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Modes of reasoning in explanatory texts and worked out examples on Linear Functions in two Grade 9 Mathematics textbooks.

A research report submitted in partial fulfilment of the requirements for the degree

of

Master of Education (Mathematics)

Research report

By

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15 February 2021

HONOUR PLEDGE

I, **Matome Edwin Matsepe**, a registered student at the University of the Witwatersrand with student number **842068**, pledge that the assessment task that I am submitting as part of **M. Ed (Mathematics Education)** is entirely my own work, except where I have indicated that I obtained information from other sources, such as books or internet sites (where allowed). I pledge that I have not consulted with anyone else, including other students, while completing the assessment task or copied information from any source without referencing it.

I acknowledge that if I am found to have cheated in any way or plagiarised other people's work, I will be subject to disciplinary action, which may lead to suspension or expulsion.

Signed 

Date 7/09/21

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Abstract

Mathematical reasoning is argued to be amongst the important learning goals (for instance, problem-solving, communicating mathematically and mathematical language) that should be learned across all grades. This research uses qualitative and textbook content analysis method to study reasoning opportunities in expository texts in two Grade 9 Mathematics textbooks on linear functions. Furthermore, the extent to which the modes of reasoning made available for learners to develop mathematical reasoning is explored.

The theoretical framework of this research is developed from the amalgamation of the following theoretical frameworks: modes of reasoning, multiple representational translations model and components of functional grammar analysis. In accordance with the developed theoretical framework, the findings reveal that the modes of reasoning through deduction are dominant across the two textbooks. Moreover, deduction using a specific case is dominant amongst the identified modes of reasoning that are deductive. It has been argued that modes of reasoning that are deductive have intentions that are intrinsic to mathematical reasoning. However, they can lead to learning without meaning since they are characterised by algorithms that are often difficult for learners to learn from.

Nonetheless, the two textbooks employ textual explanations that supplement the algorithms by explaining the mathematical principles using the everyday language. The *Mind Action Series* (MAS) textbook uses representational translations that encourage aspects of mathematical reasoning such as pattern discovery and interpretation of graphs. On the contrary, the *Platinum* textbook overlooks this aspect, and presents representational translations that encourage procedural computation. However, textual explanations in the *Platinum* textbook consider learners as active participants in doing mathematics, while the MAS does not highlight this aspect. The two textbooks, thus, display differing strengths and weakness to fulfill the desired mathematics reasoning.

CHAPTER 1: INTRODUCTION TO THE STUDY

1.1. Introduction

In this chapter, the rationale of this research is explained, and an overview is also provided. The chapter is structured as follows:

- Context in which this study is conducted as well as the background,
- Rationale and objectives of the study,
- Definition of terms and concepts that are used in this study,
- Background of the theoretical framework that is used in this research, and
- Overview of how the study was conducted.

1.2. Context of the study

This section is intended to discuss the academic rationale. The discussion on academic rationale provides discernments of the studies that have been conducted before in the topic of interest in this research.

1.2.1. Academic rationale: why research on textbooks?

It has been documented that Mathematics textbooks are commonly used as resources for learning and teaching (O'keeffev & O'donoghue, 2015). Salander (2003) points out that textbooks are advantageous to use because they can be used at home and school, necessitating them to withstand the test of time. Thus, the practice of using textbooks is old but remains relevant in the current school system (Robinson, 1981 cited in O'keeffe et al., 2015).

As a result, many researchers (For instance, Skemp, 1982; van Dormolen, 1986; Pimm, 1997; Hiebert & Carpenter, 1992; Orton, 2004; Rezat, 2006, Stacey & Vincent, 2009) have researched some of the key concerns around Mathematics textbooks. However, little research has been conducted to examine mathematical reasoning in textbooks (Valverde, Bianchi, Wolfe, Schmidt, & Houang., 2002; and Stacey et al., 2009).

Chieh, Chang and Silalahi (2017) suggest that Mathematics textbooks hold the potential to support learning and teaching in classrooms. As a result, Mathematics textbooks have been interrogated to examine their legitimacy to support teaching and learning. In a study conducted by Chieh et al.

(2017), key aspects of research in textbooks were dominantly centered around problem-solving, with a limited number of research studies in the area of mathematical reasoning and mathematical language. In their study, language was divided into two: Ideational function and interpersonal function. According to their findings, there was no evidence of research in the area of interpersonal functions. However, Chieh et al.'s (2017) work was aimed at analysing the relevant empirical education research studies from the year 1953 to 2015, and to summarise previous literature (based on 44 studies) regarding the analysis of Mathematics textbooks. The study thus, to some extent, give suggestions and an idea of the relative dominance of research in the area of problem-solving as compared to research in the area of mathematical reasoning.

The focus of research in the area highlighted above stems from the standardised based approach. This approach emphasises the development of conceptual understanding and reasoning over rote learning. This is suggested in Hasanah (2019) where problem-solving, reasoning, communication and representing mathematical content are important goals across all grades. The dominance of research in problem-solving revealed in Chieh et al.'s (2017) study is supported by previous research. For instance, the *National Council of Teachers of Mathematics* (1989) (NCTM) mentions that problem-solving is the centre of Mathematics on an international basis. The NCTM further suggests that problem-solving should be the primary focus of the Mathematics curriculum. In this regard, researchers seem to be less interested in interrogating textbooks in the context of mathematical reasoning. Hence, in this research, I have decided to focus on mathematical reasoning and some components of functional grammar analysis in two Mathematics textbooks.

Mathematical reasoning is defined as a thought process that is performed to arrive at conclusions that are based on logical-mathematical premises (Wahyudi, 2016). According to Hasanah (2019), mathematical reasoning and mathematical understanding are two things that cannot be separated. They argue that mathematical understanding is obtained through reasoning, while reasoning is understood through learning mathematics. Hasanah (2019) further argues that learners who display the ability to reason during the process of understanding, find it easier to understand Mathematics.

Sappaile (2007) in line with Hasanah (2019), has previosly argued that learners are considered to reason if they can use reasoning skills in patterns and manipulate Mathematics in generalised

mathematical ideas and statements. As such, the ability to reason is considered to be the basic ability of Mathematics in itself (Sappaile, 2007). The general definition of mathematical reasoning will be explained in detail in terms of its structural aspects and process aspects in chapter 2. Mathematical reasoning will further be defined in accodance to the notion of multiple representational translations in linear functions.

Mathematical reasoning is compartmentalised into modes of reasoning and analysed in explanatory texts and worked out examples in linear functions in this research. Textbook analysis forms an important facet that can be used to facilitate the relevance of the content presented in textbooks as the curriculum policy changes. For instance, textbook content had to change as the curriculum changed from the Outcome Based Education in the year 2005, to the current Curriculum and Policy Statement in the year 2012 in South Africa. In this, the adequacy of textbooks to carry out the learning aim and skills dictated by the curriculum should be examined. This is evident in Pepin (2008), who says textbooks are educational tools in the classroom around the world, and they impact the teaching and learning as well as how the subject matter is perceived. Adler and Dickson (2001) suggest a link between the learners' learning of Mathematics and the textbook. They advocate that Mathematics will remain at the abstract level and continue to lack structure until it is seen in a written form. Pepin (2008) adds that textbook content analysis is therefore needed to make sure that the content offered to learners is coherent and has connected learning experiences.

Pepin and Haggarty (2001) note that textbooks influence the type of questions and topics that are covered in the classroom. It is for the aforementioned reasons that textbooks need to be analysed in terms of their structure and content, as well as the role they play in the classroom. Pepin and Haggarty (2001) suggest four ways in which textbooks can be analysed in terms of their structure and content, namely:

- The mathematical intentions of textbooks
- Pedagogical intentions of textbooks
- Sociological contexts of textbooks
- The cultural traditions presented in textbooks.

The mathematical intentions of textbooks are what concern this research. Pepin and Haggarty (2001) define mathematical intention as an implicit belief about the nature of Mathematics, and how Mathematics knowledge is presented. Ensor, Dunne, Galant, Gumedze, Jaffer, Reeves and Tawodzera (2002) suggest a different approach to textbook structure and content analysis. They hold the view that textbooks have a particular content that they prioritise, and a distinctive way in which content can be sequenced, exemplified and taught.

1.3. Rationale and objectives for the research

Mckenzie (2001) revealed that learners solve Mathematics problems carelessly, and they follow textbook examples without understanding. Consequently, learners tend to ask for easier ways to solve mathematical problems from their teachers (Aineamani & Naicker, 2010). Aineamani and Naickers (2010) further argues that this suggests a gap between learners' reasoning skills and communicating mathematically when they move from written to spoken Mathematics. In this line of reasoning, Mathematics textbooks should be written in a manner that promotes mathematical reasoning. Muller and Mahar (2009) argue the that Mathematics textbooks should consider all aspects in an attempt to make the need for mathematical reasoning explicit, so that learners can learn how to reason mathematically.

Having discussed the above, Stacey and Vincent (2009) research the need for mathematical reasoning in the 8th Grade Mathematics textbooks following TIMSS Video Study (Hiebert, Gallimore, Garnier, Givin, Hollingsworth & Jacobs, 2003). The TIMSS video study revealed that mathematical reasoning is missing in the 8th Grade Australian Mathematics classrooms. As a result, Stacey and Vincent (2009) studies the need for mathematical reasoning across several topics in the 8th Grade Mathematics textbooks using the theoretical framework of modes of reasoning. The study has shown that explanations are rule-driven and offer justification in preparation for the practice of exercises rather than using explanations as thinking tools.

Silverman and Even (2016) adopted Stacey and Vincent's (2009) theoretical framework of modes of reasoning to study the place of reasoning in the 7th Grade in Israeli Mathematics textbooks. They analysed ten statements in the topic of Geometry and Algebra. It is then discovered that nearly every explanation is deductive and uses empirical justifications (Silverman & Even, 2016).

The place of mathematical reasoning in textbooks has been explored before using the theoretical framework of modes of reasoning. The theoretical framework developed by Stacey and Vincent

(2009) has proven its adequacy in describing modes of reasoning in explanatory texts and worked-out examples as evident in the work of Stacey & Vincent (2009) and Silverman & Evens (2016). However, these scholars do not explore the meaning of what the modes of reasoning make available for learners to develop mathematical reasoning. McKenzie's (2001) findings indicate that learners find it difficult to understand explanatory texts and worked examples. Part of this research's aim is to therefore, explore the meaning of what the modes of reasoning make available to develop learners' reasoning. As Muller and Mahar (2009) suggest, textbook explanations should attempt to make the need for mathematical reasoning explicit, so that learners can develop reasoning skills. In this, the linguistic features, and functions of text in explanations were explored using a component of Halliday's functional grammar analysis model.

Hong and Miching (2017) reveal that a lot of research focuses on analysing mathematical reasoning and proof in exercises. However, there seems to be a limited work on studying opportunities to reason in expository texts, and few studies have shown the theoretical frameworks that can be used to analyse reasoning in textbooks (Davis, 2012; Otten, 2014; Hong & Mi Choi, 2007; Stylianities, 2009; Thompson Senk & Johnson, 2012; Switala, 2013; Wong, 2017). A study focused on opportunities to reason in expository texts is important, mainly because expository texts have the capacity to guide learners and provide first-hand explanations, procedures, and demonstrations to solve mathematical problems (Hong & Mi Choi, 2017).

In the work of Stacey and Vincent (2009), and Silverman and Evens (2016), several topics were explored, except linear functions. As a result, I found it to be valuable to focus this research on the topic of linear functions. It is stated in the Mathematics CAPS document that the interpretation of linear functions amongst other learning aims is introduced in Grade 9 (CAPS, 2012). In this, and amongst other reasons outlined in chapter 2 and 3, I found it to be an important topic to study, especially in Grade 9.

It has been documented in the work of Hitt (1998) and Brenner (2000) that linear functions are one of the most important topics because they infuse different topics together. They require learners to use different representations, and search for similarities, patterns and interpretations of the relationship that exists between different representations (Brenner, 2000).

Furthermore, the diagnostic report of the Department of Basic Education reveals that functions form part of the topics that learners struggle with. Figure 1.1 shows the average performance per

topic in the Grade 12, 2019 final Mathematics examination. The topic on Functions is examined in questions 4. 1 to 5.5 as indicated in Figure 1.1 and they account for 23% of the total marks of the exam paper. This implies that functions consist of more marks than other topics that were examined.

Figure 1.1 shows that learners underperformed on the topic on Functions, especially in questions 4.7 (19% average performance); 5.4 (37% average performance), and 5.5 (average performance 31%) (DBE, 2019). The identified questions (4. 7, 5. 4, and 5. 5) are common in the sense that they all required, to some extent, learners to interpret graphs in relation to their equation. This was revealed in the work of Bosse, Kwaku Adu-Gyamfi (2013). They found that learners find it difficult to interpret graphs and relate them to their equations. This is because the process of graph interpretation requires pattern discovery and the ability to link characters of a graph to those of equations (Bosse, 2013).

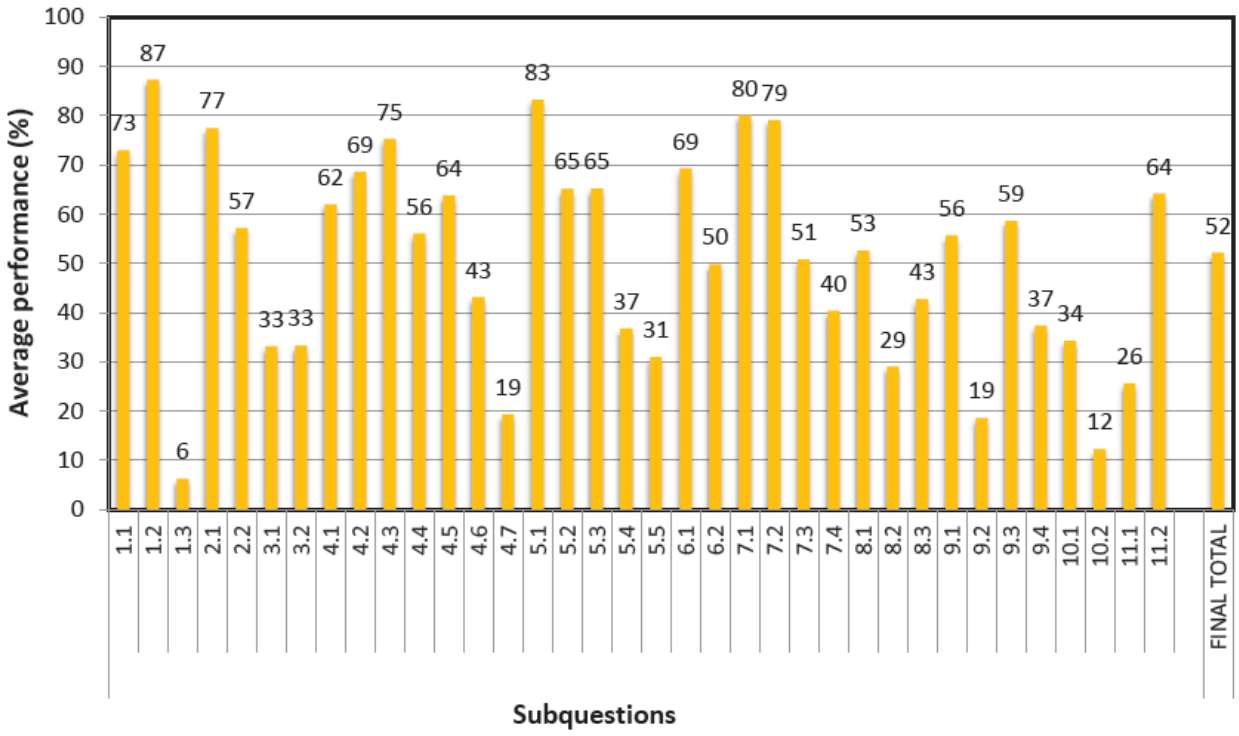


Figure 1.1. Average performance per question in the Grade 12, 2019 mathematics examination. (DBE, 2019. p. 179)

Although this diagnostic report is based on the Grade 12 examination, the foundation is laid in Grade 9 as it is evident in CAPS Mathematics document, where the interpretation of Functions is introduced. The results of the diagnostic report suggest that more still needs to be done in lower

grades to equip learners with the necessary basic reasoning skills to be able to participate in more complex Functions in higher grades.

Bearing the above data in mind, linear functions are arguably regarded as one of the important topics that learners need to master to be successful in Mathematics. One of the concerning factors is that the number of learners who wrote the Mathematics examination in 2019 decreased by 11 824, when compared to 2018 (DBE, 2019). This means there was a 30% decrease in learners who take Mathematics as a subject in Grade 12. The diagnostic report suggests that there has been a decline in performance from the year 2015 to 2019. These results suggest that learners might drop Mathematics as a subject because it is considered to be difficult. Hence, the foundation of Mathematics must be strengthened in lower grades to allow learners to participate meaningfully in higher grades where they will be faced with complex functions.

1.4. Research questions

The two textbooks used in this research are CAPS compliant and they are widely used in South African schools as the main learning and teaching resource, or supplementary teaching and learning aid. Linear Functions will be the main topic that will be analysed. This topic was chosen mainly because many researchers in the field of Mathematics see Linear Functions as a key topic, especially in secondary school (Brenner, 2000). Other researchers argue that Linear Graphs represent the foundation of the whole subject of Mathematics (Hitt, 1998).

The research based on textbook analysis seems to be relevant because textbooks are widely used in South Africa and they form part of the primary tools that inform and influence teaching and learning. Kwong-Cheong Wong (2017) notes that textbooks have an influence on what learners learn. As a result, many studies in the Western countries (for example, in the USA, Australia and Israel) have studied opportunities for learners to learn reasoning in Mathematics textbooks (Wong & Sutherland, 2016). Meanwhile, few studies have been conducted in East Asian countries and Africa to examine reasoning opportunities in textbooks (Mullis, Martin, Foy & Arora, 2012). Moreover, South Africa was the lowest performing country in the International Benchmark for Mathematics achievement in Grade 9 in the year 2012 and continued to perform poorly in Grade 8 in 2019 (Reddy, Zuze, Visser, Winnaar, Juan, Prinsloo, Arends & Rogers, 2019). The present study has, on a small scale, a complementary effect of international research knowledge by examining opportunity for learners to learn reasoning in Mathematics textbooks.

It is noted in Brodie (2010) that most Mathematics teachers in public schools were trained during the BANTU education, under the apartheid regime, however, others were not trained as Mathematics education specialists. On the other hand, Brodie (2010) argues that these teachers are forced to teach Mathematics due to a shortage of Mathematics teachers in South Africa. This means that it is likely for teachers to lack key knowledge and skills about what it takes to teach Mathematics. Following this, textbooks remain important in the Mathematics classrooms in South Africa because they can supplement teachers' content knowledge and help in situations where classrooms are overcrowded.

The type of reasoning made possible in textbooks can highlight the Mathematics that is made available to learners. It must be made clear that explanatory text and worked out examples in textbooks can be of low or high cognitive demand. However, in the implementation process, they can either decline or remain to be of high cognitive demand. This depends on how teachers and learners interact with the text in expository sections. This is argued in Stein (1996), who posits that the types of questions (high press or low press) used during classroom practice can augment or decline the cognitive demands. Without rejecting the importance of the role of teachers and learners during classroom practice, this research is not intended to make any claims or suggestions about how the content that underlies the expository texts in textbooks influences teacher practice. This research is interested in how the expository texts present modes of reasoning, and what do the identified modes of reasoning mean for the learners to develop mathematical reasoning. The following research questions are therefore of interest in this research.

1. What modes of reasoning are implied / presented in explanatory texts and worked out examples in linear functions in two Grade 9 Mathematics textbooks?
2. What do these modes of reasoning in each textbook make available as a means to develop learners' mathematical reasoning?

1.5. Theoretical framework

To answer the above-mentioned research questions, three theoretical frameworks were amalgamated to develop a theoretical framework that is specific to this research. Walshaw and de Freitas (2016) argue that researchers in the area of textbook content analysis studies normally use

previous theoretical frameworks, or they develop theoretical frameworks that are specific to their study by adopting components of theoretical frameworks that were developed before.

This research is not fully confined within the research domain boundaries that are evident in Stacey and Vincent (2009) and Silverman & Even (2016) work. It extends by exploring the meaning of the modes of reasoning, and what they make available to develop learners' reasoning as it was reiterated before. It also examines the topic of linear functions which was not explored in the work of Stacey & Vincent (2009) and Silverman & Even (2016). Thus, it would be inadequate to adopt the theoretical framework of modes of reasoning alone because there is another aspect that has been introduced.

Therefore, I saw it fit to amalgamate Stacey and Vincent's (2009) theoretical framework of modes of reasoning, Janvior's (1987) multiple representational translation model and a component of interpersonal imperative pronounce from Halliday's (1973) theoretical framework of functional grammar analysis. Chapter two will outline a detailed explanation of the above-mentioned theoretical frameworks, and why they were chosen.

1.6. Research approach

This research is centred on the domain of Vygotsky's (1978) perspectives that are assumed to influence learning and teaching. Vygotskian perspectives stem from the widely known constructivism that suggests that learning manifests when learners are part of the construction of new knowledge, based on their prior conceptual framework (Liu & Khine, 2016). Ausubel (1968) also argues that learning is influenced by what learners already know, as a result, instructional materials should be developed accordingly.

Since textbooks are widely used in teaching and learning, the effectiveness of the learners' learning process should be reflected in the textbooks' explanatory texts and worked-out examples. Liu and Khile (2016) indicate that textbooks should support learners by presenting defining features of learners' active engagement.

Since learning is assumed to require active engagement from learners, the defining features that suggest active engagement in expository texts were explored using the notion of interpersonal pronouns derived from the theoretical framework of Halliday's functional grammar analysis. This

part forms the third section of analysis that aims to explore the meaning of the modes of reasoning to learners' development of mathematical reasoning. This will be expanded further in chapter two.

1.7. Definition of terms/ key constructs/concepts

Modes of reasoning: Refers to the mathematical reasoning implied in the explanatory text and worked out examples. The meaning of the mode of reasoning in this research refers to the manner of the method in which reasoning is presented.

Explanatory texts and Worked out examples: Mean explanations of why, rather than how (Stacey & Vincent 2009). Explanatory texts and worked out examples refer to a deeper sense of explanation and connection involving justification and/or proof of new mathematical results (Stacey & Vincent, 2009). Watson and Mason (2002) define worked out examples as demonstration tools, they are normally designed to provide guidance and trajectories that can be used to solve mathematical problems. This is a section of textbooks where procedures and concepts are introduced and demonstrated (Watson & Mason, 2002). Explanatory texts and worked out examples are jointly referred to as expository texts in this research, the terms might be used interchangeably, but they mean the same thing. It is revealed in Stacey and Vincent's (2009) work that textbooks consist of expository texts that hold different reasoning opportunities. Thus, this research explores modes of reasoning in expository sections.

Learners: Refers to the readers or users of textbooks.

Characters / parameters of representations: Refers to features of representations and their function, i.e. y= $mx + c$, where m, x, y *and* c are characters / parameters with special functions attached to them.

1.8. Outline of chapters

Chapter 1

This chapter discusses the importance of Mathematics textbooks as primary learning and teaching resources. The need and relevance of textbook content analysis research in the context of mathematical reasoning in linear functions is also outlined. A brief definition of mathematical reasoning, theoretical framework, research methods and approach are outlined.

Chapter 2

This chapter elaborates further on the following.

- Theoretical underpinnings that define mathematical reasoning in terms of its structural and process aspects,
- The theoretical framework of modes of reasoning and the theoretical framework of multiple representational translations, and their relevance to studying the extent of reasoning presented in expository texts, and
- The interpersonal imperative pronouns from Halliday's functional grammar analysis and the extent of learner participation to shed light on the features and structures of text in expository text that are likely to be comprehended by learners.

Chapter 3

This chapter discusses the research design and method used to collect, organise, and analyse data. The theoretical framework used to guide the analysis of data is explained, and the reason for choosing the theoretical framework is also discussed.

Chapter 4

This chapter presents data analysis and discussion of findings. The dominant modes of reasoning in expository texts are identified and explained. To explain what the identified dominant modes of reasoning mean, for learners to develop mathematical reasoning skills, the interpersonal imperative pronouns in explanatory texts are analysed.

Chapter 5

This chapter concludes this research by discussing major findings and recommendations related to this research. Lastly, the limitations of this research are also outlined.

1.9. Conclusion

This chapter discusses the significance of research in the area of mathematical reasoning in Mathematics textbooks. It also explains the importance of studying reasoning in the topic of linear functions in the 9th Grade. A preview of the theoretical framework and research approach used in this research is outlined. Lastly, a summary of all chapters was outlined. The following chapter (chapter 2) will discuss the theoretical underpinnings, and why they are suitable in the context of

this research. Chapter 2 will also discuss the literature review, and the theoretical framework that is used in this research.

CHAPTER 2: LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1. Introduction.

According to Hasanah (2019), there are four process standards that learners need to possess when learning Mathematics:

a. Problem-solving skills
b. Reasoning and proof
c. Communication
d. Reasoning and representation

In this, reasoning and representation are amongst the important competencies that learners need to master to be successful in learning Mathematics. To expand on this, in this chapter, I defined mathematical reasoning in terms of its structure and process aspects. Thereafter, the teaching and learning of mathematical reasoning in the context of Linear Functions was discussed.

The theoretical framework that was used in this research stems from the amalgamation of the work of Stacey and Vincent's (2009) theoretical framework of modes of reasoning, Janvier (1987) theoretical framework of multiple representational translations, and some components of the theoretical framework of functional grammar analysis (Halliday, 1973).

The theoretical framework of modes of reasoning and the multiple representational translations were amalgamated and used to answer the first research question to some extent: *What modes of reasoning are implied / presented in expository texts and worked out examples in linear functions in two Grade 9 Mathematics textbooks?*

While the theoretical framework of functional grammar analysis was used to answer the second research question: *What do these modes of reasoning in each textbook make available as means to develop the reader to reason mathematically?*

In this chapter, I firstly defined mathematical knowledge as background literature to build towards defining mathematical reasoning which is the concern of this research. Secondly, I discussed how the defined mathematical reasoning emanates in the teaching and learning of linear functions.

Lastly, the theoretical frameworks that were used in this research were explained, and reasons as to why they were chosen form part of the discussion.

2.2. Conceptualisation of what constitutes Mathematical knowledge.

Scholars have differing views on what constitutes mathematical knowledge, these scholars include among others Kilpatrick, Swafford & Findell (2001); Niss (2003); Ginsburg (2008); Osterman & Brating, (2019/ 2017); Deway (1959/b); Hiebert & Lefevre (1986); and Lithner (2008). An emphasis on real-world Mathematics has been at the centre of the debate of what constitutes mathematical knowledge. Gainsburg (2008) and Gronmo (2011) among other scholars, contest the need for learners to learn something that they can use outside the school setting. They advocate that learners should learn Mathematics that is based on contextual knowledge, instead of memorizing formulas.

Brating and Osterman (2018) emphasise the importance of involving learners in doing Mathematics, with less focus on the actual doing of Mathematics through symbolic manipulation and the use of algorithms. To encourage learner involvement, Brating and Osterman (2018) suggest that instructions should ideally include what is relatable and interest learners (psychological aspect), but that which will direct learners to the end goal of gaining access to the body of principles and truths that govern the abstract mathematical knowledge (logical aspect).

The movement between learners' experience and the abstract knowledge of Mathematics defines the mathematical knowledge in the educational theory (Brating & Osterman, 2018). That is, the bridge that is used to move learners' experience to the abstract subject matter could define the competencies that learners are likely to develop.

In this, instructions could advocate procedural knowledge and or conceptual knowledge as well as reasoning. Procedural knowledge is defined as the knowledge that consists of using a chain of prescriptions to manipulate symbols, while conceptual knowledge is described as the knowledge to link relationships and pieces of information (Kilpatrick et al., 2001). This is related to Skemp (1989) cited in Wee (2015) and Hiebert & Lefevre's (1986) description of conceptual understanding. Skemp (1989) refers to conceptual knowledge as a 'connected web'.

What is common in what constitutes mathematical knowledge is the description of a situation where learners understand the use of a system of mathematical concepts. Brating and Oesterman

(2018) advocate for the use of the two strands to direct learners to the logical aspects of Mathematics, while Kilpatrick et al. (2001) contest for the use of the two strands (procedural and conceptual knowledge) together with three more strands (productive disposition, adaptive reasoning, and strategic competence) to help learners to be proficient in Mathematics.

To consider the context of this research, worked-out examples and explanatory texts should, to some extent, involve learners in the actual doing of Mathematics, and use explanations that would advocate for developing learners' mathematical reasoning.

Therefore, learning Mathematics procedurally has been argued by many scholars to impose a limited understanding of Mathematics because it leads to the computation of symbols and algorithms without reason. Algorithms and symbols are used to solve mathematical problems, not mainly for explanations and or instructions to help learners to gain access to mathematical knowledge (Broussean, 1997).

As such, instructions / explanations that single handedly promote the use of procedural knowledge have been discouraged. However, Brating and Osterman (2018) has argued that the use of procedures cannot be dismissed because they are, to some extent a part of doing Mathematics. Thus, it is advisable that both procedural and conceptual knowledge should be used in instructions.

2.2.1. Imitative and creative reasoning.

Lithner (2008) builds on the idea of procedural and conceptual knowledge by highlighting the need for reasoning in Mathematics. It is argued by Lithner (2008) that reasoning is not adequately dealt with; as a result, the theoretical framework of reasoning which draws on the distinction between imitative reasoning and creative reasoning is factored in to close this gap.

Imitative reasoning is defined as the memorization of algorithms and procedures without a considerable reason behind the choice of procedures and algorithms used (Lithner, 2008). Creative reasoning refers to the re-creation of forgotten algorithms and procedures that were once learned, and the choice of algorithms and procedures used are supported by arguments and reasons that are intrinsic to mathematical properties (Lithner, 2008). `

The description of the idea of procedural and conceptual knowledge in Brating and Osterman (2018) is closely related to the idea of imitative and creative reasoning in Lithner (2008) because both procedural knowledge and imitative reasoning refer to the memorization of facts and rules.

Conceptual knowledge and creative reasoning refer to the use of algorithms and procedures with supporting reasons behind the strategic choice and the relationship between individual stages made to solve a Mathematics problem.

The relation of ideas used to describe conceptual knowledge in Brating and Osterman (2018), Kilpatrick et al.'s (2001) adoptive reasoning, Lithner's (2008) creative reasoning and Skemp's (1989) cited in Wee (2015) description of relational understanding refer to a common aim to highlight the significance of not learning Mathematics by memorising facts and algorithms. The common idea that they encourage is to learn Mathematics by understanding and being able to apply mathematical operations in new and unfamiliar situations, as well as to understand the connections between different concepts with supporting arguments of the choice of the chosen strategies.

The desired educational aim in this regard is to develop learners' conceptual-relational understanding and reasoning, with minimal exposure to procedural knowledge. Hiebert and Routledge (2013) point out that procedural knowledge could be used to reach correct solutions to Mathematics problems. However, the ability to use procedures and algorithms does not indicate the ability to reason using principles that govern Mathematics.

On the same note, Broussean (1997) has argued that algorithmic procedures are efficient to solve mathematical problems, but not necessarily efficient to be used for learning because they avoid meaning. This is important, particularly on the idea of the structure and form in which explanations and instructions should be written to provide meaning to learners. What this suggests is that instructions in the form of algorithmic procedures might be devoid of meaning to learners, which might encourage procedural knowledge. This implies that instructions and or explanations should encompass supportive arguments for each step in the algorithmic procedures to afford some level of mathematical reasoning for learners. In the context of this research, expository texts that explain operations using algorithmic procedures without supporting arguments in the form of textual descriptions could limit learners to access principles needed for them to develop mathematical reasoning skills.

2.3. The structure and process aspects of mathematical reasoning.

Hasanah (2019) maintain that learners' understanding of Mathematics is gained through reasoning. In other words, learners who can reason during the process of understanding Mathematics will find it easier to understand Mathematics even when unfamiliar concepts are introduced.

Amidst the conception of what constitutes mathematical reasoning, and its relation to the idea of understanding and conceptual knowledge, Mathematics education theorists have a different conception of what institutes mathematical reasoning. For Arsac (1996) and Lithner (2008), mathematical reasoning is conceived as a concept that allows learners to produce new knowledge. On the other hand, Jeannotte and Kierran (2017) suggest that deductive reasoning is the only legitimate way that allows reasoning to be considered mathematical. To define the grounds on which mathematical reasoning stands in this research paper, I will define it in terms of its structural and process aspects.

2.3.1. Structural aspect of mathematical reasoning.

The structural aspect of mathematical reasoning refers to the creation of rules of mathematical discourse (Jeannotte & Kierran, 2017). The structural aspect of reasoning consists of deductive and inductive reasoning. Deductive reasoning is reasoning that can modify the epistemic value of Mathematics knowledge from being likely to being regarded true (Duval, 1995). It is a reasoning process that allows one to make a conclusion based on previously known truths. For deductive reasoning to be used, a general statement could be given about a group of things, and then a specific example could be given from there. In this, if the example given fits into the group, then deductive reasoning can be used. For instance, all men are soccer players. If it is given that Itumeleng is a man, then it could be concluded that Itumeleng is a soccer player. In this case, it is assumed that the premises, 'All men are soccer players' and 'Itumeleng is a man' are true. In this, the conclusion "Itumeleng is soccer player" is logical and true because in deductive reasoning, if something is true of a group of things in general, it is also true for all members of that group. Deductive reasoning as pointed out in Bradford (2017) therefore starts with a general statement and examines the possibilities to reach a specific, logical conclusion.

The example above highlights the importance of a premise in deductive reasoning and the fact that each step-in deductive reasoning follows logically from previous steps. Deductive reasoning is

considered to be a part of a structural aspect of mathematical reasoning because of its ability to infer a claim from data and warrant (Jeannotte & Kierran, 2017).

