MASTER OF EDUCATION

16 MARCH 2020

TOPIC:

MATHEMATICS TEACHERS' PROFESSIONAL NOTICING IN THE TEACHING OF INVERSE FUNCTIONS IN GRADE 12.

by

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STD NO. 1484157

A Research Report Submitted to The Faculty of Humanities, University of the Witwatersrand, Johannesburg, in Partial Fulfilment of the Requirements for the Degree of Master of Mathematics Education

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ETHICS CLEARANCE: 2019ECE030M

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DECLARATION

I declare that the work of this Research Report is my aided work except where indicated otherwise. All phrases, sentences, and paragraphs taken directly from other works have been cited and the reference is recorded in full in the reference list. I have followed the required conventions in referencing the thoughts and ideas of others.

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16 March 2020

ACKNOWLEDGEMENTS

To God Almighty, I give thanks for the strength and wisdom bestowed during this trying and challenging journey. God has provided wisdom and helped me regain my strength when I was feeling despondent and in despair. Honour and glory to him for his protection and love throughout this journey.

My sincerest gratitude to my supervisor Professor Judah Makonye for his guidance and support throughout this journey. As my supervisor, he taught me to be proactive and to challenge myself as I was embarking on this journey. Through his supervision, I have learned to work hard and challenge myself and work towards excellence. He instilled in me the courage and helped me to realize my capabilities, strength, and weaknesses and be able to make the best out of every situation. His support and guidance are mostly appreciated at all times.

I also wish to acknowledge, Professor Anthony Essien, my mentor, for always being there to encourage, support, challenge, and help me aim higher in life. For taking the time to make a valuable input to my academic work and personal life. I thank you for the encouragement and the drive towards my academic work and career and I sincerely appreciate you as a mentor.

I most sincerely wish to thank Dr. Livhuwani Peter Ramabulana who edited my research thesis. I am grateful for your hard work and willingness to assist and mostly for the time you put into editing my work. I appreciate the constructive feedback and the advice given to making this research a success. To you my fellow Masters' students, who made the hardest days; the most pleasant and bearable. Especially, to you Eva Katabua my dearest friend for your continuous love and support. You have always been like a sister I never had as you were always there for me; picking me up from Gautrain station and back for class. You have never had a complaint, your love and care have been amazing. May God Bless You abundantly!

To my family, my two daughters Andiswa and Sheila. I appreciate your love, support, and encouragement. Mostly, I am grateful for your understanding when I arrived late from attending classes while using Gautrain to attend as I never had a car then and returned home late at night and could not spend adequate time with you when needed. I also want to thank my siblings and family members for showing their confidence in me despite all the odds. Above all, I thank everyone who has contributed immensely and imparted their knowledge on daily basis to shape my academic journey.

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ABSTRACT

This paper discusses mathematics teachers' professional noticing in the teaching of inverse functions in Grade 12 in the South African context. The notion of 'professional teacher noticing' becomes key in teaching due to the teacher's interpretation of events becoming a challenge in a mathematics classroom (Davis, 1996). Thus, the paper looks deeper into how professional noticing may assist teachers' to notice learner(s) mental constructions, reasoning and to attend to learners' thinking strategies. Furthermore, to interpret learners' mathematical understanding, and to gather the required responding skills when teaching inverse functions and graphs. A qualitative research approach was used to conduct this study, the data was collected through learners' pre-tests and post-tests and also from the intervention video-recorded lessons. For this study, the APOS (Action, Process, Object, Schema) theory was explored as an analytical framework to determine the learners' mental constructions concerning APOS conception levels. Another objective was to investigate teachers' professional noticing using Barnhart & van Es, (2015) sophistication noticing level as an analytical framework. The study aimed to explore elements of 'professional noticing' in an enacted mathematics lessons to enhance teachers' attending, interpretation, responding, and deciding strategies in a mathematics classroom. In collecting data, mapping professional noticing and APOS theory to the curriculum documents, pre and post-test assessments were conducted with Grade 12 learners. This study used 9 male and 9 female Grade 12 learners as participants and as part of the sample for the study. Additionally, there were two teacher participants (teacher A and teacher B) who participated in analyzing the video-recorded lessons taken during the intervention classes guided by designed questionnaires. The two teacher participants were considered for the study for the reason of minimizing biases and preconceptions for data analysis and the validity of the data collected. The study established that, through correct sophistication level of noticing and interrogation of the learner conception levels using APOS theory. Professional teacher noticing can develop learners with mathematics sense-making and critical thinking skills. The study established that the implementation of professional noticing in teaching practices was key to minimizing Grade 12 challenges of learning inverses functions. As a result, the study recommends that the notion of professional noticing be considered as a means to improve, transform teaching practice, and as a platform for a concrete framework on professional noticing in the learning of inverse functions in Grade 12.

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CHAPTER ONE

INTRODUCTION AND MOTIVATION FOR THE STUDY

1.1 Introduction

Studies have recognized that a function is at the inception of mathematics in various forms referred to it as transformative, irreversible, integrative, bounded, and troublesome (Petterson,2012). The function’s various forms were explained as follows; as transformative due to understanding of the concept as leading to a new perception. A function is regarded as irreversible when the change in the observation of its nature remains unforgotten and integrative when a solid understanding of the concept evolves and relates with other mathematical topics. Moreover, a function was described as bounded based on its concepts developing the margins of other mathematical disciplinary areas, and therefore it is regarded troublesome in that it is prevalent in the learning of mathematics and therefore learners find it challenging.

Accordingly, (Pettersson et al.,2013) recognized the work related to functions in miscellaneous areas of mathematics as significant to understanding functions not only as of the actions and processes but as a mathematical object. In another study, the function concept was described as developing from the general preference of connecting two or more quantities, in mathematics (Chimhande, 2014). This means that the function concept is a phenomenon that utilizes quantities in everyday life relationships which implies that a quantity determines another one such as “input and output” or “domain and range” or “independent variable (x) and dependent variable (y)" in Mathematics. For example, in everyday life the more money you pay for some goods, the more of them you get or the better the quality you get. (Chimhande, 2014) support the notion that the function is essential to other mathematics concepts and their applications.

The function and inverse function concepts is a crucial topic for learners to understand in the Grade 12 syllabus since it is at the core of other domain topics that form a critical part of tertiary mathematics. Consequently, it is regarded as a fundamental concept in the mathematics curriculum and at the Further Education and Training (FET) phase. Similarly, other mathematics concepts are interrelated to the function. The function-related concept is however perceived as a challenging topic in mathematics generally and it has consequently received substantial preference in mathematics education (Akkus, Hand & Seymour, 2008; Ponce, 2007). A deep understanding of the concept function is also perceived as fundamental for any learner hoping to understand related concepts in algebra, calculus, trigonometry, inverse function, and many others which are critical in the mathematics course. A clear consideration of the concept of function is perceived as fundamental for future career perspectives, particularly as an access point for understanding other mathematical aspects. More research on the concept of a function and inverse function have been conducted in previous studies (Breen et al., 2012; Attorps et al., 2013; Breen, et al., 2016) among others. Eisenberg, (1992) agreed that a function concept has a significant role in the mathematics teaching discipline. Due to the prevalent nature of the function concept, it is perceived to play a role in assimilating new knowledge to the inverse function in Grade 12. Since a function germinates knowledge onto an inverse function that provoked studies to investigate mathematics learners' understanding of inverse function. The results found that most of the learners have conceptualized the inverse function from the notion of 'undoing' (Even, 1992). 'Undoing' is classified as a familiar meaning for the inverse function which emphasizes the core definition" (Even, 1992; Wilson et al.,2011).

Bayazit and Gray (2004) examined student learning of inverse functions by giving a pre-test and post-test to evaluate learners' mathematical understanding of inverse functions before and after classroom instruction. Bayazit & Gray (2004) argue that for students to grasp the concept of an inverse function, their assessment tasks need to be conceptually structured. According to Thompson (1994), the concept of a function was considered in the first three levels of APOS (Action, Process, Object, and Schema) theory. On the same notion (Breen et al., 2016), transitioned their study on the function concept inventory through the APOS levels as they describe the transmission between stages of a function concept development. Hence, this study considered APOS theory as an analysis tool to investigate learners' mental structures and how the teacher(s) notice their learners' conception of the function and inverse function concepts.

1.2 Background of the study

Developing learners' critical mathematical skills in terms of thinking, reasoning, and conceptual understanding is a pivotal outcome in Mathematics education. It is therefore argued that these skills are most fundamental in mathematics education as they are needed for identifying, investigating, and solving problems creatively and critically. Thus, the National Curriculum Statement (NCS) argues that there is a determination to develop and support mathematics sense-making and critical thinking skills among learners in the South African mathematics education fraternity. The objective of this study was to support the preceding arguments by providing strategies for mathematics teachers to professionally notice learners' mathematical thinking and understanding. Professional noticing forms an integral part of instruction for teachers to ensure that there are fundamental skills imparted to the Mathematics learners during the teaching of concept inverse function in Grade 12. Emphasis has been made by other studies that mathematics teachers must enable their learners with critical thinking skills that can be achieved through a notion of professional noticing. Many studies have defined “noticing” in various ways. For instance, Ball and Bass (2009) defined ‘noticing’ as a means to observe, to realize, or to attend, which implies that noticing is what people engage in as a daily activity. Jacobs, Phillip & Schapelle (2009) defines ‘professional teacher noticing’ as the aptitude to distinguish the performance based on key mathematical indicators.

To make mathematically appropriate decisions in a classroom, the framework of children’s mathematical thinking promoting professional noticing needs to be used as guidance for teachers’ understanding of children’s knowledge (Jacobs, Lamb & Philipp, 2010). The professional noticing framework used in this study incorporates three components of skills: attending, interpreting, and deciding (Jacobs et al., 2010). Jacobs et al., (2010) clarifies that the primary constituent of professional noticing namely: attending means to observe and identify children’s words and actions when engaging in mathematical activity. Interpreting means to relate what is attended to, with what is known about development in mathematics knowledge, and to establish what the child understands. Finally, responding and deciding is determining what is interpreted to ensure a learner is learning in alignment with their current understanding of mathematics. Most current and forthcoming teachers noticing intervention initiatives dealt with “professional noticing of children’s mathematical thinking” (Jacobs, et al., 2010). According to research, these intervention initiatives advocate noticing of learners' written tasks, for instance, (Fernández, Llinares, & Valls, 2013; Haltiwanger, & Simpson, 2014).

Another occurrence was from a short video excerpt involving the interviews of students (Schack, Fisher, Thomas, et al.,2013). For this study, it was recognized as crucial to peruse learners' mathematical thinking and understanding as a pivotal feature of professional noticing in teaching and learning of the concepts of inverse functions. The three above-mentioned components skills incorporated in the study known to provide an access point toward learner’s critical thinking and understanding skills of the concept of inverse functions. These skills are perceived as pivotal to the understanding of mathematical phenomena. Moreover, other mathematics education studies have made an impact on children’s thinking using professional noticing (Jacobs et al., 2010; Sherin, Jacobs & Philipp, 2011; Fischer et al.,2014; Philipp, 2014;

Schack, Fisher, Thomas, Tassell, Eisenhardt & Yoder, 2013; Schack, Fisher, & Wilhelm, 2017).

Thus, professional noticing may be thought of as a tightening practice aimed at funneling instruction to find key points of leverage. Equally so, other portrayals of professional noticing describe the practice more as a net aimed at collecting all pertinent information within a moment and leveraging such information for interpretations and decisions (Schack et al., 2013). In the investigation of professional noticing through the lens of justifiable teaching practice. Jong, (2017) claims that professional noticing has been the focus of recent studies. It is explicit that professional noticing has been intertwined with reasonable concepts and frameworks to better understand how teachers' activity at the moment influences learners' contribution and positioning in the mathematics classroom. Professional noticing and related practices, for example, teacher noticing and professional vision have captured the attention of mathematics education researchers for some time (Schack et al., 2017; Sherin et al., 2011).

This research report outlines the challenges that Grade 12 learners experience when dealing with the concepts of inverse functions in the South African context. The three above-mentioned interrelated components provide an access to the learners' understanding of the concept inverse functions in Grade 12 by deepening teacher noticing of learners' critical thinking skills. Sherin et al., (2011), view professional noticing as a critical constituent of mathematics teaching expertise. Hence, it can be regarded as an essential tool to advance the required mathematical skills. Other research also focuses on the dimension of professional teacher noticing resulting from a learner's mathematical thinking. However, what is key is for individual teachers to make sense of processing complex environments such as classroom events where there is a teacher-learner interaction. The process of professional noticing defines teachers' insights into learner thinking.

1.3 Statement of purpose

The study's purpose was to explore APOS framework analysis on how mathematics teachers professionally notice their learner's mathematical thinking, reasoning, and understanding of the concept inverse function in Grade 12 by examining the relationship and the connection that learners make in their written tasks. Furthermore, to investigate professional noticing during the classroom interaction with learners by deducing how mathematics teachers use the three components: attending, interpreting, and responding approaches as teaching strategies.

1.4 Rationale

This study outlines three significant notions namely, professional noticing, function and inverse concept, and the APOS analysis approach. For the rationale of this study, I have unpacked each of these key notions below.

1.4.1 Overview of professional noticing

Professional noticing is regarded holistically as a way to unpack children's mathematical thinking to the multifaceted outlook in mathematics teaching (Jacobs, Lamb & Philipp, 2010). The correlated skills comprise: a) attending to children's strategies, b) interpreting children's understandings, and c) deciding how to respond to children's understanding. Professional noticing of children's mathematical thinking was defined as a specialized type of noticing with a perspective that considers less attention from what, to how the teacher noticed (Jacobs et al., 2010). On the same notion, other researchers had also targeted teachers' proficiency towards noticing children's mathematical thinking (Kilpatrick et al., 2001; NCTM, 2000). From the perspective of teachers attending to specific aspects of instructional situations such as the learners' mental construction in the learning of mathematics, the study makes emphasis on the significance of the notion of teacher professional noticing. Some significant aspects of teaching and learning revolve around hearing and noticing what develops within teaching practices.

Focusing on these interrelated proficiencies for which this study has mainly perused, other studies argue that these distinguish "professional noticing" from other categories of noticing (Jacobs et al., 2010). The importance of using professional noticing in developing conceptual understanding was also foregrounded in this researcher's work (Jacobs et al., 2010). It is also expected to take cognizance of the crucial view of understanding mathematics education in South Africa. Moreover, there is a constant drive to pay attention to learners' understanding of mathematics concepts and ensuring that learners can connect mathematical concepts.

The notion of professional noticing enables teachers to see learner's abilities, to listen for learner's thinking through their ideas and problem solving strategies using critical thinking and deductive reasoning. Consequently, through hearing learners' conceptual understanding, being attentive to what learners say, and interpreting their mathematical thinking and reasoning capabilities. The study emphasizes the dimension of the teacher 'professionally noticing learners' mathematical thinking', the teacher's interpretation, and how teachers respond to learners' mathematical understanding (Jacobs et al., 2010). As a result, developing learners who have mathematics sense-making and critical thinking skills would be achieved.

Professional noticing was classified as a powerful conceptual understanding tool to be embedded in mathematics teaching practices (Sherin & van Es 2009). Moreover, (Sherin et al., 2011)'s the definition of noticing below resonates strongly with this study:

> "Noticing is a critical constituent of mathematics teaching expertise, and thus a better understanding of noticing could become an important tool to improve mathematics teaching and learning" Furthermore, "noticing is as important a component of the practice of teacher noticing as is the detailed ability to see and hear professionally" (Sherin et al.,2011, p.208).

More of the research initiative established that "teachers can improve their noticing by changing what they notice, for instance, moving from a focus on teacher's actions to the learners" conceptions. From this perspective, it becomes critical to explore professional noticing to facilitate the deep conceptual understanding and critical thinking skills of the specific mathematical concepts.

1.4.2 Stages in professional noticing

Attending to Children's Strategies (thinking and reasoning)

The first skill is regarded as the 'attending skill' it was recognized through teacher engagement with learners when solving mathematical problems while posing questions from video analysis of learner's written work in class (Jacobs et al., 2010). The hypothesis for learner's strategy was perceived as an important factor and it was used to evaluate teachers' professional noticing. Piaget (1970) argues that learners need to be allowed the construction and invention of their ideas. This will provide teachers with better instructional strategies to attend to learners' understanding skills which may result in meaningful teaching. Consequently, teachers became involved in how learners(s) solve mathematical problems and also engaged in analyzing learners' written work in the classrooms. Jacobs et al., (2010) states that assessing how teachers attend to learners' mathematical thinking skills and the reinforcement of interpreting learners' mathematical understanding is key for noticing.

Interpreting Children's Mathematical Understanding

Interpreting children's mathematical understanding was viewed as compulsory for teachers' consistent noticing of learners' mathematical thinking skills (Jacobs et al., 2010).Interpreting means to relate what is attended to, with what is known about development in mathematics knowledge, and to establish what the child understands. Hence, this study took an approach to explore learner's understanding of inverse functions using structured written tasks with the view to emphasize on teacher interpreting learners' mathematical understanding. Concurrently, the study used the video-recorded lesson approach to explore professional noticing from a teacher-learner interaction perspective. Davis (1997) affirms the usefulness to interpret the phenomenon in mathematics teaching, as an approach to gain insight into learner's constructivism ideas. Even & Wallach (2004) state that observing learners solve mathematical tasks during classroom practice formed an integral part of the instruction when incorporated with the required interpretive listening skills. Interpretive listening skills were considered useful skills for transforming mathematics teaching practice (Davis, 1997). Hence, the engagement of video-recorded lessons was sought to bring in the insight aspect of professional noticing. That approach was aimed to investigate the learners' interactions with the teacher while providing their interpretation of the concepts and definitions of inverse functions. Hence, the notion of professional noticing was seen as a fundamental aspect of the teachers' interpretation of learner's mathematical understanding. Similarly, this study promotes the creation of critical thinking skills in learners and embarks on assisting teachers to acquire interpretive listening skills for their mathematics teaching pedagogy.

Responding Skills

Responding skill determines what is interpreted to ensure a learner is learning in a manner that best fits their current understanding of mathematics. Even and Wallach (2004) also emphasize various assessment methods and tools for assessing learners' understanding and teachers' hearing learners' mathematical thinking. The notion that learners' assessment can inform and improve instructional decisions was found pivotal for this study. If the instruction is based on the assessments of tasks, it then provides teachers with good instructional design and better learning tasks for their learners (Even & Wallach, 2004). Further emphasis was on re-iterating that teachers assessing learner's mathematical understanding forms part of responding to learner thinking and is an integral part of instruction. Other studies have also explored the practice with a focus on the aspect of how to identify, interpret and respond to the most significant and impactful occurrences in their mathematical classrooms (Schack, Fisher & Willem, 2017). Subsequently, (Barnhart & van Es, 2015) argue that teachers responding to their learners' mathematical ideas form an integral part of instruction.

1.4.3 Function Concepts in the Mathematics Curriculum

The function concepts have been identified as the main topic in the FET Mathematics curriculum which unifies and supports other related algebraic concepts in mathematics. The classification of the concept function comes highly recommended in the mathematics curriculum and mathematics education. The function finds its existence within algebra as some of the function concepts are depicted in algebraic formulas through the equations defining one or more variables (Grinstein & Lipsey, 2001). Similarly, there is an association between patterns and sequences which is formed by a rule as a representation of algebraic formulas and equations. The function is considered a critical concept due to its popularity in the mathematics curriculum. It is also derived as involving the algebraic and geometric spheres of mathematics. The significance of functions is given recognition in the curriculum as it progresses throughout all the FET phase (Grade 10-12). The concept of function is considered quite significant in the curriculum and it has adequate time allocated in the teaching timeframe.

1.4.4 Inverse Function Concept

The inverse function was signified by a representative definition, for example, symbolic such as: $f^{-1}(x) = g(x)\ and\ g^{-1}(x) = f(x)$ (Cetiner, Kavcar & Yildiz, 2000). From the discipline perspective, when the inverse of $f(x)\ is\ h(x)$ denotes the inverse of $h(x)$ is definitely $f(x)$. Bayazit and Gray, (2004) characterize "an inverse function as undoing what a function does" (p.2). Even (1991) agrees with the assertion of 'undoing' as the fundamental sphere of the inverse function. He further indicates that the one-to-one rule is the only measure that defines that a function has an inverse. In this sense, (Bayazit & Gray, 2004) argue that the difficulty for students to comprehend cognitively a simple mathematical idea is the peculiarity in the representation. The argument that Bayazit and Gray (2004) make was subsequently agreed to, that the function concept in the South Africa context is perceived as peculiar (Pillay, 2006). Hence, Benson and Buerman's (2007) definition of the inverse function encompasses the operations performed in the relationship, providing an example of learners' understanding of inverse as reciprocals as multiplicative inverses and opposites as additives. They highlight a problem when defining the dependent and the independent variable of the function concept differing in units. In a different view, (Breen et al., 2016) found that "the conception of 'undoing' is not the only way to look upon inverse function" (p.2228). Whereas, Wilson et al., (2011) suggested ways of solving for the dependent variable, instead of interchanging x and y, to condense and improve students' conceptualization of inverse functions. Carlson et al., (2010) had a contrasting view to that of (Bayazit & Gray, 2004) and the current one founded by (Breen et, al. 2016). Carlson et, al. (2010) claim that for students to comprehend inverses as reverse processes they need to first determine and answer a wide variety of questions concerning inverse functions. In their view, they further demonstrate, $f(x)$ represents as a function, then the notation $f^{-1}(x)$, read "f inverse of x", will be used to denote the inverse of $f(x)$. Equally $g^{-1}(x)$, read "g inverse of x", will use the notation of the inverse of $g(x)$ (Carlson et al.,2010). The core of this study lies in the "inverse of a function" when knowing that the function concept is about "doing" and the inverse does the opposite.

1.4.5 APOS analytical approach

APOS theory was defined as understanding how learners construct knowledge and the levels of knowledge development (Brijlall & Ndlovu, 2013). In other studies, the development of knowledge was emphasized as critical to learning and characterized into four stages of knowledge development (Piaget, 1972). Piaget (1972) argued for the operational stage as pivotal in the development of knowledge. Furthermore, Piaget claims that the formal operation stage connects reasoning for learners to progress onto objects. In other words, within instruction learners' conception levels can be found in different operation stages therefore, it becomes the teacher's obligation to formulate practical strategies for their learners to achieve the projected level. APOS has been considered an analytical framework for the study and a way to understand learner's mathematical thinking on learning the concept of an inverse function. Therefore, this framework has been broadly and insightfully deliberated in the theoretical framework.

1.5 Problem Statement

The Curriculum and Assessment Policy of Statement (CAPS) document of South Africa reports poor performance on topics such as function and its inverse. Also, the reports show learners' performance is attributed to the failure of teachers to notice learners' mathematical thinking and understanding of abstract mathematical concepts. Davis (1997) asserts that the teacher's interpretation of events is becoming a challenge in a mathematics classroom. In this instance, interpretation of mathematical phenomenon becomes vital in pedagogy and pivotal in a mathematics classroom (Sherin et al., 2011). The root cause of learner understanding and thinking resides in teacher failure to notice professionally in a mathematics classroom (Jacobs et, al. 2010). This research study suggests for the teacher to notice professionally. They have to first understand the learners' cognitive levels of mathematical thinking and understanding. Hence, this study deploys APOS theory as an analytical framework to access mathematics learners' cognitive levels in the learning of inverse functions in Grade 12.

1.6 Aim of the study

Mathematics teachers are mandated with a national constitutional task to develop learners who have mathematics sense-making and critical thinking skills. The study was to discover elements of professional noticing enacted in mathematics lessons that boost learners 'understanding of the inverse functions in Grade 12.

1.6.1 Research Questions

Additional emphasis was made on the importance of observing learners' thinking as a primary component of noticing in teaching. The research questions that underpin this study are found as follows:

- What APOS theory levels of learners' mental constructions do teachers notice in the learners' written tasks using the concept of inverse function?
- What are the learners' mathematical thinking and understanding noticed by the teacher during intervention classroom interaction using the concept of inverse function?
- What are the teachers' levels of noticing sophistication observed when attending, analyzing, and responding to learners' mathematical thinking and reasoning during the intervention?

1.7 Significance of the Study

Research asserts that noticing becomes a constituent of mathematics teaching proficiency, and is regarded as a fundamental tool for refining mathematics teaching and learning (Sherin et, al. 2009). Professional noticing is defined as the skill to identify and perform on key parameters substantial to one's profession (Jacobs et, al. 2009). This research study intended to reinforce the contribution of professional noticing in mathematics teaching of inverse functions in Grade 12 in South Africa. Subsequently, it aims to add expertise to mathematics education with the implementation of professional noticing. This pedagogy approach is new compared to traditional chalk and talk approaches where teachers advocate mathematical knowledge to learners without being mindful of how learners process the mathematics imparted to them.

1.8 Conclusion

In the introductory chapter, the reader was provided with the significance of the concept function and inverse functions. I have also pointed out the significance of it, in the mathematics education curriculum context; the discussion was mainly on how these concepts interrelate with each other. I have also discussed the ideas that evolve from other studies and the researchers' point of view regarding the function and inverse function concepts. Accordingly, I discussed the notion of "professional noticing" which is also pivotal for this study as the study revolves around it. Furthermore, I have pointed to the prominence of the three key components concerning professional noticing as discussed in the study. In light of professional noticing and the three key components. I have provided different definitions about professional noticing and the three components mentioned above. The APOS theory approach was also indicated as the analytical framework for the study. Other key aspects of the study that are discussed in this chapter are as follows: statement of purpose, rationale, aim, and the importance of the study. The research questions were pertinent to the study because they underpin the importance of the concept of inverse functions and teacher noticing. The next chapter will explore and discuss the theoretical framework and literature review that underpin this study.

1.9 Structure of the research report

Chapter 1 Introduction and motivation of the study

Chapter 2 Theoretical Framework and Literature Review

Chapter 3 Methodology and Research design

Chapter 4 Results, Data Analysis, and Interpretations

Chapter 5 Findings, Conclusions, and Recommendations

References

Appendices

CHAPTER TWO

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 Introduction

This chapter will introduce, explore and substantiate the theoretical framework concerning this study. The chapter discusses the theoretical framework of APOS (Action, Process, Object, and Schema) that underpin the study and also served as an analytical framework. The chapter also provided the review of the literature to provide the theoretical perspectives, thus locating the study in the broader scientific arguments of the concept of professional noticing in the teaching of inverse function that aligns with the research questions (objectives) of this research.

2.2 Theoretical Framework

The theoretical framework adopted for this study is APOS theory and it was developed to understand how mathematics is learned and to investigate learners' mental constructions in the understanding of inverse functions (Arnon et al., 2014). The theory categorizes APOS theory into four types of mental constructions namely: actions, processes, objects, and schema. The use of the acronym 'APOS' is used as an abbreviation. The present study used all four conception levels to define and deliberate on APOS theory.

2.2.1 APOS Theory Unpacked

APOS theory was defined as a framework that promotes the learning process that involves the understanding of learners' mental constructions when learning problematic mathematical concepts (Weyer, 2010). Maharaj (2010) argues that, for a learner to develop critical sense-making skills for a specified mathematical concept, they must have suitable mental structures. Accordingly, Weyer, (2010) categorized learners' approaches of understanding the function concept across the conception levels, for example; action level up to the schema level by applying APOS theory to the function concept. Furthermore, APOS theory conception levels were applied during the understanding of linear programming as a related topic to the function concept (Brijlall & Ndlovu, 2013). In some derivative concept studies, there was a specific focus afforded to students' mental structures for the derivative concepts with a different representation (Borji et al., 2018; Huang, 2011; Tokgöz, 2012). Thus, APOS theory was perceived as constructivism and focussed on mathematical learning, as the main concern was on learners' conception levels (Dubinsky, 2010). Dubinsky's (2010) claim was interpreted from Piaget's constructivism. As a result, APOS can be viewed as an access point to the development of learner thinking about abstract mathematics and how learning can be assimilated and understood (Arnon et al., 2014). APOS theory was also viewed differently from the way Dubinsky and Wilson (2013) characterized learners' strategies across the development levels

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however, the key idea focused on improving learners' understanding from their initial conception level of mathematical concepts.

Summary of APOS theory levels:

APOS analysis approach will guide this study by putting information in convenient categories. There are four levels (Action, Process, Object, and Schema) for this study and they provide definitions and characteristics of the APOS approaches as found below.

2.2.2 Action level of an inverse function

APOS theory acknowledges the importance to understand the action conception as one that involves the mental construction transformation in reaction to external stimuli (Dubinsky & Harel, 1992). At this level, learners only understand how to operate at the memory level or from the external stimuli perspective as they perceive the transformation of objects in the sense of the external (Dubinsky & McDonald, 2001). Maharaj (2010) highlights that there should be a specific instruction mandatory for the learners' explicit completion of each step of transformation at this level. Meaning it is difficult for a learner to conceptualize the steps of constructing the concept on their own without external help. Therefore, there is a need for a teacher to scaffold the learners' understanding of inverse functions at the action conception level. More emphasis was by (Asiala et al.,1970) who asserts that for an action conception to take place on transforming the mathematical objects performed by an individual, learners require external motivation. Furthermore, (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996) defines the action conception level as a mental or physical operation employed unto an object to acquire new development. Hence, Bayazit & Gray, 2004 view a student's understanding as inadequate at the action conception level and argue that they could execute the rule of inverse function by creating the process algorithmically. As a result, learners can theorize each step explicitly without skipping any necessary steps (Arnon et al., 2014). In that way, learners would be able to recall the algorithm explicitly using memory (Cottrill et al., 1996). Oehrtman and Engelke (2010) attest to the instances where students experience difficulties to comprehend a function as a process thus end up unable to establish the concept of an inverse function.

Understanding action conception in a teaching environment context

Teachers found that learners at an action conception level can perform procedures without understanding the meaning of concepts. Therefore, it becomes a challenge for these learners to foresee the answers as they perform the procedures. At this stage, learners can find the answer however, they would still have difficulties in realizing if the answer makes sense towards the performed procedure. The preceding text means that a learner with an action level might have a knowledge of the procedure but that does not imply they have constructed the meaning of the procedures. For instance, given that a learner knows how to sketch a parabolic function and its inverse function but they do not know why the graph has an inverse function and why its domain needs to be restricted. They can carry out procedures to plot the graphs using the table method or point by point method but do not understand why is there an inverse function for the particular graph and not for the other.

