The definition of $|\Psi_H\rangle$, equation 2.15, may be used to rewrite the arbitrary matrix element given by 2.5 in the Heisenberg picture, as

$$\langle\Psi'_S(t)|O_S|\Psi_S(t)\rangle = \langle\Psi'_H|e^{iHt/\hbar}O_S e^{-iHt/\hbar}|\Psi_H\rangle, \tag{2.18}$$

implying:

$$O_H = e^{iHt/\hbar}O_S e^{-iHt/\hbar}. \tag{2.19}$$

Taking the derivative of equation 2.19 with respect to time, the equation of motion for $O_H(t)$ is obtained:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}O_H(t) &= e^{iHt/\hbar}[O_S, H]e^{-iHt/\hbar} \\ &= [O_H(t), H], \end{aligned} \tag{2.20}$$

and the time dependence of Heisenberg operators is demonstrated.

2.1.2 The Time Development Operator

The unitary operator $U(t,t_0)$ defined by

$$|\Psi_I(t)\rangle = U(t,t_0)|\Psi_I(t_0)\rangle, \tag{2.21}$$

describes the evolution in time of $\Psi_I(t)$.

Comparison of equations 2.7 and 2.21 yields the formal definition:

$$U(t,t_0) = e^{iH_0t/\hbar}e^{-iH(t-t_0)/\hbar}e^{-iH_0t_0/\hbar} \quad t \text{ finite}. \tag{2.22}$$

However a more practically useful equation is obtained from equations 2.11 and 2.21.

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_1(t')U(t',t_0). \tag{2.23}$$

Equation 2.23 is found, on iterating, to be equivalent to the following expansion

$$U(t,t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_1(t_1)H_1(t_2)\cdots H_1(t_n), \tag{2.24}$$

where the $n=0$ term is the unit operator. The $n=2$ term in the expansion can be rewritten as follows:

$$\begin{aligned} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_1(t_1)H_1(t_2) &= \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_1(t_1)H_1(t_2) + \frac{1}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1\, H_1(t_1)H_1(t_2) \\ &\quad - \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 [H_1(t_1)H_1(t_2)\theta(t_1 - t_2) + H_1(t_2)H_1(t_1)\theta(t_2 - t_1)] \\ &\quad = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T[H_1(t_1)H_1(t_2)] \end{aligned} \tag{2.25}$$

where the definition of the T product for bosonic operators has been used. The generalised form of equation 2.25 may be used to rewrite equation 2.24 in the form

$$U(t,t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, T[H_1(t_1)H_1(t_2)\cdots H_1(t_n)] \tag{2.26}$$

Equation 2.26 has the formal solution:

$$U(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_1(t')\right] \tag{2.27}$$

The Interaction Picture

The state vector $|\Psi_I(t)\rangle$ in the interaction picture is related by a unitary transformation to that in the Schrödinger picture $|\Psi_S(t)\rangle$.

$$|\Psi_I(t)\rangle = e^{iH_0 t/\hbar}|\Psi_S(t)\rangle \tag{2.6}$$

$$= e^{iH_0 t/\hbar}e^{-iHt/\hbar}e^{iH_0 t/\hbar}|\Psi_I(t_0)\rangle \quad t \text{ finite}. \tag{2.7}$$

where H_0 is the Hamiltonian of some soluble system such that the total Hamiltonian of the system under consideration, H, may be written as

$$H = H_0 + H_1. \tag{2.8}$$

H_0 is usually chosen describe the noninteracting system, and H_1 usually represents the contribution due to the interaction between particles in the system.

Differentiating equation 2.6 with respect to time yields the equation of motion

$$i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = H_I(t)|\Psi_I(t)\rangle. \tag{2.9}$$

where

$$H_I(t) = e^{iH_0 t/\hbar}H_1 e^{-iH_0 t/\hbar} \tag{2.10}$$

In fact, the name of this picture reflects the relation between the time dependence of the interaction state vector and $H_I(t)$.

The differential equation 2.9 is equivalent to the following integral equation

$$|\Psi_I(t_1)\rangle = |\Psi_I(t_0)\rangle - i\int_{t_0}^{t} dt' H_I(t')|\Psi_I(t')\rangle \tag{2.11}$$

Equations 2.4, 2.5, and 2.7 imply

$$\langle\Psi_S'(t)|O_S|\Psi_S(t)\rangle = \langle\Psi_I'(t)|e^{iH_0 t/\hbar}O_S e^{-iH_0 t/\hbar}|\Psi_I(t)\rangle. \tag{2.12}$$

from which the following definition is obtained

$$O_I(t) = e^{iH_0 t/\hbar}O_S e^{-iH_0 t/\hbar}. \tag{2.13}$$

Definition 2.13 had the implication

$$i\hbar\frac{\partial}{\partial t}O_I(t) = [O_I(t), H_0] \tag{2.14}$$

It is evident that in the interaction picture, both operators and state vectors have a dependence on time.

The Heisenberg Picture

As is the case in the interaction picture, the Heisenberg state vector, $|\Psi_H\rangle$, may be defined in terms of $|\Psi_S(t)\rangle$.

$$|\Psi_H\rangle = e^{iHt/\hbar}|\Psi_S(t)\rangle. \tag{2.15}$$

By differentiating equation 2.15 and making use of equation 2.3 the time independence of $|\Psi_H\rangle$ is established i.e.

$$i\hbar\frac{\partial}{\partial t}|\Psi_H\rangle = 0 \tag{2.16}$$

From equation 2.2, it becomes apparent that the Heisenberg eigenstates are stationary solutions to the Schrödinger equation, and as such they have a definite energy i.e.

$$H|\Psi_H\rangle = E|\Psi_H\rangle. \tag{2.17}$$

and they are the exact eigenstates of the interacting system.

Chapter 2

Operator Representation of Quantum Field Theory

2.1 Preliminaries

2.1.1 The Three "Pictures" in Quantum Mechanics

Quantum mechanics in the operator representation is generally formulated in either the Schrödinger, Heisenberg or interaction "picture". In the Schrödinger picture all time dependence resides in the state vectors. At the other end of the spectrum, in the Heisenberg picture it is the operators that are time dependent. In between these extremes lies the interaction picture, in which both operators and state vectors vary with time.

Since the three pictures are merely different ways of representing the same physical system, observables calculated in the different pictures must have the same values i.e.

$$\langle \Psi_S(t)|O_S|\Psi'_S(t)\rangle = \langle \Psi_I(t)|U_I(t)|O_I(t)|\Psi'_I(t)\rangle = \langle \Psi_H|O_H(t)|\Psi'_H\rangle \tag{2.1}$$

Furthermore, if it is required that the three pictures be identical at $t = 0$, then the following must hold:

$$\begin{aligned} |\Psi_H\rangle &= |\Psi_I(0)\rangle = |\Psi_S(0)\rangle \\ O_H(0) &= O_I(0) = O_S \end{aligned} \tag{2.2}$$

The choice of "picture" is determined by the methods or techniques used, as certain operations are more easily carried out in one representation than in the other two. For instance, time dependent perturbation theory is most conveniently formulated in the interaction picture. However, the Green's function is most simply defined in the Heisenberg picture. Hence, in order to apply time-dependent perturbation theory to the Green's function, a relation between the state vectors and operators in the Heisenberg and interaction pictures must be found.

The Schrödinger Picture

The time dependence of the state vectors in the Schrödinger picture is described by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = H|\Psi_S(t)\rangle, \tag{2.3}$$

which has the formal solution

$$|\Psi_S(t)\rangle = e^{-iHt/\hbar}|\Psi_S(0)\rangle. \tag{2.4}$$

Operators in this picture are time-independent and the matrix element of an arbitrary operator O_S is represented as

$$\langle \Psi'_S(t)|O_S|\Psi_S(t)\rangle. \tag{2.5}$$

Chapter 1

Introduction

The path integral method of quantization [2, 3, 4] is an extremely elegant and useful tool used in field theory, which can be applied to the typical multi-fermion systems considered in condensed matter physics [5].

In order to correctly take the statistics obeyed by fermions into account, the Fermi fields appearing in the path integral must be Grassmann (anticommuting) in nature [2]. Unfortunately, Grassmann variables are usually awkward, if not technically difficult to work with.

Bosonization involves changing to a set of commuting variables, which have the same quantum states and matrix elements as the original theory [6]. The main advantage of bosonization, and the reason it was first introduced, is that these variables provide a more natural basis for describing collective excitations. In the path integral approach these variables have the added advantage of being easier to work with than Grassmann variables.

Most of the attempts, thus far, to bosonize fermionic theories have used a transformation to local variables in $1+1$ dimensions, for example [7, 8, 9, 10, 11, 12, 13] etc.. Local bosonization in $1+1$ dimensions has already yielded interesting results, including establishing the equivalence of seemingly unrelated theories and the solution of several interacting field theories [14].

However, the two-body interaction prevalent in condensed matter theory is nonlocal in nature, and as yet there is no completely generalized extension of local bosonization to higher dimensions[1]. For this reason, the bilocal bosonization formalism valid in any dimension d developed recently [1] will be used in this dissertation.

This bosonization method is an exact rewriting of fermionic vector models in terms of bilocal time-ordered invariants. It was originally considered by Clavelli *et al.* [16], however the nontrivial Jacobian [17], introduced into the path integral by the change of variables, has only recently been calculated exactly [1].

Since it was developed in the context of the large N expansion, this approach has, thus far, only been applied relativistic theories. This bilocal bosonization is particularly useful, as it leads naturally to a $\frac{1}{N}$ expansion, with N playing the role of an inverse coupling constant.

It is the aim of this research project to investigate the applicability of bilocal bosonization to condensed matter multi-electron systems, despite the fact that for these systems $N = 1$.

In particular, the effective theory in the bilocal fields of a general nonrelativistic multi-electron system with a nonlocal potential will be obtained. It will then be investigated how the stationary configuration of the effective theory is related to more traditional second quantized approaches used for the single particle Green's function, as, for example, [18]. It will also be demonstrated how certain standard approximations, such as the ladder and the ring diagram approximations, may be obtained from specific forms of the effective action. In addition, it will be shown that the effective action can be used to generate, as corrections, the sets of diagrams neglected in the leading approximation.

[1] Although recently some progress has been made in this direction [15].

Conventions and Definitions

The convenient four dimensional notation has been used. For example, if x is a space-time variable, then

$$x = (x_0, \vec{x}) = (t, \vec{x})$$
$$\int d^4x = \int d^3x \int dt,$$

where the arrow denotes a three-dimensional vector. In momentum space,

$$k = (k_0, \vec{k}) = (\omega, \vec{k})$$
$$\int d^4k = \int d^3k \int d\omega$$
$$k \cdot x = \vec{k} \cdot \vec{x} - \omega t$$

Spin indices are labeled by Greek letters, and for fermions, can take either of the two values $\pm\frac{1}{2}$. The Einstein convention is followed, i.e. repeated spin indices imply summation.

The Fourier Transform of $F(x)$ is defined to be

$$F(x) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} F(k)$$
$$= \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\vec{k}\cdot\vec{x}} F(k),$$

and the inverse Fourier transform is defined to be

$$F(k) = \int_{-\infty}^{\infty} d^4x\, e^{ik\cdot x} F(x).$$

The function $\delta^4(x - x')$ denotes

$$\delta^4(x - x') = \delta(\vec{x} - \vec{x}')\delta(t - t')$$
$$= \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x - x')}$$
$$\delta^4(p) = \delta(\vec{p})\delta(\omega).$$

Operators are denoted by a caret, and the commutator of two operators, A and B, is defined by

$$[A, B] = AB - BA,$$

and the anticommutator by

$$\{A, B\} = AB + BA.$$

Contents

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To my parents

Diana Claire and Gerald Derek de Wet

Whose never failing belief in me
has been the mainstay of my life.

Declaration

I declare that this dissertation is my own unaided work. It is submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination in any other university.

(Juliet Diana Braemhoeg)

13 day of December, 1997.

Abstract

The bilocal bosonization technique developed in [1] is applied to nonrelativistic multi-fermion systems. It is found that certain traditional condensed matter approximations, in particular the ring diagram approximation, correspond to exact rewritings of the effective action. The bilocal bosonization technique, although developed in connection with the large N expansion, is found to be useful in explaining the origin of these approximations, and in generating higher order corrections systematically.

Bilocal Bosonization of Nonrelativistic Fermions in d Dimensions

Juliet Diana Braemhoej

A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

Johannesburg, 1997.

Figure 2.4: Form of the integral equation for U_{ab}.

2.4.3 Polarization Insertion

The exact effective interaction between two particles, U_{ab}, may always be decomposed into a series of connected diagrams. The lowest order term is $U_{\alpha\beta;\gamma\delta}$, which is defined in equation 2.80, and all the other terms are connected at both ends to a bare interaction. The sum of the parts of each diagram sandwiched between the end U's is defined to be the polarization insertion, Π. Hence, U_{ab} may be written as an integral equation in terms of U and Π. The structure of this equation is depicted in figure 2.4.

This equation is an algebraic equation in momentum space for a uniform system, and if in addition the particle-particle interaction is spin independent, the complicated matrix structure disappears, i.e.

$$U_{ab}(q) = U(q) + U(q)\Pi_{ab;cd}(q)U(q). \tag{2.95}$$

Again, the diagrams contributing to Π may be subdivided into two classes: those that become two separate diagrams when one *interaction* line is cut, and those that do not. The sum of the latter class of diagrams is defined to be the proper polarization insertion, denoted by Π^*. In terms of Π^* equation 2.95 becomes:

$$U_{ab}(q) = U(q) + U(q)\Pi^*_{ab;cd}(q)U_{cd}(q), \tag{2.96}$$

which is Dyson's equation for U_{ab}, and is solved by

$$U_{ab}(q) = \frac{U(q)}{1 - \Pi^*_{ab;cd}(q)U(q)}. \tag{2.97}$$

Clearly, U_{ab} can also be obtained from G^2U. If the system is uniform and the interaction spin independent, then, provided the approximation to the Bethe-Salpeter equation is such that V in equation 2.93 is the bare interaction potential U, M is essentially U_{ab}. This will be examined in more detail in the context of the ring diagram approximation in section 5.2.3.

If the interaction is invariant under translations, and the system is spatially uniform, it is possible to move into momentum space, using the four-dimensional Fourier transforms of G, $G^{(0)}$ and Σ^*. Therefore in momentum space Dyson's equation becomes

$$G_{\alpha\beta}(k) = G^{(0)}_{\alpha\beta}(k) + G^{(0)}_{\alpha\lambda}(k)\Sigma^*_{\lambda\mu}(k)G_{\mu\beta}(k). \tag{2.88}$$

For all cases under consideration here, the interaction is spin independent, and G, $G^{(0)}$, and Σ^* are all diagonal in the spin indices. As a result, the spin labels may be factored out of equation 2.88, and the following useful equation obtained:

$$G(k) = \frac{1}{G^{(0)-1}(k) - \Sigma^*(k)} \tag{2.89}$$

In the case of fermions, where $G^{(0)}$ is the free fermion propagator, given by equation 2.78,

$$G^{(0)-1}(k) = (\omega - \omega'_{\vec{k}}), \tag{2.90}$$

where the denominator is understood to contain the $i\eta\,\mathrm{sgn}(|k| - k_f)$ term, if necessary. Equation 2.90 may be substituted into equation 2.89, yielding

$$G(k) = \frac{1}{(\omega - \omega'_{\vec{k}}) - \Sigma^*(k)} \tag{2.91}$$

2.4 The Two-Particle Green's Function

2.4.1 Definition

The two particle Green's function is defined in analogy with equation 2.62 to be given by

$$iG^{(2)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1, x_2, x_3, x_4) = \frac{\langle\Psi_0|T[\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\psi^\dagger_{\alpha_3}(x_3)\psi^\dagger_{\alpha_4}(x_4)]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}. \tag{2.92}$$

The two particle Green's function is used in condensed matter physics to calculate, for example, the retarded density correlation function, which is used in linear response theory.

2.4.2 The Bethe-Salpeter Equation

Each diagram in the expansion of $G^{(2)}$ has four external propagators. The four point function, or two body scattering amplitude, M, is defined to be the sum of these diagrams, without their external propagators. The "insertions" contributing to M may be divided into two particle reducible and irreducible diagrams. The reducible diagrams are those that become two disconnected diagrams by cutting two particle lines, and the irreducible, those that do not separate in this way. If V is defined to be the sum of all two particle irreducible diagrams, then the two point function obeys a linear two-particle equation in terms of V, with integration over the four momentum of the internal particle pair, i.e. M is given by a Bethe Salpeter equation:

$$M_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2; P) = V_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2; P) + \int\frac{d^4k}{(2\pi)^4} V_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, k; P)G^{(2)}_{\alpha_3\alpha_4\alpha_5\alpha_6}(k, P)$$
$$\times M_{\alpha_5\alpha_6\alpha_3\alpha_4}(k, p_2; P) \tag{2.93}$$

where p_1 is the relative four momentum of the incoming particle pair, p_2 is the relative four momentum of the outgoing particle pair, and P is the total momentum of the system. V is the kernel of the equation and $G^{(2)}$ the full two-particle propagator, is given by

$$G^{(2)}(k, P) = G^{(0)}(\tfrac{1}{2}P + k)G^{(0)}(\tfrac{1}{2}P - k) \tag{2.94}$$

where $G^{(0)} = G$ is the full interacting, one particle Green's function defined by equation 2.62.

However, like Σ^*, it is in general practically not possible to calculate V, and some approximation is usually made for V which yields a corresponding approximation for M, and ultimately $G^{(2)}$. The approximations most commonly used involve replacing V with the first few terms of its perturbation expansion.

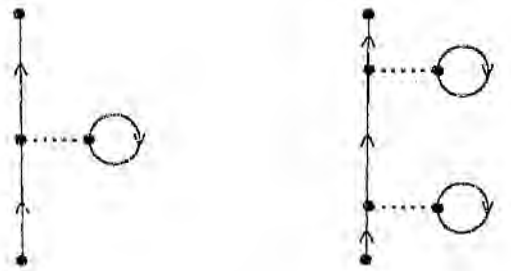

Figure 2.2: (a) A proper self-energy insertion. (b) An improper self-energy insertion.

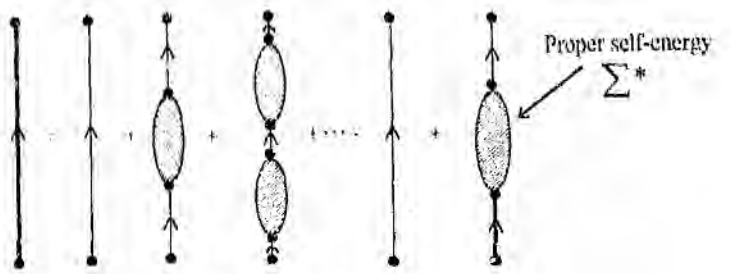

Figure 2.3: Expansion of G in Dyson's equation.

The self energy may itself be re written in terms of the proper self-energy, Σ^*, which is defined as the sum of all proper self-energy insertions.

$$\begin{aligned}\Sigma_{\alpha\beta}(x_1,x_1') &= \Sigma^*_{\alpha\beta}(x_1,x_1') + \int d^4x_2 \int d^4x_2' \Sigma^*_{\alpha\gamma}(x_1,x_2) G^0_{\gamma\rho}(x_2,x_2') \Sigma^*_{\rho\beta}(x_2',x_1') \\ &+ \int d^4x_2 \int d^4x_2' \int d^4x_3 \int d^4x_3' \Sigma^*_{\alpha\gamma}(x_1,x_2) G^0_{\gamma\rho}(x_2,x_2') \Sigma^*_{\rho\sigma}(x_2',x_3) \\ &\times G^0_{\sigma\tau}(x_3,x_3') \Sigma^*_{\tau\beta}(x_3',x_1') + \ldots \\ \Sigma^*_{\alpha\beta}(x_1,x_1') &= \int d^4x_2 \int d^4x_2' \Sigma^*_{\alpha\gamma}(x_1,x_2) G^0_{\gamma\rho}(x_2,x_2') \Sigma_{\rho\beta}(x_2',x_1') \end{aligned} \quad (2.86)$$

Dyson's equation [28, 29] is obtained when equation 2.86 is inserted into 2.85

$$\begin{aligned} G_{\alpha\beta}(x,y) &= G^0_{\alpha\beta}(x,y) + \int d^4x_1 \int d^4x_1' G^0_{\alpha\lambda}(x,x_1') \left[\Sigma^*_{\lambda\mu}(x_1,x_1') \right. \\ &\left. + \int d^4x_2 \int d^4x_2' \Sigma^*_{\lambda\nu}(x_1,x_2) G^0_{\nu\rho}(x_2,x_2') \Sigma_{\rho\mu}(x_2',x_1') \right] G_{\mu\beta}(x_1',y) \\ G_{\alpha\beta}(x,y) &= \int d^4x_1 \int d^4x_1' G^0_{\alpha\lambda}(x,x_1') \Sigma^*_{\lambda\mu}(x_1,x_1') G_{\mu\beta}(x_1',y) \end{aligned} \quad (2.87)$$

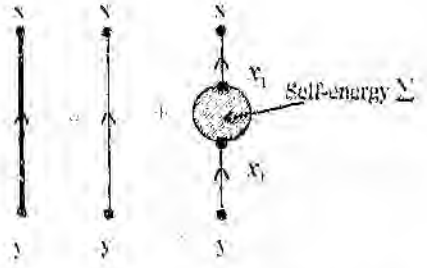

Figure 2.1: Graphical depiction of the structure of $G_{\alpha\beta}(x,y)$.

then equation 2.81, written in terms of G yields[1]

$$iG_{\alpha\beta}(x,y) = iG^{(0)}_{\alpha\beta}(x,y) - \frac{1}{2\hbar}\sum_{\substack{\lambda\lambda'\\ \mu\nu}}\int d^4x_1\int d^4x_1' V_{\lambda'\nu\mu\nu}(x_1,x_1')$$
$$\times\langle\Phi_0|T\left[\psi^\dagger_\lambda(x_1)\psi^\dagger_\mu(x_1')\psi_{\mu'}(x_1')\psi_{\nu'}(x_1)\psi_\alpha(x)\psi^\dagger_\beta(y)\right]|\Phi_0\rangle + \cdots. \quad (2.83)$$

From equation 2.83 it is apparent that each term in the expansion is the noninteracting ground-state expectation value of a T-product of interaction picture field operators. Hence, Wick's theorem can be used to reduce the first order term in equation 2.83 to

$$iG^{(1)}_{\alpha\beta}(x,y) = \frac{i}{2\hbar}\sum_{\substack{\lambda\lambda'\\ \mu\nu}}\int d^4x_1\int d^4x_1' V_{\lambda'\nu\mu\nu}(x_1,x_1')\left[iG^{(0)}_{\alpha\mu}(x,y)[iG^{(0)}_{\mu'\mu}(x_1',x_1')iG^{(0)}_{\lambda'\lambda}(x_1,x_1')\right.$$
$$iG^{(0)}_{\mu'\lambda}(x_1',x_1)iG^{(0)}_{\nu'\nu}(x_1,x_1')\} + iG^{(0)}_{\alpha\lambda}(x,x_1)\{iG^{(0)}_{\nu'\mu}(x_1,x_1')iG^{(0)}_{\mu'\nu}(x_1',y)$$
$$iG^{(0)}_{\lambda'\nu}(x_1,y)iG^{(0)}_{\mu'\mu}(x_1',x_1')\} + iG^{(0)}_{\alpha\mu}(x,x_1')\{iG^{(0)}_{\mu'\lambda}(x_1',x_1)iG^{(0)}_{\lambda'\nu}(x_1,y)$$
$$\left.iG^{(0)}_{\mu'\nu}(x_1',y)iG^{(0)}_{\nu'\lambda}(x_1,x_1)\}\right] \quad (2.84)$$

The higher order corrections may be calculated in likewise manner.

2.3.4 Dyson's Equations

In the perturbation expansion of G in section 2.3.3 it is found that each Feynman diagram has two external propagators or particle lines, both of which are *free propagators*. The part of the Feynman diagram left after these external propagators are removed is called a self-energy insertion. Hence, G may be written in terms of the self-energy Σ, the sum of self-energy insertions, and the free propagator, G^0, as follows:

$$G_{\alpha\beta}(x,y) = G^{(0)}_{\alpha\beta}(x,y) + \int d^4x_1\int d^4x_1' G^{(0)}_{\alpha\lambda}(x,x_1)\Sigma_{\lambda\mu}(x_1,x_1')G^{(0)}_{\mu\beta}(x_1',y). \quad (2.85)$$

The Feynman diagrams representing equation 2.85 are shown in figure 2.1.

There are two types of self-energy insertions, those that consist of two other self-energy insertions connected by a single particle line and those that cannot be decomposed into simpler self-energy insertions. The latter type of self-energy insertions are known as *proper* self-energy insertions. Examples of both types of self-energy insertions are contained in figure 2.2.

[1] Remember

$$iG^{(0)}_{\alpha\beta}(x,y) = \langle\Phi_0|T\left[\psi^\dagger_\alpha(x)\psi^\dagger_\beta(y)\right]|\Phi_0\rangle$$

is the single particle Green's function for the system described by H_0.