Inductive reasoning infers a warrant from the claim about the data (Reid, 2010). Inductive reasoning makes broad generalisations from specific observations. Data is given, and then conclusions are made from the data. To draw an inductive conclusion from data, many specific observations are made from data to discern a pattern that can be used to make generalisations to infer an explanation of a theory (Bradford, 2017).

Figure 2.1 shows lines A-D and E-H on the Cartesian plane which is used to draw triangles that illustrate the ratio between the vertical rise, and the horizontal rise. It is observed that the vertical and the horizontal rise are labelled using specific values, i.e. triangle D in Figure 2. 1 has a vertical rise of 6 units and a horizontal rise of 2 units. It is therefore given that the gradient $M$of line D is 3. This is because of Gradient $M = \frac{6}{2} = 3$. The same observation could be made for the rest of the lines A-D and E-H, the value of the vertical rise is always divided by the value of the horizontal rise on the triangles to find the Gradient of a line.

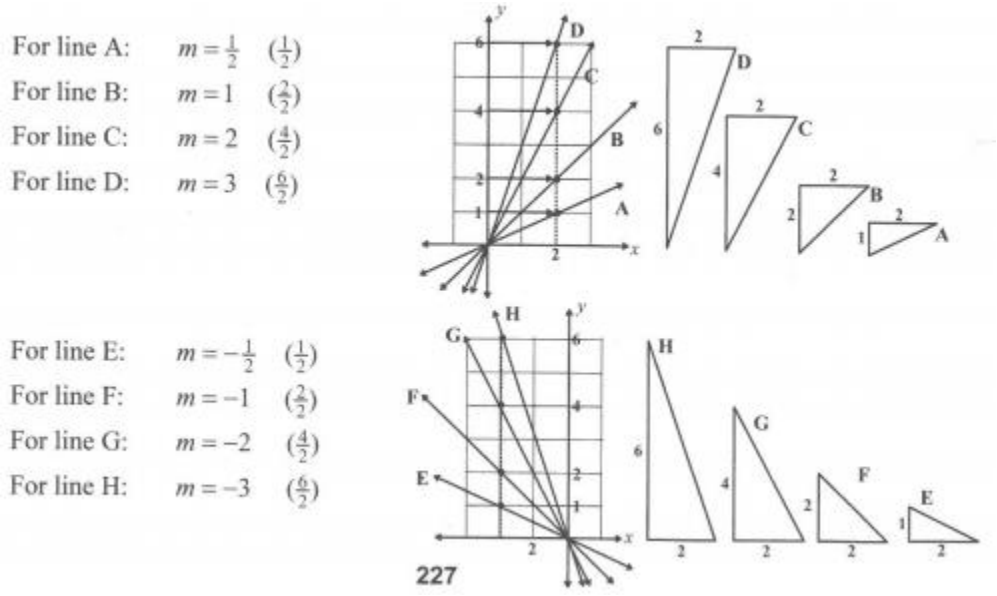


Figure 2.1: Inductive reasoning used to infer an explanation of Gradient (Basson, Phillips & Odendaal, 2014)

The prevailing pattern of having to always divide the vertical rise by the horizontal line to find the Gradient of a line could be used to formulate the formula of the Gradient, $m = \frac{vertical\ rise}{horizontal\ rise}$. In this, the prevailing uniformity of the pattern of always dividing the vertical by the horizontal rise has helped to infer an explanation that could be used to generalise, that is, to come up with a formula that could be used to find the Gradient of any straight line. Thus, specific cases have been used to explain generalisation.

2.3.2. Process aspect of mathematical reasoning.

The idea of generalisation and pattern discovery has been mentioned in the explanation of inductive reasoning, and the use of premises that should be true to infer logical steps that are used to reason deductively. In the section of the process aspect of mathematical reasoning, generalisation and the identification of patterns have been identified among other aspects of mathematical processes such as conjecturing, comparing, classifying, validating and formal and informal proving. The process aspect of mathematical reasoning is divided into two: Process aspects related to the search for similarities and differences and process aspects related to validation.

2.3.2.1. Process aspects related to the search for similarities and differences.

Ellis (2007) defines mathematical reasoning in terms of generalisation and justification. The generalization is defined as a process that consists of three categories, namely: relating, searching, and extracting. `Relating` refers to the association between two or more mathematical problems; searching refers to performing a repeated action to test if the given answer is valid for all cases, and extracting, on the other hand, refers to the removal of contextual details to test if the given answer is valid for all cases. These categories were developed when Ellis (2007) was researching different ways in which learners' reason mathematically. Generalisation is concluded to be a process of expanding one's range of reasoning beyond the case considered (Ellis, 2007). In other words, one should be able to reflect upon, and extend, regardless of context, the knowledge gained from previously learned generalisations. According to Stylianides (2008), generalisation is a process that deduces narratives about mathematical relationships between objects.

Another important aspect of reasoning is justification. Ellis (2007) recognizes justification in terms of persuasion and ascertaining. Persuasion refers to an act where one removes doubt of others while ascertaining means an act where one removes his/her doubt. Justifying is a mathematical reasoning process that allows the searching of data, warrant, and support, which allows the

modification of the epistemic value of a narrative ((Jeannotte & Kierran, 2017). These two concepts: generalisation and justification are found to influence each other in the development of mathematical reasoning (Ellis, 2007).

Learners who are not sufficiently equipped with the skill to generalise and justify their solutions are likely to lack knowledge of what to do and how to solve mathematical problems (Russell, 1999 cited in Gunhan, 2014). In this, learners will simply add, multiply, divide or subtract numbers they see without thinking much about the reasons behind their computations.

Expository texts refer to paragraphs, text boxes that normally contain formulas, definitions, theorems and worked out examples (Hong & Mi Choi, 2017). It is revealed in Stacey and Vincent (2009) that expository sections in Mathematics textbooks can be used to study opportunities to reason. In this research, I will study the expository sections to infer the modes of reasoning presented. This section of the textbook is relevant for this research because it consists of the content knowledge that is legitimised to promote certain habits of mind towards solving mathematical problems.

Conjecturing is another important process / aspect of mathematical reasoning. Conjecturing refers to the process of searching for similarities and differences and can allows mathematics to be theorised (Jeannotte & Kierran, 2017). Identifying a pattern, on the other hand, is a process that allows for the search of similarities and differences with an intent to deduce a narrative about the relationship between mathematical objects ((Jeannotte & Kierran, 2017).

2.3.2.2. Process aspects related to validation.

According to Duval (1995), Canadas (2007), and Lithner (2008), validation is a mathematical reasoning process that changes the epistemic value (the truth or the likelihood) of a mathematical narrative. This process refers to the possible changes in the epistemic value of a narrative by providing backup that supports the change (Duvel, 1995, Cabassut, 2005., and Stylianides, 2008)

For Balacheff (1988), proving is an explanation that is socially accepted in a Mathematics discourse. It is an activity that is used to remove doubt about the truth of an assertion (Harel & Sowder, 2007). Proving is used to justify claims using deductive arguments (Maher, 2009). This means, proving is a mathematical reasoning process that modifies the epistemic value of a narrative

from likely to be true and is achieved by using utterances and claims that are accepted in the Mathematics community.

Jeannotte and Kierran (2017) argue that deductive reasoning consists of a structure that is acceptable in the Mathematics discourse (Jeannotte & Kierran, 2017). However, at some point, Jeannotte and Kierran, (2017) reveals that proving does not have to be structured deductively at all times, it must, at all times rely on a set of accepted narrative as being true in the Mathematics community (Mariotti, 2005). Exemplifying is another important process of mathematical reasoning because it supports other processes of mathematical reasoning by providing examples that help in the search for the relation between mathematical objects (Jeannotte & Kierran, 2017). To sum up this section, mathematical reasoning has been described and explained in terms of its structure and process aspects.

2.4. Teaching and learning of mathematical reasoning in Linear Functions

Bosse, Kwaku and Adu-Gyamfi (2013) documented that learners who understand multiple forms of representations such as tables, graphs, and Algebra in linear functions, are most likely to understand the synergy that gives the whole picture of the concept of linear functions.

Linear Functions form one of the most important areas in Mathematics owing to its characteristic of unifying different topics in Mathematics (Mellor, Clark & Essien, 2018). It is also characterised by various representations which include graphs, tables, and equations. These representations can act to communicate mathematical ideas and relationships that are embedded in Linear Functions (Steinbring, 2006). Tripathi (2008) holds that these representations can allow functions to be communicated and discussed.

The importance of multiple representational translations and their role in promoting mathematical reasoning in Linear Functions is documented in Eisonberg (1992); Even (1998); Porzio (1999); Bosse et al. (2013). It is revealed in the work of Bosse et al. (2013) that when learners are exposed to the connections of the relationship of the variables that exist between different forms of representation, Linear Functions could be learned with minimum difficulty. Leinhardt et al. (1990) reiterate that for Linear Functions to be accessible with minimum misconceptions, the correlation between a graphical line and its algebraic equation is important, and the sequencing of knowledge thereof (Mellor et al., 2018).

It was found that learners normally compartmentalize the various representations of Linear Functions, they are conceived as 'independent objects in their own right' (Bednarz & Balanger, 1987). Eisonberg (1992) reported that learners fail to integrate information across multiple representations. Sfard, (1992); Vinner & Dreyfus, (1989) also pointed out that learners possess disconnected ideas for the meaning of Linear Functions. They argue that learners often fail to effectively use representations when learning Linear Functions. Furthermore, Tabachnek and Simon (1998) revealed that learners fail to integrate information across representations.

It was pointed out in Bosse et al. (2013) as well that learners can move from a given representation of a Linear Function to another, but still demonstrate a limited reasoning ability in Linear Functions. It is in the individual articulation of each representation that makes it difficult to develop mathematical reasoning in Linear Functions (Bosse et al., 2013).

To remediate this issue, van Dyke and Craine (1997) suggest that it is important that learners understand the equivalence that underlie the different representations of a function. They need to understand that the same function can be represented in algebraic form, tabular, and graphical form. And most importantly, they need to understand their strengths as well as their weaknesses (Rider, 2007). In so doing, learners will be able to translate the content knowledge fluently between these different representations.

The ability to make connections between different knowledge forms and representations, and to strategically select the ones that are relevant to the context of learning a certain topic in Mathematics is evident in Kilpatrick et al.'s (2001) idea of adaptive reasoning, and Ellis's (2007) definition of reasoning in terms of generalisation and justification as discussed earlier on. With the above background, the conceptualisation of reasoning in Linear Functions is therefore understood in terms of the multiple representations' translation.

2.5. Theoretical frameworks: Modes of reasoning, multiple representational translation model and functional grammar analysis.

This section discusses the theoretical frameworks that were used in this research. As reiterated before, I first discuss the theoretical framework of modes of reasoning and explained why it was chosen. I further engaged with its limitations to studying Linear Functions. Afterward, the identified limitations are dealt with by introducing the theoretical framework of multiple

representational translation model, and an explanation of choosing this theoretical framework are outlined. Lastly, some elements of the functional grammar analysis are discussed and their relevance in answering the second research question is explained.

2.5.1. Modes of Reasoning.

Stacey and Vincent (2009) analyse expository texts in the Australia 8th Grade Mathematics textbooks in response to the TIMMS video study in 2003. The TIMSS study reported that reasoning in the 8th Grade Mathematics classrooms was missing. To study the place of reasoning in the classroom, Stacey and Vincent (2009) analysed reasoning in the textbook's expository section with an interest to see how the need for mathematical reasoning is dealt with. The following six topics were analysed across nine Mathematics textbooks:

- Dividing fractions
- Multiplication of two negative numbers
- The angle sum of a triangle
- Area of a trapezium
- Distributive law
- Multiplication of powers.

In as much as the theoretical framework of modes of reasoning was used to study the modes of reasoning in the above-mentioned topics, Linear Functions was not studied. However, due to its credibility in studying modes of reasoning in Algebra and Geometry, which was highlighted in the work of Silverman and Even (2016), the theoretical framework of modes of reasoning's adequacy to describe the modes of reasoning in the topic of linear functions is therefore questionable.

It was mentioned before that Linear Function are characterised by multiple representations, which amongst others includes textual and algebraic representations. These representational aspects of Linear Functions could be adequately studied using the theoretical framework of modes of reasoning since Stacey & Vincent (2009) and Silverman & Even (2016) succeeded in describing the modes of reasoning in explanatory texts in the topic of Algebra. The aspect of translation between representations (tables, graphs, algebra, and textual descriptions) in linear functions, the direction of translation between representations, and the link between the parameters of the representations could not be described using the theoretical framework of modes of reasoning. The importance of translation between representations in linear functions, and its role to teaching and

learning was discussed earlier, as a result, reasoning in the context of linear functions has to be studied following representations and the extend of translations between them. The limitations of the theoretical framework of modes of reasoning that has been highlighted will be dealt with later in the chapter.

2.5.1.1. What are the modes of reasoning?

The theoretical framework of modes of reasoning was developed by drawing ideas from theoretical frameworks that were developed before, namely: Harel & Sawder's (2007) framework of proof schemes; Blum & Kirsch's (1991) three levels of proof; and Sierpinska's (1994) distinction between scientific explanations and didactic explanation. By combining these frameworks, Stacey and Vincent (2009) were able to construct a new theoretical framework that partitioned reasoning into three proof scheme classes (External conviction, empirical proof and deductive proof) that are subdivided into seven modes of reasoning. Table 2.1 summarises how the modes of reasoning are subdivided into three proof schemes.

Table 2.1: Classification of modes of reasoning into three proof schemes (Stacey et al., 2009)

Proof scheme class	Modes of reasoning
External conviction	Appeal to authority: null explanation or reliance on an external source of authority Qualitative analogy: reliance on a surface similarity to non-mathematical situations,
Empirical proof	Concordance of a rule with a model: Comparing specific results of a rule and a model Experimental demonstration: Identifying a pattern after checking selected examples.
Deductive proof	Deduction using a model: a model that serves to illustrate a mathematical structure. Deduction using a special case: Inferential process conducted using a specific case. Deduction using a general case: Inferential process conducted using a specific case.

The theoretical framework of modes of reasoning correlates closely with Harel and Sawder's (2007) theoretical framework of proof schemes. The word proof in Harel and Sawder (2007) refers to both formal and informal arguments. The proof is conceptualised as a process that is adopted by a "community or an individual to remove doubt about the truth of an assertion" (Harel and Sawder, 2007. p. 808). Harel and Sowder (2007) grouped proof schemes into three classes depending on the nature of reasoning involved, and each proof consists of three or more schemes, see Table 2.1. In Harel and Sowder (2007) study, it was revealed that learners' proof schemes normally belong to either external or empirical class, rather than the desired deductive class which consists of transformational and axiomatic schemes.

Stacey and Vincent (2009) used the descriptions of these proof schemes mentioned above to investigate their adequacy in describing the modes of reasoning presented in expository texts. The descriptions of the modes of reasoning are summarised as follows:

- *Deduction Using a General Case**:*** uses variables instead of specific values, and makes use of previously known facts, definitions, and knowledge (each step is deduced from previous steps)
- *Deduction Using a Specific Case**:*** Uses specific values instead of variables, and it makes use of previously known facts, definitions, and knowledge (each step is deduced from the previous step)
- *Deduction Using a Model:* Use of a model with structural similarity with the Mathematics. There is a direct correspondence between the model and the associated Mathematics. The model can be used to predict the results of calculations.
- *Concordance of a Rule with a Model:* Matching the answer derived from a model and a rule as well as alignment of answers derived from a model in a specific case to a rule.
- *Analogical Reasoning:* The Mathematics explained depends on a superficial analogy to make sense of the Mathematics envisioned to be conveyed. However, the scenario cannot be used to check the validity of Mathematics calculations.
- *Experimental Demonstration:* Results are obtained after observing a pattern in numerous cases. This has the potential to engage learners to participate to allow examination of physical or numerical regularity.

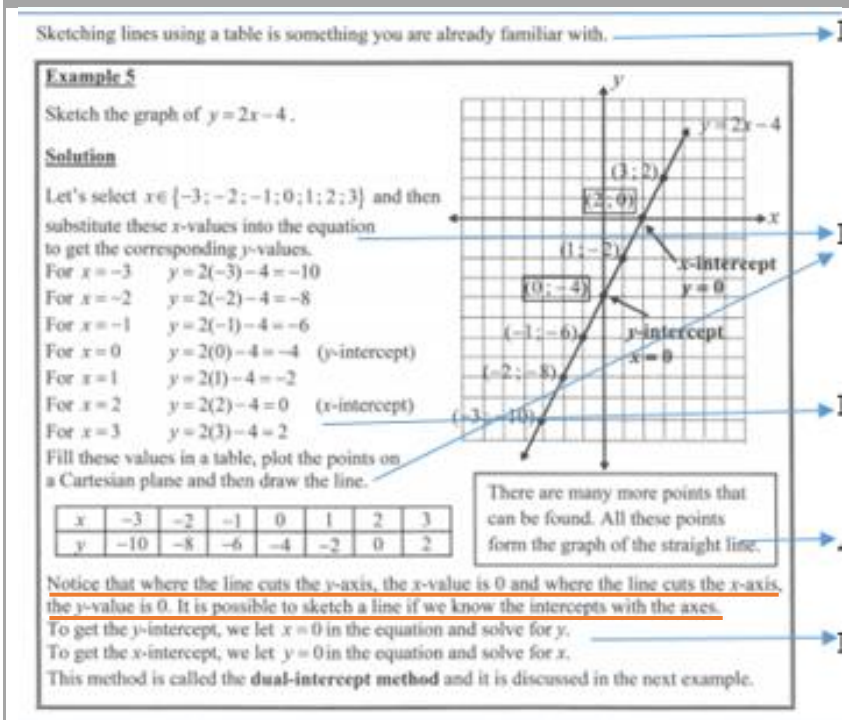
- *Appeal to Authority:* This is defined as a warrant that is given to justify that an assertion is true because it has been found to be so by an authority (e.g. Euclid, and or mathematical textbook) (Stacey & Vincent, 2009). There is some explanation power and some sense of justification. However, an explanation of the principles and processes that define the mathematical idea is not supplied.

2.5.1.2. Describing modes of reasoning in explanatory texts.

Expository texts in Stacey and Vincent (2009) are defined as units that consist of textual descriptions, worked-out examples, and demonstrations of mathematical concepts. To exemplify how the theoretical framework of modes of reasoning is used to describe and identify the modes of reasoning in explanatory text, examples 1 is given in Figure 2. 2. Figure 2.2 shows explanatory texts and worked out examples used to explain how to sketch a graph using a table.

The explanations are described and categorised into modes of reasoning. The phrase, 'sketching lines using a table is something you already know,' in Figure 2.2 is categorised as *deduction using a general case* because it makes use of previously learned facts and knowledge. The phrase assumes learners are familiar with sketching lines using a table method.

The worked-out examples and explanatory texts in Figure 2.2 are categorised as *deduction using a specific case*. An explanation of the process of how to sketch a line graph when given an equation is demonstrated using specific values that are selected and substituted into the equation. Substitution into the equation and the demonstration of algebraic procedures is previous knowledge that is deduced to suit the context of sketching a line graph using a table.

Example 1	Modes of reasoning
	Deduction using a General Case Deduction Using a Specific Case Deduction Using a Specific Case. Appeal To Authority. Deduction Using a Model

Sketching lines using a table is something you are already familiar with.

Example 5

Sketch the graph of $y = 2x - 4$.

Solution

Let's select $x \in \{-3; -2; -1; 0; 1; 2; 3\}$ and then substitute these x-values into the equation to get the corresponding y-values.

For $x = -3$ $y = 2(-3) - 4 = -10$
For $x = -2$ $y = 2(-2) - 4 = -8$
For $x = -1$ $y = 2(-1) - 4 = -6$
For $x = 0$ $y = 2(0) - 4 = -4$ (y-intercept)
For $x = 1$ $y = 2(1) - 4 = -2$
For $x = 2$ $y = 2(2) - 4 = 0$ (x-intercept)
For $x = 3$ $y = 2(3) - 4 = 2$

Fill these values in a table, plot the points on a Cartesian plane and then draw the line.

x	−3	−2	−1	0	1	2	3
y	−10	−8	−6	−4	−2	0	2

There are many more points that can be found. All these points form the graph of the straight line.

Notice that where the line cuts the y-axis, the x-value is 0 and where the line cuts the x-axis, the y-value is 0. It is possible to sketch a line if we know the intercepts with the axes.

To get the y-intercept, we let $x = 0$ in the equation and solve for y.
To get the x-intercept, we let $y = 0$ in the equation and solve for x.
This method is called the **dual-intercept method** and it is discussed in the next example.

Figure 2.2: Sketching straight line graphs using a table method (Basson et al., 2014)

The *appeal to authority* mode of reasoning was also identified. The phrase in Figure 2. 2 'There are many more points that can be found. All these points form the graph of the straight-line" suggests that more points can be found on the line without explaining how they are found. This kind of explanation that does not go to the extent of explaining why and how more points can be found on a line suggests an explanation from an external source of authority. In other words, a further explanation is not needed, therefore it is adopted as it is without the how and why because it was proven to be true.

2.5.1.3 The adequacy of the theoretical framework of modes of reasoning and its limitations to the study of modes of reasoning in Linear Functions.

Figure 2.2 demonstrates the adequacy of the theoretical framework in describing and identifying modes of reasoning in worked-out examples and explanations. It also can link worked out examples to a model. The mode of reasoning: *deduction using a model* provides description standards that could be used to link a model (graph) to the associated Mathematics. This is the case (in the highlighted part: orange colour) in Figure 2.2 which links the concept of intercepts to how they are demonstrated on a graph.

The description standards used by the theoretical framework of modes of reasoning are appropriate following the demonstration in Figure 2.2. However, the theoretical framework falls short of providing description standards that could be used to study the translations between representations in Linear Functions. It fails to provide a full view of how the combination of parameters of representation is translated to create a representational structure that promotes a certain mathematical idea. Lesh, Post and Behr (1987) suggest that in linear functions, learners should be able to selectively combine characters of representations to understand the representational structure. The 'combinations' of representational characters, as it was argued by Duval (2002), is fundamental to understand Linear Functions. The word '*function*' refers to the unique relationship that exists between two elements (Bosse et al. 2013), as such, their fundamental character and how they are related is important.

Following what has been argued thus far, the theoretical framework of multiple representational translation model provides bases and description standards that could help to study the combination of the characters that exist between representations, and the translation between them.

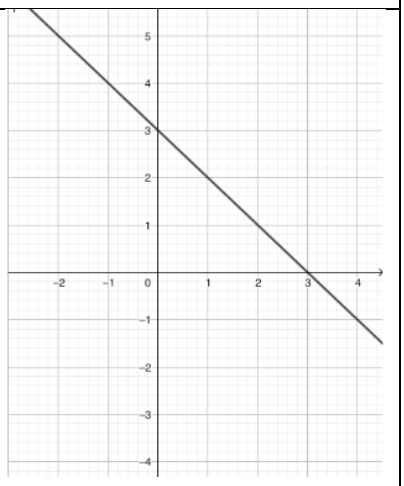
2.5.2.1 Theoretical framework of multiple representational translation model.

The literature documented in Bosse et al. (2013), Eisenberg, (1992), Even (1998), Sfard, (1992), Yerushalmy, (1991), Porzio, (1999), Tabachnek & Simon, (1998) and Yerushalmy, (1991) suggests that learners who are exposed to translations between multiple forms of representations (graphs, tables, and algebraic) in linear functions stand a better chance of accessing the whole picture of the mathematical relation between representations, which could ultimately lead to the knowledge of when, why and how to integrate information across these representations. Such a

skill is associated with the process aspects of mathematical reasoning (pattern discovery, proof, validation, search for similarities and differences and comparing) which was discussed before. For a learner to translate from one representation to another, certain underlying transformation principles between representations need to be mastered.

Duval (2006) defines two types of actions that are inherent in the process of translating / transforming from one representation to another. The first one is treatment, which refers to the transformation within one representation register, and conversion which refers to the transformation that changes the representation register without changing the meaning of the mathematical object (Duval, 2006). Lesh et al. (1987) argue that the combination of the action-treatment and conversion is fundamental to mathematical reasoning. Kaput (1987) adds that these actions (treatment and conversion) articulate the need for learners to selectively combine the perimeters of the representational register to depict a particular mathematical idea. The action of conversion resonates well in the context of Linear Functions in this research because, in the process of transforming from one representation register to another, the meaning of the mathematical object does not change. For instance, when the $x-intercepts\ (3;0)\ and\ \ y-intercept\ (0;3)$ are given in the table form as well as in a graphical form as illustrated in table 2.2. The example shows the same meaning of a mathematical object of the concept of intercepts, but the representational register has transformed from a coordinate system to a table form and a graphical form.

Table 2.2: Multiple representational registers of the concept of intercepts.

Coordinates	Table	Graph
(3; 0) (0; 3)	x | 3 | 0 y | 0 | 3	

Although the representation register is different, the mathematical idea depicted in these representational registers is the same. The idea of transformation will be referred to as translation for the purposes of this research, this will be explained shortly.

2.5.2.2. What is the theoretical framework of multiple representational translation model?

It pointed out earlier that learners' struggle to move from one representation in Linear Functions to another which limits their ability to see the underlying relationship that exists between the different representations. Janvier (1987) was interested in the psychology that is involved in the process of moving from one representation to another. It was revealed that moving from an equation to a graph involves different psychological processes than moving from a graph to an equation.

Janvier (1987) suggests that the movement from equations to graphs is a less difficult task because it is a straightforward process, as it is dominated by a series of steps that involve generating ordered pairs and plotting them on a Cartesian Plane and connecting them to make a line. Most instructions in textbooks and other learning aids proceed in the equation to graph direction whereas the movement from graphs to their equation is suggested to be more difficult because it involves interpretation and pattern discovery (Janvier, 1987).

The term representation refers to a combination of components such as symbols, physical objects, and carefully constructed arrangement of ideas in one's mind (Janvier, 1987). Representations according to Janvier (1987) are further defined as some mathematical organisation of symbols such as diagrams, graphs and schema which models different mental processes (Janvier, 1987).

Representation was modelled to resemble a star-like iceberg (see Figure 2.3) that would show one point at a time, with other representation hidden underneath. A translation would then occur from one representation to another. Translation refers to the psychological processes involved in moving from one representation to another (Janvier, 1987). Kaput (1987), defines translation as an act of seeing the same function in different forms, for example, an equation and its graph.

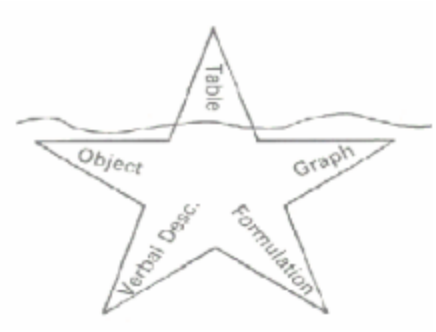


Figure 2.3: Visual resemblance between representations and a star (Janvier, 1987)

It is suggested in Janvier (1987) that using conventional representations as mathematical tools, rejecting one representation to another in each mathematical situation, making translations from one representation to another, are expected from the learner in traditional Mathematics. All these expectations suppose that "the learner has grasped the multiple representations; that he knows the possibilities, the limits, and the effectiveness of each" assumption (Janvier, 1987. p. 111). Moreover, the learner is supposed to choose the appropriate mode of representation depending on the mathematical task. For example, an equation might not be giving equal access to the same information and possibilities as a graph. The coefficient of x in an equation can give access to the direction (+/-) and the value of steepness of a graph, meanwhile a graph on a Cartesian plane cannot directly give access to the value of steepness. Hence, to meet all these expectations, instructional strategies should be improved in a way that includes a variety of representations and are flexible to use translation processes in representations (Janvier, 1987).

Table 2.3 was developed by Janvier (1987) to study the possible direct and indirect psychological translation between representations. Direct translations can be carried out from one representational mode to another without using any other kind of representational mode between these translations, for example, a translation from an equation to a representation of a table. Indirect translation refers to a translation that involves other representational modes between translations, for example, a translation from an equation to a graph then to a table (Janvier, 1987). In studying the process of direct and indirect translations between representations, the idea of source and target is important because it might change the readers' cognitive process (Janvier, 1987). In other words, the source and target could help the learner to achieve the correct translation between representational modes, because it helps to set the aim and channel learners to a trajectory that should be taken to solve a given mathematical task.

Table 2.3: Multiple representational translation model (Janvier, 1987)

From-To	Situations, pictures and verbal descriptions	Tables	Graphs	Formulas
Situations, pictures, verbal descriptions	_	Measuring	Modeling	Analytical Modeling
Tables	Reading	_	Plotting	Fitting
Graphs	Interpretation	Reading Off	_	Curve Fitting
Formulas	Parameter Recognition	Computing	Sketching	_

To illustrate Janvier's (1987) theoretical framework in use, consider Table 2. 3 which show the source (formula/equation of a line) representation and the target (sketched graph). Table 2.3 was used to study the representational translation between the source and the target representation in the example in Figure 2.4. Table 2.3 shows that the translation in Figure 2.4 started from a formula/equation to computations using a table, and thereafter a linear graph was sketched. This example illustrates an indirect representational translation because it used a mediating representation of a table between the source and the target representation. In a nutshell, three representations have been used.

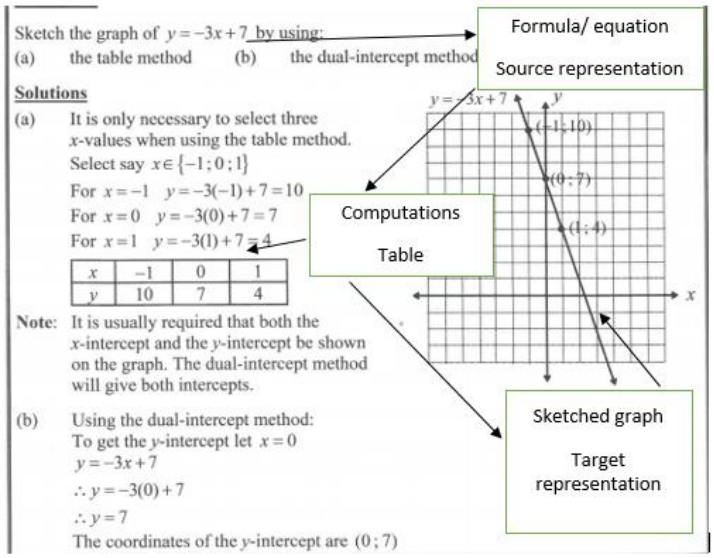


Figure 2.4: Multiple representational translations used in expository texts (Basson et al., 2014)

As discussed before, multiple representations form an important source of learning mathematical reasoning in linear functions. This is due to their primary resemblance of using tables, diagrams, equations, and graphs to communicate and or present the same mathematical concept. It is

important to study the representational modes presented in expository sections of textbooks because they are tools that are used to reason with when learning Linear Functions. It is in the multiple representational translations in Linear Functions that depict the theoretical framework of Janvier (1987) to be an essential aspect that could be used to get a better view of the reasoning that is presented in expository texts (textbook examples).

2.5.2.3. Relevance of the theoretical framework of multiple representational translation models.

Several reasons as to why the theoretical framework of multiple representational translation model was chosen to have been highlighted by using Table 2.3 and Figure 2.4. The example serves as an illustration, to show how the theoretical framework of multiple representational translation model could be used, with an intention to close the gap that has been identified in the theoretical framework of modes of reasoning.

Some of the important reasons that have led the theoretical framework of multiple representational translations to be chosen are summarised as follows: The multiple representational translation model can track the translation between representational registers, and depict the legitimacy of the conversion from a graph to its algebraic equation and vice versa, intending to study the psychological process involved. It is noted in Janvier (1987) that the psychological process that is involved in the translation from a graph to an equation is complex and requires interpretation and pattern discovery. Leinhardt et al. (1990) has reiterated this and suggest that the whole conversion between a graph and its equation captures the ultimate mathematical idea of reasoning in Linear Functions.

As such, the theoretical framework of multiple representational translations seems to provide legitimate descriptive standards that could be used to study the translations between representational registers. This theoretical framework also provides ways to study the direction of the translation which is important to analysing the psychological process involved as was illustrated in Figure 2.4 and Table 2.3. The discussion above was put in place with an assumption that mathematical reasoning in Linear Functions stems from multiple representational translations together with their conversions. The word ‘translation’ has been assumed to refer to conversion; it is more than an act of seeing the same function in different forms as it is defined by Kaput (1986).

In this research, translation refers to the movement from one representation register to another together with its characters / parameters without changing the meaning of the mathematical idea.

It is explained in the work of Bednarz and Balanger (1987) and Eisonberg (1992) that learners move from one representation to another successfully. However, it is argued that learners demonstrated a limited understanding of the link between the characters / parameters of the representation. In other words, learners compartmentalise the representational registers as independent entities. Thus, the word translation in this research has been defined in relation to the movement and the link between character / parameters of representation registers.

Although the theoretical framework of multiple representational models has the potential to study the movement between one representation register to another, it fails to provide descriptive standards that could be used to study the link between representational characters / parameters. This is not explicitly highlighted in the multiple representational models. In this, I saw it fit to discuss the systematic linguistic features using the theoretical framework of functional grammar analysis. This was done in explanatory texts with the intent to explore the position of learners in terms of doing Mathematics. The positioning of the relation between the author and the learner could shed light on the extent to which the textual descriptions provide links and relationships, if there are any, between characters / parameters of representational registers. Moreover, the theoretical framework of functional grammar analysis was introduced to provide descriptive standards that could help to understand what the modes of reasoning in explanations and worked out examples mean to learners.

2.6. Linguistic features in textual explanations/descriptions.

The work of Vygotsky (1962) on the social constructivist theory has highlighted the characteristics of learning as a social activity that requires active learner engagement. Some of the important findings in Vygotsky include the significance of the language of Mathematics, which has the potential to encourage social interaction (promoting generalisation and high order thinking) and the connection between thought and language (development of high order concepts). The two outlined findings are of interest to this research because they form part of mathematical reasoning that was discussed before.

