MSc. Dissertation
Memory in non-Abelian Gauge Theory

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DECLARATION

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the university of the Witwatersrand or any other University.

Bourgeois Biova Irénée Gadjagboui, May 25, 2017.

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DEDICATION

To my loving friend Mahugnon Sonia Brigitte Amoussou.

Abstract

This project addresses the study of the memory effect. We review the effect in electromagnetism, which is an abelian gauge theory. We prove that we can shift the phase factor by performing a gauge transformation. The gauge group is U(1). We extend the study to the nonabelian gauge theory by computing the memory in SU(2) which vanishes up to the first order Taylor expansion.

Keywords: Memory Effect, Aharonov-Bohm effect, Nonabelian Gauge Theory, Supersymmetry.

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Chapter 1

Introduction

The theory of general relativity was introduced in 1915. It provides a description for the mechanism by which gravity acts on masses. The Newtonian description of gravity suggests that gravity acts instantaneously between two masses. This violates one of the postulates of special relativity. The postulate being violated is that the speed of light sets the limit for information transfer. In general relativity, gravity arises from the geometry of spacetime. Based on this theory, in 1916, Einstein predicted gravitational radiation, which are ripples in the fabric of spacetime. Gravitational waves from a black hole merger were reported in February 2016 by the LIGO experiment [1]. A symmetry of the gravitational S-matrix is generated by Bondi-Metzner-Sachs (BMS) transformations. BMS symmetries help characterize the behavior of gravitational waves in asymptotically Minkowski spacetime [2].

Quantum Electrodynamics (QED) is an Abelian gauge theory. One of the consequences is the soft photon theorem [3]. This theorem says that when we consider an amplitude M involving some incoming and some outgoing particles and the same amplitude with an additional soft-photon coupled to one of the particles, let's say M', the two amplitudes are related.

Recent considerations have demonstrated that the soft photon theorem in abelian gauge theories, with only massless charged particles, is a Ward identity of an infinite-dimensional symmetry group (BMS). The BMS are transformations comprised of certain "large" gauge transformations which do not vanish at infinity. By large gauge transformation, we mean a gauge transformation that is not homotopic to the identity. QED has massive charged particles in the real world, not massless [3]. The Ward identity is a quantum version of the classical Noether's theorem. According to Weinberg and Witten, the soft photon theorem does not allow for massless particles with spin greater than 1/2 to carry a Lorentz-covariant current. In the same way, massless particles with spin greater than 1 cannot carry a Lorentz-covariant stress-energy tensor.

Recently, Susskind [4] has studied the electromagnetic memory effect in response to Strominger's paper [3]. Strominger, in that paper, has extended the analysis of the soft photon theorem to all the unitary $U(1)$ gauge theories. Susskind has defined the electromagnetic memory effect to be frozen in phase after charged particles have passed through the source considered [4]. Similarly, the memory effect refers to the displacement of test masses in a gravitational waves interferometer. Moreover, Strominger and collaborators [4, 5, 6] have mentioned the equivalence of gravitational memory, soft theorems, and BMS transformations.

A year ago, gravitational waves were discovered by the LIGO experiment [1]. Gravitational waves can be observed from a binary neutron star merger or core-collapse supernovae or any other similar process. In these mechanism, regular mass and momenta are radiated away in the form of gravitational waves which leave a trace in spacetime. When gravitational waves such as those measured in LIGO pass through such apparatus, test masses within the apparatus are displaced afterwards. This is called the "memory" effect.

Gravitational waves are the natural objects to study in the memory effect. They are important because they contain physical information of a gravitating system. Recently, gravitational waves have become a main subject for Strominger and collaborators [4, 5, 6]. They have established the equivalence of the soft photon theorem and the Ward identity. The Ward identity is a consequence of symmetries. Symmetries help us determine and classify the solutions to the equations of motion of a system, which is described by the action. Therefore the equations of motion select the classical solution as the solution that minimizes the action S.