The integral representation of the step function

$$\theta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega(t-t')}}{\omega + i\eta} \qquad \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega(t-t')}}{\omega - i\eta} \tag{2.74}$$

may be used to rewrite equation 2.73 as

$$iG^0_{\alpha,\alpha}(x,x') = \delta_{\alpha,\alpha} \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi i} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} \left[\frac{e^{i\omega(t-t')}\theta(|k|-k_F)}{\omega - i\eta} + \frac{e^{i\omega(t'-t)}\theta(k_F-|k|)}{\omega - i\eta} \right]$$
$$= \delta_{\alpha,\alpha} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left[\frac{\theta(|k|-k_F)}{\omega - \omega_{\vec{k}} + i\eta} + \frac{\theta(k_F-|k|)}{\omega - \omega_{\vec{k}} - i\eta} \right]. \tag{2.75}$$

i.e.

$$iG^0_{\alpha,\alpha}(x,x') = \delta_{\alpha,\alpha} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \frac{1}{\omega - \omega_{\vec{k}} + i\eta\,\mathrm{sign}(|k|-k_F)}. \tag{2.76}$$

Comparison of the four-dimensional Fourier transform of iG^0,

$$G^0_{\alpha,\alpha}(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} G^0_{\alpha,\alpha}(k) \tag{2.77}$$

with equation 2.76 yields

$$G^0_{\alpha,\alpha}(k) = \frac{1}{\omega - \omega_{\vec{k}} + i\eta\,\mathrm{sign}(|k|-k_F)}. \tag{2.78}$$

2.3.3 Perturbative Expansion of G

Although it is possible to calculate the single-particle Green's function for a free fermion system explicitly, the fully interacting single-particle Green's function can only be calculated perturbatively.

For interacting fermions,

$$H_1 = \frac{1}{2}\int d^3x \int d^3x' \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x')V_{\alpha,\alpha',\beta,\beta'}(\vec{x},\vec{x}')\psi_{\beta'}(x')\psi_{\alpha'}(x)$$
$$\frac{1}{2}\int d^3x \int d^3x' \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x')V_{\alpha\alpha',\beta\beta'}(x,x')\psi_{\beta'}(x')\psi_{\alpha'}(x), \tag{2.79}$$

where

$$V_{\alpha,\alpha',\beta,\beta'}(x,x') = \delta(x_0-x'_0)V_{\alpha\alpha',\beta\beta'}(\vec{x},\vec{x}'). \tag{2.80}$$

Assuming that H_1 may be adiabatically switched on, theorem 2.59 can be applied to equation 2.62 with the result:

$$iG_{\alpha,\beta}(x,y) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n$$
$$\frac{\langle\Phi_0|T[H_1(t_1)\cdots H_1(t_n)\psi_\alpha(x)\psi^\dagger_\beta(y)]|\Phi_0\rangle}{\langle\Phi_0|T[H_1(t_1)\cdots H_1(t_n)]|\Phi_0\rangle} \tag{2.81}$$

If G is defined as follows

$$iG_{\alpha,\beta}(x,y) = \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \langle\Phi_0|T[H_1(t_1)\cdots H_1(t_n)\psi_\alpha(x)\psi^\dagger_\beta(y)]|\Phi_0\rangle \tag{2.82}$$

where λ is the spin index. In the case of free fermions the eigenstates are plane waves, i.e.

$$\varphi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}. \tag{2.67}$$

The canonical transformation[7]

$$c_{\vec{k}\lambda} = \begin{cases} a_{\vec{k}\lambda} & |\vec{k}| > k_F \quad \text{particles} \\ b^\dagger_{-\vec{k}\lambda} & |\vec{k}| < k_F \quad \text{holes} \end{cases}$$

defines the operators a and b, where $a^\dagger$ creates particles and a destroys particles above the Fermi sea, while $b^\dagger$ creates holes and b destroys holes below the Fermi sea.

The anticommutation relations for c and $c^\dagger$

$$\begin{aligned} \{c_\lambda, c^\dagger_\mu\} &= \delta_{\lambda\mu} \\ \{c_\lambda, c_\mu\} &= \{c^\dagger_\lambda, c^\dagger_\mu\} = 0, \end{aligned} \tag{2.68}$$

imply the following anticommutators for the a, $a^\dagger$, b and $b^\dagger$ operators

$$\begin{aligned} \{a_\lambda, a^\dagger_\mu\} &= \{b_\lambda, b^\dagger_\mu\} = \delta_{\lambda\mu} \\ \{a_\lambda, b_\mu\} &= \{a^\dagger_\lambda, b_\mu\} = \{a_\lambda, b^\dagger_\mu\} = \{a^\dagger_\lambda, b^\dagger_\mu\} = 0. \end{aligned} \tag{2.69}$$

In terms of these new operators, equation 2.66 becomes

$$\begin{aligned} \psi_\lambda(\vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \theta(|k| > k_F) \varphi_{\vec{k}\lambda}(\vec{x}) a_{\vec{k}\lambda} e^{-i\omega_k t} + \int \frac{d^3k}{(2\pi)^3} \theta(|k| < k_F) \varphi^*_{\vec{k}\lambda}(\vec{x}) b^\dagger_{-\vec{k}\lambda} e^{-i\omega_k t} \\ \psi^\dagger_\lambda(\vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \theta(|k| > k_F) \varphi^*_{\vec{k}\lambda}(\vec{x}) a^\dagger_{\vec{k}\lambda} e^{i\omega_k t} + \int \frac{d^3k}{(2\pi)^3} \theta(|k| < k_F) \varphi^*_{\vec{k}\lambda}(\vec{x}) b_{-\vec{k}\lambda} e^{i\omega_k t} \end{aligned} \tag{2.70}$$

The vacuum $|\Psi_0\rangle$ consists of the filled Fermi sea, i.e. there are no holes below, and no particles above the Fermi surface. Hence,

$$b_{\vec{k}\lambda}|\Psi_0\rangle = \langle\Psi_0| a^\dagger_{\vec{k}\lambda} = 0. \tag{2.71}$$

The single particle Green's function for free fermions becomes[8]

$$\begin{aligned} i G^0_{\lambda\lambda'}(x, x') &= \langle\Psi_0| T[\psi_\lambda(x) \psi^\dagger_{\lambda'}(x')] |\Psi_0\rangle \\ &= \theta(t - t') \langle\Psi_0| \psi_\lambda(x) \psi^\dagger_{\lambda'}(x') |\Psi_0\rangle - \theta(t' - t) \langle\Psi_0| \psi^\dagger_{\lambda'}(x') \psi_\lambda(x) |\Psi_0\rangle. \end{aligned} \tag{2.72}$$

Equation 2.66 can be substituted into equation 2.72. The resulting expression reduces to, using equations 2.71:

$$\begin{aligned} i G^0_{\lambda\lambda'}(x, x') &= \theta(t - t') \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \theta(|k| - k_F) \theta(|k'| - k_F) e^{-i\omega_k t'} e^{i\omega_{k'} t'} \\ &\quad \cdot \langle\Psi_0| \varphi_{\vec{k}\lambda}(\vec{x}) a_{\vec{k}\lambda} \varphi^*_{\vec{k}'\lambda'}(\vec{x}') a^\dagger_{\vec{k}'\lambda'} |\Psi_0\rangle - \theta(t' - t) \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \theta(k_F - |k|) \\ &\quad \cdot \theta(k_F - |k'|) e^{i\omega_k t'} e^{-i\omega_{k'} t'} \langle\Psi_0| \varphi^*_{\vec{k}'\lambda'}(\vec{x}') b_{-\vec{k}'\lambda'} \varphi^*_{\vec{k}\lambda}(\vec{x}) b^\dagger_{-\vec{k}\lambda} |\Psi_0\rangle \\ &\quad - \delta_{\lambda\lambda'} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x} - \vec{x}')} e^{-i\omega_k(t - t')} [\theta(t - t') \theta(|k| - k_F) \\ &\quad - \theta(t' - t) \theta(k_F - |k|)] \end{aligned} \tag{2.73}$$

[7] $\vec{k}_F = (k_F, 0, 0)^T$ is the Fermi wavevector.

[8] For free fermions the interaction and Heisenberg pictures coincide, i.e. $\psi_{I\lambda}(\vec{x}, t) = \psi_{H\lambda}(\vec{x}, t) = \psi_\lambda(\vec{x}, t)$.

2.3 The Single-Particle Green Function

The traditional approach to the many-body fermion systems encountered in condensed matter physics has been to find the single-particle Green's function, G, perturbatively. Despite the fact that it is a ground-state average, G is related to important physical quantities. In fact, the single-particle Green's function can be used to calculate the ground-state expectation value of any single-particle operator. Furthermore, despite the fact that the interaction between particles is described by a two-particle operator, the ground-state energy may also be calculated from G. Finally, from the Lehmann representation [27] it is clear that G has simple poles at the exact excitation energies of the interacting system, i.e. the excitation spectrum of the system may also be determined from G.

2.3.1 Definition

The single-particle Green's function, also known as the electron-hole propagator, is defined as the interacting ground-state expectation value of the T-product of $\psi^\dagger_{\mathrm{H},s}(x)$ and $\psi_{\mathrm{H},s}(y)$. If $|\Psi_0\rangle$ is the interacting ground-state in the Heisenberg picture, then

$$iG_{r,s}(\vec{x}t,\vec{x}'t') = \frac{\langle\Psi_0|T[\psi_{\mathrm{H},r}(\vec{x}t)\psi^\dagger_{\mathrm{H},s}(\vec{x}'t')]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{2.62}$$

From this definition it is evident that G is simply a function of the coordinate variables $\vec{x}t$ and $\vec{x}'t'$.

The definition 2.31 of the T product may be inserted into equation 2.62, yielding

$$iG_{r,s}(\vec{x}t,\vec{x}'t') = \begin{cases} \dfrac{\langle\Psi_0|\psi_{\mathrm{H},r}(\vec{x}t)\psi^\dagger_{\mathrm{H},s}(\vec{x}'t')|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} & t > t' \\ \dfrac{\langle\Psi_0|\psi^\dagger_{\mathrm{H},s}(\vec{x}'t')\psi_{\mathrm{H},r}(\vec{x}t)|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} & t' > t \end{cases} \tag{2.63}$$

If H does not contain any time-dependent terms, and E is the ground-state energy, then equation 2.63 becomes

$$iG_{r,s}(\vec{x}t,\vec{x}'t') = \begin{cases} e^{iE(t-t')/\hbar}\dfrac{\langle\Psi_0|\psi_r(\vec{x})e^{-iH(t-t')/\hbar}\psi^\dagger_s(\vec{x}')|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} & t > t' \\ e^{iE(t-t')/\hbar}\dfrac{\langle\Psi_0|\psi^\dagger_s(\vec{x}')e^{iH(t-t')/\hbar}\psi_r(\vec{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} & t' > t \end{cases} \tag{2.64}$$

which demonstrates that, in this case, G depends only on the time difference $t - t'$, and not on t and t' separately. With such Green's functions it is convenient to take the Fourier-transform of G in terms of $t - t'$:

$$G_{r,s}(x_1,x_2,\omega) = \int_{-\infty}^{\infty} d\tau\, G_{r,s}(x_1,x_2,\tau)e^{i\omega\tau} \tag{2.65}$$

2.3.2 Free Fermion Green's Function

The field operator in the interaction representation is defined in terms of the annihilation and creation operators, c_β and $c^\dagger_\beta$, and the eigenstates, φ, as

$$\begin{aligned} \psi_\alpha(\vec{x}) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\varphi_{\vec{k}\alpha}(\vec{x})c_{\vec{k}\alpha}e^{-i\omega_k t} \\ \psi^\dagger_\alpha(\vec{x}) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\varphi^*_{\vec{k}\alpha}c^\dagger_{\vec{k}\alpha}e^{i\omega_k t} \end{aligned} \tag{2.66}$$

Theorem 2.58 *Assuming that the adiabatic application of $|\Phi_0\rangle$ is the nondegenerate ground state of the interacting system then*

$$\frac{\langle\Psi_0(t)|H_I(t)|\Psi_0(t)\rangle}{\langle\Psi_0(t)|\Psi_0(t)\rangle} = \frac{1}{\langle\Phi_0|U_I(\infty,-\infty)|\Phi_0\rangle}\langle\Phi_0|\sum_{\nu=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{\nu}\frac{1}{\nu!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_\nu \times e^{-\alpha(|t|-|t_1|-\cdots-|t_\nu|)}H_I(t_1)\cdots H_I(t_\nu)O_I(t)|\Phi_0\rangle$$

where $O_H(t)$ represents an arbitrary Heisenberg operator and, $O_I(t)$, the corresponding interaction operator.

Theorem 2.58 alone is insufficient, but it can be used to prove theorem 2.59, which provides the required relation.

Theorem 2.59 *Assuming that $\frac{\langle\Psi_0|O_H(t)|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$ is the nondegenerate ground state of the interacting system, then the expectation value of time-ordered Heisenberg operators may be written as*

$$\frac{\langle\Psi_0|T[O_H(t)O_H(t')]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} = \frac{1}{\langle\Phi_0|U_I(\infty,-\infty)|\Phi_0\rangle}\langle\Phi_0|\sum_{\nu=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{\nu}\frac{1}{\nu!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_\nu \times e^{-\alpha(|t|+|t'|+\sum_i|t_i|)}T[H_I(t_1)\cdots H_I(t_\nu)O_I(t)O_I(t')]|\Phi_0\rangle$$

2.2.2 Wick's Theorem

The definition 2.41 can be used to decompose the ground-state expectation value of a T-product in the interaction picture into N-products and pair-wise contractions.[1] Since, $\langle\Phi_0|N(A\cdots Z)|\Phi_0\rangle = 0$, and the only nonzero contractions are given by 2.49, this decomposition simplifies the evaluation of the ground-state expectation value. However, as the number of field operators is increased, it becomes increasingly more tedious to keep track of the terms involved. Fortunately, the T-product form of Wick's theorem [26] makes it possible to generate these terms automatically.

Theorem 2.60 *A T-product of the interaction picture field operators $A, B, C, \ldots, Z$ with different time coordinates can be represented as a sum of N-products, including as factors the contractions of all possible pairs of operators. The overall sign for each of these terms is determined by the number of rearrangements of the operators necessary to bring the contracted operators together.*

$$T(A\,B\,C\ldots Z) = N(A\,B\,C\ldots Z) + A\,B\,N(C\ldots Z) \pm A\,C\,N(B\ldots Z) + \cdots + B\,C\,N(A\ldots Z) \pm \cdots + N(A\,B\,C\ldots X\,Y\,Z) + N(\text{sum over all possible pairs of contractions})$$

Theorem 2.60 is based on the fact that a T-product may be rearranged into the corresponding N product, by moving the creation parts of the operators to the left, provided each time two operators fail to anticommute an additional term proportional to the contraction of the operators is added. Furthermore, the terms resulting from *all* possible contractions may be included, as from equation 2.49 it is evident that

$$\psi^{(-)}(x)\,\psi^{(-)\dagger}(y) = 0 \quad t_x > t_y\,, \tag{2.61}$$

i.e. if a creation part is originally to the left of a destruction part in the T-product, then the contraction is zero, and has no effect on the sum.

Wick's theorem still holds when one or more N product is contained within the T-product, however, in this case the contraction between operators involved in the N-product is omitted.

Wick's theorem applies to all T-products, no matter where they occur. However, the fact that the ground-state expectation value of an uncontracted N-product is zero, makes it an especially powerful technique when applied to the ground-state expectation value of a T product, such as the Green functions defined in section 2.3.

[1] Equation 2.47 was obtained in precisely this manner.

From equations 2.2 and 2.54 it would seem that the exact eigenstates may be constructed from $|\Psi_I(0)\rangle$ using U_ϵ since

$$|\Psi_H\rangle = |\Psi_I(0)\rangle = U_\epsilon(0,-\infty)|\Phi_0\rangle. \tag{2.56}$$

However, U_ϵ has an explicit ϵ-dependence as shown in equation 2.55, which causes U_ϵ to diverge like $\frac{1}{\epsilon}$ in the limit $\epsilon \to 0$. Hence, equation 2.56 cannot be used to obtain anything of physical significance.

The Gell-Mann and Low Theorem

Using the adiabatic assumption, Gell-Mann and Low [19] proved that under certain conditions, it is possible to construct $|\Psi_H\rangle$. The problem with the diverging phase is overcome by considering the ratio of two quantities proportional to $e^{-i\epsilon t}$.

Theorem 2.57 *The "adiabatic modification" of the unperturbed state Φ_0, is generated by*

$$\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = \lim_{\epsilon\to 0} \frac{U_\epsilon(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|U_\epsilon(0,-\infty)|\Phi_0\rangle},$$

If the limit exists, then

$$H\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle},$$

i.e. the adiabatic modification is the exact Heisenberg eigenstate of H with eigenvalue E. If in addition, $\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$ is nondegenerate then

$$\lim_{\epsilon\to 0} \frac{U_\epsilon(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|U_\epsilon(0,-\infty)|\Phi_0\rangle} = \lim_{\epsilon\to 0} \frac{U_\epsilon(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|U_\epsilon(0,-\infty)|\Phi_0\rangle}.$$

The energy shift due to the interaction is given by [20]

$$\begin{aligned}\Delta E &= E - E_0 \\ &= \lim_{\epsilon\to 0} \frac{\langle\Phi_0|H_I U_\epsilon(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|U_\epsilon(0,-\infty)|\Phi_0\rangle} \\ &= \frac{\langle\Phi_0|H_I|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}.\end{aligned}$$

Theorem 2.57 applies to any stationary solution of the unperturbed system. However, the excited states of many-body systems are normally highly degenerate, and become at best quasi-stationary as the interaction is switched on [21, 22, 23, 24]. Hence, in condensed matter physics, theorem 2.57 is usually applied to the unperturbed ground state. Unfortunately, the resulting adiabatic modification used is not necessarily the true ground state of the full Hamiltonian [25]. However, in order to simplify the discussion, it will be assumed that the adiabatic modification of the unperturbed ground state is indeed the true interacting ground state.

Adiabatic Heisenberg and Interaction Picture Matrix Elements

In order to use perturbation theory to expand, for example, the Green function, it must be possible to relate the matrix element of a T-product of Heisenberg operator adiabatically to its interaction picture counterpart.

The Gell-Mann and Low theorem 2.57 applied to the unperturbed ground state yields the following theorem:

A contraction may also occur within a N-product. Since it is not an operator, it may be moved outside the product once the operators have been rearranged so that the contracted operators are together.

$$N(ABC\,D\cdots QR\,S\cdots) = (-1)^P C\cdots F(AB D\cdots QS\cdots)A, \tag{2.47}$$

where, again, P is the number of permutations required to move C and R to the indicated positions.

From equation 2.15 it can be shown that the only nonvanishing contractions are:

$$\psi^{(+)}(x)\,\psi^{(+)\dagger}(y) = \begin{cases} \langle\Phi_0|T(\psi^{(+)}(x)\psi^{(+)\dagger}(y))|\Phi_0\rangle = iG^0(x,y) & t_x > t_y \\ 0 & t_y > t_x \end{cases}$$

and (2.48)

$$\psi^{(-)}(x)\,\psi^{(-)\dagger}(y) = \begin{cases} 0 & t_x > t_y \\ \langle\Phi_0|T(\psi^{(-)}(x)\psi^{(-)\dagger}(y))|\Phi_0\rangle = iG^0(x,y) & t_y > t_x \end{cases} \tag{2.49}$$

2.2 Perturbation Theory

2.2.1 The Adiabatic Hypothesis

Since H_0 represents a soluble problem, most commonly the noninteracting system, it is possible to calculate the Heisenberg eigenstates, $|\Phi_{H_0}\rangle$, of H_0 exactly. However, the objects of interest are the exact eigenstates of the interacting system.

Let $t = t_0$ and assume that in the limit $t_0 \to \pm\infty$ the system is in a nondegenerate time-independent eigenstate, $|\Phi_0\rangle$, of H_0, i.e. the interaction Hamiltonian, H_1, is zero, and

$$H_0|\Phi_0\rangle = E_0|\Phi_0\rangle. \tag{2.50}$$

Hence, from equation 2.4, the Schrödinger eigenstate is given by

$$|\Psi_S(t_0)\rangle = e^{-iE_0 t_0/\hbar}|\Phi_0\rangle, \tag{2.51}$$

and it follows from the definition 2.6 that

$$\begin{aligned} |\Psi_I(t_0)\rangle &= e^{iH_0 t_0/\hbar}|\Psi_S(t_0)\rangle \\ &= |\Phi_0\rangle. \end{aligned} \tag{2.52}$$

i.e., as $t_0 \to \pm\infty$ the interaction picture state vectors become time independent.

Also, assume that the interaction can be switched on infinitely slowly from $t = -\infty$ to $t = 0$ by replacing H with H_ϵ, where

$$H_\epsilon = H_0 + e^{-\epsilon|t|}H_1 \qquad 0 < \epsilon \ll 1. \tag{2.53}$$

As $\epsilon \to 0$ the Hamiltonian, H_ϵ, becomes the full Hamiltonian of the interacting system H, and the eigenstates $|\Psi_I(t)\rangle$ are now time dependent. Since ϵ is introduced as part of a mathematical device, any meaningful physical result must be independent of ϵ. Hence, the limit $\epsilon \to 0^+$ must be taken at the end of any calculation using H_ϵ.

The time-development operator, $U_\epsilon(t,t_0)$, corresponding to H_ϵ, in analogy to definition 2.21, is given by

$$|\Psi_I(t)\rangle = U_\epsilon(t,t_0)|\Psi_I(t_0)\rangle, \tag{2.54}$$

i.e. U_ϵ describes the evolution in time of the interaction state vector vector of H_ϵ.

An expansion of $U_\epsilon(t,t_0)$, similar to that for $U(t,t_0)$ given by equation 2.26, may also be obtained.

$$\begin{aligned} U_\epsilon(t,t_0) = &\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, e^{-\epsilon(|t_1|+\cdots+|t_n|)} \\ &\cdot T[H_{1\epsilon}(t_1)\cdots H_{1\epsilon}(t_n)]. \end{aligned} \tag{2.55}$$

The Normal-Ordered Product

The N-operator rearranges a product of annihilation and creation operators in such a way that all annihilation operators occur to the right of all creation operators. Again, a factor of -1 is included for each interchange of fermion operators required.

For example, the N-product of $\psi^{(-)}(x)\psi^{(+)}(y)$, according to the above definition, is given by

$$\begin{aligned} N(\psi^{(+)}(x)\psi^{(+)}(y)) &= -\psi^{(+)}(y)\psi^{(-)}(x) \\ &= -N(\psi^{(+)}(y)\psi^{(-)}(x)) \end{aligned} \tag{2.37}$$

which illustrates the following property of N-products:

$$N(AB(C)D\cdots) = (-1)^p N(C(AB\cdots). \tag{2.38}$$

More importantly, it follows from the definition that the unperturbed ground state expectation value of any normal-ordered product is zero. This is what makes N-products so convenient.

However, the field operator is, in general, not purely an annihilation or creation operator. Hence, in order to find the normal product of a collection of field operators, each field operator must first be decomposed into its creation part, $\psi^{(-)}$ and, $\psi^{(+)}$, the part of it which annihilates the noninteracting ground state, $|\Phi_0\rangle$ i.e.

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x), \tag{2.39}$$

where

$$\psi^{(+)}(x)|\Phi_0\rangle = 0. \tag{2.40}$$

The adjoints of equations 2.39 and 2.40 also hold, i.e.

$$\psi^{\dagger}(x) = \psi^{(+)\dagger}(x) + \psi^{(-)\dagger}(x) \tag{2.41}$$

and

$$\langle\Phi_0|\psi^{(+)\dagger}(x) = 0. \tag{2.42}$$

In addition, the normal-ordered product must be distributive, hence, for example,

$$\begin{aligned} N(\psi(x)\psi^{\dagger}(y)) &= N((\psi^{(+)}(x) + \psi^{(-)}(x))(\psi^{(+)\dagger}(y) + \psi^{(-)\dagger}(y))) \\ &= \psi^{(+)}(x)\psi^{(+)\dagger}(y) + \psi^{(+)}(x)\psi^{(-)\dagger}(y) - \psi^{(+)\dagger}(y)\psi^{(-)}(x) \\ &\quad + \psi^{(-)}(x)\psi^{(-)\dagger}(y). \end{aligned} \tag{2.43}$$

The Contraction of Operators

The contraction or pairing of two operators U and V is defined to be equal to the difference between the T product and the N-product.

$$\overbrace{U\,V} = T(UV) - N(UV). \tag{2.44}$$

From definition 2.14, and the fact that the ground-state expectation value of an N-product vanishes, it follows that

$$\begin{aligned} \langle\Phi_0|T(UV)|\Phi_0\rangle &= \langle\Phi_0|\overbrace{U\,V}|\Phi_0\rangle + \langle\Phi_0|N(UV)|\Phi_0\rangle \\ &= \overbrace{U\,V}. \end{aligned} \tag{2.45}$$

and it is evident that contractions are c-number functions i.e. not operators, but functions of the coordinate variables, which possess the property

$$\overbrace{U\,V} = \overbrace{V\,U} \tag{2.46}$$

The restriction 2.2 and the definition 2.21 of the operator U may be used to obtain the following important relation between $|\Psi_0\rangle$ and $|\Psi_I(t)\rangle$:

$$|\Psi_0\rangle = |\Psi_I(0)\rangle = U(0,t_0)|\Psi_I(t_0)\rangle. \tag{2.28}$$

This equation implies that, provided the interaction state vectors at time $t = 0$ are known, U may be used to reconstruct $|\Psi_0\rangle$. The significance of this is appreciated when it is remembered that $|\Psi_0\rangle$ is an exact eigenstate of the interacting system.

The matrix element of an operator in the Heisenberg picture may also be related its counterpart in the interaction picture by the U operator. Equations 2.13 and 2.19 imply

$$O_H(t) = e^{iHt/\hbar}e^{-iH_0t/\hbar}O_I(t)e^{iH_0t/\hbar}e^{-iHt/\hbar}. \tag{2.29}$$

Comparison with equation 2.22 finally yields

$$O_H(t) = U(0,t)O_I(t)U(t,0). \tag{2.30}$$

This equation also has important implications [illegible] ...ation theory.