Mayer (2001) has outlined how reasoning in Mathematics and problem solving can be improved if authors pay careful attention to the part of the text that provides explanations in textbooks. This suggests that the language within a text, and how it is presented is important for learners to learn the principles that the explanatory text and worked out examples intends to convey.

In the context of this research, this suggests that the language, together with its structure and form within a text, could have implications for learners' level of comprehension of the intentions embedded within an explanatory text. The research question that is also of interest in this research is to explore what the identified modes of reasoning in explanatory texts and worked out examples mean for learners to develop mathematical reasoning. In this, the linguistic structure presented in textual explanations could suggest the implication that this might have on learners.

2.6.1 Linguistic features within expository texts: theoretical framework of functional grammar analysis.

The theoretical framework of functional grammar analysis was developed by Halliday (1973) to enable the researcher to discuss the purpose and the use of language within texts. Functional grammar has the following elements amongst others:

- Ideational function
- Interpersonal function

The *ideational function* refers to the structure and logic of relationships within a text, it is also described as a category of learners' experience of the world and how they interpret it (Halliday, 1973).

The *interpersonal function* refers to the social relationship between the learners and the author (Halliday, 1973). The interpersonal function consists of three language forms (imperatives, personal pronouns, and modality). Imperatives are defined as commands or instruction to the learner on what to do and direct the attention of the learner to particular ideas that the text intents to convey (Morgan, 1998). Rotman (2006) differentiates between inclusive and exclusive imperatives. Inclusive imperatives such as explain, or prove, which normally make use of symbolic Mathematics language require the learner to engage in critical thinking (Rotman, 2006). Thus, a text that uses such imperatives suggests that the learner must be a "thinker". Exclusive imperatives such as 'write' or 'put' suggests that learners should follow instructions, they are therefore referred

to as ‘scriber’ (Rotman, 2007). In this, a learner is regarded to be a thinker when a text engages the learner to think critically, while a scriber is expected to follow instructions. Following this, Herbel-Eisenmann and Wagner (2005) advise that a text should be expected to consist of an equal mix of inclusive and exclusive imperatives because mathematicians are both scribers and thinkers.

The personal pronoun represents the learner by using words such as ‘you’ and the author by using words such as ‘I’, and equal participation of the learner and the author by using words such as ‘we’ and ‘us’ (Morgan, 1998). In other words, the personal pronoun helps to examine the relative relationship suggested in a text between the author and the learner. O’keeffe and O’donoghue (2015) allude that the presence of personal pronoun within a text indicates where the ownership and authority of the text lies.

Pimm (1987) has warned that although the pronoun ‘we’ might suggest the learner as a colleague, or a member of the Mathematics community, an additional burden is imposed on less confident learners if the pronoun ‘we’ is overused. This is also highlighted in Kamio (2001) that the overuse of ‘we’ within a text has the potential to impact negatively on learners who are not confident in Mathematics because of the expectations and responsibilities that come with being a member of the Mathematics community.

Morgan (2004) studied the presence of mathematical symbolism within a text- mathematical specialist vocabulary and imperatives. Shuard and Rothery (1984) also alluded to the role of mathematical symbols within a text. Morgan (2004) and Shuard and Rothery (1984) point out that the use of mathematical symbolism is important for the language of Mathematics. However, Godino, Font, Wilhelmi & Lurduy, (2014) warned that the excessive use of symbols in explanatory texts have proven to impose difficulties for learners because they find them to be difficult to interpret. Österholm’s (2006) study also revealed that learners need a special comprehension process to understand textual explanations that are symbol heavy. Thus, in this research, I assume that learners find it easier to understand the intentions of what a Mathematics text conveys when it does not use intensive symbolism.

The use of interpersonal pronouns and imperatives in expository texts is demonstrated in Figure 2. 5.

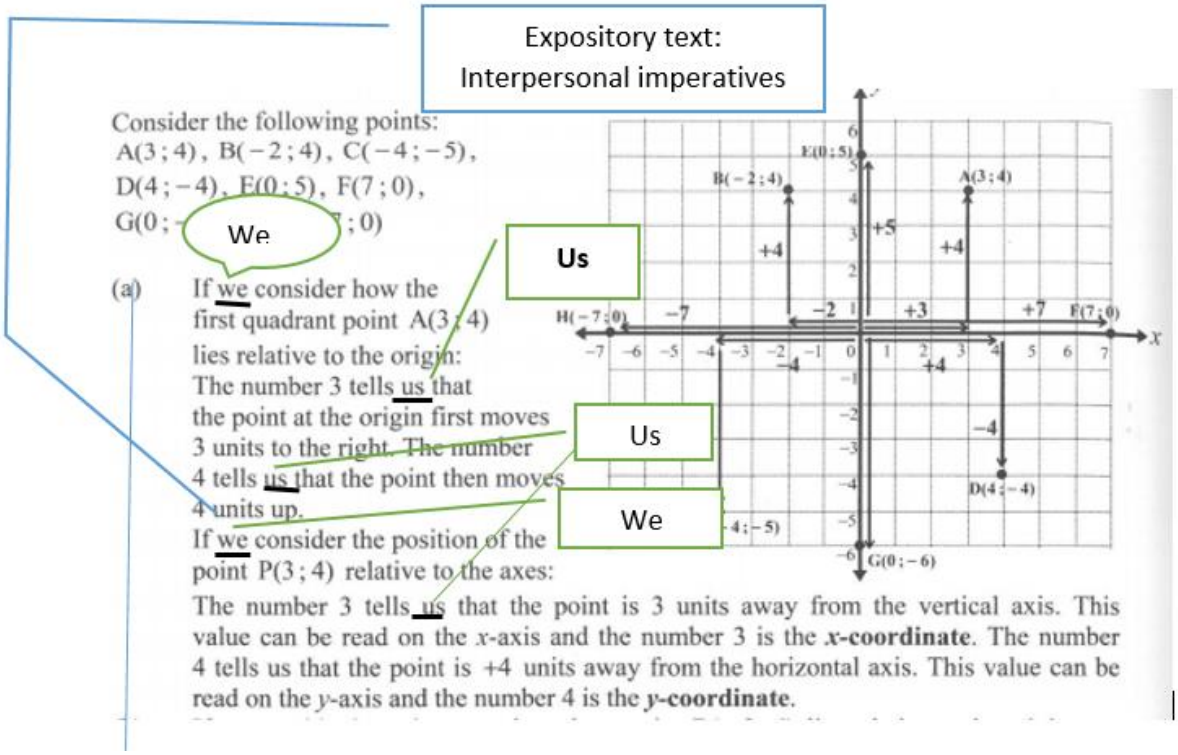


Figure 2.5: Interpersonal pronouns and imperatives (Basson et al., 2014)

The highlighted words in Figure 2.5 are examples of interpersonal pronoun that could be used to study the relationship between the author and learners. The constant use of 'us' and 'we' in the expository text suggest equal participation between the author and learners. This suggests that learners are involved (active member) in the construction of the knowledge needed to plot coordinates. However, the expository text uses interpersonal imperatives because the text directs and or instructs the learner on what to do, and where to pay attention. Although learners are regarded as participants as suggested using interpersonal pronouns 'we" and "us", this positions the instruction of learners as scribers rather than thinkers. This might be the case since the example in Figure 2.5 is part of the introductory topic of linear functions. Learners, in this regard, are assumed not to have previous knowledge of how to plot coordinates (scribers). As such, they are enculturated into principles needed to construct the knowledge to plot coordinates, by assuming an equal position in the process of doing Mathematics with the author.

It must be noted in Figure 4.5 that the expository text used textual descriptions that are drawn from the everyday language to instruct learners, rather than using a formal mathematical language that uses symbols. The everyday language might be intended to appeal to what learners are familiar

with (psychological aspect) and use this as a directive to steer learners to the end goal of gaining access to the principles that govern the idea of plotting graphs on a Cartesian Plane (logical aspect) (Osterholm, 2006).

In this research, the use of interpersonal pronouns and imperatives will be used to examine the linguistic features of the text in expository sections because it helps to study the relationship between the author and learner in doing Mathematics.

The ideational function will not be used because this research aims not to study what the learner writes, but rather, how the learner as a reader might be affected by what he / she reads. Thus, the learners' mental activities cannot be studied since a learner is a reader. In this, the likely impact of how what they read affects their access to the intentions of the explanatory text is of interest in this research.

As such, the components of interpersonal pronouns and imperatives are elements that are adopted from Holliday's theoretical framework of functional grammar analysis. These components are chosen because of their ability to provide descriptive standards of the form, structure, and function of a text. The interpersonal pronouns allude to the ideas in Vygotsky, which regard learning as a social constructivist activity. The use of pronouns has proven to shed a light on the relationship that a text implies between the author and the learner. This could provide evidence of the level of learner participation suggested in a text. While the interpersonal imperative uses the idea of learners being a scriber and a thinker. This could be used the study the level of the affordance of critical thinking suggested in a text. The interpersonal pronouns and imperatives impact directly on the analysis of a text. The aim is to examine how the modes of reasoning in the explanatory text mean to a learner as a reader. This theoretical framework will be referred to as the interpersonal imperative pronoun since it consists of components of the theoretical framework of functional grammar analysis.

2.7. Amalgamated theoretical framework.

The theoretical framework that was used in this research emanates from the amalgamation of Stacey and Vincent (2009) theoretical framework of modes of reasoning, Janvior's (1987) multiple representational translation models, and some components of Halliday's (1973) functional

grammar analysis. This was developed with the intent to construct a theoretical framework that is specific to the mathematical domain of this research.

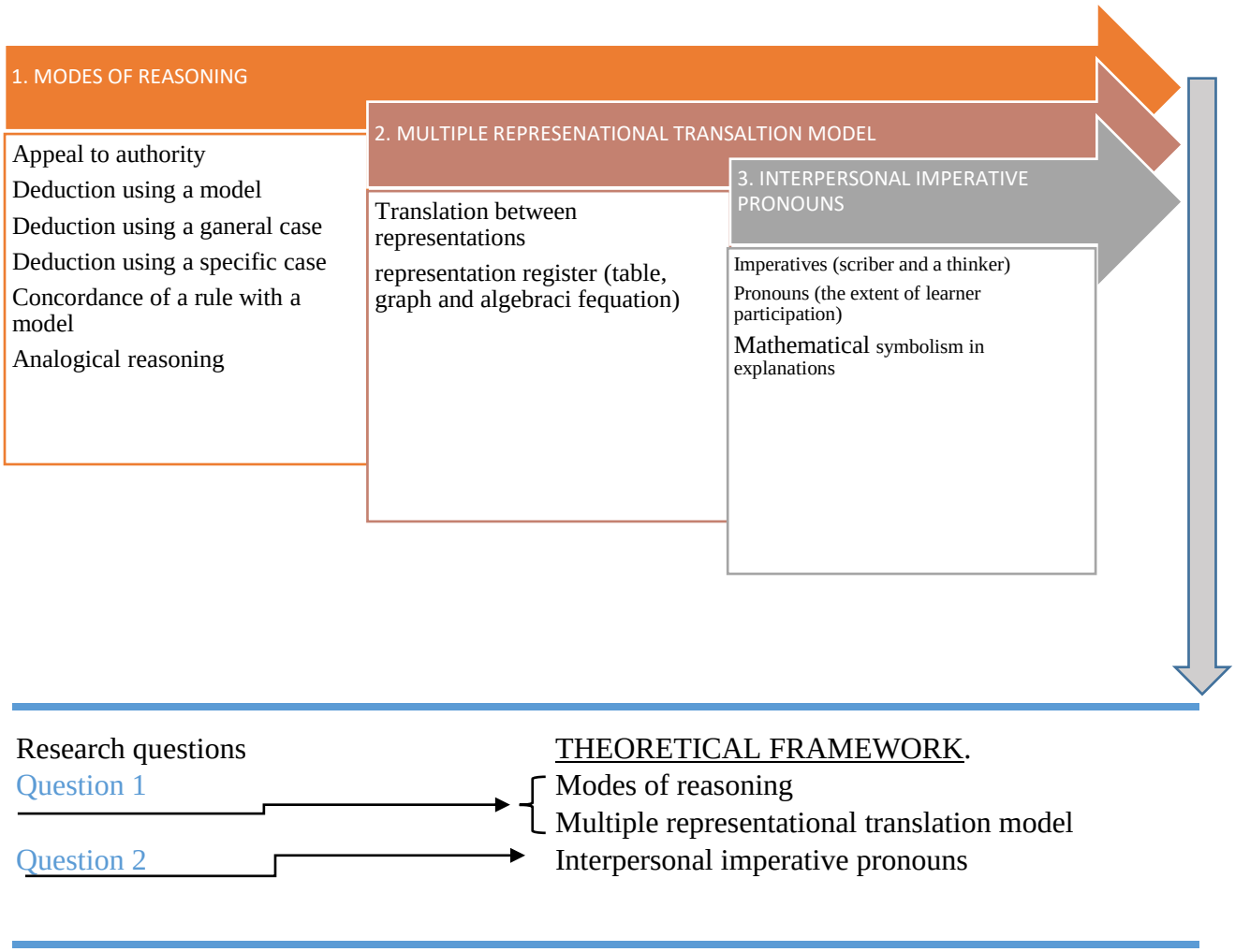


Figure 2.6: Amalgamation of theoretical frameworks used to develop a theoretical framework that is specific to the research.

Question 1: What modes of reasoning are implied/ presented in expository texts and worked out examples in linear functions in two Grade 9 Mathematics textbooks?

Question 2: What do these modes of reasoning in each textbook make available as means to develop the readers to reason mathematically?

This research integrated the work of the three theoretical frameworks of the above-mentioned scholars sequentially. The theoretical framework of this research comprises of two stages. The first stage consists of two elements that are used to study the modes of reasoning and the multiple representational translations. The second stage consists of components of the functional grammar

analysis theoretical framework, which is referred to as the interpersonal imperative pronoun for the purposes of this research. The second stage is intended to study the textual structure and form of explanation to explore what they make available for learners to develop mathematical reasoning. Figure 2.6 summarises the theoretical framework discussed above to develop a theoretical framework that is specific to this research.

2.8. Conclusion.

This chapter discussed literature on mathematical knowledge and the role of mathematical reasoning (multiple representations) in the teaching and learning of Linear Functions. Mathematical reasoning is defined in terms of its structural aspect and process aspect. Lastly, the three theoretical frameworks are also discussed and used to develop a theoretical framework that is specific to this research. Chapter three will discuss the research design and the methods used to collect and analyse data.

CHAPTER 3: RESEARCH DESIGN AND METHODOLOGY

3.1. Introduction

In this chapter, I discuss the research design and the method used to collect and analyse the data. The data for this research was collected from two Grade 9 Mathematics textbooks that are CAPS aligned. Linear Functions is the topic that is being interrogated. The research design and the methodology adopted in this research are guided by the research questions in Chapter 1, as well as the theoretical frameworks that are discussed in Chapter 2.

To reiterate, the purpose of this research is to explore the modes of reasoning implied / presented in explanatory texts and worked out examples in two Grade 9 textbooks on the topic of Linear Graphs. These textbooks are analysed in the next chapter (chapter 4). An example of how these textbooks have been analysed is given towards the end of this chapter to illustrate the methodology used.

This research follows the social constructivist epistemological assumptions. Piaget (1946) cited in Glasserfield (1998) asserts that "knowledge does not attempt to produce a copy of reality, instead, serves the purposes of adaptation" (p. 2). In the context of this research, knowledge of Mathematics is assumed to be viable when learners can reason mathematically. Rather than producing a series of steps that are memorised with no justification behind them. Knowledge is constructed, with the teacher and the learner being active participants, Piaget (1946) cited in Selden & Selden (1996). The teacher is assumed to be a facilitator of knowledge, rather than the authority of it. The ontological assumptions made in this research consider reasoning to be presented in different modes in expository texts.

3.2. Research design and methods used to collect and analyses data.

This research has adopted qualitative research methods to answer its research questions. The purpose of a qualitative research method is for the researcher to understand a social situation from the perspective of the participants (McMillan & Schumacher, 2010). This is what this research intends to do, but in the context of two Mathematics textbooks.

In following the qualitative research approach, the following aspects are important. They form the structural base of the research design, procedures and methods used to collect and analyse data in this research study:

- The setting in which the research is based.
- Description of the sampled data.
- Procedure, and the method used to organise data,
- The role of the researcher and issues pertaining to the limitations, validity, reliability, and ethical considerations.

Given that the nature of this research seeks to answer the 'what' and 'how' of textbook content, I have approached this research as a case study inquiry. Yin (2009) suggests that a case study inquiry should take the following aspects into cognizance:

- Situations in which there are many variables of interest.
- Relies on multiple sources of evidence, with data that needs to converge in a triangulated manner.
- Develops from previous theoretical frameworks to guide data collection and analyses.

Yin (2009) also points out that there are two types of case study designs, namely, single case design and multiple case designs. The single case study design studies a single case, whereas a multiple case study design studies multiple case. Within each of the two case study methods, there could be a unit of analysis or multiple units of analyses (Yin, 2009). These categorisations are important in the context of this research because they require one to be explicit about the units of analysis. In so doing, boundaries that are of interest are defined and those that are not important are excluded (see Table 3.1 below). Hatch (2002) points out that defining boundaries and or specifying the units of analysis are important key decision points in a case study design.

The current research study is hence a case study if the argument made by Yin (2009) is considered, which, amongst other aspects, posits that a case study uses theoretical frameworks to guide the collection and analysis of data. In this study, as indicated before, previous theoretical frameworks are used for data analysis. The units of analysis are also defined and referred to as worked examples and explanatory texts.

Since this research study seeks to analyse textbook content, it is therefore referred to as a textbook content analysis case study. Textbook content analysis is often used to examine text with the aim to make inferences from features of the content about the context in which the text is written (Stacey & Vincent, 2009). In the context of this research, text is examined to determine the modes of reasoning implied in explanatory texts and worked out examples. It is furthermore examined in the context of how these modes of reasoning make available as means to develop the learner to reason mathematically in Linear Functions.

The positioning of text that is of interest in this research is expository texts (worked out examples and explanatory texts). Table 3.1 shows a textbook extract and how the expository texts are partitioned into units of analysis. In so doing, the boundaries that are of research interest are defined in this instance. The selection of the units of analyses is attached to conditions of possibility that could help to answer the research questions at hand (Janks, 1997). This serves to elaborate on the reason why the examples from the Mathematics textbooks were partitioned into recognisable units that could be sequenced and analysed in an ordered manner.

Table 3. 1: Partition of explanatory texts into units of analysis (Basson et al., 2014).

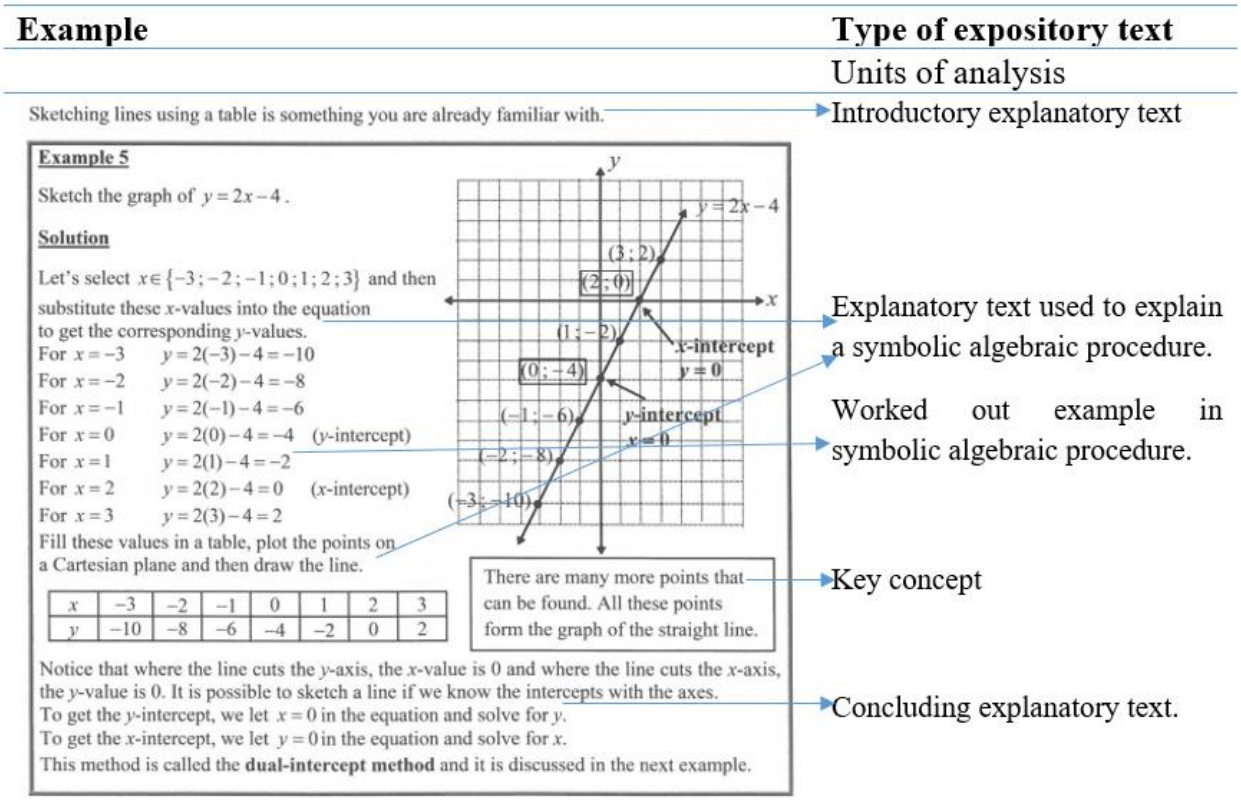


Table 3.1 shows the structure and sequencing of expository text that is observed from one textbook, but similar across both textbooks under analysis. Expository texts are presented by starting with introductory texts announcing the purpose of the text. This is followed by an instruction which says: Sketch the graph of $y = 2x - 4$, and then a procedural illustration which restricts the domain to a set of numbers beginning with negative 3 to positive 3. These are integers, however, the line that is drawn in the graphical representation is a continuous straight line. The assumption that is made is that the set of elements chosen is in fact a set of all real numbers between -3 and 3, this serves that only integers are used for illustrative purposes.

The restrictions mentioned above are explained in the key concept, which explains the taken-for-granted reasoning that is illustrated using the domain of integers. The 'key concept / word' unit is demarcated with an assumption that the reader would not miss it. The demarcation serves to highlight the presence of the unit of a key concept / word.

Concluding explanatory texts are found at the bottom of the example given. They are used to summarise the main concepts and make generalisations when it is appropriate. Representations are given in the form of a table, equation, and graphs. Representations are used as part of the questions in expository texts, and, or as demonstration tools that are used to explain a concept. The expository texts that are found in the two textbooks have a similar structure and the sequencing of explanatory texts and worked out examples is also observed to resemble a similar structure.

3.2.1. Units of analysis: Expository texts (Explanatory texts and worked out examples)

Explanatory texts and worked out examples are the units that are analysed. They are collectively referred to as expository texts as reiterated earlier. Expository texts are sections found in textbooks where procedures and concepts are introduced and demonstrated (Watson, 2006). Demonstration tools are designed to provide guidance and trajectories that can be used to solve problems. Worked out examples normally provide steps that illustrate and explain how a mathematical object operates (Ronda & Adler, 2016). Normally, learners are expected to interpret the author's solution by following the steps that draw on previously known mathematical facts (Ronda & Adler, 2016).

3.2.2. Procedure and organisation of data.

This research is a qualitative study, data is therefore analysed using qualitative methods. The typological analysis method is used to organise data into recognisable units. Typological analysis refers to data analysis that is conducted first by dividing the data into a set of categories that are based on predetermined typologies (Hatch, 2002). The "typologies are generated from theory, common-sense and objectives of the research" (Hatch, 2002. p. 243). Following Hatch's (2002) definition of typological analysis, data is first divided into sub-topics which are then divided into a set of units in this research. The topic, Linear Functions, is divided into the following sub-topics in Table 3. 2:

Table 3.2: Structure and sequence of content knowledge in the Mind Action Series.

Sub-topic	**Number of examples**
Revision of plotting points in the Cartesian plane.	1
Sketching linear graphs using the table method and dual intercept method	5
Sketching linear graphs using the point intercept method	1
Increasing and decreasing linear graphs	1
Gradient of a linear graph	5
Horizontal and vertical lines	3
Finding the equation of linear graphs given the points on the graph	3
Total number of activities	18

The topic, Linear Functions, consists of 18 examples in the *Mind Action Series* textbook (see Table 3. 2). In the examples analysed, all 18 of them have expository sections that consist of worked out examples and explanatory texts.

Table 3.3: Structure and sequence of content knowledge in the Platinum textbook.

Sub-topic	**Number of examples**
Investigate linear graphs	1
Intercepts	1
Gradient	2
Standard form of a linear line $y = mx + c$	1
Draw linear graphs	1
Total	6

From the sub-topics shown in the two Tables, 3.2 and 3.3, from two Mathematics textbooks used in this research, the sub-topic Gradient, for example, is firstly identified, and then the expository texts found in the examples that make the topic, Gradient, are divided into units of analysis which are illustrated in Table 3.1 above. The explanatory texts are divided into the following units: Introductory text, key words / concepts, concluding explanatory text, explanatory text used to explain an algebraic symbolic procedure and worked out examples which are distinctively identified by the use of algebraic symbolic notation. These units are identified to make it easier for me to know the type of expository text referred to in a particular example.

3.2.3. Selection of the topic.

This research is focused on linear graphs in Grade 9 for several reasons. Firstly, Linear Functions are introduced in Senior Phase (Grades 7-9) in the CAPS Mathematics document (CAPS, 2012). This topic is not encountered in the earlier years of learning Mathematics. Senior Phase is, therefore, a critical stage of learning linear graphs because it is in this phase where a foundation for learning Linear Functions is laid.

Secondly, Linear Functions describes the relationship between variables that can be sketched in a straight line and provides learners with their first experience of interpreting and identifying the relationship between the two variables. One variable is dependent while the other is independent (Mushipe & Ogbonnaya, 2019). It can be regarded as a topic that provides a bridge between reasoning from the concrete level, which is normally learned in earlier years of schooling

(Foundation Phase and Intermediate Phase), to abstract reasoning in Senior Phase and higher grades (Leinhardt et al., 1990). Algebra forms an important part of linear graphs, and it is known as an important field in Mathematics that introduces logical and abstract reasoning in learners (Ayber, 2017).

Finally, linear graphs are characterized by various branches of other topics in Mathematics, such as patterns, algebra, and graphs. Linear graphs can also assume different representations which can be in the form of a table, graph, and diagram (sometimes called a spider diagram), as well as in the form of algebraic equations. Kaf (2007) holds that algebra is an important skill for reasoning in Mathematics because it allows learners to use and explain the meaning of different representations to understand variables. It also provides a platform for learners to reflect on their mental activities to represent the link between algebraic relations by assigning meaning to variables (Kaf, 2007). Algebra in linear graphs is important because of its ability to attach meaning to symbolic representations; this means thought can be revealed using multiple representations, describing concrete and abstract concepts in algebraic relations which could result in the ability to reason mathematically (Kaya & Kesan, 2014 cited in Ayber, 2017).

It is for these reasons that I draw my attention on linear graphs because they are found on the integration of various areas in Mathematics. Most importantly, the conception of linear graphs in Grade 9 level (an exit grade in the Senior Phase) could give a sense of whether learners are given a solid foundation to cope with graphs that normally call for intensive abstract mathematical reasoning in Grades 10-12. This is not the concern of this research. However, this research will discuss this in the discussion section to signify the recommendation for future research studies to consider.

3.2.4 Theoretical framework and coding schemes used in this research.

In this research, data is collected on a specific section of the textbook called expository texts which are divided into units as mentioned before. This is done to categorise the extraction of data from expository texts into manageable chunks that can be analysed clearly. The main instruments that are used to analyse and categorise the modes of reasoning in expository texts are summarised in Figure 3.1

Figure 3.1 shows the descriptive rules and codes that are used during the analysis of data. It further shows the component of interpersonal imperatives adopted from Halliday's theoretical framework of functional grammar analysis. The example in Figure 3.6 is given to illustrate how the theoretical framework in Figure 3. 1 is used to analyse data. The category of mode of reasoning in Figure 3.2 serves to record the presented mode of reasoning analysed according to Stacey and Vincent's (2009) theoretical framework of modes of reasoning. The category of multiple representational translation models is used to study the relationship between representations while the component of interpersonal imperative pronouns is intended to study the meaning of the identified modes of reasoning to learners.

Theoretical framework of modes of reasoning

Modes of reasoning	Codes
Appeal to authority	ATA
Deduction using a specific case	DUSC
Deduction using a general case	DUGC
Deduction using a model	DUM
Experimental demonstration	ED
Concordance of a rule with a model	CRM
Analogical reasoning	AR

Interpersonal Imperative pronouns

- Imperatives: Inclusive (learner is a thinker) /exclusive (learner is a scriber)
- Pronouns: the use of 'I'; 'you' 'we' 'us'
- Symbolic use in expository texts

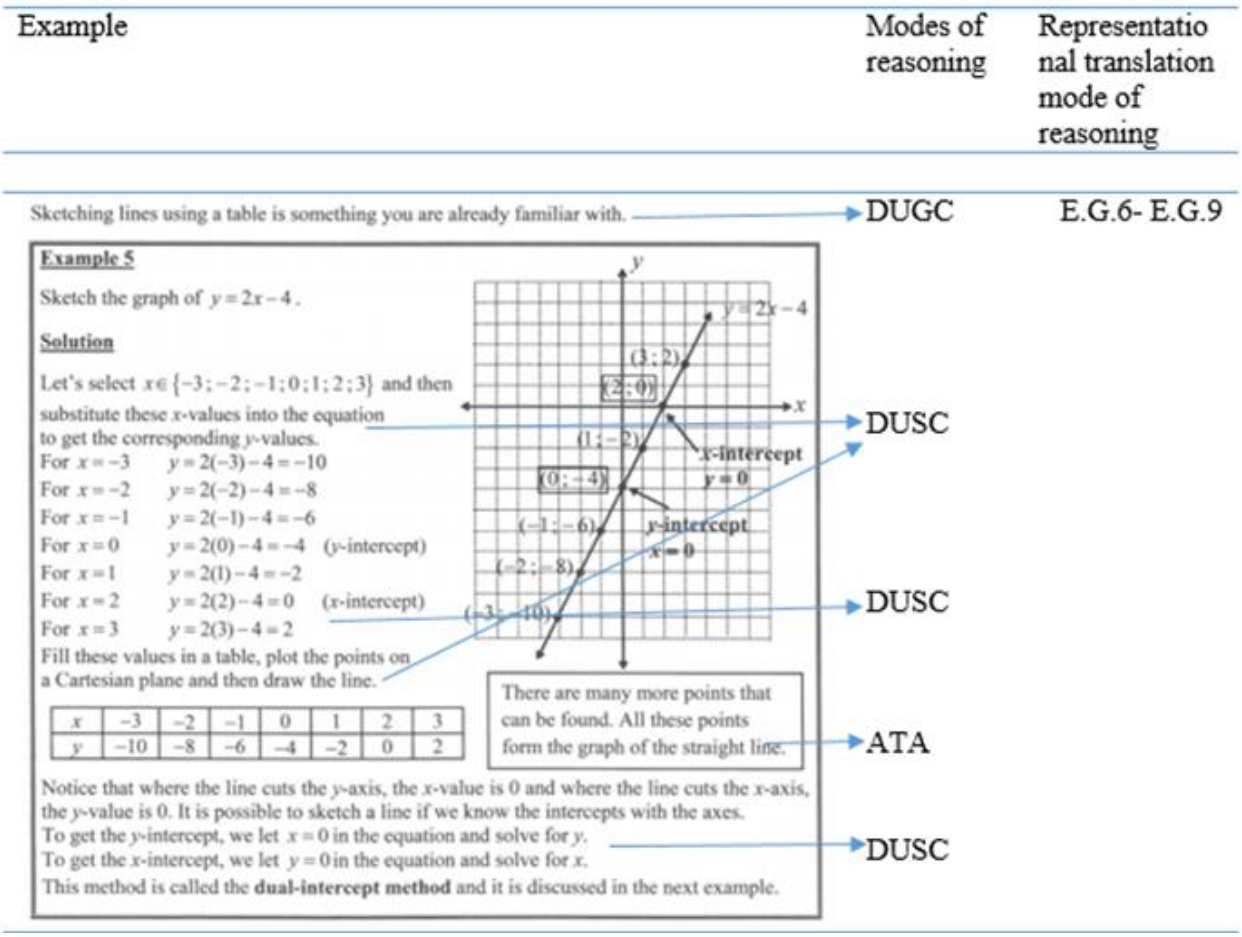
Theoretical framework of Multiple representational translation model

From-To	Descriptions in words	Tables	Graphs	Formulas/equation
Description in words	— EGO	Measuring (EG4)	Modeling (EG7)	Analytical Modeling (EG10)
Tables	Reading (EG1)	—	Plotting (EG8)	Fitting (EG11)
Graphs	Interpretation (EG2)	Reading off (EG5)	—	Linear (EG12)
Formulas/equation	Parameter Recognition (EG3)	Computing (EG6)	Sketching (EG9)	Fitting (EG13)

Figure 3.1: Theoretical framework of the research, and descriptive codes.

3.2.5 Using the theoretical framework in Figure 3.1 to analyse data.

The analysis and categorisation of modes of reasoning in Figure 3.2 was conducted in three ways following the theoretical framework in Figure 3.1. The first stage analyses and categorises modes of reasoning presented in the example. Secondly, the multiple representational translation model is used. Thirdly, the interpersonal imperative pronouns are used.



x	−3	−2	−1	0	1	2	3
y	−10	−8	−6	−4	−2	0	2

Figure 3.2: Textbook example analysed using the theoretical framework in Figure 3.1 (Basson et al., 2014).

