.11) by taking $a = t$. But by taking $a = e^t$, then

$$\bar{\bar{x}} = e^t x \tag{1.20}$$

and by (1.17), $\xi = x$ i.e. $\xi^i = x^i$. The resulting infinitesimal transformation generates the group (1.20), whose equations give the solution of

$$\frac{d\bar{x}}{dt} = \bar{x}, \quad \bar{x} \mid_{t=0} = 0.$$

Using similar ideas, we next treat any one-parameter group of transformations (1.1).

The product of (1.1) by any transformation

$$\bar{\bar{x}} = f(\bar{x}, b) \tag{1.21}$$

of the group must be a third transformation

$$\bar{\bar{x}} = f(x, c) \tag{1.22}$$

of the group. Hence there exists a function $c = \gamma(a, b)$ such that

$$f(f(x, a), b) = f(x, \gamma(a, b)) \tag{1.23}$$

identically in x, a and b. Differentiate (1.23) with respect to b

$$\frac{\partial f(\bar{x}, b)}{\partial b} = \frac{\partial f(x, c)}{\partial c} \cdot \frac{\partial \gamma(a, b)}{\partial b}. \tag{1.24}$$

Since $\gamma(a, 0) = a$, (1.24) yields

$$\frac{\partial f(\bar{x}, b)}{\partial b} \mid_{b=0} = \frac{\partial f(x, a)}{\partial a} \cdot \frac{\partial \gamma(a, b)}{\partial b} \mid_{b=0}. \tag{1.25}$$

Since $\gamma(0, b) = b$, the final factor of (1.25) is unity if $a = 0$. Thus this factor is not identically zero. It also contains the parameter a continuously. We denote it by $A(a)$ i.e.

$$A(a) = \frac{\partial \gamma(a, b)}{\partial b} \mid_{b=0}. \tag{1.26}$$

Thus

$$\xi(\bar{x}) = \frac{\partial f(x, a)}{\partial a} A(a). \tag{1.27}$$

Then

$$\frac{d\bar{x}}{da} = \frac{\partial f(x, a)}{\partial u} = \frac{\xi(\bar{x})}{A(a)}. \tag{1.28}$$

Now introduce the new parameter

$$t = \int_0^a \frac{1}{A(\alpha)} d\alpha \tag{1.29}$$

in the place of a. Then

$$dt = \frac{1}{A(a)}da$$

and

$$\frac{d\bar{x}}{dt} = \xi(\bar{x}). \tag{1.30}$$

If $a = 0$, then $\bar{x} = x$, $\bar{y} = y$, $t = 0$, which are the initial data employed in the previous section. Now we can state the following fundamental theorem:

Theorem 1.2.2 (Lie) The transformations (1.1) of any one-parameter group whose identity has the parameter $a = 0$ are put into one-to-one correspondence by the introduction of the new parameter (1.29), with the infinitesimal transformation (1.16). Namely the functions f^i are solutions of the initial value problem

$$\frac{d\bar{x}}{dt} = \xi(\bar{x}), \quad \bar{x}\,|_{t=0},$$

where t is given by (1.29).

This theorem motivates the following definition.

Definition 1.2.3 The *symbol* of the infinitesimal transformation (1.16) is also called the *symbol* or the *generator* of the group of transformations (1.1).

Lie's theorem provides a simplification to the investigation of continuous one-parameter group : It reduces the study of continuous one-parameter groups to that of infinitesimal transformations. Therefore it provides a (sort of) 'linearisation' to the continuous one-parameter group.

Theorem 1.2.3 Every continuous one-parameter group of transformations whose transformations are given by (1.1), reduces to the group of translations

$$\bar{y}^1 = y^1 + a, \quad \bar{y}^i = y^i, \quad i = 2, \ldots, n \tag{1.31}$$

under a suitable change of variables

$$y^i = y^i(x), \quad i = 1, \ldots, n. \tag{1.32}$$

The new variables y^i are called *canonical variables.* They are singular for values of x such that $\xi(x) = 0$.

Proof: Under a change of variables (1.32), the differential operator (1.18) transforms according to the formula

$$X = X(y^i)\frac{\partial}{\partial y^i}.$$

Therefore canonical variables are found from the system of linear partial differential equations

$$X(y^i) = \delta^i_1; \quad i = 1, \ldots, n;$$

where δ_j^i denotes the Kronecker symbol defined by

$$\delta_j^i = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

□

1.2.4 Equivalence of two one-parameter groups

Definition 1.2.4 Two groups whose transformations are

$$\bar{x} = f(x, a) \tag{1.33}$$

$$\bar{x} = g(x, \alpha) \tag{1.34}$$

are called equivalent if and only if there is a one-to-one correspondence $\alpha = \Phi(a)$ such that

$$f(x, a) = g(x, \Phi(a))$$

identically in x and a.

Theorem 1.2.4 Two one-parameter groups with symbols X and Y are equivalent if and only if their symbols are proportional i.e.

$\exists k \in \mathcal{R}^* = \mathcal{R} - \{0\}$ such that $X = kY$.

Proof: See for instance Dickson (1924).□

1.2.5 Group invariants

Definition 1.2.5 A function $F(x)$ is an invariant of the group of transformations (1.1) if

$$F(f(x,a)) = F(x,a), \tag{1.35}$$

identically in x, a.

Theorem 1.2.5 (Infinitesimal criterion of invariance) A function $F(x)$ is an invariant of the group of transformations (1.1) if and only if

$$XF(x) = \xi^i(x)\frac{\partial F(x)}{\partial x^i} = 0. \tag{1.36}$$

Proof:

$\Longrightarrow$):

Differentiate (1.35) with respect to a and set $a = 0$ in the result.

$\Longleftarrow$):

$$\frac{\partial[F(f(x,a))]}{\partial a} = (XF)(f(x,a)) = 0.$$

(by the chain rule and Lie's theorem)

Whence

$$F(f(x,a)) = F(f(x,0)) = F(x).$$

□

It follows from the above theorem that any continuous one-parameter group of transformation (1.1) has $(n-1)$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J^1(x) = c^1, \ldots, J^{n-1}(x) = c^{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x)} = \ldots = \frac{dx^n}{\xi^n(x)}. \tag{1.37}$$

Example 1.2.4 (Group of rotations)

$$\xi^1 = -y, \quad \xi^2 = x.$$

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

$$XF = 0 \iff -y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y} = 0.$$

The characterisric equations are

$$\frac{dx}{-y} = \frac{dy}{x}$$

i.e.

$$x^2 + y^2 = C.$$

Thus

$$F(x,y) = f(x^2+y^2).$$

Then any function $F(x,y)$ invariant under the group of rotations is function of the single variable $r^2 = x^2+y^2$.

Now, consider the system of equations

$$F^1(x) = 0, \ldots, F^s(x) = 0, \quad s < n. \tag{1.38}$$

We shall assume that the rank of the matrix $\| \frac{\partial F^k}{\partial x^i} \|$ is equal to s at all points satisfying the equation (1.38). Thus (1.38) defines an $(n-s)$ - dimensional manifold M.

Definition 1.2.6 A system of equations (1.38) is said to be invariant with respect to the group G of transformations (1.1) if each point $x \in M$ on the surface M is moved by G along the surface, that is, if $x \in M$ implies $\bar{x} \in M$.

Theorem 1.2.6 The system of equations (1.38) is invariant with respect to the group G of transformations (1.1) with symbol

$$X = \xi^i \frac{\partial}{\partial x^i}$$

if and only if

$$XF^k \mid_M = 0, \quad k = 1, \ldots, s. \tag{1.39}$$

Proof:

i) Suppose that (1.38) is invariant with respect to G.

Let $x \in M$, thus $\bar{x} = f(x,a) \in M$ i.e.

$$F^k(f(x,a)) = 0, \quad k \in \{1,\ldots,s\}. \tag{1.40}$$

Differentiate (1.40) with respect to a

$$\frac{\partial f^i(x,a)}{\partial a} \cdot \frac{\partial F^k}{\partial x^i}(f(x,a)) = 0, \quad k = 1,\ldots,s;$$

i.e.

$$\xi^i(f(x,a))\frac{\partial F^k}{\partial x^i}(f(x,a)) = 0, \quad k = 1,\ldots,s. \tag{1.41}$$

Set $a = 0$ in (1.41):

$$XF^k(x) = 0, \quad k = 1,\ldots,s.$$

Thus

$$XF^k \mid_M = 0 \quad k = 1,\ldots,s.$$

ii) Now assume that $XF^k \mid_M = 0$; $k = 1,\ldots,s$. Thus the restriction of each F^k to M is an invariant function i.e.

$$F^k(f(x,a)) = F^k(x),$$

for all $x \in M$ and for $k = 1, \ldots, s$. Whence

$$F^k(f(x,a)) = 0, \quad k = 1, \ldots, s;$$

i.e. $\bar{x} \in M$.□

Theorem 1.2.7 (Representation via invariant) Let the equations (1.38) be invariant under the group G and let the symbol of the group X not vanish on the surface M defined by (1.38). Then the surface M can be represented by equations of the form

$$\phi^k(J^1(x), \ldots, J^{n-1}(x)); \quad k = 1, \ldots, s; \tag{1.42}$$

where $J^1, \ldots, J^{n-1}$ is a basis of invariants of the group G.