2.1.3 Operator Product Definitions for Fermions

In order to present some of the theorems and results of quantum field theory in the most concise form possible, it is necessary to introduce the following definitions. Since, the system under consideration is fermionic, the operator product definitions for *fermions*, only, are presented.

<u>The Time-Ordered Product</u>

The T-product of a number of field operators orders the operators in ascending time order, so that field operator with the latest time is on the left, and that with the earliest time occurs on the right. During the rearrangement process, the all anticommutators are taken to be equal to zero, i.e. effectively, a factor of -1 is added for each interchange of fermion operators needed to rearrange the product.

$$T(O(t_1)O(t_2)\cdots O(t_n)) = (-1)^P O(t_a)O(t_b)\cdots O(t_p) \qquad t_a > t_b > \cdots > t_p \tag{2.31}$$

where P is the number of interchanges performed.

As an example, the T-ordered product of $\psi(x)\psi(y)$ is

$$T(\psi(x)\psi(y)) = \begin{cases} \psi(x)\psi(y) & x_0 > y_0 \\ -\psi(y)\psi(x) & y_0 > x_0 \end{cases} \tag{2.32}$$

$$= \psi(x)\psi(y), \tag{2.33}$$

while that of $\psi^\dagger(x)\psi^\dagger(y)$ is,

$$T(\psi^\dagger(x)\psi^\dagger(y)) = \psi^\dagger(x)\psi^\dagger(y). \tag{2.34}$$

More interestingly,

$$T(\psi(x)\psi^\dagger(y)) = \begin{cases} \psi(x)\psi^\dagger(y) & x_0 > y_0 \\ -\psi^\dagger(y)\psi(x) & y_0 > x_0 \end{cases} \tag{2.35}$$

From definition 2.31 the following property is found:

$$T(AB(C)\cdots) = (-1)^P T(C\cdots BA) \tag{2.36}$$

Chapter 4

Bilocal Bosonization for Fermions

4.1 Changing Variables to Bilocal Invariants

Traditional nonrelativistic multi-electron systems considered in condensed matter involve the complex spinor fields ψ_α and $\psi^\dagger_\alpha$. The classical action S, of such theories, is $U(1)$ invariant i.e. S is unchanged by the transformation

$$\begin{aligned} \psi_\alpha &\to \psi'_\alpha = U\psi_\alpha \\ \psi^\dagger_\alpha &\to \psi'^\dagger_\alpha = \psi^\dagger_\alpha U^\dagger \end{aligned} \tag{4.1}$$

where U is an arbitrary $U(1)$ element.

The identity

$$0 = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \frac{\delta K_\alpha[\psi,\psi^\dagger]}{\delta\psi_\alpha(x)} \tag{4.2}$$

holds for K_α an arbitrary functional of ψ and $\psi^\dagger$. In particular let

$$K_\alpha = \psi_\alpha(y) I[\psi,\psi^\dagger] e^{i\int \psi^\dagger\psi} \tag{4.3}$$

where, for some integer n, I is of the form

$$I[\psi,\psi^\dagger] = \psi^\dagger_{\alpha_1}(x_1)\psi_{\alpha_1}(y_1)\cdots\psi^\dagger_{\alpha_n}(x_n)\psi_{\alpha_n}(y_n). \tag{4.4}$$

In the particular case identity 4.2 becomes

$$0 = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \frac{\partial}{\delta\psi_\alpha(x)}\left[\psi_\alpha(y) I[\psi,\psi^\dagger] e^{i\int\psi^\dagger\psi}\right]. \tag{4.5}$$

which yields the following Schwinger-Dyson equation

$$\begin{aligned} 0 = &\int \mathcal{D}\psi\mathcal{D}\psi^\dagger \frac{\partial}{\delta\psi_\alpha(x)}[\psi_\alpha(y) I[\psi,\psi^\dagger] e^{i\int\psi^\dagger\psi}] \\ &\int \mathcal{D}\psi\mathcal{D}\psi^\dagger[\delta_{\alpha\beta}\delta^4(x-y) I[\psi,\psi^\dagger] + \psi_\alpha(y)\frac{\delta I}{\delta\psi_\alpha(x)} - \frac{i}{\hbar}\psi_\alpha(y)\frac{\delta S}{\delta\psi_\beta(x)} I[\psi,\psi^\dagger]] e^{i\int\psi^\dagger\psi} \end{aligned} \tag{4.6}$$

If variables are changed, inside the path integral, from the original Grassman ψ and $\psi^\dagger$ to the invariant antikommutian bosonic bilocals, σ, defined by

$$\sigma_{\alpha\beta}(x,y) = \psi^\dagger_\alpha(x)\psi_\beta(y), \tag{4.7}$$

then in terms of the σ invariants I becomes

$$I(\sigma) = \prod \sigma_{\alpha_1\beta_1}(x_1,y_1)\cdots \tag{4.8}$$

$$\sum_{n=0}^{\infty}\frac{1}{2h}^{n}\frac{1}{n!}\left[\int d^{4}x\int d^{4}x' V(\vec{x}-\vec{x}')\delta(x_{0}-x_{0}')\right.$$

$$\left.\int D\psi D\psi^{\dagger}\psi_{\alpha}(x_{1})\psi_{\alpha}^{\dagger}(x)\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x')\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\right]^{n}. \tag{3.68}$$

Approximating $G^{(n)}$ by the first two terms yields

$$\langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x_{2})\rangle \approx \langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}+\frac{1}{2h}\int d^{4}x\int d^{4}x' V(\vec{x}-\vec{x}')\delta(x_{0}-x_{0}')\times$$
$$\langle\psi_{\alpha}(x_{1})\psi_{\alpha}^{\dagger}(x)\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x')\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\rangle \tag{3.69}$$

The connected part of the second term in equation 3.69 is

$$\langle\psi_{\alpha}(x)\psi_{\alpha}^{\dagger}(x)\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x')\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\rangle_{conn} = \langle\psi_{\alpha}(x_{1})\psi_{\alpha}^{\dagger}(x)\rangle_{F}[\langle\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x')\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}$$
$$\langle\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x')\rangle_{F}]$$
$$+\langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x')\rangle_{F}[\langle\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\alpha}^{\dagger}(x)\rangle_{F}$$
$$\langle\psi_{\alpha}(x')\psi_{\alpha}^{\dagger}(x)\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}]. \tag{3.70}$$

(or, to first order in V)

$$\langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x_{2})\rangle = \langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}+\frac{1}{2h}\int d^{4}x\int d^{4}x' V(\vec{x}-\vec{x}')\delta(x_{0}-x_{0}')$$
$$\times[\langle\psi_{\alpha}(x_{1})\psi_{\alpha}^{\dagger}(x)\rangle_{F}[\langle\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x')\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}-\langle\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}$$
$$\times\langle\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x')\rangle_{F}]+\langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x')\rangle_{F}[\langle\psi_{\alpha}(x')\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\alpha}^{\dagger}(x)\rangle_{F}$$
$$\langle\psi_{\alpha}(x')\psi_{\alpha}^{\dagger}(x)\rangle_{F}\langle\psi_{\alpha}(x)\psi_{\beta}^{\dagger}(x_{2})\rangle_{F}]. \tag{3.71}$$

Using equation 3.38, equation 3.71 can be rewritten as follows

$$\langle\psi_{\alpha}(x_{1})\psi_{\beta}^{\dagger}(x_{2})\rangle = \int\frac{d^{4}k}{(2\pi)^{4}}e^{ik(x_{1}-x_{2})}G_{\psi_{\alpha}\psi_{\beta}}^{(0)}(k)+\frac{1}{h}\int\frac{d^{4}k_{1}}{(2\pi)^{4}}\int\frac{d^{4}k}{(2\pi)^{4}}e^{ik_{2}(x_{1}-x_{2})}$$
$$\times G_{\psi_{\alpha}\psi_{\alpha}}^{(0)}(k_{1})[\delta_{\alpha\beta}V(0)G_{\psi_{\alpha}\psi_{\beta}}^{(0)}(k_{2})$$
$$-V(\vec{k}_{1}-\vec{k}_{2})G_{\psi_{\alpha}\psi_{\alpha}}^{(0)}(k_{2})]G_{\psi_{\alpha}\psi_{\beta}}^{(0)}(k_{1})$$
$$\times G_{\psi_{\beta}\psi_{\beta}}(x_{1},x_{2}) \tag{3.72}$$

comparison with equation 2.84 shows that this is indeed the *first order correction to the single particle Green's function*.

The higher Green's functions can be calculated in a similar manner.

3.3.3 Interacting Fermions

Adding a perturbation, H_1 to the free Hamiltonian, H_0, results in a classical action of the form

$$S = S_0 + S_1 \tag{3.62}$$

The generating functional, Z for the perturbed system is obtained by substituting equation 3.62 into 3.30:

$$Z[\eta,\eta^\dagger] = \frac{\int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}\int d^4x[\psi^\dagger(x)i\hbar\partial_{x_0}\psi(x)+\psi^\dagger(x)\eta(x)+\eta^\dagger(x)\psi(x)]}}{\int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S}} \tag{3.63}$$

The series expansion for the exponential involving S may be used to expand Z, provided the space-time integral of the potential is small in comparison with $\hbar$, i.e.

$$\begin{aligned}
Z[\eta,\eta^\dagger] &= \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\left(1+\frac{i}{\hbar}S_1[\psi,\psi^\dagger]+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2(S_1[\psi,\psi^\dagger])^2+\cdots\right)\\
&\quad e^{\frac{i}{\hbar}S_0-\frac{i}{\hbar}\int d^4x[\psi^\dagger(x)i\hbar\partial_{x_0}\psi(x)+\psi^\dagger(x)\eta(x)+\eta^\dagger(x)\psi(x)]}\\
&\quad \frac{}{\int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S}}\\
&= \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n\frac{1}{n!}\int \mathcal{D}\psi\mathcal{D}\psi^\dagger(S_1[\psi,\psi^\dagger])^n e^{\frac{i}{\hbar}S_0-\frac{i}{\hbar}\int d^4x[\psi^\dagger(x)i\hbar\partial_{x_0}\psi(x)+\psi^\dagger(x)\eta(x)+\eta^\dagger(x)\psi(x)]}\\
&\quad \frac{}{\int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S}}\\
&= \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n\frac{1}{n!}(S_1[\frac{\delta}{\delta\eta^\dagger},\frac{\delta}{\delta\eta}])^n\\
&\quad \int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}\int d^4x[\psi^\dagger(x)i\hbar\partial_{x_0}\psi(x)+\frac{\hbar^2\nabla^2}{2m}\psi(x)+\psi(x)\eta(x)+\eta^\dagger(x)\psi(x)]}\\
&\quad \frac{}{\int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S}}
\end{aligned} \tag{3.64}$$

where the last line is obtained using equations 3.39, 3.40 and 3.57.

The Green's Functions of Identical Interacting Fermions

As an example, consider a system of identical interacting fermions, where the interaction potential is scalar, and the perturbing Hamiltonian in equation 2.8 is given by

$$\begin{aligned}
H_1 &= \frac{1}{2}\int d^3x\int d^3x'\psi_\alpha^\dagger(x)\psi_\beta^\dagger(x')V(x-x')\psi_\alpha(x')\psi_\alpha(x)\\
&\quad \frac{1}{2}\int d^3x\int d^3x'V(x-x')\psi_\alpha^\dagger(x)\psi_\alpha(x')\psi_\beta^\dagger(x')\psi_\beta(x)
\end{aligned} \tag{3.65}$$

Substituting equation 3.65 into 3.62 yields the classical action

$$\begin{aligned}
S &= \int d^4x\,\psi_\alpha^\dagger(x)[i\hbar\partial_{x_0}+\frac{\hbar^2\nabla^2}{2m}]\psi_\alpha(x)+\frac{1}{2}\int d^3x\int d^3x'V(x-x')\delta(x_0-x_0')\\
&\quad \psi_\alpha^\dagger(x)\psi_\alpha(x')\psi_\beta^\dagger(x')\psi_\beta(x)
\end{aligned} \tag{3.66}$$

$$S_0+S_1 \tag{3.67}$$

Hence using equation 3.64, the full interacting single particle Green's function is given by

$$\begin{aligned}
\langle\psi_\alpha(x)\psi_\beta^\dagger(x')\rangle &= \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\psi_\alpha(x)\psi_\beta^\dagger(x')\\
&\quad \int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}\int d^4x\psi_\alpha^\dagger(x)i\hbar\partial_{x_0}\psi_\alpha(x)}[1+\frac{i}{2\hbar}\int d^3x\int d^3x'V(x-x')\delta(x_0-x_0')\\
&\quad \psi_\alpha^\dagger(x)\psi_\alpha(x')\psi_\beta^\dagger(x')\psi_\beta(x)+\cdots]e^{\frac{i}{\hbar}S_0}
\end{aligned}$$

The multi now arbitrary constants ϵ_α^* and ϵ_α are chosen such that

$$(ih\partial_{x_0} - \frac{h^2\partial_{\vec{x}}^2}{2m})\psi_\alpha^\dagger(x) = \eta_\alpha^\dagger(x)$$
$$(ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})\psi_\alpha(x) = \eta_\alpha(x), \tag{3.53}$$

i.e.

$$\psi_\alpha(x) = -i\int d^4y\,(ih\partial_{y_0} + \frac{h^2\partial_{\vec{y}}^2}{2m})^{-1}\eta_\alpha(y). \tag{3.54}$$

Equations 3.47, 3.49, 3.53, together with 3.54, result in the following expression

$$Z_0[\eta,\eta^\dagger] = N\cdot\int d^4x\int d^4y\,\eta_\alpha^\dagger(x)\,ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})^{-1}\eta_\alpha(x)\int D\psi\int D\psi^\dagger\,e^{i\int d^4x\,\psi_\alpha^\dagger(x)ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})\psi_\alpha(x)}. \tag{3.55}$$

If N is chosen such that $Z[\eta,0] = 1$, i.e.

$$N = \frac{1}{\int D\psi\int D\psi^\dagger\,e^{i\int d^4x\,\psi_\alpha^\dagger(x)ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})\psi_\alpha(x)}}, \tag{3.56}$$

then equation 3.55 simplifies to

$$Z_0[\eta,\eta^\dagger] = e^{i\int d^4x\int d^4y\,\eta_\alpha^\dagger(x)ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})^{-1}\eta_\alpha(y)}. \tag{3.57}$$

The free single particle Green's function is obtained by applying equation 3.41 to 3.57.

$$G_{\alpha\beta}^{(2)}(x_1,y_1) = h^2\frac{\delta}{\delta\eta_\alpha^\dagger(x)}\frac{\delta}{\delta\eta_\beta(y)}e^{i\int d^4x\int d^4y\,\eta_\alpha^\dagger(x)ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})^{-1}\eta_\alpha(y)}\Big|_{\eta=\eta^\dagger=0}$$
$$= ih\delta_{\alpha\beta}\int d^4x\int d^4y\,\delta^4(x_1-x)(ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})^{-1}\delta^4(y_1-y). \tag{3.58}$$

From the definition of the inverse of an operator it follows that

$$\int d^4y\,(ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})(y)(ih\partial_{x_0} + \frac{h^2\partial_{\vec{y}}^2}{2m})^{-1}(x) = \delta^4(x-y), \tag{3.59}$$

i.e. $G^{(2)}$ is indeed the Green's function of the operator $(ih\partial_{x_0} + \frac{h^2\partial_{\vec{x}}^2}{2m})$.

Wick's Theorem

From equation 3.57 it is clear that Z_0 has the following structure

$$Z_0[\eta,\eta^\dagger] = e^{i\int d^4x\int d^4y\,\eta_\alpha^\dagger(x)M^{-1}(x,y)\eta_\alpha(y)}, \tag{3.60}$$

where M^{-1} represents the operator in equation 3.59.

Equation 3.36 has been used to obtain $G^{(2)}$ in the previous section. It may also be used to obtain higher Green's functions, for example $G^{(4)}$.

$$G^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1,x_2,y_1,y_2) = h^4\frac{\delta}{\delta\eta_{\alpha_1}^\dagger(x_1)}\frac{\delta}{\delta\eta_{\alpha_2}^\dagger(x_2)}\frac{\delta}{\delta\eta_{\alpha_3}(y_1)}\frac{\delta}{\delta\eta_{\alpha_4}(y_2)}e^{i\int d^4x\int d^4y\,\eta_\alpha^\dagger(x)M^{-1}(x,y)\eta_\alpha(y)}\Big|_{\eta=\eta^\dagger=0}$$
$$\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}M^{-1}(x_1,y_1)M^{-1}(x_2,y_2)$$
$$+\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}M^{-1}(x_2,y_1)M^{-1}(x_1,y_2)$$
$$+\langle T[\psi_{\alpha_1}(x_1)\psi_{\alpha_3}^\dagger(y_1)]\rangle_0\langle T[\psi_{\alpha_2}(x_2)\psi_{\alpha_4}^\dagger(y_2)]\rangle_0$$

$$+\langle T[\psi_{\alpha_1}(x_1)\psi_{\alpha_4}^\dagger(y_2)]\rangle_0\langle T[\psi_{\alpha_2}(x_2)\psi_{\alpha_3}^\dagger(y_1)]\rangle_0 \tag{3.61}$$

which is identical to the result obtained using theorem 2.60, i.e. the Wick expansion follows naturally from the definition of the path integral.

The Single-Particle Green's Function for Noninteracting Fermions

A system of identical nonrelativistic fermions is described by the Hamiltonian

$$H_1(x) = \int d^3x \frac{\hbar^2}{2m} \vec{\nabla} v_\alpha^\dagger(x) \cdot \vec{\nabla} v_\alpha(x). \tag{3.44}$$

The corresponding Lagrangian L_1 is given by

$$L_1(t) = \int d^3x \mathcal{L}_1(x)$$
$$\int d^3x v_\alpha^\dagger(x)(i\hbar\partial_{x_0} + \frac{\hbar^2\partial_x^2}{2m}) v_\alpha(x), \tag{3.45}$$

and the classical action is given by

$$S_1 = \int dt\, L_1(t)$$
$$\int d^4x v_\alpha^\dagger(x)(i\hbar\partial_{x_0} + \frac{\hbar^2\partial_x^2}{2m}) v_\alpha(x). \tag{3.46}$$

where m is the mass of the fermion described by the field v. S is quadratic in v and $v^\dagger$, and, since the particles are free, their mass need not be renormalised.

Substituting S_1 into equation 3.30 one obtains

$$\mathcal{Z}(\eta, \eta^\dagger) = N \int \mathcal{D}v\, \mathcal{D}v^\dagger e^{i \int d^4x [v_\alpha^\dagger(x)(i\hbar\partial_{x_0} + \frac{\hbar^2\partial_x^2}{2m}) v_\alpha(x) + \eta_\alpha^\dagger(x) v_\alpha(x) + v_\alpha^\dagger(x)\eta_\alpha(x)]}. \tag{3.47}$$

Consider the transformation

$$v_\alpha(x) \to v_\alpha(x) + v_{0_\alpha}(x)$$
$$v_\alpha^\dagger(x) \to v_\alpha^\dagger(x) + v_{0_\alpha}^\dagger(x). \tag{3.48}$$

where v_0 and $v_0^\dagger$ are constants with respect to v and $v^\dagger$. Since Grassman integration is defined to be translation invariant, the path integral is left unchanged by the transformation 3.48.

Under this transformation the exponent in equation 3.47 becomes,

$$\int d^4x \mathcal{L}_1(x) + i\eta_\alpha^\dagger(x) v_\alpha(x) + i v_\alpha^\dagger(x)\eta_\alpha(x) \to \int d^4x (v_\alpha^\dagger(x) + v_{0_\alpha}^\dagger(x))(i\hbar\partial_{x_0} - \frac{\hbar^2\partial_x^2}{2m})$$
$$(v_\alpha(x) + v_{0_\alpha}(x)) + i\eta_\alpha^\dagger(x)(v_\alpha(x) + v_{0_\alpha}(x))$$
$$+ i(v_\alpha^\dagger(x) + v_{0_\alpha}^\dagger(x))\eta_\alpha(x). \tag{3.49}$$

Integration by parts yields

$$\int d^4x\, v_{0_\alpha}^\dagger(x)\partial_x^2 v_\alpha(x) = \int d^4x\, \partial_x^2 v_{0_\alpha}^\dagger(x) v_\alpha(x). \tag{3.50}$$

and, in addition,

$$\int d^4x\, v_{0_\alpha}^\dagger(x)\partial_{x_0} v_\alpha(x) = \int d^4x\, \partial_{x_0} v_{0_\alpha}^\dagger(x) v_\alpha(x). \tag{3.51}$$

Use

$$\int d^4x\, v_{0_\alpha}^\dagger(x)(i\hbar\partial_{x_0} + \frac{\hbar^2\partial_x^2}{2m}) v_\alpha(x) = \int d^4x\, (i\hbar\partial_{x_0} v_{0_\alpha}^\dagger(x) - \frac{\hbar^2\partial_x^2}{2m} v_{0_\alpha}^\dagger(x)) v_\alpha(x). \tag{3.52}$$

$$\eta^\dagger_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\eta^\dagger_{\alpha_3}(x_3)\psi_{\alpha_4}(x_4)\psi^\dagger_{\beta_1}(y_1)\eta_{\beta_1}(y_1)\psi^\dagger_{\beta_2}(y_2)\eta_{\beta_2}(y_2)\psi^\dagger_{\beta_3}(y_3)\eta_{\beta_3}(y_3)+\ldots)$$

$$N\sum_{n=0}^{\infty}\left(-\frac{1}{h}\right)^{2n}\frac{1}{(n!)^2}\int d^4x_1\ldots\int d^4x_n\int d^4y_1\ldots\int d^4y_n\,\eta^\dagger_{\alpha_1}(x_1)\ldots\eta^\dagger_{\alpha_n}(x_n)$$

$$\eta_{\beta_1}(y_1)\ldots\eta_{\beta_n}(y_n)\int\mathcal{D}\psi\mathcal{D}\psi^\dagger\,e^{ih^{-1}S}\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\psi^\dagger_{\beta_1}(y_1)\ldots\psi^\dagger_{\beta_n}(y_n)$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{h}\right)^{2n}\frac{1}{(n!)^2}\int d^4x_1\ldots\int d^4x_n\int d^4y_1\ldots\int d^4y_n\,\eta^\dagger_{\alpha_1}(x_1)\ldots\eta^\dagger_{\alpha_n}(x_n)$$

$$\eta_{\beta_1}(y_1)\ldots\eta_{\beta_n}(y_n)G^{(2n)}_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n}(x_1,\ldots,x_n,y_1,\ldots,y_n), \tag{3.36}$$

where $G^{(2n)}$ is the $2n$ point Green's function i.e.

$$G^{(2n)}_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n}(x_1,\ldots,x_n,y_1,\ldots,y_n) \equiv N\int\mathcal{D}\psi\mathcal{D}\psi^\dagger\,e^{ih^{-1}S}\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\psi^\dagger_{\beta_1}(y_1)\ldots\psi^\dagger_{\beta_n}(y_n)$$
$$\frac{\langle 0|T[\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\psi^\dagger_{\beta_1}(y_1)\ldots\psi^\dagger_{\beta_n}(y_n)]|0\rangle}{\langle 0|0\rangle}. \tag{3.37}$$

Comparison of equation 3.37 with the *generalisation* of definition 2.62 yields the following relation between the Green's functions in the path integral formalism and those in the operator formalism:

$$G^{(2n)}_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n}(x_1,\ldots,x_n,y_1,\ldots,y_n) = iG^{(2n)}_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n}(x_1,\ldots,x_n,y_1,\ldots,y_n). \tag{3.38}$$

Now

$$h\frac{\delta}{\delta\eta_\alpha(z)}e^{-\frac{i}{h}\int d^4x\,\psi^\dagger_\beta(x)\eta_\beta(x)} = \psi^\dagger_\alpha(z)e^{-\frac{i}{h}\int d^4x\,\psi^\dagger_\beta(x)\eta_\beta(x)} \tag{3.39}$$

and

$$h\frac{\delta}{\delta\eta^\dagger_\alpha(z)}e^{-\frac{i}{h}\int d^4x\,\eta^\dagger_\beta(x)\psi_\beta(x)} = -\psi_\alpha(z)e^{-\frac{i}{h}\int d^4x\,\eta^\dagger_\beta(x)\psi_\beta(x)}. \tag{3.40}$$

Therefore

$$G^{(2n)}_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n}(x_1,\ldots,x_n,y_1,\ldots,y_n) = (-1)^n(h)^{2n}\frac{\delta}{\delta\eta^\dagger_{\alpha_1}(x_1)}\ldots\frac{\delta}{\delta\eta^\dagger_{\alpha_n}(x_n)}$$
$$\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\ldots\frac{\delta}{\delta\eta_{\beta_n}(y_n)}Z[\eta,\eta^\dagger]\Big|_{\eta,\eta^\dagger=0} \tag{3.41}$$

Now Z generates *all* the Green's functions, both connected and disconnected. However, the disconnected Green's functions describe particles moving independently, and hence are not very interesting. It is possible to define W such that

$$Z[\eta,\eta^\dagger] = e^{iW[\eta,\eta^\dagger]}, \tag{3.42}$$

i.e.

$$W[\eta,\eta^\dagger] = -i\ln Z[\eta,\eta^\dagger], \tag{3.43}$$

and W generates only connected diagrams.

3.3.2 Free Fermions

In the case of free fermions Z_0 separates into two factors[3], one which depends on the fields only, and the other depending only on the sources. Since the functional integral is taken over the fields, the integral of the factor depending on fields is *just a number*. In addition each term in the series given by equation 3.36 is a *free* Green's function. These free Green's functions can be calculated from Z_0 using equation 3.41, which in turn reproduces the Wick expansion term for term. These comments will be clarified by considering a specific example, namely, free noninteracting nonrelativistic fermions.