The following modes of reasoning are identified in Figure 3.2:

- Deduction using a specific case (DUSC)
- Deduction using a general case (DUGC)
- Deduction using a model (DUM)

DUGC is identified in the introductory unit section. The unit assumes that learners already know how to sketch straight line graphs. This suggests the knowledge of previously learned facts about sketching lines. The previously learned knowledge and facts are not specific, but general since a

generic example was not given. The explanatory text is therefore considered to be deductive since it refers learners to previous known facts and is considered a general case since it does not use a generic example.

DUSC is identified based on the use of specific values such as the domain of integers from *-3* to *3* which are substituted in the formula and computed using a series of algebraic procedures. It is specific because it uses a generic example and it is considered to be deductive since it uses a chain of previously known facts.

DUM is identified because the graph is assumed to be a model that illustrates the *y* and *x* intercepts. The graph (model) holds the same mathematical validity as the procedure used to find intercepts. There is a direct correspondence between the model and the associated Mathematics. The model can be used to predict the results of calculations (Stacey and Vincent, 2009).

The following multiple representational translations are identified in Figure 3.1:

A translation from computations using a formula (EG6) to a table (EG1), and then to sketching a graph (EG9) was identified. Note that the order of the codes (EG6-EG1-EG9) represents the order of translation. It is already established that the order of translations between representations is important and is one of the critical elements of mathematical reasoning in linear graphs.

Interpersonal imperative pronouns

- Imperatives: Explanatory texts are written to direct and instruct learners on how the worked-out examples operate. The language used is elaborate and uses a normal language with few symbols. This means learners are regarded to be 'scribers' since they are instructed on what to do. This follows Morgan's definition of imperatives; they command or instruct the learner on what to do and direct the attention of the learner to particular ideas that the text intends to convey (Morgan, 1998). However, the instructional texts in Figure 3.7 establish learners to be scribers rather than thinkers.
- Pronouns: As stated earlier, the presence of pronouns in a text suggests the ownership and authority and of the text (O'keeffe & O'donoghue, 2015). It suggests the extent of learners' relative engagement and role in doing Mathematics. The example in Figure 3.6 does not directly use pronouns but the function of text in explanations is written from an authoritative point of view. This is because of directions and instructions are suggestions

to the learner in explanatory text. There is no evident of explicit pronouns (for instance; 'you' 'we') that suggest learners to be part of the process of constructing knowledge.

In summary, the examples in Figure 3.2 represent modes of reasoning that are dominantly deductive. This suggests a dominant use of previously known facts that includes a chain of procedures and definitions. Deductive reasoning is regarded as a legitimate way of reasoning in Mathematics. However, Godino, Batanero, Cañadas and Contreras (2015) argue that it is often difficult for learners to understand or follow explanatory texts that are accompanied by heavy mathematical symbols. As such, the function of text, and its structure in explanatory texts suggests leaners to be scribers. The explanatory text does not highlight the role of learners in doing Mathematics. However, the author writes from a position of authority of the mathematics knowledge. This suggests that learners' engagement, being part of the construction of mathematical knowledge, is overlooked in this example. The representational translation identified in Figure 3.2 suggests a procedural way of learning Linear Functions (Janvior, 1987).

3.3 Sampling.

Data has been collected from two Grade 9 Mathematics textbooks that are CAPS compliant. The *Platinum Mathematics* Textbook is widely used in public schools, while the *Action Mind Series* textbook is prescribed by the head of the mathematics department in the school that I work in. This textbook is much more expensive than the *Platinum Mathematics* Textbook. As a result, only a few learners can access the *Mind Action Series* Textbook, meanwhile *Platinum Mathematics* Textbook is given to all learners in the school that I work in, and in other neighboring public schools that fall under the Ekurhuleni North district.

3.4 Validity.

A case study is a type of a qualitative approach which is, like any other type of a qualitative research, threatened by instances of invalid conclusion because qualitative data can be interpreted wrongly due to the researcher's conception that might impose bias, and or wrong interpretation of the theory guiding the interpretation and analysis of qualitative data (Maxwell, 1992. p. 89). He further defines validity as the correctness or credibility of a description, conclusion, explanation or interpretation. He asserts that validity does not imply the existence of any objective truth to which an account can be compared (Maxwell, 1992. p. 87).

To rule out the threats of validity, Maxwell (1992) suggests the use of feedback. In this research, a theoretical framework is used and the validity of identifying and categorising the modes of reasoning is done by using the strategy of inter-rater reliability and feedback. Thereafter, consultation with my supervisors for feedback and my classmates (Masters Students) is done, and their feedback is considered to guide the process of categorising and identifying the modes of reasoning.

The methods and procedures applied in this research study are important to the process of ruling out the validity threats and increasing the credibility of the analysis and conclusions made from the data. As already established, it is important to use the topology strategy, with the aim to compartmentalise the expository texts into chunks of units that can be identified and analysed individually.

3.5 Ethical consideration.

This is a textbook content analysis research study. Therefore, it does not impose harm to the society. Textbooks are documents that are readily available to the public domain. This means that they can be analysed and interrogated. The ethical clearance letter is attached in the appendices, as well as approval letters from the authors of the two textbooks.

3.6 Conclusion.

The theoretical framework used in this research is developed from three theoretical frameworks. The units of analysis and why they were chosen is explained. This research is concluded to adopt qualitative methods to analyse data. It is also discussed to be a textbook content analysis case study. An example is given to illustrate how the unit sections are partitioned and then analysed.

The sampling methods are discussed as well as the topic and the grade in which the research is based on. The issues of ethical considerations and validity are discussed as well. The following chapter focuses on data analysis. The two textbooks are analysed separately, and the categorised modes of reasoning are therefore summarised quantitatively.

CHAPTER 4: DATA ANALYSIS AND DISCUSSION

4.1. Introduction.

This chapter focuses on the analysis of Linear Functions in two Mathematics textbooks (*Mind Action Series* and *Platinum*). The analysis and discussion are sequenced and structured in the following sections:

1. Broad features: Description of features found in the chapter, Linear Functions, in the two textbooks.
2. Macro-section analysis: General analysis of all examples contained in the chapter, Linear Functions, in the two textbooks.
3. Micro-section analysis: Analysis of sampled examples with sufficient unit sections that could be studied to identify the modes of reasoning presented, and what opportunities do the identified modes of reasoning present to the learner / reader to develop mathematical reasoning skills.

In the first section, 'broad features', I examine the features that characterise the chapter, Linear Functions. I look at the number of examples presented in each textbook and the sequence of sub-topics. In the second section 'Macro-section analysis", I analyse all examples contained in the chapter, Linear Functions, focusing on expository texts. This section is called the Macro-section analysis and I used the theoretical framework called modes of reasoning (Stacey and Vincent, 2009) and multiple representational translations (Janvier, 1987) to recognise, identify and categorise the modes of reasoning presented. The Macro-section analysis is set out to partially answer the first research question: *What modes of reasoning are implied / presented in explanatory texts and worked out examples in Linear Functions in two Grade 9 Mathematics textbooks?*

Lastly, the Micro-section analysis is set out to account for the shortfalls that are encountered in the Macro-section analysis during the process of analysis. For instance, it is clear, during the analysis, that not all examples presented in each textbook have all categories of unit sections that can be analysed. As such, these kinds of examples are not included in the sampled examples contained in the Micro-section analysis. The Micro-section analysis is also put forth with the intention to partially answer the second research question: *What do these modes of reasoning in each textbook make available as means to develop learners to reason mathematically?* The Micro-section is more detailed and explores the multiple explanations found in expository texts.

The unit sections within the individual explanations in expository texts are analysed using the theoretical frameworks to identify and categorise the modes of reasoning presented. The literature on reasoning in Mathematics, multiple representational translations, linguistics features, and the role of textual descriptions in expository texts is adopted as background literature to elaborate on the findings put forth using the two theoretical frameworks to answer the two research sub-questions.

A distinguishing aspect of studying Linear Functions is its integration of different representations which include graphical, tabular, and symbolic descriptions of the concept, which allow for it to be interpreted, discussed, and communicated (Tripathi, 2008). As such, the theoretical framework of multiple representational translations (see Figure 4.1, numbers 2 and 3), is used as a thread that intertwines the theoretical framework of modes of reasoning to define and recognise the modes of reasoning in expository sections. The sequence and structure of the analysis is summarized in figure 4.1.

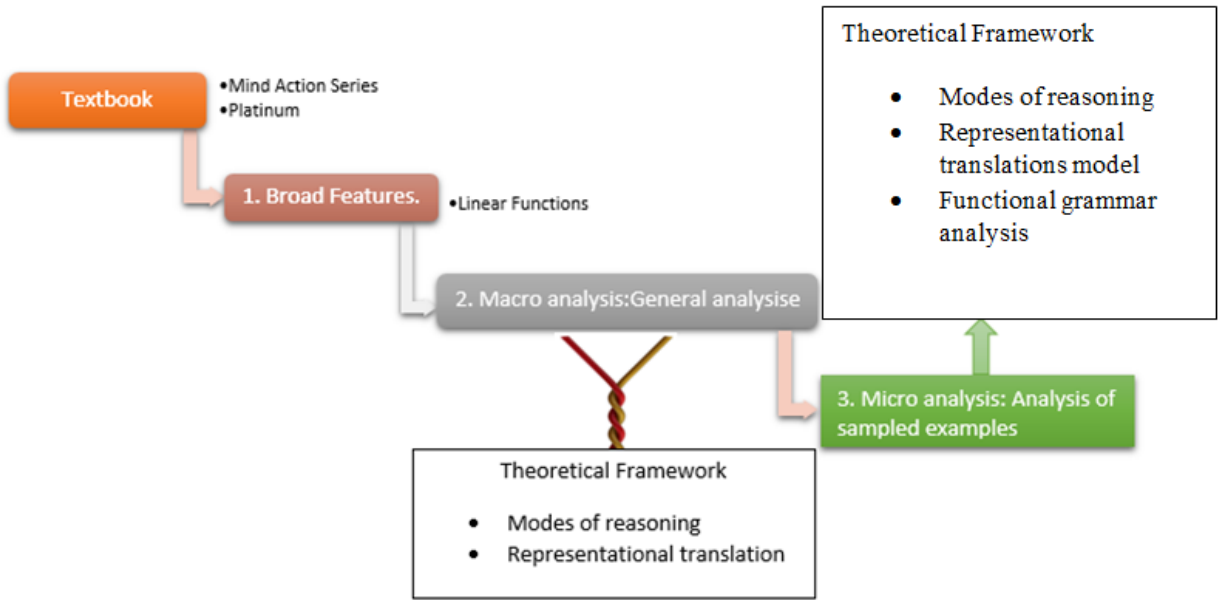


Figure 4.1: Sequence and structure of the analysis.

Figure 4.1 illustrates the sequence of the analysis from the description of features of the Linear Functions of the two textbooks to the Macro-section, and to the Micro-section.

4.2. Analysis and discussion of broad features of the chapter on linear functions in the two textbooks.

In both textbooks, expository texts usually precede the exercises associated with them. In occasional instances, expository texts are encountered after the exercises. The expository texts which follow the exercises are mostly given to provide an analogy or generalisation of the concept the exercise is focused on. For instance, an expository text in the *Mind Action Series* that is provided after an exercise is found in examples 5, 7 and 12, see Appendix 1. In the *Platinum* textbook, this is encountered in examples 2, 3 and 6, see Appendix 2.

Linear Functions in the two textbooks is broadly structured in the following manner:

1. An introduction section which introduces the reader to the contents of the chapter.
2. A list of intended outcomes for the reader.
3. The chapter is then segmented into several sections. Each section in the chapter consists of examples, sometimes accompanied by expository texts, intended to introduce learners to new mathematical ideas.
4. Each section has exercises intended for the practice of mathematical concepts.
5. The chapter, in both textbooks has enrichment sections where high order questions are asked, and additional explanations are offered.

The sections that consist of examples intended to introduce the reader to new mathematical concepts are the concern of this research. The enrichment sections are considered only if they provide explanations instead of exercises. The two textbooks use a different approach to explicate the topic, Linear Functions, by using sub-topics with different terminologies. The number of examples used to explain Linear Functions is different. The *Mind Action Series* has 18 examples while the *Platinum* textbook has 6 examples (see Table 4.1).

Table 4.1: Sub-topics and number of examples in the MAS and Platinum textbook.

Mind Action Series mathematics textbook	
Sub-topics	**Examples**
Sketch linear graphs using the table method and dual intercept method	5
Sketching linear graphs using the point intercept method.	1
Increasing and decreasing graph	1
Gradient of a linear graph	5
Horizontal and vertical lines	3
Finding the equation of linear graphs given the points on the Graph	2
Total number of examples	**18**

Platinum mathematics textbook	
Sub-topics	**Examples**
Investigate linear graphs	1
Intercepts	1
Gradient	2
Standard form of a linear Graph ($y = mx + c$)	1
Drawing linear graphs	1
Total number of examples	**6**

Although the total number of examples found in each sub-topic is different, the sequencing of knowledge is the same. The concept of Gradient in the *Mind Action Series*, for instance, is divided into two sub-topics (increasing and decreasing graphs, and Gradient of a line) which has 7 examples (see Table 4.1). In the *Platinum* textbook, the concept of Gradient has 2 examples (see Table 4.1). The sub-topic of 'finding the equation of a linear graph is not encountered in the *Platinum* textbook. This distinction is not intended to foreground the differences between the two textbooks, but rather to describe the preferred arrangement of knowledge that the two textbooks used. The *Mind Action Series* used an example that explained the concept of increasing and decreasing graph prior to examples that explained the concept of Gradient. Meanwhile, in the *Platinum* textbook, the sub-topic of Gradient dealt with the idea of increasing and decreasing graphs together with the concept of Gradient simultaneously. This could be one of the reasons why these two textbooks have a different number of examples.

4.3. Macro-section analysis: General analysis of all expository texts contained in Linear Functions in the textbooks.

Table 4.2 is used to record the identified modes of reasoning presented in expository texts across the chapter, Linear Functions, in the two textbooks. The criteria for recognizing each mode of reasoning is established. The judgment standard that I used to categorise the adequacy of explanations in expository texts is informed using the two theoretical frameworks (modes of reasoning and multiple representational translations). Other aspects such as mathematical correctness and completeness of explanations are not examined and or judged.

In order to get the overall modes of reasoning presented in expository texts, the list of the identified modes of reasoning in Table 4.2 is quantified, and the proportion of the modes of reasoning that are deductive is calculated. The main body of this section explains the identified modes of reasoning in detail, thereby defining them. In so doing, I will be providing evidence for inferences relating to the criteria used to recognise and categorise the identified modes of reasoning.

Table 4.2: Identified modes of reasoning and representational translations across the two textbooks.

***Mind Action Series Textbook:* Identified modes of reasoning**		
Modes of reasoning	**Codes**	**Representational Translation.**
Appeal to Authority	ATA	**<u>Indirect translations</u>** From descriptions in words **(EG0)** to modelling on a graph **(EG10)** and sketching a graph **(EG7)**
Deduction Using a Specific Case	DUSC	From reading from a table **(EG1)** to computations **(EG6)** and fitting values on the linear equation **(EG13).**
Deduction Using a General Case	DUGC	From interpreting a graph **(EG2)** to recognizing parameters of an equation **(EG3),** to determining the equation of a linear graph. **(EG12)**

Deduction Using a Model	DUM	From a sketched graph **(EG9)** to descriptions in words **(EG0)** and to recognizing parameters of a linear equation **(EG3).**
Concordance of a Rule with a Model.	CRM	**Direct translations** From descriptions in words **(EG0)** to modelling on a linear graph **(EG).** From interpreting a graph **(EG2)** to determining a linear equation **(EG12).**
***Platinum* textbook: Identified modes of reasoning.**		
Modes of reasoning	**Codes**	**Representational Translation**
Appeal to Authority	ATA	**Indirect translations** From reading a table **(EG1)** to fitting on the formula (EG110, to plotting a graph **(EG8)**
Deduction Using a Specific Case	DUSC	
Deduction Using a General Case	DUGC	**Direct translation** From interpretation of a sketched graph **(EG2)** to formulating a linear equation **(EG12)**
Deduction Using a model	DUM	
Concordance of a Rule with a Model.	CRM	From formula/equation **(EG3)** to sketching a graph **(EG9)**

Table 4.2 shows the modes of reasoning presented in expository texts of all examples found in Linear Functions in the *Mind Action Series* and *Platinum* textbooks. Examples are divided into unit sections, and Table 3. 4 and 3. 5 in chapter 3 are used to recognise the modes of reasoning presented in expository texts. The discussion in this section is general and explores all examples in Linear Functions. It is observed that not all examples have expository texts that could be used

to study the modes of reasoning. As a result, it is necessary to sample examples with sufficient expository text as unit sections that could be used to study the modes of reasoning. Thus, the micro-section analysis has been put forth with the intention to analyse elaborately the sampled examples with sufficient unit sections of expository texts. The importance of sampling examples with appropriate unit sections that could be studied has been discussed before in Fairclough (1989). It is pointed out that it is important for the researcher to pay attention to the important parts of the unit sections that could be recognised in order to understand the conditions of possibility that could help to answer the research questions at hand (Jank, 1997).

4.3.1 Explanation of the proportion of the identified modes of reasoning in the *Mind Action Series* and *Platinum* textbooks.

Similar modes of reasoning are identified from both textbooks (see Table 4.1). However, the proportion of the identified modes of reasoning presented in the expository texts is different. It is clear that both textbooks also used a variety of representational translations that are direct and indirect. Yet again, the use of representations was different.

I. Mind Action Series

Table 4.2 summarises the direct and indirect translations between representations that are presented in expository texts. The aim is to record the possible uniformity that might emerge, if there is any, from the translation of the source (refers to a representation that is used as the starting point) to the targeted representation (representation aimed to be achieved). It is evident that seven instances of indirect translations (the use of three or more representations) between representations were presented in expository texts. Two (28, 5%) representational translations used textual descriptions as the source, and sketching a graph was the target. It is further observed that there are two (28, 5%) instances where a sketched graph is used as a source to determine the equation of a linear graph, which is the targeted representation. Two (28, 5%) representations used the interpretation of a graph as a source to determine the equation of a linear graph which is the target. One (14, 5%) translation from a tabular representation is used as the source to determine the equation of linear graph (targeted representation).

A total of nine direct translations (the use of two representations) between representations was recorded, and five out of nine (55, 5%) representational translations used the recognition of parameters of a linear equation as a source, to the sketch of a linear graph, which is the target. Two

instances (22%) of interpreting graphs are used as the source to determine the equation of a linear graph (target). Lastly, one (11%) representational translation used descriptions in words as a source to model concepts of a linear graph. In total, 16 representational translations are recorded and 43, (75%) accounted for indirect representational translations while 56, 25% accounted for direct representational translations.

Interrogation of a sketched graph is dominant and accounts for 40% of representational translations that used a sketched graph as a representational source (see Figure 4.2). The representational source that started with a linear equation is presented 33.3%, while a representational source that uses descriptions in words accounted for 20%. Lastly, a representational source that used a table is encountered 6.7%. These trends of representational sources suggest a fair balance between examples that used a sketched graph as a source and those that used other representations such as descriptions in words, linear equations, and tables. The linear equation of a linear graph (33.3%), as well as a representational source that uses a sketched graph (40%), seems to stand out.
This might stem out from the idea of the importance of identifying patterns to search for similarities

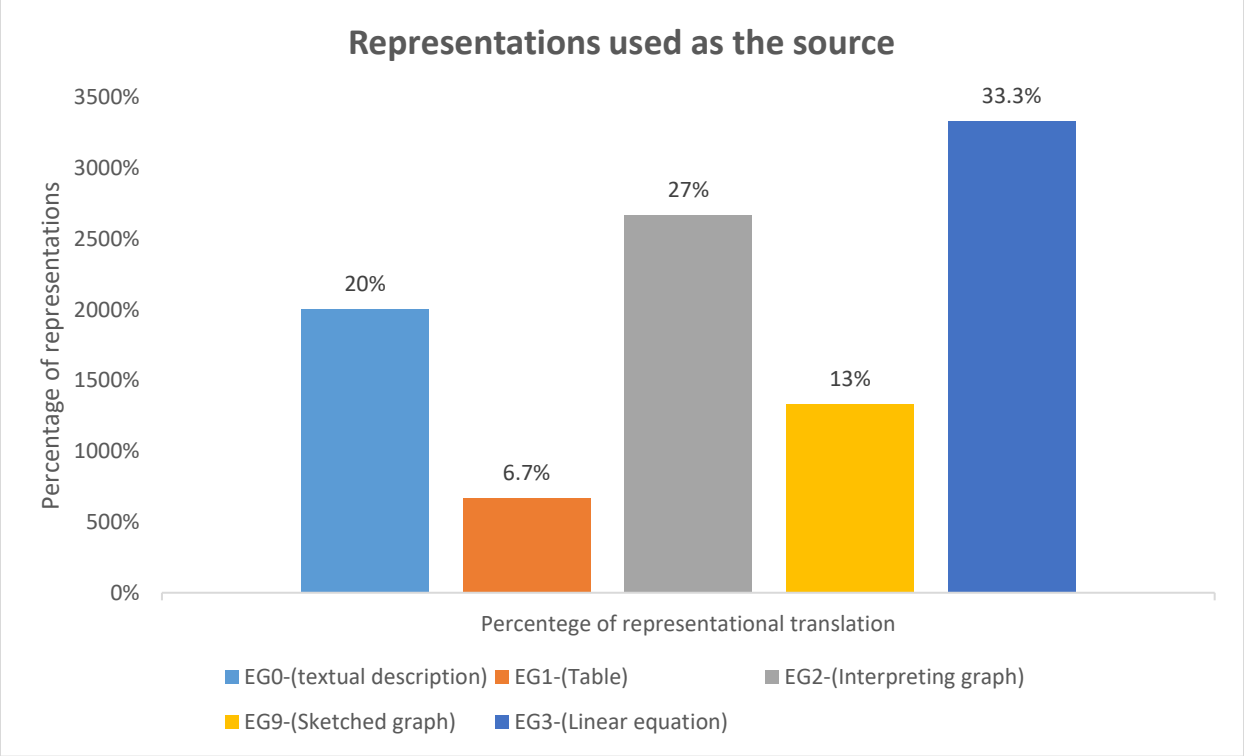


Figure 4.2: Representation used as a source.

and differences that exist between two mathematical objects, which are, in this case, a sketched graph and a linear equation. In the words of Jeannotte (2005), identification of a pattern infers a narrative about a recursive relation between mathematical objects. 'Identifying a pattern' is regarded as one of the process aspects of mathematical reasoning. In this case, the algebraic equation representational source and sketched graphs seem to have been the main concern, which echoes the Leinhardt, Zaslavsky and Stein's (1990) notion that suggests that linear graphs and their linear equation define the whole concept of Linear Functions. This might suggest the reason why the focus of the source representations is based on linear graphs and linear equations.

The explanations in expository texts encountered in examples in this regard might be aimed to address the symbolic correspondence of features, and the movement from a graph to a linear equation and vice versa, that exists between the graph and its linear equation. On the same note, Leichardt et al. (1990) argue that a limited connection between parameters encountered in the algebraic equation and their graphs might result in a limited understanding of linear graphs because these two representations jointly define the mathematical concept of Linear Functions. The relationship of representations features encountered in linear graphs and their linear algebraic equations will be interrogated further in the Micro-section analysis. Having discussed the representational translations presented in the *Mind Action Series* - which is analysed to have a balanced perception between the graph to equation and equation to graph translation.

Consider Figure 4.3 which represents the modes of reasoning identified in expository texts. The findings indicate that Deduction Using a Specific Case (DUSC) mode of reasoning is dominant, and accounts for 58% of the modes of reasoning presented in expository texts. Deduction Using a General Case' (DUGC) accounts for 11% of the modes of reasoning presented in explanatory texts. This mode of reasoning (DUGC) is presented in the unit section, 'concluding explanatory text' (see Table 4.2).

'Concordance of a rule with a model' (CRM) mode of reasoning is encountered twice (5%) in the unit section, 'Explanation of a worked-out example (symbolic algebraic form)'. Appeal to Authority accounts for 13% of modes of reasoning encountered in the expository text in the unit section, 'introductory text'. Deduction Using a Model (DUM) is 13%. Analogical reasoning and experimental demonstration are not encountered.

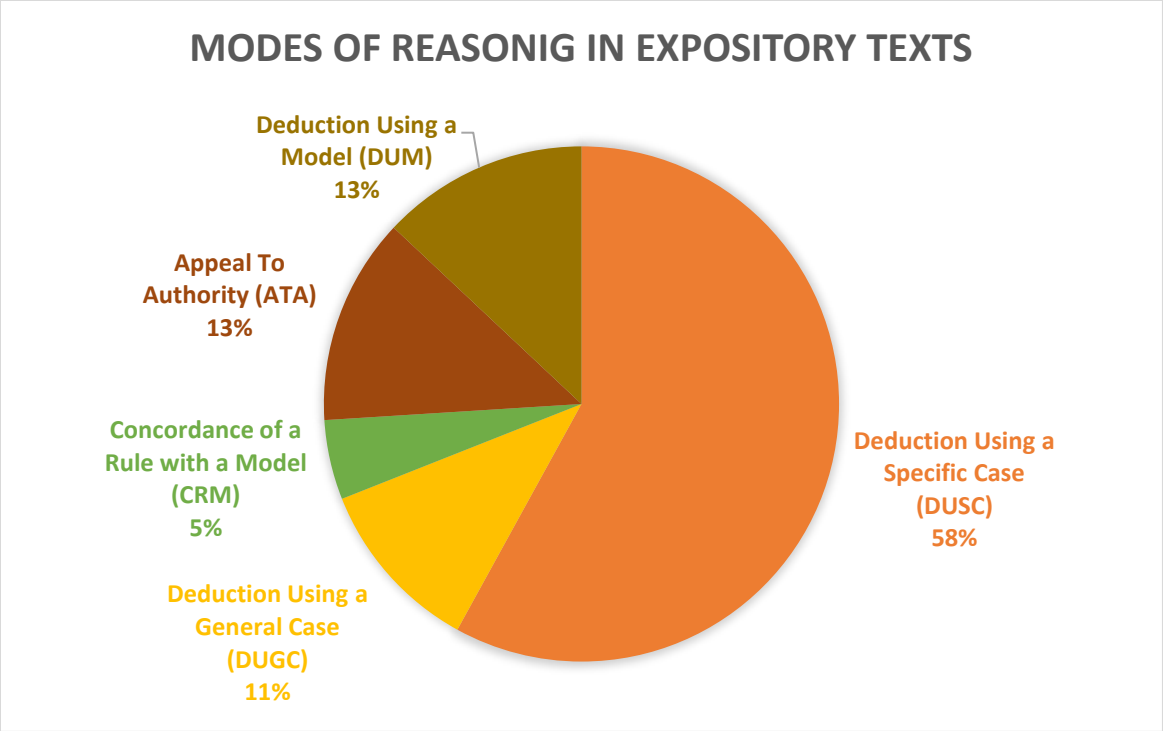


Figure 4.3: Summary of modes of reasoning of the Mind Action Series textbook.

In summary, the modes of reasoning reveal that deduction using a specific case is dominant (58%), and it is commonly encountered in the unit section of explanation of a worked-out example and symbolic algebraic procedure. In the following section, I will discuss the general analysis of the modes of reasoning and the multiple representational translations in the *Platinum* textbook.

II. Platinum Textbook.

As established earlier, the *Platinum* textbooks have a total of six examples in the chapter, Linear Functions. The identified modes of reasoning are as follows:

- Deduction Using a Specific Case (DUSC)
- Deduction Using a General Case (DUGC)
- Concordance of a Rule with a Model (CRM)
- Appeal to Authority (ATA)
- Deduction Using a Model. (DUM)

It is evident that Deduction Using a Specific Case (DUSC) accounted for 37% of the modes of reasoning presented in the *Platinum* textbook. Appeal to Authority (ATA) accounted for 16% of

the identified modes of reasoning. Meanwhile, Deduction Using a Model (DUM) and Deduction Using a General Case (DUGC) are encountered equally at 21%.

A similar trend as in the *Mind Action Series* could be observed in the *Platinum* textbook in terms of the dominant mode of reasoning (Deduction Using a Specific Case) that is identified. A notable high proportion of modes of reasoning that are deductive are generally prevalent in expository texts.

The sequential unpacking of content when learning Linear Functions is usually from an equation to a graph. This, according to Janvier (1987) is a psychological process that is different from the sequential content that is from a graph to an equation. Janvier (1987) has revealed that the movement from an equation to a graph is dominated by a series of steps that consist of ordered pairs of coordinates and plotting them on the Cartesian plane to make a line. The process aforesaid is straightforward, and most textbooks proceed in the equation to graph direction (Janvier, 1987). The findings in the *Platinum* textbook suggest that 57% of representations proceed from an equation as the starting representation. This might suggest a procedural way of learning Linear Functions according to Janvier (1987).

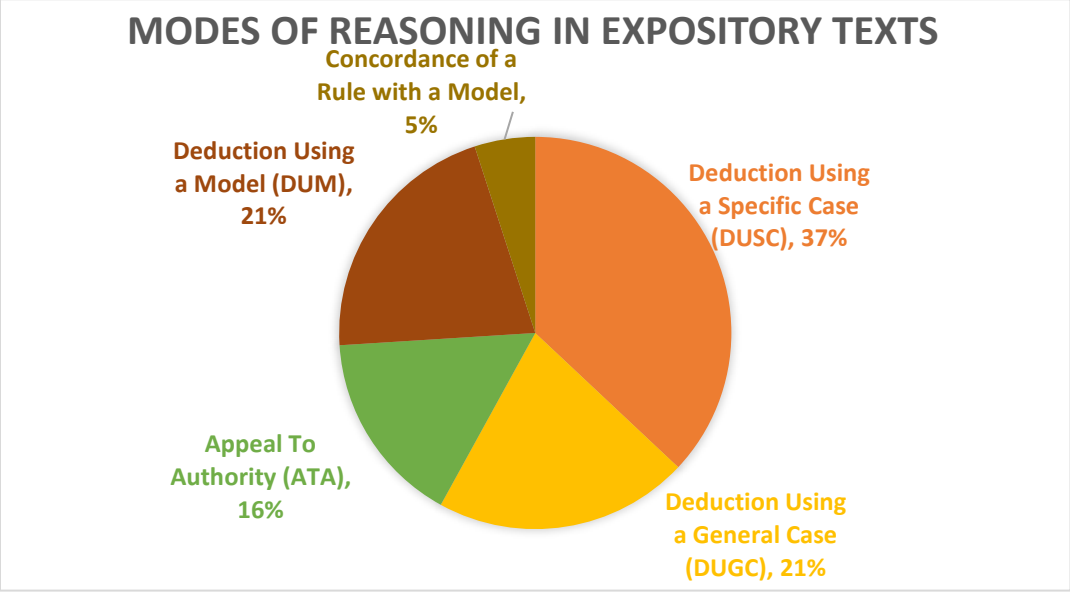


Figure 4.4: Percentage of modes of reasoning in the Platinum textbook.

The graph to an equation direction is suggested to be a complicated process because it involves pattern discovery (Janvier, 1987). Having spoken about the importance of pattern discovery in

Mathematics reasoning before, the *Platinum* textbook uses a lower proportion of representational translations from a graph to an equation direction (see Figure 4.5). The representational translation that uses a graph as a source accounted for 14% of representations presented in expository texts (see Figure 4. 5).

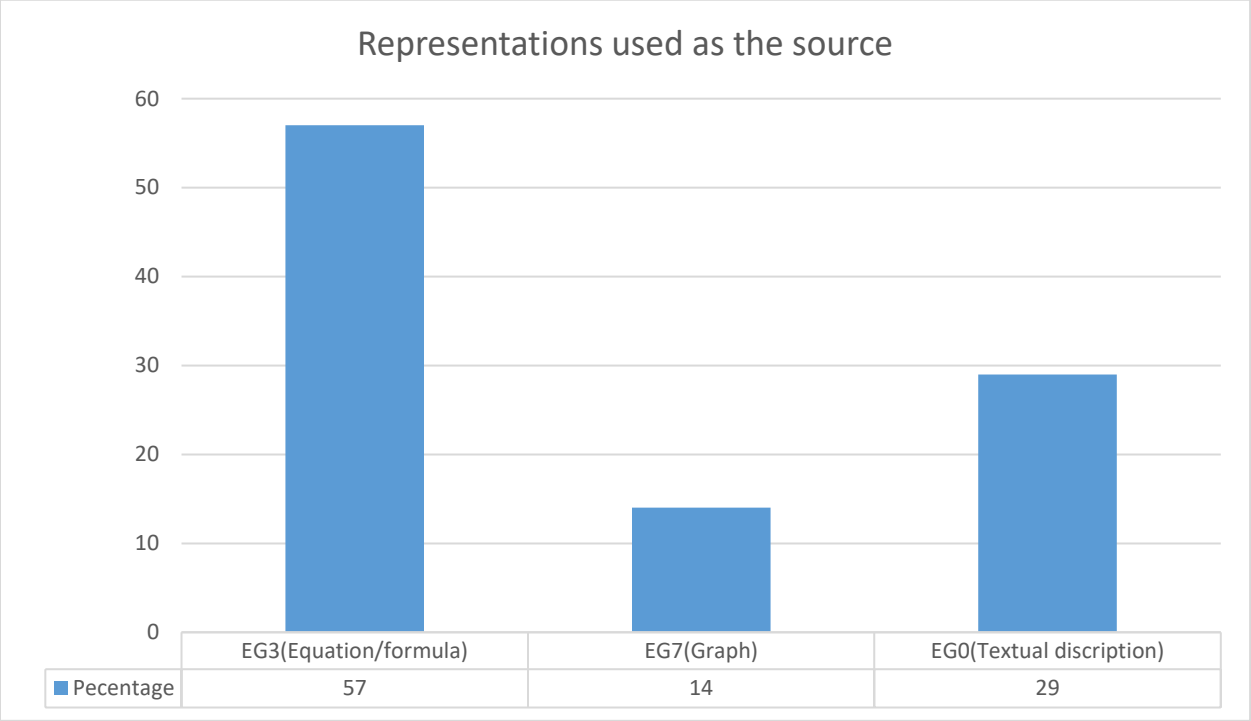


Figure 4.5: Representations used as a source in the Platinum textbooks.