In the interacting theory, bosons are considered to be mediators of interaction, while fermions are the constituents of matter. The scattering matrix (S-matrix) of a physical system is a unitary matrix that describes the evolution of initial particle states to final particle states at asymptotic infinity. The symmetries of the S-matrix include the familiar Poincaré symmetries, which involve translations of the origin and Lorentz transformations (rotations and boosts). The largest possible symmetry, however, relates fermions to bosons. This is supersymmetry. It predicts that bosonic and fermionic degrees of freedom pair up. A particle has a superpartner with opposite statistics and the same mass. Supersymmetry is a broken symmetry in nature. It has not been experimentally observed. However, it is a useful model for exploring new physics.

The outline of this dissertation is as follows: Chapter 2 presents the electromagnetic memory effect. In Chapter 3, we study a particular four dimensional supersymmetric quantum field theory called $\mathcal{N} = 4$ super-Yang–Mills gauge theory. The gauge symmetry of the theory is $SU(N)$. This theory is conformal, which means that it is scale invariant. In particular, unlike Quantum Chromodynamics (QCD), it is a non-confining theory. It makes sense to think of excitations at asymptotic infinity and examine the S-matrix. In Chapter 4, we examine the phenomenon of memory in the nonabelian quantum field theory. Our study focuses on the theory with $SU(2)$ gauge group. Finally, we have presented a summary of the document in Chapter 5.

Chapter 2

The Electromagnetic Memory Effect

2.1 Introduction to Field Theory

Quantum field theory provides a description of how elementary particles interact [7]. It combines special relativity and quantum mechanics. Special relativity describes the physics of systems at speeds comparable to that of light. Quantum mechanics describes the physics of systems at small scales with small speeds [8]. In this section, we review notation and provide some background concepts in special relativity as well as in quantum mechanics and quantum field theory. Any theory of a fundamental nature must be consistent with special relativity and with quantum theory. Therefore, we start with special relativity by recalling the Lorentz transformation and the idea of four-vector. Thereafter, we derive the equations of motion of free particles in Klein-Gordon and Dirac fields. Finally, we write down the equations of motion, that is the Euler-Lagrange equations in the quantum field theory.

2.1.1 Special relativity and relativistic notations

The theory of special relativity is based on two postulates:

1) The laws of physics are the same in any inertial reference frame.

2) The speed of light c is constant.

Two inertial reference frames are for example related by :

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{2.1}$$

where coordinates are denoted in spacetime by (ct, x, y, z) in the initial reference frame and (ct', x', y', z') in the relative reference frame. The coordinates characterize an event in special relativity [9]. The angle ϕ is called rapidity and is given by:

$$\tanh\phi = \frac{v}{c}. \tag{2.2}$$

Define

$$\Lambda^{\mu}{}_{\nu} := \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

$\Lambda^{\mu}{}_{\nu}$ is a Lorentz transformation called a "boost" along x–axis. To see how we obtain the boost along the x–axis, let us consider two inertial reference frames $\mathbf{R}$ and $\mathbf{R}'$, where the reference frame $\mathbf{R}'$ moves relative to $\mathbf{R}$ with a constant speed v along x–axis. Since the relative motion is along the x–axis, then y and y' (respectively, z and z') are the same in both reference frames, where coordinates in $\mathbf{R}'$ carry the prime label. For this reason,we only consider transformations of the coordinates x and t from the reference frame $\mathbf{R}$ to x' and t' in the reference frame $\mathbf{R}'$ so that

$$\begin{cases} x' &= \gamma(x+\beta t) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t+\beta x) \end{cases}, \tag{2.4}$$

where $\gamma = (1-\beta^2)^{-\frac{1}{2}}$ and $\beta = \frac{v}{c}$. One can see that $\gamma^2 - \beta^2\gamma^2 = 1$. Therefore, we may choose $\gamma = \cosh\phi$ and $\beta\gamma = \sinh\phi$.