2.2.3 Process level of an inverse function

APOS theory defines the process stage in a "form of understanding a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under their control" (Dubinsky & Harel, 1992). A process occurs as a complete object in the mind of a learner. Unlike with the action conception level, where a learner is expected to perform each step explicitly however in the process stage a learner is expected to perform the same action not necessarily explicitly (Arnon et al.,2014). For instance, at the process stage, the learners are expected to identify the x and y-intercept coordinates without performing the procedure step by step. In that way, the action conception level has transformed to a process level through interiorization understanding (Dubinsky, 2010). Interiorization means that working out the x and y-intercept of the graph is regarded as an operation on a lower level of mathematics objects. Therefore, these lower-level operations should enable a learner to be familiar with the processes that will ultimately progress their understanding into a new concept and this takes place in the interiorization process. At this level, learners are expected to accomplish the performance of the same transformation without external stimuli. The learners have at this particular stage internalized the procedure. Learners are also expected to "think of performing a process without doing it" to be able to reverse the process in conjunction with other processes (Dubinsky & McDonald, 2001). Consequently (Breidenbach, et al.,1992) makes an inference to a process conception being considered crucial in the understanding of inverse function. In Bayazit & Gray (2004) view learners who have developed a process conception find it meaningful to construct the concept of inverse function in other situations besides operational formula (Bayazit & Gray, 2004). The arguments relate the process level of a function in a form of learners understanding the function and inverse functions concepts such that they are fully conversant with the equations of different functions such as exponential, hyperbola, and parabolic functions, they can also be able to reverse these functions.

2.2.4 Object-level of an inverse function

The object conception level remains "a form of understanding a concept as a phenomenon to which actions and processes may be applied" (Dubinsky & Harel, 1992). Learners at this level recognize the process as complete with the understanding that further transformation can be performed on it (Dubinsky & McDonald, 2001)."Encapsulation is the term used to define mental construction of a process (transformed by some action) into a cognitive object that can be seen as a total entity (or the coherent totality) and which can be acted upon (mentally) by actions or processes. The only way to mentally construct a mathematical object" (Dubinsky & Harel, 1992). An object can be seen as a mathematical entity when it possesses the characteristics of an actual thing. Sfard (1991) argues that an object can also be recognized as the idea at a glance' and be operated as complete, without detailing it. Hence, a process will be summarized into an object and subsequently, the object can be re-condensed into a process from which it originally developed. Therefore, the encapsulation and de-encapsulation formulation get created as action and reversal (see Figure 1).

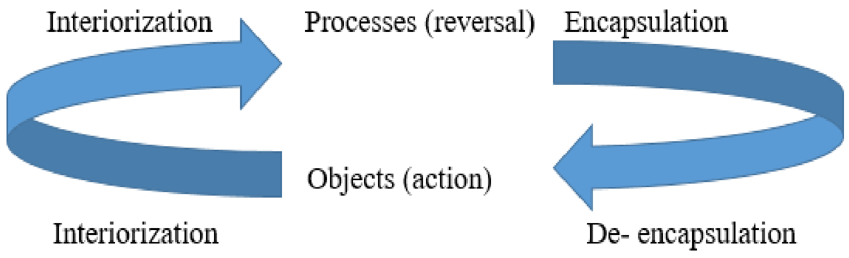

Figure 1: APOS levels of constructing mathematical knowledge (Adapted: Arnon et al, 2014:18)

The APOS levels are characterized as mental structures created for sense-making of the abstract and complex mathematical concepts to aid learners' mathematical understanding and thinking. The APOS theory is argued on the basis that a learner can independently learn any mathematical concepts when the critical structures necessary for the understanding of the mathematical concepts are constructed (Arnon et al, 2014). The teaching sequence in figure 1 gives the learner's an opportunity of realizing that the construction of a process can be developed by a certain mathematical action. This may be formed as a mental object that can be transformed coherently and the process known as encapsulation can be formed which can also be de-encapsulation or a reversal process. The preceding text considers the view that 'an inverse function undoes what a function does' (Even, 1991). In this sense, the notion of 'undoing' could be considered as the definition for the inverse function (Even, 1991). However, further research argues that the conception of 'undoing' cannot be considered merely a way to

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view inverse functions. Even (1992) established "a solid understanding of the concept of inverse function cannot be limited to an immature conceptual understanding of 'undoing", which she indicates may impose unfitting conclusions concerning inverse functions.

2.2.5 Schema level of an inverse function

At a schema phase, the learners have "a collection of actions processes, objects, and other schemas, together with their relationships, that the individual understands" (Dubinsky & Harel, 1992). Similarly, the schema has been characterized as a logical mental construction that has developed all the actions, processes, objects, and other schemas of which the assemblage of all these stages can be regarded as a problem-solving development (Cottrill et al.,1996).An individual at the schema level "can jump back and forth between the levels of action, process, object, and schema about the inverse function concept" (Weyer, 2010). At a schema stage, it is recognized that the learner has gained adequate skills to connect algorithmic forms to graphical forms as well as to construct abstract and complex mathematical concepts. As a result, the learner who is solving an inverse function should have gathered the knowledge of the function concept and the various representation concerning it that connects with the inverse function (Bennet, 2009).

2.2.6 Overview of APOS theory:

APOS conception levels are measured by mental constructions that have developed for the sense-making of complex mathematical concepts. The premise that any mathematics learner is enabled to learn complex concepts can be recognized as possible when the key structures for comprehending these concepts have been assembled (Arnon et al, 2014). APOS is used as an acronym and an abbreviation for the organized structures known as schemas which are organized through the observation and development of the actions, processes, and objects When an individual is dependent on external stimuli to perform an explicit algorithm, the individual's mental construction is considered to be at an action stage (Asiala et al., 1997). Arnon et al., (2014), suggest the explicit performance of each step according to the action conception level be given consideration. Accordingly, (Cottrill et al., 1996), refer to the explicit performance by the learner as the act of recalling from the memory. Subsequently, the process is considered to be the occurrence of what is complete in an individual imagination while the actions are performed. This preceding text means a learner has conceptualized the steps for performing the algorithm therefore, it is not necessary to execute the procedure explicitly (Arnon et al., 2014). Thus (Dubinsky, 2010) refers to action as converted to a process through interiorization. When the learner can consider a process as an object and act upon it, then APOS

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theorists claim that the process has been encapsulated and is now considered an object. Whereas an application of an object conception could be referred to as creating a set of functions (Fowler, 2014).

Teachers' participation in the learners' construction of knowledge

In a mathematics classroom, a teacher plays a role of a facilitator by scaffolding for each learners' learning. Scaffolding learning in a mathematics classroom involves the construction of knowledge for each process. These learning opportunities provided by a mathematics teacher become fundamental for the development of APOS theory levels towards learning inverse functions. When the teacher scaffolds for learners and makes an emphasis on procedural fluency that would largely place the learner at action level because learning procedures imply that learning does not entail making connections between concepts. Learning organized for the learner to make connections encourage internally thinking which encourages interiorization of action into a process thereafter, encapsulation into an object. Therefore, the teacher can pay attention to the conception level that the learner possesses at that precise point in time so to provide the necessary scaffolding to help the learner reach the next level. The basic level is the action stage where learners should construct knowledge of concepts then develop it to the process level. In essence, teachers are expected to help learners construct mental structures that enable them mathematical sense-making of concepts. Hence, it is required that teachers give their learners' assessment tasks that provide the external cues to generate learners' thinking process (Brijlall & Ndlovu, 2013).

Steps to Determine the Inverse of a function

Step 1: Determine which function has an inverse. Find out the one-on-one relation. From the deduction conclude if the function has an inverse or not.

↓

Step 2: Change $f(x)$ to y

↓

Step 3: Switch x and y

↓

Step 4: Solve for y

↓

Step 5: Change y back to f^{-1}

Figure 2: Finding the inverse of a function (Adaption of Brijlall & Ndlovu, 2013)

The inverse function is considered to show a particular relation; thus it is important to determine when there is an inverse. To determine the inverse functions, there is a need to identify the one-to-one function. To ascertain the one-to-one function, each input (x −value) should have a distinctive output (y −value). Mathematically the x − value is known as the input or a dependent variable and the y −value is called an output or an independent variable. A one-on-one function has categorized a function when it passes the horizontal line testing. By applying the horizontal line testing we can show that there is one–to –one function when the horizontal line intersects the graph of the function at more than one point. More importantly, an inverse function could be established and defined as a function when it provides the total aspects of a procedure its inputs, a process, and its outputs (Weyer, 2010).

When does a function have an inverse?

Activity 1:

Explain if the function $f(x) = -\frac{3}{4}x + 2$ and $g(x) = 2x - 1$ are one-to-one relations. The equation of the graphs of f (x) is a line with a slope -$\frac{3}{4}$ and a y-intercept of (0;2) and g(x) is a line with a slope of value 2 and a y-intercept (0; -1). Both graphs are linear functions that are tested using a horizontal line test for f(x) is shown in red and for g(x) is shown in blue, the individual graphs have the horizontal line touching the graphs only once. Therefore, the functions of $f(x)$ and $g(x)$ are one-to-one functions and they have been tested to prove that the horizontal line test touches the graph once, as a result, they have an inverse. Weyer (2010) suggests that learners need to recognize a relationship even when a relationship is not stated. Furthermore, Weyer (2010) makes an inference to the learner that has fully developed APOS levels of conception, saying that a particular learner is enabled to establish a one-to-one or a many-to-one function and what qualifies such a function to be an inverse or not (see Figure(s) 3 and 4 as sketched using Geogebra graphing).

$g(x) = 2x - 1$ $\qquad$ $f(x) = -\frac{3}{4}x + 2$

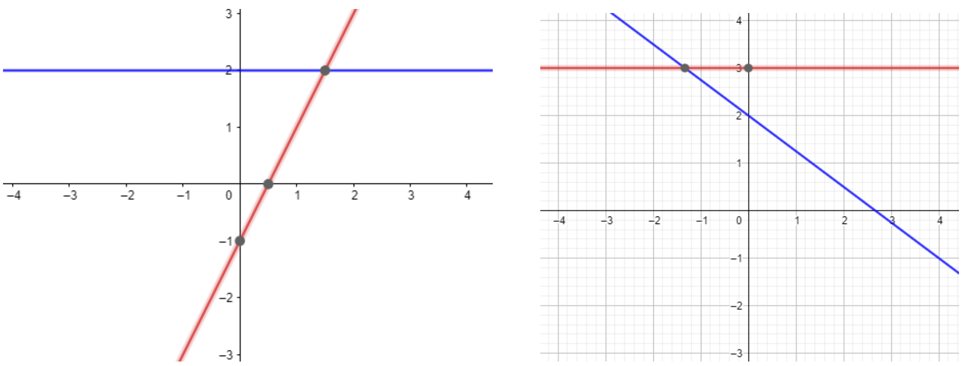

Figure 3: Illustrate examples of the horizontal line test for one-to-one functions

Activity 2: Explain if the function of $f(x) = x^2$ and $f(x) = (x+1)^2$ are one-to-one functions or not. The functions f(x) are denoted as $y = x^2$ and $y = (x+1)^2$, they are not one-to-one because the horizontal line testing touches both graphs more than once. Therefore, the graphs have failed the horizontal test. Reed, (2007) claims that there is a difference between a one-to-one function and having a one-to-one correspondence.

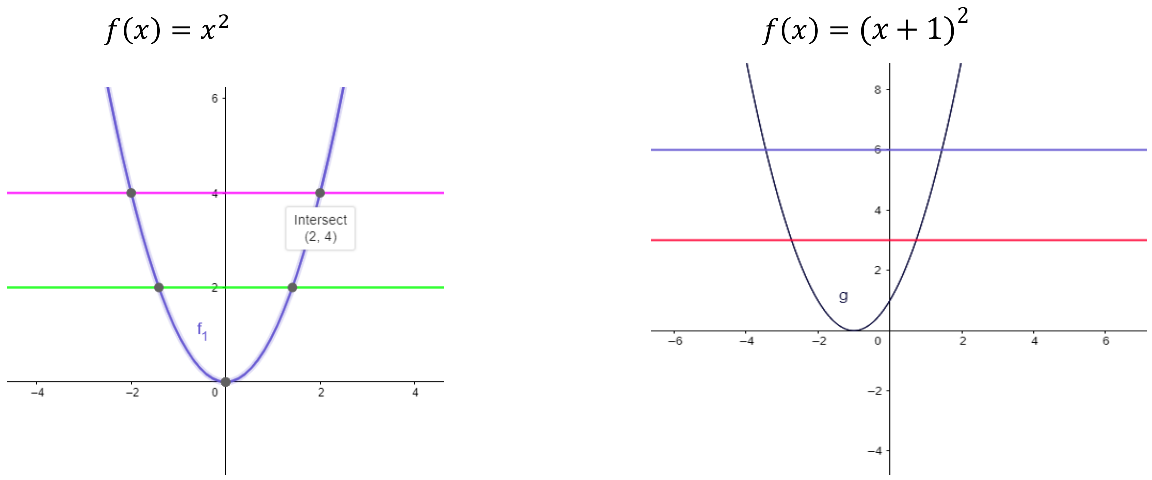

Figure 4: Illustrate examples of a non (one-to-one function)

2.2.7 The Genetic Decomposition of the Inverse Function

For the formation of the genetic decomposition, there is a need for prerequisite knowledge which is essential for learners basic understanding of the concepts of a function and inverse function such as, determine the one-to-one function, many –to – one function, and one-to-many function, horizontal line testing, domain, a range for a linear, exponential, or a parabolic graph. In most studies where the APOS theory was explored, a genetic decomposition was considered a framework,that could be modeled for the mental constructions that a learner requires for learning and understanding a concept (Arnon et al.,2014). Genetic decomposition is classified as the structured set of mental structures, which can provide an understanding of how the algorithm process can be executed in the learners' minds (Brijlall & Ndlovu, 2013).

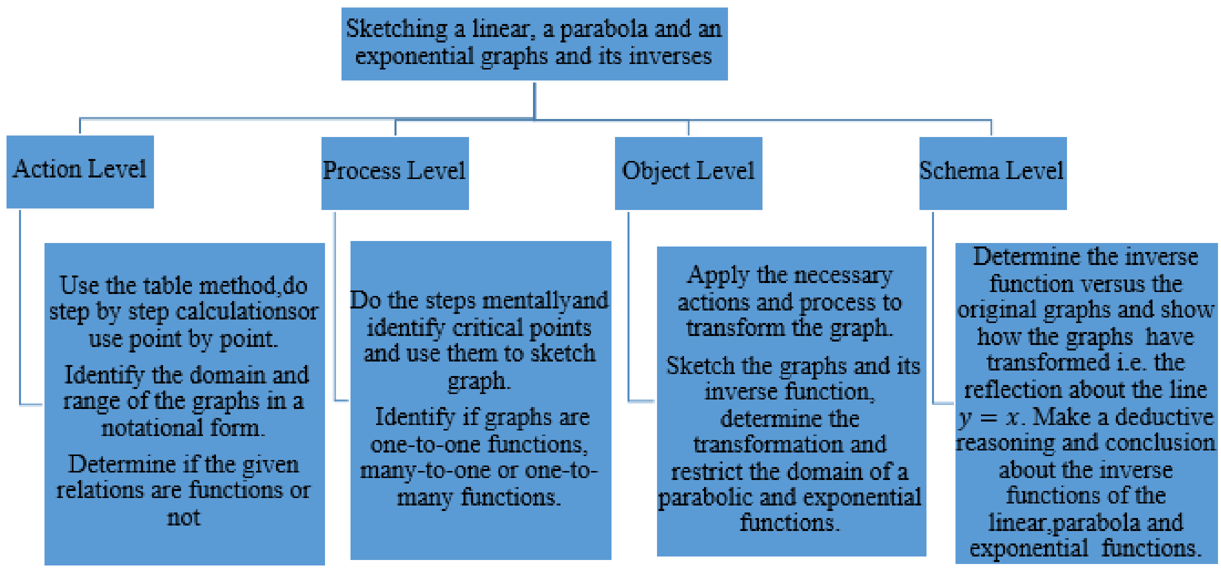

Figure 5: Genetic decomposition for sketching a linear, a parabola and an exponential graphs

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The genetic decomposition explored in understanding that the concept of inverse function often deals with graphs, table method, point to point method or the x and y-intercept method, domain and range, and so forth. These are identified as critical steps that a learner needs to decompose to be able to transform the inverse function. Thus, Figure 5 on genetic decomposition for sketching a linear, exponential and parabola, illustrates how a learner can formulate a genetic decomposition for understanding the inverse function and to build the pertinent mental constructions needed. The steps on finding the inverse of a function are subsequently illustrated in Figure 2, the adapted figure from (Brijlall &Ndlovu, 2013). Accordingly, the action, process, object, and schema levels for genetic decomposition are also discussed below constructing the ideas and point of discussion from Figure 5 of genetic decomposition.

a) Action Level for Genetic Decomposition

1) Learners are expected to understand the action of using either a table method or a point-by-point method to plot a linear or a parabolic graph.
2) Also, learners should construct the action of identifying the domain and range on both graphs individually and correctly write down the domain and range using the set notation.
3) Learners should construct the action of determining if the given relation is a function or not.
4) The action discussed in number (1) will be interiorized as a process when a table method or point by point plotting of the x –values provide the y-values and create x and y coordinates that will be used to perform the process of plotting and sketching the graphs. Both graphs will indicate the dual intercepts and show gradients of the graphs.
5) Action discussed in (2) will be interiorized into the process when the slopes or gradients of the graphs are formed that will indicate and clarify to the learners the domain and range of the linear and parabolic functions. For example, looking at the below figures. Learners can identify the domain for (Figure 6) as $x \in (-\infty;\infty)$ and range $y \in (-\infty;\infty)$ and (Figure 7) domain as $x \in (-\infty;\infty)$ and range as $y \in [-4;\ \infty)$ **(Refer to Annexure B)**.

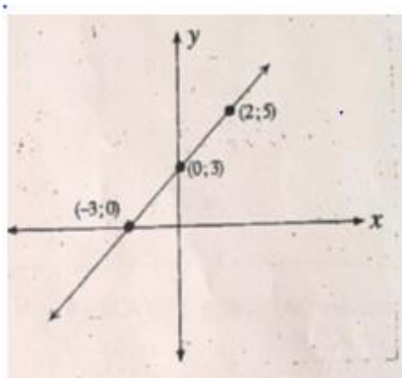

Figure 6: Pre-test Question Paper

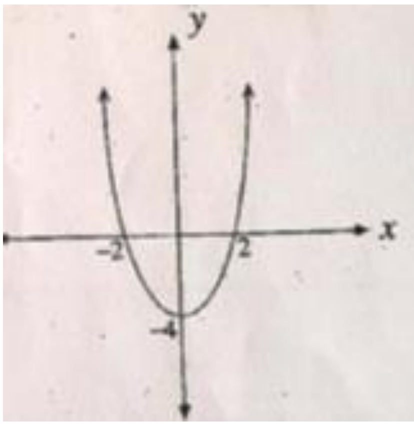

Figure 7: Pre-test Question Paper

6) Learners should be able to interiorize action in (3) by determining the relation for each function correctly. For example, learners should correctly construct the action of horizontal line testing on Figure 3 and 4 and easily identify that Figure 3 (linear function) is a function because the horizontal line touches the graph once and that Figure 4 (parabolic function) is not a function because it fails the horizontal line test.

b) Process Level for Genetic Decomposition

1) Learners should construct a process of calculating the x and y-intercept to identify the critical points to sketch the graph by using the dual intercept method or table method.
2) When the graph is sketched, learners can construct the process of using the horizontal line testing to find out if a relation is a function or not.
3) Further to the horizontal line testing of both the linear function and non-linear (parabolic function). Learners should identify if the graph is a one-to-one, many-to-one, or one-to-many. The processes have to be determined by making the connection of the type of relation for function.
4) Processes discussed in (1) have to be coordinated by understanding the action to take to correctly apply the table method, point by point, or the dual intercept method for the x and y-intercept. Also, both graphs to be sketched using the correct x and y-intercepts determined from any of the three methods.
5) Processes discussed in (2) are encapsulated as an object when the horizontal line test is drawn across each sketched graph to test which graph is the one–to–one function or not. Once, learners find out that the function is one-to-one function, then they should automatically make a connection to it having an inverse function. The processes in (5) will be encapsulated into an object when the graphs are sketched and identified as a function or not.

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c) Object and Schema Level for Genetic Decomposition

1) The objects established in the process level (4) and (5) can be de-capsulated into processes from which comes from (1), (2), and (3).
2) In the object stage, the learners has acquired adequate skills to sketch the graphs and their inverse function. Subsequently, the learners should identify that how that domain can be restricted and see the transformation of these graphs taking place once the domains of the parabolic and exponential graphs have been restricted.
3) Coherence schema for an inverse function can be considered when there are connections between the first three stages of APOS theory. The schema stage could be demonstrated by solving the applied problems. For example, if the learner was able to transform the algebraic algorithm knowledge such as working out the x and y-intercepts of the function and translating it into the geometrical representation. This requires showing the ability to sketch the graph using the dual intercepts, determining the function using the horizontal line test and sketching the inverse function, and restricting the domain for a non-function. Furthermore, the learner's understanding of the function and inverse function processes created from the actions that produced them and the objects created from the processes that produced them should be related to producing the coherent schemas. The learners should be able to make a deductive reasoning about the inverse functions of a linear, parabolic and exponential functions and make conclusions regarding the transformation of these particular graphs. The learners should be able to conclude about how these graphs transform to become inverse functions through the action, process, object up to the schema level or stages.

2.2.8 APOS Strategies of Professional Noticing

The above-mentioned conception levels of APOS theory were explored for the learner's understanding of inverse functions concepts and to determine the level at which specific learner(s) conceptions are functioning. APOS theory was implemented in this study as a framework of analysis to explore learners' mental structures in the construction of the inverse function concepts. The researcher has therefore categorized their responses into APOS levels and the analysis is given as evidence thereof. The theory on how learning a mathematical concept take place provides a framework for understanding learners' conception of function concepts (Dubinsky & McDonald, 2001). The framework gives the vision of, "subdividing the understanding of concepts related to functions into conception levels corresponding to specific mental constructions that a learner might make to develop an understanding of that concept". Thus, learners' task questions and their responses helped to transform instructional strategies for the study as it considers how to transform learners from a low conception level to the acceptable one. These conception levels provide the teacher(s) with a clearer idea with regards to learner mathematical thinking and understanding. Subsequently, it sharpens the teachers' "professional noticing skills" in terms of them hearing learners and paying attention to the required concepts and definitions. It is crucial to take into cognizance that the APOS conception stages or levels will only be used as a guideline for this study and not as an ultimate decider for the study. This is due to the awareness, that the APOS analysis was an adopted approach that was used by other researchers before.

2.3 Literature Review

2.3.1 Relevant studies on professional noticing

The professional noticing framework used in this study incorporates three components of skills namely: attending, interpreting, and deciding (Jacobs et al., 2010). Jacobs et al., (2010) explains the primary component of professional noticing is attending means to observe and identify children's words and actions when engaging in mathematical activity. Interpreting means to relate what is attended to, with what is known about development in mathematics knowledge, and to establish what the child understands. Finally, deciding is determining what is interpreted to ensure a learner is learning in a manner that best fits their current understanding of mathematics. Hence, professional noticing is positioned as a specific type of notice because the study incorporates three interrelated and key components as discussed above. The above-mentioned key components have been considered eminence and distinct of interest within mathematics researchers (Schack, Fisher, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011). Professional noticing is acknowledged as a multifaceted and thought-provoking notion (van Es, 2011), hence there are differences in perspectives unto its complexity. Thomas (2017) has suggested that professional noticing "be used as a filter to identify only the most impactful moments" (p.508). Whereas differing views regarding noticing may be "focused on capturing and interpreting as much of the landscape as possible" (Thomas, 2017, p.508).

However, (Schack, Fisher & Wilhelm, 2017) argue against teacher noticing as deeply rooted in fundamental mathematical contexts, and comparing it to sciences in schools. Other studies have also explored the practice with a focus on the aspect of how to identify, interpret and decide about the most significant and impactful occurrences in their mathematical classrooms. Consecutively, Schack et al., (2015) support professional noticing by exemplifying the instructional occurrences related to it and referring to the work of Steffe's et al., (1988) who used the notion of noticing in a brief video –recorded interaction between a teacher and a learner with the focus given to arithmetic reasoning. In the study referred to, the participants

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used attending and interpretation of the child’s mathematical activity as the key component skills. Regarding the instructional next steps that followed the video-recorded occurrences (Fisher et al., 2017), make an emphasis on how participants theorized it. In a similar study conducted for preservice teachers’ attending capabilities, it was found that the inquiry was constructed to extensively accommodate professional noticing (Star, Lynch, and Perova, 2011). Thus, studies have demonstrated through the notion of professional noticing that pre-service teachers across practice levels engaged in vigorous noticing (Floro & Bostic, 2017; Krupa, Huey, Leiseg, Casey & Monson, 2017). There was establishing evidence found in another study that professional noticing can be self-possessed with the constituents of attending, other skills, and so forth (Schack et al., 2013). In using the video-based instrument, (Fisher et al., 2017) found statistical evidence regarding the performance gains related to the three-component skills in particular the deciding component.

2.3.2 Research on professional noticing

Professional noticing was established as effective on intended instructional practices organized around progressions of “children’s mathematical development” (Jacobs, et al., 2010). Jacobs et al., (2010) agree with Mason’s view of attending to “the mathematical details in children’s strategies,” (Mason, 2011). Jacobs et al., (2010) also argue that “professional noticing” is the skill teachers use to identify and act upon salient mathematical actions of children. It is noted that there is also growth in several inquiries into professional noticing in recent years (Jacobs et al.,2010). Jong, (2017) also agrees that the notion of professional noticing has been the focus of recent studies. The professional noticing framework used in this study incorporates three interrelated components: attending, interpreting, and deciding (Jacobs et al., 2010). Jacobs et al., (2010) explains that the first component of professional noticing, attending, is to observe and identify children’s words and actions when engaging in mathematical activity. Interpreting means to relate what is attended to, with what is known about development in mathematics knowledge, and to establish what the child understands. Finally, deciding is determining what is interpreted to ensure a student is learning in a way that best fits their current understanding of mathematics. Some studies explored noticing using teacher video clubs as a way to analyze student thinking (Sherin & van Es 2009). Participants provided videos of their classrooms and analyzed the videos through a noticing lens during the professional learning sessions. Thus, found that focusing on noticing children’s thinking impacted the teachers’ instructional practices positively. Therefore, they used video vignettes to improve the professional noticing skills of elementary preservice teachers and found positive changes within one semester of instruction focused on professional noticing (Fisher et al., 2018). (Sherin et al.,2011) found evidence for the value of professional noticing within systems learning and teaching. Hence, the importance of using noticing as a conceptual tool to attending, seeing, hearing, and making sense of particular classroom events in a complex teaching environment (Sherin et al.,2011). On the notion Mathematically Significant Pedagogical to Build on Student Thinking (MOST) research study it is evident that Stockero et al., (2017) support the ideas of Sherin et al., (2011) of using classroom video-based development to gain practice in identifying key opportunities for pedagogical action within a mathematics lesson. Prediger et al., (2008) assert that the researchers distinguish the advantage of distinctive frameworks that embrace approachable teaching practices such as professional noticing. Explicitly, there was a theoretical space identified "that would help the researchers to examine the nature of teachers' mathematical-pedagogical reasoning about learners" (Hill, Ball & Schilling 2008). Moreover, (Erickson, 2011) affirmed that professional noticing has been associated with (Dewey's, 1904) description of attention that was observed using behavioural cues. Finally, the deliberations regarding professional noticing are organized around pedagogy through learner and teacher interaction.

2.3.3 Research on the inverse of the function

Studies conducted by Attorps, Björk, Radic, and Viirman (2013) were advocated towards GeoGebra teaching inverse functions. In the outcome for their study, it was indicated that some of their students demonstrated an instinctive comprehension of inverse functions in terms of reflection. However, these students did not fully comprehend the reasons for performing reflection in an inverse function. Wilson, Adamson, Cox, and O'Bryan (2011) also considered with regards to their algebraic and geometric perspectives. Their argument was concerning the mutual techniques on exchanging the x and y variables to find the inverse which creates uncertainty for the investigation, which means the outcome is concealed in the context of real-world problems. It was further argued that exchanging the variables and sketching the reflection in the line $y = x$ does not take into cognizance the imperativeness of interchanging the domain and range of the function and vice versa. The function concept is the idea of a correspondence between two sets of objects. Wilson et al., (2011) rather proposed a different approach to solving the dependent variable, instead of swapping x and y. This proposed view suggests improving and develop learners' conceptual understanding of inverse functions.

On the other hand, Vidakovic (1996) placed significance on combining the function and the inverse to be identified despite the exchange in x and y. Carlson and Oehrtman (2005) categorized different conceptions of inverse function: inverse as algebra, inverse as geometry, and inverse as a reversal process. Carlson et al., (2010) illustrated that students who conceived of inverses as reverse processes were more familiar with answer different questions concerning inverses. If $f(x)$ represents a function, then the notation $f^{-1}(x)$, read "f inverse of x", will be used to denote the inverse of $f(x)$. Similarly, the notation $g^{-1}(x)$, read "g inverse of x", will be used to denote the inverse of $g(x)$. The core of this study lies in the "inverse of a function and learners' conceptualizing it as an integral part of this study. Concerning inverse function definition, it is considered a concept that "undoes what a function does" (Bayazit & Gray, 2003). It is significant to note that an inverse function only holds if the function under discussion is one-to-one or many-to-one. The concept of an inverse function, which is further, defined as a process of doing and undoing operations. This study has drawn broadly on the concepts of function and its inverse, moreover on how "teacher professional noticing" impacts learners' understanding and critical thinking skills in a mathematics classroom.