[3] for example see equation 3.35

of equation 3.31 is evaluated when the sources vanish, i.e.

$$\frac{\delta}{\delta\eta^\dagger_{\alpha_1}(u)}\frac{\delta Z}{\delta\eta_{\alpha_2}(v)}\Bigg|_{\eta=\eta^\dagger=0} = \frac{\int\mathcal{D}\psi\mathcal{D}\psi^\dagger\,e^{iS/\hbar}\,T(\psi_{\alpha_1}(u)\psi^\dagger_{\alpha_2}(v))}{\hbar^2\int\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{iS/\hbar}} = \frac{\langle 0|T(\psi_{\alpha_1}(u)\psi^\dagger_{\alpha_2}(v))|0\rangle}{\hbar^2\langle 0|0\rangle} \tag{3.32}$$

This illustrates another useful property of Z, namely, that functional derivatives of Z with respect to the sources evaluated when the sources are zero yield vacuum expectation values of time ordered products, where the fields appear in the same order in which the functional derivatives of their corresponding sources is taken. This is particularly useful, since the Green's functions or n point functions are vacuum expectation values of time ordered products of operators. Comparing equation 2.62 with equation 3.32, it is evident that the Green's functions can be obtained from Z.

In fact Z is the generating functional of the Green's functions, in that, Z may be expanded as a series, each term of which is a Green's function.

3.3.1 The Generating Functionals of the Green's Functions

The functional expansion of Z may be obtained using the series expansions :

$$e^{\frac{i}{\hbar}\int d^4x\,\eta^\dagger_\alpha(x)\psi_\alpha(x)} = 1 + \frac{1}{\hbar}\int d^4x\,\eta^\dagger_\alpha(x)\psi_\alpha(x) + \left(\frac{1}{\hbar}\right)^2\frac{1}{2}\int d^4x_1\int d^4x_2\,\eta^\dagger_\alpha(x_1)\psi_\alpha(x_1)\eta^\dagger_\beta(x_2)\psi_\beta(x_2) + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\hbar}\int d^4x\,\eta^\dagger_\alpha(x)\psi_\alpha(x)\right)^n \tag{3.33}$$

and

$$e^{-\frac{i}{\hbar}\int d^4x\,\psi^\dagger_\alpha(x)\eta_\alpha(x)} = 1 - \frac{1}{\hbar}\int d^4x\,\psi^\dagger_\alpha(x)\eta_\alpha(x) + \left(\frac{1}{\hbar}\right)^2\frac{1}{2}\int d^4x_1\int d^4x_2\,\psi^\dagger_\alpha(x_1)\eta_\alpha(x_1)\psi^\dagger_\beta(x_2)\eta_\beta(x_2) + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{\hbar}\int d^4x\,\psi^\dagger_\alpha(x)\eta_\alpha(x)\right)^n \tag{3.34}$$

At first glance this appears extremely complicated. However, luckily for most systems

$$\langle\Phi_0|T(\psi_\alpha(x)\psi^\dagger_\beta(y)\psi^\dagger_\gamma(z))|\Phi_0\rangle = N\int d^4x\int d^4y\int d^4z\int\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{iS/\hbar}\psi_\alpha(x)\psi^\dagger_\beta(y)\psi^\dagger_\gamma(z) = 0 \tag{3.35}$$

in fact, in most normal field theories all transition amplitudes involving an odd number of field operators vanish. For the multi-fermion system under consideration, because charge is conserved, the *only* nonzero transition amplitudes are those involving an equal number of ψ and $\psi^\dagger$ operators. Hence,

$$Z[\eta,\eta^\dagger] = N\int\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{iS/\hbar}\left(1 + \left(\frac{1}{\hbar}\right)^2\int d^4x_1\int d^4y_1\,\eta^\dagger_{\alpha_1}(x_1)\psi_{\alpha_1}(x_1)\psi^\dagger_{\beta_1}(y_1)\eta_{\beta_1}(y_1)\right.$$
$$+\left(\frac{1}{\hbar}\right)^4\frac{1}{4}\int d^4x_1\int d^4x_2\int d^4y_1\int d^4y_2\,\eta^\dagger_{\alpha_1}(x_1)\psi_{\alpha_1}(x_1)\eta^\dagger_{\alpha_2}(x_2)\psi_{\alpha_2}(x_2)$$
$$\psi^\dagger_{\beta_1}(y_1)\eta_{\beta_1}(y_1)\psi^\dagger_{\beta_2}(y_2)\eta_{\beta_2}(y_2)$$
$$+\left(\frac{1}{\hbar}\right)^6\frac{1}{(3!)^2}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4y_1\int d^4y_2\int d^4y_3\,\eta^\dagger_{\alpha_1}(x_1)\psi_{\alpha_1}(x_1)$$

where c_i and c_j are both Grassmann in nature. The last relation has the consequence that

$$\frac{\partial^2}{\partial c^2} = 0. \tag{3.23}$$

i.e. differentiation has an inverse operation, and hence integration can not merely be defined as the inverse to differentiation.

Instead another useful property of integration of real functions is used to define the integration of Grassmann functions, namely, translational invariance. Demanding translational invariance results in the following relations:

$$\int dc_i = 0$$

$$\int dc_i c_i = 1 \tag{3.24}$$

The above relations hold for the generators of a *finite* Grassmann algebra. However, the fermion field variables are functions of the space-time variable, x, hence they must behave like generators of an *infinite* Grassmann algebra, obeying the relations:

$$\{c(x), c(y)\} = 0 \tag{3.25}$$

$$\frac{\partial^{(n)} c(x)}{\partial c(y)} = \delta(x - y) \tag{3.26}$$

$$\int dc(x) = 0 \tag{3.27}$$

$$\int dc(x)\, c(x) = 1 \tag{3.28}$$

These relations can be used to prove the following important identity:

$$\int \mathcal{D}c\, \mathcal{D}c^\dagger e^{c^\dagger A c}\, dc^\dagger = \det A, \tag{3.29}$$

where A is an operator. This result is often used in the path integral formalism.

3.2.2 Definition of $\mathcal{Z}$

Consider a multi fermion system described by the fields c and $c^\dagger$. Associate the external sources η and $\eta^\dagger$ with $c^\dagger$ and c respectively. Then, in analogy with equation 3.18, the vacuum to vacuum transition amplitude in the presence of the relevant sources is defined to be

$$\mathcal{Z}[\eta, \eta^\dagger] = \frac{\int \mathcal{D}c\, \mathcal{D}c^\dagger e^{\frac{i}{\hbar}\int dx [c^\dagger(x) \mathcal{O} c(x) + \eta^\dagger(x) c(x) + c^\dagger(x) \eta(x)]}}{\int \mathcal{D}c\, \mathcal{D}c^\dagger e^{\frac{i}{\hbar} S}}, \tag{3.30}$$

where it must be remembered that c, $c^\dagger$, η, and $\eta^\dagger$ have two-components and are Grassmann in nature.

3.3 Path Integrals and The Green's Functions

In order to have any practical use the vacuum-to-vacuum transition amplitude written as a path integral, $\mathcal{Z}$, must be related to physical observables.

Consider

$$\frac{\delta \mathcal{Z}}{\delta \eta_\alpha(x)} = \frac{\int \mathcal{D}c\, \mathcal{D}c^\dagger e^{\frac{i}{\hbar}\int dx [c^\dagger(x) \mathcal{O} c(x) + \eta^\dagger(x) c(x) + c^\dagger(x) \eta(x)]} c^\dagger_\alpha(x)}{\hbar \int \mathcal{D}c\, \mathcal{D}c^\dagger e^{\frac{i}{\hbar} S}}$$

$$\frac{\delta}{\delta \eta^\dagger_\alpha(x)} \frac{\delta \mathcal{Z}}{\delta \eta_\beta(x)} = \frac{\int \mathcal{D}c\, \mathcal{D}c^\dagger e^{\frac{i}{\hbar}\int dx [c^\dagger(x) \mathcal{O} c(x) + \eta^\dagger(x) c(x) + c^\dagger(x) \eta(x)]} T(c_\alpha(x) c^\dagger_\beta(x))}{\hbar^2 \int \mathcal{D}c\, \mathcal{D}c^\dagger e^{\frac{i}{\hbar} S}}$$

$$\frac{\langle 0 | T(c_\alpha(x) c^\dagger_\beta(x)) | 0 \rangle_\eta}{\hbar^2 \langle 0 | 0 \rangle} \tag{3.31}$$

be added to the integrand in the exponent. This term also causes the integral to converge. At the end of the calculation ϵ is eliminated by taking the limit $\epsilon \to 0$. Or, secondly, if equation 3.17 is continued into Euclidean space by Wick rotation (i.e. rotation of the time axis), then $t \to it$ and the product it is real, and hence, the integrand is no longer oscillatory. This method can only be used if no singularities are encountered when the contour is rotated. The final answer in real time is then obtained by analytic continuation. Both methods result in the same pole prescriptions.

Generally, in field theory the number of particles fluctuates as particles are created and destroyed. This can be taken into account by using source terms [36]. A source is associated with each field, and the products of each source with its associated field and $\hbar$, are added to the Lagrangian in the vacuum-to-vacuum transition amplitude. For example, if the field $\phi(x)$ has the corresponding source $J(x)$, then Z, the vacuum-to-vacuum transition amplitude in the presence of J (which acts as an external driving force), is given by:

$$Z[J] = N \int \mathcal{D}\phi e^{\frac{i}{\hbar}\int d^4x[\mathcal{L}(x) + J(x)\phi(x)]} dx_0 dx_1 dx_2 dx_3 \tag{3.18}$$

where the constant N is a normalisation factor. If Z is normalised such that $Z[J = 0] = 1$, then $N = \frac{1}{Z[0]}$. From equation 3.18 it is clear that Z is a functional integral of J.

As will be demonstrated in the next section, Z is the generating functional of the Green's functions of the system. Usually perturbation theory is used to extract the Green's functions from Z. These functions are then used to calculate the physical consequences of the theory.

3.2 Path Integrals and Fermionic Field Theories

As stated before, the path integral description of fermionic field theories is obtained by analogy with that of bosonic field theories. However, whereas the field variables appearing in the bosonic field path integral are treated as normal c number functions, in order to preserve fermion statistics, fermionic field variables must behave like Grassmann variables in the path integral formalism.

3.2.1 Grassmann Variables

Anticommuting c numbers are also known as Grassmann numbers, because they were first considered in a paper written by Hermann Grassmann in 1855. They remained relatively obscure, until their usefulness in connection with fermionic path integrals was realised.

In order to be a Grassmann variable ϵ must satisfy the following anticommutation relation:

$$\{\epsilon, \epsilon'\} = 2\epsilon\epsilon' = 0, \tag{3.19}$$

from which it immediately follows that

$$\epsilon^2 = 0 \quad \forall \quad \epsilon \in \mathbb{R}, \tag{3.20}$$

i.e. any function of a Grassmann variable can at most be linear. This has important implications for, amongst other things, the definition of differentiation and integration of functions of Grassmann variables.

Firstly, it will be noticed, by considering, for example, the product of two Grassmann variables, that left and right differentiation with respect to a Grassmann variable are not necessarily equivalent, and the following anticommutation relations must hold [2, 3]:

$$\left\{ \frac{\partial}{\partial\epsilon_i}, \epsilon_j \right\} = \delta_{ij} \tag{3.21}$$

$$\left\{ \frac{\partial}{\partial\epsilon_i}, \frac{\partial}{\partial\epsilon_j} \right\} = 0 \tag{3.22}$$

where the δ-functional is defined to be the product of the δ-function at each point in the three-dimensional space x.

$$\begin{aligned}\delta[\phi_{i+1} - \phi_i] &= \prod_{\vec{x}} \delta[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})] \\ &= \prod_{\vec{x}} \int_{-\infty}^{\infty} \frac{d\pi_i}{2\pi} e^{i\pi_i(\vec{x})[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})]} \\ &= \int D\pi_i \, e^{i \int d^3x \, \pi_i(\vec{x})[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})]}, \end{aligned} \tag{3.12}$$

the second line is obtained by replacing each δ-function with its Fourier transform, and use is made of the following definition:

$$\int D\pi_i = \prod_{\vec{x}} \int_{-\infty}^{\infty} \frac{d\pi_i}{2\pi}. \tag{3.13}$$

Equation 3.12 is the functional Fourier transform of the δ-functional.

Since $\exp(i \int d^3x \, \pi(\vec{x})\phi(\vec{x}))$ is a field momentum eigenfunctional,

$$\langle \phi_{i+1} | H | \phi_i \rangle = \int D\pi_i \, H[\pi_i, \phi_i] e^{i \int d^3x \, \pi_i(\vec{x})[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})]}. \tag{3.14}$$

Note: H is not an operator, but a functional of π and ϕ, where $\phi_i = \frac{\phi_{i+1}+\phi_i}{2}$ defines an operator ordering.

It then becomes obvious that G has the form

$$\begin{aligned} G[\phi_f, t_f; \phi_i, t_i] &= \lim_{N \to \infty} \int \prod_{i=1}^{N} D\phi_i \int \prod_{i=0}^{N} D\pi_i \, e^{i \sum_{i=0}^{N} \int d^3x \, \pi_i(\vec{x}) \frac{\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})}{\epsilon} - H[\pi_i, \phi_i]} \\ &= \int D\phi \int D\pi \, e^{i \int_{t_i}^{t_f} dt \, \int d^3x \, \pi(x)\dot{\phi}(x) - H[\pi,\phi]}, \end{aligned} \tag{3.15}$$

where

$$D\phi = \lim_{N \to \infty} \prod_{i=1}^{N} \prod_{\vec{x}} d\phi(\vec{x}, t) \left(\frac{1}{2\pi i \epsilon} \right)^{\frac{1}{2}}. \tag{3.16}$$

Provided H is quadratic in π, and the interaction term does not have a π-dependence, it is possible to perform the functional integration over π by completing the square in the exponent, and then using the result for Gaussian integrals generalised to include functional integrals.[1] This yields

$$G[\phi_f, t_f; \phi_i, t_i] = \int D\phi \, e^{iS}. \tag{3.17}$$

Here $S = \int d^4x \, \mathcal{L}(x)$ is again the classical action, and the boundary conditions are taken into account by requiring that the each function ϕ in the functional integral takes on the values ϕ_i at t_i, and ϕ_f at t_f. However, since the integrand is purely oscillatory, the path integral in equation 3.17 is ill defined. In addition, in order to be able to use equation 3.17, as written, the physical states of the system ($|\phi_i\rangle$ and $|\phi_f\rangle$) must be known. Unfortunately this is not always possible.

These problems are avoided if the propagator (or tansition amplitude) of interest is the *vacuum-to-vacuum* transition amplitude[2]. Since, irrespective of the physical states of the system, a state of least energy can always be found.

The vacuum to vacuum transition amplitude, may be isolated from equation 3.17 by a order of two methods [3, 35]. Firstly, a damping term, $-\frac{1}{2}\epsilon\phi^2$ (where $\epsilon \ll 1$, $\epsilon > 0$), may

[1] In this thesis, all systems considered are described by Hamiltonians which are quadratic in the conjugate momentum of the fields, and the interaction potentials considered are independent of the conjugate momentums.

[2] The transition amplitude of the system from the vacuum state (state of least energy) at $t = -\infty$ to the vacuum state at $t = \infty$.

described by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H|\Phi(t)\rangle \tag{3.4}$$

If the state of the system at time t_0 is $|\Phi(t_0)\rangle$ then equation 3.4 yields the following formal solution for $|\Phi(t)\rangle$:

$$|\Phi(t)\rangle = e^{-iH(t-t_0)/\hbar}|\Phi(t_0)\rangle \tag{3.5}$$

If the field theory is formulated in terms of the basis vectors $|\phi\rangle$, then $|\phi\rangle$ obey the functional version of the completeness relation

$$\mathbf{1} = \int \mathcal{D}\phi\, |\phi\rangle\langle\phi|, \tag{3.6}$$

where $\int \mathcal{D}\phi$ denotes an integral over the space of all maps from three-dimensional space into the reals.

The wave functional describing the state of the system at t, $|\Phi(t)\rangle$, is defined to be

$$\Phi[\phi(\vec{x}),t] \equiv \langle\phi|\Phi(t)\rangle. \tag{3.7}$$

Equations 3.5 and 3.6 may be inserted into equation 3.7 yielding the functional relation

$$\begin{aligned}\Phi[\phi(\vec{x}),t] &= \int \mathcal{D}\phi_0 \langle\phi|e^{-iH(t-t_0)/\hbar}|\phi_0\rangle \Phi[\phi_0(\vec{x}),t_0] \\ &= \int \mathcal{D}\phi_0\, G[\phi,t;\phi_0,t_0]\Phi[\phi_0(\vec{x}),t_0].\end{aligned} \tag{3.8}$$

where $G[\phi,t;\phi_0,t_0]$ is the matrix element of the time-evolution operator, and is equivalent to a sum over all paths through the space of all mappings from three-dimensional space into the reals starting at $\phi_0(\vec{x})$ at t_0 and ending at $\phi(\vec{x})$ at t.

The time interval $t - t_0$ in the exponent of the time evolution-operator may be divided into $N+1$ equal time intervals, and the completeness relation inserted N times to give

$$\begin{aligned}&\prod_{i=1}^{N}\langle\phi|e^{-iH(t-t_N)/\hbar}\cdots e^{-iH(t_1-t_0)/\hbar}|\phi_0\rangle = \\ &\int\prod_{i=1}^{N}\mathcal{D}\phi_i\langle\phi|e^{-iH(t-t_N)/\hbar}|\phi_N\rangle\cdots\langle\phi_1|e^{-iH(t_1-t_0)/\hbar}|\phi_0\rangle\end{aligned} \tag{3.9}$$

where $\phi_i = \phi(\vec{x},t_i)$. In the limit $N\to\infty$ equation 3.9 tends to G [illegible]

$$G[\phi,t;\phi_0,t_0] = \lim_{N\to\infty}\int\prod_{i=1}^{N}\mathcal{D}\phi_i\, G[\phi_{i+1},t_{i+1};\phi_i,t_i], \tag{3.10}$$

where the definitions $\phi_{N+1}\equiv\phi$ and $t_{N+1}\equiv t$ have been made. As $N\to\infty$ the interval $t_{i+1}-t_i = \epsilon\to 0$, and the exponential in G may be expanded, and then approximated by the first two terms, as follows:

$$\begin{aligned}G[\phi,t;\phi_0,t_0] &= \lim_{N\to\infty}\int\prod_{i=1}^{N}\mathcal{D}\phi_i\,\langle\phi_{i+1}|e^{-iH(t_{i+1}-t_i)/\hbar}|\phi_i\rangle \\ &\approx \lim_{N\to\infty}\int\prod_{i=1}^{N}\mathcal{D}\phi_i\,\langle\phi_{i+1}|\left[1-\frac{i}{\hbar}H(t_{i+1}-t_i)\right]|\phi_i\rangle \\ &= \lim_{N\to\infty}\int\prod_{i=1}^{N}\mathcal{D}\phi_i\,\delta[\phi_{i+1}-\phi_i]-\frac{i}{\hbar}\epsilon\langle\phi_{i+1}|H|\phi_i\rangle.\end{aligned} \tag{3.11}$$

Chapter 3

Path Integral Formulation of Nonrelativistic Field Theory

Feynman developed the path integral approach to quantum mechanics in his 1942 Princeton thesis, and published the theory in 1948 [31]. However, the basic observation underlying path integrals for time evolution amplitudes was made by Dirac as early as 1933 [32]. The crucial observation was that the *short-time* propagator can be written as the exponential of $\frac{i}{\hbar}S$ where S is the classical action.

In its original form, the path integral formalism is applied to the quantum mechanical description of a single point particle moving in a Cartesian co-ordinate system, and is used to calculate transition amplitudes for the time evolution operator of the particle [33]. In this formalism the object of central interest is the propagator, K, defined to be such that:

$$\psi(q_f,t_f) = \int dq_i\, K(q_f t_f; q_i t_i)\psi(q_i,t_i) \quad \text{Huygen's principle} \tag{3.1}$$

i.e. given the wave function of the particle at a specific point in time, using the propagator, the wave function of the particle at some later time can be calculated. Using elementary quantum mechanics it can be established that

$$K(q_f t_f; q_i t_i) = \langle q_f t_f | q_i t_i \rangle \tag{3.2}$$

i.e. the propagator is actually a transition amplitude. But most importantly the propagator may be expressed as a path integral:

$$\langle q_f t_f | q_i t_i \rangle = \int \frac{\mathcal{D}q\,\mathcal{D}p}{h} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} p\dot{q} - H(p,q)\,dt} \tag{3.3}$$

Provided $H = \frac{p^2}{2m} + V(q)$ [34], the p-integration in equation 3.3 can be performed, and the exponent is shown to be $\frac{i}{\hbar}S$.

Hence, the transition amplitude is the sum over all possible paths that start at q_i at time t_i and end at q_f at t_f, weighted by the exponential of $\frac{i}{\hbar}$ times the action evaluated for that particular path. As $\hbar \to 0$, the integrand oscillates with greater frequency, unless $\frac{\delta S}{\delta q} = 0$, i.e. only the contribution from the classical path survives in the classical limit. The path integral formalism eloquently connects classical and quantum mechanics.

However, in this thesis path integral quantisation applied to *field theory* [2] is of interest, and is presented below. Bosonic fields are briefly considered first, since the path integral approach to fermionic fields is arrived at by analogy with the bosonic case. A review is given of only those aspects of field theory which have been applied in chapter 6. Hence more complicated topics [7, 3], like renormalisation, etc., have been omitted.

3.1 Path Integrals and Bosonic Field Theories

Let H be the Hamiltonian describing a multi-bosonic system, and $|\Phi(t)\rangle$ be the state of the system at time t in the Schrödinger representation. Then the evolution in time of $|\Phi(t)\rangle$ is

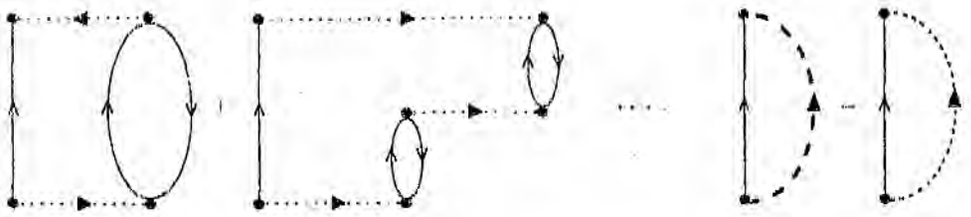

Figure 5.8: The ring diagram approximation to the pp per self-energy, Σ_0^*.

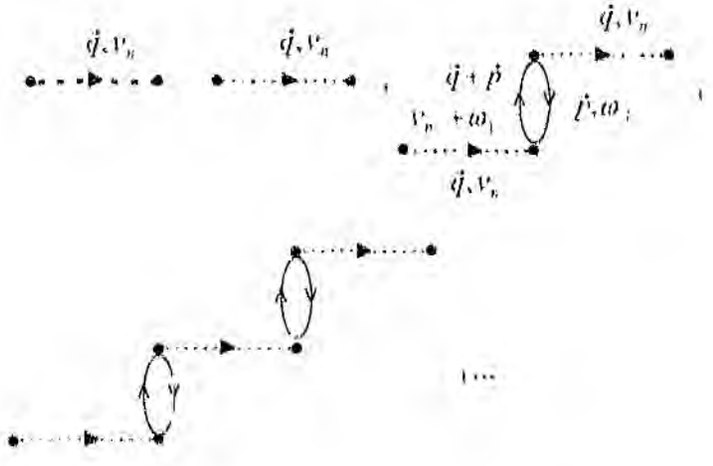

Figure 5.9: The effective two body interaction in the Ring Diagram approximation.