Another useful Lorentz transformation is the rotation. The Lorentz transformation of a rotation for example in the $x-y$ plane for a vector V about the $z-$ axis in a four spacetime dimension is given by:

$$\begin{pmatrix} V_t' \\ V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_t \\ V_x \\ V_y \\ V_z \end{pmatrix}, \tag{2.5}$$

where θ is the angle of rotation.

It is convenient to recall the Einstein summation convention as we will make use of it in the future. In fact, the Einstein summation represents a sum in a given expression when we have an index repeated once down and once up. It is given in the example as follows:

$$x_\mu x^\mu = \sum_{\mu=0}^{3} x_\mu x^\mu = x_0x^0 + x_1x^1 + x_2x^2 + x_3x^3, \tag{2.6}$$

see [8].

The principle of special relativity requires that the action should be invariant under the transformations of Poincaré group. The transformations in the Poincaré group are composed of translations in spacetime and Lorentz transformations. The action of the Poincaré group on spacetime coordinates is:

$$x^\mu \longrightarrow x'^\mu = \Lambda^{\mu}{}_{\nu}x^\nu + a^\mu, \tag{2.7}$$

where a^μ characterizes translations in spacetime, see [7, 9]. Therefore, each element of the Poincaré group is identified by a Lorentz transformation and a translation. The Poincaré group's transformations leave invariant the interval between two nearby points x^μ and $x^\mu + dx^\mu$. The interval is denoted ds^2 and is given by

$$ds^2 \quad = \quad \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.8}$$

where $\eta_{\mu\nu}$ is the metric in Minkowski spacetime. The differential form dx^μ, transforms as follows:

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu = \Lambda^\mu{}_\nu dx^\nu. \tag{2.9}$$

The relation above defines the transformation of a contravariant vector under the Lorentz transformations. The interval invariance is a consequence of the second postulate. Then, it is clear that two observers in two different inertial reference frames must measure the same distance, as the speed of light is constant so that:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2, \tag{2.10}$$

see [8].

The Minkowski metric is given by:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.11}$$

It can be used to lower or raise indices. The inverse metric $\eta^{\mu\nu}$ takes the same form as $\eta_{\mu\nu}$ so that

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho, \tag{2.12}$$

where δ^μ_ρ is the Kronecker delta function defined as

$$\delta^\rho_\mu = \begin{cases} 1 & if \quad \mu = \rho \\ 0 & if \quad \mu \neq \rho \end{cases} \tag{2.13}$$

More generally, instead of making use of the metric $\eta^{\mu\nu}$, we use $g^{\mu\nu}$. The metric $\eta^{\mu\nu}$ is used in flat spacetime (special relativity), while $g^{\mu\nu}$ (used in general relativity) generalizes the notion of metric in curved spacetime. It contains all the information about the geometry of the space. As an example, we can lower or raise indices using $g^{\mu\nu}$ as follows:

$$x^\mu = g^{\mu\nu} x_\nu \quad \text{and} \quad x_\mu = g_{\mu\nu} x^\nu. \tag{2.14}$$

The invariance condition of interval, $ds^2 = ds'^2$, requires that

$$g_{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu}, \tag{2.15}$$

which characterizes completely the Lorentz transformations. Likewise, the Lorentz transformation $\Lambda^\mu{}_\nu$ has an inverse $\Lambda_\mu{}^\nu$ such that

$$\Lambda_\mu{}^\nu = g_{\mu\rho} g^{\nu\sigma} \Lambda^\rho{}_\sigma. \tag{2.16}$$