2.3.4 Research on the function concept

The function concept involves the idea of a correspondence between sets of objects. A function is defined as a rule for the domain in conjunction with the range as illustrated in Figure 12 adapted from (Phillip, Basson & Botha 2012). This figure demonstrates the relationship between the (domain) and the (range). Two can also denote the x –variable and the y-variable of a function or set of x and set of y as described in the text.

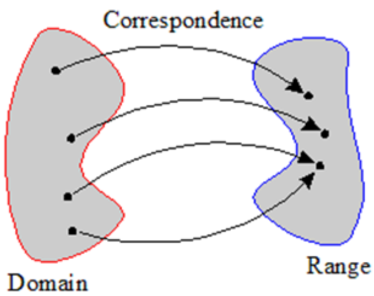

Figure 8: Correspondence of Domain and Range (Adaptation of Phillips, Basson & Botha, 2012)

The function concept is a phenomenon that utilizes quantities in everyday life relationship which means a quantity determines another one such as "input and output" or "domain and range" or "independent variable (x) and dependent variable (y)" in Mathematics. For example, in everyday life the more money you pay for some goods, the more of them you get or the better the quality you get. It is also critical for learners to adopt the understanding of concept images of inverse functions. Findings in the study established that learners were deliberating on the concepts involving functions and their inverse (Cansiz, Kücük, & Isleyen, 2011) These findings brought the researchers to a conclusion of experiencing difficulties in understanding the function concept. From other studies, the researchers made conclusions that learners were

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still experiencing difficulties with the function concepts Other studies viewed the understanding of functions in terms of relational thinking supporting a high level of mathematical thinking and reasoning (Celik & Guzel,2017; Cansiz et al.,2011).

2.4 Conclusion

This chapter has outlined the APOS theory as an analytical framework for learners' written pre and post-tests. In this chapter APOS theory was defined by other research studies, the APOS levels or stages are unpacked and the summary of the levels is also provided for the reader to understand. The overview of APOS theory was discussed and an example of the genetic decomposition has been demonstrated and unpacked under the theoretical framework. Under literature review, the study has broadly explored the relevant studies on professional noticing, thereafter engaged with the research on professional noticing, research on inverse function, and finally the research on the function concept. The challenges that learners continue facing regarding the understanding of concepts on the topic of inverse functions were discussed in this literature review. Outlining those challenges led to the learner's difficulties in building the necessary mental structures to deliberate with concepts and definitions of inverse functions.

Thus, APOS theory and the noticing sophistication level for teacher noticing are used as the analysis frameworks. APOS is used for analyzing learners' mental constructions on their written tasks and noticing sophistication level for analyzing the teacher noticing level using questionnaires and the video-recorded lessons derived from intervention. Therefore, the data analysis for this study will be in two forms; firstly, data analysis was from the learners' written tasks, and secondly, data analysis was from the video recording to investigate teacher sophistication levels during intervention lessons. The next chapter will engage with the study's methodology, the research approach, design, and thorough examination of how data was collected validated, the reliability of data, and also the ethical considerations. Then chapter 4 will discuss data analysis and findings.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This chapter also discusses and justifies the qualitative research approach used in the study, highlight the sample selection, the data collection instruments, as well as the data analysis. This chapter also details the procedure used for the trustworthiness of the study and maintenance of rigor in terms of validity and reliability. Furthermore, it explains what ethical consideration measures are taken for the study. Finally, the study explicitly provides the design of the study.

3.2 Research Method and Approach

This research was informed by an APOS theory framework that focused on learners' mathematical thinking and understanding of the inverse function concept. This study was exploratory in developing a deeper understanding of Grade 12 learners' mental constructions on inverse function concept. This study also used the approach of teacher "professional noticing" to further investigate how APOS theory can be developed into learners' mental construction.

In recognition of this study's exploratory and interpretative nature, the qualitative approach was adopted and found relevant for data collected. Creswell (2012, p.70) defines qualitative study as "inquiry process of understanding a social or human problem, based on building a complex, a holistic picture, formed with words, reporting detailed views of informants and conducted in a natural setting". Yin's (2016) argument concerning qualitative research, highlights that a researcher can conduct a thorough investigation of the phenomenon in their everyday context. This research was informed by APOS theory that focuses on learners' understanding of the inverse function concepts and questionnaires to analyze teacher noticing.

3.3 Design of the Study

3.3.1 Context of study and sampling

The research for the study was conducted during three different teaching sessions of mathematics at a school in Gauteng Province. Participants of the study were chosen as a sample size from a single Grade 12 class of 33 learners. The sample size chosen was manageable and chosen based on time constraints and availability of learners before the commencement of the study. A total of 18 learners were involved and two teacher participants namely (teacher A and teacher B). The research methods were aimed at providing the exploratory and narrative perspectives of the study regarding teacher noticing. The term "professional noticing" uses the epistemological and ontological as these are drawn from the nature and board of knowledge. Ontology is described as the nature of knowledge and epistemology as the broad of knowledge and “the deeper meaning behind research questions lies in the ontological and epistemological perspectives” (Trede & Higgs, 2009). The research for the study was designed to explore “professional noticing” as a phenomenon that can be understood and enhanced towards the transformation of pedagogy.

3.3.2 Data Collection

Various methods of data collection were used in this study. Firstly, the structured pre and post-test questions on the inverse function concepts. Data was gathered using learners' written tasks. Subsequently, from the video recorded analysis of teacher professional noticing learners’ mathematical understanding through interaction with learners during the teaching of inverse function concepts. The questions aligned with the CAPS curriculum syllabus were considered for the pre and post-test on the inverse function concept **(refer to Appendix B)**. Subsequently, the video recorded lesson with the semi-structured questionnaire of the lessons are a source of data collection for the objectives of professional noticing study. In terms of teacher participants, the questionnaires allowed these participants to voice their opinions and views without the confinement of the researcher’s prejudiced ideas (Creswell, 2012; Yin, 2016). Data was quantified through questionnaires arising from video evaluation and pre and post-tests. This study used different tools to collect data. The initial assessment tool (pre and post-tests) measures learners’ conception levels of the concepts, definitions, and representations on the inverse function topic. The learner's mental construction of mathematical concepts of inverse function was perceived as key in this study due to difficulties that learners were experiencing in this topic (DBE, 2018) diagnostic report. APOS theory explored the individual learner's thinking structures in pursuit to make sense of the mental constructions used in learning of the inverse function (Maharaj, 2010). The second assessment tool measures levels of sophistication on teachers noticing their learners’ attending, analyzing, and responding to their understanding skills. The tools for assessment entail the pre-test component before the intervention on functions and inverse after intervention, the post-test assessment component of functions and inverse. Table 1 gives a summary of the research design matrix concerning the posed research questions, the research instruments used, data collection methods, to establish the assessment purpose and method of analysis used. The table is used as a guideline and as a summary of the research design used in the study.

Research Questions	Research Instruments	Data Collection Methods	Assessment purpose	Method of Analysis
1.What APOS theory levels of learners' mental constructions do teachers notice in the learners' written tasks on the concept of inverse function?	1 Pre and post-tests. (Written tasks) and Memorandum.	1.Teacher assess learners using pre-tests on inverse function. 2. Learners display their thinking by attempting to answer questions related to inverse function, definitions and concepts. 3. Teacher use the memo as marking tool to evaluate learners' mental constructions.	1.To determine learners understanding of concepts, definitions, representations and difficulties. 2.Learners responses to the pre and post-tests questions in terms of the APOS mental constructions.	1. Marked. Pre and post – test and APOS analytical framework to analyse. Table and graphs(pie charts and bar graphs)
2.What are the learners' mathematical thinking and understanding noticed by the teacher during intervention classroom interaction on the concept of inverse function?	2.Intervention classes or sessions and video recorded lessons.	2.Learners pre-tests provides learners' conception levels and informs intervention prior to post-tests	2. To determine the teacher noticing sophistication level from intervention.	2.Constant comparative method of analysis was used to compare teachers' sophistication levels of noticing.
3.What are the levels of noticing sophistication observed when the teacher(s), attend, analyse and respond to learners' mathematical thinking and reasoning during intervention?	3.Intervention classes or sessions and video recorded lessons. Questions and Answer to determine attending, analysing and responding.	3.Application of sophistication level analysis framework to analyse teacher noticing. Assess from the transcripts from the video recorded lesson the teachers attending, interpreting and deciding skills.	3. To establish if professional noticing can be recommended as a framework for attending, analysing and deciding for concept inverse function	3. Barnhart & van Es, 2015 adapted framework.

Table 1: Adaptation of Research design matrix

a) Pre and post-tests

Learners were given a maximum of 40 minutes to complete the pre-test and a few days later an intervention session was conducted to address learners' errors and issues noted in the pre-tests. Thereafter, two days later learners were allowed to write post-tests which they were not aware was similar to the pre-test. The pre-tests and post-test questions were structured questions with a focus on the concepts and definitions of inverse functions (**refer Appendix B**):

1. Learner's mathematical knowledge and identification of what is a function and what is not a function.
2. Learners providing the correct concepts of the domain and range on each of the given functions.

3. Learners mapping out the correct identification of varied functions e.g. one-on-one function, many-to-one function, or one-to-many function.
4. Learners considering an equation and sketching the correct function and its inverse from the given equations i.e. Linear, Quadratic and Exponential equations.
5. Also, learners restricting the required functions accordingly to produce an inverse function and provide the correct interval for the restriction of functions.

b) Video Recorded (Intervention Lessons):

Secondly, the video recorded lessons were conducted by the researcher, mathematics educator during teaching practice as another integral part of data collection. The video recorded lessons were used to analyze the researcher's intentions on professional noticing during classroom practice and helped to inform her successes and/ or failure in noticing. Some of the recordings were conducted during her intervention session before the post-test session. The summary of video details as outlined below and the semi-structured questions answered by the teacher participants analyzing the video recordings are also included below.

1. Video recording of 45 minutes to an hour lesson was conducted with a class of 18 male and female learners.
2. The researcher conducted a normal lesson on inverse functions highlighting all the important concepts and definitions to the learners.
3. The teacher, as a researcher was focused on providing insight to the learners without being biased to the fact that the lesson had anything to do with the research, however, the lesson was conducted fairly and was done in a normal classroom setting.
4. Learners' faces were not included in the video recording as it was clung onto the researcher's part only.

c) Questionnaire for teacher participants:

These are the open –ended semi-structured questions posed on the questionnaire summarizing the video recorded analysis and intervention classes held. Responded to by two teacher participants, their responses attached **(see Appendix D, annexure 3).**

3.4 Validity and Reliability

Cohen et al, (2000) argue that validity and researcher bias is closely connected. Cohen et al., also suggest that the "most practical way of achieving greater validity is to minimize the amount of bias as much as possible" (Cohen et al., 2000, p.121). Cohen et al, (2000) classify the bias sources as the characteristics of the interviewer, of the respondent, and the substantive content of questions. Validity is explained as "the extent to which the results can be generalized beyond the sample used in the study" (Burns & Grove, 2010). For the validity of the study, there was a need to choose a convenient sample. To also ascertain the validity of the study, in the context of this study the teaching sessions were recorded, and thereafter the unedited recordings were provided to the two teacher participants individually together with the questionnaires to observe, transcribe and give an analysis of the lessons without any preconceptions. As a result, the participants' analysis was used in making sense of the data, that was analyzed and the findings were what teacher participants expressed. Similarly, with the pre and post-test, the findings were from individual learner's interpretation and construction of their understanding of the mathematical concepts.

Reliability is defined as "the extent to which a test or procedure produces similar results under constant conditions" (Bell, 1999, p.103) in (Opie, 2004, p.66). Cohen et al., (2011) caution researchers from allowing their prejudiced ideas to interfere with the data collection. Arguing towards misinterpreting the answers from the respondent so that they suit the predetermined notions of what the researcher expected to find in the field of study. Cohen et al., (2011) further argue that "a mere presence of the researcher as an observer can cause those being observed to act differently to how they would usually act" as the current situation that is being observed may include the preceding events. This text implies that the observer or researcher must take into consideration that there may be more to what is being observed. Therefore, for the reliability of the study as a researcher, what I observe during my investigation may be all I can make extrapolation to. As a result, to address the misconceptions triangulation of data sources and methodologies was proposed as a reliable instrument for the study (Cohen et al., 2011). For this study reliability was ascertained by designing a questionnaire for teacher participants as observers to the video recordings on teacher professional noticing, along with intervention and structured pre and post-test which the researcher designed a memo to establish learners' mental constructions of the inverse function concepts. Consequently, this study explored the work of Maxwell (1992) to ascertain rigor in the qualitative study.

3.5 Ethical Considerations

Ethical clearance was requested from the Wits' Faculty of Humanities Research Ethics Committee accordingly. The principles that guide this process entail the following: confidentiality and anonymity. The principle of beneficence encompassed freedom from harm and exploitation. All participants who were involved in the study were fully informed about the procedures, expectations, and intentions of the study. Participants under the age of 18 and more, were provided with consent forms for their parents to give consent for their participation as a requirement and precautionary measure. A teacher's consent form was also provided to the two teacher participants that were taking part in the study. All participants who participated in the study were fully informed about the procedures, expectations, and intentions of the study. Each participant voluntarily signed a consent form. A letter to GDE requesting permission for the study to be conducted at a particular school was submitted to GDE and approval was received, and permission was granted accordingly. On the same notion, a letter to the Principal of the school outlining the purpose of the study as well as consent and information regarding terminating the study was also submitted accordingly and permission was granted by the principal. The participants' confidentiality and anonymity were maintained and pseudonyms were used in this study. Samples of the information and consent forms including ethics approval letters are made available in the Annexures of this study. All protocols were observed according to the ethics requirements and the application to all relevant stakeholders was correspondingly done.

3.6 Data Analysis

The analysis was measured using the APOS theory whereby the theory was used as an analytical framework and the learners' mental constructions in the understanding of inverse function were perused into details using each level accordingly. The APOS theory was used to conduct data analysis as a consideration to the reliability and validity of the study. Utilizing the relevant indicators such as learners' responses in line with the APOS levels as units of analysis of their mental structures. APOS theory was applied in the data analysis according to the four-category stage. These categories have been unpacked in the theoretical framework in Chapter 2. Data analysis is presented randomly based on learner participants' pre and post-tests as outlined below. Each data was analyzed guided by APOS theory conception of each learner in mind while describing individual learner's analysis as an excerpt in the study using learner's scripts from their pre and post-tests to analyze what level they are operating at and their difficulties. Also providing a constant comparative of each learner conception levels from their pre and post-tests. The adapted framework is represented in Figure 13 (See the pre and post-test tasks in **Appendix B**)

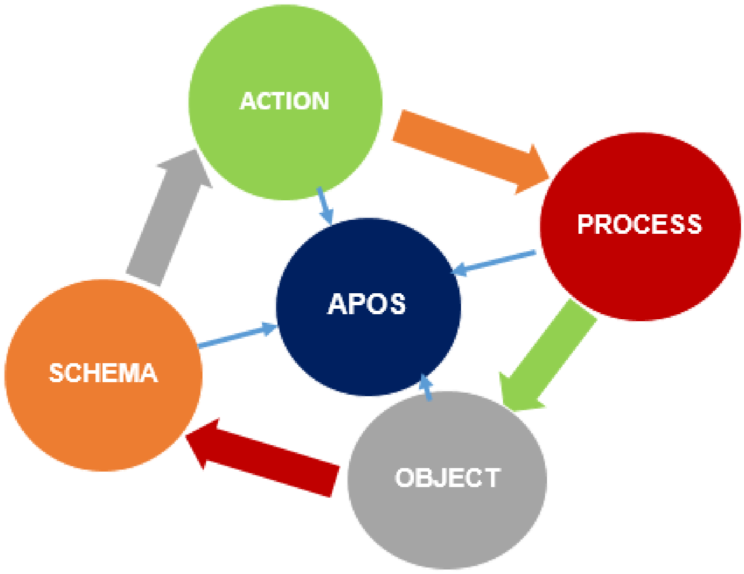

Figure 9: APOS Analytical Framework

In analyzing the data gathered through pre-tests and post-tests, the learners were given these tasks in a quest for the teacher to examine some important mathematical concepts related to functions and their inverse. To determine learners' mental constructs from their individual written responses, I refer to APOS theory indicators as indicated in Table 2. Learners' critical mathematical skills and understanding of the inverse function concepts becomes crucial in answering the pre and post-tests allocated to the Grade 12 participants. There were specific questions mapped out for this particular study in a pursuit to ascertain learners' mental constructs of thinking and understanding of inverse function concepts. There are important aspects that the study focused on, to assess the participant's level of conception through the use of the APOS analytical framework. APOS analytical framework was chosen for the study to provide learners with structured activities that provide the basis of action, process, and object levels and coherently developed into schemas of function and its inverse. Accordingly, "the role of teachers is to create situations in which learners are likely to construct these actions, processes, objects, and schemas for themselves" (Weyer, 2010, p.10). This implies that APOS analytical framework for the current study is intended to helping learners build the fundamental mental structures that can be organized into coherent schemas. Hence, from this perspective of APOS theory, I perused this study with an attempt to identify the pertinent mental structures that were constructed in the learning of inverse function concepts by Grade 12 learners.

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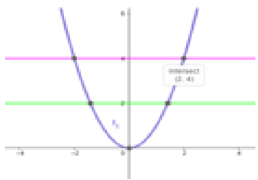
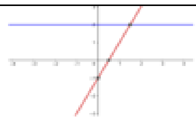
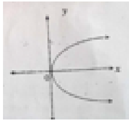
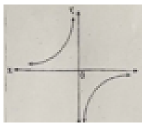
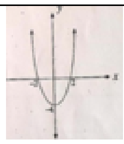
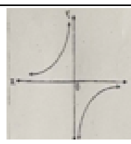
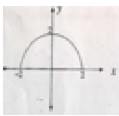
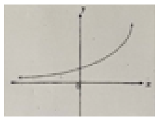
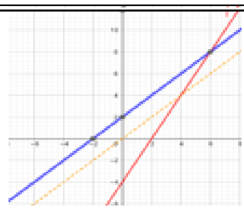
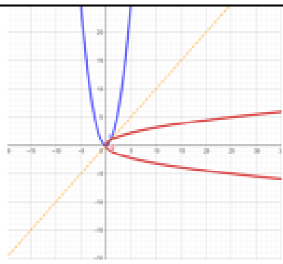
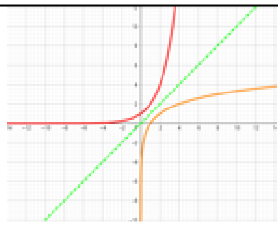
Task	Concepts covered and assessed	Questions asked to analyse data	Diagrams Illustration	APOS levels / indicators
1.	Determine if each of the following relations is a function or not	Will a learner be able to recognize if the given graphs are functions or not?		**A**
2.	If the relation is a function, determine if it is a one-to-one or many-to –one or not a function.	What cues will the help learners to know the difference in terms of the relations and type of functions?	  	**A & P**
3.	Write down the domain and range for each relation.	What helps the learner to identify the domain and range for each relation. How does the teacher notice that the learner has correctly identified the domain and range for each function?	   	**A & P**
4.	Consider the function and sketch of and its inverse function $f(x) = 2x - 4$	What mental constructions a learner need to have to sketch the graph of $f(x) = 2x - 4$ and it's inverse.		**A, P &O**
5.	Consider the many to one function. Draw the graph of this function and its reflection about the line.	What does a learner understand by a many-to-one function and what makes it different from a one-to one function Can a learner sketch the reflection of this function and what is important to know about the reflection of this function?		**A, P, O &S**
6.	Consider the function of $f(x) = 2^x$, since the graph is a one-to-one exponential function. Its inverse should also be a one-to-one function for the domain not to be restricted. Sketch the graph and its inverse on the same set of axes.	What mental constructions does a learner need to be able to sketch the function of $f(x) = 2^x$ and it's inverse function on the same set of axes.		**A,P & O**

Table 2: Adaptation of Chimhande's (2014) Table on Content analysis using APOS levels

3.7 Pre and Post-test content analysis

The learners were each provided with 40 minutes to complete the pre-test and the post-test. **Question 1** was structured to assess learners attending strategies (conceptual understanding and analyzing skills (definition understanding). Question 2, 3, and 4 were designed to assess learners attending strategies, analyzing skills, and interpretation and responding skills (representation understanding). The first three questions asked in question 1 provided insight into analyzing learners' written work which gave a good understanding of the action and process conception levels of learners:

a) Learners' understanding of when a relation is a function; how learners identify the difference in relations?

b) And how they determine that meaning can they recall the vertical and horizontal test to further determine if it's one–to–one or many-to-one.

c) On the same notion if learners can write down the domain and range of each relation and represent it accordingly. Moreover, data collected was necessary for the researcher to determine if learners can relate to the domain as a set of x-variables and range as a set of y-variables also as both elements of real numbers.

Question 2, 3, and 4 were used to assess learners' actions, processes, object up to the schema conception levels possible. Research has shown that APOS theory was used by other researchers to characterize learners' ways of understanding the function concept from the action level up to the schema level (Weyer, 2010). Furthermore, APOS theory was used to explore individual mental structures given a complex mathematical concept (Maharaj, 2010).

a) Question 2 was structured to assess the learner's (x and y-intercept) attending strategies or understanding of plotting point by point to sketch the required graph.

b) The same question was used to create an opportunity for learners to sketch the prescribed graphs and their inverses.

c) For question 3 & 4, learners written work was used to assess learners restrict the function $f(x) = x^2$ to make it a function. Also, I used the learner's written work to determine the interpretation and responding skills (representation understanding).

As a result, the formative assessment tasks were used to establish conceptual understanding, definitions, and representations of functions and inverse. On the same notion, it was an opportunity to establish learners' mathematical thinking and reasoning in the topic of inverse functions. Similarly, to determine the teacher's sophistication level of noticing during the teaching practice (**see Table 3**).

Levels of Sophistication for Noticing Skills			
Skill	**Low Sophistication**	**Medium Sophistication**	**High Sophistication**
Attending	Highlights classroom event, teacher pedagogy, student behaviour, and/or classroom climate. No attention to student thinking.	Highlights student thinking with respect to the collection of data from the scientific enquiry (scientific procedural focus).	Highlights student thinking with respect to the collection, analysis and interpretation of data from scientific inquiry (scientific conceptual focus)
Analyzing	Little or no sense making of highlighted events: mostly descriptions. No elaboration or analysis no use of evidence to support claims.	Begins to make sense of highlighted events. Some use of evidence to support claims.	Consistently makes sense of highlighted events. Consistent use of evidence to support claims.
Responding	Does not identify or describe acting on specific student ideas as topics of discussion; offers disconnected or vague ideas of what to do differently next time.	Identities and describes acting on a specific idea during the lesson; offers ideas about what to do differently next time	Identifies and describes acting on a specific student idea during the lesson and offers specific ideas of what to do differently next time in response to evidence; makes logical connections between teaching and learning.

Table 3: Adaptation Framework for Teacher Noticing (Barnhart & van Es, 2015)

3.8 Sophistication Levels for Noticing

The sophistication levels for noticing are outlined and clarified in Table 3 as the adaptation analysis approach for teacher noticing. It was considered for this study to guide and inform the teachers' instruction, and review levels of sophistication between low, medium to high sophistication for teacher noticing. While, this analysis approach peruses in-depth, the components of attending, analyzing, and responding skills that the learners' entails during mathematical tasks and the classroom interactions with their teachers. This approach carries a huge significance to analyzing the data collected from video-recorded lessons which pertain to the teaching practices and the intervention sessions. The objective of implementing this approach into the study was to ensure the validity of this study and affirm that the instruments used do provide the teacher noticing part of the study with the analysis guideline for the collected evidence. Similarly, that guideline will subsequently review the teacher noticing that concepts are used in understanding the inverse and its functions. This analysis guideline together with the APOS analytical framework was discussed in detail in the data analysis chapter six of this study. The sophistication for noticing framework will facilitate teacher noticing levels according to the coding matrix constructed. Hence, APOS theory will explore the learner's mental structures according to their conception levels (Maharaj, 2010). APOS theory embarks on analyzing the internal mental structures and strategies constructed by individual learners when learners think about a mathematical concept (Dubinsky & Wilson, 2013). The APOS theory has also been used to effectively develop learners' process conception of the function concept (Breidenbach, Dubinsky, Hawks & Nichols, 1992) and to evaluate the learners' understanding of the function concept (Dubinsky & Wilson, 2013). Consequently, it was asserted that APOS can be concurrently used to develop and evaluate learners' understanding of the inverse function concept (Weller, Arnon & Dubinsky, 2011) of which was also the best-suited approach for this study.

3.9 Conclusion

For this study, APOS theory was used to characterize learners' ways of thinking about the function concept from the action level up to the schema level (Weyer, 2010). This implies that the learner's understanding of the concept of functions and inverse may not be limited to action and process levels and they can be used up to the schema level. APOS was also adopted to examine learner's mental constructions in mathematical thinking and understanding of a concept "inverse function" from their knowledge of a "function" concept. Research has shown that other researchers have used the APOS theory to investigate conception levels up to object-level to evaluate grade 12 learners' understanding of mathematical concepts similar to functions prior (Dubinsky & Wilson, 2013; Brijlall & Ndlovu, 2013). Consequently, the adopted theory and its application to mathematics teaching can adhere to two assumptions: The first assumption claims that learners' mathematical knowledge shows "ways of responding to perceived mathematical problem situations and their solutions by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes, and object and organizing these in schemas to use in dealing with situations" (Dubinsky, 2001). The second assumption claims that learners are not learning mathematical concepts directly. "They apply mental structures (critical thinking) to make sense of concepts" (Piaget, 1977). Therefore, mathematics learning is facilitated if learners retain critical thinking and understanding of a particular concept. If the required thinking structures are not applied, therefore learning a mathematical concept becomes impossible. APOS clarifies those mental/thinking structures required as actions, processes, objects, and schemas. For this study, consideration was given to the four levels of APOS which informed the analyses of this study. APOS framework will guide this study as an analytical framework that helps to put information in convenient categories to analyze the data. The researcher has expanded on the four levels of this study to provide definitions and characteristics of what these levels entail and below there are integrated conceptual and analytical frameworks outlined below.

CHAPTER FOUR

RESULTS, ANALYSIS, AND INTERPRETATION

4.1 Introduction

This chapter presents the results, analysis, and interpretation of the data collected from the participants of this study's written tasks before and after the intervention. The participants were assessed for the topic of inverse functions which is in the CAPS curriculum. The other results and analysis presented pertain to teacher noticing learners attending strategies, interpretation of learners' mathematical understanding, and the teachers responding skills based on their sophistication levels of noticing during the intervention lesson in the teaching of inverse functions. The learners written tasks were analyzed using APOS theory (conception levels) analytical framework and the teachers' sophistication noticing levels guided by Barnhart and van Es, (2015) framework.

This research aimed to examine the professional noticing sophistication levels that the teacher uses towards learners' mathematical understanding and thinking. Furthermore, the study investigated the learners' conception levels about their mathematical thinking and understanding of concepts, definitions, and representations in inverse functions. To analyze, interpret and make inferences towards the findings of the data analysis, the research questions used in the study remains pertinent:

- What APOS theory levels of learners' mental constructions do teachers notice in the learners' written tasks using the concept of inverse function?
- What are the learners' mathematical thinking and understanding noticed by the teacher during intervention classroom interaction using the concept of inverse function?
- What are the teachers' levels of noticing sophistication observed when attending, analyzing, and responding to learners' mathematical thinking and reasoning during the intervention?

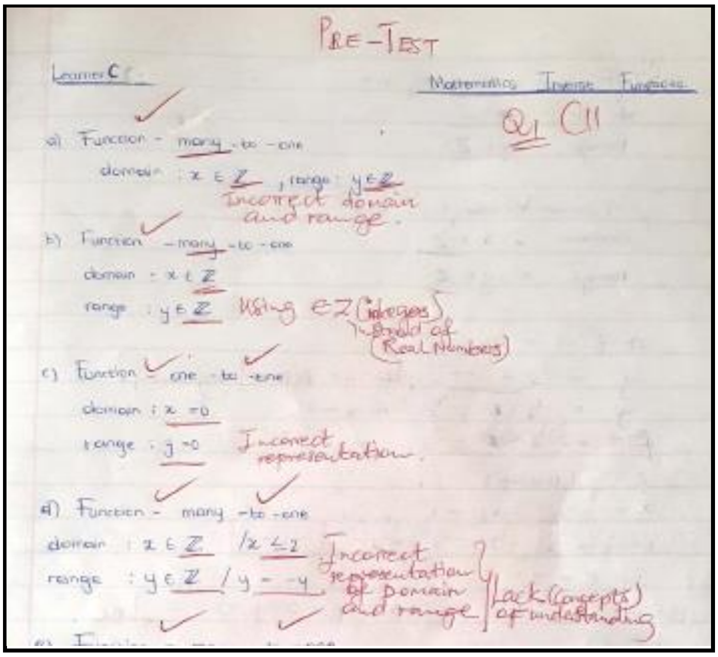
4.2 Results and analysis on Pre and Post Test (APOS conception levels)

From the emerging study, it has been made evident that the learners' mathematical thinking and understanding of the inverse function are pivotal to this study. Professional noticing and the three key components: attending to the learners' conceptions, interpreting learners' mathematical understanding, and deciding on how teachers respond are the foundational and fundamental aspects of this study. Using APOS levels as an analytical framework to ascertain learners' conception levels and mental constructions were considered prominent and crucial to achieving the objective, conclusion, and recommendations of this study. The first part of the findings emerged from the APOS analysis which solely analyzes the learners' mental constructions based on their written tasks (pre and post-tests). This data was organized and presented using the table, and bar graphs and it was further explained using the four APOS stages of learners' conception.

Excerpt 1, represents learner C's pre-test and the object of focus was for this learner to answer six questions as tabulated in Table 2 of the content analysis. Question 1 was assessing learners' action level and Question 2 was designed to assess the process. Both questions were intended to assess two different conception levels.