To sum up the discussion of representational translation background, Figure 4.6 illustrates an example of an equation to a graph direction that is dominantly used across the textbook (see examples 3 in Appendix 2). Figure 4.6 shows an example from the *Platinum* textbook that requires the following graphs to the sketched:

- $y = 4x + 8$
- $y = 4x - 8$
- $y = 4x - 4$

The equations are followed by a textual description that is given with the intent to generalise what has been learned thus far. The textual description is aimed at explaining the parameters of the straight-line equation ($y = mx + c$) in relation to the sketched graph (see Figure 4.6). The example in Figure 4.6 captures the dominant representational translations that are found in the

Platinum textbook. The procedure of the explanation is given, which follows a procedural sequence of learning Linear Functions from an equation to a table then a sketched graph. It is notable that this example's intent is to summarise all that has been explained in the previous examples. However, the example (Figure 4.6) and 57% of other examples (see figure 4. 5) resemble an equation to graph direction that Janvier (1987) cautions us about.

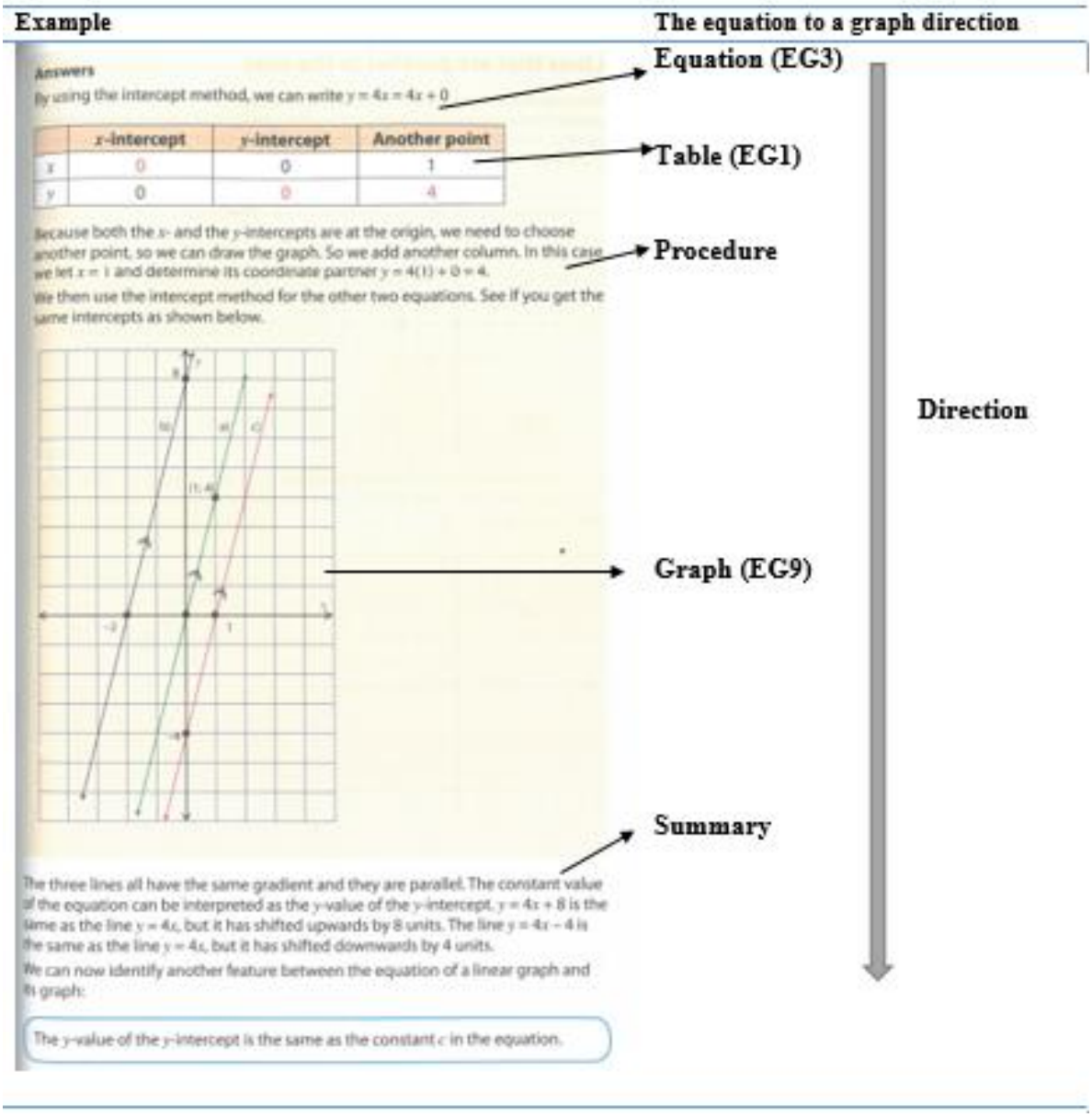


Figure 4.6 : Translation from equation to a graph (Campbell, Maritz, Rossouw & Willers, 2013).

The summary section has intentions of identifying features between linear equations and their graph. Although the links between parameters of the equation of a straight line $y = mx + c$ and

their sketched graphs have been presented, it is an important aspect of reasoning, specifically the ability to identify a pattern and represent a concept in different representations.

Eisonberg's (1992) report has put into view suggestions that encourage the integration of information across multiple representations. It is revealed that learners tend to possess a limited understanding of the connections between parameters of linear equations and their graphs (Eisonberg, 1992). The summary section in Figure 4.6 suggests a positive response to the suggestion put forth by Eisonberg (1992). Against what has been revealed in Figure 4. 6 thus far, remains a concern about the connections between representations in the direction from an equation to a table then a graph. This might suggest that although the representational translations are presented in a procedural manner (from the linear equation to a table then a graph) the textual descriptions (summary section in Figure 4. 6) could be presented to guide the reader on the connections that need to be observed from the different representations in order gain more understanding.

The idea of interpreting a graph and having to draw information to formulate a linear equation is not sufficiently stressed across the examples given in the chapter on Linear Functions, in the *Platinum* textbook. The example in figure 4.6 among other examples (see examples 1, 2, 4, 5 in Appendix 2) shows the dominant sequential outline of the representational translations that are presented in expository texts across Linear Functions. To support what has been explored thus far, Figure 4.6 shows that there is 14% presence of representational translation that uses a graph as a source. This suggests a limited opportunity for learners to develop interpreting and pattern discovery skills from a sketched graph to formulate an equation. In this view, the idea of pattern discovery, direction from a graph to an equation advocated for by Janvier (1987), which is an important aspect of mathematical reasoning, seems to be limited or overlooked.

An attempt to draw information from a graph is encountered in an example that explains the formula of a Gradient. In this example (Figure 4.7 *Platinum* textbook), the construction of the formula of Gradient is explained by interpreting a graph and using its coordinates to define the change in y over the change in x. The graph is used to model and define the formula of a Gradient. In the context of defining the concept of Gradient in the *Platinum* textbook, it is notable that the language used to explain the link between the formulae of a Gradient and how it is derived from a

straight-line graph has attempted to use concepts from the everyday language such as rate and change.

Österholm (2008) notes that learners are likely to read and understand a mathematical text better when they have a concrete prior knowledge of the concepts and symbols that are found in the Mathematics text and that they comprehend a mathematical text better when it does not use symbols, but rather concepts from the everyday language. On the same note, Rotman (1988) refers to these learners as scribers because they are assumed to possess some mathematical facts from previous encounters.

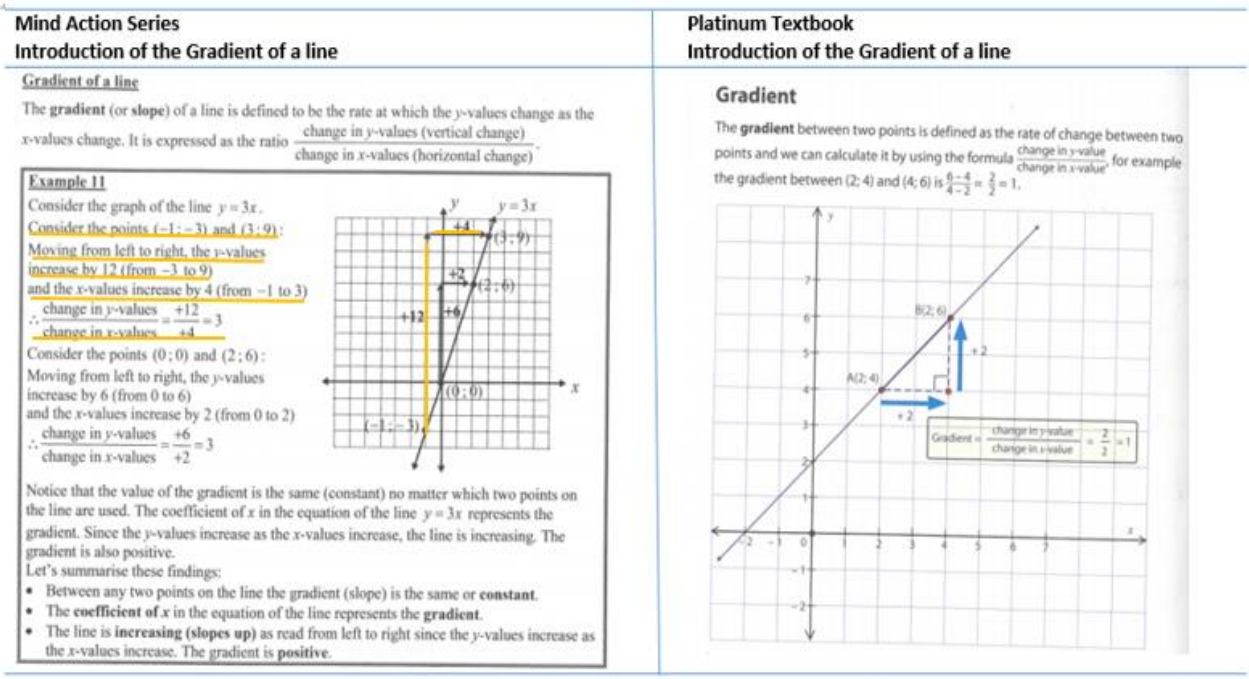


Figure 4.7: Language used in expository text in the Mind Action Series and Platinum textbook Campbell et al., 2013; Basson et al., 2014).

It is notable that the mathematical text in figure 4.7 (*Platinum* textbook) has used a relatively brief textual explanation to define Gradient. Regardless of the text being brief, which is likely to be difficult to comprehend depending on the prior knowledge of the learners (Österholm, 2008), an effort to minimise the use of mathematical symbols seems to have been prioritised. For instance, the textual description used concepts such as 'rate of change'; and '$\frac{change\ in\ y}{change\ in\ x}$'. The word 'rate' and 'change' are concepts that are used in the everyday language, as such, the language used in explanatory texts seems to use less mathematical symbols. The language of the explanation (Figure

4.7 *Platinum* textbook) would have been considered to use intensive symbolism if, for example, the' $\frac{change\ in\ y}{change\ in\ x}$, was written in the form $\frac{y2-y1}{x2-x1}$ or rather $\frac{\Delta y}{\Delta x}$. In this particular example, the concept of 'rate of change' is symbolised by $'\Delta'$ in the latter example, while the 'change' in the former example is symbolised by the use of '–'.

The adjacent example from the *Mind Action Series* (Figure 4.7) is sampled to highlight the relative use of language, as well as the guiding principles that these two textbooks use in the expository texts. Having mentioned the brief explanation given to introduce and define the concept of Gradient in the *Platinum* textbook, it is perceivable that the *Mind Action Series* uses explanations that are elaborate and attempts to use the everyday language appropriately to guide learners in the process of learning Linear Functions.

To exemplify the notion of gradient, the *Mind Action Series* has used the word 'slope' as an alternative term. Stump (2001) reveals that the use of words that are derived from everyday language could relate the concept being learned to the context of the learners, and to some extent, this can lead to intuitive understanding. Stump cautions that although intuitive understanding that is derived from the everyday language is important, learners who are exposed entirely to the everyday language struggle to conceptualise 'slope' as a ratio or rate in the context of Mathematics. However, learners who intuitively understood the concept of 'slope' and used the formulae correctly, demonstrated a better understanding of 'slope' as a measure of steepness or as a rate of change between two points on the line graph.

Mevarech and Kramarsky (1997) in line with Österholm (2008) warn that the use of formal language (text with mathematical symbols) imposes additional difficulties on learners' comprehension of an expository text. This is commonly encountered when a word (slope) has a different meaning in Mathematics and normal language.

To sum up this discussion, the introduction of Gradient in figure 4.7 (*Mind Action Series*) used the word 'slope' deliberately to appeal to learners' intuition from the use of the word "slope' in the everyday language. It is furthermore observed that the word slope is then explained and defined using everyday language devoid of algebraic symbolic system in the expository text by writing the formula of the gradient in the form $\frac{change\ in\ y-values(vertical\ rise)}{change\ in\ x-values(horizontal\ rise)}$. The expository texts used in

figure 4.7 (*Mind Action Series*) seem to be elaborate, and guide learners using an authoritative position from the author on the important aspects that need attention and how the formula $\frac{change\ in\ y-values(vertical\ rise)}{change\ in\ x-values(horizontal\ rise)}$ is constructed. Learners are scribers because they are instructed to the important aspects that need attention as it is discussed in Herban-Eisenmann (2007). This might be the case because learners might be in the process of being inculcated to facts, ways of reasoning and procedures needed in the coming examples. The explanatory text goes to the extent of identifying exposed patterns from the graph that learners need to focus on to understand the construction of the formulae of the gradient.

The concept of vertical and horizontal rise is derived from the everyday language in the formulae $\frac{change\ in\ y-values(vertical\ rise)}{change\ in\ x-values(horizontal\ rise)}$, the explanatory texts might have used the words (vertical and horizontal rise) deliberately with the intention to illustrate them on the straight-line graph alongside (figure 4. 7, *Mind Action Series*). What is of interest is the use of arrows on the straight-line graph, and the explanatory text that refers learners to these arrows because they show the vertical and the horizontal rise referred to in the formula of the gradient (see the highlighted arrows on the graph and their adjacent explanation in Figure 4.7, *Mind Action Series*). A combination of the everyday language that is elaborate, and with less symbolism, with appropriate explanations that point learners on the important aspects is used. This echoes the findings by stump (2001) that revealed that understanding because of the use of the everyday language, and the appropriate use of the formulae of gradient tend to equip learners to demonstrate a better reasoning capacity of the concept of slope.

The use of expository text in the *Platinum* textbook is brief, however, it is worth noting that more examples follow the introduction and definition of Gradient in figure 4.7 (*Platinum* textbook).

Figure 4.8 shows an example that follows the introduction of Gradient in the *Platinum* textbook in figure 4.8. An attempt to explain the uniformity of the gradient between any coordinates on the straight line has been given as a summary. The explanation of the gradient between two points on the straight line given in Figure 4.8 was explained procedurally, and learners were asked what it is that they noticed regarding the solutions of the worked-out procedures given. The question highlighted in figure 4.8 is important because it directs learners' reasoning on discovering

prevailing patterns that would help them to realise that the gradient of any two points on the straight-line graph is constant.

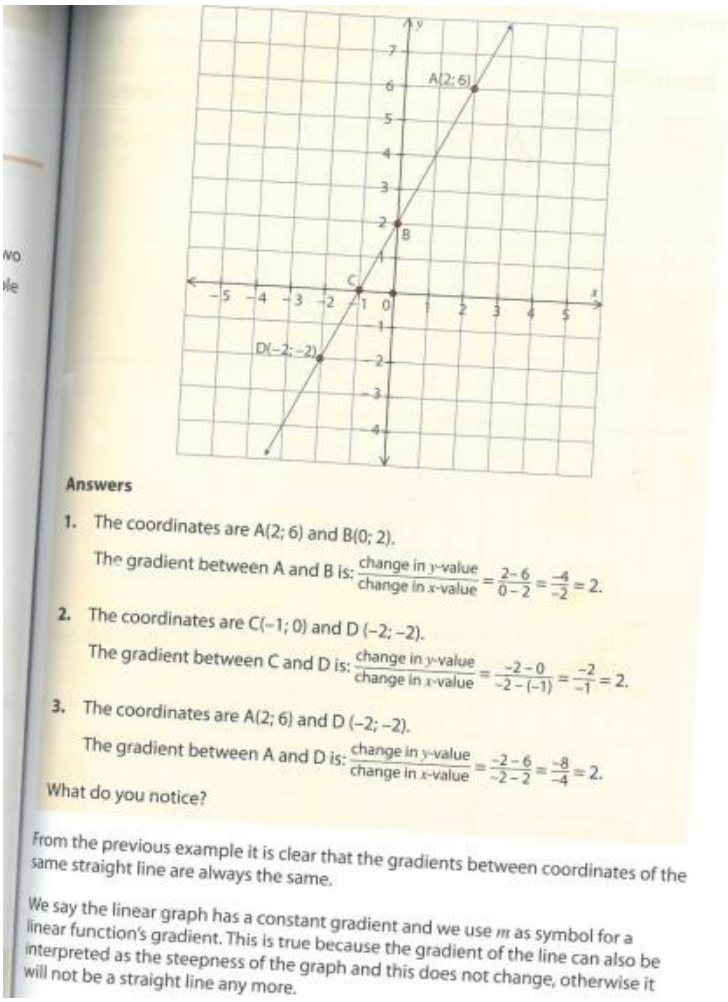


Figure 4. 8: Gradient of a straight-line graph (Campbell et al., 2013).

Although the explanations attempted to use the everyday language, the briefness of these explanations left other aspects of what needed to be understood by the learners. The following questions among others were asked as part of the explanatory text in examples found in the *Platinum* textbook:

- What do you notice?
- Why does this graph have a negative gradient?
- What do you notice about the gradient and the equation of the linear graph?"

The above questions have the potential to engage the readers' reasoning to the extent of searching for uniformity that exists between representations. It is notable that the third bullet point has the potential to steer the reader to search for relative consistencies of the features of the gradient and the equation of the linear graph. Included in the questions is the 'why' question which required reasoned justification. These example questions, among others in various examples found in the *Platinum* textbook, might explain the briefness of explanations encountered. Stump's (2001) report on the use of the everyday and formal language has revealed that the exposure to technical language and formal instructions such as procedurally worked out examples (worked out example in Figure 4.9) does not guarantee the ultimate aim of developing the readers' mathematical reasoning capacity. In summary, the language used in the expository text has attempted to avoid the intensive use of symbols. In some instances, the expository texts have questions as part of the explanation process to aid learners to search for the relationships that might exist between different representations, thus the explanations turned to be brief.

4.4. Summary of macro-section analysis.

The dominance of modes of reasoning that are deductive-the structural aspect of mathematical reasoning-which is divided into three modes of reasoning (deduction using in specific case, generalisation and using a model) is evident in both textbooks. The formal conceptualisation of deduction captured in the work of Harel and Weber (2020) suggests a formal and symbolic computation that appeals to the expert mathematician. This cannot be easily decoded by the learners when considering their position as individuals who are in the initiation stage of learning reasoning principles in Linear Functions.

It was discussed that deductive reasoning is characterised by theorem, facts and formal computations, as such, Curcio, Frances, Sydney and Schwartz (1998) cites in Russell (2018) that this type of reasoning is ideal for solving mathematical tasks rather than for learning purposes. This backs the question, what do the identified modes of reasoning, which are dominantly deductive in this instance, mean to the reader to develop reasoning skills in Linear Functions? The question of the readers' level of comprehension comes to the fore since deductive reasoning is formal and uses symbolic system which is proven by Österholm (2008) to reduce learner comprehension of a Mathematics text.

Mathematical expository text with symbols is not easily understood by learners (Österholm, 2008). The linguistic features embedded in expository text have been identified in literature that they can help in terms of exploring the 'voice' that the textbook uses. This will be interrogated in the micro level analysis section in detail. Instances of using the everyday language as descriptive text were evident in guiding the reader on the important aspects that needs attention - for instance, key word in example, Appendix 2. The explanation uncovers the taken for-granted concepts that are key to accessing the much-needed connections between features of representations. The use of the everyday language might have been used to appeal to what learners find to be familiar, and this is used to direct learners to the much-needed goal to access the principles that govern Mathematics reasoning in Linear Functions.

The language used in the expository sections must be explored with the intention to highlight what the identified deductive modes of reasoning mean for the learners in the process of developing mathematical reasoning skills. The micro section has been put into place to analyse examples that were sampled. The sampling process of examples had minimum requirements; examples had to have sufficient expository unit sections that could be studied to explore the modes of reasoning presented. Secondly, the examples had to explain the connection of the features between linear graphs and their equation. This was part of the requirements because I had to study the fluency of the representational translation between a graph and its equation / formula and the translation between an equation / formula and its graph. The transition between a graph and its equation is important because Schoenfeld (cited in Leinhardt et al., 1990) argues that a deep understanding of the relationship between a graphical line and its algebraic equation is required for learners to reason in Linear Functions.

4.5. Micro-section analysis: Analysis of sampled examples.

As indicated previously, the sampled examples included in this section had to adhere to the minimum requirements needed to study the modes of reasoning presented in expository texts. It was realized during the analysis in the Macro-section analysis that not all examples had appropriate expository texts that could be studied adequately to reveal the modes of reasoning presented. As such, I saw it fit to sample examples that had sufficient unit sections of expository texts that would allow me to analyse the modes of reasoning presented.

The sampled examples had to somehow deal with the relationship between a straight-line graph and its formula of gradient / algebraic equation. One of the selection criteria amongst the ones that were mentioned before is an example that consists of representational translation that used a sketched linear graph as a source, and an example that used a formula of gradient / linear equation as a source. The criteria were formulated with the guidance from Schoenfeld's (cites in Leinhardt et al., 1990) idea that suggests that the connection between a linear graph and its equation defines the mathematical concept of Linear Functions. The intention is to examine the corresponding symbolic parameters, if there are any, between linear graphs and their algebraic equation / formula of the gradient. The importance of identifying the corresponding parameters of representations in learning Linear Functions is discussed in Leinhardt et al. (1990). The sampled examples might help to explore the extent to which the modes of reasoning presented, and the representational translation used, mean for learners to develop mathematical reasoning skills. The following points will be followed to guide the development of the argument that this section is set out to make about the identified modes of reasoning:

- The meaning of the identified modes of reasoning to the reader.
- The meaning of the fluency of the identified representational translation to the development of mathematical reasoning in Linear Functions.
- The effects of linguistic features in expository text on the learners' level of comprehension of what the expository text intents to convey, and the suggested positioning (using the interpersonal linguistic imperatives) to explore the degree of involvement of the learner and the author in doing mathematics.

The points will be the focus when analysing the sampled examples to answer the main research sub-questions to some extent. The aim is to interrogate the expository texts of the sampled examples and to examine the contingencies with which the identified modes of reasoning could provide access to the principles needed to develop the need to reason mathematically.

4.5.1. Sampled examples of the *Mind Action Series* and *Platinum* textbook.

Sub-topics:

- Finding the equations of lines using points on the graph. (*Mind Action Series*)
- Determining the gradient of the straight-line graph (*Platinum* textbook)

4.5.1. a) Finding an equation of a line from a sketched graph.

Figure 4.9 shows an example that illustrates how to find a linear equation when given a sketched linear graph. It is part of the last three examples at the end of the chapter of Linear Functions in the *Mind Action Series*. This means most of the concepts needed to learn Linear Functions have been explained in the previous examples in this stage, a summary of important concepts has been dealt with at this stage. In this, the author might assume that certain aspects of ideas and concepts have been learned. In Herbal-Eisenmann's (2007) terms, readers in this stage are regarded as 'thinkers'

The identified modes of reasoning in figure 4.10 are as follows:

- Deduction Using a General Case (DUGC)
- Deduction Using a Specific Case (DUSC)
- From a sketched graph (EG9) to a formula/linear equation (EG3)

The phrase 'given the *y*-intercept and a point' is a previously learned fact that supposes that learners at this stage should be familiar with procedures required because this has been dealt with in previous examples. In Figure 4.9, the previously learned facts are used to explain how to derive a linear equation from a sketched graph when given the *y*-intercept and an equation.

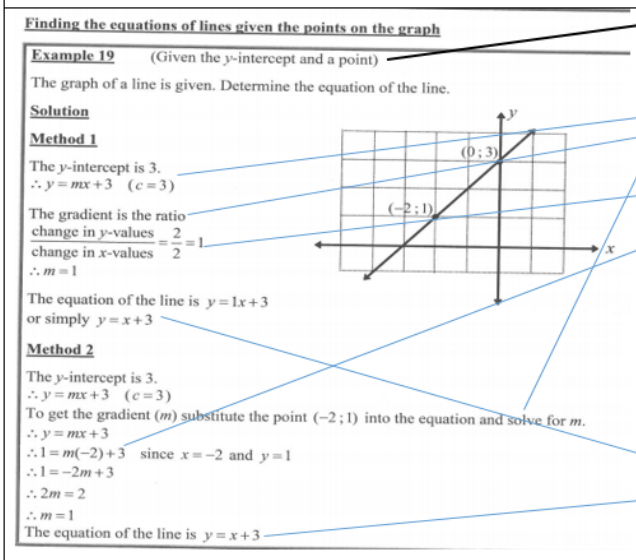
Example 19: Finding equations on the graph given the points on the graph.	**Unit section**	**Modes of reasoning**	**Representational translation.**
	Introductory text	DUGC	(EG9-EG3) From a sketched graph to computing using a formula/equation
	Explanation of a worked out example	DUSC	
	Symbolic algebraic procedure	DUSC	
	Key words/concepts	N/A	
	Concluding explanatory text	DUSC	

Figure 4 .9: Finding equation on the graph given a point (Basson et al., 2014)

The phrase makes use of words such as "*y*-intercepts' and 'a point' to illustrate generalisation. The unit section (introductory text) is categorised as Deduction on Using a General Case (DUGC) because it used variables (*y*-intercept, and a point) that suggest generalisation, and the assumption that learners are familiar with the previously learned facts about the *y*-intercept and a given equation.

The previously learned facts are presented in the unit sections of 'explanation of a worked-out example,' 'symbolic algebraic procedure' and 'concluding explanatory text'. These facts (algorithms and procedures) have been learned in detail from previous examples, as such, it is assumed that the reader has the prior knowledge needed to understand the explanations.

The use of specific values in Figure 4.9, for instance; '*y*-intercept is 3', (*c*=3) $\therefore y = mx + 3$, suggests that the variable (*c*) is substituted with the value that suggests the specificity of this example. This means the answer is only true for this specific example, thus the unit sections ('explanation of a worked-out example,' 'symbolic algebraic procedure' and 'concluding explanatory text') are categorised as Deduction Using a Specific Case (DUSC).

The identified model of reasoning: Deduction Using a Specific Case (DUSC) is the dominant mode of reasoning in Figure 4. 9. This echoes the general findings in the Macro-section analysis which also revealed that Deduction Using a Specific Case dominates across the examples in the two textbooks. This mode of reasoning suggests the use of previous facts and procedures that are derived from previous mathematical knowledge (Stacey & Vincent, 2009). This mode of reasoning is characterised by a formal way of doing and learning Mathematics ((Duvel, 1995).

The linguistic features in the expository text used in Figure 4.9 suggests an authoritarian positioning of the author mainly because of the use of expository texts such as "to get the gradient '*m*', substitute the point (-2; 1) into the equation and solve for" The reader is commanded and or instructed on what to do and procedures that need to be computed. Such utterances also suggest the positioning of the reader as a scriber due to the instruction on the algorithms and procedures that should be followed. The reader is not involved in the construction of the Mathematics in this instance because of the absence of interpersonal pronouns such as 'you'. In line with this, Morgan (1998) has put forth interpersonal imperatives, they are defined as instructions or commands in the expository texts that directs the reader on what to do and suggests directives on the aspects of ideas

that the text intends to convey. The personal pronoun such as 'you' represents the reader, and the author is presented in pronouns such as 'I' in expository texts (Morgan, 2004).

However, this was not limited to the example in Figure 4.9, a considerable number of examples in the *Mind Action Series* textbook were observed to express utterances in the expository text that assume an authoritative position of the author (examples 5, 6, 8, 10, 12 in Appendix 1). That is, the author uses an authoritative position where readers are scribers with little or no involvement in the construction of the Mathematics. In the example in figure 4.7 (*Mind Action Series*) and other examples (see examples 5, 6, 8 10 in Appendix 1), it was observed that there is a consistent evidence of an authoritarian position of the author, with no evidence of direct interpersonal pronouns that suggest the involvement of learners. In this stage, learners might be in the enculturation stage of mathematical facts since the examples formed part of the demarcation of the beginning of a new sub-topic. Thus, learners are considered to be scribers instead of the desired scriber and a thinker.

The text used in the example in Figure 4.9 uses symbols with brief textual descriptions as compared to examples 2, 5, 8 and 9 in Appendix 1. The position of this example within the chapter might suggest that the reader is considered to have facts that are learned from previous examples. As a result, the language structure seems to change from being elaborate and lengthy in examples that are used to introduce a new concept (examples 7, 9, 10, 11) to being brief and dominated by algorithmic procedures in examples that are found in the middle and towards the end of the chapter. In Rotman (1988), learners are thinkers in the Mathematics community when the text assumes certain operations and facts are known to the reader.

The representational translations encountered in these examples represent a 33.7% of representational transformation that uses a graph as a source representation. The need for the use of representations that use a graph as a source seems to be dealt with adequately. The use of a graph as a source representation to other representations such as a linear equation supports the need for the identification of patterns which is an important aspect of reasoning (Jeannotte & Kieran, 2017). The translation in figure 4.9 is a difficult process as suggested by Janvier (1987) because it entails pattern discovery. The identification of parameters of a linear equation $y = mx + c$ to its graph is illustrated in the examples in figure 4. 9. The y-intercept on the graph is

indicated to be 3. This is one of the important proficiencies that readers need to understand to see the corresponding meaning of features of a linear equation $y = mx + c$ and its graph.

The example in Figure 4.9 used two methods to determine the equation of a line. It is worth noting that the first method refrained from using a chain of algorithmic procedure and used an interpretative approach of the sketched graph. This is an important skill that might enforce the ability to choose appropriate methods and representations that are convenient to solve mathematical problems. Such alternative methods could help learners not to rely on working in algebraic form even when a graphical approach could be less computationally intense (Knuth, 2000).

In summary, I found that the dominant modes of reasoning are deductive: Deduction Using a Specific Case 58%, Deduction Using a General Case 11% and Deduction Using Model 13%. In total, the modes of reasoning that are deductive accounts for 82% of the modes of reasoning presented. A high proportion of representational translation that used a graph as a source representation was identified and accounts for 33, 7% of representations used as a source. The use of a graph as a source suggests an opportunity for pattern discovery and interpretation.

The expository texts were examined to suggest a language structure that adopted the use of the everyday language, especially the explanations in examples used to introduce a new concept. The interpersonal linguistic imperatives used to instruct learners were analysed to regard learners as a scriber. This is the case in examples that were used to introduce new concepts. The reason behind this might be because of the intent to enculturate learners into principles and facts that would be needed in the next examples. I also noticed that there was a gradual change of the language structure to brief explanations that are dominantly in the form of computational algebraic procedures. The shift from introductory examples aimed to introduce new concepts, to examples in the middle and towards the end of the chapter suggest a change in the language form - from the use of the everyday language to the formal language that dominantly uses symbols and computational algebraic procedures as mentioned before.

This change in the language structure might suppose the reader as a' thinker' mainly because some of the symbols and concepts have been explained in the preceding examples used to introduce concepts. In this sense, the reader is regarded to be a scriber and gradually to a thinker - since certain facts have been explained and the reader in this stage is assumed to possess these facts.

The proportion of the modes of reasoning mentioned earlier suggests an inclination towards formal structural processes of doing Mathematics. Duvel (1995) notes that deductive reasoning is a form of the structural aspects of mathematical reasoning that is concerned with formal processes of proof, and it 'changes the value of Mathematics knowledge from likely to being regarded as true'. The use of deduction has been regarded to impose difficulties for learners to access the ultimate principles needed to do Mathematics (Sierpinska, 1994). In this, Sierpinska (1994) reveals that explanations turn to use the everyday language with the intention to appeal to learners' intuition. In the language of Brating and Österholm (2018), everyday language is used to appeal to what the reader is familiar with (psychological aspect), and this language is then used to direct the reader to the logical aspect of Mathematics - mathematical reasoning (deductive reasoning) since it is the mode of reasoning that is dominantly presented in expository texts.

With the aforesaid, the gradual change of the use of the everyday language towards the formal language of Mathematics might suggest a reduced anxiety towards formal Mathematics (Urquhart, 2009). This means that learners who find it hard to comprehend explanations that are dominated by symbols are gradually inculcated into principles of reasoning in Linear Functions. The findings suggest that the reader should be considered a scriber and gradually to a thinker towards the end of the chapter. The reader, in this case, might be supposed to have prior knowledge that qualifies them to be part of the discourse to some extent. Thus, the language, form, and function change from the introductory examples to the proceeding examples.

However, the interpersonal linguistic function seems to suggest an authoritative position of the author, with no evidence of interpersonal pronouns that suggest learner involvement in doing Mathematics. The idea of constructing mathematical knowledge or regarding the reader to be an active member in the construction of mathematical concepts seems to be overlooked and that has stressed the importance of learner active participation to develop the required competencies-reasoning in the context of this research.