Equations 5.18 and 5.16 may be used to obtain

$$\Sigma^*(p) = \frac{i}{\hbar}\delta_{\alpha\alpha'}\int\frac{d^4k}{(2\pi)^4}G^0(k)\left[V(0)+\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}V(q)G^0(p-q)G^0(k+q)\Gamma(p-q,k+q;p,k)\right]$$
$$+\frac{i}{\hbar}\int\frac{d^4k}{(2\pi)^4}G^0(k)\left[V(k-p)+\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}V(q)G^0(k-q)G^0(p+q)\right.$$
$$\left.\Gamma(p-q,p+q;p,k)\right]. \tag{5.19}$$

which in turn may be substituted into equation 2.87, with the result that

$$G_1(x,y)(k) = G^0(k) + \frac{i}{\hbar}\int\frac{d^4k}{(2\pi)^4}[G^0(k)]^2\left[V(0)\delta_{\alpha\alpha'}+\frac{i}{\hbar}\delta_{\alpha\alpha'}\right.$$
$$\left.\int\frac{d^4q}{(2\pi)^4}V(q)G^0(p-q)G^0(k+q)\Gamma(p-q,k+q;p,k)\right.$$
$$\left.V(k-p)-\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}V(q)G^0(k-q)\right.$$
$$\left.G^0(p+q)\Gamma(k-q,p+q;p,k)\right]G_1(k). \tag{5.20}$$

5.2.3 The Ring Diagrams Approximation

In the degenerate gas model, the system considered is an electron gas in a uniform positive background. The positive background has the effect of cancelling the $q=0$ component of $V(q)$, i.e. all tadpole diagrams vanish. The ring diagram approximation to the proper self energy is obtained by retaining the most divergent diagram in each order in V in the expansion of Σ^*. Hence,

$$\Sigma^* \approx \Sigma^*_{(1)} + \Sigma^*_{(2)} + \sum_{n=2}^{\infty}\Sigma^*_{(n)ring}$$
$$\Sigma^*_{(1)} + \Sigma^*_{(2)ring} + \Sigma^*_{ring} \tag{5.21}$$

where Σ^*_n is the sum of all proper self energy insertions with the structure shown in figure 5.8, $\Sigma^*_{(1)}$ is the lowest order term, and $\Sigma^*_{(2)}$ is the second order contribution to Σ^*, without the contribution from the first ring diagram. The Feynman rules can be used to show

$$\Sigma^*_{ring\,\alpha\alpha'}(x_1,x_1') = \frac{1}{\hbar^0}\delta_{\alpha\alpha'}G^0_{\alpha\beta}(x_1',x_1)\int d^4x_2\int d^4x_2' V(x_1-x_2)G^0_{\beta\gamma}(x_2',x_2)G^0_{\gamma\delta}(x_2,x_2')V(x_1'-x_2')$$
$$\frac{1}{\hbar^3}G^0_{\alpha\beta}(x_1',x_1)\int d^4x_2\int d^4x_2'\int d^4x_3\int d^4x_3' V(x_1-x_2)$$
$$G^0_{\beta\gamma}(x_2'-x_2)G^0_{\gamma\delta}(x_2,x_2')V(x_2'-x_3)$$
$$G^0_{\delta\epsilon}(x_3'-x_3)G^0_{\epsilon\alpha'}(x_3,x_3')V(x_1'-x_3')+\cdots \tag{5.22}$$

Moving into momentum space, equation 5.22 becomes

$$\Sigma^*_{ring\,\alpha\alpha'}(p) = \frac{1}{\hbar}\int\frac{d^4q}{(2\pi)^4}G^0_{\alpha\alpha'}(p-q)\int\frac{d^4k_1}{(2\pi)^4}V(q)G^0_{\beta\gamma}(k_1)G^0_{\gamma\beta}(k_1+q)V(q)$$
$$\frac{i}{\hbar^3}\int\frac{d^4q}{(2\pi)^4}G^0_{\alpha\alpha'}(p-q)\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}V(q)G^0_{\alpha\beta}(k_1)$$
$$G^0_{\beta\gamma}(k_1+q)V(q)G^0_{\gamma\delta}(k_2)G^0_{\delta\beta}(k_2+q)V(q)+\cdots \tag{5.23}$$

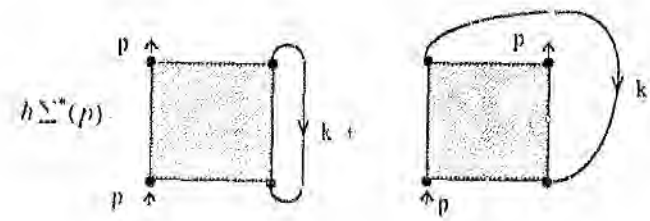

Figure 5.5: The integral equation for the ladder diagram approximation to the proper self-energy.

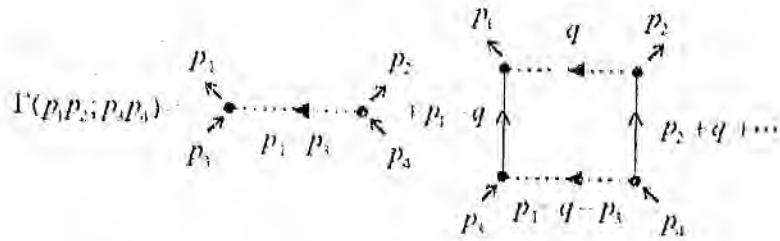

Figure 5.6: The two particle effective interaction in the ladder diagram approximation.

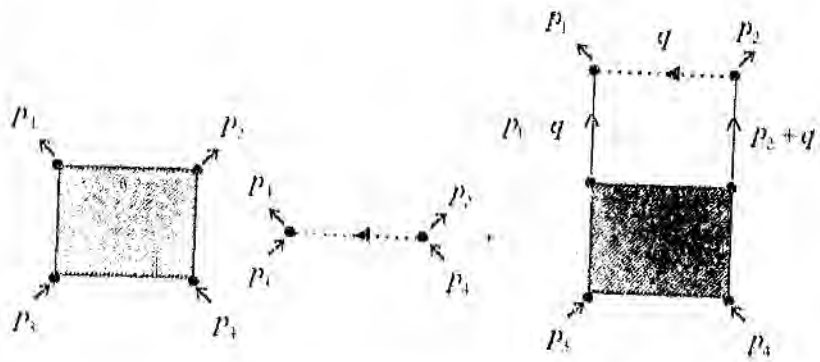

Figure 5.7: The Bethe Salpeter equation for Γ.

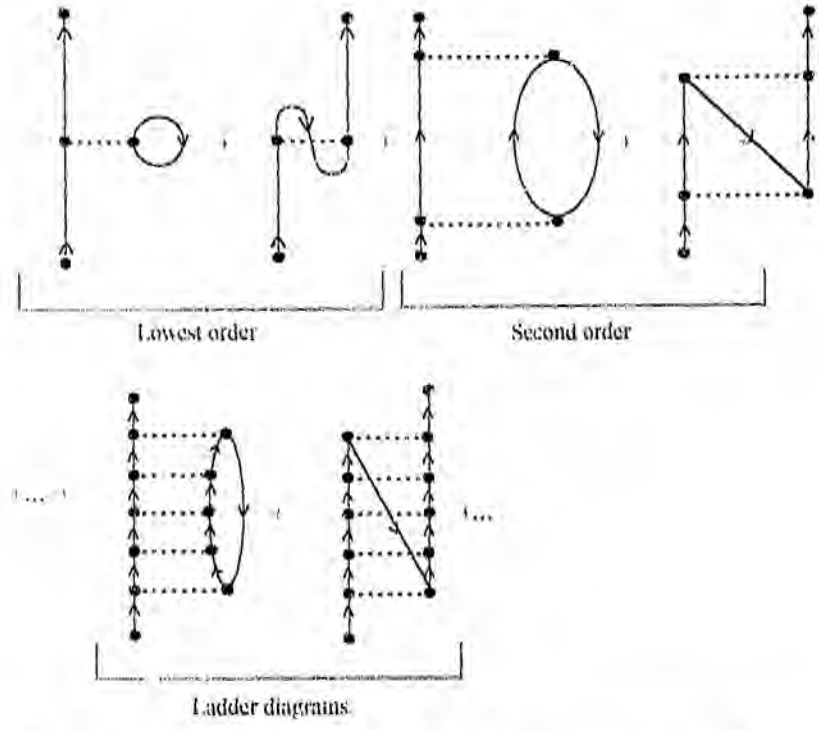

Figure 5.1: Single particle Green's function in the ladder diagram approximation.

In momentum space equation 2.14 becomes

$$\Sigma^{n}_{\lambda_a}(p) = \frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}\left[\delta_{\lambda_a}V(0)G^{n}_{\lambda'\lambda'}(q) - V(\vec{k}-\vec{q})G^{n}_{\lambda\lambda}(q)\right]$$

$$+\frac{1}{\hbar^2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\delta_{\lambda_a}G^{n}_{\lambda_1\lambda_2}(k)V(q)G^{n}_{\lambda_3\lambda_4}(p-q)G^{n}_{\lambda_2\lambda_4}(k+q)V(-q)$$

$$-\frac{1}{\hbar^2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}G^{n}_{\lambda_3\lambda_2}(k)V(q)G^{n}_{\lambda_2\lambda_4}(p+q)G^{n}_{\lambda_4\lambda_1}(k-q)V(\vec{k}-q'-p) \tag{5.15}$$

$$+O(V^4)+\ldots$$

Assuming that V is spin independent, this can be rewritten as the following integral equation:

$$\Sigma^{n}_{\lambda_a}(p) = \frac{i}{\hbar}\delta_{\lambda_a}\int\frac{d^4k}{(2\pi)^4}\delta_{\lambda_a}G^{n}(k)V(p,k;p,k)+\frac{i}{\hbar}\int\frac{d^4k}{(2\pi)^4}G^{n}(k)\Gamma(k,p;p,k), \tag{5.16}$$

where the effective two particle interaction in the ladder diagram approximation, Γ, is defined by the relation

$$\Gamma(p_1,p_2;p_3,p_4) = V(p_1-p_3,p_4)+\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}V(\vec{q})G^{n}(p_1-q)G^{n}(p_2+q)V(p_1-q-p_3,p_4)+\ldots \tag{5.17}$$

Equation 5.17 may also be rewritten as an integral equation,

$$\Gamma(p_1,p_2;p_3,p_4) = V(p_1-p_3,p_4)+\frac{i}{\hbar}\int\frac{d^4}{(2\pi)^4}qV(q)G^{n}(p_1-q)G^{n}(p_2+q)\Gamma(p_1-q,p_2+q;p_3,p_4). \tag{5.18}$$

If the kernel of equation 2.33 is approximated by the lowest order two particle interaction, V, then equation 5.18 is obtained, i.e. equation 5.18 is the ladder diagram approximation to the Bethe Salpeter equation.

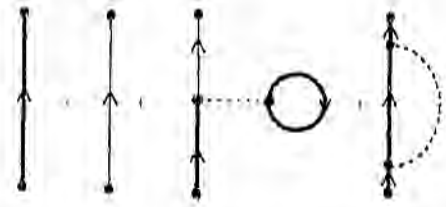

Figure 5.3: The single particle Green's function in the Hartree-Fock approximation.

The single particle Green's function for the Hartree-Fock approximation is obtained by substituting equation 5.8 into 2.87, yielding

$$\begin{aligned} G_{m_s m_s}(x,y) &= G^0_{m_s m_s}(x,y) + \frac{i}{\hbar}\int d^4x_1 \int d^4x_1' \, G^0_{m_s m_s}(x,x_1)\delta(x_{1_0} - x'_{1_0})[\delta^{(3)}(\vec{x}_1 - \vec{x}_1')\delta_{\lambda\mu} \\ &\quad - \int d^3x_2 \, V(\vec{x}_2 - \vec{x}_2')G_{m_\lambda m_\lambda}(x_2,x_2) \quad V(\vec{x}_1 - \vec{x}_1')G_{m_\lambda m_\mu}(x_1,x_1')] \\ &\quad * G_{m_\mu m_s}(x_1',y), \end{aligned} \tag{5.10}$$

which is shown in figure 5.3

Moving into momentum space, since the interaction is invariant under translations and the system is spatially uniform, G_{m_s}, $G^0_{m_s}$, $\Sigma^*_{m_s}$ and V may all be replaced by their four dimensional Fourier transforms. Therefore, equation 5.8 becomes

$$\Sigma^*_{m_\lambda m_\mu}(x_1,x_1') = \frac{i}{\hbar}\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x_1 - x_1')} \int \frac{d^4q}{(2\pi)^4}\left[\delta_{\lambda\mu}V(0)G_{m_\lambda m_\lambda}(q) - V(\vec{k}-\vec{q})G_{m_\lambda m_\mu}(q)\right], \tag{5.11}$$

Hence

$$\Sigma^*_{m_\lambda m_\mu}(k) = \frac{i}{\hbar}\int \frac{d^4q}{(2\pi)^4}\left[\delta_{\lambda\mu}V(0)G_{m_\lambda m_\lambda}(q) - V(\vec{k}-\vec{q})G_{m_\lambda m_\mu}(q)\right], \tag{5.12}$$

which, when substituted into 2.91 yields

$$\begin{aligned} G_{m_\lambda m_\mu}(k) &= G^0_{m_\lambda m_\mu}(k) + \frac{i}{\hbar}G^0_{m_\lambda m_\lambda}(k)\int \frac{d^4q}{(2\pi)^4}\,[\delta_{\lambda\mu}V(0)G_{m_\lambda m_\lambda}(q) \\ &\quad - V(\vec{k}-\vec{q})G_{m_\lambda m_\mu}(q)]G_{m_\mu m_\mu}(k). \end{aligned} \tag{5.13}$$

5.2.2 The Ladder Diagrams Approximation

In the ladder diagram approximation, the subclass of proper self-energy insertions, consisting of all the ladder diagrams between Green's functions with arrows running in the same direction, is retained.

$$\begin{aligned} \Sigma^*_{m_\lambda m_\mu}(x_1,x_1') &= \frac{i}{\hbar}\delta_{\lambda\mu}\delta(x_{1_0} - x'_{1_0})[\delta^{(3)}(\vec{x}_1 - \vec{x}_1')\int d^4x_2 \, V(\vec{x}_1 - \vec{x}_2')G^0_{m_\mu m_\mu}(x_2,x_2) \\ &\quad - V(\vec{x}_1 - \vec{x}_2')G^0_{m_\lambda m_\lambda}(x_1',x_1)] \\ &\quad \frac{1}{\hbar^2}\delta_{\lambda\mu}\int d^4x_2 \int d^4x_2' \, G^0_{m_\lambda m_\lambda}(x_2',x_2)V(x_1 - x_2)G^0_{m_\mu m_\mu}(x_2,x_2')G^0_{m_\mu m_\lambda}(x_2,x_2)V(x_1' - x_2') \\ &\quad - \frac{1}{\hbar^2}\int d^4x_2 \int d^4x_2' \, G^0_{m_\lambda m_\lambda}(x_2',x_1')V(x_1 - x_2')V(x_1' - x_2)G^0_{m_\lambda m_\lambda}(x_2,x_2')G^0_{m_\lambda m_\mu}(x_1,x_2) \\ &\quad + \ldots \end{aligned} \tag{5.14}$$

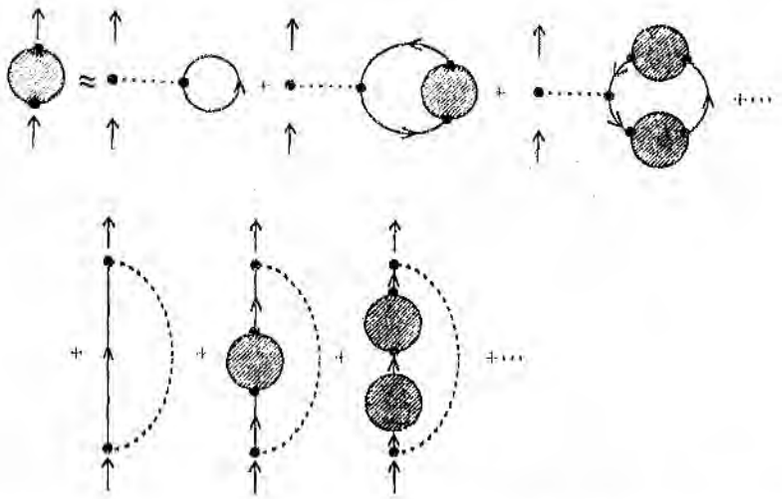

Figure 5.1: The series expansion of the proper self-energy in the self-consistent Hartree-Fock approximation.

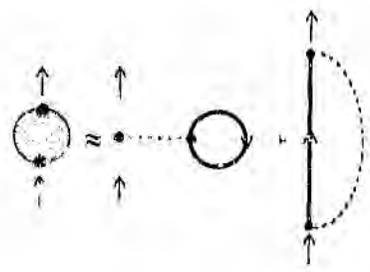

Figure 5.2: Integral representation of N_m^{σ}.

and

$$V(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} V(\vec{k}), \tag{5.6}$$

where $k = (\omega, \vec{k})$.

Substituting equation 5.3 into 5.79, yields the following expression for H_1:

$$H_1 = \frac{1}{2}\int d^3x \int d^3x' \psi^\dagger_\alpha(\vec{x})\psi^\dagger_\alpha(\vec{x}')V(\vec{x}-\vec{x}')\psi_\alpha(\vec{x}')\psi_\alpha(\vec{x}). \tag{5.7}$$

5.2 Approximations

Using perturbation theory it is possible to calculate the Green's function, and hence other observables, order by order. However, in most cases of interest, in order to obtain reasonable results, the perturbation expansion would have to be continued to infinite order. A practically impossible task, unless simplifying approximations can be made. The approximation used is usually dictated by experiment.

Approximations making it possible to calculate G to infinite orders in the interaction potential, can be roughly divided into two classes:

1. In the first class of approximations, the proper self-energy, Σ^* is replaced by the sum of a small number of proper self-energy insertions in which the noninteracting Green's function, G^0 is replaced by the full Green's function, G. Since, G is used to find Σ^*, which in turn is used to find G, such an approximation is self consistent.

2. The second class differs from the first, in that Σ^* is replaced by the sum of an *infinite* subset of unmodified proper self-energy insertions.

In both cases Dyson's equation 2.87 can be used to find the single-particle Green's function corresponding to the approximation. Either the Bethe-Salpeter equation 2.93, or the exact effective interaction given by equation 2.97 may be used to find the two-particle Green's function.

5.2.1 The Hartree-Fock Approximation

The Hartree-Fock approximation [37, 38, 39] belongs to the first class of approximations described above. This approximation is obtained by replacing the proper self-energy by the sum of its two first order contributions, with G^0 replaced by G, i.e.

$$\Sigma^*_{\alpha_1\alpha_2}(x_1,x_1') = \frac{i}{\hbar}\delta(x_{1_0}-x_{1_0}')\{\delta^{(3)}(\vec{x}_1-\vec{x}_1')\delta_{\alpha_1\alpha_2}\int d^3x_2 V(\vec{x}_1-\vec{x}_2)G_{\alpha_2\alpha_2}(x_2,x_2)$$

$$- V(\vec{x}_1-\vec{x}_1')G_{\alpha_1\alpha_2}(x_1,x_1')\}, \tag{5.8}$$

as depicted in figures 5.1 and 5.2.

Remembering that $G_{\alpha\beta}(x_2,x_2)$ is just the particle density at the point $\vec{x}_2$ at time $t = x_{2_0}$, equation 5.8 becomes

$$\Sigma^*_{\alpha_1\alpha_2}(x_1,x_1') = \frac{i}{\hbar}\delta(x_{1_0}-x_{1_0}')\left[\delta^{(3)}(\vec{x}_1-\vec{x}_1')\delta_{\alpha_1\alpha_2}\int d^3x_2 V(\vec{x}_1-\vec{x}_2)n(x_2)\right.$$

$$\left. - V(\vec{x}_1-\vec{x}_1')G_{\alpha_1\alpha_2}(x_1,x_1')\right]. \tag{5.9}$$

I.e. in this approximation the proper self energy consists of a local term proportional to the particle density and a nonlocal term. The local term is known as the direct term, while the nonlocal term is called the exchange term.

Chapter 5

Traditional Condensed Matter Approaches to the Many-Body Problem

In most condensed matter calculations, the observable of interest is usually related to the single particle Green's function. The Green's function is then calculated perturbatively in the operator representation of the particular second quantized field theory under consideration.

In this thesis only the zero-temperature formalism is considered.

5.1 The Typical System

The typical system considered is an interacting multi-electron system in a static spin-independent external potential. In the bilocal formalism [1] an external potential can be introduced straightforwardly, since it couples to the density, i.e. to σ. However, for the sake of simplicity, external potentials will be ignored, and the system considered will be translationally invariant. Such a system is usually described by a Hamiltonian, H, given by

$$H = H_0 + H_1, \tag{5.1}$$

where

$$H_0 = \int d^3x \, \psi^\dagger_\alpha(\vec{x}) \frac{\hbar^2 \nabla^2}{2m} \psi_\alpha(\vec{x}), \tag{5.2}$$

and H_1 is of the form shown in equation 2.79. For simplicity it is also assumed that the interparticle potential is spin independent and static in time, i.e.

$$\begin{aligned} V_{\lambda\lambda'\mu\mu'}(\vec{x}, \vec{x}') &= V(\vec{x} - \vec{x}')\delta_{\lambda\lambda'}\delta_{\mu\mu'} \\ &= V(\vec{x}' - \vec{x})\delta_{\lambda\lambda'}\delta_{\mu\mu'}. \end{aligned} \tag{5.3}$$

Using equation 2.80, this corresponds to

$$\begin{aligned} U_{\lambda\lambda'\mu\mu'}(x, x') &= \delta(x_0 - x'_0)V(\vec{x} - \vec{x}')\delta_{\lambda\lambda'}\delta_{\mu\mu'} \\ &= U(x - x')\delta_{\lambda\lambda'}\delta_{\mu\mu'} \\ &= U(x' - x)\delta_{\lambda\lambda'}\delta_{\mu\mu'}. \end{aligned} \tag{5.4}$$

The corresponding Fourier transforms of V and U are:

$$V(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} V(\vec{k}), \tag{5.5}$$

vertices are given by

$$
\begin{aligned}
S_{tp} &= i\delta_{\alpha\gamma} L^4 \delta^4(0) \frac{1}{\sqrt{N}} C_1 \\
&= i\delta_{\alpha\gamma} L^4 \delta^4(0) \frac{h}{\sqrt{N}} \operatorname{tr} \operatorname{Tr} \left((\sigma^0)^{-1} \eta \right) && (4.33) \\
&= i\delta_{\alpha\gamma} L^4 \delta^4(0) \frac{h}{\sqrt{N}} \int d^4p \, \sigma^{0^{-1}}_{\alpha\beta}(p) \eta_{\beta\alpha}(p, p), && (4.34)
\end{aligned}
$$

and,

$$
\begin{aligned}
S_c &= \frac{ih}{3\sqrt{N}} C_3 \\
&= \frac{i^h}{3\sqrt{N}} \operatorname{tr} \operatorname{Tr} \left((\sigma^0)^{-1} \eta \right)^3 \\
&= \frac{ih}{3\sqrt{N}} \int d^4k_1 \int d^4k_2 \int d^4k_3 \, \sigma^{0^{-1}}_{\alpha_1\beta_1}(k_1) \eta_{\beta_1\alpha_2}(k_1, k_2) \\
&\quad \times \sigma^{0^{-1}}_{\alpha_2\beta_2}(k_2) \eta_{\beta_2\alpha_3} \sigma^{0^{-1}}_{\alpha_3\beta_3}(k_3) \eta_{\beta_3\alpha_1}(k_3, k_1). && (4.35)
\end{aligned}
$$

Corrections of order $\frac{1}{\sqrt{N}}$ to the single-particle Green's function are given by the linear tadpole from S_{tp}, and the cubic tadpole from S_c. These corrections to $G^{(2)}$ are considered in more detail in section 5.2.3.

The other terms in equation 4.30, $n > 3$, together with S_{tp} and S_c, yield higher order corrections to the to *all* the Green's functions.

implies that the "classical" configuration σ^0 obeys the standard gap equation for the single-particle Green's function, given by the definitions 2.62 and 4.7

$$\begin{aligned} G_{\alpha\beta}(x,y) &= -i\frac{\langle\Psi_H|T(\psi_\alpha(x)\psi^\dagger_\beta(y))|\Psi_H\rangle}{\langle\Psi_H|\Psi_H\rangle} \\ &= i\frac{\langle\Psi_H|T(\psi^\dagger_\beta(y)\psi_\alpha(x))|\Psi_H\rangle}{\langle\Psi_H|\Psi_H\rangle} \\ &= \frac{i\int D\sigma\, e^{iS_{\rm eff}}\sigma^{-1}_{\beta\alpha}(y,x)}{\int D\sigma\, e^{iS_{\rm eff}}}. \end{aligned} \tag{4.27}$$

4.3 Green's Functions

It is now possible to use perturbation theory to obtain the mass spectrum of the effective field theory, by perturbing around σ_0. This yields an expression which, in terms of the original fermionic variables, is the two-particle propagator, $G^{(2)}$. Let

$$\sigma_{\alpha\beta}(x,y) = \sigma^0_{\alpha\beta}(x,y) + \frac{1}{\sqrt{N}}\eta_{\alpha\beta}(x,y). \tag{4.28}$$

Substituting this into equation 4.20 yields

$$S_{\rm eff} = ihN\,\delta_{\gamma\gamma}L^d\delta^d(0))\,{\rm tr}\,{\rm Tr}\,\ln(\sigma^0_{\gamma\beta}(x,y) + \frac{1}{\sqrt{N}}\eta_{\alpha\beta}(x,y)) + S, \tag{4.29}$$

which may be functionally Taylor expanded as

$$S_{\rm eff} = NS_0[\sigma^0] + S_1[\sigma^0] + \frac{i}{2}C_2 + \frac{1}{2}D_2 + \sum_{n=1}^{\infty}\frac{1}{\sqrt{N}^n}\left[i(-1)^{n+1}\{\frac{C_{n+2}}{n+2} + \delta_{\gamma\gamma}L^d\delta^d(0)\frac{C_n}{n}\} + \frac{1}{(n+2)!}D_{n+2}\right], \tag{4.30}$$

where

$$\begin{aligned} C_n &= h\,{\rm tr}\,{\rm Tr}((\sigma^0)^{-1}\eta)^n \\ D_n &= \int d^dx_1 \cdots \int d^dx_n \int d^dy_1 \cdots \int d^dy_n \left.\frac{\delta^n S}{\delta\sigma_{\alpha_1\beta_1}(x_1,y_1)\cdots\delta\sigma_{\alpha_n\beta_n}(x_n,y_n)}\right|_{\sigma_0} \\ &\quad \cdot\eta_{\alpha_1\beta_1}(x_1,y_1)\cdots\eta_{\alpha_n\beta_n}(x_n,y_n) \end{aligned} \tag{4.31}$$

The leading quadratic part of the effective action, S_2 is given by

$$\begin{aligned} S_2 &= -\frac{i}{2}C_2 + \frac{1}{2}D_2 \\ &= \frac{1}{2}\int d^dx_1\int d^dx_2\int d^dx_3\int d^dx_4\left[\left.\frac{\delta^2 S}{\delta\sigma_{\alpha_1\beta_1}(x_1,x_2)\delta\sigma_{\alpha_2\beta_2}(x_3,x_4)}\right|_{\sigma^0} - ih\sigma^{0-1}_{\beta_1\alpha_2}(x_1,x_2)\sigma^{0-1}_{\beta_2\alpha_1}(x_3,x_4)\right] \\ &\quad \cdot\eta_{\alpha_1\beta_1}(x_1,x_2)\eta_{\alpha_2\beta_2}(x_3,x_4). \end{aligned} \tag{4.32}$$

Equation 4.32 can be used to find the propagator $\Delta_{\alpha_1\beta_1\alpha_2\beta_2}$ of the field σ. Specific examples have been considered in section 4.2.3.