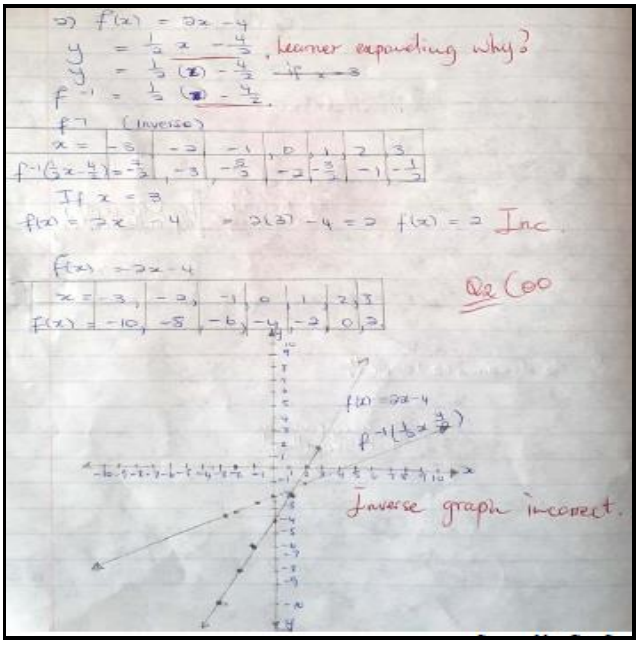
Learner C (Pre-test)

Task	Concepts covered and assessed	Questions asked to analyze data.	APOS level/ indicators
1.	Determine if each of the following relations is a function or not?	Will a learner be able to recognize if the given graphs are functions or not?	A
2.	If the relation is a function. Determine, whether it is a one-to-one or many-to-one.	What cues will help the learner to know the difference in terms of the relations and functions.	A & P
3.	Write down the domain and range for each relation.	What helps the learner to identify the domain and range for each relation. How does the teacher notice that the learner has correctly identified the domain and range for each function.	A & P

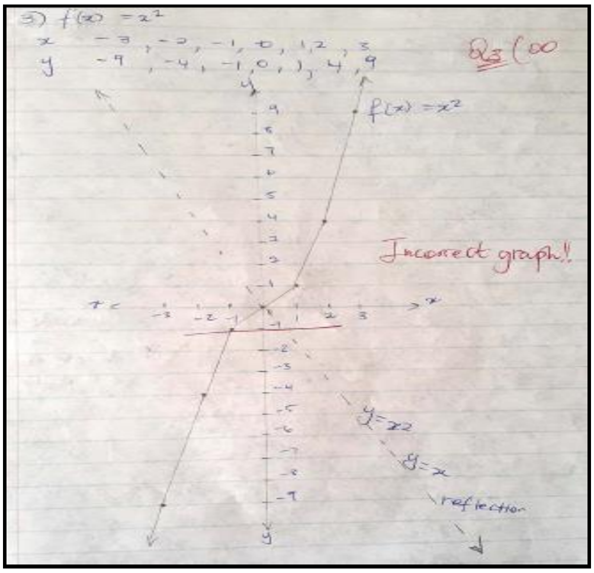


Excerpt 1: Learner C (Pre-test)

The learner experienced some difficulties in understanding that the domain and range are elements of real numbers ($\mathbb{R}$). Referring to (excerpt 1) it is noticeable from this learner's written task, according to APOS analysis that the learner understands the relation and they could distinguish between a function and not a function. They could also respond correctly to in many instances to determining which function it is known to be depending on their understanding of a function. The evidence shows that the learner had a challenge with understanding a domain and range even in terms of writing it as a set notation. Learner C denoted the domain and range as the elements of integers($\mathbb{Z}$) instead of a real number ($\mathbb{R}$). In this analysis, it is evident that the learner is at the action level, due to how the learner is unable to gather his or her thoughts without the external stimuli instruction, which says that the learner only knows how to operate memory or from given instructions (Dubinsky & McDonald, 2001). In the action, stage learners are perceived recalling concepts from memory (Cottrill et al., 1996). The learner recalled the element of integer for domain and range which is not a correct symbol and notation. Moreover, they are unable to recall the correct way of expressing the set-builder notation. Meaning this learner has not conceptualized these concepts accordingly. Hence, this learner was stagnant in the action stage of conception.

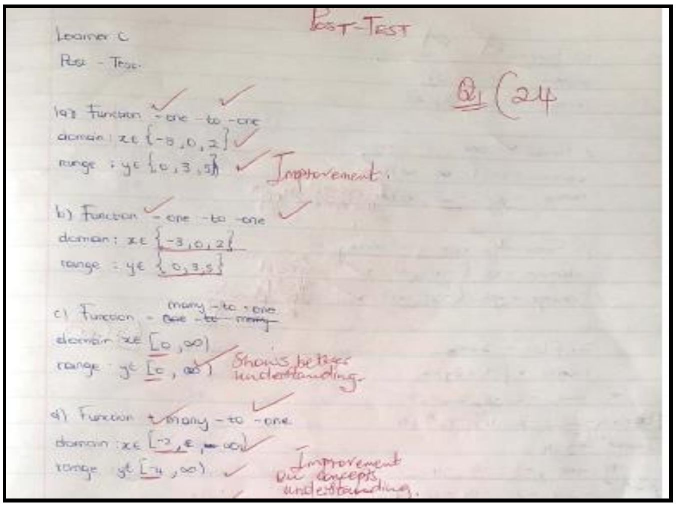


Similarly, in Question 2 the learner could not comprehend working out x and y –intercepts, to sketch the correct function and its inverse. For Question 2 and 3, the learner was therefore unable to sketch the graphs and inverse graphs of $f(x) = 2x - 4 \; and \; f(x) = x^2$ correctly. Evidence in the pre-test has shown that the learners' mental construction stage is at the action level of APOS analysis as the learner was having no understanding of solving the problem. In this instance, the learner has not gained adequate skills to connect to other mathematical forms to construct a precise symbolization. The learner shows limited mental constructs on the concept and the various representation of inverse functions linked together (Bennet, 2009). The learner could neither formulate the x and y-intercept method correctly nor work out the dual intercept method so to sketch the graph. Bayazit and Gray, (2004) argued concerning students whose understanding is limited to the action conception level, that they would have difficulties working out the rule of an inverse function. In Question 2 and 3 of the learner script, it is clear that the learner could not work out the function step by step as there lacked good conception level regarding the determining the coordinates of the function of $f(x) = 2x - 4$ and same with the function of $f(x) = x^2$, the coordinates are incorrect.



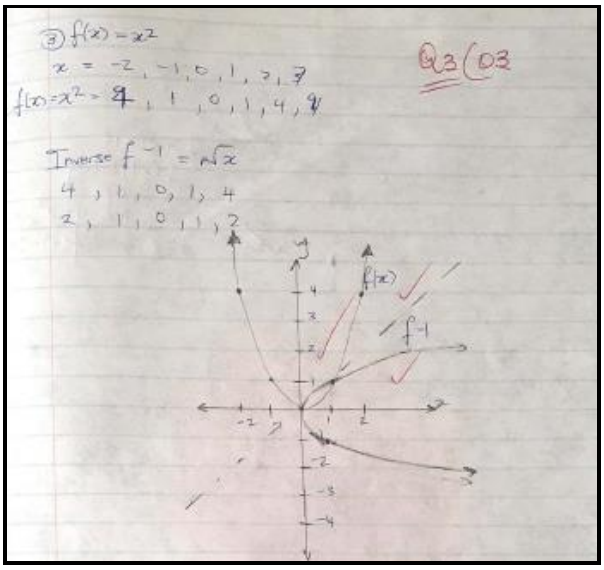
Explicit performance of each step in an algorithm process is encouraged (Arnon et al., 2014). Following (Arnon et al., 2014) argument regarding explicitly performing each step, it is evident that this is not demonstrated in learner C’s written task. Thus, it was even difficult for the learner to sketch the graph because there were no coordinates.

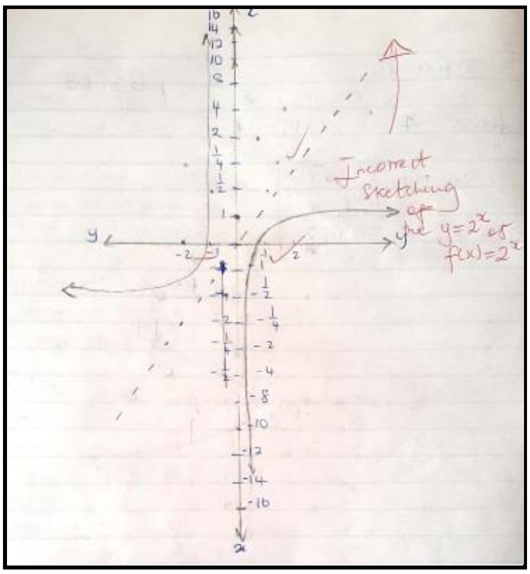
Learner C (Post-test)



Excerpt 2: Learner C (Post-test)

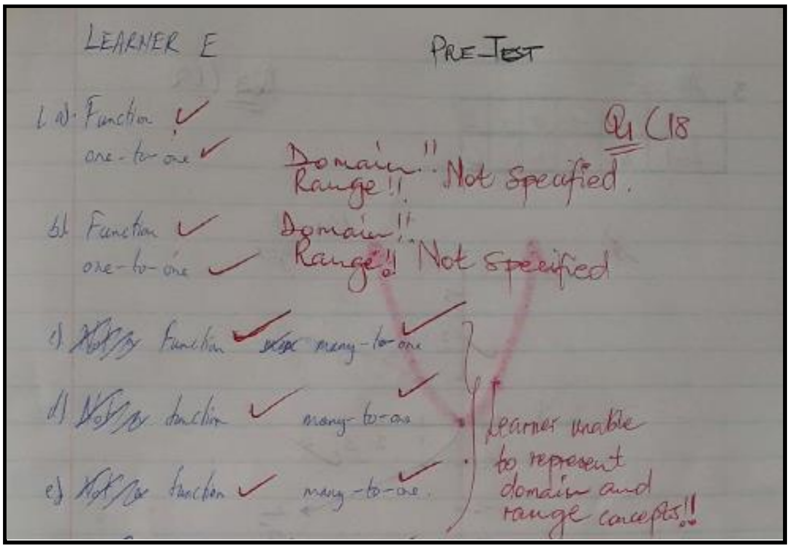
The post-test shows an improvement regarding the understanding of the domain and range. The learner shows evidence of understanding that domain is mathematically denoted as $x \in \{\ \}$ or $x \in [\ \ ]$ and $x \in (\ )$ and range in the form of y ∈ { } or y ∈ [} and y ∈ ( ). This shows evidence of the development of the concepts. Moreover, the learner's thinking has developed from the activity to the process stage of APOS. The learner at this stage has internalized the procedure for working out the domain and range which could be recalled from the memory (Cottrill et al., 1996) because he/she was able to perform each step explicitly, by showing the domain elements and using the correct symbol and notations (Arnon et al., 2014). It also shows that the learner has now assimilated new information from the correct mapping of a function, identifying the domain and range concepts correctly.





In Questions 3 and 4, learner C was able to plot and sketch the function correctly with minor errors which also shows that the learner has been orientated to learning and making mathematical connections due to understanding teacher utterances during the intervention session. The post-test also shows some improvement on this particular learner's assessment. Hence, (Dubinsky, 2010) would refer to this transformation as an action converted to a process through interiorization. It also shows that the learner recognizes the process as a whole as the learner was now able to sketch the graph and its inverse function correctly and restricted the function in the line $y = x$ accordingly. Therefore, the learner has now comprehended that transformation can be performed (Dubinsky & McDonald, 2001) as the process of sketching the function of $f(x) = x^2$ and the inverse of $f^{-1} = \sqrt{x}$ was correctly performed. This excerpt 2 shows the improvement in Questions 3 and 4 of the post-test. The teacher played a significant role to construct learners' mental structures so to strengthen the adaptive reasoning on concepts (Brijlall & Ndlovu, 2013). The teacher has identified and described how learners can act on some ideas during the intervention session which helped learners to execute their ideas differently in their post-tests.

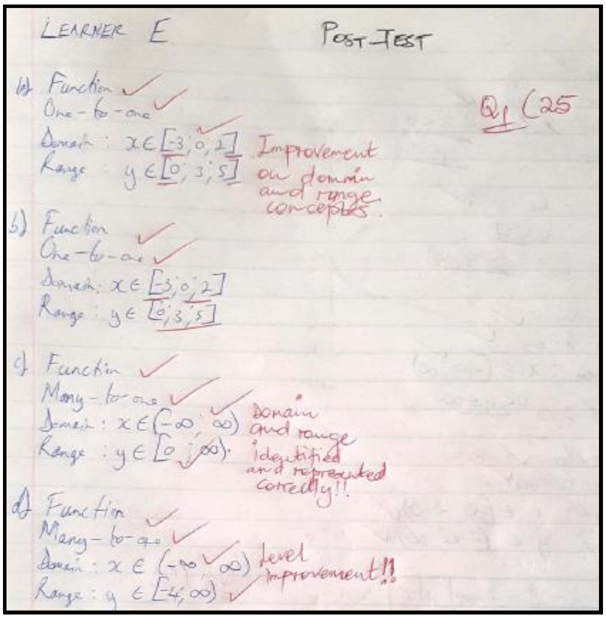
Learner E (Pre-test)



Excerpt 3: Learner E (Pre-test)

Excerpt 3, learner E had difficulties in representing the functions in the pre-test. This shows that the learners' mental constructions on the concepts of identifying the domain and range correctly have not transformed to the mathematical objects as needed to be performed by an individual (Asiala et al.,1997). Therefore, learner E was functioning at the action level when she/he was performing the pre-test task.

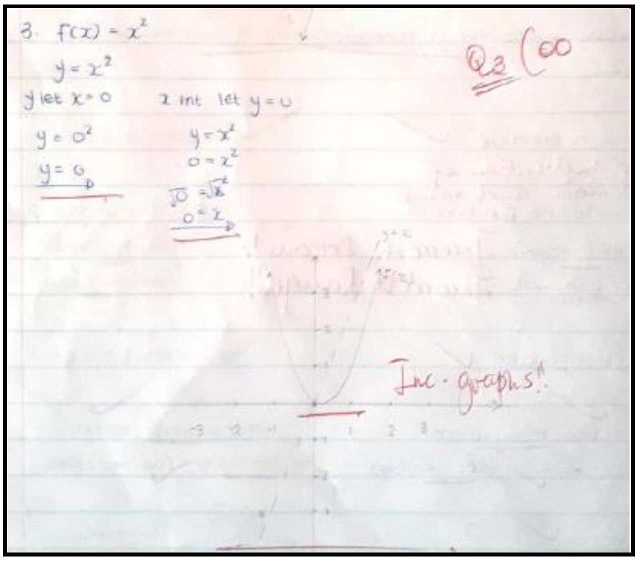
Learner E (Post-test)

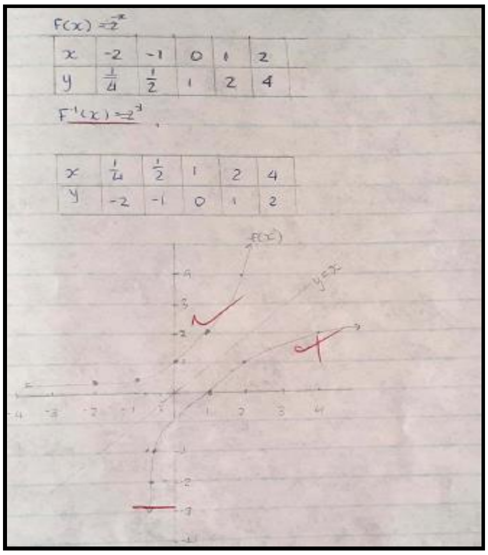


Excerpt 4: Learner E (Post-test)

In the analysis of excerpt 4, learner E data shows a huge improvement in the assessed concepts. Unlike in the pre-test, learner E was able to identify the domain and range with more understanding and he or she was able to recall some concepts (Cottrill et al., 1996), and a process was completely developed into an object in learner E's mind. Subsequently, the process was explicitly performed which demonstrates the development of necessary mental structures (Arnon et al.,2014).

Learner A (Pre-test)





Excerpt 5: Learner A (Pre-test)

The analysis of this excerpt for learner A pre-test shows a lack of understanding when sketching the function $f(x) = x^2$ and its inverse function as well as the $f(x) = 2^x$ graph and its

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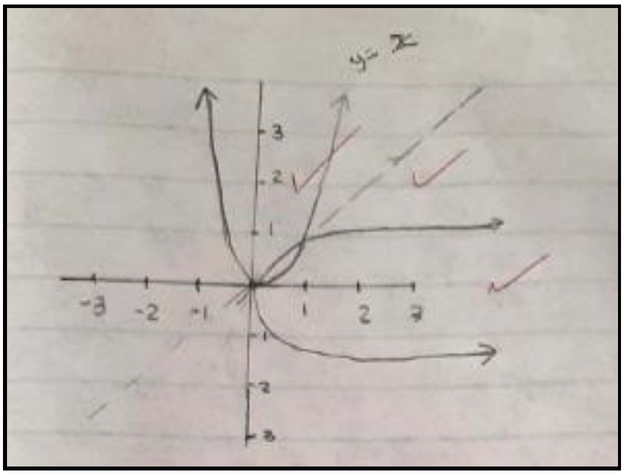
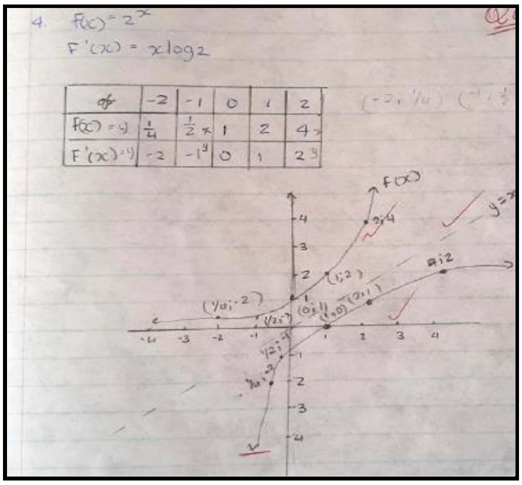
inverse function. Similarly, a learner could not construct the algorithm of working out the coordinates of both functions, whether using the x and y dual intercept method or the table method. An encapsulation of the process had not transformed into a cognitive object that can be identified as a coherent entity (Dubinsky & Harel,1992) claims is the only way to mentally construct a mathematical object. Thus, learner A was incapable of referring to the object as a real thing and being able to recognize the idea at a glance and manipulate it as a whole (Sfard,1991).

Learner A (Post-test)

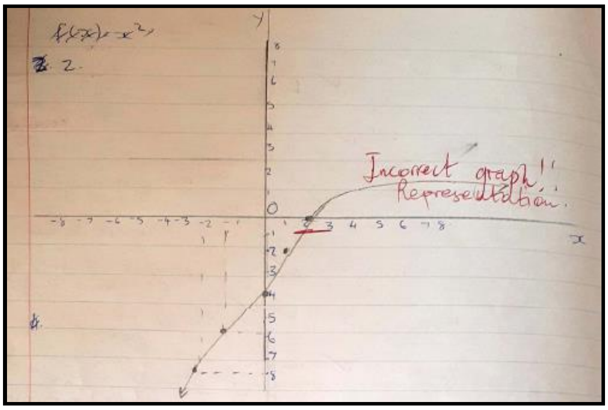


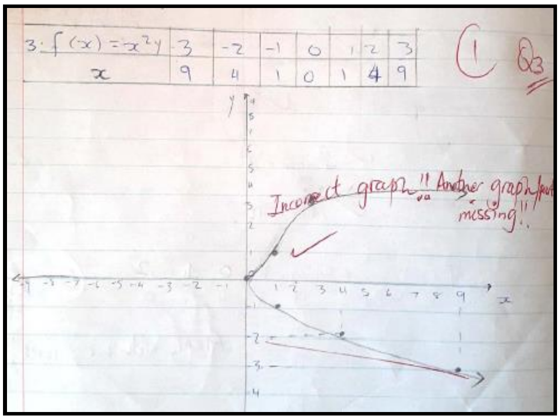


Excerpt 6: Learner A (Post-test)

In excerpt 6, learner A post-test shows significant improvement when sketching functions of $f(x) = x^2$ and of $f(x) = 2^x$. Learner E is considered at the object stage of APOS theory. Learners recognize the process as a whole and comprehend the transformation of the process (Dubinsky & McDonald, 2001). They have constructed the process incoherent totality. Learner E has transformed from the action, process, and object-level and it's ready to construct schema level. For this study, we view the schema level as a coherent collection of actions, processes, objects, and schemas of understanding the function and its inverse (Cottrill et al.,1996). Accordingly, the learner has gained adequate skills to connect other mathematical forms to construct precise representations concerning the functions.

Learner O (Pre-test)

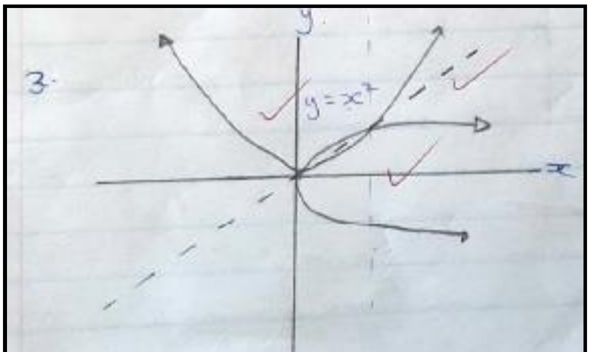


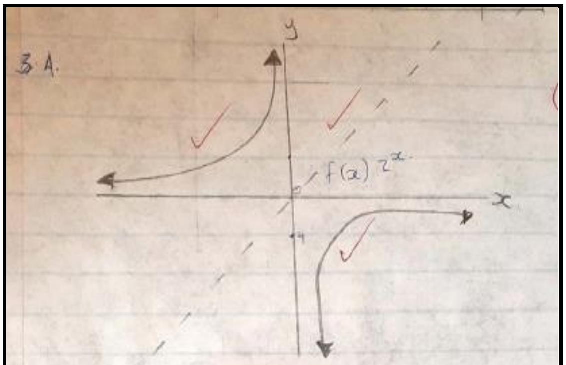


Excerpt 7: Learner O (Pre-test)

Excerpt 7 representing learner O, pre-test shows that the learner was having difficulties with constructing the concepts of sketching both functions as from the function of $f(x) = x^2$ it becomes apparent that the learner did not know the steps to use in sketching the graph as there is no evidence of how the graph was sketched. The learner was stagnant at the action conception level which shows that understanding of the concepts that involve a mental or physical transformation is perceived to be relatively external ((Dubinsky & Harel,1992). This learner is considered to be operating on memory or could rather operate by given the instructions ((Dubinsky & McDonald, 2001).

Learner O (Post-test)





Excerpt 8: Learner O (Post-test)

In excerpt 8, learner O post-test scoring has significantly improved in all the questions from Question 3 and 4. In Question 3, the learner shows improvement in the representation of the functions and the inverses. In Question 3, learner O was able to use point by point plotting to sketch the graph of f and its inverse f^{-1} the quadratic function. As well as sketching of the exponential function and its inverse. This shows improved mental constructions and transformation from action, process and object stages. The teachers' intervening before the post-test produced an effective outcome looking at the improvement as shown in the results.

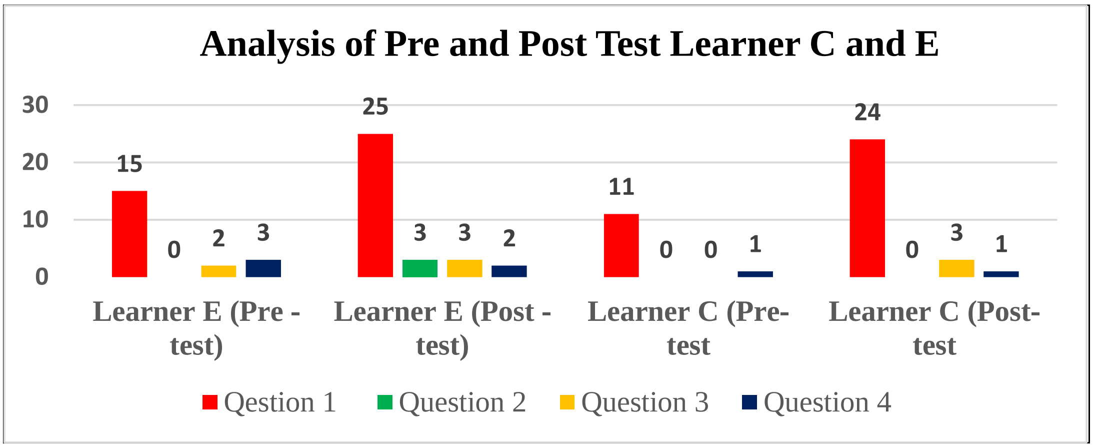


Figure 10: Summary of the pre and post

It emerged that two distinct objects on the focus of this study are the learners' conception levels, and the teacher noticing sophistication levels. For the learner conception levels, APOS was used as an analytical framework for assessing the pre and post-test for the study. As for the teacher sophistication level, there were questionnaires formulated for the teacher participants to comment on the researcher noticing during the lesson and intervention lesson for inverse function **(see Appendix D)**. This data analysis will focus only on the APOS theory for learners' mental conception as the teacher sophistication levels would be analyzed under the intervention lesson.

The first results to be interpreted emerge from the learner pre and post-tests of learners' C and E in Figure 14 as following: Learner C, pre-test seemed to lack knowledge and understanding of concepts such as domain and range. This learner denoted the domain and range using the integer (Z) rather than the real number (R). On the same notion, it emerged that Learner E also had difficulties in the pre-test to identify the same concepts of domain and range. In this case, learner E showed no understanding of these concepts at all. There is a lack of conceptual understanding and both learners' APOS level of conception is based on the action level. The learners demonstrate a lack of comprehension concerning forms of understanding the concept that involves mental structures to transform mental or physical objects (Dubinsky & Harel, 1992).

In the post-tests, learners' C and E both show a remarkable improvement in their Question 1. These learners have both developed mental structures to perform logical connections and they also developed the conceptual understanding of domain and range, function, and not a function. They have managed to move from the action stage to the process stage and further to the objects stage based on their progressive mental conception levels as they have correctly ascertained the domain and range and conceptualized it into symbols and notations. These learners' mental constructions of a process have transformed into a cognitive object (Dubinsky & Harel,1992). After the development of their mental constructions, there is also significant improvement demonstrated in Questions 3 and 4 on their post-tests thereof.

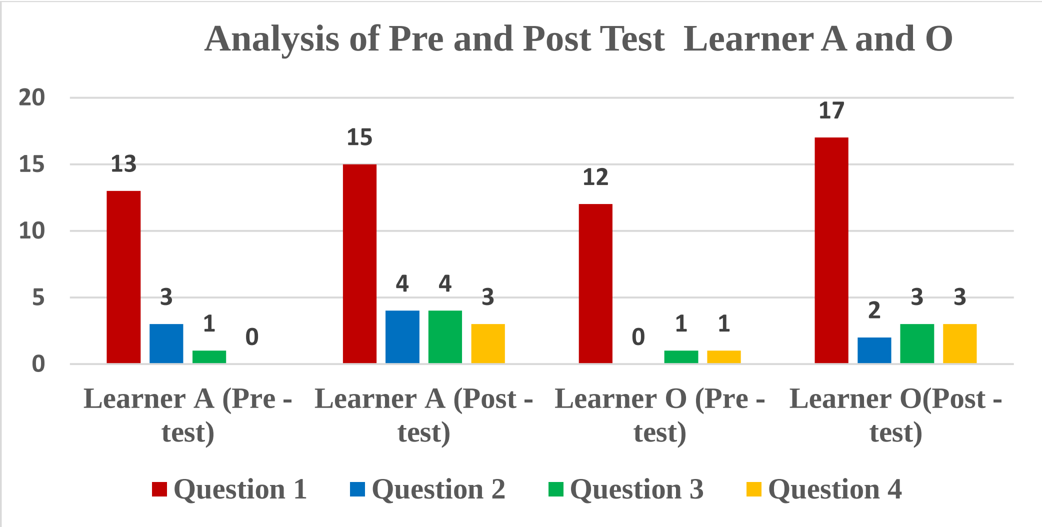


Figure 11: Summary of the pre and post test

In the succeeding analysis, the pre-test results of learners' A and O determine that the learner experienced some difficulties with the representation of the function $f(x) = x^2$ and its inverse function as well as the $f(x) = 2^x$ graph and its inverse function in their pre-tests. However, the analysis of the post-test established some improvement when sketching the parabolic and the exponential functions. Both learners initially were having challenges with performing each step explicitly without imagined or skipped step (Arnon et al., 2014). As they could not comprehend a function to be a process and rather take it as an action, they experienced difficulties (Oehrtman & Engelke, 2010).

4.3 Discussions on Pre and Post-test (APOS theory)

From the data analysis, it emerges that some learners were experiencing difficulties with understanding concepts for example; domain, range, x *and* y intercept, function, or not a function. While some learners had a challenge of conceptualizing the steps to follow to sketch

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the parabolic and exponential functions and their inverse. This study has deduced that most learners have difficulties with their conception levels in constructing mathematics concepts in learning functions. For this study, we used APOS theory to explore the Grade 12 learners' mental constructions through APOS theory conception levels. The investigation of learners' mental conception levels was pivotal for this study to analyze the written work of the participants to establish how these mental constructs occur and prevent learners from understanding the concepts of inverse functions. The data analysis conducted in this study established that indeed there are mental constructs that are crucial in the learning of inverse functions.

Hence, the APOS theory was used by other researchers to characterize learners' ways of understanding the function concept from the action level up to the schema level (Weyer, 2010). In this study, the same theory was used to categorize learners' conceptual understanding of functions and inverse as well as "to explore individual mental structures to make sense of given mathematical concepts" (Maharaj, 2010). For this study, data analyzed learners' difficulties and their APOS conception levels together with some content questions extracted from the textbook as exemplified in the pre and post-test to identify the possible hypothetical learning trajectories.