Proof: See Ovsiannikov (1982).□

1.3 Symmetries of differential equations and their use

1.3.1 Prolongation of point transformations

Consider a continuous one parameter group G of transformations

$$\bar{x} = f(x, u, a), \quad f(x, u, 0) = x,$$

(1.43)

$$\bar{u} \;=\; \varphi(x,u,a),\;\; \varphi(x,u,0)=u,$$

in the space $\mathcal{R}^{n+m}$ with variables $x=(x^1,\ldots,x^n)$ and $u=(u^1,\ldots,u^m)$. Now take variables $u_{(1)}=\{u_i^\alpha \mid \alpha=1,\ldots,m; i=1,\ldots,n\}$ and subject them to the transformation

$$\begin{aligned} \bar{u}_i^\alpha &= \Phi_i^\alpha(x,u,u_{(1)},a), \\ u_i^\alpha &= \Phi_i^\alpha(x,u,u_{(1)},0). \end{aligned} \tag{1.44}$$

We require that (1.44) together with the transformations of the derivatives $\frac{\partial u^\alpha}{\partial x^i}$ under the change of variables (1.43) be compatible with the quantities

$$u_i^\alpha = \frac{\partial u^\alpha(x)}{\partial x^i} \tag{1.45}$$

for any $u^\alpha = u^\alpha(x)$. This condition uniquely defines the transformation (1.44) for any one-parameter group of transformations (1.43) and (1.43)-(1.44) define a unique one-parameter group $G_{(1)}$ of transformations acting in the space $\mathcal{R}^{n+m+nm}$ with variable $(x,u,u_{(1)})$ (see Dickson (1924)). The equations (1.43) are called *point transformations* whereas (1.44) are called *prolongation of point transfomations.* $G_{(1)}$ is termed *the first prolongation* of

the group G.

Let

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{1.46}$$

be the symbol of G. The symbol of $G_{(1)}$ is

$$X_{(1)} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}.$$

The additional coordinates $\zeta_i^\alpha = \frac{\partial \Phi_i^\alpha}{\partial a}\,|_{a=0}$ of this operator can be determined from the compatibility condition of (1.44) and (1.45).

Using quantities $\omega = (\omega^1, \ldots, \omega^m)$, where

$$\omega^\alpha = du^\alpha - u_i^\alpha dx^i, \tag{1.47}$$

one can rewrite the equalities (1.45) in the form

$$\omega = 0. \tag{1.48}$$

The compatibility conditions imposed on the prolongation states that (1.48) defines a manifold which is invariant under the group $\tilde{G}_{(1)}$ of transformations (1.43), (1.44) and

$$\begin{aligned} d\bar{x} &= \frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial u^\alpha}du^\alpha, \\ d\bar{u} &= \frac{\partial \Phi}{\partial x^i}dx^i + \frac{\partial \Phi}{\partial u^\alpha}du^\alpha, \end{aligned} \tag{1.49}$$

acting in the space $(x, u, u_{(1)}, dx, du)$. Introduce the infinitesimal operator $\tilde{X}$ of $\tilde{G}_{(1)}$

$$\tilde{X} = X_{(1)} + \tilde{\xi}^i \frac{\partial}{\partial dx^i} + \tilde{\eta}^\alpha \frac{\partial}{\partial du^\alpha}, \tag{1.50}$$

where

$$\begin{aligned} \tilde{\xi} &\equiv \frac{\partial d\bar{x}}{\partial a} \Big|_{a=0} \\ &= \frac{\partial \xi}{\partial x^i} dx^i + \frac{\partial \xi}{\partial u^\alpha} du^\alpha, \\ \tilde{\eta} &\equiv \frac{\partial d\bar{u}}{\partial a} \Big|_{a=0} \\ &= \frac{\partial \eta}{\partial x^i} dx^i + \frac{\partial \eta}{\partial u^\alpha} du^\alpha. \end{aligned}$$

The criterion for invariance can be written as

$$\tilde{X}\omega^\alpha \Big|_{\omega=0} \equiv (\tilde{\eta}^\alpha - u_i^\alpha \tilde{\xi}^i - \zeta_i^\alpha dx^i) \Big|_{\omega=0} = 0,$$

$$\alpha = 1, \ldots, m.$$

This gives, after inserting the above expressions of $\tilde{\xi}$ and $\tilde{\eta}$, the following formula for the first prolongation of the operator (1.46)

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j). \tag{1.51}$$

Here

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha}$$

is the operator total differential with respect to x^i. Prolongations of higher order are obtained by defining the action of the group G on the variables $u_{(1)}, u_{(2)}, \ldots$, where

$$u_{(s)} = \{u^\alpha_{i_1 \ldots i_s} \mid \alpha = 1, \ldots; \quad i_1, \ldots, i_s = 1, \ldots, n\}.$$

Define

$$u^\alpha_J = \frac{\partial^k u^\alpha}{\partial x^{j_1} \ldots x^{j_k}} \ , \tag{1.52}$$

where

$$\alpha = 1, \ldots, m$$

$$J = (j_1, \ldots, j_k), \quad 1 \leq j_k \leq n.$$

Define also

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_{J,j} \frac{\partial}{\partial u^\alpha_J}, \tag{1.53}$$

where

$$u^\alpha_{J,j} = \frac{\partial u^\alpha_J}{\partial x^j} \ , \quad 1 \leq j \leq n.$$

The symbol $X_{(s)}$, $s \geq 1$, of the sth prolongation of the group G is

$$X_{(s)} = X + \zeta^\alpha_J \frac{\partial}{\partial u^\alpha_J},$$

where summation is over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq n, 1 \leq k \leq s$. The coefficients ζ^α_J are given by the recursion

relations

$$\zeta_j^\alpha = D_j(\eta^\alpha) - u_i^\alpha D_j(\xi^i), \tag{1.54}$$

$$\zeta_{J,j}^\alpha = D_j(\zeta_J) - u_{J,i}^\alpha D_j(\xi^i), \tag{1.55}$$

with $J = (j_1, \ldots, j_k)$, $1 \leq k \leq s-1$.

1.3.2 Group admitted by a differential equation

We consider a system of s–order partial differential equations

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \tag{1.56}$$

where

$$F = (F^1, \ldots, F^p).$$

Definition 1.3.1 If the local manifold defined by (1.56) in the $(x, u, u_{(1)}, \ldots, u_{(s)})$-space is invariant under the action of the sth prolongation $G_{(s)}$ of G, we say that the differential equation (1.56) admits G or G is the *symmetry group* of (1.56). Each generator of G is termed a *symmetry* of (1.56).

Let X be the symbol of G. Then, the infinitesimal criterion of invariance says that (1.56) is invariant under the group G if and only if

$$X_{(s)}F \mid_{F=0} = 0 \tag{1.57}$$

Definition 1.3.2 (1.57) is called *defining* or *determining equation* of (1.56).

The following fundamental theorem enables us to construct some solutions of (1.56) from known one.

Theorem 1.3.1 A symmetry of (1.56) transforms any classical solution of (1.56) into another classical solution of the same equation.

Proof: This comes from the fact that a symmetry of an equation leaves invariant that equation. □

Definition 1.3.3 (Lie algebra) A Lie algebra of operators

$$X = \xi^i \frac{\partial}{\partial x^i}$$

is a vector space L with the following property

$$\forall X_1,\ X_2 \in L,\ [X_1, X_2] \equiv X_1(X_2) - X_2(X_1) \in L$$

Theorem 1.3.2 The set of all solutions of any determining equation forms a Lie algebra.

Proof: See for example Ovsiannikov (1982).□

Definition 1.3.4 (invariant solutions) A solution $u = f(x)$ of (1.56) is an invariant solution if the manifold M whose equation is $u - f(x) = 0$ is invariant under the group admitted by (1.56).

From this definition, we infer via the infinitesimal criterion of invariance that an invariant solution of (1.56) must satisfy the equations

$$\frac{dx^1}{\xi^1(x,f)} = \frac{dx^2}{\xi^2(x,f)} = \ldots = \frac{dx^n}{\xi^n(x,f)} = \frac{df^1}{\eta^1(x,f)} = \ldots = \frac{df^m}{\eta^m(x,f)}. \quad (1.58)$$

When we restrict the class of solutions we are looking for to invariant solutions, we reduce the number of independent variables of the equation under consideration by one. Almost all well-known exact solutions which occur in applications are invariant solutions. So the importance of invariant solutions can not be overstated.

1.4 Illustration of the concept group admitted: The heat equation

1.4.1 Symmetries of the heat equation

We consider the classical diffusion equation

$$u_t = u_{xx}, \tag{1.59}$$

where subscripts denote differentiation. Let

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{1.60}$$

(1.59) admits X if and only if

$$X_{(2)}(u_t - u_{xx}) \mid_{u_t - u_{xx}=0} = 0. \tag{1.61}$$

But

$$X_{(2)} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{11}\frac{\partial}{\partial u_{tt}} + \zeta_{12}\frac{\partial}{\partial u_{tx}} + \zeta_{22}\frac{\partial}{\partial u_{xx}}.$$

Thus (1.60) is equivalent to

$$[\zeta_1 - \zeta_{22}]_{u_t - u_{xx}=0} = 0, \tag{1.62}$$

where

$$\zeta_1 = \eta_t - \xi_t^2 u_x + (\eta_u - \xi_t^1)u_t - \xi_u^2 u_x u_t - \xi_u^1 u_t^2, \tag{1.63}$$

$$\zeta_{22} = \eta_{xx} + (2\eta_{xu} - \xi^2_{xx})u_x - \xi^1_{xx}u_t + (\eta_{uu} - 2\xi^2_{xu})u_x^2 - 2\xi^1_{xu}u_xu_t - \xi^2_{uu}u_x^3$$

$$- \xi^1_{uu}u_x^2u_t + (\eta_u - 2\xi^2_x)u_{xx} - 2\xi^1_xu_{tx} - 3\xi^2_uu_xu_{xx} - \xi^1_uu_tu_{xx} - 2\xi^1_uu_xu_{tx}. \quad (1.64)$$