4.4 Perturbation Theory in the Effective Theory

Assuming that the theory under consideration is quadratic in the original fermion variables, then $D_{n>2} = 0$. However, C_n exists for all values of n. As a result the effective theory has an infinite number of vertices. Despite this, with N reinstated there exists a natural "coupling constant" $\frac{1}{\sqrt{N}}$. For example, from equation 4.30 it is apparent that the $\frac{1}{\sqrt{N}}$

The formal solution to equation 4.15 is

$$\begin{aligned}\ln J &= -(N + L^d\delta^d(0)\delta_{x,x})\delta_{x,y}\int d^dx \int d^dy\, \delta^d(x-y)\sigma_{x,x}(x,y) \\ &\equiv -(N + L^d\delta^d(0)\delta_{x,x})\ \mathrm{Tr}\ \mathrm{tr}\ \ln\sigma. \end{aligned} \tag{4.16}$$

The bilocal bosonization formalism was developed in the context of the large N expansion, where the following approximation is justifiable,

$$\ln J \approx -N\ \mathrm{Tr}\ \mathrm{tr}\ \ln\sigma. \tag{4.17}$$

Although, $N = 1$ in this case, this approximation will still be used to obtain the gap equation. This is justifiable, in the sense that, the resulting two-point propagator is found to satisfy the standard gap equation, as will be shown in chapter 6.

4.2 The Effective Action and Perturbation Theory

The path integral in the original variables is rewritten as

$$\begin{aligned}\int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S[\psi,\psi^\dagger]} &\equiv \int \mathcal{D}\sigma\, J e^{\frac{i}{\hbar}S[\sigma]} \\ &= \int \mathcal{D}\sigma\, e^{\ln J + \frac{i}{\hbar}S} \\ &\equiv \int \mathcal{D}\sigma\, e^{\frac{i}{\hbar}S_{\mathrm{eff}}}. \end{aligned} \tag{4.18}$$

where the "effective action", S_{eff} is defined to be

$$S_{\mathrm{eff}} \equiv -i\hbar \ln J + S. \tag{4.19}$$

In the arbitrary N case, equation 4.16 may be substituted into equation 4.19, yielding

$$S_{\mathrm{eff}} = i\hbar(N + \delta_{x,x}L^d\delta^d(0))\ \mathrm{Tr}\ \mathrm{tr}\ \ln\sigma + S. \tag{4.20}$$

If the fields are rescaled i.e.

$$\begin{aligned}\psi_\alpha(x) &\to \sqrt{N}\psi_\alpha(x) \\ \psi^\dagger_\alpha(x) &\to \sqrt{N}\psi_\alpha(x). \end{aligned} \tag{4.21}$$

then $S \to NS$ and equation 4.20 becomes

$$\begin{aligned}S_{\mathrm{eff}} &= i\hbar N\ \mathrm{tr}\ \mathrm{Tr}\ \ln\sigma + i\hbar\delta_{x,x}L^d\delta^d(0)\ \mathrm{tr}\ \mathrm{Tr}\ \ln\sigma + NS \\ &= NS_0 + S_1 \end{aligned} \tag{4.22}$$

where

$$S_0 = i\hbar\ \mathrm{tr}\ \mathrm{Tr}\ln\sigma + S \tag{4.23}$$

and

$$S_1 = i\hbar\delta_{x,x}L^d\delta^d(0)\ \mathrm{Tr}\ \mathrm{tr}\ \ln\sigma. \tag{4.24}$$

In the limit $N \to \infty$

$$S_{\mathrm{eff}} \to NS_0 \tag{4.25}$$

and the leading contribution to σ is determined by the stationary point of S_0 i.e. by the field minimising S_0. As will be demonstrated later, the condition

$$\left.\frac{\delta S_0}{\delta\sigma_{x,x}(x,y)}\right|_{\sigma_0} = 0 \tag{4.26}$$

and

$$\int \mathcal{D}\psi \mathcal{D}\psi^\dagger K[\psi,\psi^\dagger] \longrightarrow \int \mathcal{D}\sigma J K[\sigma] \tag{4.9}$$

where it is assumed that J, the functional Jacobian associated with the change of variables, is well defined and exists [1]. In addition, since the bilocals were defined as the product of fermionic fields at *different* points (equation 4.7), they are in general *not* nilpotent and form a much larger set than the density variables[3] used in more traditional approaches to bosonization. As a result, the integration measure in equation 4.9 is well-defined.

Using the functional form of the chain rule, the following equation is obtained:

$$\begin{aligned}\frac{\delta}{\delta\psi^\dagger_\alpha(x)} &= \int d^dy \int d^dz \frac{\delta\sigma_{\gamma\beta}(y,z)}{\delta\psi^\dagger_\alpha(x)} \frac{\delta}{\delta\sigma_{\gamma\beta}(y,z)} \\ &= -\int d^dy\, \psi^\dagger_\beta(y) \frac{\delta}{\delta\sigma_{\alpha\beta}(y,x)}\end{aligned} \tag{4.10}$$

Using equations 4.9, 4.10, and 4.6,

$$\begin{aligned}0 &= \int \mathcal{D}\sigma J \left[N\delta_{\alpha\beta}\delta^d(x-y)F[\sigma] - \int d^dz\, \sigma_{\gamma\alpha}(z,y)\frac{\delta F}{\delta\sigma_{\gamma\beta}(z,x)} - \frac{i}{\hbar}\int d^dz\, \sigma_{\gamma\alpha}(z,y)\frac{\delta S}{\delta\sigma_{\gamma\beta}(z,x)}F[\sigma]\right] e^{\frac{i}{\hbar}S} \\ &= \langle N\delta_{\alpha\beta}\delta^d(x-y)F[\sigma]\rangle - \int d^dz \langle \sigma_{\gamma\alpha}(z,y)\frac{\delta F}{\delta\sigma_{\gamma\beta}(z,x)}\rangle - \frac{i}{\hbar}\int d^dz \langle \sigma_{\gamma\alpha}(z,y)\frac{\delta S}{\delta\sigma_{\gamma\beta}(z,x)}F[\sigma]\rangle,\end{aligned} \tag{4.11}$$

is obtained.

A similar identity to 4.2 in terms of the invariant bilocal variables may also be obtained

$$\begin{aligned}0 &= \int \mathcal{D}\sigma \int d^dz \frac{\delta}{\delta\sigma_{\gamma\beta}(z,x)}\left[\sigma_{\gamma\alpha}(z,y)J F[\sigma]e^{\frac{i}{\hbar}S}\right] \\ &= \int \mathcal{D}\sigma \left[\delta_{\gamma\gamma}\delta_{\alpha\beta}L^d\delta^d(0)\delta^d(x-y)F[\sigma] + \int d^dz \frac{\sigma_{\gamma\alpha}(z,y)}{J}\frac{\delta J}{\delta\sigma_{\gamma\beta}(z,x)}F[\sigma]\right. \\ &\quad \left. + \int d^dz\, \sigma_{\gamma\alpha}(z,y)\frac{\delta F}{\delta\sigma_{\gamma\beta}(z,x)} + \frac{i}{\hbar}\int d^dz\, \sigma_{\gamma\alpha}(z,y)F[\sigma]\frac{\delta S}{\delta\sigma_{\gamma\beta}(z,x)}\right] Je^{\frac{i}{\hbar}S} \\ &= \langle\delta_{\gamma\gamma}\delta_{\alpha\beta}L^d\delta^d(0)\delta^d(x-y)F[\sigma]\rangle + \int d^dz \langle\sigma_{\gamma\alpha}(z,y)\frac{\delta \ln J}{\delta\sigma_{\gamma\beta}(z,x)}F[\sigma]\rangle \\ &\quad + \int d^dz \langle\sigma_{\gamma\alpha}(z,y)\frac{\delta F}{\delta\sigma_{\gamma\beta}(z,x)}\rangle + \frac{i}{\hbar}\int d^dz \langle\sigma_{\gamma\alpha}(z,y)F[\sigma]\frac{\delta S}{\delta\sigma_{\gamma\beta}(z,x)}\rangle\end{aligned} \tag{4.12}$$

where $\int d^dx = L^d$ is the volume of the system.

Adding the Schwinger Dyson equations 4.11 and 4.12 together yields the following differential equation for the Jacobian, J:

$$\int d^dz\, \sigma_{\gamma\alpha}(z,y)\frac{\delta \ln J}{\delta\sigma_{\gamma\beta}(z,x)} = -(1+L^d\delta^d(0))\delta_{\gamma\gamma}\delta_{\alpha\beta}\delta^d(x-y). \tag{4.13}$$

The above discussion has been for the $N=1$ case. Had there been N species, i.e. had the field operators been given by $\psi^a_\alpha(x)$, and $\psi^{a\dagger}_\beta(x)$ where $a=1,\ldots,N$, then definition 4.7 would have been

$$\sigma_{\alpha\beta}(x,y) = \psi^{a\dagger}_\alpha(x)\psi^a_\beta(y), \tag{4.14}$$

i.e. σ would have been defined to be invariant in the space labelled by the a index. In this case, equation 4.13 would become

$$\int d^dz\, \sigma_{\gamma\alpha}(z,y)\frac{\delta \ln J}{\delta\sigma_{\gamma\beta}(z,x)} = -(N+L^d\delta^d(0))\delta_{\gamma\gamma}\delta_{\alpha\beta}\delta^d(x-y). \tag{4.15}$$

[3] which are defined as the product of fermionic fields at the same point in space

6.1.2 Free Green's Function

If $|\Phi_0\rangle$ is the normalised noninteracting ground state in the position basis, then the single-particle *Green's function* is, by *definition*,

$$G^0_{\sigma_1\sigma_2}(x,x') = -i\langle\Phi_0|T(\psi_{\sigma_1}(x)\psi^\dagger_{\sigma_2}(x'))|\Phi_0\rangle. \tag{6.19}$$

Using the bilocal invariants (equation 6.3) it is easily seen that

$$\begin{aligned} G^0_{\sigma_1\sigma_2}(x,x') &= i\langle\Phi_0|T(\psi^\dagger_{\sigma_2}(x')\psi_{\sigma_1}(x))|\Phi_0\rangle \\ &\quad i\langle\Phi_0|\sigma^0_{\sigma_1\sigma_2}(x',x)|\Phi_0\rangle \\ &= i\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}\sigma^0_{\sigma_1\sigma_2}(k) \qquad (6.20) \\ &= \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{\omega-\omega^*_k}\delta_{\sigma_1\sigma_2} \qquad (6.21) \end{aligned}$$

However, the free Green's function, $G^0_{\sigma\sigma}$ in the conventional Fermi vacuum, as derived in equation 2.78, is given by

$$G^0_{\sigma_1\sigma_2}(x,x') = \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{\omega_k+i\varepsilon\,\mathrm{sign}(|k|-k_F)}\delta_{\sigma_1\sigma_2}. \tag{6.22}$$

Hence, the correct prescription is

$$\omega_k \to \omega_k - i\varepsilon\,\mathrm{sign}(|k|-k_F), \tag{6.23}$$

which ensures that the constraint

$$\int d^4x\,\psi^\dagger_\sigma(x)\psi_\sigma(x) = N, \tag{6.24}$$

where N, is the number of particles in the system, is correctly taken into account.

6.2 Interacting Fermions

The Hamiltonian describing interacting fermions is given by equation 5.7, and the corresponding classical action in the path integral is given by

$$\begin{aligned} S = &\int d^4x\,\psi^\dagger_\sigma(x)(i\hbar\partial_{x_0}+\frac{\hbar^2\nabla^2}{2m})\psi_\sigma(x) - \frac{\lambda}{2}\int d^4x\int d^4x'\,V(\vec{x}-\vec{x}')\delta(x_0-x'_0)\times \\ &\psi^\dagger_\sigma(x)\psi_\sigma(x)\psi^\dagger_{\sigma'}(x')\psi_{\sigma'}(x), \end{aligned} \tag{6.25}$$

where ψ and $\psi^\dagger$ are *Grassman* variables.

The variables in which the bilocal formalism was originally developed had a degree of freedom, namely colour, not present in normal condensed matter nonrelativistic theories. As originally defined, the bilocals were colour invariants i.e. contracted over the colour index. In the present case, since there are no such indices ($N=1$), there exists an ambiguity in the way in which S is rewritten in terms of the bilocal variables. Since,

$$\psi^\dagger_\sigma(x)\psi^\dagger_{\sigma'}(x')\psi_{\sigma'}(x')\psi_\sigma(x) \qquad \psi^\dagger_\sigma(x)\psi_\sigma(x')\psi^\dagger_{\sigma'}(x')\psi_{\sigma'}(x)$$
$$\sigma_{\sigma\sigma}(x,x')\sigma_{\sigma'\sigma'}(x',x)$$

and

$$\psi^\dagger_\sigma(x)\psi^\dagger_{\sigma'}(x')\psi_{\sigma'}(x')\psi_\sigma(x) \qquad \psi^\dagger_\sigma(x)\psi_{\sigma'}(x')\psi^\dagger_{\sigma'}(x')\psi_\sigma(x')$$
$$\sigma_{\sigma\sigma'}(x,x')\sigma_{\sigma'\sigma}(x',x')$$

Chapter 6

Bilocal Bosonization of Condensed Matter Systems

6.1 Free Fermions

In the case of noninteracting nonrelativistic fermions, the Hamiltonian in d-dimensions takes the form

$$\begin{aligned} H_0 &= \int d^d x \frac{\hbar^2}{2m} \vec{\nabla} \psi_\alpha^\dagger(x) \cdot \vec{\nabla} \psi_\alpha(x) \\ &= -\int d^d x \, \psi_\alpha^\dagger(x) \frac{\hbar^2 \nabla^2}{2m} \psi_\alpha(x) \end{aligned} \tag{6.1}$$

and the action functional is

$$S_0 = \int d^{1+d} x \, \psi_\alpha^\dagger(x) [i\hbar\partial_{x_0} + \frac{\hbar^2 \vec{\nabla}^2}{2m}] \psi_\alpha(x) \tag{6.2}$$

In terms of the bilocal invariants introduced in chapter 4

$$\sigma_{\alpha\alpha}(x, x') = \psi_\alpha^\dagger(x) \psi_\alpha(x') \tag{6.3}$$

the action is rewritten as

$$\begin{aligned} S_0 &= \int d^{1+d} x \int d^{1+d} x' \delta^{1+d}(x - x') (i\hbar\partial_{x'_0} + \frac{\hbar^2 \vec{\partial}^2_{\vec{x}} - \vec{\partial}^2_{\vec{x}'}}{2m}) \psi_\alpha^\dagger(x) \psi_\alpha(x') \\ &= \int d^{1+d} x \int d^{1+d} x' \delta^{1+d}(x - x') (i\hbar\partial_{x'_0} + \frac{\hbar^2 \vec{\partial}^2_{\vec{x}} - \vec{\partial}^2_{\vec{x}'}}{2m}) \sigma_{\alpha\alpha}(x, x') \\ &= -\int d^{1+d} x \int d^{1+d} x' \sigma_{\alpha\alpha}(x, x') (-i\hbar\partial_{x'_0} + \frac{\hbar^2 \vec{\partial}^2_{\vec{x}} - \vec{\partial}^2_{\vec{x}'}}{2m}) \delta^{1+d}(x - x'), \end{aligned} \tag{6.4}$$

where the identity

$$\int d^{1+d} x \, f(x) \partial_x \delta^{1+d}(x - y) = -\partial_y f(y), \tag{6.5}$$

has been used.

Substituting equation 6.4 into equation 4.22, the effective action for nonrelativistic fermions is obtained

$$\begin{aligned} S_{\text{eff}} &= ih \operatorname{tr} \operatorname{Tr} \ln \sigma + S_0 + ih\delta_{x_0, L^{1+d}}(0) \operatorname{tr} \operatorname{Tr} \ln \sigma \\ &= S_0 + S_1 \end{aligned} \tag{6.6}$$

where S_0 and S_1 are given by equations 4.23 and 4.24 respectively, i.e.

$$S_0 = ih \operatorname{tr} \operatorname{Tr} \ln \sigma + S_0 \tag{6.7}$$

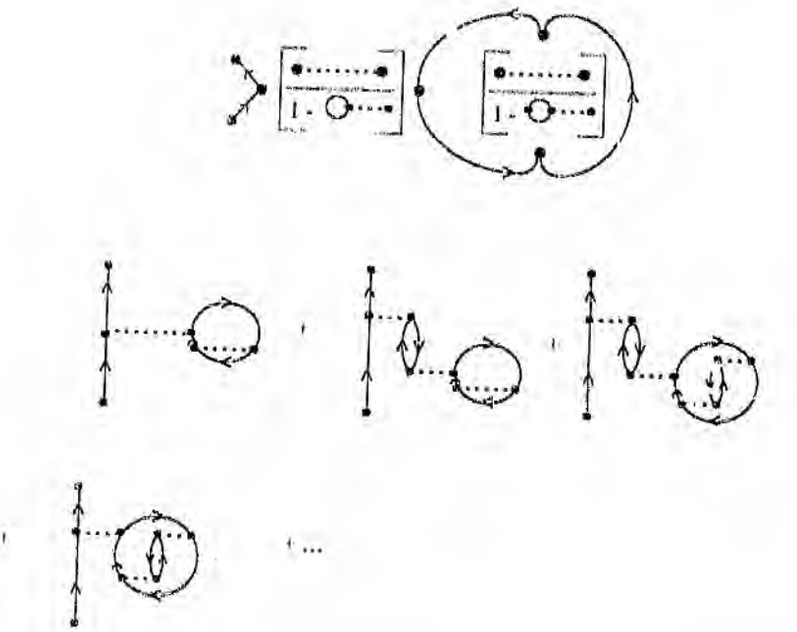

Figure 6.1: Structure of the second term in equation 6.62 in the ring diagram approximation

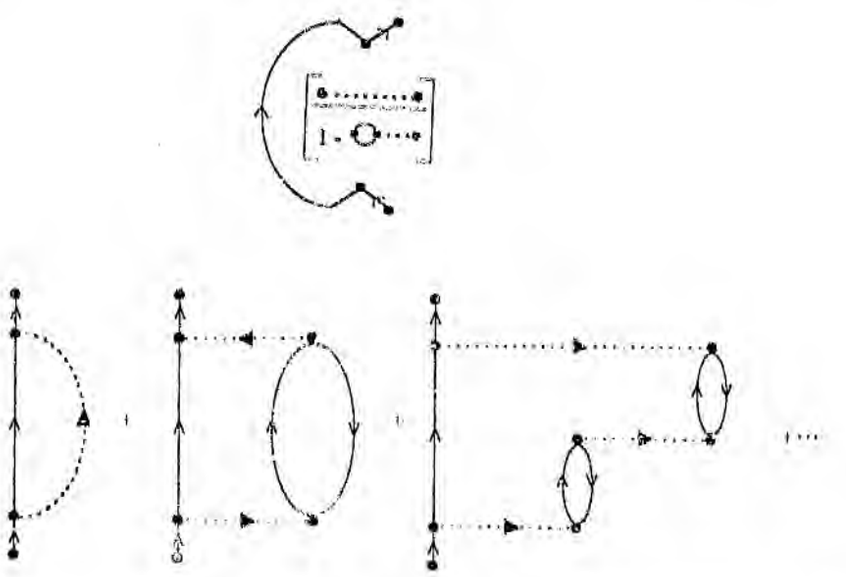

Figure 6.2: Lowest order corrections to the single-particle Green's function in the ring diagram approximation

$$+\left(\frac{i}{\hbar(2\pi)^3}\right)^3 V^3(\vec{p}_1 - \vec{p}_2')\int d^3q\,\sigma^0_{F_{\alpha\alpha_1}}(p_1 - p_2' + q)\sigma^0_{F_{\alpha_1\alpha_2}}(q)$$
$$\int d^3q_1\sigma^0_{F_{\alpha_2\alpha_3}}(p_1 - p_2' + q_1)\sigma^0_{F_{\alpha_3\alpha_2}}(q_1) + \left(\frac{i}{\hbar(2\pi)^3}\right)^3 V^3(\vec{p}_1 - \vec{p}_2')$$
$$\int d^3q\,\sigma^0_{F_{\alpha\alpha_1}}(p_1 - p_2' + q)\sigma^0_{F_{\alpha_1\alpha}}(q)$$
$$\int d^3q_1\sigma^0_{F_{\alpha\alpha_2}}(p_1 - p_2' + q_1)\sigma^0_{F_{\alpha_2\alpha}}(q_1)$$
$$\int d^3q_2\sigma^0_{F_{\alpha\alpha_3}}(p_1 - p_2' + q_2)\sigma^0_{F_{\alpha_3\alpha}}(q_2) + \cdots$$
$$\frac{i}{\hbar(2\pi)^3}V(\vec{p}_1 - \vec{p}_2')\sigma^0_{F_{\alpha\alpha}}(p_1)\sigma^0_{F_{\alpha\alpha}}(p_2)\sigma^0_{F_{\alpha'\alpha}}(p_3)\sigma^0_{F_{\alpha\alpha'}}(p_4)$$
$$+\frac{i}{\hbar(2\pi)^3}V(\vec{p}_1 - \vec{p}_2')\int d^3q\,G^0_{F_{\alpha\alpha_1}}(p_1 - p_2' + q)G^0_{F_{\alpha_1\alpha}}(q) \tag{6.66}$$

Equation 6.66 can be rewritten in terms of the two body effective interaction, V_a, and the lowest order polarization insertion, Π^0, given by equations 5.27 and 5.25, respectively.

$$G_{\alpha\alpha'\alpha'\alpha}(p_1, p_2; p_3, p_4) = \frac{i}{\hbar(2\pi)^3}\sigma^0_{F_{\alpha\alpha}}(p_1)\sigma^0_{F_{\alpha\alpha}}(p_2)\sigma^0_{F_{\alpha'\alpha'}}(p_3)\sigma^0_{F_{\alpha'\alpha'}}(p_4)$$
$$V(\vec{p}_1 - \vec{p}_2')$$
$$+\frac{i}{\hbar(2\pi)^3}V(\vec{p}_1 - \vec{p}_2')\int d^3q\,G^0_{F_{\alpha\alpha_1}}(p_1 - p_2' + q)G^0_{F_{\alpha_1\alpha}}(q)$$
$$\frac{i}{\hbar(2\pi)^3}\sigma^0_{F_{\alpha\alpha}}(p_1)\sigma^0_{F_{\alpha\alpha}}(p_2)\sigma^0_{F_{\alpha'\alpha'}}(p_3)\sigma^0_{F_{\alpha'\alpha'}}(p_4)\left[1 - V(\vec{p}_1 - \vec{p}_2')\Pi^0_{F_{\alpha\alpha_1}}(p_1 - p_2)\right]$$
$$\frac{i}{\hbar(2\pi)^3}\sigma^0_{F_{\alpha\alpha}}(p_1)\sigma^0_{F_{\alpha\alpha}}(p_2)\sigma^0_{F_{\alpha'\alpha'}}(p_3)\sigma^0_{F_{\alpha'\alpha'}}(p_4)V_a(p_1 - p_2). \tag{6.67}$$

Comparison with equation 5.30, shows that equation 6.67 is indeed the ring diagram approximation to the two point Green's function.

Higher Order Corrections

In the ring diagram approximation, the term in equation 6.62 containing the factor $G_{\alpha\alpha'\alpha'\alpha}(p_1, p_3; k_1, k_1)$, yields the series of Feynman diagrams depicted in figure 6.1. As is evident from the diagrams this term is proportional to $V(0)$, and hence, it vanishes in the ring diagram approximation. Therefore, equation 6.62 reduces to

$$\langle\eta\rangle_0 = \delta^3(p_1 - p_3)\int d^3p\,\sigma^0_{F_{\alpha\alpha}}(p)G_{\alpha\alpha'\alpha'\alpha}(p_1, p; p, p_4) \tag{6.68}$$

Substituting equation 6.67 into 6.68, and using the relation 6.37 yields

$$\langle\eta\rangle_0 = \frac{i}{\hbar}V_a(q)G^0(p)G^0(p)G^0(p - q), \tag{6.69}$$

which may be depicted graphically as shown in figure 6.2. Comparing $\langle\eta\rangle_0$ with equation 5.29 make it obvious that $\langle\eta\rangle_0$ is the single particle Green's function in the ring diagram approximation.