4.4. Results and analysis on Professional Noticing:

4.4.1 Levels of sophistication for noticing skills

The sophistication levels for noticing framework adopted by Barnhart & van Es (2015) was considered due to its concern with the attending taking place during their classroom practice, how teachers analyze what is interpreted, and make sense of the classroom events. Subsequently, it is concerned about teachers responding to their learners' mathematical ideas that form an integral part of instruction. These sophistication levels for noticing skills are similar to the components skills (Jacobs et al., 2010), that is attending, analyzing, and responding **(Refer to table 3)** in the study.

4.4.2 Intervention Session using video-based teaching

The objective of the intervention was to determine evidence on learners' thinking as it unfolds in the lesson. To also deduce the effectiveness of teacher professional noticing as a pedagogical strategy. Furthermore, it was to analyze while reflecting on my teaching and learning ways (Santagata & van Es, 2010).The pivotal factor of the intervention session was to use the analysis generated from the lessons to develop professional noticing in the teaching practice. The analysis collected from the intervention lessons was crucial for determining how a teacher can; a) attend to learners thinking strategies that unfold between the teacher and the learners b) to interpret learners understanding arising from the intervention interactions, and c) to decide next instructional strategies based on the collected analysis as a result, aiming at collecting the skills as an initiative for noticing (Jacobs, Lamb & Philipp, 2010).

4.4.3 Analysis of intervention

The lesson on functions and inverse functions was conducted with the Grade 12 learners a week after the pre-test tasks. From the pre-test written tasks, the study also established that learners had limited knowledge on how to sketch the parabolic and exponential graphs and their inverses and to determine the domain and the range concepts. These challenges were also identified during the intervention sessions as the researcher (teacher) was attending to learners' strategies in response to the activities given in class. There was also a lack of interpretation about inverse function concepts as some learners were having difficulties in reflecting a parabolic function in the line $y = x$. Some learners did not have an understanding of why the domain of a parabolic function has to be restricted. Thus, the level of sophistication for noticing for the teacher becomes decisive and key. The analysis of teacher noticing was deduced from the video recorded lessons (the intervention lesson). The conducted lesson was by the researcher who collected this data for the study while teaching the topic of inverse function in Grade 12. The researcher was mainly concerned with how teachers notice during their teaching practice. The aim was to investigate teacher noticing and what learner mathematical thinking and understanding do teachers notice during classroom interaction with learners. Furthermore, the data analysis of the teacher noticing during the intervention was provided in the evidence by teacher A and teacher B **(see Appendix D, annexure 3**).

4.5 Discussions on Professional Noticing

From the data provided on teacher noticing and evidence from teacher A and teacher B, it became evident that the researcher did not only use learners' pre-and post-tests to uncover learners' challenges and difficulties about the topic of inverse functions. The researcher was also concern about the level of noticing that the teacher applies during the teaching practice. Hence, the researcher did self-study in a quest to discover how she as a teacher notice during her classroom interaction with learners. The researcher did not only do this video recording analysis only for herself but to mirror other teachers in the mathematics discipline. Evidence was produced from the video recording analysis which was transcribed by the two teacher participants (teacher A and B) who were not involved when the video was recorded. Also, they were doing this analysis independently from each other, they saw the video separately and at their own accord. The two teacher participants recognized what was on the video and they analyzed it and answered based on the questionnaire as guidance. However, they used their discretion based on what they saw to respond and provide feedback. The researcher used these two teacher participants to minimize the biases and preconceptions in analyzing the video herself. The video analysis provides this study with validity and reliability of the study as the researcher was consistent in using the same video to be analyzed by two other participants except herself.

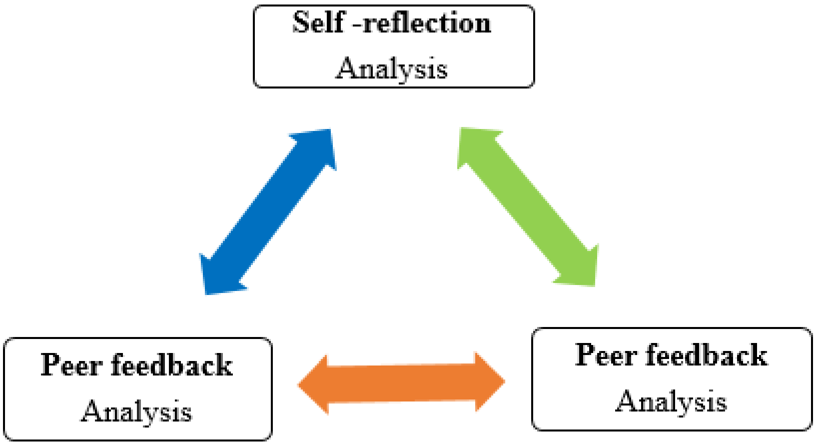

Figure 12: Enactment of Triangulation Process

The method of analysis known as triangulation involves seeking advice from other parties to analyze the data while it eliminates biased behavior. The triangulation process also provides validity and rigor for the study while it enables the researcher the required data for the study. In their discussion, Pegg and Tall (2005) emphasize the importance of theoretical triangulation as, "the process of compiling relevant theoretical perspectives and practitioner explanations, assessing their strengths, weaknesses, and appropriateness, and using some subset of these as the focus of the empirical investigation" (p. 43). It was also asserted that it is significant to "provide checks and balance to maintain acceptable standards of scientific inquiry by addressing the need for rigorous data collection and methods of analysis" (Bowen, 2005, p.214).

4.6 Framework Suggestions

The suggested framework for this study is the APOS theory which was used as an analytical framework for the pre-and post-test data analysis. Another framework was the (Barnhart& van Es 2015) framework for sophistication levels of noticing skills for teachers. The afore-mentioned framework was used to determine data for video-recorded lessons and data collected through this framework was validated for this study. For this study, it can be suggested that other studies take into consideration the video recorded lessons idea and draw broadly from it as this data collection method can be deemed beneficial for similar studies. Similarly, evidence showed that the teachers' learning on noticing what occurred during the "professional vision" case study could be extended to teaching instruction (Sherin & van Es, 2009). Moreover, some studies examined video-based professional development accomplishments and established that the noticing skills had an impactful effect on instructional development (Sherin and van Es 2009). Perhaps the video-based framework can also be developed further in other studies. This study has established that APOS theory determines learners' conception levels and assist to develop rational evidence for the research.

4.7 Conclusions

The chapter shared the findings that emanated from the study. The results from the assessments (pre and post-tests) as well as from the video-recorded analysis have been taken into consideration in this study. These findings have been presented following the analysis of the study. The analysis and presentation of data were done in taking into consideration the theoretical and analytical frameworks chosen for the study. The frameworks used were following making sense of the data gathered for the study. The next and last chapter will discuss the findings using the research questions for the study.

CHAPTER FIVE

CONCLUSIONS, FINDINGS, AND RECOMMENDATIONS

5.1 Introduction

The study explored the teaching of mathematics through professional noticing of learner thinking on the topic of inverse functions in grade 12. This study aimed to research if this new pedagogical approach could improve the teaching and learning of the topic. The theoretical framework used in the study was informed by the APOS theory (Dubinsky et al., 2011) and (Barnhart& van Es 2015), sophistication level of noticing framework. These frameworks have informed the analysis of this study and they were both discussed in previous chapters. A qualitative methodology was adopted for this study and the analysis was drawn from the learners written tasks the concept of inverse functions and from the analysis of video-recorded lessons **(see Appendix D, annexure 3)** for the Questionnaire response from the teacher(s) A and B. These tests were important for the researcher to identify learner thinking patterns. In the particular pre-test, script analysis helped the researcher to design a teaching intervention that would address the thinking patterns noticed in the pre-tests. The post-tests served to help assess if the teaching with noticing has affected the learner outcomes.

5.2 Findings

The study presents the finding of the research question by question. This chapter considered qualitative data analysis to answer the first question which pertains to data analysis of the pre and post-tests determined using APOS theory as an analytical framework.

Research Question 1:

- What APOS theory levels of learners' mental constructions do teachers notice in the learners' written tasks using the concept of inverse function?

The findings shown in the data analysis demonstrate that from the learners' written work, the learner's conception level can either be in the action, process object, and /or schema levels. Data analysis also shows that learner mathematical thinking cannot be determined instantly, it can be established in different stages of mental constructs as APOS theory suggests. The teacher and the learner have to put measures and strategies in place to determine these thinking skills or strategies. Hence, for this study, the researcher had put in place the intervention lessons for the topic under investigation, subsequently, engaged learners in writing the pre-test and post-tests as part of the scientific investigation for this study. APOS analytical framework has established different mental constructions from the learners' written work, which vary in levels from action to process, the process to object, object to schemas levels. Consequently, these results show that some learners demonstrated vague and disconnected ideas before intervention regarding their understanding and reasoning of the inverse functions. Whereas, post-intervention, there was a huge improvement in the results for most of the learners. Their mental construction conception levels had significantly improved and understanding and reasoning of inverse function concepts also at a higher level of APOS. Responding to research Question 1 above, from learner's written work, the study noticed that learners' conception level varied according to their thinking and understanding of the concepts taught. The study further established the importance of teachers noticing the concepts and definitions that learners may find challenging and difficult to address through intervention where the teacher applied the correct measures of noticing. For instance, the teacher being attentive to what learners say, do and hear in the classroom. While noticing occurred, the learners written work in conjunction with APOS levels of conception transformed significantly from the action, process, object, and up to the schema levels. Therefore, learners' mathematical thinking and mental conception levels, and cognitive capacity were changed from what it was initially to being improved significantly. Some learners had a low conception level of thinking and they were unable to move across the conception stages. Post-intervention, the learners who were challenged, showed significant progress and were transformed across the conception stages accordingly. Intervention which included the element of teacher noticing had a meaningful impact on boosting learners' mathematical thinking and mental constructs.

Research Question 2:

- What are the learners' mathematical thinking and understanding noticed by the teacher during intervention classroom interaction using the concept of inverse function?

The study revealed that during classroom interaction, there was learner participation in the classroom. The interaction was also weighed by the nature of questions the teacher-initiated to learners and how the teacher elicits responses shown to be pivotal for this study (Lobato et al., 2005). The learner represented different conception levels during the intervention class and lessons on their understanding of the function and inverse function concepts. Most of the learners' mental constructions were shown below as they could not recall the symbol and notation for domain and range, somehow they could not recall the difference between domain and range. Furthermore, learners were having challenges with sketching the functions and their inverses e.g. the parabolic and exponential functions. They had more difficulties understanding how to perform the procedure of restricting the parabolic function in the line y = x. From this, it was evident that learners' mental constructions were at an auction or process stage and they have not comprehended the concept of inverse functions and they were struggling to see these functions as objects with external stimuli. As a result, the importance of applying some of the suggested mathematics teaching strategies into the classroom environment such as intervention had become crucial. Questions posed to learners must be questions that engage learners' construction process for effective classroom interaction. New knowledge must be assimilated to germinate old knowledge to make learning meaningful. Piaget (1970) states that "knowledge is the result of construction and invention, rather than discovery". Learners should be allowed an opportunity to construct their knowledge and to make errors in their discovery. For the teacher to allow learners the opportunity to do that, they should initiate questions and elicit learners' responses which in turn allows teachers the opportunity to hear learners' mathematical ideas and thinking (Lobato et al., 2005). It is asserted that teachers responding to learners' understanding of the phenomena should be predominantly prioritized to hearing learner's mathematical ideas, attributing to their thinking and understanding (Davis,1996). Barnhart and van Es, (2015) made emphasis on teachers' awareness of their sophistication levels when using the attending, analyzing, and responding skills to learners during teaching practice. The emphasis was on how the sophistication levels should advance from low, medium to high for all the three skills to help improve the learners' mental conception levels improve in mathematics. Consequently, interpreting a phenomenon of mathematics in teaching becomes vital to a teacher noticing in a mathematics classroom (Jacobs et al.,2010). These arguments and deliberation made in this study answer Research Question 2.

Research Question 3:

- What are the teachers' levels of noticing sophistication observed when attending, analyzing, and responding to learners' mathematical thinking and reasoning during the intervention?

According to (Barnhart & van Es 2015), there are three levels of noticing sophistication for attending, analyzing, and responding. All three levels were perused through analysis of data and determining teacher noticing for the interrelated skills. The first level of noticing entails low sophistication noticing level. This was found to be the level at which the majority of teachers apply their teaching strategies. Hence, there is a constant noticing problem realized which pertains to learner mathematical thinking. The level of sophistication level that the teacher uses to observed has to highlight learner thinking for conceptual focus. It has to consistently make sense of highlighted events, as identified and described based on specific learner ideas during the lesson (Barnhart & van Es 2015). It also has to be specific to what the learner can do differently next time when making logical connections in their learning. Observed teachers should be able to dig deep to access learner mathematical thinking, source strategies that they can use to instill conceptual understanding in their learners. Subsequently, guide learners on how to attend to the strategies, how to analyze and respond accordingly while developing critical thinking skills and deep conceptual understanding. This concludes in response to Question 3 of this study.

5.3 Limitations of the study

The limitation of the study was getting the learners to attend the intervention sessions as it was after the Preliminary exams. Some of the learners were not around school around the scheduled times. I had to reschedule the intervention several times to accommodate the majority of the learners as it was imperative to do those sessions to address the gap established in the pre-tests. If it was not for this limitation, I could have done more intervention sessions with the learners and made more emphasis on the concepts they were finding challenging.

5.4 Reflection

This study helped me gain deeper insight into learners' mental conception levels on the concepts of the inverse function and the notion of professional noticing. The knowledge I have gained throughout this study has deepened and transformed beyond measure. Through the use of APOS theory, I was able to understand learners' different capabilities and thinking levels. I have gained a lot of insight from the learner's pre and post-test assessment, from the video analysis, and teacher participants' opinions about my teaching. This has helped my view about what and how I notice during the teaching practice. Also, it gave me a sense of reflection into what I do during my teaching practice and how I can transform my teaching strategies. Mathematics teaching and learning need improvement in professional teachers noticing as to be fully conversant with the learner's mathematical understanding.

5.5 Recommendations

For further studies and effective classroom interaction and the development of critical thinking skills within learners. I would like to recommend that teachers transform their classroom pedagogy and develop their learners thinking skills and conceptual understanding by applying the notion of professional noticing. Professional noticing was defined by Jacobs, Lamb, & Philipp (2010) who refers to teachers' capacity to (a) attend to the student's mathematical conceptions and practices as they occur, (b) interpret these conceptions and practices, and (c) decide upon a productive instructional course of action based on this interpretation. As other studies consider professional noticing to be a powerful conceptual understanding tool that can be embedded in the mathematics practice, it is subsequently the study's recommendation. Like other studies, Jacobs, Phillip, and Schapelle (2009) define 'professional noticing' as the ability to recognize and act on key indicators significant to mathematics studies. Professional noticing is highly recommended in this study and based on the research conducted for this study. For this study, it is recommended that teachers take cognizance of the notion of professional noticing as a tool that can help transform their teaching practice. That also helps to boost learner's mathematical thinking and understanding skills. Moreover, to provide the teachers with the attending strategies, analyzing skills, and responding skills. In essence to build teaching strategies that will transform teaching and learning while ensuring a positive impact on learner's results. For this study, the suggestion is that teachers respond to learners' understanding of the phenomena that should be predominantly be prioritized to hearing learner's mathematical ideas, attribute to their thinking and understanding. The study recognizes that when teachers reflect on their teaching strategies, they can notice the skills necessary for their learner's learning of mathematics. Hence, teachers can improve their learner's mathematical thinking and understanding during classroom interaction as they will notice these during their teaching and learning processes.

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References

Akkoc, H., & Tall, D. (2005). *In Proceedings of the sixth British Congress of Mathematics Education* (pp. 1-8). Warwick: University of Warwick.

Akkus, R., Seymour, J., & Hand, B. (2007). Improving dialogical interaction in classrooms through improving teacher pedagogical practices: Impact on students' scores. *In Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of mathematics Education, NV: University of Nevada, Reno.*

Arnon, L., Cotrill, J., Dubinsky, E., Oktac, A., Roa, S., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education.* Springer. https://doi.org/10.1007/978-1-4614-7966-6.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior,* 16(4), 399–431. https://doi.org/10.1016/S0732-3123(97)90015-8.

Attorps, I., Björk, K., Radic, M., & Viirman, O. (2013). Teaching inverse functions at tertiary the level. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 2524–2533). Ankara: Middle East Technical University and ERME.

Ball, D., & Bass, H. (2003). Towards a practice-based theory of mathematics knowledge for teachers. *Proceeding of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton: AB: CMESG /GCEDM.

Ball, D., & Bass, H. (2009). With an eye on the mathematical horizon. In B. Bass, *With an eye on the mathematical horizon: Knowing mathematics teaching learners' mathematical futures.* (pp. 11-29). Munster: Neubrand (Ed).

Ball, D.L., Thames, M.H., and Phelps, G. (2008) Content knowledge for teaching: What makes it special? *Journal of Educator Education, 59*(5), 389-407.

Barnhart, T., & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze, and respond to student thinking. *Teacher and Teacher Education, 45, 83 -93.*

Bayazit, I., & Gray, E (2004). Understanding inverse functions: the relationship between teaching practice and student learning. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 (2), 103-110.*

Bayazit, I. (2011). Prospective teachers' inclination to single representation and their development of the function concept. *Educational Research and Reviews 6(5), 436.*

Bennet, A. (2009). *APOS Theory*. REESE Project.

amkgosi

Borji, V., Alamolhodaei, H., & Radmehr, F. (2018). Application of the APOS-ACE theory to improve students' graphical understanding of derivatives. *Eurasia Journal of Mathematics, Science, and Technology Education, 14(7),* 2947–2967. https://doi.org/10.29333/ejmste/91451.

Bowen, G. A. (2005). Preparing a qualitative research-based dissertation: Lessons learned. *The qualitative report, 10*(2), 208-222.

Breidenbach, D.E., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function, *Educational Studies in Mathematics 23, 247 -285.*

Brijlall, D. & Ndlovu, Z.(2013). High school learners' mental construction during solving optimization problems in Calculus: a South African case study. *South African Journal of Education;* 33(2), http://www.sajournalofeducation.co.za.

Bryman , A. (2004). *Social Research Methods.* Oxford: Oxford University Press.

Burns, N., & Grove, S. K. (2010). *Understanding nursing research-eBook: Building an evidence-based practice*. Elsevier Health Sciences.

Cansiz, S., Kücük, B., & Isleyen, T. (2011). Identifying the secondary school students' misconceptions and functions. In S. a. Sciences, *Identifying the secondary school students' misconceptions and functions.* (pp. 3837 -3842). Social and Behavioural Sciences, 15.

Carlson, M. P. (1998). *A cross-sectional investigation of the development of the function concept.* CBMS issues in mathematics education, 114-162.

Carlson, M., & Oehrtman, M. (2005). Key aspects of knowing and learning the concept of function. *Research Sampler, 9.* (Mathematical Association of America).

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understanding. *Cognition and Understanding, 28*(2), 113–145.

Celik, A. O., & Guzel, E. B. (2017). Revealing Ozgur's thoughts of a quadratic function with a clinical interview: Concepts and their underlying reasons. *International Journal of Research in Education and Sciences, 3(1),* 122-123.

Cetiner, Z., Kavcar, M. & Yildiz, Y. (2000). *High School Mathematics 1*. Istanbul: Ministry of Education Publishing Company.

Chappuis, J. (2015) Seven strategies for Assessment for learning. Second Edition. USA: Pearson.

Chimhande, T. (2014). *Design research towards improving understanding of functions.*Pretoria: South African Case Study (University of Pretoria).

Cohen, M. J., & Stanczak, D. E. (2000). On the reliability, validity, and cognitive structure of the Thurstone Word Fluency Test. *Archives of Clinical Neuropsychology*, *15*(3), 267-279.

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Cottrill, J., Dubinsky, Ed., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the Limit Concept: Beginning with a Coordinated Process Scheme. *Journal for Mathematical Behaviour,* 15 (2), p: 167-192.

Creswell, J. W. (2012). *Planning, conducting, and evaluating quantitative and qualitative research.* NY : Pearson .

Davis, B. (1996). *Mathematics teaching: Toward a Sound Alternative.* New York: Garland Publishing Inc.

Davis, B. (1994)."Mathematics teaching: Moving from telling to listening". *Journal of Curriculum and Supervision 9*, 267 -267.

Davis, B. (1997). "Listening for differences: An evolving conception of mathematics teaching". *Journal for research in Mathematics Teaching*, 355-376.

Department of Basic Education. (2018). *National Senior Certificate Diagnostic report*. Pretoria, South Africa: Department of Basic Education Retrieved from https:// www.education.gov.za.

Dewey, J. (1904). The relation of theory to practice in education. In National Society for Scientific Study of Education. *The relation of theory to practice in the education of teachers (Vol. 3, Part 1). Bloomington,* IL: Public School Publishing.

Dubinsky, Ed. (1991). Reflective Abstraction in Advanced Mathematical Thinking, in D. Tall (Ed.), *Advanced Mathematical Thinking,* Netherlands: Kluver Academic Publisher, p: 95-123.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. Aspects of epistemology and pedagogy. In M. N. Vol.25, *The nature of the process conception of function:* (pp. 85-106). Washington DC: Mathematics Association of America.

Dubinsky, E., & MacDonald, M. (2001). APOS: a constructivist theory of learning in undergraduate mathematics education research. In: D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study (273 -280).* Dordrecht: Kluwer Academic Publishers.

Dubinsky, E. (2010, January). The APOS theory of learning mathematics: Pedagogical applications and results. In *Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education. Durban, South Africa*.

Dubinsky, E., & Wilson, R.T. (2013). High School students' understanding of the function concept. *The Journal of Mathematics Behavior, 32, 83-101.*

Erickson, F. (2011) On noticing teacher noticing. In M.G. Sherin, V.R. Jacobs, & R.A. Philipp (Eds.), *Mathematics teacher noticing*: *Seeing through teachers' eyes*. New York: Routledge.

Eisenberg, T., & Sfard, A. (1992). On the development of a sense for functions. In *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 153-174). Washington, DC: Mathematics Association of America.

Even, R. (1991). The Inverse Function: Prospective Teachers' Use of 'Undoing'. *International Journal of Mathematics Education Science and Technology*, 23 (4), p: 557-562.

Even, R., & Wallach, T. (2004). Between student observation and student assessment: A critical reflection. *Canadian Journal of Science, Mathematics and Technology Education, 4(4), 483-495*.

Grinstein, L., & Lipsey, S. I. (2001). *Encyclopedia of mathematics education.* Routledge.

Fernȧndez, C., Llinares, S.,& Valls, J.(2013). Primary school teacher's noticing of students' mathematical thinking in problem solving. *The Mathematics Enthusiast, (10)1 & 2*, 441 -468.

Fisher, M. H., Thomas, J., Jong, C., Schack, E. O., & Tassell, J. (2018). Noticing numeracy now! Examining changes in preservice teachers' noticing, knowledge, and attitudes. *Mathematics Education Research Journal, 30*(2), 209 –232.

Floro, B., & Bostic, J. D. (2017). A case study of middle school teachers' noticing during modeling with mathematics tasks. In *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 73-89). Springer, Cham.

Fowler, B. (2014). *An Investigation of the Teaching and Learning of Function Inverse.* Arizona State University.

Haltiwanger, L. & Simpson, A. (2014). Secondary mathematics preservice teachers' noticing of students mathematical thinking. In G. T. Matney, & S. M. Che (Eds.), *Proceedings of the 41st annual meeting of the Research Council on Mathematics Learning* (pp. 49-56). San Antonio, TX.

Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 153-174). Washington, DC: Mathematical Association of America.

Hill, H.C., Ball., D.L., & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics in Mathematics Education, 39, 372-400.*

Huang, C. (2011). Engineering students' conceptual understanding of the derivative in calculus. *World Transactions on Engineering and Technology Education,* 9(4), 209–214.http://www.wiete.com.au/journals/WTE%26TE/Pages/Vol.9,%20No.4%20(2011)/01-06-Huang-C-H.pdf.

Jacobs, V. R., Lamb, L. C., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 41, 168–202.

Jong, C. (2017). Extending equitable practices in teacher noticing. In E. O. Schack, M. H. Fisher, & J. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 2017–214). Springer. J, A. &. (2006). Opening another black box. *Journal for Research In Mathematics Education*, 270-296.

Kalchman, M., & Koedinger, K. R. (2005). Teaching and learning functions. *How students learn: History, mathematics, and science in the classroom,* 351-396.

Kalchman, M., Kelly, W., & Case, R. (2002). Learning to Teach Mathematical Functions: Sixth-grade Students and their Teacher Engage in an Innovative Curriculum. *Curriculum and Teaching*, 17(2), 5-18.

Kilpartick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington: National Academy Press.

Klein, D. (1998, 2005). *Epistemology*. London: Routledge.

Krupa, E. E., Huey, M., Lesseig, K., Casey, S., & Monson, D. (2017). Investigating secondary preservice teacher noticing of students' mathematical thinking. In *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 49-72). Springer, Cham.

Lamb, L.C., Philipp, R.A., Jacobs, V.R., & Schappelle, B.P.(2009). Developing teachers' stances of inquiry: Studying teachers' involving perspectives (STEP). In D. Slavit, T. Holmlund Nelson, & A. Kennedy (Eds.), *Perspectives on supported collaborative teacher inquiry* (p.16 -45). New York: Taylor & Francis.

Leatham, K.R., Peterson, B.E., Stockero, S.L, & Van Zoest, L.R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build student thinking. *Journal of Research in Mathematics Education, 46(1), 88-124.*

Leedy, P.D. 2005. Practical research: *Planning and design*. *London*. Prentice-Hall.

Lobato, J., Clarke, D. & Burns Ellis, A. (2005). Initiating and eliciting in teaching: a reformulation of telling. *Journal for research in mathematics education*, 36 2, 101 -136.

Maharaj, A. (2010). An APOS analysis of students' understanding of the concept of a limit of a function. *Pythagoras, 71, 41-52*.

Mason, J. (2011). *Noticing: Roots and branches.* In M.G. Sherin, V.R. Jacobs, & R.A. Philipp (Eds.), Mathematics teacher noticing: *Seeing through teachers' eyes.* New York: Routledge.

Maxwell, J. (1992). Understanding and validity in qualitative research. *Harvard educational review*, 62(3), 279-301.

National Council of Teachers of Mathematics (NCTM). (2014, April 8). Access and equity in mathematics education: *A position of the National Council of Teachers of Mathematics.*

https://www.nctm.org/uploadedFiles/Standards_and_Positions/Position_Statements/Access_and_Equity.pdf.

Opie, C. (2004). *Doing Educational Research -A Guide to First Time Researchers*. London.Sage Publications.

O'Shea, A., Breen, S., & Jaworski, B. (2016). The development of a function concept inventory. I*nternational Journal of Research in Undergraduate Mathematics Education, 2(3), 279-296.*

Pegg, J & Tall, D. (2005). Fundamental cycles of cognitive growth. In Anne D. Cockburn & Elena Nardi (Eds), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education: 4, 41-48. Norwich: UK.*

Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University Press.

Piaget, J. (1972). *Some aspects of operations.*

Piaget, J. (1977). *Epistemology and psychology of functions*. Dordrecht, Netherlands: D. Reidel Publishing Company.

Pillay, V. (2006). *Mathematical knowledge for teaching functions*. Paper presented at the 14th Annual meeting of the Southern African Association for Research in Mathematics, Science and Technology Education, University of Pretoria, South Africa.

Phillip, M.D., Basson, J., & Botha, C. (2012). Mathematics: *Textbooks and workbooks. Grade 11.* All copy.

Philipp, R.A. (2014). *Research on teachers' focusing on children's thinking in learning to teac*h: Teacher noticing and learning trajectories. In J.J. Lo, K.R Leatham, & L.R. Van Zoest (Eds.), Research Trends in Mathematics Teacher Education. New York: Springer.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). *Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework*. ZDM, 40, 165-178.

Pettersson, K., Standler, E., & Tambour, T. (2013). *Development of students' understanding of the threshold concept.* Retrieve from http://cerme8.metu.edu.tr.

Reed, B.(2007).*The effects of studying the history of the concept of the function on student understanding of the concept.* (Doctoral dissertation, Kent State University, 2007). Retrieved from:http://www.ohiolink.edu/eytc/sendpdf.cgi/Reed%20Beverly%20M.pdf?acc%5Fnum=kent1195713522

Roblyer, M. D., & Doering, A. H. (2013). *Integrating Educational Technology into Teaching.* New Jersey: Pearson Education.

Santagata, R., Zannonni, C., & Stingler, J.W. (2007). *The role of lesson analysis in pre-service teacher education:* An empirical investigation of teacher learning from a virtual video field experience. *Journal of Mathematics Teacher Education, 10,* 123 -140.

Santagata, R. (2011). From teacher noticing to a framework for analyzing and improving classroom lessons. In Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.), *Mathematics Teacher Noticing: Seeing through Teachers' Eyes* (p. 152-168). New York: Routledge.

Schack, E., Fisher, M.Thomas, J., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Preservice teachers professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education, 16*(5), 379–397.

Schack, E. O., Fisher, M. H., Wilhelm, J. (Eds.) (2017). *Teacher noticing — bridging and broadening perspectives, contexts, and frameworks.* Springer.

Schoenfeld, A. (2011). Noticing matters. A lot. Now what? In M.G. Sherin, V.R. Jacobs, & R.A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eye*s. New York: Routledge.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 18, 371–397. https://link.springer.com/article/10.1007/BF00302715.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20–37.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes.* Routledge.

Star, J. R., Lynch, K. H., & Perova, N. (2011). Using video to improve mathematics' teachers' abilities to attend to classroom features: A replication study. *Mathematics teachers' noticing: Seeing through teachers' eyes.*

Stockero, S.L. & Van Zoest, L.R. (2013). Characterizing pivotal teaching moments in beginning mathematics teacher' practice. *Journal of Mathematics Teacher Education, 16(2), 125-147.*

Stockero, S. L., Leatham, K. R., Van Zoest, L. R., & Peterson, B. E. (2017). Noticing distinctions among and within instances of student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 467–480). Springer.