Substituting (1.63) and (1.64) into (1.62) and treating t, x, u_t, u_x, u_{tt}, u_{tx}, u_{xx} as 'independent variables', we obtain

$$\begin{aligned}
\xi^1_u &= 0,\\
\xi^1_x &= 0,\\
\xi^1_{uu} &= 0,\\
\xi^2_u &= -\xi^1_x,\\
\xi^1_t &= \xi^1_{xx} + 2\xi^2_x,\\
\xi^2_{uu} &= 0,\\
\eta_{uu} &= 2\xi^1_{xu},\\
\xi^2_t &= -2\eta_{xu} + \xi^2_{xx},\\
\eta_t &= \eta_{xx}.
\end{aligned} \quad (1.65)$$

(1.65) is an overdetermined system of linear partial differential equations whose solution is

$$\eta = (-\frac{1}{8}c_1x^2 - \frac{1}{2}c_4x - \frac{1}{4}c_1t + c_5)u + \beta(t,x),$$

$$\begin{aligned} \xi^1 &= \frac{1}{2}c_1t^2 + c_2t + c_3, \\ \xi^2 &= \frac{1}{2}c_1xt + \frac{1}{2}c_2x + c_4t + c_5. \end{aligned} \tag{1.66}$$

Where c_i, $i = 1, \ldots, 6$, are constants of integration and β is an arbitrary solution of the heat equation. The Lie algebra admitted by (1.59) is therefore spanned by the following operators

$$\begin{aligned} X_1 &= 4t^2\frac{\partial}{\partial t} + 4xt\frac{\partial}{\partial x} - (x^2 + 2t)u\frac{\partial}{\partial u}, \\ X_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial t}, \\ X_4 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \\ X_5 &= \frac{\partial}{\partial x}, \\ X_6 &= u\frac{\partial}{\partial u}, \\ X_\beta &= \beta(t,x)\frac{\partial}{\partial u}, \end{aligned} \tag{1.67}$$

where β is any solution of (1.59).

The operators (1.67) are obtained from (1.66) by equating a given contant to a fixed number (say one) and the remaining constants to zero. This procedure is justified by the fact that the operators admitted by a differential equation form a Lie algebra and *a fortiori* a vector space.

1.4.2 Brief comment on the meaning of symmetries

Using Lie's equations, the transformations of the one-parameter group generated by the symmetries X_i, X_β; $i = 1, \ldots, 6$; are given by

$$
\begin{aligned}
X_1 \;&:\; \bar{t} = \frac{t}{1-4\lambda t}, \quad \bar{x} = \frac{x}{1-4\lambda t}, \\
&\quad\; \bar{u} = u\sqrt{1-4\lambda t}\exp\left(\frac{-\lambda x^2}{1-4\lambda t}\right), \\
X_2 \;&:\; \bar{t} = \lambda^2 t, \quad \bar{x} = \lambda x, \quad \bar{u} = u, \\
X_3 \;&:\; \bar{t} = t + \lambda, \quad \bar{x} = x, \quad \bar{u} = u, \\
X_4 \;&:\; \bar{t} = t, \quad \bar{x} = x + 2\lambda t, \quad \bar{u} = e^{-\lambda x - \lambda^2 t}u, \\
X_5 \;&:\; \bar{t} = t, \quad \bar{x} = x + \lambda, \quad \bar{u} = u, \\
X_6 \;&:\; \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = \lambda u, \\
X_\beta \;&:\; \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + \lambda\beta.
\end{aligned}
$$

Thus X_3 and X_5 express the fact that (1.59) does not depend explicitly on the independent variables i.e. it is an autonomous equation: If $u(t,x)$ is a solution of (1.59), $u(t+\lambda, x)$, $u(t, x+\lambda)$ are also solutions of the same equation. The symmetries X_6 and X_β descend from the linearity and homogeneity of the heat equation: They summarise the superposition principle. The symmetry X_2 describes a scaling : If $u(t,x)$ is a solution of (1.59), $u(\lambda^2 t, \lambda x)$ is also

a solution of the same equation. Finally only X_1 and X_4 are non-obvious symmetries. Note that under X_1, the trivial solution $u(t,x)=1$ of the heat equation transforms into the less trivial one

$$\bar{u} = \sqrt{1-4\lambda t}\exp\left(\frac{-\lambda x^2}{1-4\lambda t}\right)$$

i.e.

$$\bar{u} = \frac{1}{\sqrt{1+4\lambda\bar{t}}}\exp\left(-\frac{\lambda\bar{x}^2}{1+4\lambda\bar{t}}\right).$$

Dropping the bars, we obtain

$$u = \frac{1}{\sqrt{1+4\lambda t}}\exp\left(-\frac{\lambda x^2}{1+4\lambda t}\right).$$

1.5 Conclusion

In this preliminary chapter, we have given a brief account of Lie group analysis of differential equations. We have omitted the classical result of Lie concerning the reduction of order of ordinary differential equations via symmetries because we will not need this theory in the forthcoming chapters. Further applications of Lie group analysis are presented in the next chapter.

Chapter 2

Group Theoretic Determination of Fundamental Solutions

2.1 Introduction

In the initial value problem for linear differential equations, obtaining a fundamental solution is generally the last nail in the coffin. The Fourier transform method is commonly used to obtain fundamental solutions in the case of linear differential equations with constant coefficients. But in the case of

variable coefficients, the Fourier transform method is almost of no use. In 1989, N. H. Ibragimov (Ibragimov (1989)) noticed that if instead of seeking for fundamental solutions in general, we look for the fundamental solution of a given Cauchy problem, Lie symmetry analysis can render a useful service. Namely, he found that if the group admitted by an equation is sufficiently large, one can construct the fundamental solution of an initial value problem as an invariant solution. Almost all well-known fundamental solutions of mathematical physics can be derived using this method. The main concern of this chapter is to present Ibragimov's idea. In order to make this chapter self-contained, we shall begin with a brief introduction to generalised functions theory.

2.2 Preliminaries on the theory of distributions

Here we give a short account of the theory of distributions. We restrict ourselves to the notions that we shall need in forthcoming p .ts. For more details on generalised functions theory, consult for instance Schwartz (1966)

or Gel'fand and Shilov (1964).

2.2.1 Some introductory definitions

Let $x = (x^1, \ldots, x^n) \in \mathcal{R}^n$, $\mid x \mid = ((x^1)^2 + \cdots + (x^n)^2)^{1/2}$. Let α be an n–tuple of nonnegative integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$. We define

$$\mid \alpha \mid = \alpha_1 + \cdots + \alpha_n$$

and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial (x^1)^{\alpha_1} \ldots \partial (x^n)^{\alpha_n}}$$

Definition 2.2.1 A function $f(x)$ is locally integrable in $\mathcal{R}^n$ if $\int_K \mid f(x) \mid dx$ exists for every bounded region K in $\mathcal{R}^n$ (here $\int$ is the integral in the Lebesgue sense).

Another concept that plays a crucial role in the theory of distribution is the *support* of a function defined as follows:

Definition 2.2.2 The *support* of a function f is the closure of the set of all points x such that $f(x) \neq 0$. We shall denote the support of f by *supp* f.

2.2.2 Test functions

Consider the space $\mathcal{D}$ consisting of real-value functions $\varphi(x) = \varphi(x^1, \ldots, x^n)$, such that the following holds:

1. $\varphi(x)$ is an indefinitely differentiable function defined at every point of $\mathcal{R}^n$. This means that $D^\alpha\varphi$ exists for all multi-indices α and is continuous. Such a function is called a C^∞*–function.*

2. There exists a number $A > 0$ such that $\varphi(x)$ vanishes for $\mid x \mid > A$. This means that $\varphi(x)$ has a compact support.

Then φ is called a *test function.*

The prototype test function is

$$\varphi(x) = \begin{cases} \exp\left(-\frac{a^2}{a^2-|x|^2}\right), & \mid x \mid < a \\ 0, & \mid x \mid > a \end{cases}$$

Definition 2.2.3 A sequence $\{\varphi_m\}$, $m = 1, 2, \ldots$, where $\varphi_m \in \mathcal{D}$, converges to φ if the following two conditions are satisfied:

1. All φ_m as well as φ vanish outside a common region.

2. $D^\alpha\varphi_m \longrightarrow D^\alpha\varphi$ uniformly over $\mathcal{R}^n$ as $m \longrightarrow \infty$ for all multi-indices α.

2.2.3 Linear functionals and the theory of distributions

Definition 2.2.4 A linear functional T on the space $\mathcal{D}$ is an operation (or rule) by which we assign to every test function φ a real number denoted by $< T, \varphi >$, such that

$$< T, c_1\varphi_1 + c_2\varphi_2 > \; = \; c_1 < T, \varphi_1 > + c_2 < T, \varphi_2 >,$$

for every arbitrary test functions φ_1 and φ_2 and real numbers c_1 and c_2.

The next concept is that of the continuity of linear functionals.

Definition 2.2.5 A linear functional on $\mathcal{D}$ is continuous if and only if the sequence of numbers $< T, \varphi_m >$ converges to $< T, \varphi >$ when the sequence of test functions $\{\varphi_m\}$ converges to the test function φ (in the sense of the convergence defined in the previous section).

We now have all the tools for defining the concept of distribution.

Definition 2.2.6 A continuous linear functional on the space $\mathcal{D}$ of test functions is called a *distribution*. The space of all distributions on $\mathcal{R}^n$ is denoted $\mathcal{D}'$.