4.5.1. b) Determining the gradient of the straight-line graph (Platinum textbook)

Figure 4.10 illustrates how to find the gradient of a line from a sketched graph and its equation: $y = -3x + 4$. It is one of the 14% of examples that use a graph as a source representation. It was found that the textbook did not have an example that explains how to determine a linear equation from a

sketched graph; as a result, I chose an example that requires interpretation of a sketched graph to determine the Gradient of a line.

The following modes of reasoning were identified in figure 4. 10:

- Deduction Using a Model (DUM)
- Deduction Using a General Case (DUGC)
- Deduction Using a Specific Case (DUSC)
- Interpreting a sketched graph (EG20) to fitting coordinates on the formula /equation (EG11)

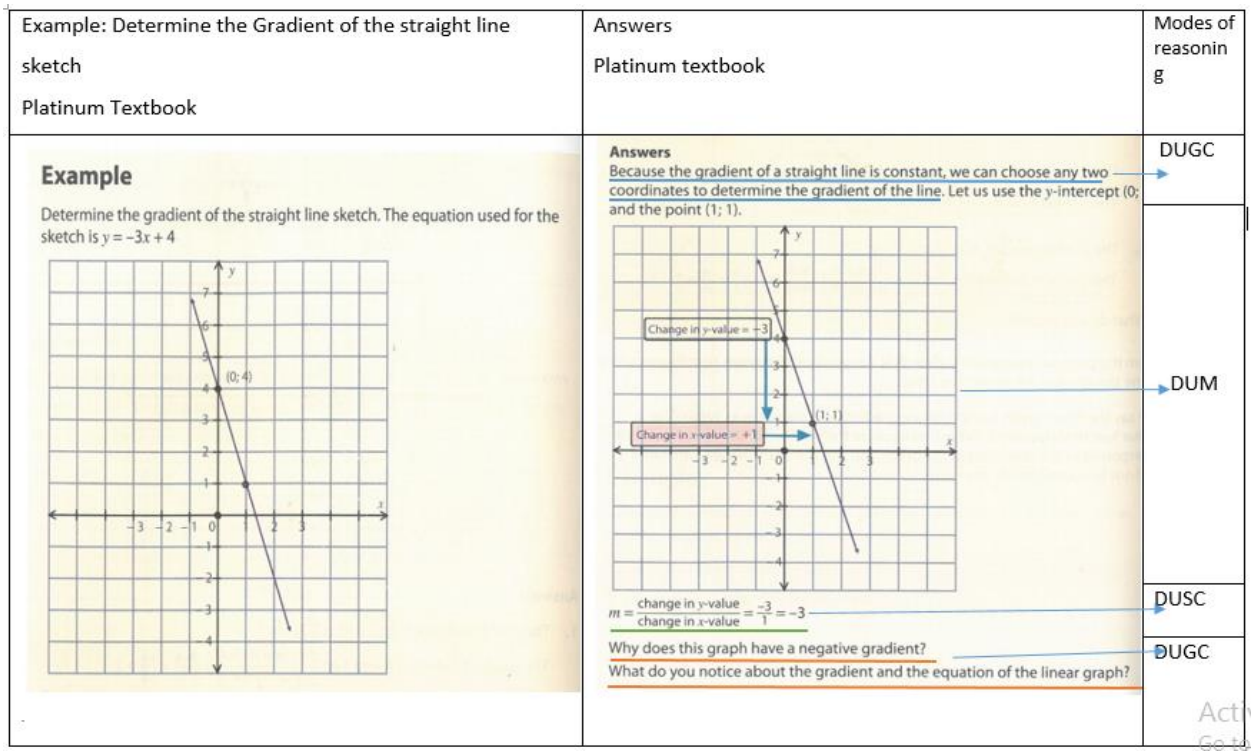


Figure 4.10: Determining the gradient of a sketched straight-line graph (Campbell et., 2013)

Deduction Using a Model (DUM) accounts for 21% of the modes of reasoning that were identified in the *Platinum* textbook. In figure 4.10, the explanatory text highlighted in blue was categorised as Deduction Using a General Case primarily because it makes use of previously learned facts from examples that precede the example in figure 4. 10. The previous generalised facts; for instance: 'Because the gradient of a line is constant' are used, and this is applied as a justification to choose any point that lays on the adjacent sketched line graph to determine the gradient. The

phrase ‘we can choose any two coordinates’ suggests generalisation because any point that lies on the sketched line can be used.

The coordinates are sourced from a sketched graph that is assumed to be a model that could be used to reason with in learning and communicating Linear Functions. The model can be used to reason with because it resembles the same mathematical structure as the formula that is used to calculate the gradient. Stacey and Vincent (2009) define Deduction Using a Model (DUM) as the mode of reasoning that has the direct correspondence of the model and its associated Mathematics, and the model can be used to predict the results of the calculations.

In the context of the example in Figure 4.10, the model showing arrows that explain the ratio between the change in *y* and change in *x* results in the same calculation and solution to when coordinates are substituted in the gradient formula. Deduction Using a Model is identified to account for 21% of the modes of reasoning in the *Platinum* textbook. It is noticeable that Deduction Using a Model required, to some extent, the interpretation of a line graph. In this, it must be pointed that although representations that used a graph as a source representation accounted for 14%, a substantial number of examples involved the interpretation of a straight line was evident in the examples (Figure 4.10, appendix examples 3, 5 and 6).

Deduction Using a Specific Case (DUSC) was identified in the unit section that is highlighted in green in Figure 4.10. The formula of the gradient of a line is a previous mathematical fact that was learned before, and the use of specific values of coordinates used means the unit section presented a Deduction Using a Specific Case Mode of reasoning. To reiterate, the definition of Deduction Using a Specific Case, according to Stacey and Vincent (2009), is an expository text that uses values instead of variables and it makes use of the previously known mathematical facts. Such expository texts are characterised by a series of procedures that use mathematical symbols (see the highlighted green section in Figure 4.10.).

The representational translation that is observed in Figure 4.10 is from the interpretation of a sketched line graph to fitting coordinates on the formula of the Gradient (EG2-EG11). The use of a sketched graph as the source representation has been encouraged before because it is perceived to foster the idea of identifying patterns (Leinhardt et al., 1990).

I observed that expository texts were given in the form of questions (see Figure 4. 10, the questions highlighted in orange). This instance of expository texts which were written in a question form is encountered in example 3 in Appendix 2 as well. They highlight the positioning of the learner / reader and the author (Morgan, 1995). The use of words such as 'we' 'us' as evident in Figure 4.10 are interpersonal linguistic features that suggest the degree of the learner and author's involvement in the construction of Mathematics (Morgan, 1995). The questions in Figure 4.10: "why does this graph have a negative gradient? And what do you notice about the gradient of a line and the equation of a linear graph?" form part of the interpersonal imperatives that instructs the learner / reader to search for connections between the parameters of the linear graph and the gradient. This supposes the learner / reader as a scriber. The learner / reader is considered to be a scriber when the text in expository sections directs, commands or instructs the learner to follow particular algorithms or to focus on particular ideas (Rotman, 1988).

The expository text in Figure 4.10 that reads: 'Because the gradient of a line is constant, we can choose any two points to determine the gradient of the line' supposes the reader to have prior knowledge about the gradient of a line, and the use of the interpersonal 'we' suggests an equal involvement of the author and the reader in the construction of Mathematics. The assumption of the learner / reader being in possession of mathematical facts regarding gradient suggests that the reader is regarded as a thinker. To support this, Rotman (1988) reveals that a reader is a thinker when an expository text assumes that the reader possesses some previously known operations and facts. For readers to learn from an expository text, in this instance, they should be positioned both as a scriber and a thinker (Rotman, 1998).

To summarise the findings in the *Platinum* textbook thus far, the dominant modes of reasoning presented were deductive (79%): Deduction Using a Specific Case 37%; Deduction Using a Model (21%) and Deduction Using a General Case (21%). Deductive reasoning has been comprehended by Siepenska (1994) to be characterised by formal mathematical processes. As mentioned before, it is often difficult for the reader to decode the formal language that comes with the formal mathematical processes such as the use of symbols in algorithmic procedures (Sierpinska, 1994). In this sense, the language structure and linguistic features are explored to assess the likelihood of the comprehension of the explanations through which the identified dominant mode of reasoning is conveyed.

The language structure used brief explanations as compared to the *Mind Action Series* textbook, and there is evidence of the use of the everyday language especially in examples that are used to introduce a new concept. There is evidence of the use of interpersonal pronounce such as 'we' in explanations which suggest the equal involvement of the learner / reader and the author in doing Mathematics.

In general, the explanations used a language that suggests an authoritative position of the author- this is as a result of the use of interpersonal imperatives that directed / instructed the reader on the facts and ideas that need to be learned. In this, learners are positioned as scribers because the explanations suppose a process of acculturating learners into principles and facts that would be needed to understand Linear Functions. The briefness of explanations suggests dominant algebraic procedural explanations in expository texts; thus, Deduction Using a Specific Case is the dominant mode of reasoning among the modes of reasoning that are deductive. The high relative use of symbols is found to dominate in explanations. As already established, a language that uses mathematical symbols was not found to be easy to comprehend by learners.

The representational translation that uses a graph as a source was identified to account for 14% of representational translations. This suggests that there are fewer examples that encouraged the translation process of graph-equation direction. The *Platinum* textbook dominantly (86%) adopted the equation-graph translation process which has been documented to encourage a procedural way of learning linear functions since it is a straightforward process that does not require pattern discovery. This style of explanation (from equation-graph) is a common approach used in textbooks to explain Linear Functions.

4.6. Conclusion and summary of data analysis.

The modes of reasoning that are deductive are found to be dominantly apparent in the two textbooks. Deduction Using a Specific Case is the most encountered among the modes of reasoning that are deductive. The representational translation is not the same, the *Mind Action Series* presented 33, 7% of representations that used a graph (interpretations of a graph) as a source. While the *Platinum* textbook presented 14% of representational translations that used a graph as a source.

To explain what the modes of reasoning and representational translations mean for the learners to develop reasoning skills, I used literature that would help to explicate the linguistic structure of

the explanations that are used to convey the identified modes of reasoning. I found this to be important because the linguistic structure could suggest the reader and the author's position, and the level of involvement in doing Mathematics. This could then be used to evaluate the likelihood of an explanation being comprehended by the reader and eventually gaining access to the identified modes of reasoning.

The language structure used in the two textbooks is similar to some extent. Both textbooks attempt to use everyday language in examples that were used to introduce a new concept. The *Mind Action Series* relatively uses lengthy explanations that apply everyday language in examples used to introduce new concepts, and gradually used brief explanations that dominantly use symbols, and algebraic procedures in the middle and towards the end of the chapter (formal language). The same language structure is prevalent in the *Platinum* textbook. However, there are instances of the use of interpersonal linguistic pronouns such as 'we' in explanations that suggested an equal involvement of the learner and the author in doing Mathematics in the *Platinum* textbook. An authoritative position is dominantly presented in explanations across the two textbooks. In the *Mind Action Series*, there is no evidence of interpersonal pronouns that suggested the involvement of learners.

The aforesaid background suggests how learners might comprehend the language used in explanations and eventually what the identified modes of reasoning ultimately mean. The style of writing in the *Mind Action Series* is authoritative - it uses imperatives that are devoid of interpersonal pronouns, which suggests an unequal relationship between the author and the learner. The author's role is to inculcate learners into the principles and facts needed to learn Linear Functions by directing their activity through the use of commands in explanatory texts. Thus, the learners' role is that of a scribe as they are instructed to follow certain algorithms and features of representations that need to be understood. Leaners are then considered to be thinkers in examples that are found in the middle and towards the end of the chapter. In these examples, learners are assumed to have relevant prior knowledge of facts that were explained in the preceding examples. Thus, the language structure in explanations changed from the everyday to formal language. These findings are also apparent in the Platinum textbook.

The findings in this regard, seem to align with Herbel-Eisenmann's (2007) suggestion that the representation of Mathematics in a written form is not an easy task, especially when certain

ideological principles of mathematical reasoning need to be conveyed. The author is faced with a task to help the reader to learn certain mathematical ideas first, then to assume that some certain ideas have been learned to move to the next sub-topic (Herbel-Eisenmann, 2007).

The next chapter will conclude this research by discussing what the identified modes of reasoning, the linguistic functions and language structures in expository texts suggest to the readers for them to comprehend what is conveyed. Recommendations will be discussed as well as suggestions pertaining to a field that needs further research in the study of textbook explanations and mathematical reasoning.

CHAPTER 5: SUMMARY, FINDINGS, CONCLUSION AND RECOMMENDATIONS

5.1. Introduction.

This chapter is set out to conclude this research report by focusing on the following aspects:

- Major findings of the study.
- Recommendations relating to the study, and recommendations for further studies.
- Limitations of the study.
- Concluding remarks.

5.2. Overview of the study

The main aim of this research was to investigate the modes of reasoning presented in expository texts and worked out examples (referred collectively as expository texts) in two Grade 9 Mathematics textbooks on the topic of Linear Functions. The two basic research questions were formulated as follows:

- What are the modes of reasoning presented / implied in expository texts in two Mathematics textbooks on Linear Functions?
- What do the identified modes of reasoning mean for learners to develop mathematical reasoning in Linear Functions?

The preceding chapters discussed the relevance of these study questions which could be answered within the broad debate of mathematical reasoning in the context of textbook analysis on Linear Functions and examined in different ways in the aforementioned research.

5.3. Major findings of the study.

5.3.1. Findings based on the identified modes of reasoning and representational translations in the *Mind Action Series* and *Platinum* textbook.

The modes of reasoning that are deductive (Deduction Using a Specific Case (58%), Deduction Using a General Case (11%) and Deduction Using a Model (13%)) have been identified to account for 82% of the modes of reasoning presented in expository texts in the *Mind Acton Series*.

A similar trend of the modes of reasoning that are deductive was identified in the *Platinum* textbook. The modes of reasoning that are deductive (Deduction Using a Specific Case (37%), Deduction Using a Model (21%) and Deduction Using a General Case (21%)) accounted for 79% in the Platinum textbook.

Although the identified modes of reasoning that are deductive were presented in differing proportions, the notion of deduction suggests a legitimate way to reason in Mathematics as it was argued before. Kilpatrick et al. (2001, p. 231) opinions that deductive reasoning 'is used to settle disputes and disagreements' in deductive reasoning, solutions are achieved if and when they follow assumptions and a series of logical steps that are agreed upon in the Mathematics discourse (Kilpatric et al., 2001).

The two textbooks attempt to provide explanations for every rule given to some limited extent. There are some instances where rules are presented without deductive explanations. However, it should be considered that explanations may have been given in the previous 8th grade textbook. For instance, the *Platinum* textbook simply states that plotting coordinates is done in the previous grade (appendix 2, example 1). Meanwhile the *Mind Action Series* textbook presents deductive explanations of how to plot coordinates on a Cartesian plain (appendix 1, example 1).

Stacey and Vincent (2009) advise that the preview process of concepts and rules should consist of underlying reasoning behind the rules if Mathematics is to be seen as more than rules.

It was generally found that most explanations were presented to derive rules with the aim to prepare learners for practice exercises rather than to give explanations that might be used by learners as thinking tools to solve subsequent exercises. Inclusive imperatives such as explain or prove, which normally make use of symbolic mathematical language, as it was discussed before, have the potential to engage learners to think critically. Rotman (2007) argues that a text that uses such imperatives suggests the learner to be a "thinker". It has been established that explanations across the two textbooks present instructions to learners on what to do, rather than engaging them to critical thinking. Rotman (2007) refers to such instruction as suggesting learners to be scribers.

Furthermore, exclusive imperatives such as 'write' or 'put' in explanations suggest learners to follow instructions, they are therefore referred to as 'scriber' (Rotman, 2007).

Although explanations across the two textbooks suggest learners to be scribers, a differentiated effort has been made to promote the practice of reasoning, especially in the *Mind Action Series*

(MAS). The MAS has prominent instances of the practice of reasoning in expository texts. For examples, the MAS has a section where the concept of gradient is explained elaborately (comparing the steepness of lines using the value of the vertical and horizontal change, and used representations to guide the process of how the formula of gradient is generated, (see appendix 1, enrichment section). This example, amongst a few in the *Mind Action Series*, can help with the link between the parameters of a straight-line graph (steepness and orientation of line graph) and that of a gradient. This example highlights the idea of guided discovery since the formula of a gradient is discovered using a generic example, and guide learners to see the pattern and aspects that are used to generate the formula of a gradient.

This example (enrichment section) uses deductive explanations with a model (line graphs), using generic examples with an intention to generate a formula that can be applied generally. This might highlight the author's interest to deliberately use the combination of the modes of reasoning that are deductive (DUSC, DUGC and DUM) to additively contribute to learners' understanding and reasoning skills.

In the *Platinum* textbook, an effort has been made to present explanations to promote the process aspects of mathematical reasoning. For instance, (see appendix 1, example 5) the example presents textual explanations in the form of questions, and learners are expected to search for links between representations. It can be inferred that the author has an intention to require some form of thinking from learners. However, this is the only example that presents explanations in the form of a question. Generally, the *Platinum* textbook presents explanations that are short as compared to those in the MAS. There are some instances where basic reasoning is omitted, and learners are simply reminded that a certain concept is dealt with in previous grades. This finding suggests the need for teachers to elaborate in such instances. A shortened style of explanation that is common in the *Platinum* textbook is suggestive of 'proof with few words' which is normally used by mathematicians (Nelsen, 1993). However, it should be noted that this finding highlight that these explanations are presented to Grade 9 learners, who, because of their embryotic stage of Linear Functions, might not be successful in understanding the symbolic mathematical language.

A variety of modes of reasoning cross the two textbook explanations are evident, especially the ones that are deductive. The variety is evident between different concepts as well as between the textbooks. Rotman (2007) suggests that text in explanations should consider learners as both scribers and thinkers if the motive is to develop learners' reasoning skills. Following Rotman's

argument, it may be expected that readily accessible pre-formal deductive explanations (use of everyday language) would be used to introduce concepts, and short mathematical symbolic explanations (symbolic mathematical language) can follow. However, this is not always the case, especially in the *Platinum* textbook where explanations are generally brief and use mathematical symbolism.

At Grade 9 level, learners are still at the early stages of using algebraic notations and lack full control of the language for expressing generalisation in Linear Functions since this topic is introduced in Grade 9. Where deductive using specific cases were used, this might reflect the author's interest to expect learners to see the generalisation in the generic deductive explanations given. In fact, the two textbooks try to present sections that summarise the main concepts with the intention to generalise the application of the concept learned.

Stacey and Vincent (2009) advise that deduction using a specific case and deduction using a general case are not necessarily of different value in lower grades. What is important is that learners learn that deductive explanations apply generally and not just in specific cases. The difficulty is in how the textbooks make an effort to present explanations in a manner that would help learners to transition from looking at specific cases to regarding them to be application in general cases. This, regardless of the sections meant to summarise, with intent to generalise what has been learned, the deductive explanations aimed at generalising remain limited in the main section of expository texts across the two textbooks. In other words, it is detached from the main section of expository text. It is, therefore, a concern since expository texts found in the main sections, are prioritised sections that are used to present principles of doing Mathematics to learners. In this, the summary and enrichment sections should be adopted in the main sections.

Ayalon and Even (2010) converse that deductive reasoning does not have to be limited to formal mathematical language, algorithms, and mathematical representations that could only be accessed by mathematicians. Ayalon and Even (2010) argues further that deductive reasoning can be presented in informal ways provided it complies with the agreed-upon assumptions. In the context of this research, a series of logical steps and assumptions are inherent in representational translations that are used to reason within Linear Functions. Representations include (tables / diagrams, algebraic equations and graphs). As such, the representational translations must be

studied to explore the possible reasoning opportunities that are afforded to learners in the context of Linear Functions.

A set of assumptions are adopted from Janvier's (1987) theoretical framework of representational translation to study the reasoning presented in expository texts:

- the psychological aspect of representational translations,
- the role of a source representation and a targeted representation,
- the connections between the parameters of representations.

The psychological process involved in representational translations across the two textbooks is different. By psychological process, I refer to the mathematical reasoning skills required to translate from a graph to an equation and vice versa.

The *Mind Action Series* has made it a priority to include representational translations that start with a graph as a source representation (encountered 46, 3%). The translation that proceeds from a graph to other representations has been argued by Adu-Gymfi et al. (2013) to encourage pattern discovery and interpretation-which has been identified earlier as the process aspects of mathematical reasoning while the translation that proceeds from an equation / formula to other representation accounted for 20% of representation translations presented.

In the *Platinum* textbook, a low proportion (14%) of representational translations that proceed from a graph to other representations was encountered. The translation from an equation / formula to other representations accounted for 57% of representational translations presented. The dominance of representations that proceed from an equation / formula to other representations suggests a series of straightforward steps and is most likely to promote computation of algebraic procedures that are devoid of reasoning (Adu-Gymfi et al., 2013). Janvier (1987) has cautioned against the dominant use of expository texts that use representations that proceed from an equation / formula to other representations because it limits learners to acquire mathematical reasoning skills such as interpretation, pattern discovery and conjecturing. Stylianides (2009) mentions pattern discovery and conjecturing as one of the most important mathematical reasoning activities required. Therefore, expository texts should make means to prioritise the presentation of representations that expose these activities to learners.

The findings imply that the *Mind Action Series* makes an effort to present expository texts that require learners to interpret and discover patterns. Meanwhile the *Platinum* textbook presents expository texts that are procedural and require a step-by-step execution of algorithms from an equation to graphs.

The dominance of equation to a graph translation in the *Platinum* textbook suggests that learners might rely on procedural algebraic computations even when it is not necessary to do so. This means learners are likely to lack the ability to adopt alternative procedures and methods that do not require algebraic computations but rather require interpretation. The ability to choose methods, facts and procedures appropriately according to the mathematical problem given is captured well in Kilpatric et al.'s (2001) notion of adaptive reasoning. Kilpatrick et al. asserts that adaptive reasoning could allow one to navigate through many facts, procedures, methods and concepts, and to see that they fit together to make sense given the context of the mathematical problem that need to be solved.

To sum up the findings thus far, both textbooks dominantly present the modes of reasoning that are deductive, and a greater proportion that is deductive using a specific case. This implies that instructions explained the execution of algorithms that used generic examples. Bert, Yagmur, Kulaksiz and Lithner (2016) argue that algorithms are quick, reliable and save time when a sequence of instructions explains the execution of the algorithms. The fact that algorithms are quick, means that they are designed to avoid meaning, and do not give rise to the deeper understanding of the principles of mathematical reasoning (Bert et al., 2016). Yet again, if all or almost all expository texts are presented using pre-defined procedures without considering the underlying intrinsic mathematical properties, which are representational translations and the link between properties of the representations in the context of this research, then there is a risk that learners' task-solving procedures will be based on the superficial characteristics of the tasks, leading learners to engage in the un-reflected use of those algorithms (Boesen, Helenius & Bergqvist, 2014). As such, being able to recall the algorithms in their original form, without any conceptual understanding of them, does not promote the type of learning that promotes mathematical reasoning.

It has been argued that modes of reasoning that are deductive have intentions that are intrinsic to mathematical reasoning. However, they can lead to learning without meaning. Thus, the intrinsic

mathematical properties that are characteristic to mathematical reasoning in Linear Functions: the psychological aspect of representational translations, the role of a source representation and a targeted representation, and the connections between the parameters of representations, as mentioned before are interrogated.

To get a better understanding of what the findings mean for learners to develop mathematical reasoning skills, I discuss the findings of the linguistic features used in expository texts to convey the identified modes of reasoning and representational translations. This is done with the intention to examine the level of the leaners' likely comprehension of the presented modes of reasoning in expository texts.

5.3.2. Linguistic features and the language structure of textual explanations in expository sections.

Some teaching and learning assumptions have been found to be amongst the important elements to learn mathematical reasoning. Active participation amongst other teaching and learning methods has been identified by Segerby and Chronaki (2018**)** to be an important aspect that needs to be considered when the aim is to develop learners' mathematical reasoning skills. This must be adopted in the context of a textual explanation in Mathematics textbooks to suit this research. This has, however, backed the question, how does one assess learners' level of participation suggested in textbooks' textual explanations?

The linguistic features (interpersonal imperative pronouns) are, as a result, examined to assess the level of learner participation that the explanation suggests. The following main findings are found in the two textbooks:

Consistent use of brief textual descriptions that use the everyday language in the *Platinum* textbook is evident across all examples that were analysed. This is dominantly encountered when a new concept is introduced. Thereafter, algebraic procedures and symbols are used preceding the introduced concept.

The *Mind Action Series* dominantly uses elaborate and lengthy textual descriptions using the everyday language to direct learners whenever a new concept is introduced. This is then followed by algebraic procedures and symbolism with brief textual descriptions.

The linguistic features encountered in the *Platinum* textbook explanations use interpersonal imperative pronouns such as 'you' and 'we' that suggest learner involvement in doing Mathematics. Such pronouns position the learner as an active member during the learning process.

The *Mind Action Series*, on the other hand, consistently uses interpersonal imperative pronouns that are devoid of learner involvement. Instead, the explanations are consistently used to direct learners on what to do and aspects that need focus. Such explanations suggest a passive positioning of the learner.

The use of everyday language is evident in both textbooks, especially when a new concept is introduced. The everyday language does not use algebraic symbolism; the learner, in this regard, is most likely to comprehend what the text intents to convey. In line with this finding, Österholm (2006) reveals that learners comprehend explanations that do not use algebraic symbolism better. However, Österholm (2006) has warned that explanations that dominantly use the everyday language to explain mathematical concepts run a risk of leading learners to partially understand the desired mathematical concept. Instead, explanations should use the everyday language to make sense of the desired end goal to gain access to mathematical reasoning (Österholm, 2006).

Brating et al. (2018) reckon that explanations should proceed from the psychological aspect (what learners can relate to) to the logical aspect (the desired mathematical reasoning with symbolism). In the context of the findings of the linguistic features in explanations, the everyday language seems to have been used as the psychological aspect with which learners can relate, especially in sections where new concepts are introduced. The gradual progression towards the use of algebraic symbolism in examples preceding the introduced explanation of new concepts, assumes that the learners have gained facts and other principles needed to understand the language with symbolic systems.

Rotman (1988) captures well the idea of explanations that use the everyday language then gradually move towards the formal language. Explanations, according to Rotman (1988), should consider the learner as a scriber and a thinker. The use of the everyday language in the two textbooks is used to explain concepts suggest that learners are given facts and principles that are needed to reason with and to understand the proceeding explanations. At this stage, learners are considered to be 'scribers'. The language that uses algebraic symbolism in the proceeding examples assumes learners to have some kind of knowledge, facts and principles that regard them

as ‘thinkers’. The language progression from every-day to the formal suggests learners to be both ‘scribers’ and ‘thinkers’, which is the desired goal of how explanations should be structured to promote competencies that the explanation aims to achieve.

To sum up this section, both textbooks attempt to use the language that appeals to what learners can relate to and gradually move towards a formal language. This suggests that both textbooks have tried to use textual explanations that learners could comprehend. Although these textbooks adopt different trajectories, the *Platinum* textbook used brief explanations and considers learners to be active participants while the *Mind Action Series* (MAS) uses lengthy and elaborate explanations that use the everyday language that dominantly instructs learners on what to do. Therefore, the explanations in the MAS suggest a passive position of learners.

5.3.3. The meaning of the identified modes of reasoning to learners.

Considering what has been found thus far, the *Platinum* textbook has expository texts that dominantly present modes of reasoning that are deductive. However, the representational translations that are presented encourage procedural computations. This is because of the dominant representational translation that proceeds from an equation to a graph. In this sense, process aspects of reasoning (pattern discovery and interpretation amongst others) are limited to develop the learners’ reasoning skills.

The linguistic features in expository texts have interpersonal pronouns that position learners as active members. In some instances, explanations have questions as part of the expository texts. Therefore, learners can participate in the discovery of links between representations and their parameters.

In the *Mind Action Series*, the modes of reasoning that are deductive are identified as well. However, the representational translations that are presented encourage pattern discovery and interpretation. This is because of the dominant representational translations that proceed from a graph to an equation. This is revealed in the work of Bosse et al. (2013). They found that learners find it difficult to interpret graphs and relate them to their equations. This is because the process of graph interpretation requires pattern discovery and the ability to link characters of a graph to those of equations (Bosse et al., 2013). As a result, an effort to promote mathematical reasoning to learners is made since a graph to equation translation is dominantly presented in MAS.

To explore the extent to which the above finding means for learners, the linguistic features in expository texts were examined. The analysis reveals that explanations in MAS did not make an effort to involve learners in doing Mathematics, as such; learners are regarded as passive participants who are instructed on what to do. However, the textual explanations used the everyday language that could be comprehensible to learners because less symbols are used as part of explanations.

An effort has been made to provide modes of reasoning and representational translations that encourage reasoning. Textual explanations, on the other hand, do not regard learners as active member who should be involved in doing Mathematics. In this, I argue that textual explanations in the MAS textbook are not, to some extent, structured in an ideal manner that could easily allow learners to realise the intentions of the expository texts, and the modes of reasoning and representational translations they convey, regardless of the identified modes of reasoning and representational translations that encourage mathematical reasoning.

5.4. Recommendations

5.4.1. Recommendations relating to the study.

Expository texts should present explanations that prioritise mathematical reasoning in Linear Functions by using representations and textual explanations that encourage the graph to equations / formula translation. This has the potential to promote the idea of mathematical inquiry, and identification of patterns.

Representational explanations should be accompanied by elaborate textual explanations to direct learners to important aspects of learning mathematical reasoning. The textual explanations should use the everyday language to appeal to what learners are already familiar with throughout the explanations. This is to ensure that they are not lost along the course of the knowledge sequence the explanations intend to build. Furthermore, it is reflected in Brousseau (1997) that representational and algebraic explanations could be difficult to comprehend by learners because they are meant to solve mathematical problems, and not ideal to be used for learning purposes. As a result, the linguistic features, as well as the language structure of the textual explanation, should limit the use of algebraic symbolic system and use interpersonal linguistic pronouns that involve learners in the process of learning Linear Functions. The use of pronouns such as ‘we’ and ‘you’

is important to include in textual explanations because they position the learners as active members in learning Mathematics.

Although a high proportion of modes of reasoning that are deductive have been identified in the two textbooks, having to decode these modes of reasoning for learning purposes remains an overlooked issue. A balance, therefore, is needed to make learners realise the modes of reasoning the expository text intends to convey. Recommendations could, therefore, be drawn from strengths displayed by the two textbooks' method of presenting the identified dominant modes of reasoning (deductive reasoning). Textual explanations should use questions as part of their explanation method because the learners would be encouraged to make connections and discover patterns.

Expository texts should regard learners as thinkers, and scribers. That is, to use a balanced form of textual explanations and representations that appeal to those which learners are familiar with. Thereafter, this should be used as a transition mechanism towards the desired mathematical reasoning that uses symbolic notations. The aspect of guided discovery which is evident to a limited extent across the two textbooks should be prioritised. These examples are structured in a manner that explains how rules are derived. Furthermore, guided discovery helps learners to understand how operational principles underlying the links between concepts and representations are related, and why certain translations are appropriate in certain contexts.

This would afford learners to not only rely on procedural computations to find solutions, but to also interpret and connect features found in different representations to define the same concept. The link between different features of representation would also afford learners the opportunity to use appropriate methods and trajectories that are convenient and relevant to the context of the mathematics problem given. Having to see one concept represented in different ways has the potential to afford learners to apprehend the strength and weaknesses of representations and their features in certain mathematical contexts.

5.4.2. Recommendations for further research.

The reality of South African public schools point to the fact that the majority of schools rely on textbooks as their main learning and teaching resource, especially in rural remote areas around the country. Mathematics textbooks must be written to speak to the demanding call for mathematical reasoning; this will most probably provide a platform that might enhance learners' and teachers'

mathematical reasoning skills. It is worth mentioning that reading a Mathematics text requires a specialised skill for one to comprehend the rigor of what the text intents to convey. As such, research on how to write and structure the mathematics text in explanations with intentions to develop mathematical reasoning in the context of Linear Functions must be interrogated, and mitigation measures on how to write and structure mathematical text explanations into a reader-friendly manner should be prioritised.

5.5. Limitations of the study.

This research paper interrogated two Grade 9 Mathematics textbooks on Linear Functions. As such, generalisation based on the findings of this research could not be made since one topic (small selection of topics, not all topics included in all the textbooks) was explored in the two textbooks. The selection of two textbooks out of many Mathematics textbooks might have imposed limitations because I had a choice to choose from a wider scope of textbooks.

The units of analysis in expository texts are not presented in all of the examples that were analysed, especially the unit of 'key concepts/words' and 'textual explanations. As a result, not all examples had appropriate units that could be analysed. Thus, few examples with appropriate units had to be sampled across the two textbooks. This might have limited the extent to which the interrogations of expository texts could have been carried out.

Recent articles and journals in mathematical reasoning in mathematics textbooks are limited as it was argued earlier. This has resulted in the relative use of old articles to some extent.