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Thomas, J. & Tabor, P.D. *(2012). Developing Quantitative Mental Imagery. T*eaching Children Mathematics, 19, 174–183.

Thomas, J. N., Eisenhardt, S., Fisher, M. H., Schack, E. O., Tassell, J., & Yoder, M. (2014). Professional noticing: D*eveloping responsive mathematics teaching. Teaching Children Mathematics, 21(5),* 294-303.

Thomas, J., Jong, C., Fisher, M. H., & Schack, E. O. (2017). Noticing and Knowledge: Exploring Theoretical Connections between Professional Noticing and Mathematical Knowledge for Teaching. *The Mathematics Educator*, 26(2), 3-25.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. E*ducational studies in mathematics, 26(2),* 229-274.

Trede, F. & Higgs, J. (2009). Framing research questions and writing philosophically. In J. Giggs, D. Horsfall & S. Grace (eds.), *Writing qualitative research on practice,* (2009), 13-25. Sense Publishers.

Tokgöz, E. (2012). Numerical method/analysis students' conceptual derivative knowledge. *International Journal of New Trends in Education and Their Implications*, 3(4), 118–127. http://ijonte.org/FileUpload/ks63207/File/11.tokgoz1.pdf.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club.*Teaching and Teacher Education, 24*(2), 244–276.

van Es, E.A., & Sherin, M.G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10,* 571-596.

Vidakovic, D. (1996). Learning the concept of inverse function. *Journal for* Computers *in Mathematics and Science Teaching, 15*(3), 295–318.

Weller, K., Arnon, I., & Dubinsky, E. (2011). Preservice teachers' understandings of the relationship between a fraction or integer and its decimal expansion: Strengthen and stability of belief. *Canadian Journal of Science, Mathematics, and Technology Education 11(2)*, 129-159.

Weyer, S.R. (2010). *APOS Theory as a conceptualization for understanding mathematical learning*. Summation. 9-15.

Wilson, F. C., Adamson, S., Cox, T., & O'Bryan, A. (2011). Inverse functions: What our teachers didn't tell us. *Mathematics Teacher, 104*(7), 500–507.

Yin, R. K. (2016). *Qualitative Research From Start to Finish, Second Edition.* New York: The Guilford Press.

APPENDICES

Appendix A

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	22 July 2019
Validity of Research Approval:	04 February 2019 – 30 September 2019 2019/155
Name of Researcher:	Kgosi A.M
Address of Researcher:	
Telephone Number:	
Email address:	
Research Topic:	Mathematics Teachers Professional Noticing in teaching of Inverse Functions in Grade 12.
Type of qualification	Masters' Degree in Education
Number and type of schools:	One Secondary School
District/s/HO	Ekurhuleni North

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

22/07/2019

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
Tel: (011) 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gpg.gov.za

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1. Letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.
2. The District/Head Office Senior Manager/s must be approached separately, and in writing, for permission to involve District/Head Office Officials in the project.
3. A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.
4. A letter / document that outline the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.
5. The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.
6. Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.
7. Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.
8. Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.
9. It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.
10. The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.
11. The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.
12. On completion of the study the researcher/s must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.
13. The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.
14. Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

....

Mr Gumani Mukatuni
Acting CES: Education Research and Knowledge Management

DATE: 22/07/2019

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
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WITS SCHOOL OF EDUCATION

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

SCHOOL OF EDUCATION ETHICS COMMITTEE
CONSTITUTED UNDER THE UNIVERSITY HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

CLEARANCE CERTIFICATE **PROTOCOL NUMBER: 2019ECE030M**

PROJECT TITLE Mathematics teachers professional noticing in teaching of inverse functions and graphs in grade 12.

INVESTIGATOR Annie Mamoretsi Kgosi

SCHOOL/DEPARTMENT OF INVESTIGATOR WITS SCHOOL OF EDUCATION

DATE CONSIDERED 15 July 2019

DECISION OF THE COMMITTEE Approved unconditionally

EXPIRY DATE Date of submission of the project report

ISSUE DATE OF CERTIFICATE 29 July 2019 **CHAIRPERSON** (Dr. Paul Goldschagg)

cc: Supervisor: Dr. Judah Makonye

DECLARATION OF INVESTIGATOR

To be completed in duplicate and **ONE COPY** emailed to the Ethics Office: Matsie.Mabeta@wits.ac.za .

I fully understand the conditions under which I am authorized to carry out the abovementioned research and I guarantee to ensure compliance with these conditions. Should any departure be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee.

MKgosi
Signature

02 / 08 / 2019
Date

PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES

Dear Ethics Committees,

Re: Covering Letter regarding changes made on the Ethics Application Form

Kindly note that I have made the required changes as follows:

1) In Section 8, data collection anonymity has been changed to YES and highlighted in yellow.
2) Risk level have been changed to low risk and changes are highlighted in yellow.
3) A principal's consent form was not provided for with the rest of the other documents. Only the teacher and learner consent forms has been provided.
4) My Supervisor Dr P Makonye have seen and approved the changes made to the ethics submission.

Herewith the hardcopy of the signed letter by supervisor (Dr P Makonye) as requested.

Yours Sincerely,

Annie M Kgosi

(Student Number-1484157)

Signature: MKgosi

Supervisor: Dr P Makonye

Signature:

Mathematics Teacher Professional Noticing in teaching of Inverse Functions for Grade 12.

(Acting)

I ---------am the principal of ----------------------------, Have read and understood the content of the letter seeking permission to do research on

....

1 permit/ ~~do not permit~~ Ms. A. Kgosi .... to do the above named research at this school

Signature ____________________

Signed at Rhodesfield ____________ on this day of 5/8/ 2019

Cell Number____________________

Tel: 011 970 1209 Fax: 011 ...

e-mail: admin@rhodesfield.co.za

2019 -08- 05

Privaatsak / Private Bag X05
KEMPTON PARK 1620

University of the Witwatersrand, Wits School of Education, 97 St Andrews Rd, Parktown, Johannesburg

INFORMATION SHEET LEARNERS

02nd August 2019

Dear Learner

My name is Annie Mamoretsi Kgosi and I am a Master of Degree in Education Student in the School of Education at the University of the Witwatersrand.

I am doing research on **Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12).**

My investigation involves professional teacher noticing, to determine if Mathematics teachers do notice children's mathematical thinking and reasoning within the teaching practices. The aim of my research is to ascertain that the learners' mathematical thinking and reasoning are attended to, as learning strategies by the teacher(s) and to know how teachers' attend to these learner's strategies. On the same note, to assess interpretation of learner's mathematical understanding based on teachers' consistent reasoning. Moreover, to investigate teacher reasoning towards learners, if teacher(s) reasoning provides listening for a difference which implies that the teacher might have different type of listening skill which surpasses learner mathematical thinking and/ or understanding, or rather teachers have what is known as "intended listening skill". My research intends to contribute to the South Africa's Mathematics education context through exploring ways that professional teacher noticing can become effective towards interpretation of classroom phenomenon as well as how teachers attends to a stimuli.

Would you mind if I request that you participate in my study and assist me in carrying this research by participating in doing pre and post tests and by giving your comments regarding my teaching lesson of inverse functions during re –teaching of this topic using questionnaires. You will not be part of the video recording that I am taking as part of this research study, your face will not be in the visible in the video. The video will be clung onto myself and my lesson, your interaction will not be affected by the video recording. As a result, the required participation is required in writing of pre and post-tests that entails the topic of inverse functions and also your comment regarding my lesson implementation using questionnaires.

Remember, this is not a test, it is not for marks and it is voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way.

I will not be using your own name but I will make one up so no one can identify you. All information about you will be kept confidential in all my writing about the study. Also, all collected information will be stored safely and destroyed between 3-5 years after I have completed my project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you

X MKgosi

Annie M Kgosi
University of Witwatersrand - Student

Ms Annie M Kgosi

Learner Consent Form **Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12).**

Please fill in the reply slip below if you agree to participate in my study called:

My name is ____________

Permission for questionnaire/test
I agree to fill in a question and answer sheet or write a test for this study. YES/NO

Permission to be videotaped
I agree to be videotaped in class. YES/NO
I know that the videotapes will be used for this project only. YES/NO

Informed Consent
I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign ____________ Date 10/08/2019

Teacher's Consent Form [**Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12)**].

Please fill in and return the reply slip below indicating your willingness to be a participant in my voluntary research project called:

I, _______________ give my consent for the following:

Circle one

Permission for questionnaire/test

I agree to fill in a questionnaire in evaluation of the researchers study. YES/NO

Permission to be videotaped

I agree to analyze the researchers video tape in assistance to give him/ her required feedback in her study. YES/NO

I understand that the contents of his/her videotapes will be used for this project only and will be kept confidential. YES/NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign_______________ Date 13/08/2019

Teacher's Consent Form [**Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12)**].

Please fill in and return the reply slip below indicating your willingness to be a participant in my voluntary research project called:

I, [redacted] give my consent for the following:

Circle one

Permission for questionnaire/test

I agree to fill in a questionnaire in evaluation of the researchers study. YES/NO

Permission to be videotaped

I agree to analyze the researchers video tape in assistance to give him/ her required feedback in her study. YES/NO

I understand that the contents of his/her videotapes will be used for this project only and will be kept confidential. YES/NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign__________ Date 2019. 08. 07.

Parent's Consent Form [**Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12)]**.

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:

I, ______________________ the parent of ______________ ____

Circle one

Permission for questionnaire/test

I agree that my child may be fill in a question and answer sheet or write a test for this study. YES/NO

Permission to be videotaped

I agree my child may be videotaped in class. YES/NO

I know that the videotapes will be used for this project only. YES/NO

Informed Consent

I understand that:

- my child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- he/she does not have to answer every question and can withdraw from the study at any time.
- he/she can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign______________________ Date 12/08/2019

Parent's Consent Form [**Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12)**].

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:

I, ______________ the parent of ______________

Circle one

Permission for questionnaire/test

I agree that my child may be fill in a question and answer sheet or write a test for this study. YES/NO

Permission to be videotaped

I agree my child may be videotaped in class. YES/NO

I know that the videotapes will be used for this project only. YES/NO

Informed Consent

I understand that:

- my child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- he/she does not have to answer every question and can withdraw from the study at any time.
- he/she can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign______________ Date 06 . 08 . 2019

amkgosi

University of the Witwatersrand, Wits School of Education, 97 St Andrews Rd, Parktown, Johannesburg

NFORMATION SHEET PARENTS

01st July 2019

Dear Parent

My name is Annie Mamoretsi Kgosi and I am a Master of Degree in Education Student in the School of Education at the University of the Witwatersrand.

I am doing research on **Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12).**

My research involves professional teacher noticing, to determine if Mathematics teachers do notice children's mathematical thinking and reasoning within the teaching practices. The aim of my research is to ascertain that the learners' mathematical thinking and reasoning are attended to, as learning strategies by the teacher(s) and to know how the teachers' attend to these learner's strategies. On the same note, to assess interpretation of learner's mathematical understanding based on teachers' consistent reasoning. Moreover, to investigate teacher reasoning towards learners, if teacher reasoning provides listening for a difference which implies that the teacher might have different type of listening skill which surpasses learner mathematical thinking and/ or understanding, or rather teachers have what is known as "intended listening skill". My research intends to contribute to the South Africa's schools Mathematics education through exploring ways that professional teacher noticing can become effective towards interpretation of classroom phenomenon. As a result, to help teachers to reflect on their pedagogical actions during teaching practice and to make better teaching subject and content knowledge through a notion of teacher professional noticing.

The reason why I have chosen your child's class is because it is learners that I have been teaching since Grade 8 up to now Grade 12. It is a class that I am aware of their strength and weaknesses regarding topic of Functions and I have a thorough understanding of where they may be having difficulties with the topic of inverse functions based on their Grade 10 and 11 work.

Would you mind if they can assist me in carrying this research by participating in doing pre and post tests and by giving your comments regarding my teaching lesson of inverse functions during re –teaching of this topic using questionnaires. Your child will not be part of the video recording that I am taking as part of this research study, his/ her face will not be in the video. I will be the only participant conducting the study who will be involved in the video recording. Whereas learners will only be interacting in the classroom through their voices only.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

Annie M Kgosi
University of Witwatersrand - Student

Ms Annie M Kgosi
[redacted]

University of the Witwatersrand, Wits School of Education, 97 St Andrews Rd, Parktown, Johannesburg

Corner Catalina and Ventura Street

Kemptonpark

02nd August 2019

Dear Sir/Madam,

REF: Permission to conduct research in your school.

My name is Annie Mamoretsi Kgosi, I am studying for a Master of Education Degree in Mathematics in the School of Education at the University of the Witwatersrand. I am seeking permission to do research at your school.

I am doing research on Mathematics teachers' professional noticing in teaching of inverse functions and graphs in Grade 12 level. I have employed a self –study on the topic of inverse functions and graphs whereby I will conduct the research during my lesson taking a video camera which will be focused to myself and my teaching. Pertaining to the video recording, two chosen peer teachers who have agreed to be participants in my study from your school will review and analyze the video recording based on the lesson accordingly. I have chosen this topic due to its relevance and significance to other mathematical domains. There is also a gap in the topic that was caused over the years, where past research shows that learners have been progressed while they have been struggling to grasp the concept of functions from earlier grades. Therefore, functions in Grade 12 becomes problematic for learners to understand, due the progression gap created previously.

The focus of my research is on public schools in Gauteng Province. I want to explore professional teacher noticing, to determine if Mathematics teachers do notice children's mathematical thinking and reasoning within the teaching practices. The aim of my research is to ascertain that the learners' mathematical thinking and reasoning are attended to, as learning strategies by the teacher(s) and to know how the teachers' attend to these learner's strategies. On the same note, to assess interpretation of learner's mathematical understanding based on teachers' consistent reasoning. Moreover, to investigate teacher reasoning towards learners, if teacher reasoning provides listening for a difference which implies that the teacher might have

amkgosi

Would you mind if they can assist me in carrying this research by participating in doing pre and post tests and by giving your comments regarding my teaching lesson of inverse functions during re –teaching of this topic using questionnaires. Your child will not be part of the video recording that I am taking as part of this research study, his/ her face will not be in the video. I will be the only participant conducting the study who will be involved in the video recording. Whereas learners will only be interacting in the classroom through their voices only.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

Annie M Kgosi
University of Witwatersrand - Student

Ms Annie M Kgosi

amkgosi

02nd August 2019

Dear Teachers

My name is Annie M Kgosi and I am a Masters Degree in Education Student in the School of Education at the University of the Witwatersrand.

I am doing research on **Mathematics Teachers Professional Noticing in Teaching of Inverse Functions (Grade12).**

My research involves professional teacher noticing, to determine if Mathematics teachers do notice children's mathematical thinking and reasoning within the teaching practices. The aim of my research is to ascertain that the learners' mathematical thinking and reasoning are attended to, as learning strategies by the teacher(s) and to know how the teachers' attend to these learner's strategies. On the same note, to assess interpretation of learner's mathematical understanding based on teachers' consistent reasoning. Moreover, to investigate teacher reasoning towards learners, if teacher reasoning provides listening for a difference which implies that the teacher might have different type of listening skill which surpasses learner mathematical thinking and/ or understanding, or rather teachers have what is known as "intended listening skill". My research intends to contribute to the South Africa's schools Mathematics education through exploring ways that professional teacher noticing can become effective towards interpretation of classroom phenomenon. As a result, to help teachers to reflect on their pedagogical actions during teaching practice and to make better teaching subject and content knowledge through a notion of teacher professional noticing. Firstly, I intend to re-teach the topic of inverse functions as part of intervention while I have a video recording of my teaching, secondly to provide learners with pre-test and post-tests to evaluate their understanding of the topic prior and after re-teaching. Furthermore, engage learners by using questionnaires to assess and comment on my teaching and provide constructive feedback on where I can improve in my professional teacher noticing based on the teaching of function concept.

amkgosi

The reason why I have chosen your school is because it is convenient school for me as I am an Educator and my participants will be my own learners and colleagues that I have built rigor with over the years.
Would you mind if I request that you participate in my study. I need your help in analyzing my video recording and for you to comment on my professional teacher noticing during my teaching of inverse functions. Please take note that the video recording that you would be requested to analyze, I would have taken it myself during classroom interaction with my own learners during teaching practice. This video entails will entail an hour lesson of inverse functions and interaction with learners accordingly. You will be provided with questionnaire that you will be required to complete your comments and recommendations when analyzing this video recording. Your name and identity will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the project.

You will not be advantaged or disadvantaged in any way. Your participation is voluntary, so you can withdraw your permission at any time during this project without any penalty. There are no foreseeable risks in participating and you will not be paid for this study.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

X AMkgosi

Annie M Kgosi
University of Witwatersrand - Student

Ms Annie M Kgosi

Appendix B

MATHEMATICS INVERSE FUNCTIONS

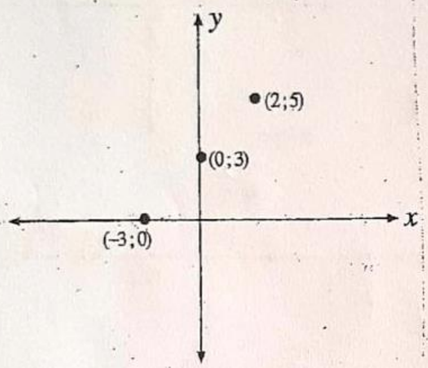
GRADE 12 – INTERVENTION SESSION

(PRE–TEST)

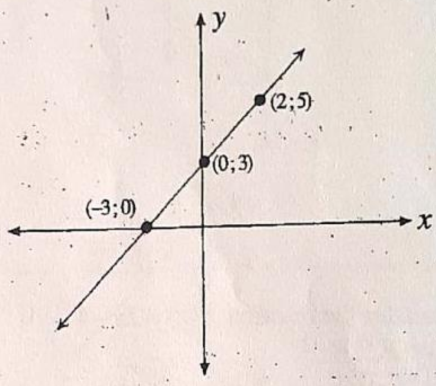
DURATION: 2O -30 MINUTES

Determine whether each of the following relations is a function or not. If the relation is a function, determine whether it is one-to-one or many-to-one. Also write down the domain and range for each relation.

1 (a)



(b)



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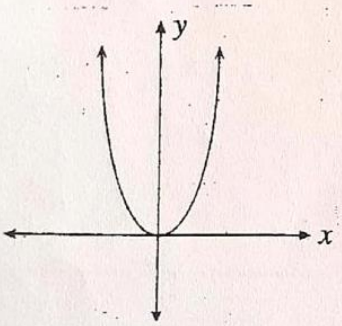
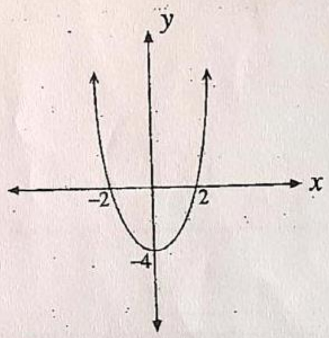
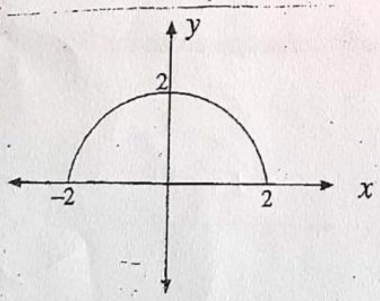


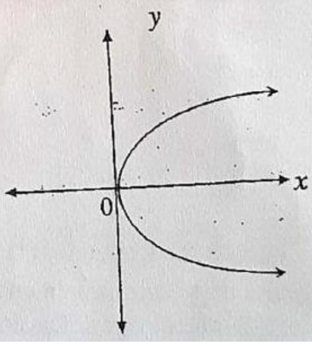
(d)



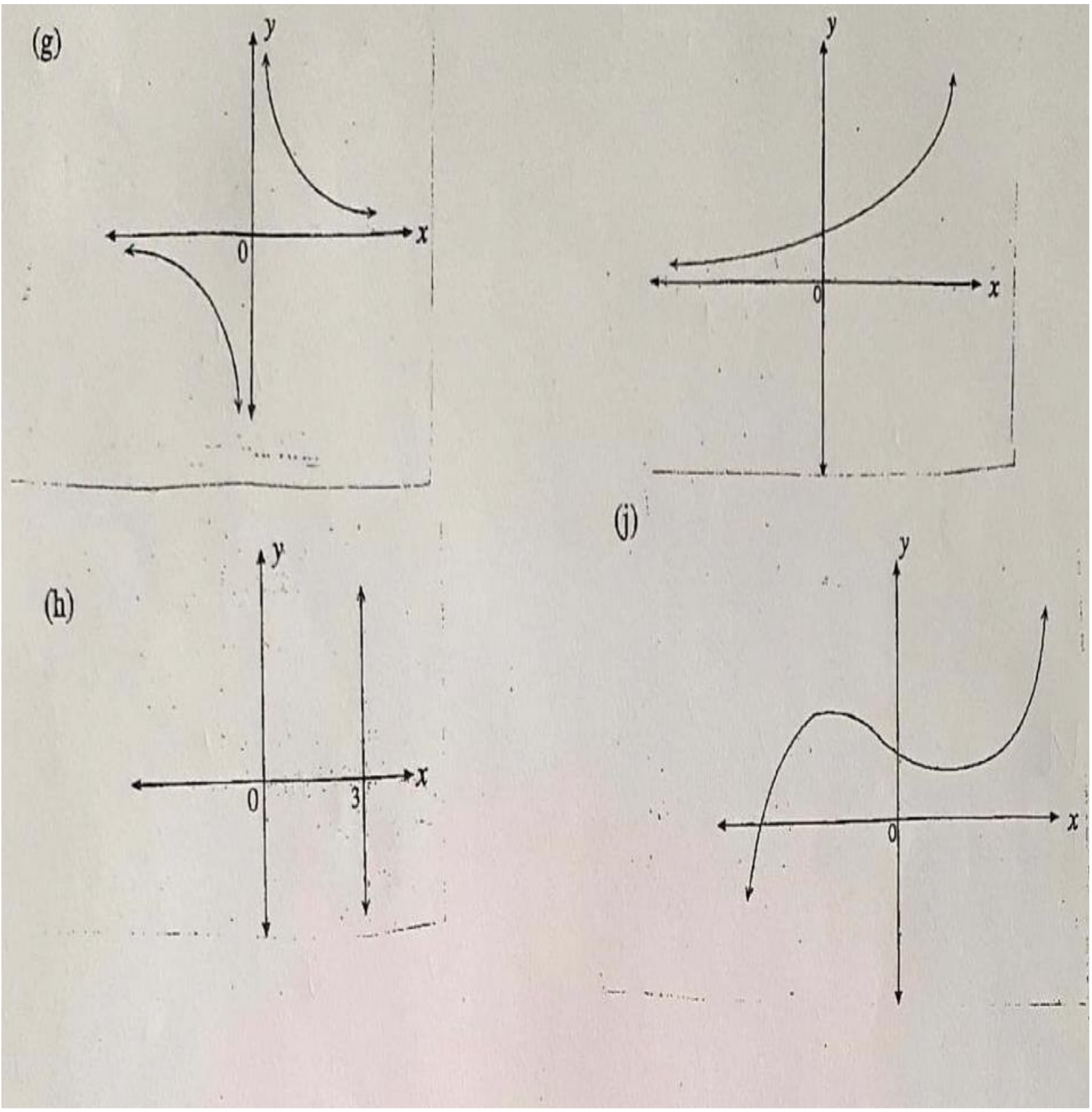
(e)



(f)



amkgosi


(g)
y
x
0
y
x
0
(h)
y
x
0
3
(j)
y
x
0

(2) Consider the function $f(x) = 2x - 4$.

(3) Consider the many-to-one function $f(x) = x^2$.

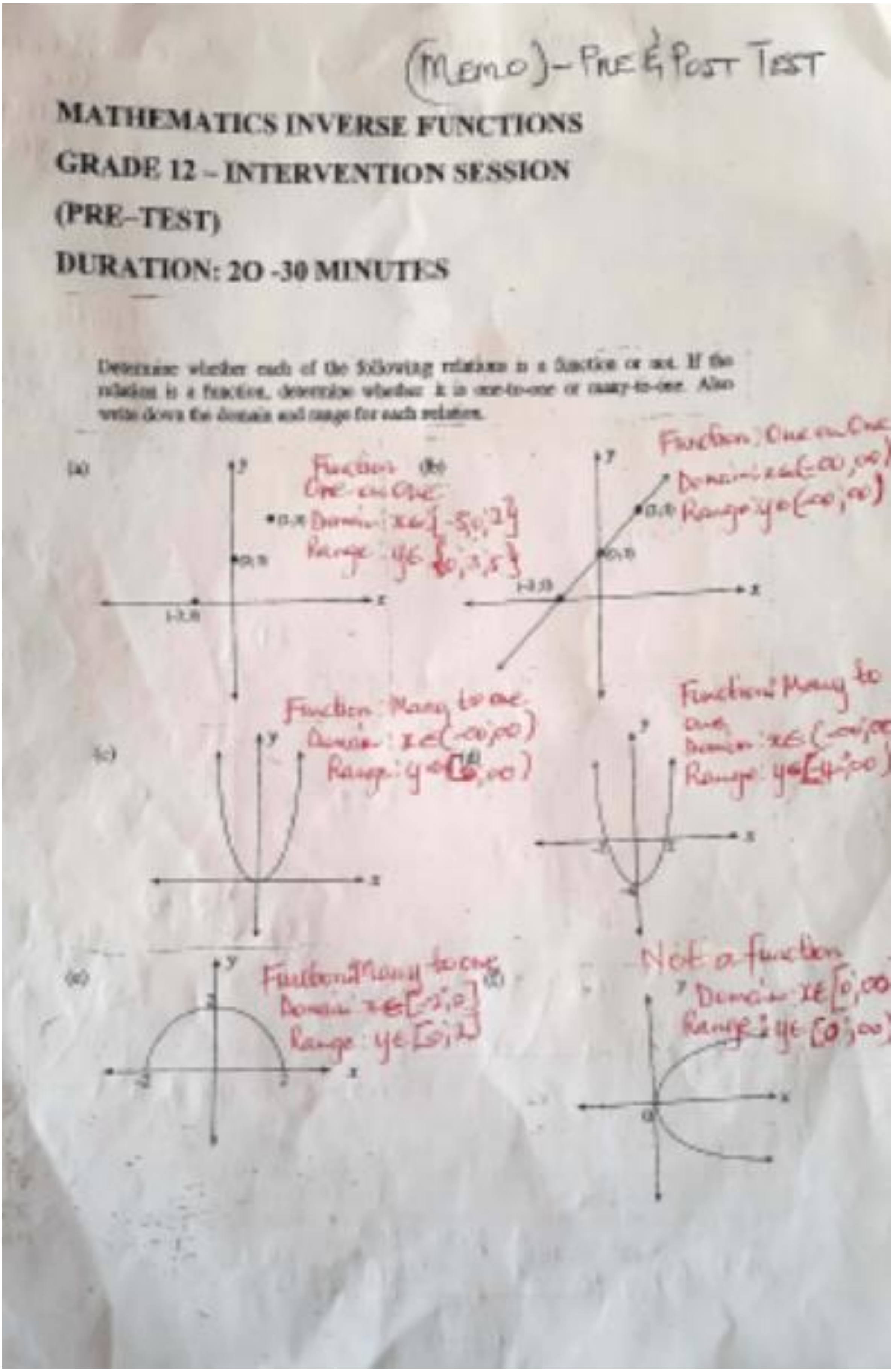
We will now draw the graph of this function as well as its reflection about the line $y = x$.

$y = x^2$

(4) Consider the function $f(x) = 2^x$.

Since this function is a one-to-one exponential function, its inverse will also be a one-to-one function. The domain will not need to be restricted in any way.

We will now sketch the graph of this function and its inverse on the same set of axes.



(Memo) – PRE & POST TEST

MATHEMATICS INVERSE FUNCTIONS

GRADE 12 – INTERVENTION SESSION

(PRE–TEST)

DURATION: 2O -30 MINUTES

Determine whether each of the following relations is a function or not. If the relation is a function, determine whether it is one-to-one or many-to-one. Also write down the domain and range for each relation.