2.2.4 Some examples of distributions

Regular distributions

The set of distributions that are most useful are those generated by locally integrable functions. Indeed, every locally integrable function $f(x)$ generates a distribution through the formula

$$< f, \varphi > = \int_{R^n} f(x)\varphi(x)dx. \tag{2.1}$$

Linearity of this functional is obvious. To prove its cortinuity, we observe that

$$|< f, \varphi >| \leq \max_{x \in supp\varphi} | \varphi(x) | \int_{supp\varphi} | f(x) | dx < \infty. \tag{2.2}$$

Thus, if the sequence $\{\varphi_m\}$ converges to zero, then so does $< f, \varphi_m >$. Hence the functional defined by f is continuous.

Definition 2.2.7 Distributions defined by (2.1) are called *regular distributions*. All other distributions are called *singular distributions*. However, we may use formula (2.1) symbolically for a singular distribution as well.

The dirac distribution

Consider the functional $\delta(x-\xi)$ defined by

$$< \delta(x-\xi), \varphi > \; = \; \varphi(\xi), \quad \text{for all } \varphi \in \mathcal{D}. \tag{2.3}$$

It is straightfoward to check that the functional defined by (2.3) is a distribution. This distribution is called *Dirac's distribution.* It plays a crucial role in the theory of partial differential equations.

2.2.5 Algebraic operation on distributions

Theorem 2.2.1 $\mathcal{D}'$ is a vector space.

In general the multiplication of two singular distributions is not defined. But we can define the product of a smooth function by a distribution as follows: Let $\Phi \in C^\infty$ and $T \in \mathcal{D}'$, ΦT is the distribution defined by

$$< \Phi T, \varphi > \; = \; < T, \Phi\varphi >, \quad \text{for all } \varphi \in \mathcal{D} \tag{2.4}$$

Example 2.2.1 (Important) For all $\Phi \in C^\infty$,

$$\Phi\delta(x-\xi)' = \Phi(\xi)\delta(x-\xi). \tag{2.5}$$

2.2.6 Differentiation of distributions

Let α be a multi-index. The partial derivative of order $\mid \alpha \mid$ of a distribution T is the distribution $D^{\alpha}T$ defined by

$$< D^{\alpha}T, \varphi > \; = \; (-1)^{|\alpha|} < T, D^{\alpha}\varphi >, \tag{2.6}$$

for all test functions φ.

2.2.7 The support of a distribution

Definition 2.2.8 A distribution T vanishes on an open set U if

$$< T, \varphi > \; = \; 0,$$

for all test functions φ such that $supp\varphi \subset U$.

Let ω be the union of all open sets on which T vanishes.

Definition 2.2.9 supp $\mathrm{T} = \mathcal{R}^n - \omega$

Example 2.2.2 supp $\delta(x - \xi) = \{\xi\}$

2.2.8 Transformation of distributions

Consider the transformation

$$\bar{x} = \Phi(x), \tag{2.7}$$

where Φ is a C^∞–diffeomorphism. We denote the Jacobian of this transformation by J i.e.

$$J = \det(\partial \bar{x}^i / \partial x^j).$$

Definition 2.2.10 Let T be a distribution on $\mathcal{R}^n$, The image of T under the tranformation (2.7) is the distribution $\bar{T}$ defined by

$$< \bar{T}(\bar{x}), \varphi(\bar{x}) > \; = \; < T(x), \varphi \circ \Phi(x) > . \tag{2.8}$$

Consider the case where T is a regular distribution generated by a locally integrable function f:

$$< f(x), \varphi \circ \Phi(x) > \; = \int_{\mathcal{R}^n} f(x) \varphi \circ \Phi(x) dx. \tag{2.9}$$

From the usual change of variables formula in the integral, we have

$$\begin{aligned} \int_{\mathcal{R}^n} f(x) \varphi(x) \circ \Phi(x) dx \; &= \; \int_{\mathcal{R}^n} \mid J \mid^{-1} f(x) \varphi \circ \Phi^{-1}(\bar{x}) d\bar{x} \\ &= \; <\mid J \mid^{-1} f \circ \Phi^{-1}(\bar{x}), \varphi(\bar{x}) > . \end{aligned} \tag{2.10}$$

Whence

$$< \bar{f}(\bar{x}), \varphi(\bar{x}) > \; = \; <\mid J \mid^{-1} f \circ \Phi^{-1}(\bar{x}), \varphi(\bar{x}) > .$$

Thus, a regular distribution transforms as follows

$$\bar{f} = \mid J \mid^{-1} f. \tag{2.11}$$

This formula suggests that an arbitrary distribution should transform the same way. To prove this rigorously, we need some preparation.

Definition 2.2.11 (Convergence of distributions) A sequence of distributions $T_m \in \mathcal{D}'$, $m = 1, 2, \ldots$, is said to converge to a distribution $T \in \mathcal{D}$ if

$$\lim_{m \to +\infty} < T_m, \varphi > \; = \; < T, \varphi >, \tag{2.12}$$

for all test functions φ.

Theorem 2.2.2 $\mathcal{D}$ is dense in $\mathcal{D}'$ i.e. for all distributions T, there is a sequence $\{T_m\}$, $m = 1, 2, \ldots$, of test functions such that

$$\lim_{m \to +\infty} < T_m, \varphi > \; = \; < T, \varphi >,$$

for all test functions φ.

Proof: See for instance Schwartz (1966).□

Theorem 2.2.3 Under the transformation (2.7), an arbitrary distribution T transforms according to the following formula

$$\tilde{T} = \mid J \mid^{-1} T \tag{2.13}$$

Proof: Use Theorem 2.2.2 and (2.11).□

Corollary 2.2.1 (Transformation of the delta function)

$$\bar{\delta}(\bar{x} - \bar{x}_0) = (\mid J \mid^{-1})_{x=x_0} \delta(x - x_0). \tag{2.14}$$

Now, let consider a one-parameter transformation group. Writing instead of (2.7) the infinitesimal transformation

$$\bar{x}^i = x^i + a\xi^i(x), \quad i = 1, 2, \ldots, n; \tag{2.15}$$

of this group and expanding J in Taylor serie about $a = 0$ to the first order of precision

$$J \approx 1 + aD_i(\xi^i), \tag{2.16}$$

where $D_i = \partial/\partial x^i$. We obtain from (2.13), when we perform the substitution $T = \delta(x - x_0)$,

$$\bar{\delta}(\bar{x} - \bar{x}_0) \approx \delta(x - x_0) - aD_i(\xi^i) \mid_{x=x_0} \delta(x - x_0). \tag{2.17}$$

This last equation will play an important role in what follows.

2.3 Invariance principle

Given a linear differential operator L, consider a boundary value (in particular, initial value) problem

$$Lu = f(x), \tag{2.18}$$

$$u\,|_M \;=\; \varphi(x), \tag{2.19}$$

where the data φ is defined on the manifold $M \subset \mathcal{R}^n$.

Definition 2.3.1 The problem (2.18)-(2.19) is said to be invariant under a group G if the following hold:

(i) The differential equation (2.18) admits G.

(ii) The manifold M together with (2.19) are invariant under the group G.

The invariance principle: Let the boundary value problem (2.18)-(2.19) be invariant under a group G. Then we should seek the solution among the functions (distributions) invariant under G.

2.4 Construction of fundamental solutions using symmetries

For the sake of clarity and simplicity, we shall consider only linear second-order parabolic equations

$$Lu = u_t + a^{ij}u_{ij} + b^i u_i + cu = 0, \quad x \in \mathcal{R}^n, \quad t > t_0, \tag{2.20}$$

where the subscripts denote partial derivatives, a, b, c are function of t and x, $t_0 \in \mathcal{R}$. Cauchy's problem for L is to find the solution $u(x,t)$ of (2.20) which satisfies the additional condition (initial condition)

$$u(x,t_0) = \varphi(x), \tag{2.21}$$

where φ is given.

Definition 2.4.1 A fundamental solution of the Cauchy problem (2.18)-(2.19) is a distribution $S(x,t;\xi,t_0)$ which satisfies (2.18) and the initial condition

$$S(x,t_0;\xi,t_0) = \delta(x-x_0). \tag{2.22}$$

Note that if the coefficients of L are sufficiently regular, it can be proved that the fundamental solution of a Cauchy problem is unique in the space of tempered distributons (Mizohata (1973)). When the coefficients of L are constant, the procedure for the construction of fundamental solutions is well-defined: The Fourier transform is commonly used to transform the Cauchy problem

$$Lu = 0, \quad x \in \mathcal{R}^n, \quad t > t_0, \tag{2.23}$$

$$u(x,t_0;x_0,t_0) = \delta(x-x_0), \tag{2.24}$$

into an initial value problem which is much more simple to solve. After solving this initial value problem, one has to invert the solution in order to obtain the fundamental solution. When L has variable coefficients, there is not a unified method to tackle the problem as in the case of constant coefficients. But Ibragimov (1989) noticed that if (2.23) has enough symmetries, one can construct the fundamental solution of (2.18)-(2.19) using the invariance principle mentioned above. In the sequel we shall examine in detail what invariance of (2.24) under the group admitted by (2.23) means.

Let us begin with the following fundamental property of the symmetry Lie algebra admitted by (2.23).

Theorem 2.4.1 (Ovsiannikov) If the matrix (a^{ij}) is nonzero and equation (2.23) is not very degenerate, then the Lie algebra admitted by (2.23) has the form

$$\mathcal{L} = \mathcal{L}^r \oplus \mathcal{L}^\infty, \tag{2.25}$$

where the finite dimensional subalgebra $\mathcal{L}^r$ consists of operators of the form

$$X = \xi^0(x,t)\frac{\partial}{\partial t} + \xi^i(x,t)\frac{\partial}{\partial x^i} - \eta(x,t)u\frac{\partial}{\partial u}, \tag{2.26}$$

satisfying the determining equation and $\mathcal{L}^\infty$ is generated by operators of the form $v(x,t)\frac{\partial}{\partial u}$, where v is an arbitrary solution of (2.23).