6.3.2 Ladder Diagram Approximation

On the other hand if $A = 1$ and $B = 0$ then the equation 6.35 becomes

$$\sigma_{-1}(k) = \sigma^0_{-1}(k) + \frac{i}{\hbar}\int\frac{d^3q}{(2\pi)^3}V(k - q)\sigma^0_{-1}(q) \tag{6.70}$$

In this case $\sigma^0 = \sigma^0_F$ and equation 6.40 becomes

$$G_{\alpha\alpha'\alpha'\alpha}(p_1, p_2; p_3, p_4) = \frac{i}{\hbar(2\pi)^3}\sigma^0_{F_{\alpha\alpha}}(p_1)\sigma^0_{F_{\alpha\alpha}}(p_2)\sigma^0_{F_{\alpha'\alpha}}(p_3)\sigma^0_{F_{\alpha\alpha'}}(p_4)[1 - (p_3' - p_2')$$
$$\int d^3q\,[1 - \sigma^0_{F_{\alpha\alpha}}(p_1 - p_2' + q, q, p_3, p_4)]. \tag{6.71}$$

Therefore the $\frac{1}{N}$ correction is given by

$$\frac{1}{N}\int d^4k_1\int d^4k_2\,\sigma^{(0)}_{\alpha_1\beta_1}(k_1)\sigma^{(0)}_{\alpha_2\beta_2}(k_1)\sigma^{(0)}_{\alpha_3\beta_3}(k_2)$$
$$\times\Lambda_{\alpha_1\alpha_2\alpha_3}(p_1,p_2;k_1,k_1)G_{\beta_2\alpha_4\alpha_5\beta_3}(k_1,k_2;k_2,k_1) \tag{6.61}$$

notice that the correction from the linear tadpole exactly cancels one of the terms resulting form the cubic tadpole. Substituting equation 6.54 into 6.61, and setting $N=1$, the leading order correction to the single particle Green's function, is given by

$$\eta^{(1)}_{\alpha\beta}(p_1,p_2)=\int d^4k_1\int d^4k_2\,\sigma^{(0)}_{\alpha_1\beta_1}(k_1)\sigma^{(0)}_{\alpha_2\beta_2}(k_1)\sigma^{(0)}_{\alpha_3\beta_3}(k_2)\,\Lambda_{\alpha\alpha_1\alpha_2}(p_1,p_2;k_1,k_1)$$
$$\times\big[\delta^4(p_1-k_1)\delta^4(p_2-k_1)\sigma^{(0)}_{\alpha_3\beta}(p_1)\sigma^{(0)}_{\beta_3\beta}(p_2)+\delta^4(p_1-p_2;k_1,k_1)\big]$$
$$-\delta^4(p_1-p_2)\int d^4p\,\sigma^{(0)}_{\alpha\beta}(p)G_{\beta\alpha_1\alpha_2\alpha}(p_1,p,p,p_1)$$
$$\delta^4(p_1-p_2)\int d^4k_1\int d^4k_2\,\sigma^{(0)}_{\alpha_1\beta_1}(k_1)\sigma^{(0)}_{\alpha_2\beta_2}(k_1)\sigma^{(0)}_{\alpha_3\beta_3}(k_2)$$
$$\times G_{\alpha_1\alpha_2\alpha_3\alpha}(k_1,k_2;k_2,k_1)G_{\beta\alpha_4\alpha_5\beta}(p_1,p_1;k_1,k_1) \tag{6.62}$$

6.3 Traditional Approximations in the Context of the Effective Theory

In this section, the possibility that certain traditional condensed matter physics approximations may by obtained by choosing values for A and B is investigated.

6.3.1 Ring Diagram Approximation

Let $A=0$ and $B=1$, then the gap equation (equation 6.35) becomes

$$\sigma^{-1}_{\alpha\beta}(k)=(\omega-\epsilon)\delta_{\alpha\beta}-\frac{1}{\beta}\int\frac{d^4q}{(2\pi)^4}V(0)\sigma^{0}_{\alpha\beta}(q) \tag{6.63}$$

Furthermore, if the number of electrons in the system is N_e and a uniform positive background with charge $N_e e$ is assumed, then all terms involving $V(0)$ are zero (as discussed in section 5.2.3), i.e. the tadpole diagrams vanish. The gap equation reduces to

$$\sigma^{-1}_{\alpha\beta}(k)=(\omega-\epsilon)\delta_{\alpha\beta} \tag{6.64}$$

i.e. $\sigma^{-1}=\sigma^{0}_{f}$, in this case.

Therefore, equation 6.60 reduces to

$$G_{\alpha\alpha_1\alpha_2\beta}(p_2,p_1;p_1,p_2)=\frac{i}{\hbar(2\pi)^4}\lambda(p_1-p_2)\sigma^{0}_{f\alpha\beta}(p_1)\sigma^{0}_{f\alpha_1\alpha_2}(p_2)[\sigma^{0}_{f\alpha_3\beta}(p_2)\sigma^{1}_{f\alpha_4\beta}(p_1)$$
$$-\int d^4q\,G_{\alpha_3\alpha_4\alpha_5\alpha_6}(p_1,p_2+q,q,p_1)\sigma^{0}_{f\alpha_5\beta}(p_2+q)\sigma^{0}_{f\alpha_6\beta}(q)] \tag{6.65}$$

Iterating equation 6.65 yields

$$G_{\alpha\alpha_1\alpha_2\beta}(p_2,p_1;p_1,p_2)=\frac{i}{\hbar(2\pi)^4}\lambda(p_1-p_2)\sigma^{0}_{f\alpha\beta}(p_1)\sigma^{0}_{f\alpha_1\alpha_2}(p_2)\sigma^{0}_{f\alpha_3\beta}(p_2)\sigma^{0}_{f\alpha_4\beta}(p_1)$$
$$-\left(\frac{i}{\hbar(2\pi)^4}\right)^2\lambda(p_1-p_2)\int d^4q\,\sigma^{0}_{f\alpha\alpha_1}(p_1-p_2+q)\sigma^{0}_{f\alpha_2\alpha_3}(q)$$

Assuming that it is possible to write

$$\Lambda_{\alpha\alpha'\beta\beta'}(p_1,p_2,p_3,p_4) = \langle \eta_{\alpha\alpha'}(p_1-p_2)\eta_{\beta\beta'}(p_3,p_4)\rangle$$
$$\delta^4(p_1-p_1)\delta^4(p_2-p_2)U_{\alpha\alpha'\beta\beta'}(p_1,p_2)+\delta^4(p_1-p_2+p_3-p_4)$$
$$\cdot G_{\alpha\alpha'\beta\beta'}(p_1-p_2,p_3,p_4). \tag{6.54}$$

and substituting this ansatz into equation 6.53 gives a solution for F and G:

$$U_{\alpha\alpha'\beta\beta'}(p_1,p_2) = \sigma^0_{\alpha\beta}(p_1)\sigma^0_{\alpha'\beta'}(p_2) \tag{6.55}$$

and

$$G_{\alpha\alpha'\beta\beta'}(p_1,p_2,p_3,p_4) = \frac{Ni}{\hbar(2\pi)^4}\sigma^0_{\alpha\alpha}(p_1)\sigma^0_{\beta\beta}(p_2)[\sigma^0_{\alpha'\mu}(p_3)\sigma^0_{\nu\beta'}(p_4)V(p_3-p_2)$$
$$-\int d^4q\, V(q-p_2)G_{\mu\alpha\nu\beta'}(p_3-p_2+q,q,p_3,p_4)]$$
$$\frac{Ni}{\hbar(2\pi)^4}V(p_1-p_3)\sigma^0_{\alpha\alpha}(p_1)\sigma^0_{\beta\beta}(p_2)[\sigma^0_{\alpha'\nu}(p_3)\sigma^0_{\mu\beta'}(p_4)$$
$$\int d^4q\, G_{\mu\alpha\nu\beta'}(p_1-p_2+q,q,p_3,p_4)]. \tag{6.56}$$

6.2.4 Higher Order Corrections

Reinstating N-corrections of order $\frac{1}{\sqrt{N}}$ to σ^c (i.e. to $G^{(2)}$) are obtained from $S_{\eta\eta}$ and S_{η^3} which are given by equations 4.34 and 4.35, respectively. This follows from the fact that

$$\int D\eta e^{\frac{i}{\hbar}[S_0+\frac{1}{\sqrt{N}}S_{\eta\eta}+\frac{1}{N}S_{\eta^3}]} \int D\eta(1+\frac{i}{\hbar}\frac{1}{\sqrt{N}}S_{\eta\eta}+\ldots)(1+\frac{i}{\hbar}\frac{1}{N}S_{\eta^3}+\ldots)e^{\frac{i}{\hbar}S_0}$$
$$\int D\eta(1-\frac{\delta_{\alpha\alpha}L^4\delta^4(0)}{\sqrt{N}}\mathrm{tr}\,\mathrm{Tr}((\sigma^0)^{-1}\eta)-\frac{1}{3\sqrt{N}}\mathrm{tr}\,\mathrm{Tr}((\sigma^0)^{-1}\eta)^3+\ldots)e^{\frac{i}{\hbar}S_0}$$
$$\sigma^0-\frac{\delta_{\alpha\alpha}L^4\delta^4(0)}{\sqrt{N}}\mathrm{tr}\,\mathrm{Tr}((\sigma^0)^{-1}\langle\eta\rangle)-\frac{1}{3\sqrt{N}}\mathrm{tr}\,\mathrm{Tr}((\sigma^0)^{-1}\langle\eta\rangle)^3+\ldots \tag{6.57}$$

Hence, the linear tadpole is given by

$$\frac{1}{\sqrt{N}}\delta_{\alpha\alpha}L^4\delta^4(0)\int d^4p\,\sigma^{0\,-1}_{\beta\beta'}(p)\Lambda_{\alpha\alpha'\beta\beta'}(p_1,p_2;p_3,p_4). \tag{6.58}$$

and the cubic tadpole[1] by

$$\frac{1}{\sqrt{N}}\int d^4k_1\int d^4k_2\int d^4k_3\,\sigma^{0\,-1}_{\alpha_1\alpha'_1}(k_1)\sigma^{0\,-1}_{\alpha_2\alpha'_2}(k_2)\sigma^{0\,-1}_{\alpha_3\alpha'_3}(k_3)$$
$$\Lambda_{\alpha\alpha'\alpha_1\alpha'_1}(p_1-p_2;k_1,k_2)\Lambda_{\alpha_2\alpha'_2\alpha_3\alpha'_3}(k_2,k_3;k_3,k_1). \tag{6.59}$$

Substituting equations 6.54, 6.55, and 6.56 for $\Lambda_{\alpha\alpha'\alpha_1\alpha'_1}(k_2,k_3;k_3,k_1)$ into 6.59 yields

$$\delta^4(k_1-k_2)G_{\alpha\alpha'\alpha_1\alpha'_1}(k_2,k_3;k_3,k_1)$$
$$\frac{\delta_{\alpha\alpha}L^4\delta^4(0)}{\sqrt{N}}\int d^4p\,\sigma^{0\,-1}_{\beta\beta'}\Lambda_{\alpha\alpha'\beta\beta'}(p_1,p_2;p,p)$$
$$\frac{1}{\sqrt{N}}\int d^4k_1\int d^4k_2\,\sigma^{0\,-1}_{\alpha_1\alpha'_1}(k_1)\sigma^{0\,-1}_{\alpha_2\alpha'_2}(k_1)\sigma^{0\,-1}_{\alpha_3\alpha'_3}(k_2)$$
$$\Lambda_{\alpha\alpha'\alpha_1\alpha'_1}(p-p_2,k_1-k_1)G_{\alpha_2\alpha'_2\alpha_3\alpha'_3}(k_1,k_2;k_2,k_1). \tag{6.60}$$

[1]Remember there are two ways to obtain the tadpole from the cubic vertex.

Now if η is chosen such that it minimises $S_{\rm eff}[\eta]$,

$$\left.\frac{\delta S_{\rm eff}[\eta]}{\delta \eta_{\alpha\beta}(p_1,p_2)}\right|_{\eta_0} = 0. \tag{6.45}$$

then equation 6.44 implies that

$$\begin{aligned} ihJ_{\beta\alpha}(p_1,p_2) = & \int d^4k_1 \int d^4k_2 \Big[\tfrac{1}{4}\lambda\delta_{\alpha\beta}\delta_{\mu\nu}V(\vec{k}_2-\vec{p}_1) - B\,\delta_{\alpha\mu}\delta_{\beta\nu}V(\vec{p}_2-\vec{p}_1)\Big)\frac{\delta^4(p_1+p_2+k_1-k_2)}{(2\pi)^4} \\ & ih\delta^4(k_1-p_2)\delta^4(k_2-p_1)\sigma^{0\,-1}_{\alpha\mu}(p_1)\sigma^{0\,-1}_{\beta\nu}(p_2)\Big]\,\eta_{\mu\nu}(k_1,k_2). \end{aligned} \tag{6.46}$$

It is convenient to define the matrix M as

$$\begin{aligned} M_{\alpha\beta\gamma\delta}(k_1,k_2,p_1,p_2) = & \tfrac{1}{4}\lambda\delta_{\alpha\beta}\delta_{\gamma\delta}V(\vec{k}_2-\vec{p}_1) - B\,\delta_{\alpha\gamma}\delta_{\beta\delta}V(\vec{p}_2-\vec{p}_1)\Big)\frac{\delta^4(p_1+p_2+k_1-k_2)}{(2\pi)^4} \\ & ih\delta^4(k_1-p_2)\delta^4(k_2-p_1)\sigma^{0\,-1}_{\alpha\gamma}(p_1)\sigma^{0\,-1}_{\beta\delta}(p_2) \\ & M_{\alpha\beta\gamma\delta}(p_1,p_2,k_1,k_2), \end{aligned} \tag{6.47}$$

then equation 6.46 becomes

$$\eta_{\alpha\beta}(k_1,k_2) = -\int d^4p_1 \int d^4p_2\, M^{-1}_{\alpha\beta\gamma\delta}(k_1,k_2,p_1,p_2)J_{\gamma\delta}(p_2,p_1). \tag{6.48}$$

Expanding about $\eta = 0$ i.e. writing η as

$$\eta_{\alpha\beta}(k_1,k_2) = \eta_{\alpha\beta}(k_1,k_2) + \phi_{\alpha\beta}(k_1,k_2). \tag{6.49}$$

it is found that

$$\begin{aligned} S_{\rm eff} = & \frac{1}{2}\int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4 \Big[\tfrac{1}{4}\lambda\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}V(\vec{k}_2-\vec{k}_3) - B\,\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}V(\vec{k}_3-\vec{k}_4)\Big] \\ & \frac{\delta^4(k_1-k_2+k_3-k_4)}{(2\pi)^4}\Big[ih\delta^4(k_1-k_4)\delta^4(k_2-k_3)\sigma^{0\,-1}_{\alpha_1\alpha_2}(k_3)\sigma^{0\,-1}_{\alpha_3\alpha_4}(k_4)\Big] \\ & \phi_{\alpha_1\alpha_2}(k_1,k_2)\phi_{\alpha_3\alpha_4}(k_3,k_4) - J_{\alpha_1\alpha_2}(k_2,k_1)M^{-1}_{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1,k_2,k_3,k_4) - \\ & J_{\alpha_3\alpha_4}(k_3,k_4). \end{aligned} \tag{6.50}$$

Hence, the Feynman path integral for the quadratic part of the action is

$$\begin{aligned} \mathcal{Z}[J] = & \int \mathcal{D}\eta_{\alpha\beta}\, e^{\frac{i}{h}S_{\rm eff}} \\ \mathcal{Z}[J=0]e^{-\frac{i}{h}\int d^4p_1\int d^4p_2\int d^4p_3\int d^4p_4 J_{\alpha_1\alpha_2}(k_2,k_1)M^{-1}_{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1,k_2,k_3,k_4)J_{\alpha_3\alpha_4}(k_4,k_3)} = & \\ e^{-\frac{i}{h}\int d^4p_1\int d^4p_2\int d^4p_3\int d^4p_4 J_{\alpha_1\alpha_2}(k_2,k_1)M^{-1}_{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1,k_2,k_3,k_4)J_{\alpha_3\alpha_4}(k_4,k_3)}. & \end{aligned} \tag{6.51}$$

where $\mathcal{Z}[J=0]$ is normalised to 1. Hence, in the effective theory, the two point correlation function is

$$\begin{aligned} \langle\eta_{\alpha\beta}(p_1,p_2)\eta_{\gamma\delta}(p_3,p_4)\rangle = & h^2\frac{\delta}{\delta J_{\rho_1\sigma_1}(p_2,p_1)}\frac{\delta}{\delta J_{\rho_2\sigma_2}(p_4,p_3)}\mathcal{Z}[J]\Big|_{J=0} \\ & ihM^{-1}_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4). \end{aligned} \tag{6.52}$$

which implies that

$$\begin{aligned} ih\delta_{\beta\alpha}\,\delta_{\gamma\delta}\delta^4(k_1-p_2)\delta^4(k_2-p_1) = & \int d^4p_1\int d^4p_2\Big[\tfrac{1}{4}\lambda\delta_{\alpha\mu}\delta_{\nu\nu}V(\vec{p}_2-\vec{k}_1) - B\,\delta_{\alpha\nu}\delta_{\mu\nu}V(\vec{k}_2-\vec{k}_1)\Big) \\ & \delta^4(k_1-k_2+p_1-p_2)\,ih\delta^4(p_1-k_2)\delta^4(p_2-k_2)\sigma^{0\,-1}_{\gamma\mu}(k_1)\sigma^{0\,-1}_{\beta\nu}(k_2) \\ & \langle\eta_{\mu\nu}(p_1,p_2)\eta_{\rho\sigma}(p_3,p_4)\rangle \end{aligned} \tag{6.53}$$

6.2.2 The Single-Particle Green's Function for Interacting Fermions

From the equation for σ^{-1} analogous to 6.20,

$$G^{-1}_{\alpha\beta}(x,x') = i\delta_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}\sigma^0(k) - \delta_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-x')}\Omega(k), \tag{6.36}$$

it becomes apparent that

$$\Omega(k) = i\sigma^0(k) \tag{6.37}$$

Hence, equation 2.91 becomes

$$\sigma(k) = \frac{i}{(\omega-\omega_{\vec{k}})-\Sigma^*(k)}. \tag{6.38}$$

Comparison of equations 6.35 and 6.38 shows that

$$\Sigma^*_{\alpha\beta}(k) = -\frac{1}{\hbar}\int\frac{d^4q}{(2\pi)^4}\left[A\sigma^0_{\alpha\beta}(k)V(\vec{k}-\vec{q}) - B\delta_{\alpha\beta}V(0)\sigma^0_{\mu\mu}(q)\right], \tag{6.39}$$

is a self consistent approximation to Σ^*. Different values of A and B correspond to different approximations.

6.2.3 The Two-Point propagator in the Effective Theory

The leading contribution, σ^0, is determined by equation 6.35. Perturbation theory can now be used to calculate the next order corrections.

From the Taylor expansion in equation 4.30, the quadratic portion of the effecti... ...tion is

$$S_2 = -\frac{i}{2}C_2 + \frac{1}{2}D_2. \tag{6.40}$$

Using the definitions of C_2 and D_2 given in equation 4.31,

$$\begin{aligned}S_2 = \frac{1}{2}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4\Big\{&-A\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta^4(x_1-x_3)\delta^4(x_2-x_4)V(\vec{x}_1-\vec{x}_3)\delta(x_{1_0}-x_{3_0})\\ &B\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\delta^4(x_1-x_2)\delta^4(x_3-x_4)V(\vec{x}_1-\vec{x}_3)\delta(x_{1_0}-x_{3_0}) - i\hbar\sigma^{0^{-1}}_{\alpha_1\alpha_3}(x_4,x_1)\sigma^{0^{-1}}_{\alpha_4\alpha_2}(x_2,x_3)\Big]\\ &\cdot\eta_{\alpha_1\alpha_2}(x_1,x_2)\eta_{\alpha_3\alpha_4}(x_3,x_4)\end{aligned} \tag{6.41}$$

Assuming that η has the symmetric Fourier transform

$$\eta_{\alpha\beta}(x_1,x_2) = \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}e^{i(p_1x_1+p_2x_2)}\eta_{\alpha\beta}(p_1,p_2) \tag{6.42}$$

and σ^0 has the Fourier transform given in equation 6.33, then in momentum space

$$\begin{aligned}S_2 = \frac{1}{2}\int d^4k_1\int d^4k_2\int d^4k_3\int d^4k_4\Big\{&(A\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}V(\vec{k}_1-\vec{k}_4) - B\delta_{\alpha_1\alpha_3}\delta_{\alpha_4\alpha_2}V(\vec{k}_2-\vec{k}_4))\\ &\delta^4(k_1+k_2-k_3-k_4)\\ &-i\hbar\delta^4(k_1-k_4)\delta^4(k_2-k_3)\sigma^{0^{-1}}_{\alpha_1\alpha_3}(k_1)\sigma^{0^{-1}}_{\alpha_4\alpha_2}(k_2)\Big]\\ &\cdot\eta_{\alpha_1\alpha_2}(k_1,k_2)\eta_{\alpha_3\alpha_4}(k_3,k_4)\end{aligned} \tag{6.43}$$

Let J be an arbitrary source and add a source term to S_2, then

$$\begin{aligned}S_2 = \frac{1}{2}\int d^4k_1\int d^4k_2\int d^4k_3\int d^4k_4\Big\{&(A\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}V(\vec{k}_1-\vec{k}_4) - B\delta_{\alpha_1\alpha_3}\delta_{\alpha_4\alpha_2}V(\vec{k}_2-\vec{k}_4))\\ &\delta^4(k_1+k_2-k_3-k_4)\\ &-i\hbar\delta^4(k_1-k_4)\delta^4(k_2-k_3)\sigma^{0^{-1}}_{\alpha_1\alpha_3}(k_1)\sigma^{0^{-1}}_{\alpha_4\alpha_2}(k_2)\Big]\\ &\eta_{\alpha_1\alpha_2}(k_1,k_2)\eta_{\alpha_3\alpha_4}(k_3,k_4) + \int d^4k_1\int d^4k_2\,J_{\alpha_1\alpha_2}(k_2,k_1)\eta_{\alpha_1\alpha_2}(k_1,k_2).\end{aligned} \tag{6.44}$$

If $A + B = 1$ then S can be rewritten exactly as a linear combination of the above.

$$S = \int d^4x \int d^4x' \sigma_{\alpha\alpha}(x,x')\Big(ih\partial_{x_0} + \frac{\hbar^2\partial^2_{\vec{x}}}{2m}\Big)\delta(x - x') + \frac{1}{2}V(\vec{x}-\vec{x}')\delta(x_0 - x_0') \times$$
$$\big[A\sigma_{\alpha\alpha}(x,x')\sigma_{\alpha\alpha}(x',x) - B\sigma_{\alpha\alpha}(x,x)\sigma_{\alpha\alpha}(x',x')\big] \tag{6.26}$$

and the effective action in the bilocals is:

$$S_{eff} = ih\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma + S + ih\delta_{\alpha\alpha}L^3\delta^4(0)\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma \tag{6.27}$$
$$= S_0 + S_1, \tag{6.28}$$

where

$$S_0 = ih\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma + \int d^4x \int d^4x' \sigma_{\alpha\alpha}(x,x')\Big[ih\partial_{x_0} + \frac{\hbar^2\partial^2_{\vec{x}}}{2m}\Big)\delta(x - x') + \frac{1}{2}V(\vec{x}-\vec{x}')\delta(x_0 - x_0') \times$$
$$[A\sigma_{\alpha\alpha}(x,x')\sigma_{\alpha\alpha}(x',x) - B\sigma_{\alpha\alpha}(x,x)\sigma_{\alpha\alpha}(x',x')]. \tag{6.29}$$

and

$$S_1 = ih\delta_{\alpha\alpha}L^3\delta^4(0)\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma. \tag{6.30}$$

6.2.1 Gap Equation for Interacting Fermions

The condition

$$\frac{\delta S_0}{\delta\sigma_{\alpha\alpha}(y,y')}\Big|_{\sigma^0} = 0, \tag{6.31}$$

yields,

$$-ih\sigma^{0\,-1}_{\alpha\alpha}(y',y) = \delta_{\alpha\alpha}\Big(ih\partial_{y_0} + \frac{\hbar^2\partial^2_{\vec{y}}}{2m}\Big)\delta^4(y - y') + A\sigma^0_{\alpha\alpha}(y',y)V(\vec{y}-\vec{y}')\delta(y_0 - y_0')$$
$$B\delta_{\alpha\alpha}\delta^4(y - y')\int d^4x\,\sigma^0_{\alpha\alpha}(x,x)V(\vec{x}-\vec{y})\delta(x_0 - y_0). \tag{6.32}$$

Again, it is clear that σ^0 is translationally invariant. Hence,

$$\sigma^0_{\alpha\alpha}(x,y) = \delta_{\alpha\alpha}\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\sigma^0(k). \tag{6.33}$$

This together with condition 6.15, which also applies to $\sigma^{0\,-1}$, yields the Fourier transform

$$\sigma^{0\,-1}_{\alpha\alpha}(x,y) = \int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\sigma^{0\,-1}_{\alpha\alpha}(k)$$
$$= \int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\sigma^{0\,-1}(k)\delta_{\alpha\alpha}. \tag{6.34}$$

which may be substituted into equation 6.32, and the spin indices factored out, yielding the gap equation for interacting, nonrelativistic fermions.

$$\sigma^0(k) = (\omega - \omega_\xi) + \frac{1}{h}\int\frac{d^4q}{(2\pi)^4}\big[A\,V(\vec{k}-\vec{q})\sigma^0(q) - B\delta_{\alpha\alpha}V(0)\sigma^0(q)\big]^{-1}, \tag{6.35}$$

which justifies the use of approximation 1.17, despite the fact that N cannot be considered large.

and

$$S_{11} = -ihL^{-1}\delta^3(0)\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma. \tag{6.8}$$

The choice of separating S_{σ} into these two terms is motivated in the large N limit, as discussed in chapter 4.