(a)

Function
One on One
Domain: $x \in \{-5; 0; 2\}$
Range: $y \in \{0; 3; 5\}$

(b)

Function: One on One
Domain: $x \in (-\infty; \infty)$
Range: $y \in (-\infty; \infty)$

(c)

Function: Many to one
Domain: $x \in (-\infty; \infty)$
Range: $y \in [0; \infty)$

(d)

Function: Many to one
Domain: $x \in (-\infty; \infty)$
Range: $y \in [-4; \infty)$

(e)

Function: Many to one
Domain: $x \in [-2; 2]$
Range: $y \in [0; 2]$

(f)

Not a function
Domain: $x \in [0; \infty)$
Range: $y \in [0; \infty)$

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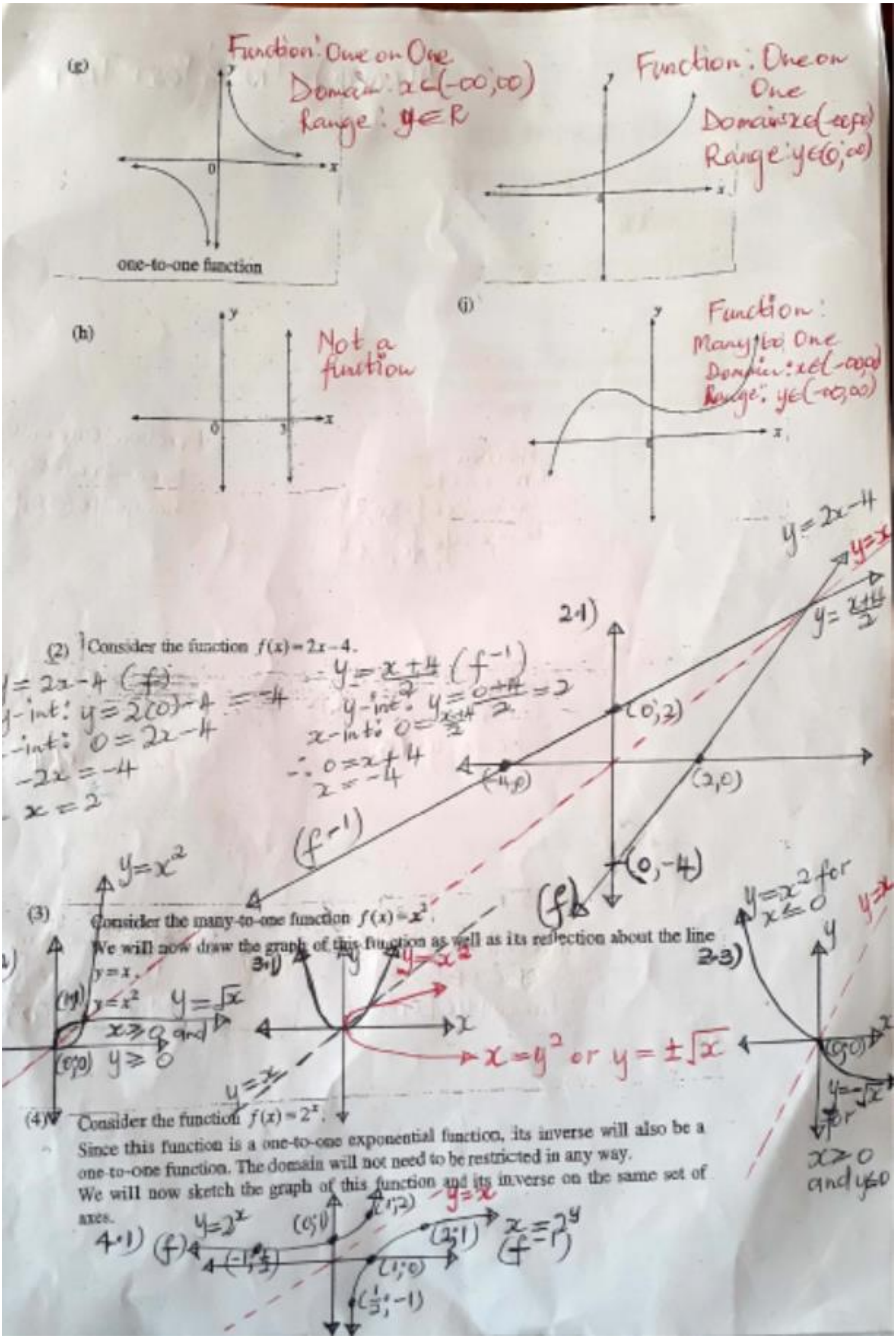

(g)

Function: One on One
Domain: $x \in (-\infty;\infty)$
Range: $y \in \mathbb{R}$

one-to-one function

Function: One on One
Domain: $x \in (-\infty;\infty)$
Range: $y \in (0;\infty)$

(h)

Not a function

(i)

Function: Many to One
Domain: $x \in (-\infty;\infty)$
Range: $y \in (-\infty;\infty)$

(2) Consider the function $f(x) = 2x - 4$.

$y = 2x - 4$ (f)
y-int: $y = 2(0) - 4 = -4$
x-int: $0 = 2x - 4$
$-2x = -4$
$x = 2$

$y = \frac{x+4}{2}$ (f^{-1})
y-int: $y = \frac{0+4}{2} = 2$
x-int: $0 = \frac{x+4}{2}$
$\therefore 0 = x + 4$
$x = -4$

2·1) $y = 2x - 4$, $y = x$, $y = \frac{x+4}{2}$, $(0;2)$, $(-4;0)$, $(2;0)$, $(0;-4)$, (f^{-1}), (f)

(3) Consider the many-to-one function $f(x) = x^2$.
We will now draw the graph of this function as well as its reflection about the line $y = x$.

$y = x^2$, $(1;1)$, $y = \sqrt{x}$, $x \geq 0$ and $y \geq 0$, $(0;0)$

3·1) $y = x^2$, $y = x$, $x = y^2$ or $y = \pm\sqrt{x}$

2·3) $y = x^2$ for $x \leq 0$, $y = x$, $(0;0)$, $y = -\sqrt{x}$ for $x \geq 0$ and $y \leq 0$

(4) Consider the function $f(x) = 2^x$.
Since this function is a one-to-one exponential function, its inverse will also be a one-to-one function. The domain will not need to be restricted in any way.
We will now sketch the graph of this function and its inverse on the same set of axes.

4·1) (f) $y = 2^x$, $(0;1)$, $(1;2)$, $(-1;\frac{1}{2})$, $y = x$, $(2;1)$, $(1;0)$, $(\frac{1}{2};-1)$, $x = 2^y$ (f^{-1})

amkgosi

Appendix C

Research Questions	Research Instruments	Data Collection Methods	Assessment purpose	Method of Analysis
1.What APOS theory levels of learners' mental constructions do teachers notice in the learners' written tasks on the concept of inverse function?	1 Pre and post-tests. (Written tasks) and Memorandum.	1.Teacher assess learners using pre-tests on inverse function. 2. Learners display their thinking by attempting to answer questions related to inverse function, definitions and concepts. 3. Teacher use the memo as marking tool to evaluate learners' mental constructions.	1.To determine learners understanding of concepts, definitions, representations and difficulties. 2.Learners responses to the pre and post-tests questions in terms of the APOS mental constructions.	1. Marked. Pre and post – test and APOS analytical framework to analyse. Table and graphs(pie charts and bar graphs)
2.What are the learners' mathematical thinking and understanding noticed by the teacher during intervention classroom interaction on the concept of inverse function?	2.Intervention classes or sessions and video recorded lessons.	2.Learners pre-tests provides learners' conception levels and informs intervention prior to post-tests	2. To determine the teacher noticing sophistication level from intervention.	2.Constant comparative method of analysis was used to compare teachers' sophistication levels of noticing.
3.What are the levels of noticing sophistication observed when the teacher(s), attend, analyse and respond to learners' mathematical thinking and reasoning during intervention?	3.Intervention classes or sessions and video recorded lessons. Questions and Answer to determine attending, analysing and responding.	3.Application of sophistication level analysis framework to analyse teacher noticing. Assess from the transcripts from the video recorded lesson the teachers attending, interpreting and deciding skills.	3. To establish if professional noticing can be recommended as a framework for attending, analysing and deciding for concept inverse function	3. Barnhart & van Es, 2015 adapted framework.

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Task	Concepts covered and assessed	Questions asked to analyze data.	APOS level/ indicators
1.	Determine if each of the following relations is a function or not?	Will a learner be able to recognize if the given graphs are functions or not?	A
2.	If the relation is a function. Determine, whether it is a one-to-one or many-to-one.	What cues will help the learner to know the difference in terms of the relations and functions.	A & P
3.	Write down the domain and range for each relation.	What helps the learner to identify the domain and range for each relation. How does the teacher notice that the learner has correctly identified the domain and range for each function.	A & P
4.	Consider the function and sketch the graph of and its inverse function $f(x) = 2x - 4$.	What mental contructions a learner need to have to sketch the function of $f(x) = 2x - 4$ and its inverse.	A, P & O
5.	Consider the many-to-one function. Draw the graph of this function and its reflection about the line.	What does a learner understand by a many-to-one function and what makes it different from a one-to-one function? Can a learner sketch the reflection of this function and what is important to know about the inverse of reflection of this function.	A, P O & S
6.	Consider the function $f(x) = 2^x$, since the graph is a one-to-one exponential function. Its inverse should also be a one-on-one function, the domain will not be restricted and its inverse in any way. Now sketch the graph of this function and its inverse on the same set of axes.	What mental contructions a learner need to be able to sketch the function of $f(x) = 2^x$ and its inverse function on the same set of axes.	A, P & O

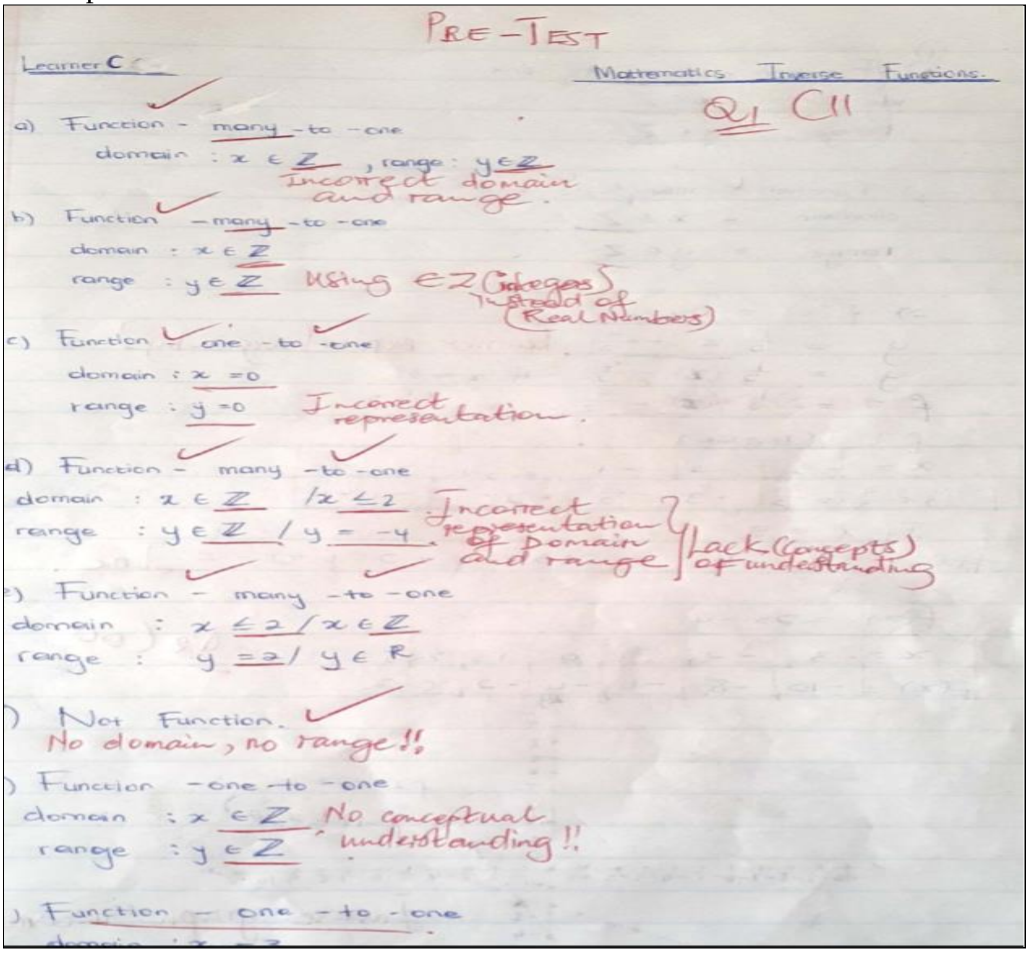
Levels of Sophistication for Noticing Skills			
Skill	**Low Sophistication**	**Medium Sophistication**	**High Sophistication**
Attending	Highlights classroom event, teacher pedagogy, student behaviour, and/or classroom climate. No attention to student thinking.	Highlights student thinking with respect to the collection of data from the scientific enquiry (scientific procedural focus).	Highlights student thinking with respect to the collection, analysis and interpretation of data from scientific inquiry (scientific conceptual focus)
Analyzing	Little or no sense making of highlighted events: mostly descriptions. No elaboration or analysis no use of evidence to support claims.	Begins to make sense of highlighted events. Some use of evidence to support claims.	Consistently makes sense of highlighted events. Consistent use of evidence to support claims.
Responding	Does not identify or describe acting on specific student ideas as topics of discussion; offers disconnected or vague ideas of what to do differently next time.	Identities and describes acting on a specific idea during the lesson; offers ideas about what to do differently next time	Identifies and describes acting on a specific student idea during the lesson and offers specific ideas of what to do differently next time in response to evidence; makes logical connections between teaching and learning.

amkgosi

Appendix D

Annexure 1

Excerpt 1


PRE-TEST

Learner C

Mathematics Inverse Functions.

Q1 C11

a) Function - many -to -one
domain : $x \in \mathbb{Z}$, range : $y \in \mathbb{Z}$
Incorrect domain and range.

b) Function - many -to -one
domain : $x \in \mathbb{Z}$
range : $y \in \mathbb{Z}$ Using $\in \mathbb{Z}$ (Integers) Instead of (Real Numbers)

c) Function - one -to -one
domain : $x = 0$
range : $y = 0$ Incorrect representation.

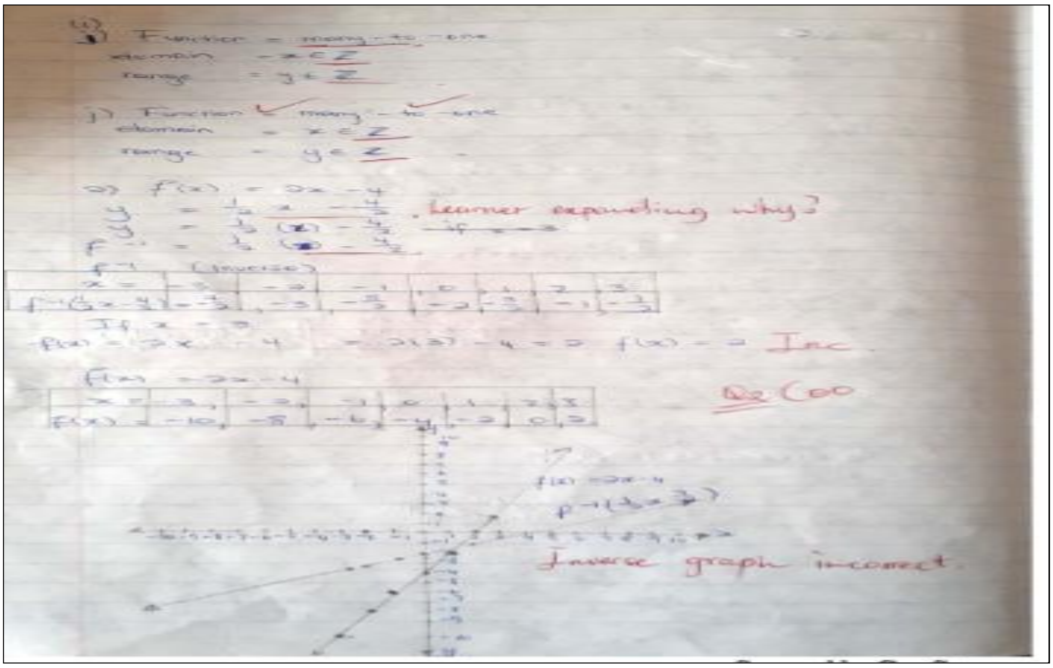
d) Function - many -to -one
domain : $x \in \mathbb{Z}$ / $x \leq 2$
range : $y \in \mathbb{Z}$ / $y = -4$ Incorrect representation of Domain and range
Lack (Concepts) of understanding

e) Function - many -to -one
domain : $x \leq 2$ / $x \in \mathbb{Z}$
range : $y = 2$ / $y \in \mathbb{R}$

) Not Function.
No domain, no range!!

) Function -one -to -one
domain : $x \in \mathbb{Z}$ No conceptual
range : $y \in \mathbb{Z}$ understanding!!

) Function - one -to -one


Function - many -to -one
domain $x \in \mathbb{Z}$
range $y \in \mathbb{Z}$

j) Function - many -to -one
domain $x \in \mathbb{Z}$
range $y \in \mathbb{Z}$

$f(x) = 2x - 4$

Learner expanding why?

f^{-1} (inverse)

Inverse graph incorrect.

Q2 C00

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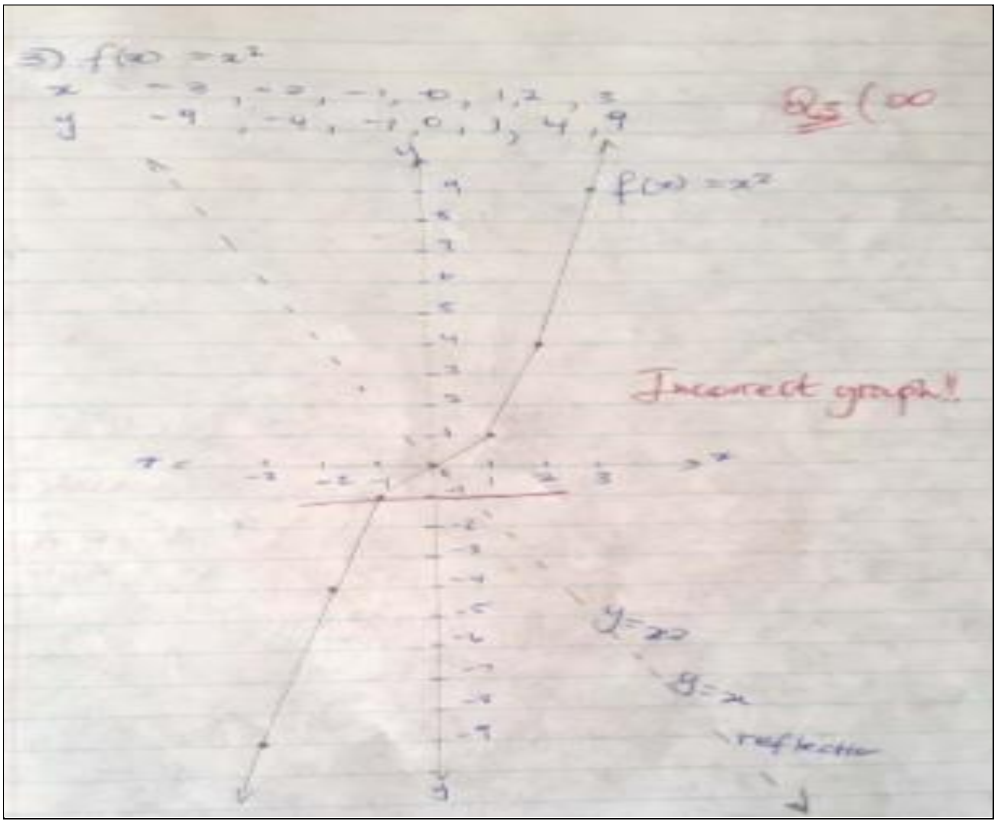
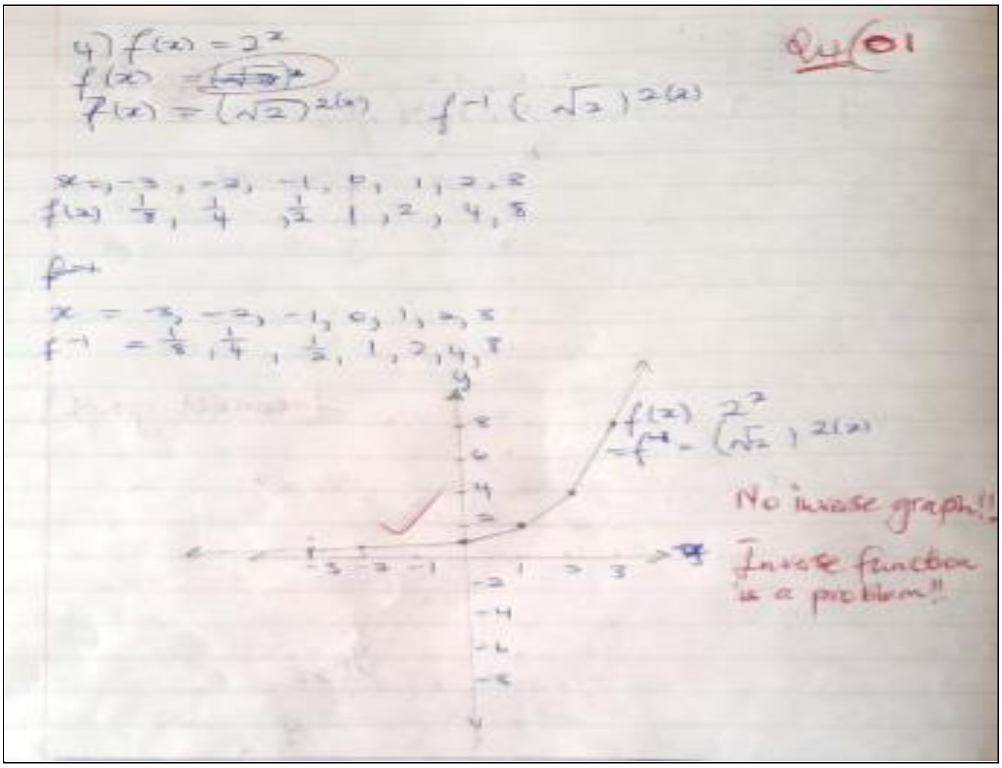
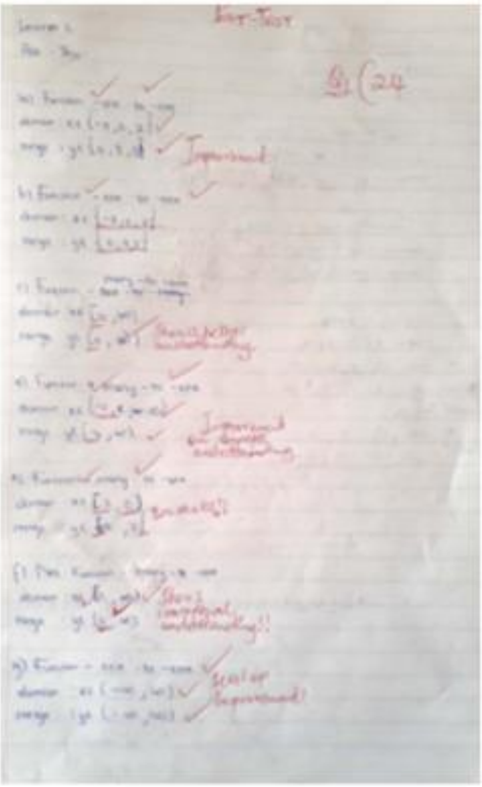
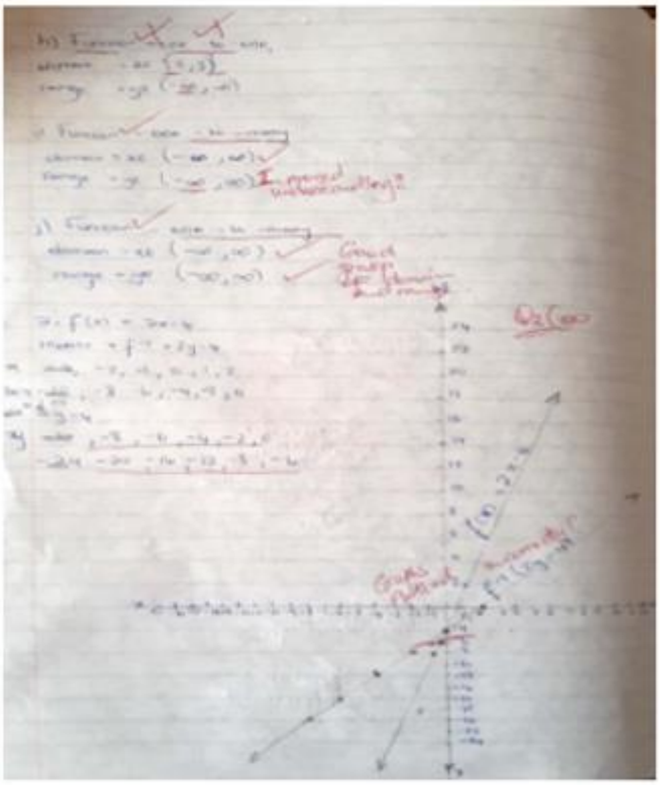
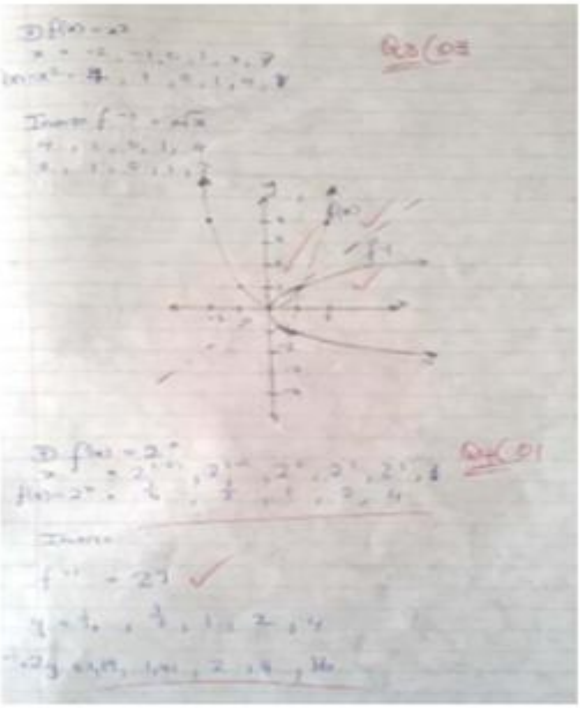

3) f(x) = x²
Q5(00
f(x) = x²
Incorrect graph!!
y = x²
y = x
reflection


4) f(x) = 2ˣ
Q4(01
f(x) 2ˣ
No inverse graph!!
Inverse function is a problem!!

Excerpt 2







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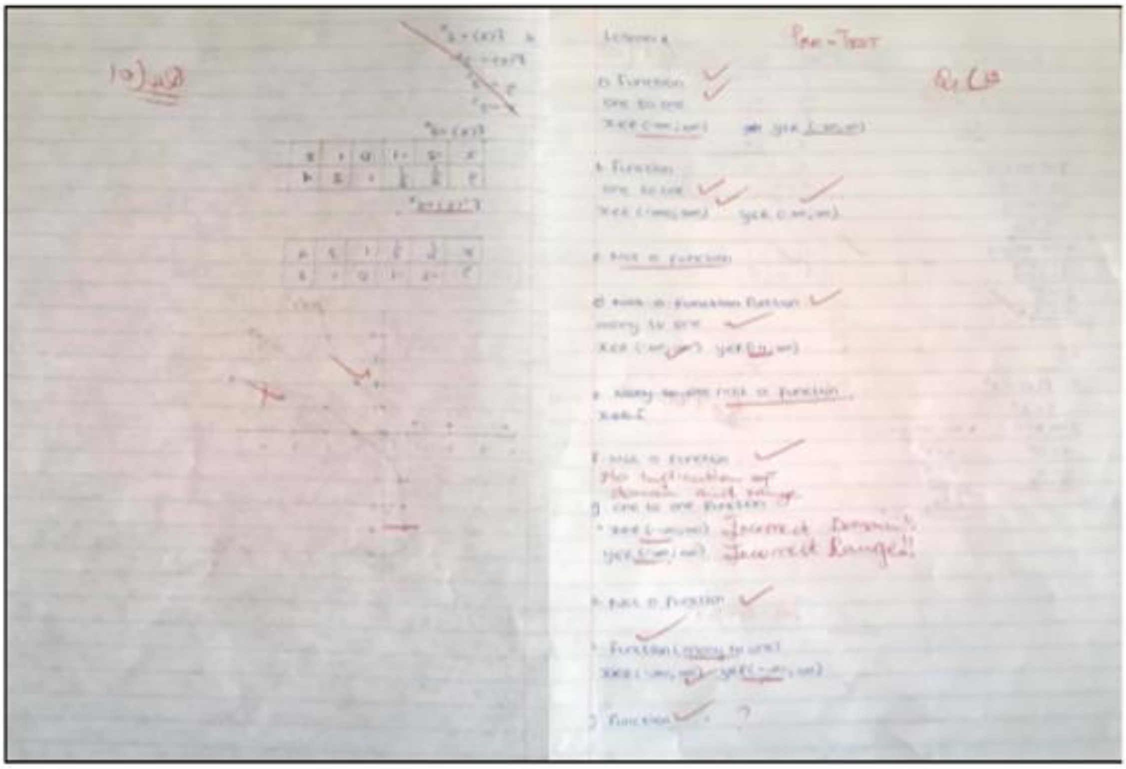

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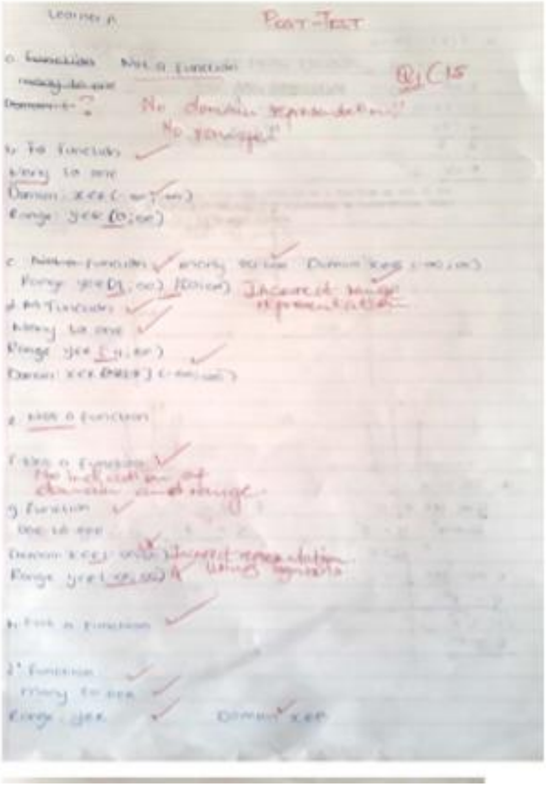