Proof: Consult Ovsiannikov (1982).□

Note that $\mathcal{L}^{\infty}$ is of no use for the integration of (2.23). Rather the part of the symmetry Lie algebra that will play an important role is $\mathcal{L}^r$. The next theorem tells us under which circumstances $\mathcal{L}^r$ is not reduced to the trivial algebra $\{0\}$.

Theorem 2.4.2 Equation (2.23) admits a nonzero operator of the form (2.26) if and only if it is equivalent to an equation all of whose coefficients are independent of one of the coordinates.

Proof: See Ovsiannikov (1982).□

In what follows, we assume that the hypotheses of the preceding theorems are satisfied.

Theorem 2.4.3 Let

$$X = \xi^0(x,t)\frac{\partial}{\partial t} + \xi^i(x,t)\frac{\partial}{\partial x^i} - \eta(x,t)u\frac{\partial}{\partial u} \in \mathcal{L}^r.$$

Equation (2.24) is invariant under this operator if and only if

$$\xi^0(x,t_0) = 0, \quad \xi^i(x,t_0) = 0, \quad i = 1,\ldots,n, \tag{2.27}$$

$$\eta(x_0,t_0) - \sum_{i=1}^{n}\frac{\partial \xi^i}{\partial x^i}(x_0,t_0) = 0. \tag{2.28}$$

The set of all $X \in \mathcal{L}^r$ leaving equation (2.24) invariant forms a subalgebra $\mathcal{F}$ of $\mathcal{L}^r$.

Proof: The invariance of (2.24) under the operator X means the invariance of the initial manifold i.e. $t = t_0$, the invariance of the frame of (2.24) and the invariance of the support of $\delta(x - x_0)$ i.e $x = x_0$ under the restriction of X to $t = t_0$, X_{t_0}.

$t = t_0$ is invariant under X if and only if

$$X(t - t_0) \mid_{t=t_0} = 0,$$

i.e.

$$\xi^0(x, t_0) = 0.$$

$$X_{t_0} = \xi^i(x, t_0)\frac{\partial}{\partial x^i} - \eta(x, t_0)u\frac{\partial}{\partial u}.$$

$x = x_0$ is invariant under X_{t_0} if and only if

$$X_{t_0}(x - x_0) \mid_{x=x_0} = 0$$

i.e.

$$\xi^i(x_0, t_0) = 0, \quad i = 1, \ldots, n.$$

Under the operator X_{t_0}, $u(x,t_0,x_0,t_0)$ and $\delta(x-x_0)$ are subjected to the infinitesimal transformations (see equation (2.17))

$$\bar{u}(\bar{x},\bar{t}_0;\bar{x}_0,\bar{t}_0) \approx u(x,t_0;x_0,t_0) - a\eta(x,t_0)u(x,t_0;x_0,t_0),$$

$$\bar{\delta}(\bar{x}-\bar{x}_0) \approx \delta(x-x_0) - aD_i(\xi^i)(x_0,t_0)\delta(x-x_0).$$

It follows that

$$\begin{aligned}\bar{u}(\bar{x},\bar{t}_0;\bar{x}_0,\bar{t}_0) - \delta(\bar{x}-\bar{x}_0) &= u(x,t_0;x_0,t_0) - \delta(x-x_0) \\ &\quad -a\left\{\eta(x,t_0)u(x,t_0;x_0,t_0) - D_i(\xi^i)(x_0,t_0)\delta(x-x_0)\right\}.\end{aligned}$$

Thus the infinitesimal criterion of invariance

$$\bar{u}(\bar{x},\bar{t}_0;\bar{x}_0,\bar{t}_0) - \delta(\bar{x}-\bar{x}_0)\mid_{u(x,t_0;x_0,t_0)=\delta(x-x_0)} = o(a)$$

is equivalent to

$$\eta(x,t_0)\delta(x-x_0) - D_i(\xi^i)(x_0,t_0) = 0$$

i.e.

$$\eta(x_0,t_0) - D_i(\xi^i)(x_0,t_0) = 0.$$

Now let us prove that $\mathcal{F}$ is a Lie algebra. It is obvious that $\mathcal{F}$ is a vector space. Let $X_1,\ X_2 \in \mathcal{F}$,

$$X_j = \xi_j^0(x,t)\frac{\partial}{\partial t} + \xi_j^i(x,t)\frac{\partial}{\partial x^i} - \eta_j(x,t)u\frac{\partial}{\partial u}, \quad j=1,2.$$

We must show that $[X_1, X_2] \in \mathcal{F}$.

$$[X_1, X_2] = \xi^0(x,t)\frac{\partial}{\partial t} + \xi^i(x,t)\frac{\partial}{\partial x^i} - \eta(x,t)u\frac{\partial}{\partial u},$$

where

$$\begin{aligned}
\xi^0(x,t) &= X_1(\xi_2^0) - X_2(\xi_1^0), \\
\xi^i(x,t) &= X_1(\xi_2^i) - X_2(\xi_1^i), \\
\eta(x,t) &= X_1(\eta_2) - X_2(\eta_1).
\end{aligned}$$

A direct calculation shows that (2.27) and (2.28) are satisfied. □

Assume that $\mathcal{F} \neq \{0\}$, $\mathcal{F} = < X_1, \ldots, X_k >$. According to the invariance principle, we must find the invariants of $\mathcal{F}$. That is, we have to solve the system of first-order partial differential equations

$$X_1 I = 0, \ldots, X_k I = 0. \tag{2.29}$$

Suppose that (2.29) admits a non trivial solution [1], $I(x,t,u)$ say. An invariant solution of (2.23)-(2.24) is obtained by solving for u the equation

$$I(x,t,u) = C.$$

[1] For a rigorous study of the integrability of a system of first-order partial differential equations, see for instance Campbell (1903) or Goursat (1921).

Let

$$u = J(x, t, C).$$

The constant C is obtained from the initial condition (2.24).

2.5 Conclusion

In this chapter, we have shown how group methods can be used to find a fundamental solution. The group approach has a great advantage compared to the classical method (the Fourier transform method): In the group approach we are spared of the inversion. Further, only elementary operations (integration of first order partial differential equations...) lead to the fundamental solution. We shall implement the method described in this chapter on a concrete equation arising in the mathematics of finance.

Chapter 3

Representation of the Solution of Cauchy's Problem

3.1 Introduction

In the preceding chapter, we introduced the notion of the fundamental solution of the Cauchy problem and we showed how the group approach can be helpful for the construction of such a fundamental solution. The present chapter is aimed at obtaining an integral representation of the solution of Cauchy's problem using the fundamental solution. This representation throws light on the finer structure of the solution (smoothness ...). Existence and

uniqueness of the solution of the Cauchy problem are asssumed throughout this chapter. Without any loss of generality, we shall restrict ourselves to linear second-order parabolic equations.

3.2 Motivation

Consider the Cauchy problem for the heat equation:

$$u_t - u_{xx} = 0, \quad x \in \mathcal{R}^n, \quad t > 0, \tag{3.1}$$

$$u(x,0) = \varphi(x), \tag{3.2}$$

where φ is a function which is sufficiently smooth. If we denote the fundamental solution of this problem by $S(x,t;\xi,\tau)$, it can be written as $S(x,t;\xi,\tau) = E(x-\xi,t-\tau)$, where E satisfies the following system

$$E_t - E_{xx} = 0, \quad x \in \mathcal{R}^n, \quad t > 0 \tag{3.3}$$

$$E(x,0) = \delta(x). \tag{3.4}$$

The solution to (3.1)-(3.2) is given by the formula

$$u(x,t) = \int_{\mathcal{R}^n} E(x-\xi,t)\varphi(\xi)d\xi. \tag{3.5}$$

Definition 3.2.1 (Convolution) Let f and g be two functions defined on $\mathcal{R}^n$. The convolution of f and g is the function $f * g$ defined by the formula

$$(f * g)(x) = \int_{\mathcal{R}^n} f(x-\xi)g(\xi)d\xi \tag{3.6}$$

We can thus rewrite (3.5) as

$$u = E * \varphi. \tag{3.7}$$

Now, consider the problem

$$Lu \equiv u_t + a^{ij}u_{ij} + b^i u_i + cu = f, \tag{3.8}$$

$$u(x,0) = \varphi(x), \tag{3.9}$$

with sufficiently smooth data $a^{ij}(x,t)$, $b^i(x,t)$, $c(x,t)$, $\varphi(x,t)$ and $f(x,t)$. Can we obtain an integral representation of the solution of (3.8)-(3.9) similar to (3.5)? We shall give an answer to this question in the next section.

3.3 Integral representation of the solution of Cauchy's problem

The theory we shall develop in this part was initiated by Hadamard (1923). To begin, we recall the following definition given in Chapter 2.

Definition 3.3.1 The *fundamental solution* of the Cauchy problem (3.8)-(3.9) is a distribution $S(x,t;\xi,\tau)$ that satisfies

$$LS = 0, \quad x \in \mathcal{R}^n, \quad t > \tau \tag{3.10}$$

$$S(x,\tau;\xi,\tau) = \delta(x-\xi) \tag{3.11}$$

Another concept playing an important role in the representation of the solution is that of the *radiation kernel.*

Definition 3.3.2 The *radiation kernel* of the Cauchy problem (3.8)-(3.9) is a distribution $R(x,t;\xi,\tau)$ that satisfies

$$L^*R = 0, \quad x \in \mathcal{R}^n, \quad t > \tau, \tag{3.12}$$

$$R(x,\tau;\xi,\tau) = \delta(x-\xi), \tag{3.13}$$

where L^* denotes the adjoint operator to L defined by

$$L^*u = -u_t + (a^{ij}u)_{ij} - (b^i u)_i + cu \tag{3.14}$$

Theorem 3.3.1 S (resp. R) is regular except along the characteristic $t = \tau$ and vanishes for large values of x.