Substituting equation 6.4 into equation 6.7 yields

$$S_{0_1} = ih\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma + \int d^3x \int d^4x'\,\sigma_{\alpha\alpha}(x,x')[-ihi\partial_{x'_0} + \frac{h^2\partial^2_{\vec{x}'}}{2m}]\delta^4(x-x'), \tag{6.9}$$

6.1.1 Gap Equation

The "classical" solution, σ^0_1, is obtained by minimising S_{0_1} with respect to σ, i.e. σ^0_1 must satisfy the relation

$$\frac{\delta S_{0_1}}{\delta\sigma_{\gamma\delta}(y_1,y_2)}\Big|_{\sigma^0_1} = 0, \tag{6.10}$$

Now,

$$\frac{\delta}{\delta\sigma_{\gamma\delta}(y_1,y_2)}\,\mathrm{tr}\,\mathrm{Tr}\ln\sigma = \sigma^{-1}_{\delta\gamma}(y_2,y_1), \tag{6.11}$$

therefore,

$$\frac{\delta S_{0_1}}{\delta\sigma_{\gamma\delta}(y_1,y_2)} = ih\sigma^{-1}_{\delta\gamma}(y_2,y_1) - (ihi\partial_{y_{20}} - \frac{h^2\partial^2_{\vec{y}_2}}{2m})\delta^4(y_1-y_2)\delta_{\delta\gamma}, \tag{6.12}$$

which together with 6.10 implies

$$ih\sigma^{0\,-1}_{\gamma\delta}(y_2,y_1) = (-ihi\partial_{y_{20}} + \frac{h^2\partial^2_{\vec{y}_2}}{2m})\delta^4(y_1-y_2)\delta_{\gamma\delta}, \tag{6.13}$$

from which it is obvious that σ^0_1 is translationally invariant. Hence, the four-dimensional Fourier transform for σ^0_1 may be used.

$$\sigma^0_{1\gamma\delta}(x,y) = \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}\sigma^0_{1\gamma\delta}(k). \tag{6.14}$$

The inverse of σ^0_1, necessarily satisfies

$$\int d^4x\,\sigma^{0\,-1}_{1\alpha\beta}(y,x)\sigma^0_{1\gamma}(x,z) = \delta_{\alpha\gamma}\delta^4(y-z). \tag{6.15}$$

Equations 6.14 and 6.15 imply

$$\sigma^{0\,-1}_{1\gamma\delta}(x,y) = \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}\sigma^{0\,-1}_{1\gamma\delta}(k). \tag{6.16}$$

Substituting the Fourier transform 6.16 into equation 6.13 yields

$$-ih\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x_0-y_0)}\sigma^{0\,-1}_{1\gamma\delta}(k) = \int\frac{d^4k}{(2\pi)^4}(hk_0 - \frac{h^2\vec{k}^2}{2m})\delta_{\alpha\beta}e^{ik\cdot(x_0-y_0)}, \tag{6.17}$$

from which it follows that

$$\sigma^0_{1\gamma\delta}(k) = \frac{i}{k_0 - \frac{h\vec{k}^2}{2m}}\delta_{\gamma\delta} = \frac{i}{\omega - \omega_{\vec{k}}}\delta_{\gamma\delta} \tag{6.18}$$

where $\omega = k_0$ and $\omega_{\vec{k}} = \frac{h\vec{k}^2}{2m}$ have been inserted. However, equation 6.18 is ill-defined, as no prescription is given as to how the integration in equation 6.17 is to be performed around the poles in the complex ω plane.

Substitution of equation 5.32 into equation 5.31 yields the following differential equation for A:

$$\frac{\partial A(I)}{\partial t} = e^{iH_0 t/\hbar} H^{ex}(t) e^{-iH_0 t/\hbar} A(I)$$
$$= H_0^{ex}(I) A(I), \tag{5.33}$$

which has the solution:

$$A(I) = A(I_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_0^{ex}(t') A(t') \quad t > t_0. \tag{5.34}$$

Iterating equation 5.34 and substituting the result into 5.32 yields

$$|\Psi(I)\rangle = e^{-iH_0 t/\hbar}|\Psi(0)\rangle - \frac{i}{\hbar} e^{-iH_0 t/\hbar}\int_{t_0}^{t} dt' H_0^{ex}(t')|\Psi_S(0)\rangle + \ldots. \tag{5.35}$$

The general expectation value of an operator in the Schrödinger representation, $O_S(I)$, after t_0 is defined to be

$$\langle O_S(I)\rangle_{ex} = \langle \Psi^I(t)|O_S(I)|\Psi_S(I)\rangle. \tag{5.36}$$

may be rewritten, using equations 5.35 and 2.2, as

$$\langle O_S(I)\rangle_{ex} = \langle \Psi^I(0)|[1 + \frac{i}{\hbar}\int_{t_0}^{t} dt' H_0^{ex}(t') + \ldots] O_0(I) [1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_0^{ex}(t') + \ldots]|\Psi_S(0)\rangle$$
$$= \langle \Psi_0^I | O_0(I)|\Psi_0\rangle + \frac{i}{\hbar}\int_{t_0}^{t} dt' \langle \Psi_0^I|[H_0^{ex}(t'), O_0(I)]|\Psi_0(0)\rangle + \ldots. \tag{5.37}$$

In the linear response approximation only the linear term in equation 5.37 is retained.

If the external potential is scalar, and the system consists of particles with a charge e, then H^{ex} has the form

$$H^{ex}(I) = e\int d^3x\, n_0(x)\varphi^{ex}(x), \tag{5.38}$$

and the corresponding contribution to the classical action is

$$S^{ex} = -e\int dt\, n_0(x)\varphi^{ex}(x). \tag{5.39}$$

The complete change in the ground-state expectation value of the charge density is given by

$$\delta\langle n(x)\rangle = \langle n_0(x)\rangle_{ex} - \langle n_0(x)\rangle, \tag{5.40}$$

and the linear response, using equation 5.37, and 5.38 is given by

$$\delta\langle n(x)\rangle = \frac{ie}{\hbar}\int_{t_0}^{t} d^3x'\, \varphi^{ex}(x')\langle \Psi_0|[n_0(x'), n_0(x)]|\Psi_0\rangle. \tag{5.41}$$

Now,

$$\langle \Psi_0|[n_0(x'), n_0(x)]|\Psi_0\rangle = \langle \Psi_0|n_0(x')n_0(x)|\Psi_0\rangle - \langle \Psi_0|n_0(x)N_0(x')|\Psi_0\rangle$$
$$= \langle \Psi_0|\psi_\alpha^\dagger(x')\psi_\alpha(x')\psi_\beta^\dagger(x)\psi_\beta(x)|\Psi_0\rangle$$

$$\langle \Psi_0|\psi_\beta^\dagger(x)\psi_\alpha(x)\psi_\alpha^\dagger(x')\psi_\beta(x')|\Psi_0\rangle. \tag{5.42}$$

i.e. the linear response is related to $G^{(2)}$ defined by equation 2.92. In fact, any approximation for $G^{(2)}$ immediately yields an approximation for the linear response, and *vice versa*.

Introducing an approximate effective two-body interaction V_0 which includes the polarization of the medium associated with the closed loop in $\Sigma^*_{\alpha\beta}$, i.e.

$$V_0(q) = V(\vec{q}) + V(\vec{q})\Pi^{*0}(q)V(\vec{q}) + \ldots \tag{5.24}$$

where, Π^{*0} is the lowest order approximation to the polarization insertion defined by equation 2.96,

$$\Pi^{*0}(q) = -\frac{i}{\hbar}\int\frac{d^4k}{(2\pi)^4}\mathcal{G}^0_{\alpha\gamma}(k)\mathcal{G}^0_{\gamma\alpha}(k+q). \tag{5.25}$$

Equation 5.23 can be rewritten as

$$\Sigma^*_{\alpha_1\alpha}(p) = -\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}\mathcal{G}^0_{\gamma\alpha}(p-q)[V_0(q) - V(\vec{q})]. \tag{5.26}$$

Now, using the binomial expansion of $\frac{1}{1+x}$, equation 5.24 becomes

$$V_0 = \frac{V(\vec{q})}{1 - V(\vec{q})\Pi^{*0}(q)}, \tag{5.27}$$

which, when substituted into equation 5.26 yields

$$\begin{aligned}\Sigma^*_{\alpha_1\alpha}(k) &= -\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}\mathcal{G}^0_{\gamma\alpha}(k-q)\left[\frac{V(\vec{q})}{1 - V(\vec{q})\Pi^{*0}(q)} - V(\vec{q})\right] \\ &= -\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}\mathcal{G}^0_{\gamma\alpha}(k-q)\frac{[V(\vec{q})]^2\Pi^{*0}(q)}{1 - V(\vec{q})\Pi^{*0}(q)}.\end{aligned} \tag{5.28}$$

Equation 5.28 can be substituted into 2.88 yielding

$$\mathcal{G}_{\alpha_1\alpha}(k) = \mathcal{G}^0_{\alpha_1\alpha}(k) + \frac{i}{\hbar}\mathcal{G}^0_{\alpha_1\lambda}(k)\int\frac{d^4q}{(2\pi)^4}\mathcal{G}^0_{\gamma\lambda}(k-q)\frac{[V(\vec{q})]^2\Pi^{*0}(q)}{1 - V(\vec{q})\Pi^{*0}(q)}\mathcal{G}_{\alpha\alpha}(k). \tag{5.29}$$

The two-particle Green's function in the ring diagram is given by,

$$\begin{aligned}\mathcal{G}^{(2)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1,p_2;p_3,p_4) &= \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_4}{(2\pi)^4}\delta^4(p_1 - p_2 + p_3 - p_4) \\ &\quad \cdot \mathcal{G}^0_{\alpha_1\alpha'_1}(p_1)\mathcal{G}^0_{\alpha_2\alpha'_2}(p_2)\mathcal{G}^0_{\alpha_3\alpha'_3}(p_3)\mathcal{G}^0_{\alpha_4\alpha'_4}(p_4) \\ &\quad \cdot V_0(p_1 - p_2).\end{aligned} \tag{5.30}$$

where V_0 is defined by 5.25

5.3 Linear Response Theory

Suppose the interacting many-body system, described by the Hamiltonian in equation 5.1, is placed at $t = t_0$ into a time-dependent external potential. Then, an extra term, H^{ex} must be added to the Hamiltonian. If $|\Psi(t)\rangle$ is a Schrödinger state vector of the perturbed system then the Schrödinger equation becomes

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = [H + H^{ex}]|\Psi(t)\rangle \quad t > t_0. \tag{5.31}$$

Assume that the solution to equation 5.31 has the form

$$|\Psi(t)\rangle = e^{-iH(t-t_0)/\hbar}\chi(t)|\Psi(0)\rangle, \tag{5.32}$$

where $\chi(t) = 1$ if $t < t_0$

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Chapter 7

Conclusions

After reviewing second quantized field theory in the operator formalism, the corresponding formulation using path integrals was presented. The bilocal bosonization technique, which was recently set out in detail in [1], was also reviewed, together with certain traditional approximations, as defined in the operator formalism.

In this dissertation, the manner in which the bilocal bosonized effective action, originally developed for relativistic fermionic models with N species, may be applied to nonrelativistic multi-electron systems is investigated.

Even though $N = 1$ for these systems, it has been shown that the splitting of the effective action, natural for large N, remains justifiable in the case $N = 1$. This is due to the fact that, using this splitting, the standard gap equations are still obtained. The gap equations, together with Dyson's equation, yield self-consistent approximations for the single-particle Green's function.

The effective action is an exact rewriting in terms of bilocals and it has the advantage of providing a systematic way of generating corrections to the solutions of the self-consistent gap equations.

In particular, the quadratic fluctuations of particular rewritings of the effective action are shown to correspond to standard approximations to the Bethe-Salpeter equation. Namely, the ring approximation to the vacuum polarization and the ladder diagram Bethe-Salpeter equation have been identified for different rewritings of the original theory.

In the case of the ring diagram approximation, an example of a field theoretic calculation of the leading correction to the single-particle Green's function, using the vertices generated by the effective action has been carried out, and shown to generate the standard ring diagram proper self energy.

In conclusion, the bilocal effective action used in this dissertation can be used to understand the origin of several traditional self consistent approximations used in non-relativistic multi-electron systems, and it provides a systematic way of identifying and including corrections (or sets of diagrams) to these approximations.

then equation 6.81 yields the following condition for η

$$i\sigma^{(1)}_{\alpha_3\beta_3}(k_3)\eta_{\alpha_2\beta_2}(k_4, k_4 - k_3)\sigma^{-1}_{\alpha_4\alpha_2}(k_3) = \frac{1}{(2\pi)^4}\int d^4q\,[A\,\delta_{\alpha_4\alpha_2}\delta_{\alpha_2\beta_4}V(\vec{k}_4 - \vec{q}) - B\,\delta_{\alpha_4\beta_4}\delta_{\alpha_2\beta_2}V(\vec{k}_3 - \vec{k}_4)] \times \eta_{\alpha_4\beta_4}(q, k_4 - k_3) \tag{6.83}$$

i.e.

$$\eta_{\mu\nu}(k_4, k_4 - k_3) = -\frac{i}{(2\pi)^4}\sigma^0_{\mu\alpha_2}(k_4)\sigma^0_{\alpha_2\nu}(k_3)\int d^4q\,[A\,\delta_{\alpha_4\alpha_2}\delta_{\alpha_2\beta_4}V(\vec{k}_4 - \vec{q}) - B\,\delta_{\alpha_4\beta_4}\delta_{\alpha_2\beta_2}V(\vec{k}_3 - \vec{k}_4)] \times \eta_{\alpha_4\beta_4}(q, k_4 - k_3)]. \tag{6.84}$$

In the ring diagram approximation equation 6.84 becomes:

$$\eta_{\mu\nu}(k_4, k_4 - k_3) = i\sigma^0_{\mu\alpha}(k_4)\sigma^0_{\alpha\nu}(k_3)\int\frac{d^4q}{(2\pi)^4}V(\vec{k}_3 - \vec{k}_4)] \times \eta_{\alpha\alpha}(q, k_4 - k_3). \tag{6.85}$$

Linear response theory was developed in order to investigate collective excitations, which appear as poles in $G^{(1)}$. The ring diagram approximation is, in general used to find the two-particle Green's function. It can be straightforwardly demonstrated that these collective excitation states may also be obtained from the solution to the homegeneous Bethe-Salpeter equation, i.e from equation 6.85.

Taking the trace of equation 6.85 and integrating over k_4, keeping $k_4 - k_3$ constant, the following equation is obtained:

$$1 = i\int\frac{d^4k}{(2\pi)^4}\sigma^0_{\alpha\beta}(k_4)\sigma^0_{\beta\alpha}(k_4 - k)V(\vec{k}_4) = \Pi^*(k_4)V(\vec{k}_4), \tag{6.86}$$

which is precisely the equation obtained for the poles of equation 2.92. In particular, by standard treatment of the poles of $G^{(1)}_0$ in [18], the plasmon is obtained.

6.4 Linear Response and the Effective Theory

By definition,

$$\begin{aligned} i\mathcal{D}^{(+)}_{\alpha\alpha}(x,x) &= \lim_{t'\to t^+}\langle\Phi_0|T[\psi_\alpha(t,\vec{x})\psi^\dagger_\alpha(t',\vec{x})]|\Phi_0\rangle \\ &= -\langle\Phi_0|n_H(x)|\Phi_0\rangle \\ &= \int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S}\psi_\alpha(x)\psi^\dagger_\alpha(x) \\ &= \int \mathcal{D}\sigma e^{\frac{i}{\hbar}S_{\rm eff}}\sigma_{\alpha\alpha}(x,x). \end{aligned} \tag{6.77}$$

Using the definition of σ and the path integral formalism, equation 5.40 becomes

$$\begin{aligned} \delta\langle n(x)\rangle &= \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\,(e^{-\frac{i\epsilon}{\hbar}\int d^4x'\psi^\dagger_\beta(x')\varphi^{ext}(x')}-1)e^{\frac{i}{\hbar}S}\psi_\alpha(x)\psi^\dagger_\alpha(x) \\ &= \int \mathcal{D}\sigma\,(e^{-\frac{i\epsilon}{\hbar}\int d^4x'\sigma_{\beta\beta}(x',x')\varphi^{ext}(x')}-1)e^{\frac{i}{\hbar}S_{\rm eff}}\sigma_{\alpha\alpha}(x,x) \\ &= \frac{i\epsilon}{\hbar}\int d^4x_1\,\varphi^{ext}(x_1)\int \mathcal{D}\sigma\, e^{\frac{i}{\hbar}S_{\rm eff}}\sigma_{\beta\beta}(x_1,x_1)\sigma_{\alpha\alpha}(x,x) \\ &\quad+\frac{i^2\epsilon^2}{2\hbar^2}\int d^4x_1\int d^4x_2\,\varphi^{ext}(x_1)\varphi^{ext}(x_2)\int \mathcal{D}\sigma\, e^{\frac{i}{\hbar}S_{\rm eff}}\sigma_{\beta_1\beta_1}(x_1,x_1)\sigma_{\beta_2\beta_2}(x_2,x_2)\sigma_{\alpha\alpha}(x,x) \\ &\quad+\frac{i^3\epsilon^3}{3!\hbar^3}\int d^4x_1\int d^4x_2\int d^4x_3\,\varphi^{ext}(x_1)\varphi^{ext}(x_2)\varphi^{ext}(x_3)\int \mathcal{D}\sigma\, e^{\frac{i}{\hbar}S_{\rm eff}}\sigma_{\beta_1\beta_1}(x_1,x_1) \\ &\quad\times\sigma_{\beta_2\beta_2}(x_2,x_2)\sigma_{\beta_3\beta_3}(x_3,x_3)\sigma_{\alpha\alpha}(x,x)+\ldots. \end{aligned} \tag{6.78}$$

and the linear response is:

$$\begin{aligned} \delta\langle n(x)\rangle_{\rm linear} &= \frac{i\epsilon}{\hbar}\int d^4x_1\,\varphi^{ext}(x_1)\int \mathcal{D}\sigma\, e^{\frac{i}{\hbar}S_{\rm eff}}\sigma_{\beta\beta}(x_1,x_1)\sigma_{\alpha\alpha}(x,x) \\ &= \frac{i\epsilon}{\hbar}\int d^4x_1\,\varphi^{ext}(x_1)\langle\sigma_{\beta\beta}(x_1,x_1)\sigma_{\alpha\alpha}(x,x)\rangle \end{aligned} \tag{6.79}$$

which is identical to equation 5.37, obtained in the operator formalism.

6.5 The Homogeneous Bethe-Salpeter Equation

The quadratic part of the effective action is given by equation 6.43, and behaves like a harmonic oscillator, yielding a spectrum which may be interpreted as belonging to a system of free particles.

Thus, the homogeneous Bethe-Salpeter equation following from equation 6.43,

$$\begin{aligned} &\tfrac{1}{2}\int d^4k_1\int d^4k_2\,[(A\,\delta_{\alpha_1\alpha_2}\delta_{\alpha_2\alpha_1}V(\vec{k}_1-\vec{k}_1)-B\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}V(\vec{k}_2-\vec{k}_1))\tfrac{\delta^4(k_1-k_2,k_1-k_2)}{(2\pi)^4} \\ &\quad -i\delta^4(k_1-k_1)\delta^4(k_2-k_2)\sigma^{(+)}_{\beta_2\alpha_1}(k_1)\sigma^{(+)}_{\alpha_2\beta_1}(k_2)]\eta_{\alpha_1\beta_1}(k_1,k_2)=0, \end{aligned} \tag{6.80}$$

yields collective bound states of the system.

Equation 6.80 implies

$$\begin{aligned} -i\sigma^{(+)}_{\beta_1\alpha_1}(k_1)\eta_{\alpha_1\gamma_1}(k_1)\sigma^{(+)}_{\gamma_2\beta_2}(k_2) &= \frac{1}{(2\pi)^4}\int d^4q\,[A\,\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\gamma_2}V(\vec{k}_1-\vec{q})-B\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}V(\vec{k}_2-\vec{k}_2)]\times \\ &\quad \eta_{\gamma_1\gamma_2}(q,q+k_2-k_1) \end{aligned} \tag{6.81}$$

If the field is redefined to be given by

$$\eta_{\alpha_1\alpha_2}(k_1,k_2)\quad\eta_{\alpha_1\alpha_2}(k_1,k_1-k_2) \tag{6.82}$$

Iterating equation 6.71 yields

$$\begin{aligned}
G_{\alpha\alpha_1\alpha_2\alpha}(p_1,-p_2,p_3,-p_3) &= \frac{i}{\hbar(2\pi)^4}\sigma^0_{\alpha\alpha}(p_1)\sigma^0_{\alpha_2\alpha}(p_2)\sigma^0_{\alpha_3\alpha_1}\sigma^0_{\alpha_4\alpha}(p_3)\left[\delta_{\alpha_1\alpha}\delta_{\alpha_2\alpha}V(p_2-p_3)\right.\\
&\quad -\frac{i}{(2\pi)^4}\int d^4q\,V(\vec{q})\sigma^0_{\alpha\alpha_2}(p_1-q)\sigma^0_{\alpha_1\alpha}(p_2+q))V(\vec{p}_2-\vec{q}-\vec{p}_3)\\
&\quad -\left(\frac{i}{\hbar(2\pi)^4}\right)^2\int d^4q\,V(\vec{q})\sigma^0_{\alpha\alpha_2}(p_1-q)\sigma^0_{\alpha_2\alpha}(p_2+q)\\
&\quad \cdot\int d^4q_1\,V(\vec{q}_1)\sigma^0_{\alpha_2\alpha_1}(p_1-q-q_1)\sigma^0_{\alpha_1\alpha_2}(p_2+q+q_1)V(\vec{p}_2-\vec{q}-\vec{q}_1-\vec{p}_3)\\
&\quad -\left(\frac{i}{\hbar(2\pi)^4}\right)^3\int d^4q\,V(\vec{q})\sigma^0_{\alpha\alpha_2}(p_1-q)\sigma^0_{\alpha_2\alpha}(p_2+q)\\
&\quad \cdot\int d^4q_1\,V(\vec{q}_1)\sigma^0_{\alpha_2\alpha_3}(p_1-q-q_1)\sigma^0_{\alpha_3\alpha_2}(p_2+q+q_1)\\
&\quad \cdot\int d^4q_2\,V(\vec{q}_2)\sigma^0_{\alpha_3\alpha_1}(p_1-p_2-q-q_1)\sigma^0_{\alpha_4\alpha_1}(p_2+q+q_1+q_2)\\
&\quad \cdot V(\vec{p}_2-\vec{q}-\vec{q}_1-\vec{q}_2-\vec{p}_3)+\ldots\Big]\\
&= \frac{i}{(2\pi)^4}\sigma^0_{\alpha\alpha}(p_1)\sigma^0_{\alpha_2\alpha}(p_2)\sigma^0_{\alpha_3\alpha_1}\sigma^0_{\alpha_4\alpha}(p_3)\Gamma^0_{\alpha\alpha_1\alpha_2\alpha_1}(p_2,p_1;p_3,p_4) \qquad (6.72)
\end{aligned}$$

where Γ^0 has the same structure as the effective two-particle interaction, Γ discussed in section 5.2.2, with the important exception that where σ^0_F would appear in Γ, σ^0 appears instead. Hence, Γ^0 is a self-consistent approximation to Γ, and obeys the integral equation

$$\begin{aligned}
\Gamma^0_{\alpha\alpha_1\alpha_2\alpha_1}(p_1,p_2;p_3,p_4) &= \delta_{\alpha\alpha_1}\delta_{\alpha\alpha_1}V(\vec{p}_2-\vec{p}_3)-\frac{i}{\hbar(2\pi)^4}\int d^4q\,V(\vec{q})\sigma^0_{\mu_1\alpha_2}(p_1+q)\sigma^0_{\nu_2\alpha_1}(p_2-q)\\
&\quad\times\Gamma^0_{\alpha\alpha_2\alpha\alpha_1}(p_1-q,p_2+q;p_3,p_4). \qquad (6.73)
\end{aligned}$$

Again, comparison with equation 5.18 makes it clear that 6.73 is $G_1^{(0)}$.

6.3.3 Hartree-Fock Approximation

As discussed in 5.2.1, the Hartree-Fock approximation corresponds to approximating the proper self-energy by

$$\Sigma^*_{\alpha\alpha_1\alpha_2\alpha}(k) = -\frac{i}{\hbar}\int\frac{d^4q}{(2\pi)^4}\left[\delta_{\alpha_1\alpha}V(0)G_{\alpha\alpha_2\alpha}(q)-V(\vec{k}-\vec{q})G_{\alpha\alpha_2\alpha_1}(q)\right]. \qquad (6.74)$$

Using equation 6.20, $\Sigma^*_{\alpha\beta}$ can be rewritten in terms of σ^0

$$\Sigma^*_{\alpha\alpha_1\alpha_2\alpha}(k) = -\frac{1}{\hbar}\int\frac{d^4q}{(2\pi)^4}\left[V(\vec{k}-\vec{q})\sigma^0_{\alpha\lambda}(q)-\delta_{\lambda\mu}V(0)\sigma^0_{\mu\nu}(q)\right]. \qquad (6.75)$$

Comparison with

$$\Sigma^*_{\alpha\beta}(k) = -\frac{1}{\hbar}\int\frac{d^4k}{(2\pi)^4}\left[A\sigma^0_{\alpha\beta}(k)V(\vec{k}-\vec{q})-B\delta_{\alpha\beta}V(0)\sigma^0_{\nu\nu}(q)\right], \qquad (6.76)$$

shows that Σ^{*0} is equal to $\Sigma^*_{\rm HF}$ if and only if $A=1$ and $B=1$. However, remember that an exact rewriting of the original theory requires that $A+B=1$. This means that, strictly speaking, the Hartree-Fock approximation cannot be obtained as a stationary solution of the bilocal effective action. The gap equation yields only half the contribution needed for the Hartree-Fock approximation. The remaining contribution would be generated by the higher order corrections to the effective action.