5.6. Concluding remarks

Explanations in documents, including textbooks, need to be interrogated given the growing need for E-learning which might lead to independent learning, and or remote learning which requires amongst others, learning from a text. As such, expository texts should be carefully written to encourage a better comprehension of the mathematical reasoning principle embedded in them. In the context of Linear Functions, representations are important to reasoning in this area of Mathematics. In this, a balance from a graph to an equation translation and vice versa should be prioritised, and textual explanations should regard the reader as a scriber, a thinker, and an active member to encourage comprehension of the text and its intentions. The analysed textbooks displayed differing strengths and weaknesses in their attempt to explain Linear Functions. As such,

using one of the two textbooks individually might disadvantage the reader from developing mathematical reasoning competencies. It was then suggested, after having discussed the major findings in the two textbooks, that it would be best to combine the strengths of the two textbooks. In this manner, the reader might gain access to the modes of reasoning implied in expository texts

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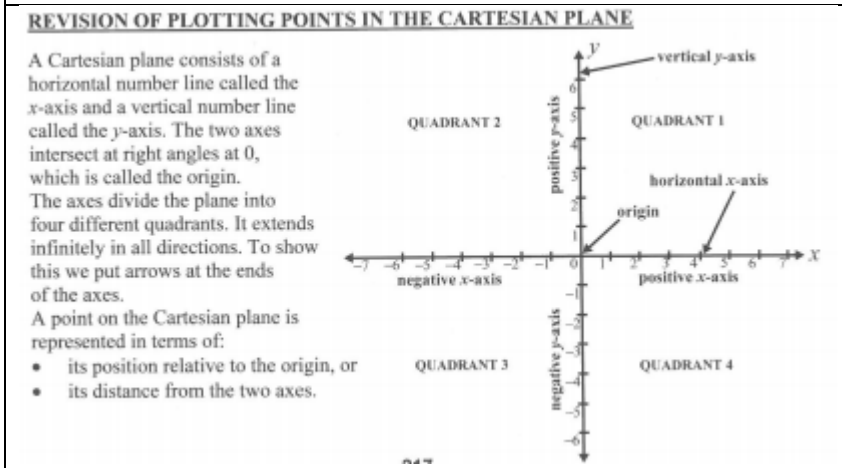
Appendices

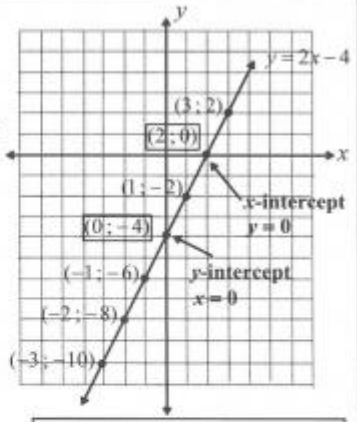
Textbook: Mind Action Series. Appendix 1

1. Revision of plotting points in the Cartesian plane.

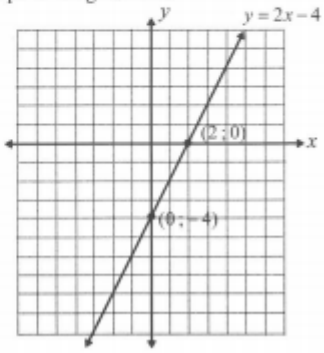
Example Number 1	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text	ATA	
	• Explanation of a workedout example (symbolic algebraic form)	DUM	**EG0-EG7**
	• Symbolic algebraic procedure	N/A	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	

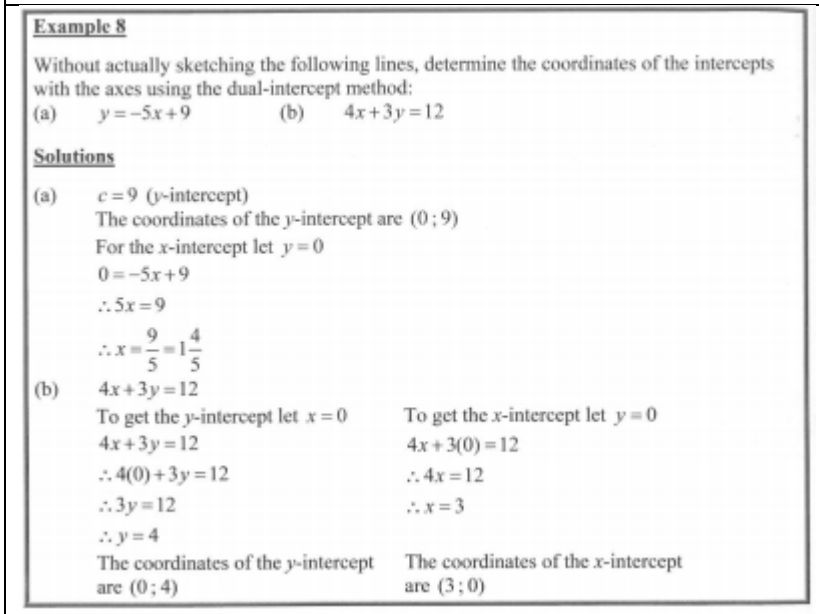
2. Sketching linear graphs.

Example Number 2	**Expository section**	**Mode of reasoning**	**Representational translation**

<table>
<tr>
<td><u>Example 5</u>
Sketch the graph of $y = 2x - 4$.
<u>Solution</u>
Let's select $x \in \{-3; -2; -1; 0; 1; 2; 3\}$ and then substitute these x-values into the equation to get the corresponding y-values.
For $x = -3$ $y = 2(-3) - 4 = -10$
For $x = -2$ $y = 2(-2) - 4 = -8$
For $x = -1$ $y = 2(-1) - 4 = -6$
For $x = 0$ $y = 2(0) - 4 = -4$ (y-intercept)
For $x = 1$ $y = 2(1) - 4 = -2$
For $x = 2$ $y = 2(2) - 4 = 0$ (x-intercept)
For $x = 3$ $y = 2(3) - 4 = 2$
Fill these values in a table, plot the points on a Cartesian plane and then draw the line.
<table>
<tr><td>x</td><td>−3</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>y</td><td>−10</td><td>−8</td><td>−6</td><td>−4</td><td>−2</td><td>0</td><td>2</td></tr>
</table>


There are many more points that can be found. All these points form the graph of the straight line.
Notice that where the line cuts the y-axis, the x-value is 0 and where the line cuts the x-axis, the y-value is 0. It is possible to sketch a line if we know the intercepts with the axes.
To get the y-intercept, we let $x = 0$ in the equation and solve for y.
To get the x-intercept, we let $y = 0$ in the equation and solve for x.
This method is called the dual-intercept method and it is discussed in the next example.</td>
<td>• Introductory text</td>
<td>ATA</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Explanation of a worked out example (symbolic algebraic form)</td>
<td>DUSC</td>
<td>EG3-EG6-EG8</td>
</tr>
<tr>
<td></td>
<td>• Symbolic algebraic procedure</td>
<td>DUSC</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Key word/key concept</td>
<td>DUGC</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Concluding explanatory text</td>
<td>DUM</td>
<td></td>
</tr>
<tr>
<td>Example Number 3</td>
<td>Expository section</td>
<td>Mode of reasoning</td>
<td>Representational translation</td>
</tr>
<tr>
<td></td>
<td>• Introductory text</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Explanation of a worked out example (symbolic algebraic form)</td>
<td>DUSC</td>
<td>EG3-EG9</td>
</tr>
<tr>
<td></td>
<td>• Symbolic algebraic procedure</td>
<td>DUSC</td>
<td></td>
</tr>
</table>

<table>
<tr><td>Example 6
Sketch the graph of $y = 2x - 4$ using the dual-intercept method.
Solution
From the previous example, we know the following:
To get the y-intercept let $x = 0$
To get the x-intercept let $y = 0$
Using this dual-intercept method, let's draw the graph of the given line.
To get the y-intercept let $x = 0$
$y = 2x - 4$
$\therefore y = 2(0) - 4$
$\therefore y = -4$
The coordinates of the y-intercept are $(0\,;-4)$
To get the x-intercept let $y = 0$
$y = 2x - 4$
$\therefore 0 = 2x - 4$
$\therefore 4 = 2x$
$\therefore x = 2$
The coordinates of the x-intercept are $(2\,;0)$
Using the two intercept points, we can draw the graph of the line.
</td><td>• Key word/key concept</td><td>N/A</td><td></td></tr>
<tr><td></td><td>• Concluding explanatory text</td><td>DUSC</td><td></td></tr>
<tr><td>Example Number 4</td><td>Expository section</td><td>Mode of reasoning</td><td>Representational translation</td></tr>
<tr><td>• Concluding explanatory text</td><td>N/A</td><td></td><td></td></tr>
<tr><td></td><td>• Explanation of a worked out example (symbolic algebraic form)</td><td>ATA</td><td>EG3-EG6-EG8</td></tr>
<tr><td></td><td>• Symbolic algebraic procedure</td><td>DUSC</td><td></td></tr>
<tr><td></td><td>• Key word/key concept</td><td>N/A</td><td></td></tr>
</table>

	•		
Example Number 5	**Expository section**	**Mode of reasoning**	**Representational translation**
 **Example 8** Without actually sketching the following lines, determine the coordinates of the intercepts with the axes using the dual-intercept method: (a) $y=-5x+9$ (b) $4x+3y=12$ **Solutions** (a) $c=9$ (y-intercept) The coordinates of the y-intercept are $(0;9)$ For the x-intercept let $y=0$ $0=-5x+9$ $\therefore 5x=9$ $\therefore x=\frac{9}{5}=1\frac{4}{5}$ (b) $4x+3y=12$ To get the y-intercept let $x=0$ $4x+3y=12$ $\therefore 4(0)+3y=12$ $\therefore 3y=12$ $\therefore y=4$ The coordinates of the y-intercept are $(0;4)$ To get the x-intercept let $y=0$ $4x+3(0)=12$ $\therefore 4x=12$ $\therefore x=3$ The coordinates of the x-intercept are $(3;0)$	• Introductory text	**N/A**	
	• Explanation of a workedout example (symbolic algebraic form)	**DUSC**	**EG6** **NO translation**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	

Summary: Representational translation modes or reasoning.

EG3-EG6-EG8	**From recognizing perimeters of an equation- to filling a table- and then to sketching a graph**
EG3-EG9	**From recognizing perimeters of an equation – to sketching a graph.**
EG3-EG6-EG8	**From recognizing perimeters of an equation- to filling a table- and then to sketching a graph**
EG6	**Computing- no translation was observed.**

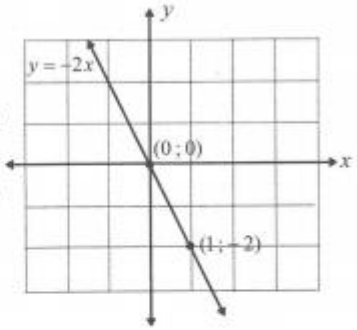
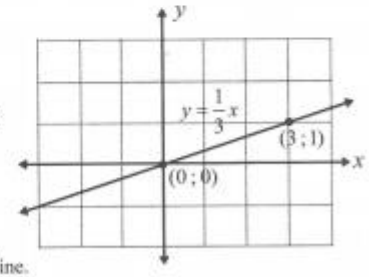
Note: Two codes show direct translation

Three or more codes show indirect translation.

3. Sketching linear graphs using the point intercept method.

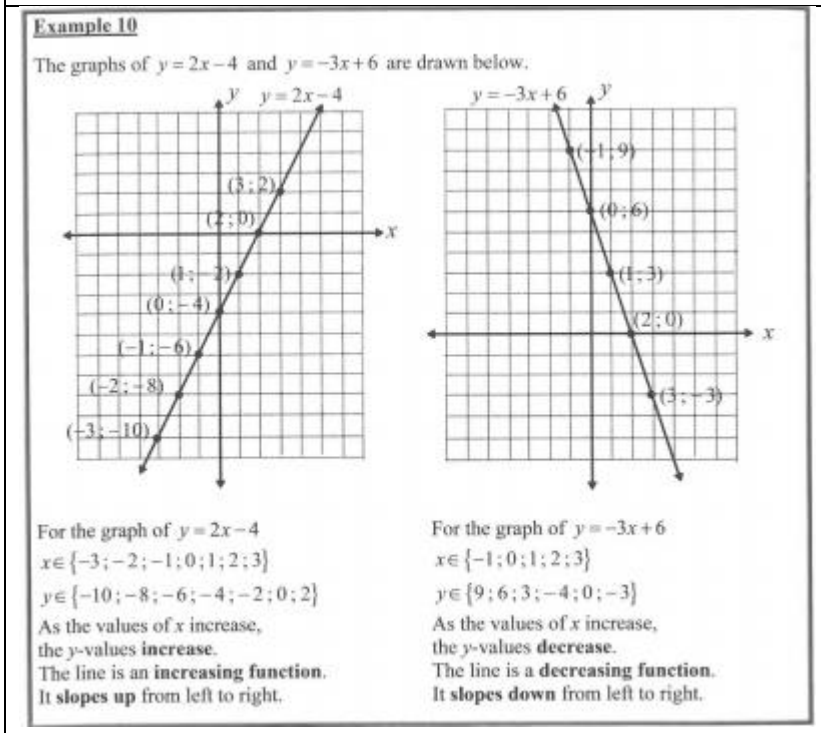
Sketching linear graphs using the intercept method.

Example Number 6	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text	**ATA**	
	• Explanation of a workedout example (symbolic algebraic form)	**DUSC**	**EG0-EG10-EG9**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	

Example 9 Sketch the graphs of the following lines: (a) $y=-2x$ (b) $y=\frac{1}{3}x$ **Solutions** (a) $y=-2x+0$ (Add in 0) The y-intercept is $c=0$ The x-intercept is therefore also 0. The dual-intercept method is not helpful here at all since you need two different intercepts to be able to draw the graph. With the point-intercept method, simply select any value for x and substitute it into the equation to get the y-value. Choose $x=1$ say. $\therefore y=-2(1)+0=-2$ Another point on the line is $(1;-2)$ Plot the two points on a Cartesian plane and then draw the line.  (b) $y=\frac{1}{3}x+0$ (Add in 0) The y-intercept is $c=0$ Now select a value for x and substitute into the equation to get the y-value. Choose $x=3$ $\therefore y=\frac{1}{3}(3)+0=1$ Another point on the line is $(3;1)$ Plot the two points and then draw the line.  Notice that we chose $x=3$ to avoid getting a fraction for y. By the way, you could have also used the table method to draw the graph. However, the point-intercept method saves a lot of time.	• Concluding explanatory text	**DUSC**	

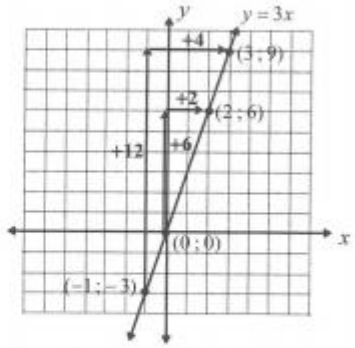
4. Increasing and decreasing linear graphs.

Increasing and decreasing graph.

Example Number 7	Expository section	Mode of reasoning	Representational translation
	• Introductory text	**ATA**	
	• Explanation of a worked out example (symbolic algebraic form)	**DUM**	**EG9-EG6-EG0**
	• Symbolic algebraic procedure	**N/A**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	

5. Gradient of a linear graph

Example Number 8	Expository section	Mode of reasoning	Representational translation

<table>
<tr>
<td rowspan="5"><u>Gradient of a line</u>
The gradient (or slope) of a line is defined to be the rate at which the y-values change as the x-values change. It is expressed as the ratio $\frac{\text{change in } y\text{-values (vertical change)}}{\text{change in } x\text{-values (horizontal change)}}$.
<u>Example 11</u>
Consider the graph of the line $y = 3x$.
Consider the points $(-1;-3)$ and $(3;9)$:
Moving from left to right, the y-values increase by 12 (from -3 to 9)
and the x-values increase by 4 (from -1 to 3)
$\therefore \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{+12}{+4} = 3$
Consider the points $(0;0)$ and $(2;6)$:
Moving from left to right, the y-values increase by 6 (from 0 to 6)
and the x-values increase by 2 (from 0 to 2)
$\therefore \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{+6}{+2} = 3$


Notice that the value of the gradient is the same (constant) no matter which two points on the line are used. The coefficient of x in the equation of the line $y = 3x$ represents the gradient. Since the y-values increase as the x-values increase, the line is increasing. The gradient is also positive.
Let's summarise these findings:
• Between any two points on the line the gradient (slope) is the same or constant.
• The coefficient of x in the equation of the line represents the gradient.
• The line is increasing (slopes up) as read from left to right since the y-values increase as the x-values increase. The gradient is positive.</td>
<td>• Introductory text</td>
<td>ATA</td>
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<tr>
<td>• Explanation of a workedout example (symbolic algebraic form)</td>
<td>DUM</td>
<td>EG0-EG10-EG7</td>
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<td>• Symbolic algebraic procedure</td>
<td>DUSC</td>
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<td>• Key word/key concept</td>
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<td>• Concluding explanatory text</td>
<td>DUGC</td>
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<tr>
<td>Example Number 9</td>
<td>Expository section</td>
<td>Mode of reasoning</td>
<td>Representational translation</td>
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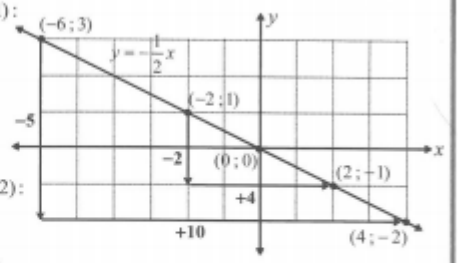
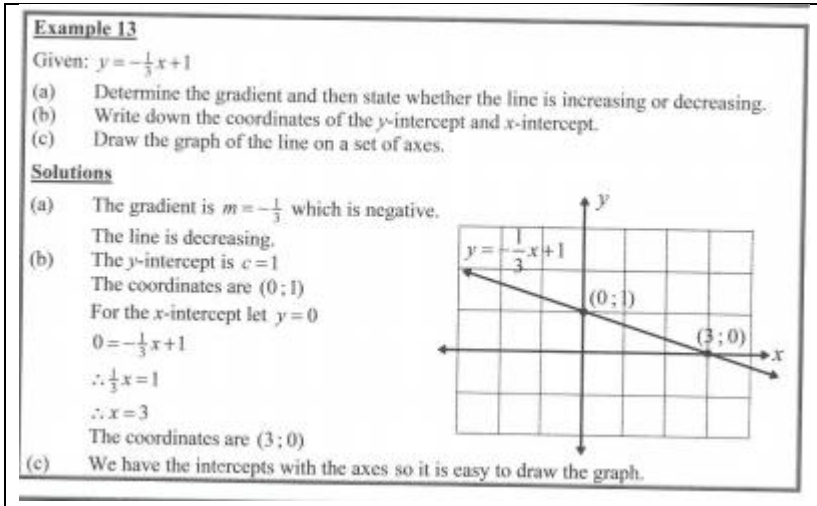
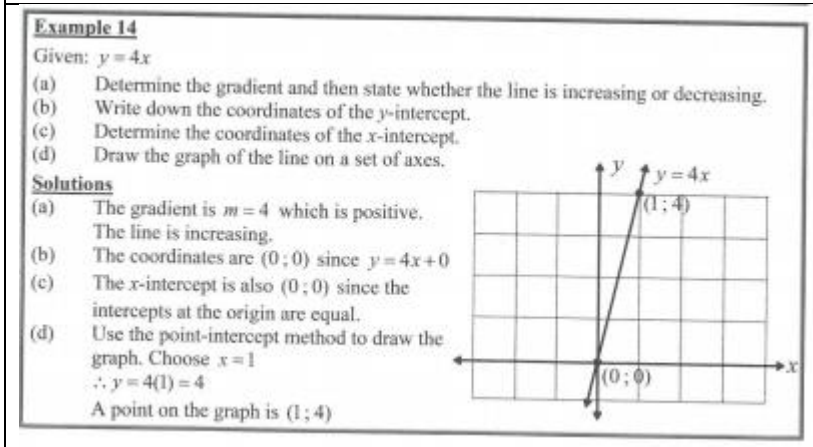
<td>EG9-EG0-EG3</td>
</tr>
<tr>
<td>• Symbolic algebraic procedure</td>
<td>DUSC</td>
<td></td>
</tr>
</table>

Example 12 Consider the graph of the line $y = -\frac{1}{2}x$. Consider the points $(-2;1)$ and $(2;-1)$: Moving from left to right, the y-values decrease by 2 (from 1 to -1) and the x-values increase by 4 (from -2 to 2) $\therefore \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{-2}{+4} = -\frac{1}{2}$ Consider the points $(-6;3)$ and $(4;-2)$: Moving from left to right, the y-values decrease by 5 (from 3 to -2) and the x-values increase by 10 (from -6 to 4) $\therefore \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{-5}{10} = -\frac{1}{2}$ Notice that the value of the gradient is the same (constant) no matter which two points on the line are used. The coefficient of x in the equation of the line $y = -\frac{1}{2}x$ represents the gradient. Since the y-values decrease as the x-values increase, the line is decreasing. The gradient is also negative. 	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**DUGC**	
Example Number 10	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text	**N/A**	
	• Explanation of a workedout example (symbolic algebraic form)	**DUSC**	**EG3-EG9**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	

	• Concluding explanatory text	**N/A**	
Example Number 11	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text		
	• Explanation of a worked out example (symbolic algebraic form)	**DUSC**	**EG3-EG9**
	• Symbolic algebraic procedure	**N/A**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	

Example Number 12	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text	**N/A**	**EG3-EG9**
	• Explanation of a workedout example (symbolic algebraic form)	**DUSC**	
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	

Example 15 (Rewriting equations in the form $y = mx + c$)

Given: $2x + 3y = 6$

(a) Determine the gradient and then state whether the line is increasing or decreasing.
(b) Determine the coordinates of the y-intercept.
(c) Determine the coordinates of the x-intercept.
(d) Draw the graph of the line on a set of axes.

Solutions

(a) The given equation is not in the form $y = mx + c$ so it is not possible to determine the gradient. However, if we get y on its own it will be possible to find the gradient.

$2x + 3y = 6$

$\therefore 3y = -2x + 6$

$\therefore \frac{3y}{3} = \frac{-2x}{3} + \frac{6}{3}$

$\therefore y = -\frac{2}{3}x + 2$

The gradient is $m = -\frac{2}{3}$ which is negative. The line is decreasing.

(b) There are two equations now available to use when finding the y-intercept.

$2x + 3y = 6$ or $y = -\frac{2}{3}x + 2$. You can use either one of these equations.

• Using $2x + 3y = 6$:

For the y-intercept let $x = 0$

$2(0) + 3y = 6$

$\therefore 3y = 6$

$\therefore y = 2$

The coordinates are $(0\,;2)$

For the x-intercept let $y = 0$

$2x + 3(0) = 6$

$\therefore 2x = 6$

$\therefore x = 3$

The coordinates are $(3\,;0)$

• Using $y = -\frac{2}{3}x + 2$:

The y-intercept is $c = 2$.

The coordinates are $(0\,;2)$

For the x-intercept let $y = 0$

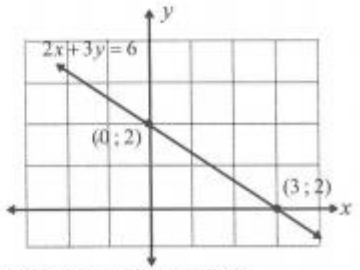
$0 = -\frac{2}{3}x + 2$

$\therefore 0 = -2x + 6$

$\therefore 2x = 6$

$\therefore x = 3$

The coordinates are $(3\,;0)$



(d) We have the intercepts with the axes so it is easy to draw the graph.

	• Concluding explanatory text	**N/A**	

Summary: Representational translations modes of reasoning

MIND ACTION SERIES	
Translation	**Translation between representations**
EG0-EG10-EG7	**From descriptions in words, to analytical modeling of a formula, to modeling a graph**
EG9-EG0-EG3	**From a sketched graph- to descriptions in words, to recognizing parameters of a formula.**
EG3-EG9	**From recognition of parameters of a formula to sketching a graph.**
EG3-EG9	**From recognition of parameters of a formula to sketching a graph.**
EG3-EG9	**From recognition of parameters of a formula to sketching a graph.**

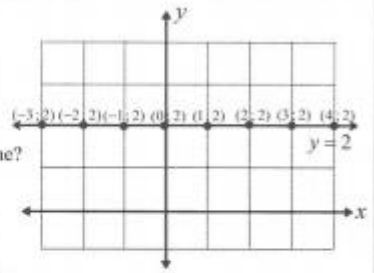
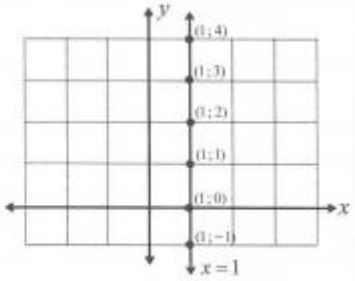
Note: Two codes show direct translation

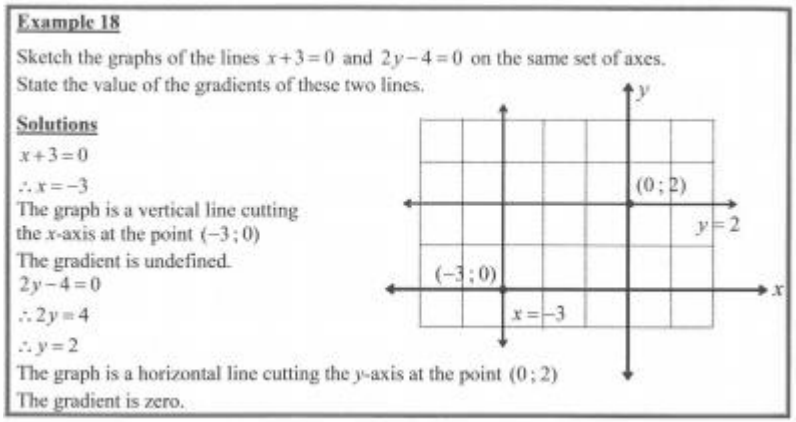
Three or more codes show indirect translation.

5. Horizontal and vertical lines

.

Example Number 13	Expository section	Mode of reasoning	Representational translation
	• Introductory text	**N/A**	

Horizontal and vertical lines **Example 16** Consider the points shown on the given graph. (a) What do you notice about the y-values? (b) What do you notice about the x-values? (c) Write down the equation of the line joining these points. (d) What is the gradient of this horizontal line? **Solutions** (a) The y-values are the same value ($y = 2$) (b) The x-values are different and $x \in R$ (c) The equation of the horizontal line joining the points is written as $y = 2$. We say that the equation of the line is $y = 2$ (d) Take any two points on the line. (1 ; 2) and (3 ; 2) $\therefore \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{0}{2} = 0$ The gradient of the horizontal line $y = 2$ is zero. 	• Explanation of a workedout example (symbolic algebraic form)	**DUM**	**EG2-EG12**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	
Example Number 14	**Expository section**	**Mode of reasoning**	**Representational translation**
Example 17 Consider the points shown on the given graph. (a) What do you notice about the x-values? (b) What do you notice about the y-values? (c) Write down the equation of the line joining these points. (d) What is the gradient of this vertical line? **Solutions** (a) The x-values are the same value ($x = 1$) (b) The y-values are different and $y \in R$ (c) The equation of the vertical line joining the points is written as $x = 1$. We say that the equation of the line is $x = 1$ 	• Introductory text	**N/A**	
	• Explanation of a workedout example (symbolic algebraic form)	**DUM**	**EG2-EG12**
	• Symbolic algebraic procedure	**N/A**	
	• Key word/key concept		

	• Concluding explanatory text	**DUGC**	
Example Number 15	**Expository section**	**Mode of reasoning**	**Representational translation**
 **Example 18** Sketch the graphs of the lines $x+3=0$ and $2y-4=0$ on the same set of axes. State the value of the gradients of these two lines. **Solutions** $x+3=0$ $\therefore x=-3$ The graph is a vertical line cutting the x-axis at the point $(-3;0)$ The gradient is undefined. $2y-4=0$ $\therefore 2y=4$ $\therefore y=2$ The graph is a horizontal line cutting the y-axis at the point $(0;2)$ The gradient is zero. y $(0;2)$ $y=2$ $(-3;0)$ x $x=-3$	• Introductory text	**N/A**	
	• Explanation of a workedout example (symbolic algebraic form)	**ATA**	**EG3-EG9**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	

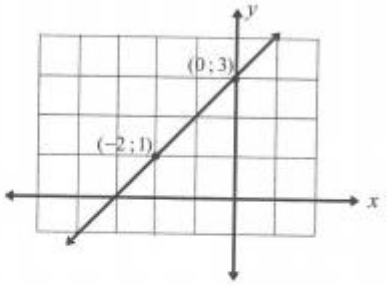
Summary: Representational translation.

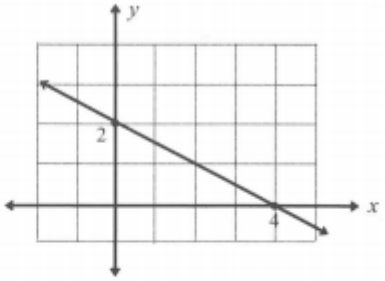
MIND ACTION SERIES	
Translation	**Translation between representations**
EG2-EG12	**From interpreting graph in words to determining the equation of a linear graph.**
EG2-EG12	**From interpreting graph in words to determining the equation of a linear graph.** **From recognition of parameters of a linear graph to sketching a graph a graph.**
EG3-EG9	

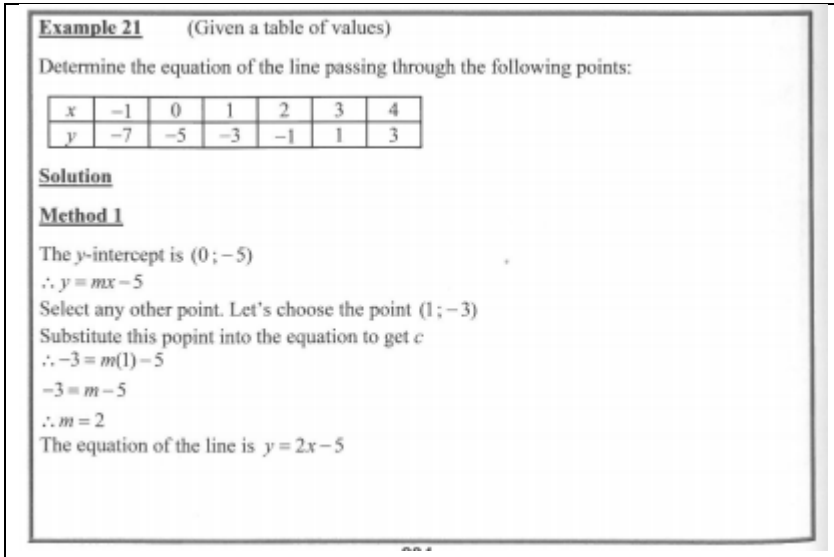
6. Finding the equation of linear graphs given the points on the graph.

Equation of a linear graph given the points on the graph.

Example Number 16	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text		
	• Explanation of a workedout example (symbolic algebraic form)	**DUSC**	**EG2-EG3-EG12**
	• Symbolic algebraic procedure	**DUSC**	

<table>
<tr><td rowspan="2"><u>Finding the equations of lines given the points on the graph</u>
<u>Example 19</u> (Given the y-intercept and a point)
The graph of a line is given. Determine the equation of the line.
<u>Solution</u>
<u>Method 1</u>
The y-intercept is 3.
$\therefore y = mx + 3 \quad (c = 3)$
The gradient is the ratio
$\frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{2}{2} = 1$
$\therefore m = 1$
The equation of the line is $y = 1x + 3$
or simply $y = x + 3$
<u>Method 2</u>
The y-intercept is 3.
$\therefore y = mx + 3 \quad (c = 3)$
To get the gradient (m) substitute the point $(-2\,;1)$ into the equation and solve for m.
$\therefore y = mx + 3$
$\therefore 1 = m(-2) + 3$ since $x = -2$ and $y = 1$
$\therefore 1 = -2m + 3$
$\therefore 2m = 2$
$\therefore m = 1$
The equation of the line is $y = x + 3$
</td><td>• Key word/key concept</td><td>N/A</td><td></td></tr>
<tr><td>• Concluding explanatory text</td><td>N/A</td><td></td></tr>
<tr><td>Example Number 17</td><td>Expository section</td><td>Mode of reasoning</td><td>Representational translation</td></tr>
<tr><td rowspan="2"></td><td>• Introductory text</td><td></td><td></td></tr>
<tr><td>• Explanation of a workedout example (symbolic algebraic form)</td><td>DUM</td><td>EG2-EG3-EG12</td></tr>
</table>

Example 20 (Given the y-intercept and the x-intercept) The graph of a line is given. Determine the equation of the line.  **Method 1** The y-intercept is 2. $\therefore y = mx + 2 \quad (c = 2)$ The gradient is the ratio $\frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \frac{-2}{4} = -\frac{1}{2}$ $\therefore m = -\frac{1}{2}$ The equation of the line is $y = -\frac{1}{2}x + 2$ **Method 2** The y-intercept is 2. $\therefore y = mx + 2 \quad (c = 2)$ To get the gradient (m) substitute the x-intercept $(4\,;0)$ into the equation and solve for m. $\therefore y = mx + 2$ $\therefore 0 = m(4) + 2$ since $x = 4$ and $y = 0$ $\therefore 0 = m(4) + 2$ $\therefore 0 = 4m + 2$ $\therefore -4m = 2$ $\therefore m = -\frac{2}{4} = -\frac{1}{2}$ The equation of the line is $y = -\frac{1}{2}x + 2$	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	
Example Number 18	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text		
	• Explanation of a workedout example (symbolic algebraic form)	**DUSC**	**EG1-EG6-EG13**
	• Symbolic algebraic procedure	**DUSC**	

 **Example 21** (Given a table of values) Determine the equation of the line passing through the following points: x: −1, 0, 1, 2, 3, 4 y: −7, −5, −3, −1, 1, 3 **Solution** **Method 1** The y-intercept is $(0;-5)$ $\therefore y = mx - 5$ Select any other point. Let's choose the point $(1;-3)$ Substitute this popint into the equation to get c $\therefore -3 = m(1) - 5$ $-3 = m - 5$ $\therefore m = 2$ The equation of the line is $y = 2x - 5$	• Key word/key concept • Concluding explanatory text	**N/A** **N/A**	

Multiple representational translation.