Proof: The proof of the first assertion of the theorem is based on the decomposition of the delta function in plane waves (see Courant-Hilbert (1962)).

The second assertion comes from the uniqueness of the solution to the Cauchy problem. □

Lemma 3.3.1 Let u and v be sufficiently smooth functions, then

$$vLu - uL^*v = \frac{\partial uv}{\partial t} + \frac{\partial P^i}{\partial x^i}, \tag{3.15}$$

where

$$P^i = va^{ij}\frac{\partial u}{\partial x^j} - \frac{(va^{ij})}{\partial x^j}u + b^i uv \tag{3.16}$$

Note that this result holds for any linear differential operator [1] $\Lambda = \sum_{|\alpha|\leq m} a^\alpha D^\alpha$. In this case $\Lambda^* = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}a^\alpha D^\alpha$ and it can be proved (Courant-Hilbert(1962)) that $v\Lambda u - u\Lambda^* v = \nabla P$ (divergence of P), where P is a polynomial of u, v and their derivatives.

Now, we are ready to derive the main result of this chapter. Integrate (3.15) over the slab $\mathcal{R}^n \times [0, \tau - \epsilon]$, where $0 < \epsilon < \tau$. Then use Green's formula in the right-hand side:

$$\begin{aligned}\int_{\mathcal{R}^n\times[0,\tau-\epsilon]}(vLu - uL^*v)dxdt &= \int_{\mathcal{R}^{n-1}\times[0,\tau-\epsilon]}[P^i]_{-\infty}^{+\infty}d\bar{x}^i dt + \int_{t=\tau-\epsilon} uvdx \\ &\quad - \int_{t=0} uvdx, \end{aligned}\tag{3.17}$$

[1] See Chapter 2, for the notations.

where $d\tilde{x}^i = dx^1 \dots dx^{i-1} dx^{i+1} \dots dx^n$.

If u or v vanish for large values of x, (3.17) becomes

$$\int_{\mathcal{R}^n \times [0,\tau-\epsilon]} (vLu - uL^*v) dx dt = \int_{t=\tau-\epsilon} uv dx - \int_{t=0} uv dx. \tag{3.18}$$

If we substitute for v a solution of $L^*v = 0$ and if $Lu = f$, then (3.18) becomes

$$\int_{t=\tau-\epsilon} uv dx = \int_{t=0} uv dx + \int_{\mathcal{R}^n \times [0,\tau-\epsilon]} vf dx dt. \tag{3.19}$$

In particular take $v = R$. Then (3.19) yields

$$\begin{aligned} \int_{\mathcal{R}^n} u(x, \tau-\epsilon) R(x, \tau-\epsilon; \xi, \tau) dx \;\; &= \;\; \int_{\mathcal{R}^n} u(x,0) R(x,0;\xi,\tau-\epsilon) dx \\ &+ \int_{\mathcal{R}^n \times [0,\tau-\epsilon]} R(x,t;\xi,\tau) f(x,t) dx dt. \end{aligned} \tag{3.20}$$

Now, take the limit of (3.20) when $\epsilon \to 0$ and use (3.9) and (3.13)

$$u(\xi,\tau) = \int_{\mathcal{R}^n} R(x,0;\xi,\tau) \varphi(x) dx + \int_{\mathcal{R}^n \times [0,\tau]} R(x,t;\xi,\tau) f(x,t) dx dt. \tag{3.21}$$

By integrating (3.15) over $\mathcal{R}^n \times [\tau_1, \tau]$ and taking $u = S$, $v = R$, we obtain the interesting property

$$S(\xi,\tau;\xi_1,\tau_1) = R(\xi_1,\tau_1;\xi,\tau). \tag{3.22}$$

Thus we can rewrite (3.21) as follows

$$u(x,t) = \int_{\mathcal{R}^n} S(x,t;\xi,0)\varphi(\xi)d\xi + \int_{\mathcal{R}^n\times[0,t]} S(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau. \quad (3.23)$$

Now, let us show that u defined by (3.23) satisfies (3.8) and (3.9).

From (3.23) we have

$$u(x,0) = < \delta(\xi - x), \varphi(\xi) > = \varphi(x).$$

Assume that the data are chosen such that the differentiation under the sign $\int$ is possible. Then

$$Lu = \int_{\mathcal{R}^n} L[S]_{\tau=0}\, \varphi(\xi)d\xi + \int_{\mathcal{R}^n\times[0,t]} L[S]f(\xi,\tau)d\xi d\tau = 0.$$

In summary, we have obtained the following result:

Theorem 3.3.2 If we assume the existence and the uniqueness for the Cauchy problem (3.8)-(3.9), then its solution is given by (3.23).

Note that if a^{ij}, b^i, c are constant then

$$S(x,t;\xi,\tau) = E(x-\xi, t-\tau),$$

where E satisfies

$$\begin{aligned} LE &= 0, \quad x \in \mathcal{R}^n, \quad t > 0, \\ E|_{t=0} &= \delta(x). \end{aligned}$$

If we suppose futher that $f = 0$, then (3.23) reduces to

$$u(x,t) = \int_{R^n} E(x-\xi,t)\varphi(\xi)d\xi \equiv (E * \varphi)(x,t)$$

3.4 Conclusion

In this chapter, we have shown that solving Cauchy's problem for linear operators reduces to the construction of the fundamental solution. That is, we have obtained a formula for the solution of the Cauchy problem in terms of the fundamental solution and the data.

Chapter 4

Solution of the Cauchy Problem for the Black-Scholes Equation using its Symmetries

4.1 Introduction

In this chapter, we implement the methods described in the previous chapters on a linear partial differential equation—the Black-Scholes equation. We solve explicitly an arbitrary initial value problem for this equation which ays a central role in the mathematics of finance (see Ibragimov and Wafo

Soh (1997)). This equation was solved by its inventors for specific initial data (see Black and Scholes(1973)).

4.2 The Black-Scholes equation

In order to grasp the importance of the Black-Scholes equation, we need to get acquainted with some concepts of the mathematics of finance.

4.2.1 What is an option?

An *option* is a contract giving its holder the right to buy or sell an asset, subject to certain conditions, within a specified period of time. The option to buy an asset is known as a *call option* and the option to sell an asset is known as a *put option*. Options may be classified as *American options* and *European options*: An American option is one that can be exercised at any time up to the date the contract expires. A European option is one that can be exercised only on a specified date in the future. There exists also options which are difficult to deal with mathematically (option of an option ...).
The price that is paid for the asset when the option is exercised is called the *exercise price* or *striking price*. The last day on which an option *may* be

exerised is called the *expiration date* or *maturity date*. The word '*may*' in this description implies that for the holder of the option, this contract is a right and not an obligation. The other party to the contract who is known as the *writer*, does have a potential obligation: He must sell the asset if the holder of the option chooses to buy it. Since the option confers on its holder a right with no obligation, it has a value. Moreover, it must be paid for at the time of the opening of the contract. Conversely, the writer of the option must be compensated for the obligation he has assumed. Therefore the main concern of the mathematics of finance is to answer the following questions:

- How much would one pay for the right to exercise an option?
- How can the writer minimise the risk associated with his obligation?

It is obvious that the option price depends on the current date and on the stock price. The stock price in its turn depends in a non-trivial way on the time. This complicates the situation further! Before Black and Scholes (1973), many authors tried to model the evolution of the option price. But all their models led to option prices involving one or more arbitrary parameters (see Black and Scholes (1973)). Black and Scholes were the first to propose a model involving only known parameters.

4.2.2 The model

The work of Black and Scholes (1973) opened a new era in the mathematical modelling of problems in finance. In the derivation of their model, the key assumption is that the stock price changes according to a *Markov process*. This leads to a stochastic differential equation. Under further restrictive assumptions, they obtained a linear evolutionary partial differential equation with variable coefficients

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \quad x > 0, \quad t < t^*, \tag{4.1}$$

where $A \neq 0$, B, C (parameters of the model) are constants. In (4.1), u denotes the option price, x is the stock price, t is the current time and t^* is the maturity date. Using an *ad hoc* method, Black and Scholes (1973) showed that (4.1) reduces to the classical heat equation

$$v_\tau = v_{yy}, \tag{4.2}$$

provided that $D = B - 1/2A^2 \neq 0$. Thanks to this connection, they gave an explicit formula for the solution defined in $-\infty < t \leq t^*$ of the Cauchy problem with specific data at $t = t^*$ (this initial data is called the *profit on exercise*). It was shown recently (Gazizov and Ibragimov (1996)) that the

link between (4.1) and (4.2) is a mere consequence of Lie's result on the group classification of second-order linear partial differential equations with variable coefficients.

For more complicated options, the end condition $u(x,t^*)$ may depend in a non-trivial way on the stock price, x and the maturity date, t^*. Thus it is important to solve (4.1) subject to an arbitrary end condition.

4.3 Symmetries of the Black-Scholes equation

The symmetries of the Black-Scholes equation are given in Gazizov and Ibragimov (1996). Here we carry out the calculations in detail.