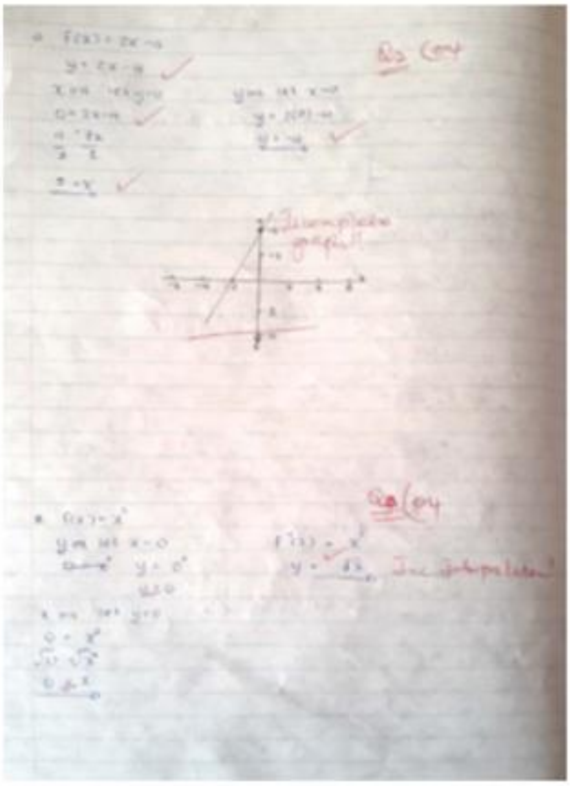

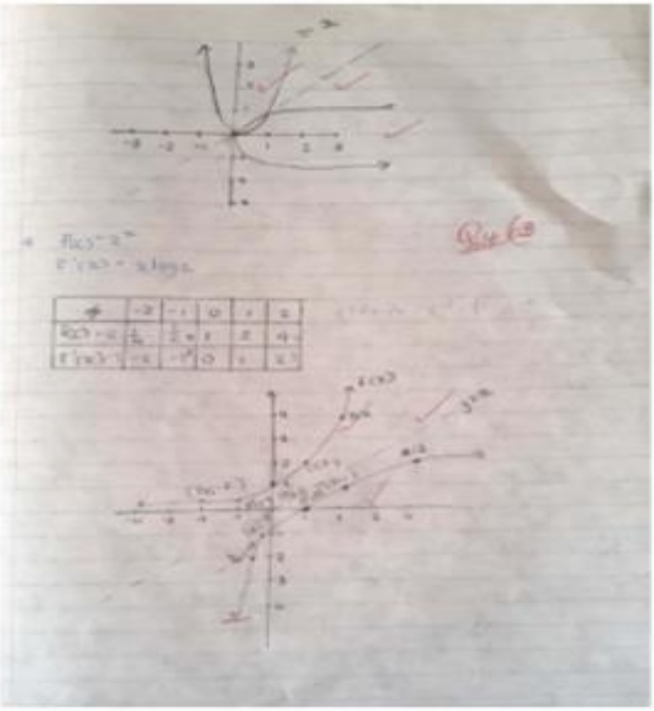

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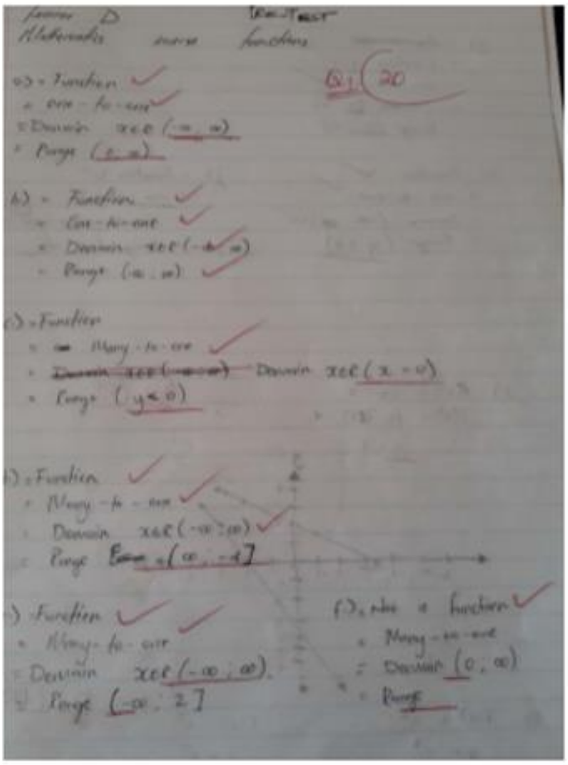
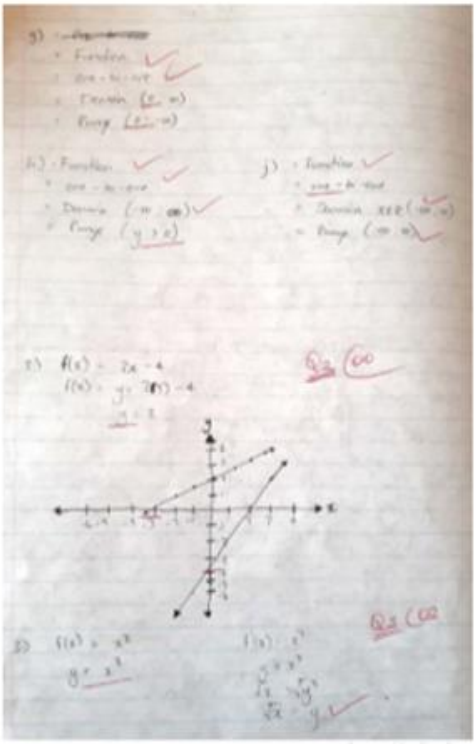
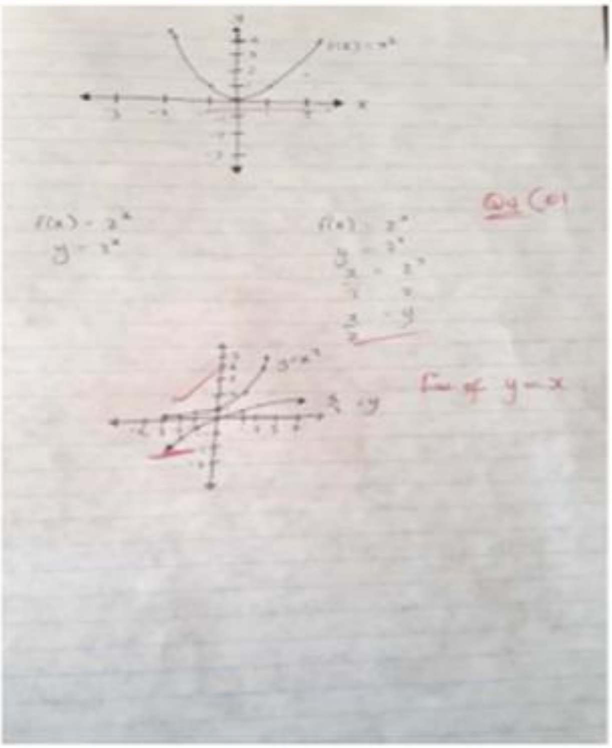
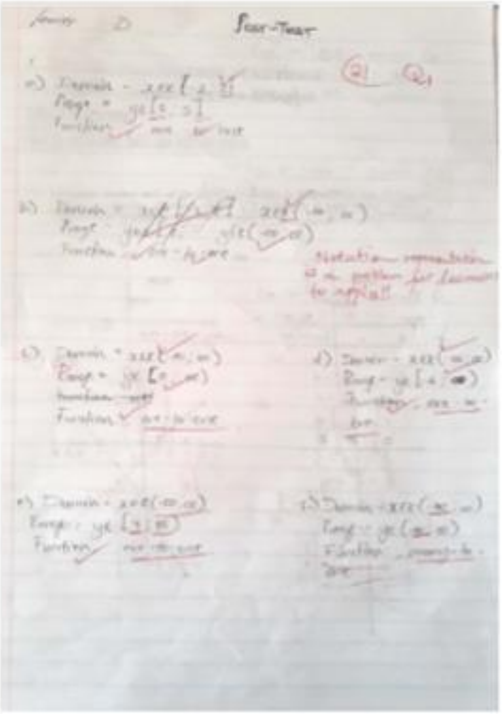


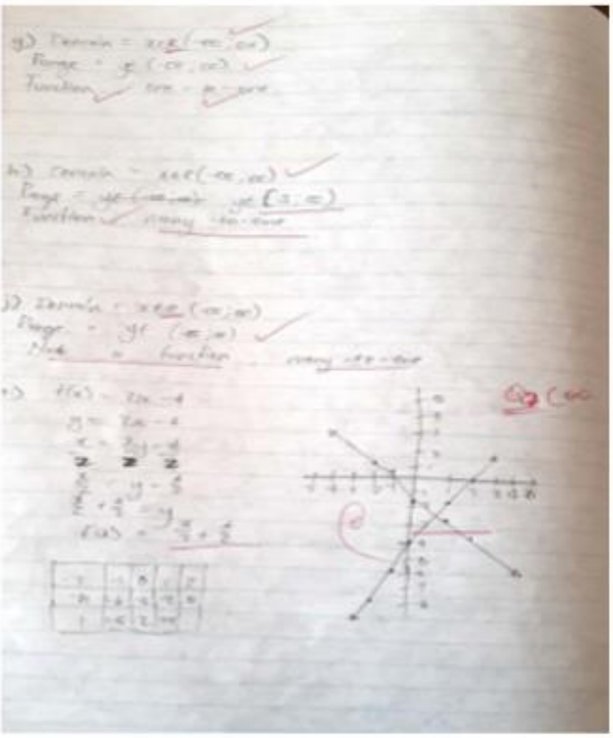


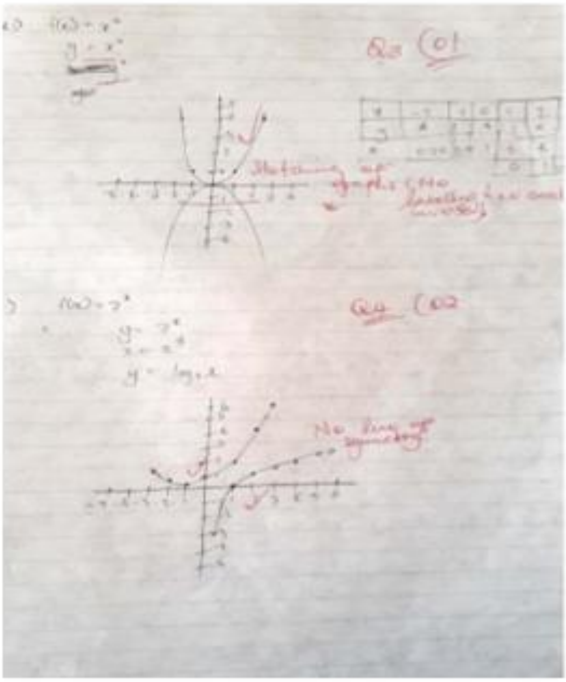


Excerpt 6

Post-test 3– Learner D

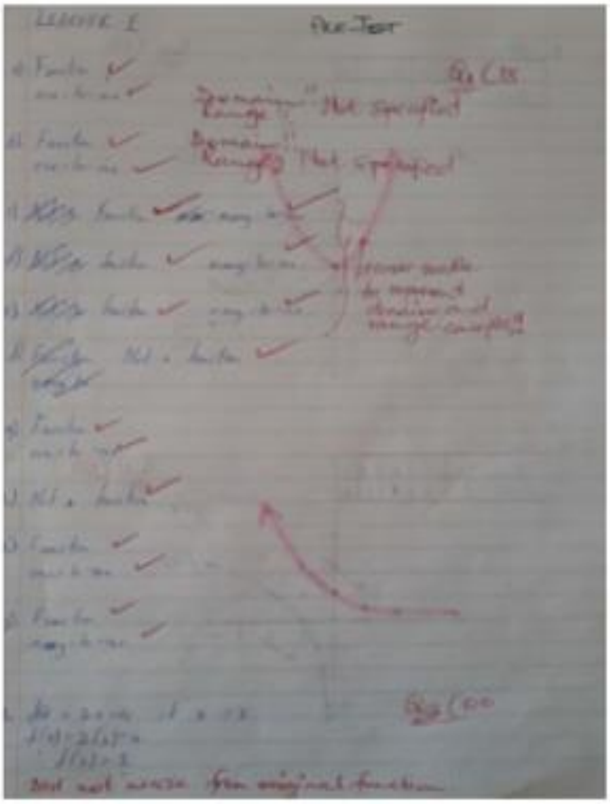


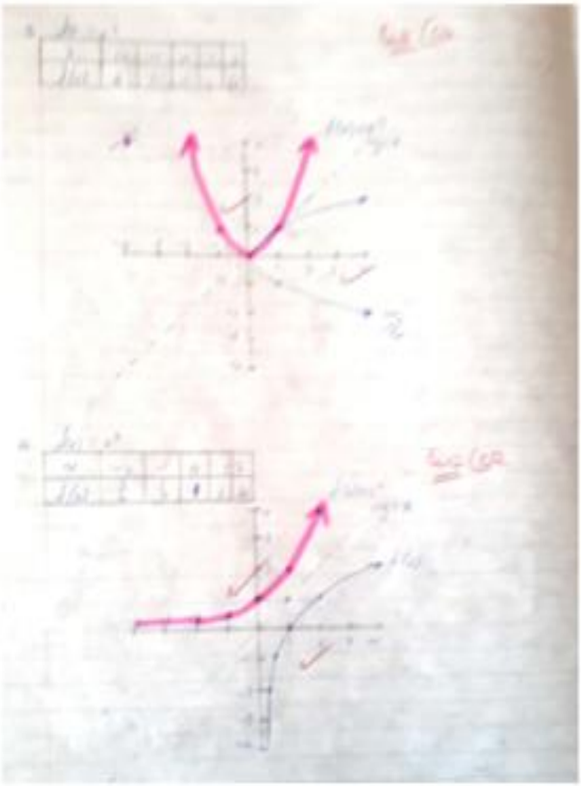




Excerpt 7

Pre-test 5– Learner E





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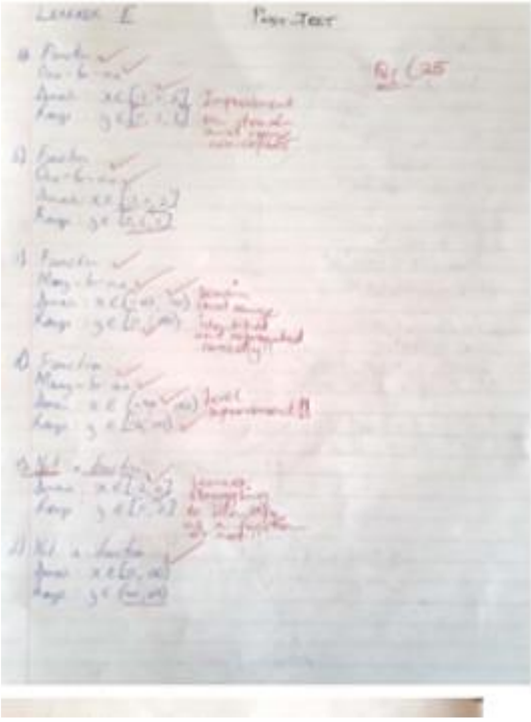
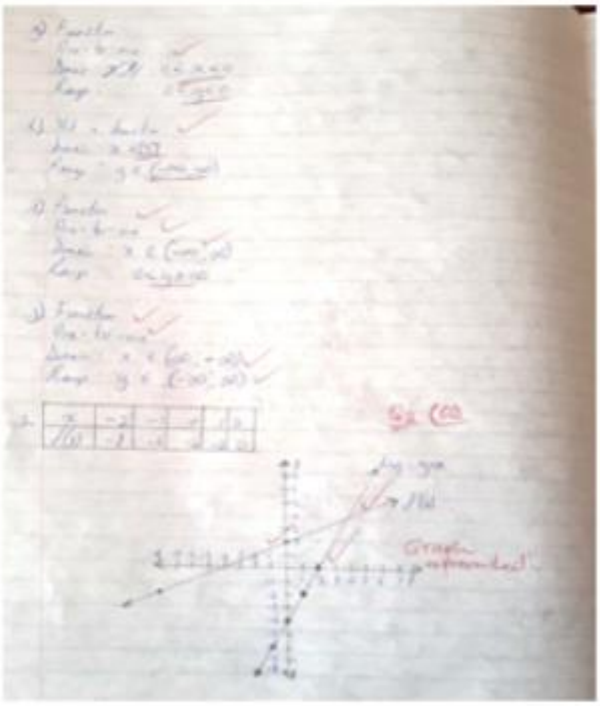
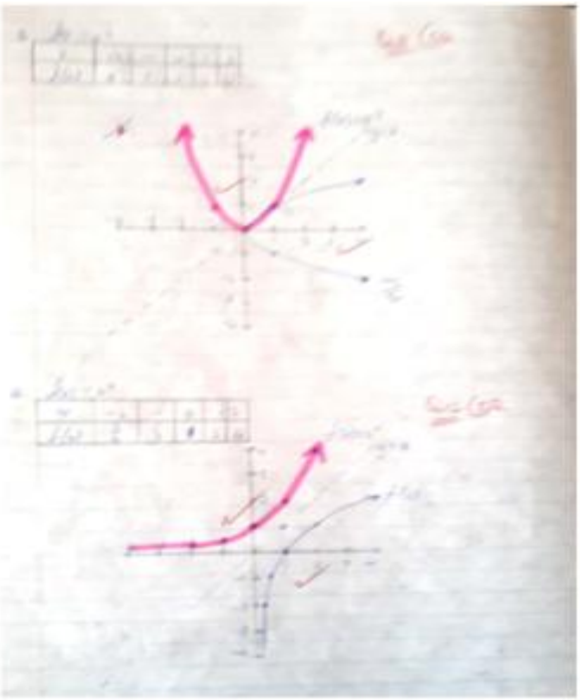
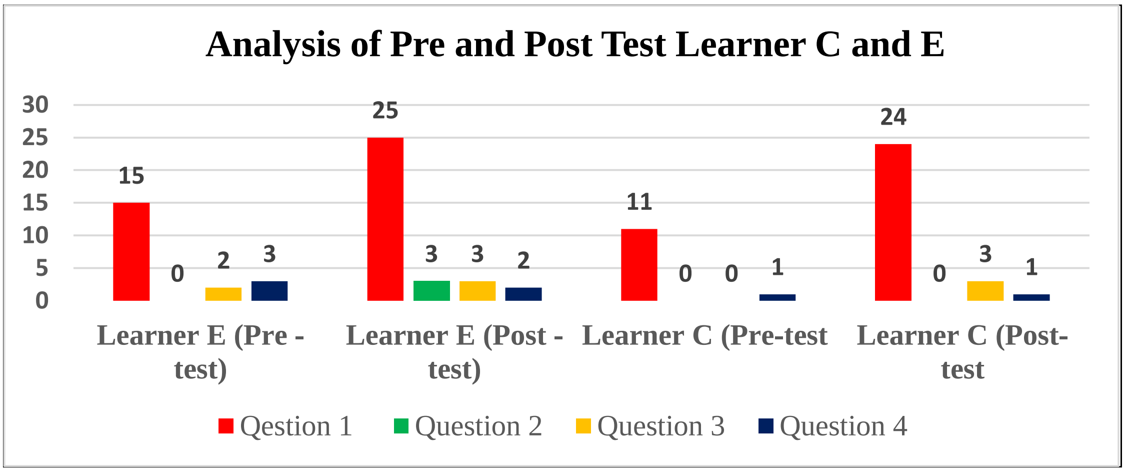
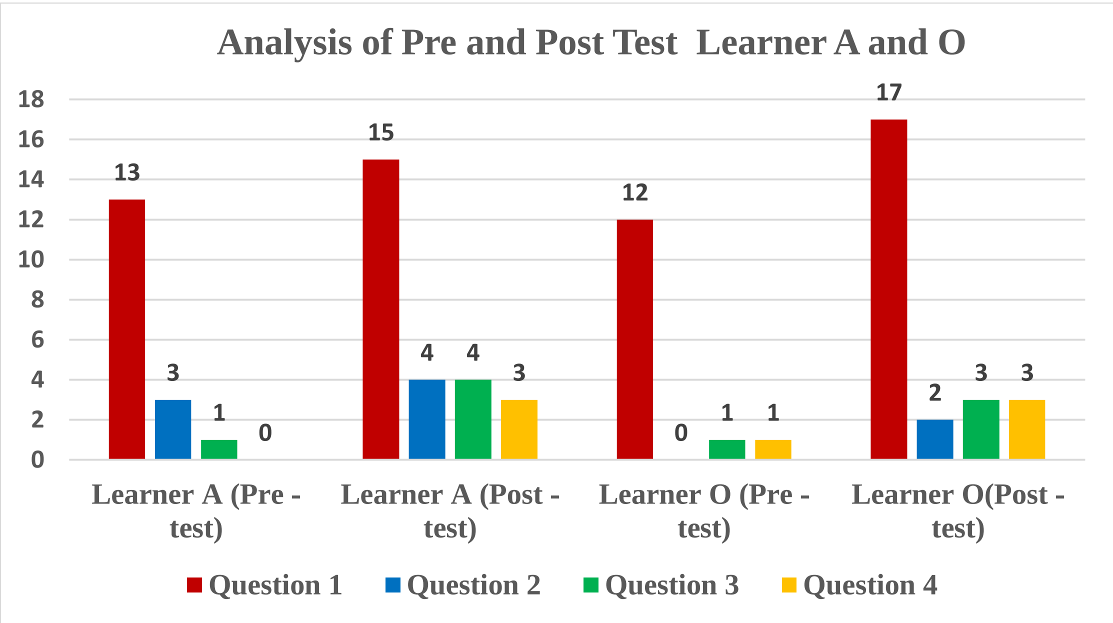






Annexure 2





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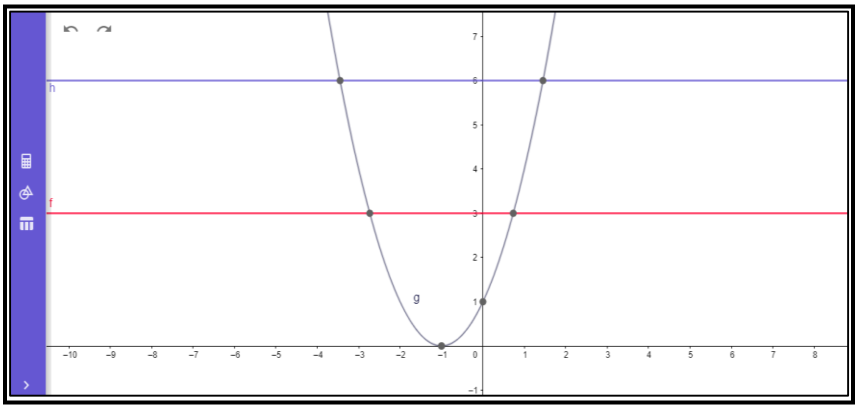
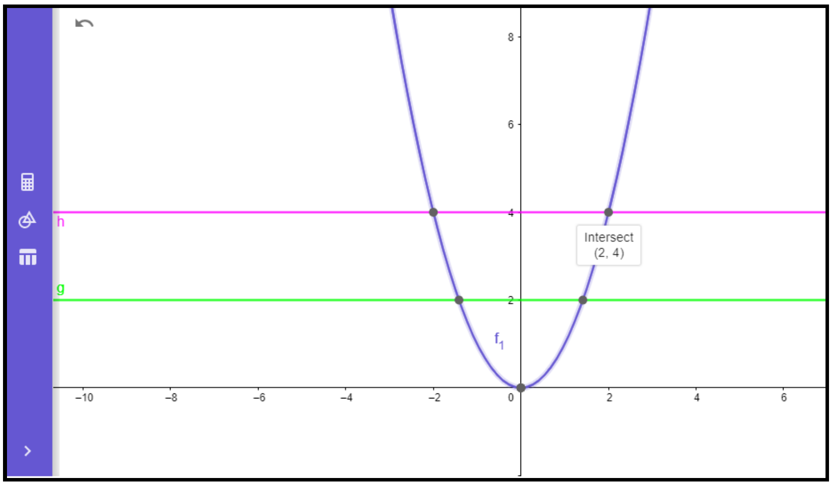
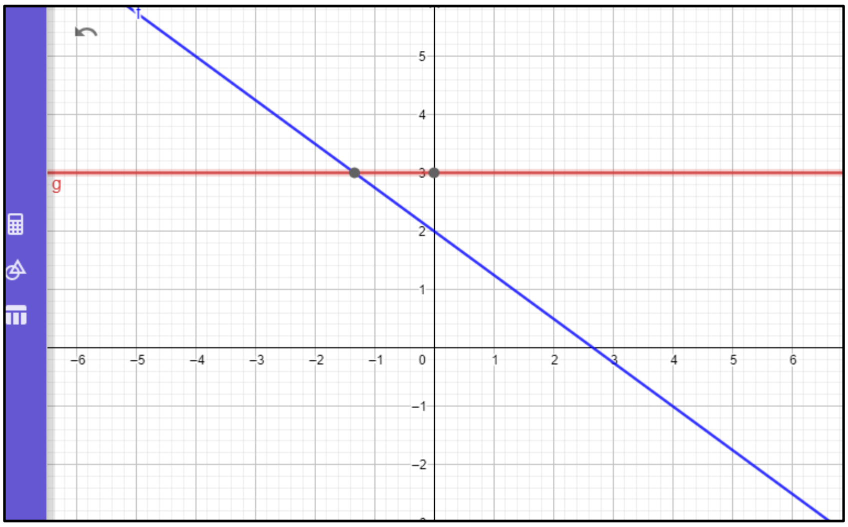
Geogebra Graphing in Figure 5







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Annexure 3

INPUT BY TEACHER B ( ) –VIDEO ANALYSIS ON MASTERS STUDENT

1.1 The researcher was very passionate and assertive about teaching the grade 12 topic "Inverse Functions". She first introduced the topic by talking about the original quadratic function and its properties. Thereafdter, she effectively demonstrated to the learners how to sketch the inverse graph of the parabolic function, which is not a function.

The researcher emphasised how you could restrict the domain of the parabolic function so that the inverse is also a function.

Hence, the topic was adequately addressed by the researcher.

1.2 The researcher as indicated was teaching the topic "Inverse Functions" to a sample of grade 12 learners of High School.

The topic was adequately covered.

1.3 The video recording was well recorded and presented. However, there should have been more intense interaction between the researcher and the sample of learners in the intervention class.

1.4 The researcher was fully aware of what the learners already know and what they did not know. She used inductive reasoning from the known to the unknown in her didactical method of teaching. Hence, there was adequate evidence of noticing skills during the researcher's teaching practice according to the video.

1.5 The researcher made use of the question and answer approach to illicit their thinking and reasoning ability on Inverse Functions. However, there should have been a more collaborative approach to lend itself to the topic of Inverse Functions.

1.6 The researcher allowed for adequate discussion and input. This assisted her to gauge the learner's conceptual understanding of the Maths

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1.7 In the first video recording, the researcher taught the topic. She gave the learners the pre test to see how much they have understood the said topic. From the test results, she selected a few learners who did not perform well to attend the intervention class. Hence, the responses in the test

informed the researcher's professional noticing and intervention strategy during the teaching of Inverse Functions.

1.8 It did to a certain extent. In the intervention class, the learner's insight into the topic was much more greater than the actual lesson in class. The researcher guided the learner's conceptual understanding and insight into the topic.

1.9 The researcher did gather adequate data on the topic Inverse Functions. However, a role play could have also taken place to enhance the learner's understanding, and to avoid any misconceptions.

1.10 Through the video recording, the researcher is seen in action during her teaching practice. Her strengths and weaknesses can easily be observed. She should have given the learners more opportunity to engage on the topic.

1.11 "Professional noticing" is an effective tool to see the teacher in action during the practice of teaching. The video recording should have also been focussed on the learners so that "professional noticing" becomes an effective and reliable tool to assess a researcher/educator. A complete picture is drawn if there is participation on both sides – the educator and the learner. Yes, I will recommend "professional noticing" to assess and critique educators.

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Masters Thesis (Masters in Education Degree)- Research Project

Researcher/ Student : Ms Annie Mamoretsi Kgosi

Student Number : 1484157

Ethics Clearance : 2019ECE030M

Analysis conducted by : Teacher A

Institution: University of Witwatersrand (Humanities Faculty)

Topic : Mathematics Teacher's Professional Noticing in teaching of Inverse Functions in Grade 12

1. Research Questions Related to the Video Analysis of the teaching and intervention classes

1.1 Was the teacher (researcher) addressing the relevant topic of the study? Please elaborate on your answer.

- Yes, after presentation of the functional graph which is the base line to do the inverse graph then she did offer the lesson on the inverse functions.

1.2 What was the topic that the researcher teaching and to what extent was this topic covered and for what grade?

- The topic was Inverse Functions for Grade 12 learners which was done to cover all the functions.

1.3 What was your analysis of the video recording regarding the lesson?

- What I have analysed is that the lesson was not just based on one type of Functions but different kinds of graph were done, e.g linear function inverse, exponential function inverse and also quadratic function inverse.

1.4 Did the researcher apply any noticing skills during her teaching practice according to the video analysis?

- Yes, the researcher did apply noticing skills by involving/engaging learners during the lesson.

1.5 What skills did the researcher use in her teaching to notice her learner's thinking and reasoning during the classroom interaction?

- The skill the researcher used was question and answer.

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1.6 How did the researcher interpret learner's Mathematical understanding during teaching practice?

- It was through class activity that was monitored to interpret learners Mathematical understanding during teaching practice.

1.7 According to your opinion, do you think that the researcher's interpretation of the learner's Mathematical understanding informed her professional noticing and intervention strategy during teaching of inverse functions? Elaborate on your answer.

- The researcher's interpretation did notice her from the learner's responses during the lesson and the activity the was given.

1.8 Did the researcher's professional noticing transform learner's thinking and understanding in accordance to the video analysis? If not how so? Please give more insight.

- Yes, through the coordinates of the function that yes to determine the coordinates of the inverse you just swop the x and the y intercepts.

1.9 According to the analysis made based on the video recordings would you say that the researcher gathered adequate data in relation to the above mentioned topic? If not why?

- Yes, adequate data was done as much as the definition of what a function is and when not a function.

1.10 Was the video recording a reflective analysis of how teacher's professionally notice during their classroom practices and interaction with learner's? Please clarify and give more details.

- More learner's did asks questions that were answers during the lesson for exam purpose.

1.11 As a Mathematical Educator yourself would you say in your opinion that the topic of "Mathematical Teacher Professional Noticing" during teaching practice can be recommended for Mathematics teachers and why? Elaborate

Yes , very impressed that it should be done by all teachers.