MIND ACTION SERIES	
Translation	**Translation between representations**
EG2-EG3-EG12	**From interpreting graph in words t recognizing parameters of a linear equation to determining the equation of a linear graph.**
EG2-EG3-EG12	**From interpreting graph in words t recognizing parameters of a linear equation to determining the equation of a linear graph.**
EG1-EG6-EG13	**From reading table to computing and fitting on a linear equation to determine the linear equation.**

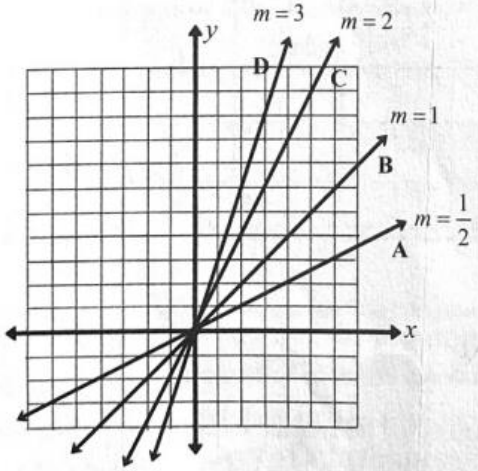
Summary of representational translations modes of reasoning.

MIND ACTION SERIES	
Translation	**Translation between representations (From source to a target representation)**
Indirect translations	.
EG0-EG10-EG7	**From descriptions in words to modelling on a graph and sketching a graph.**
EG0-EG10-EG9	**From reading from a table to fitting values on the linear equation.**
EG1-EG6-EG13	**From interpreting a graph to determining the equation of a linear graph.**
EG2-EG3-EG12	**From a sketched graph to recognizing parameters of a linear equation**
EG2-EG3-EG12	
EG9-EG0-EG3	
EG9-EG6-EG0	**From descriptions in words to modelling on a linear graph**
Direct translations	**From interpreting a graph to determining a linear equation**
EG0-EG7	
EG2-EG12	
EG2-EG12	**From recognizing parameters of a linear equation to sketching a linear graph.**
EG3-EG9	
EG3-EG9	
EG3-EG9	**No translation recorded.**
EG3-EG9	
EG3-EG9	
EG6	

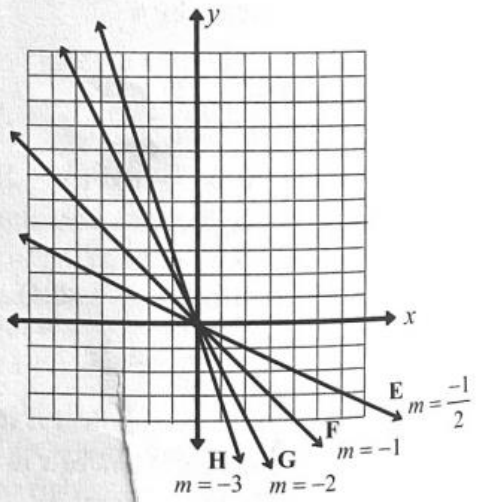
Enrichment section

The steepness of a line (optional enrichment if time permits)

Before discussing the steepness of a line, let's briefly summarise what we know about the gradient of a line thus far. The graphs of the lines in the previous exercise are drawn on the same set of axes below. Graphs A – D will be drawn on one set of axes and graphs E – G will be drawn on another set of axes.

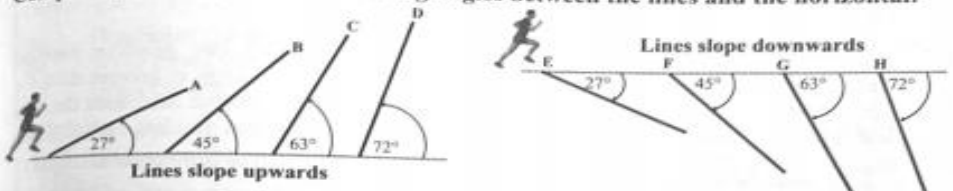


The gradients of the lines are positive and the lines are increasing from left to right.



The gradients of the lines are negative and the lines are decreasing from left to right.

Comparing the steepness of lines using angles between the lines and the horizontal:

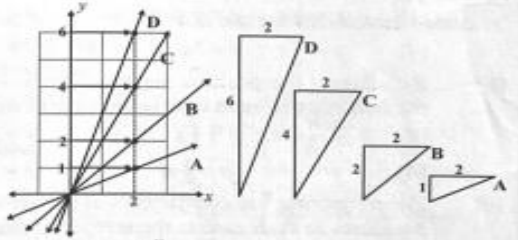


As the angle between the lines and the horizontal (x-axis) increase in size, the steeper the lines become. Line D is steeper than line C which, in turn is steeper than line B. Line B is steeper than line A. The athlete would find it easy to run up line A, less easy to run up line B. However, due to lines C and D being too steep upwards, it would be impossible for him to run up these lines. Likewise, line H is steeper than line G which, in turn, is steeper than line F. Line F is steeper than line E. The athlete would find it easy to run down line E, less easy to run down line F. However, due to lines G and H being too steep downwards, it would be dangerous for him to run down these lines!

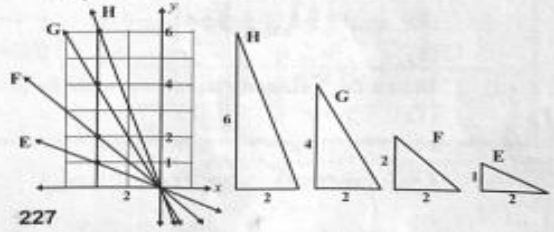
Comparing the steepness of lines using the values of the vertical and horizontal changes:

(a) You can compare the steepness of different lines by **comparing the positive value of the vertical changes**. The horizontal changes must be kept constant.
The gradients of the previous lines will be changed into fractions that can be used to compare the positive vertical changes. Negative signs are ignored for lines E – H.

For line A: $m = \frac{1}{2}$ $(\frac{1}{2})$
For line B: $m = 1$ $(\frac{2}{2})$
For line C: $m = 2$ $(\frac{4}{2})$
For line D: $m = 3$ $(\frac{6}{2})$



For line E: $m = -\frac{1}{2}$ $(\frac{1}{2})$
For line F: $m = -1$ $(\frac{2}{2})$
For line G: $m = -2$ $(\frac{4}{2})$
For line H: $m = -3$ $(\frac{6}{2})$



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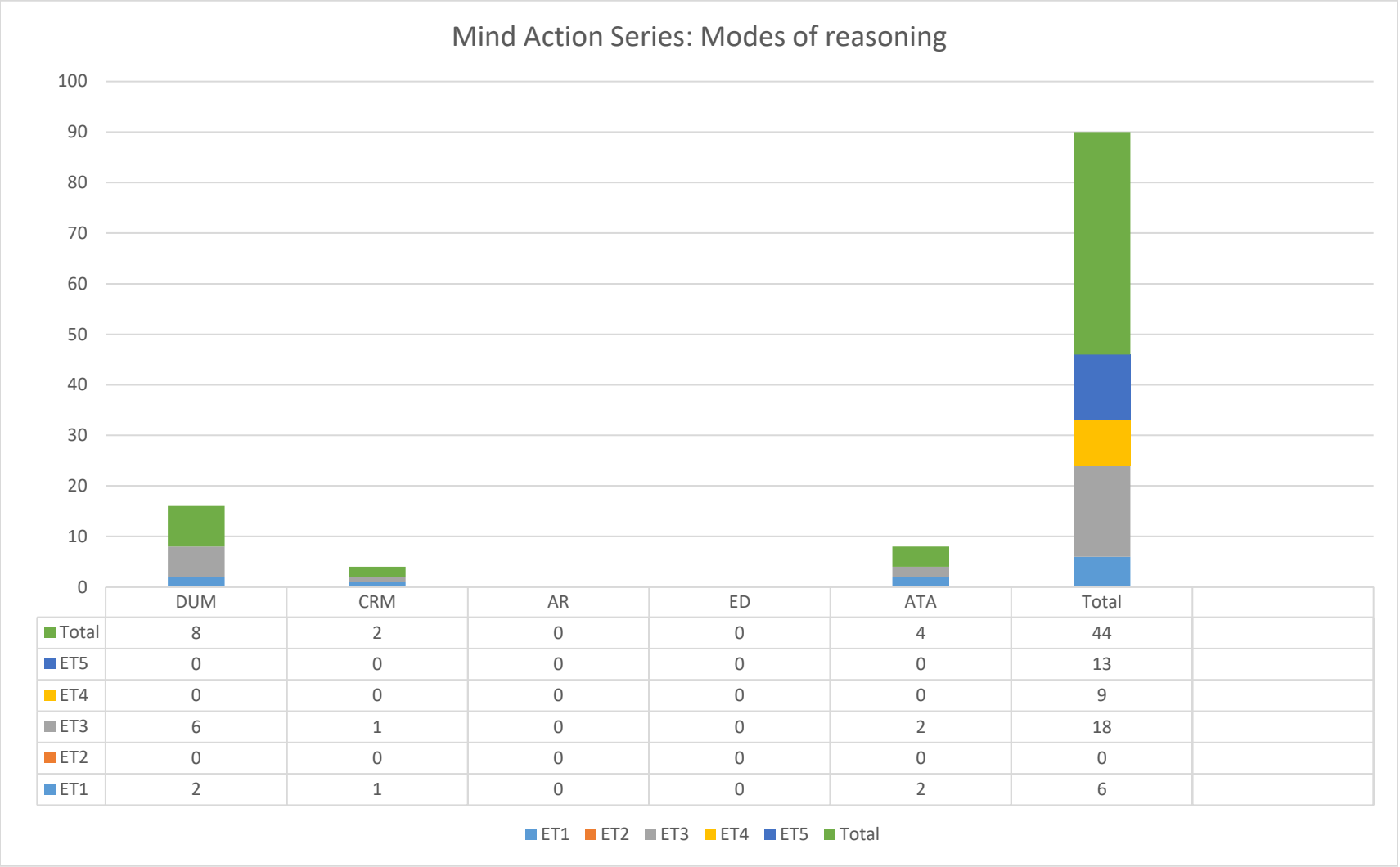
Summary modes of modes of reasoning



	DUM	CRM	AR	ED	ATA	Total	
Total	8	2	0	0	4	44	
ET5	0	0	0	0	0	13	
ET4	0	0	0	0	0	9	
ET3	6	1	0	0	2	18	
ET2	0	0	0	0	0	0	
ET1	2	1	0	0	2	6	

Textbook analysis: Platinum. Appendix 2

A total of 6 examples where analysed. The analysis was conducted according to the sequence of sub-topic as follows:

- Gradient
- Standard form of a straight line $y = mx + c$
- Lines that are parallel to the axes
- Draw linear graphs

1. Investigating linear graphs.

Example Number 1	Expository section	Mode of reasoning	Representational translation
	• Introductory text	**ATA**	
	• Explanation of a workedout example (symbolic algebraic form)	**DUGC**	**EG0-EG6-EG8**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	

Investigate linear graphs

Straight line graphs are also called linear graphs, as there is a linear relationship between the dependent and independent variables. In Grade 8 we used tables to help us plot and sketch graphs. We plot these points on a Cartesian plane and each point we plot is called a coordinate pair $(x; y)$. These coordinate pairs are determined by the equation of the graph.

Example

Complete the table below for x- and y-values of the equation $y = x + 1$:

x	−2		0	1	
y		0			3

Answers

We complete the table by substituting the values of x or y into the equation and then solving for the unknown variable:

For $x = -2$

$\therefore y = -2 + 1 = -1$

For $y = 0$

$0 = x - 1$

$\therefore x = 1$

For $x = 0$

$\therefore y = 0 + 1 = 1$

For $x = 1$

$\therefore y = 1 + 1 = 2$

For $y = 3$

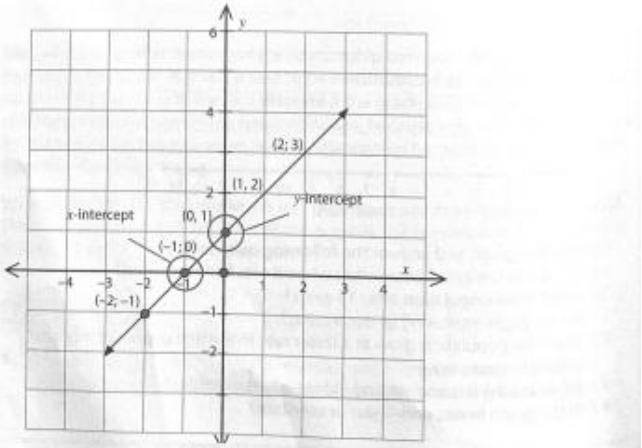
$3 = x + 1$

$\therefore x = 2$

The completed table will then look like this:

x	−2	−1	0	1	2
y	−1	0	1	2	3

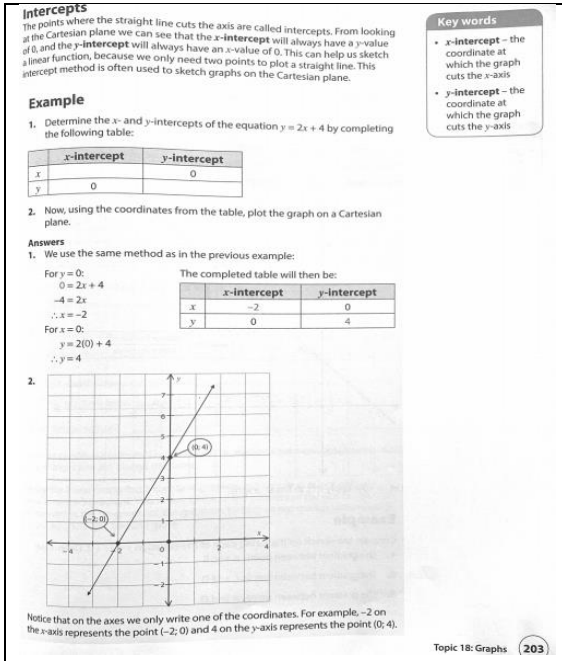
Using the table from the example we can plot the graph on a Cartesian plane. Notice the intercept points and their coordinates.



• Concluding explanatory text	**DUM**	

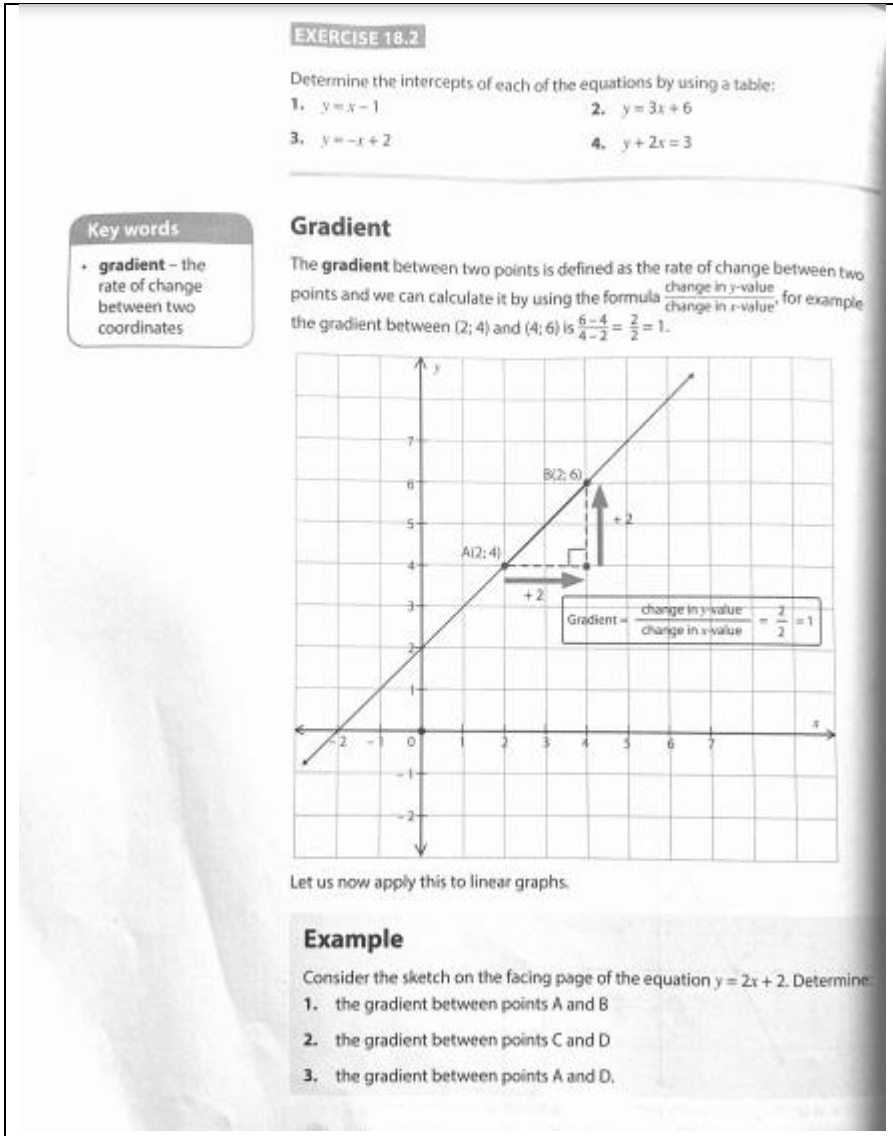
2. Intercepts

Example Number 2	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text	**DUGC**	
	• Explanation of a worked out example (symbolic algebraic form)	**N/A**	**EG0-EG6-EG8**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**ATA**	

 **Intercepts** The points where the straight line cuts the axis are called intercepts. From looking at the Cartesian plane we can see that the ***x*-intercept** will always have a y-value of 0, and the ***y*-intercept** will always have an x-value of 0. This can help us sketch a linear function, because we only need two points to plot a straight line. This intercept method is often used to sketch graphs on the Cartesian plane. **Key words** • ***x*-intercept** – the coordinate at which the graph cuts the x-axis • ***y*-intercept** – the coordinate at which the graph cuts the y-axis **Example** 1. Determine the x- and y-intercepts of the equation $y = 2x + 4$ by completing the following table: 2. Now, using the coordinates from the table, plot the graph on a Cartesian plane. **Answers** 1. We use the same method as in the previous example: For $y = 0$: $0 = 2x + 4$ $-4 = 2x$ $\therefore x = -2$ For $x = 0$: $y = 2(0) + 4$ $\therefore y = 4$ The completed table will then be: 2. Notice that on the axes we only write one of the coordinates. For example, –2 on the x-axis represents the point (–2; 0) and 4 on the y-axis represents the point (0; 4). Topic 18: Graphs 203	• Concluding explanatory text	**DUSC**	

3. Gradient

Example Number 3	**Expository section**	**Mode of reasoning**	**Representational translation**
	• Introductory text	**DUM**	



EXERCISE 18.2

Determine the intercepts of each of the equations by using a table:

1. $y = x - 1$
2. $y = 3x + 6$
3. $y = -x + 2$
4. $y + 2x = 3$

Key words

- **gradient** – the rate of change between two coordinates

Gradient

The **gradient** between two points is defined as the rate of change between two points and we can calculate it by using the formula $\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}}$, for example the gradient between (2; 4) and (4; 6) is $\frac{6-4}{4-2} = \frac{2}{2} = 1$.

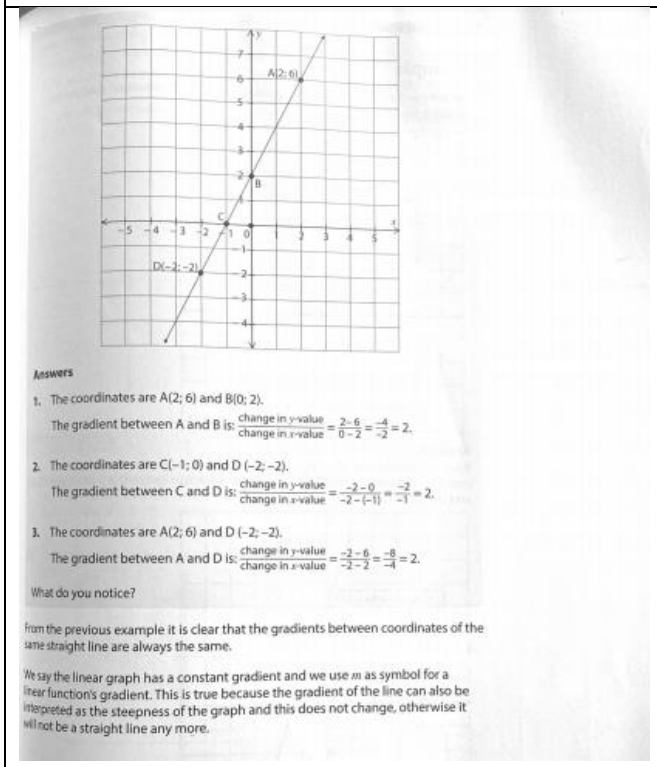
Let us now apply this to linear graphs.

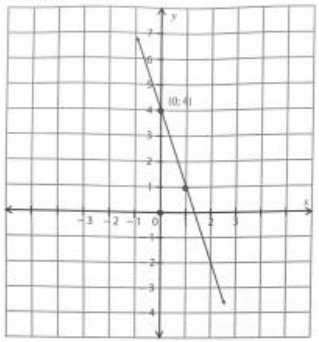
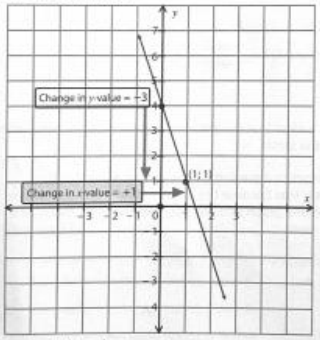
Example

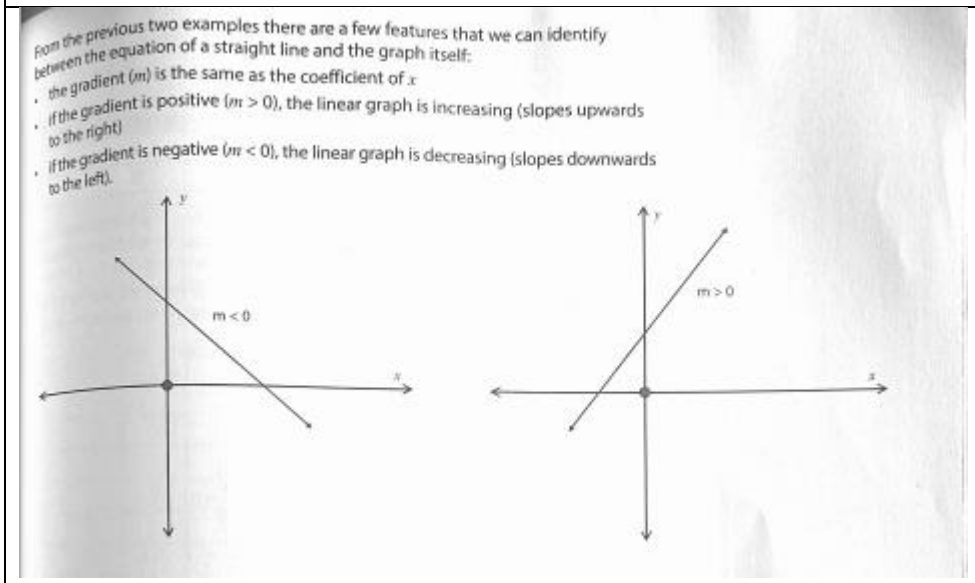
Consider the sketch on the facing page of the equation $y = 2x + 2$. Determine

1. the gradient between points A and B
2. the gradient between points C and D
3. the gradient between points A and D.

• Explanation of a worked out example (symbolic algebraic form)	**DUM**	**EG0-EG7**
• Symbolic algebraic procedure	**DUSC**	
• Key word/key concept	**ATA**	
• Concluding explanatory text	**DUGC**	

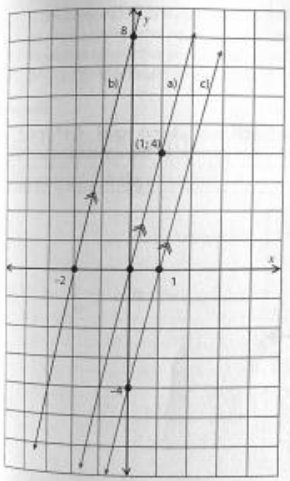
Example Number 4 Gradient	Expository section	Mode of reasoning	Representational translation
 Answers 1. The coordinates are A(2; 6) and B(0; 2). The gradient between A and B is: $\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{2-6}{0-2} = \frac{-4}{-2} = 2$. 2. The coordinates are C(−1; 0) and D (−2; −2). The gradient between C and D is: $\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{-2-0}{-2-(-1)} = \frac{-2}{-1} = 2$. 3. The coordinates are A(2; 6) and D (−2; −2). The gradient between A and D is: $\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{-2-6}{-2-2} = \frac{-8}{-4} = 2$. What do you notice? From the previous example it is clear that the gradients between coordinates of the same straight line are always the same. We say the linear graph has a constant gradient and we use *m* as symbol for a linear function's gradient. This is true because the gradient of the line can also be interpreted as the steepness of the graph and this does not change, otherwise it will not be a straight line any more.	• Introductory text	**N/A**	
	• Explanation of a worked out example (symbolic algebraic form)	**DUGC**	**EG0-EG7-EG10**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text		

Example 5	Modes of reasoning	Representational translation
Example Determine the gradient of the straight line sketch. The equation used for the sketch is $y = -3x + 4$  **Answers** Because the gradient of a straight line is constant, we can choose any two coordinates to determine the gradient of the line. Let us use the y-intercept (0; 4) and the point (1; 1).  $m = \frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{-3}{1} = -3$ Why does this graph have a negative gradient? What do you notice about the gradient and the equation of the linear graph?	DUGC DUSC	EG2-EG12

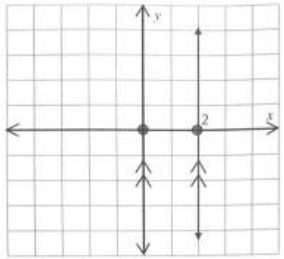
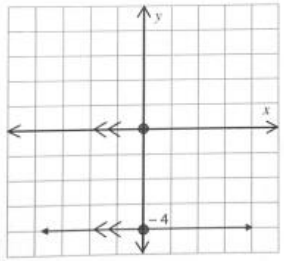
Example 5 continues	Modes of reasoning	Representational translation
	**DUGC** **DUGC**	EG0-EG10

4. Standard form of a straight line

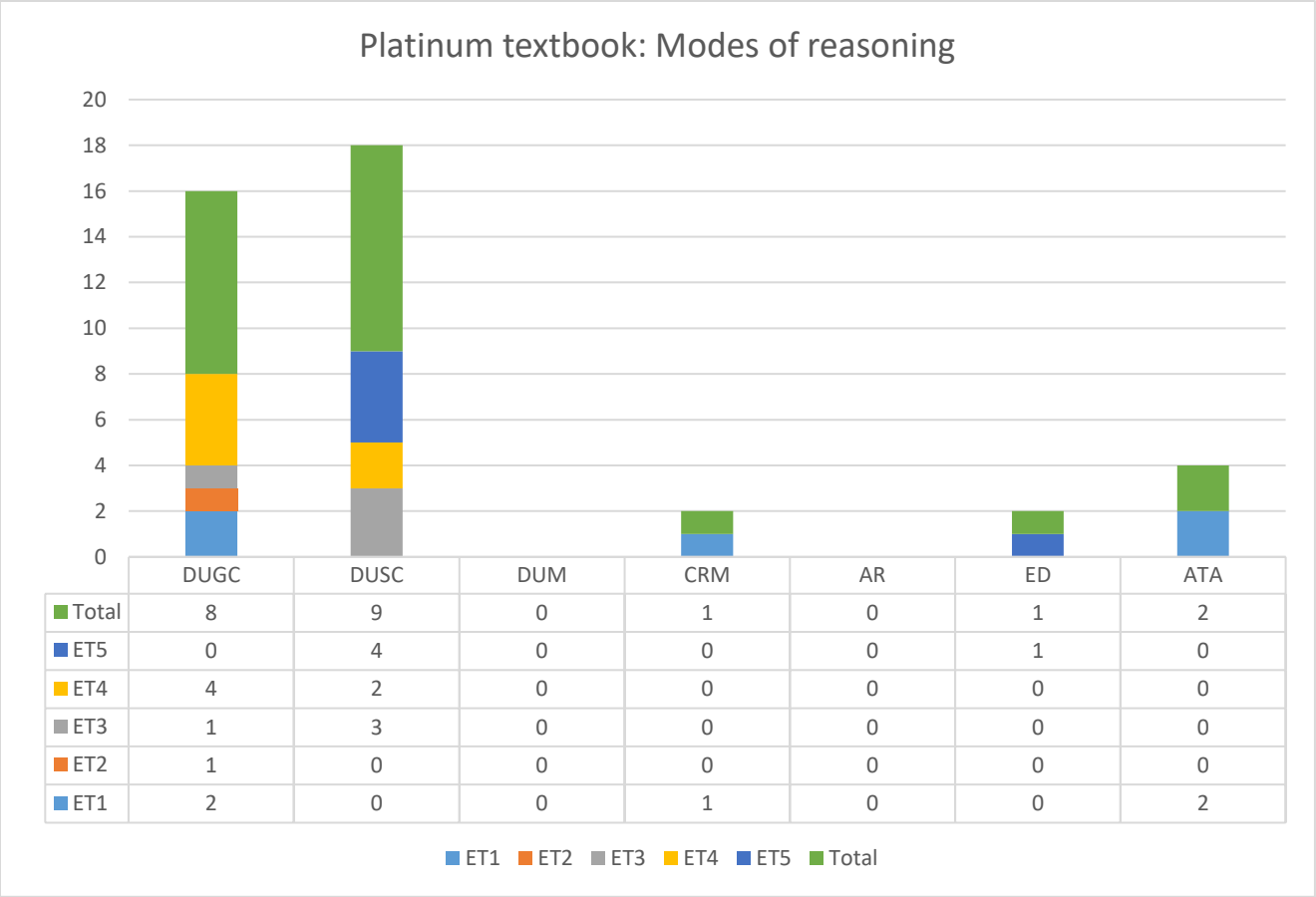
Example Number 5	Expository section	Mode of reasoning	Representational translation
	• Introductory text		
	• Explanation of a worked out example (symbolic algebraic form)	**DUSC**	**EG0-EG6-EG8-EG0**
	• Symbolic algebraic procedure	**DUSC**	
	• Key word/key concept	**N/A**	

Answers By using the intercept method, we can write $y = 4x = 4x + 0$ |	x-intercept | y-intercept | Another point | | x | 0 | 0 | 1 | | y | 0 | 0 | 4 | Because both the x- and the y-intercepts are at the origin, we need to choose another point, so we can draw the graph. So we add another column. In this case we let $x = 1$ and determine its coordinate partner $y = 4(1) + 0 = 4$. We then use the intercept method for the other two equations. See if you get the same intercepts as shown below.  The three lines all have the same gradient and they are parallel. The constant value of the equation can be interpreted as the y-value of the y-intercept. $y = 4x + 8$ is the same as the line $y = 4x$, but it has shifted upwards by 8 units. The line $y = 4x - 4$ is the same as the line $y = 4x$, but it has shifted downwards by 4 units. We can now identify another feature between the equation of a linear graph and its graph: The y-value of the y-intercept is the same as the constant c in the equation.		• Concluding explanatory text	**N/A**	

5. Lines that are parallel to the axis

Example Number 6	Expository section	Mode of reasoning	Representational translation
Lines that are parallel to the axes There are two special cases of straight lines: $x = b$ and $y = c$. b and c; are constant values. For example, if $x = 2$ it means that the x-values of the graph must always equal 2, while the y-value can have any value. We then get a line parallel to the y-axis intersecting only the x-axis at (2; 0):  If $y = -4$ it means that the y-value of the graph must always equal -4, while the x-value can have any value. We then get a line parallel to the x-axis intersecting only the y-axis at (0; -4).  	• Introductory text	**N/A**	
	• Explanation of a workedout example (symbolic algebraic form)	**CRM**	**EG0-EG7**
	• Symbolic algebraic procedure	**N/A**	
	• Key word/key concept	**N/A**	
	• Concluding explanatory text	**N/A**	

Summary modes of reasoning



	DUGC	DUSC	DUM	CRM	AR	ED	ATA
Total	8	9	0	1	0	1	2
ET5	0	4	0	0	0	1	0
ET4	4	2	0	0	0	0	0
ET3	1	3	0	0	0	0	0
ET2	1	0	0	0	0	0	0
ET1	2	0	0	1	0	0	2

Ethics clearance

Matsie Mabeta <Matsie.Mabeta@wits.ac.za> 28 Jun 2018, 16:42

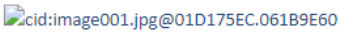
to me

Dear Matome

Your protocol number will be **2018ECE028M**

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

Regards



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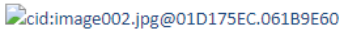

Application for exemption from completion of Ethics Application.

To the ethics committee,

My name is Matome Edwin Matsepe, student number 842068. I am a student in the School of Education at the University of the Witwatersrand.

For my Masters Research report I intend to conduct a research that explores the modes of reasoning in explanatory texts and worked out examples in two Grade 9 textbooks. This research is based on textbook content analysis only. That is, the only data that will be collected, analysed or presented is the textbook content. No data will be collected from people, nor will any data be in the form of video tapes, audio tapes or personal written texts. Therefore, there is no possibility of harm or risk to any person.

In this regard, I request that the Ethics Committee waives my need to apply for Ethics approval and or grants me an ethics protocol number without me having to apply for ethics approval.

If the committee needs more documents to support my application, please contact me on:

Cell: 081 0390522

Email: Edmatsepe@gmail.com

I would appreciate it if my application is considered.

Regards,

Matome Edwin Matsepe

842068

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