Let

$$X = \xi^0(x,t,u)\frac{\partial}{\partial t} + \xi^1(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u},$$

$$X_{(2)} = X + \zeta_0\frac{\partial}{\partial u_t} + \zeta_1\frac{\partial}{\partial u_x} + \zeta_{01}\frac{\partial}{\partial u_{tx}} + \zeta_{00}\frac{\partial}{\partial u_{tt}} + \zeta_{11}\frac{\partial}{\partial u_{xx}}.$$

According to the infinitesimal criterion of invariance (see Chapter 1), the equation (4.1) admits X if an only if

$$X_{(2)}(u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu)\mid_{(4.1)} = 0. \tag{4.3}$$

This equation is equivalent to

$$(\zeta_0 + \frac{1}{2}A^2x^2\zeta_{11} + Bx\zeta_1 - C\eta + A^2x\xi^1 u_{xx} + B\xi^1 u_x) \mid_{(4.1)} = 0, \qquad (4.4)$$

where according to the prolongation formulas (see Chapter 1), the functions ζ_0, ζ_1, ζ_{11} are given by

$$\zeta_0 = \eta_t + (\eta_u - \xi^0_t)u_t - \quad , \qquad (4.5)$$

$$\zeta_1 = \eta_x + (\eta_u - \xi^1_x)u_x - u_t\xi_x \quad . \qquad (4.6)$$

$$\begin{aligned}\zeta_{11} = {} & \eta_{xx} + (2\eta_{xu} - \xi^1_{xx})u_x - \zeta_{xx}u_t + (\eta_{uu} - \xi^1_{xu})u_x^2 \\ & -2\xi^0_{xu}u_tu_x - \xi^1_{uu}u_x^3 - \xi^0_{uu}u_tu_x^2 + (\eta_u - 2\xi^1_x)u_{xx} \\ & -2\xi^0_x u_{tx} - \xi^0_u u_t u_{xx} - 2\xi^0_u u_x u_{tx} - 3\xi^1_u u_x u_{xx}. \end{aligned} \qquad (4.7)$$

Now, substitute (4.5) to (4.7) into (4.4). Then treat t, x, u and its derivatives as " independent functions". We obtain the following overdetermined system of linear partial differential equations

$$\begin{aligned} \xi^0_x &= 0, \\ \xi^0_u &= 0, \\ \xi^0_{uu} &= 0, \\ \xi^1_{uu} &= 0, \end{aligned}$$

$$
\begin{aligned}
-A^2x^2\xi^0_{xu} + 2\xi^1_u &= 0,\\
4B\xi^1_u + A^2x\eta_{uu} - 2A^2x\xi^1_{xu} &= 0,\\
-2Bx^2\xi^0_x - 2Cux\xi^0_u - 2x\xi^0_t + 4x\xi^1_x - A^2x^3\xi^0_{xx} - 4\xi^1 &= 0,\\
2Bx\xi^1_x - 2B\xi^1 - 6Cu\xi^1_u + 2A^2x^2\eta_{xu} - A^2x^2\xi^1_{xx} - 2\xi^1_t &= 0,\\
2Bx^2\eta_x + 2Cux\eta_u - 4C \quad + 4Cu\xi^1 - 2Cx\eta + 2x\eta_t & \\
+A^2x^3\eta_{xx} &= 0.
\end{aligned}
$$

The solution to this system is given by

$$
\begin{aligned}
\xi^0 &= a_1 + 2ta_3 + 2A^2t^2a_5,\\
\xi^1 &= a_2x + (\ln x + Dt)xa_3 + 2A^2tx\ln xa_5 + A^2txa_4,\\
\eta &= 2Ctua_3 + (\ln x - Dt)ua_4 + ((\ln x - Dt)^2 + 2A^2Ct^2 - A^2t)ua_5\\
&\quad +a_6 + \varphi(x,t),
\end{aligned}
$$

where $a_1 \ldots, a_6$ are arbitrary constants of integration, $D = B - A^2/2$ and φ is an arbitrary solution of (4.1). Thus the Lie algebra admitted by (4.1) is spanned by the following operators

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x},\\
X_3 &= 2t\frac{\partial}{\partial t} + (\ln x + Dt)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u},
\end{aligned}
$$

$$
\begin{aligned}
X_4 &= A^2tx\frac{\partial}{\partial x} + (\ln x - Dt)u\frac{\partial}{\partial u}, \\
X_5 &= 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} \\
&\quad + \left((\ln x - Dt)^2 + 2A^2Ct^2 - A^2t\right)u\frac{\partial}{\partial u}, \\
X_6 &= u\frac{\partial}{\partial u}, \quad X_\varphi = \varphi(x,t)\frac{\partial}{\partial u},
\end{aligned}
$$

where $D = B - A^2/2$ and φ is an arbitrary solution of (4.1).

4.4 The fundamental solution of the Black-Scholes equation

Here we find the fundamental solution for the equation (4.1) using the group theoretic approach presented in Chapter 2.

We recall that the fundamental solution to the Cauchy problem associated to (4.1) is defined as follows:

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \quad x > 0, \quad t < t_0, \tag{4.8}$$

$$u\,|_{t=t_0} = \delta(x - x_0), \tag{4.9}$$

where $\delta(x - x_0)$ is the Dirac measure at x_0.

According to the *invariance principle* (see Chapter 2), we first find the sub-

algebra $\mathcal{F}$ of the Lie algebra $\mathcal{L}^6$ spanned by the operators $X_1, \ldots, X_6$, such that this subalgebra leaves invariant the initial condition (4.9). Let

$$X = \xi^0 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} - \eta u \frac{\partial}{\partial u} \in \mathcal{L}^6,$$

where

$$\xi^0 = a_1 + 2ta_3 + 2A^2t^2a_5, \tag{4.10}$$

$$\xi^1 = a_2x + (\ln x + Dt)xa_3 + A^2txa_4 + 2a_5A^2tx\ln x, \tag{4.11}$$

$$\eta = -2Cta_3 - (\ln x - Dt)a_4,$$

$$-((\ln x - Dt)^2 + 2A^2Ct^2 - A^2t)a_5 - a_6, \tag{4.12}$$

$a_1, \ldots, a_6$ are constants.

According to Theorem 2.4.3, X leaves invariant the initial condition (4.9) if and only if

$$\begin{aligned} \xi^0(x, t_0) &= 0, \\ \xi^1(x_0, t_0) &= 0, \\ \eta(x_0, t_0) - \frac{\partial \xi^1}{\partial x}(x_0, t_0) &= 0. \end{aligned}$$

i.e.

$$a_1 = -2t_0a_3 - 2A^2t_0^2a_5, \tag{4.13}$$

$$a_2 = -(\ln x_0 + Dt_0)a_3 - A^2 t_0 a_4 - 2a_5 A^2 t_0 \ln x_0, \tag{4.14}$$

$$-a_6 = (2Ct_0 + 1)a_3 + (\ln x_0 - Dt_0)a_4$$

$$+((\ln x_0 - Dt_0)^2 + 2A^2 t_0^2 + A^2 t_0)a_5. \tag{4.15}$$

Substitute (4.13) to (4.15) into (4.10) to (4 12):

$$\xi^0 = 2(t - t_0)a_3 + 2A^2(t^2 - t_0^2)a_5,$$

$$\xi^1 = (\ln x - \ln x_0 + D(t - t_0))xa_3 + A^2(t - t_0)xa_4$$

$$+2A^2(t \ln x - t_0 \ln x_0)xa_5,$$

$$\eta = -(2C(t - t_0))a_3 - (\ln x - \ln x_0 - D(t - t_0)a_4$$

$$- \left[(\ln x - Dt)^2 - (\ln x_0 - Dt_0)^2 + 2A^2 Ct_0^2 - A^2(t + t_0)\right] a_5.$$

Whence the subspace $\mathcal{F}$ of $\mathcal{L}^6$ is spanned by the following operators

$$Y_1 = 2(t - t_0)\frac{\partial}{\partial t} + (\ln x - \ln x_0 + D(t - t_0))\, x\frac{\partial}{\partial x} + (2C(t - t_0) - 1)u\frac{\partial}{\partial u},$$

$$Y_2 = A^2(t - t_0)x\frac{\partial}{\partial x} + (\ln x - \ln x_0 - D(t - t_0))u\frac{\partial}{\partial u},$$

$$Z = 2A^2(t^2 - t_0^2)\frac{\partial}{\partial t} + 2A^2(t \ln x - t_0 \ln x_0)x\frac{\partial}{\partial x}$$

$$+ \left((\ln x - Dt)^2 - (\ln x_0 - Dt_0)^2 + 2A^2C(t^2 - t_0^2) - A^2(t + t_0)\right) u\frac{\partial}{\partial u}.$$

Since $\mathcal{F}$ is a vector space, we can replace Z by

$$Y_3 = Z - 2A^2 t_0 Y_1 + 2Dt_0 Y_2,$$

i.e.

$$\begin{aligned} Y_3 &= 2A^2(t-t_0)^2\frac{\partial}{\partial t} + 2A^2(t-t_0)x\ln x\frac{\partial}{\partial x} \\ &\quad + \left(\ln x - D(t-t_0)^2 - \ln^2 x_0 + 2A^2C(t-t_0)^2 - A^2(t-t_0)\right)u\frac{\partial}{\partial u}. \end{aligned}$$

Invariants of $\mathcal{F} = < Y_1, Y_2, Y_3 >$ are defined by the system

$$Y_1 I = 0, \quad Y_2 I = 0, \quad Y_3 I = 0.$$

Since

$$Y_3 = A^2(t-t_0)Y_1 + (\ln x - \ln x_0 - D(t-t_0))Y_2,$$

it suffices to solve only the first two equations. We obtain

$$I = f(ux^{\sigma(t)}\sqrt{t_0 - t}e^{\omega(x,t)}), \tag{4.16}$$

where

$$\sigma(t) = \frac{D}{A^2} - \frac{\ln x_0}{A^2(t_0-t)},$$

$$\omega(x,t) = \frac{\ln^2 x + \ln^2 x_0}{2A^2(t_0-t)} + \left(\frac{D^2}{2A^2} + C\right)(t_0 - t).$$

The invariant solution is therefore given by $I =$ constant and hence has the form

$$u = K\frac{x^{-\sigma(t)}}{\sqrt{t_0 - t}}e^{-\omega(t)}, \tag{4.17}$$

where σ, ω are defined above and K is a constant to be found from the initial condition (4.9). We will use the well known limit [1]

$$\lim_{s\to 0^+} \frac{1}{\sqrt{s}} \exp(-\frac{(x-x_0)^2}{4s}) = 2\sqrt{\pi}\delta(x-x_0), \tag{4.18}$$

and the formula of change of the variable $z = z(x)$ in the Dirac measure (see Chapter 2 or Courant-Hilbert (1962), p.790):

$$\delta(x-x_0) = \mid \frac{\partial z}{\partial x} \mid_{x=x_0} \delta(z-z_0). \tag{4.19}$$

For the function (4.17), we have

$$\begin{aligned}\lim_{t\to t_0^-} u &= \lim_{t\to t_0^-} \frac{K}{\sqrt{t_0-t}} e^{-\omega(x,t)-\sigma(t)\ln x} \\ &= \lim_{t\to t_0^-} \frac{K}{\sqrt{t_0-t}} \exp\left(-\frac{(\ln x - \ln x_0)^2}{2A^2(t_0-t)} - \frac{D\ln x}{A^2}\right),\end{aligned}$$

or setting $s = t_0 - t$, $z = \frac{\sqrt{2}\ln x}{A}$,

$$\begin{aligned}\lim_{t\to t_0^-} u &= K\exp(-\frac{D\ln x}{A^2}) \lim_{s\to 0^+} \frac{1}{\sqrt{s}} \exp\left(-\frac{(z-z_0)^2}{4s}\right) \\ &= 2\sqrt{\pi}K\exp\left(-\frac{D\ln x}{A^2}\right)\delta(z-z_0),\end{aligned}$$

[1] In the sense of distributions.

where $x_0 = z(x_0)$.

By virtue of (4.19),

$$\delta(z - z_0) = \frac{Ax_0}{\sqrt{2}}\delta(x - x_0),$$

and hence

$$\lim_{t \to t_0^-} u = \sqrt{2\pi} K A x_0 \exp\left(-\frac{D \ln x}{A^2}\right) \delta(x - x_0).$$

Therefore, the initial condition (4.9) yields

$$K = \frac{1}{Ax_0\sqrt{2\pi}} \exp\left(\frac{D \ln x_0}{A^2}\right).$$

Thus , we arrive at the following fundamental solution of the Cauchy problem for the equation (4.1):

$$u = \frac{1}{\sqrt{2\pi} A x_0 \sqrt{t_0 - t}} \exp\left[-\frac{(\ln x - \ln x_0)^2}{2A^2(t_0 - t)} - \left(\frac{D^2}{2A^2} + C\right)(t_0 - t) - \frac{D}{A^2}(\ln x - \ln x_0)\right] \tag{4.20}$$

This fundamendal solution was derived in Gazizov and Ibragimov (1996).

4.5 The solution to the Cauchy problem

In this part, we solve the Cauchy problem for the Black-Scholes equation. Consider the Cauchy problem

$$u_t + \frac{1}{2}A^2 x^2 u_{xx} + Bxu_x - Cu = 0, \quad x > 0, \quad t < t^*, \tag{4.21}$$

$$u\mid_{t=t^*}=\varphi. \tag{4.22}$$

Theorem 4.5.1 If φ is such that any integral involving it is finite [2], then (4.21)-(4.22) has a unique classical solution given by

$$u(x,t)=\int_0^{+\infty} S(x,t;\xi,t^*)\varphi(\xi)d\xi, \tag{4.23}$$

where

$$S(x,t;\xi,t^*)=\frac{1}{\sqrt{2\pi}A\xi\sqrt{t_*-t}}\exp\left[-\frac{(\ln x-\ln\xi)^2}{2A^2(t_*-t)}-\left(\frac{D^2}{2A^2}+C\right)(t_*-t)\right.$$
$$\left.-\frac{D}{A^2}(\ln x-\ln\xi)\right]$$

Proof: Apply Hadamard's method (see Chapter 3) with

$$\begin{aligned} L &= \frac{\partial}{\partial t}+\frac{1}{2}A^2x^2\frac{\partial^2}{\partial x^2}+Bx\frac{\partial}{\partial x}-C,\\ L^* &= -\frac{\partial}{\partial t}+\frac{1}{2}A^2\frac{\partial^2}{\partial x^2}(x^2.)-B\frac{\partial}{\partial x}(x.)-C. \end{aligned}$$

□

[2]This is satisfied if for instance φ is a function with a polynomial growth.

Proposition 4.5.1 The equation (4.23) generalises the Black-Scholes formula. We recall that the Black-Scholes formula is the solution to (4.21) with

$$\varphi(x) = \max(x - E, 0), \quad E > 0, \quad B = C.$$

Proof:

$$\begin{aligned} u(x,t) &= \int_0^{+\infty} S(x,t;\xi,t^*)\max(\xi - E, 0)d\xi \\ &= \int_E^{+\infty} \xi S(x,t;\xi,t^* d\xi - E\int_E^{+\infty} S(x,t;\xi,t^*)d\xi. \end{aligned} \tag{4.24}$$

Since $B = C$,

$$\begin{aligned} &\int_E^{+\infty} \xi S(x,t;\xi,t^*)d\xi \\ = \; &\frac{1}{A\sqrt{2\pi(t^*-t)}}\int_E^{+\infty} \exp\left[-\frac{1}{2A^2(t^*-t)}\{\ln x - \ln\xi \right. \\ &\left. +(B+\frac{1}{2}A^2)(t^*-t)\}^2 + \ln x - \ln\xi\right] d\xi. \end{aligned}$$

i.e.

$$\begin{aligned} &\int_E^{+\infty} \xi S(x,t;\xi,t^*)d\xi \\ = \; &\frac{x}{A\sqrt{2\pi(t^*-t)}}\int_E^{+\infty} \frac{1}{\xi}\exp\left[-\frac{1}{2A^2(t^*-t)}\{\ln x - \ln\xi \right. \\ &\left. +(B+\frac{1}{2}A^2)(t^*-t)\}^2\right] d\xi. \end{aligned}$$

Make the change of variable

$$\eta = \frac{1}{A\sqrt{t^* - t}}(\ln x - \ln \xi + (B + \frac{1}{2}A^2)(t^* - t)).$$

Thus

$$\int_E^{+\infty} \xi S(x, t; \xi, t^*) d\xi = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp(-\frac{1}{2}\eta^2) d\eta, \tag{4.25}$$

where

$$d_1 = \frac{\ln(x/E) + (B + 1/2A^2)(t^* - t)}{A\sqrt{t^* - t}}. \tag{4.26}$$

$$\begin{aligned} &\int_E^{+\infty} S(x, t; \xi, t^*) d\xi \\ = \; &\frac{e^{-B(t^*-t)}}{A\sqrt{2\pi(t^* - t)}} \int_E^{+\infty} \frac{1}{\xi} \exp\left[-\frac{1}{2A^2(t^* - t)}\{\ln x - \ln \xi \right. \\ &\left. + (B - \frac{1}{2}A^2)(t^* - t)\}^2\right] d\xi. \end{aligned}$$

Make the change of variable

$$\eta = \frac{1}{A\sqrt{t^* - t}}(\ln x - \ln \xi + \left(B - \frac{1}{2}A^2\right)(t^* - t)).$$

Thus

$$\int_E^{+\infty} S(x, t; \xi, t^*) d\xi = \frac{e^{-B(t^*-t)}}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \exp(-\frac{1}{2}\eta^2) d\eta, \tag{4.27}$$

where

$$d_2 = \frac{\ln(x/E) + (B - 1/2A^2)(t^* - t)}{A\sqrt{t^* - t}}. \tag{4.28}$$

Now, substitute (4.25) and (4.27) into (4.24):

$$u(x,t) = xN(d_1) - Ee^{-B(t^*-t)}N(d_2), \tag{4.29}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-1/2\eta^2} d\eta$$

and d_1, d_2 are given by (4.26) and (4.28). Equation (4.29) is the well-known Black-Scholes formula.□

4.6 Conclusion

For more complicated options (e.g. option of an option, exotic options ...), the end condition $\varphi(x)$, also known as the *profit on exercise* may depend in a non trivial way on the stock price x and the maturity date t^*. Thus the Black-Scholes formula becomes useless in such a situation and the only alternative is to use numerical methods to solve the Cauchy problem. Hence, our formula (4.23) may be used to deal with such options or to test the validity of numerical methods.

Summary

In this work we have combined Lie group methods and Hadamard's method to solve explicitly Cauchy's problem for linear differential operators. Lie symmetry analysis has intervened mainly in the construction of the fundamental solution. Hadamard's method has enabled us to represent explicitly the solution to the Cauchy problem as a linear functional of the initial data. The two methods aforesaid have been combined to solve explicitly an arbitrary initial value problem for the Black-Scholes equation. Our solution generalises the well-known Black-Scholes formula and can be used to handle more complicated options in the mathematics of finance.

The above is a basic summary of our study. There still remains open questions: Are fundamental solutions invariant solutions? How useful is our integral formula compared to numerical methods? The first question is related

to the problem of contructing all equations whose solutions are invariant.

We shall deal with the questions mentioned above in a future work.

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