It follows that
$$\Lambda^{\mu}{}_{\nu}\Lambda_{\mu}{}^{\sigma} = \Lambda^{\mu}{}_{\nu}g_{\mu\rho}g^{\sigma\tau}\Lambda^{\rho}{}_{\tau} = g_{\nu\tau}g^{\sigma\tau} = \delta^{\sigma}_{\nu}. \tag{2.17}$$
Like in (2.9), we can write the Lorentz transformation of the covariant partial derivative vector by
$$\partial'_{\mu} = \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}} = \Lambda^{\nu}{}_{\mu}\partial_{\nu} \tag{2.18}$$
In special relativity, and more generally, we denote vectors in spacetime by V^{μ}, and we call them, four-vectors. Consider ds^2. It is invariant under Lorentz transformations. The scalar (dot) product or square magnitude or simply, magnitude of an arbitrary four vector V^{μ} given by
$$V \cdot V = V_{\mu}V^{\mu} = V_0V^0 - V_1V^1 - V_2V^2 - V_3V^3 \tag{2.19}$$
is Lorentz invariant. The magnitude is important because it is frame-independent. It is convenient to mention some four-vectors which will be used in the next sections and chapters:

(i) The energy-momentum, which combines the energy and the momentum and is given by
$$p^{\mu} = (\frac{E}{c}, \mathbf{p}) = (\frac{E}{c}, p_1, p_2, p_3). \tag{2.20}$$
When we compute the magnitude of the energy-momentum four-vector, we get Einstein's famous relation which connects energy, momentum and mass given by
$$E^2 = p^2c^2 + m^2c^4. \tag{2.21}$$
It follows for a particle at rest that
$$E = mc^2. \tag{2.22}$$

(ii) The current four-vector, a conserved quantity given by
$$j^{\mu} = (\rho, j_1, j_1, j_1). \tag{2.23}$$
Its conservation can be written as
$$\partial_{\mu}j^{\mu} = 0, \tag{2.24}$$
which is what we know as continuity equation as shown by:
$$\partial_{\mu}j^{\mu} = \frac{\partial\rho}{\partial t} + \nabla.\mathbf{j} = 0. \tag{2.25}$$
One extends the notion of Laplacian from three dimensions to four dimensions and it is called the d'Alembertian operator as written here:
$$\partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box. \tag{2.26}$$

(iii) The stress tensor $T_{\mu\nu}$.

2.1.2 Dynamics of free particles

The simplest relativistic equation describing a free field is the Klein-Gordon equation. In the Schrödinger equation below, the operator $i\hbar\frac{\partial}{\partial t}$ and the hamiltonian H commute.The Schrödinger equation is not consistent with the principles of special relativity. Therefore, we refer to (2.21) to make use of the matching principle $E = -\hbar\partial_t$ and $\mathbf{p} = -i\hbar\nabla$. We use the fact that if two operators A and B commute, and in addition, if $A\psi = B\psi$, then $A^2\psi = B^2\psi$. Therefore, we can rewrite the Schrödinger equation as

$$\left[\frac{\partial^2}{c^2\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0. \tag{2.27}$$

When one introduces the d'Alembertian operator and takes $\hbar = c = 1$, it becomes simply

$$\left(\Box + m^2\right)\psi = 0. \tag{2.28}$$

This is called the Klein-Gordon equation. The complex conjugate field $\bar{\psi}$ also satisfies this Klein-Gordon equation

$$\left(\Box + m^2\right)\bar{\psi} = 0. \tag{2.29}$$

Now, when one multiplies the equations (2.28) and (2.29) respectively by $\bar{\psi}$ and ψ, one gets after combination:

$$\partial^\mu\left(\bar{\psi}\overleftrightarrow{\partial_\mu}\psi\right) = 0, \tag{2.30}$$

where we have introduced

$$\bar{\psi}\overleftrightarrow{\partial_\mu}\psi = \bar{\psi}\partial_\mu\psi - \psi\partial_\mu\bar{\psi}\ , \tag{2.31}$$

see [9, 10, 11]. We deduce the conservation equation of Klein-Gordon field when we multiply (2.30) by $\frac{i}{2m}$ as it follows:

$$\partial^t\left(\frac{i}{2m}\bar{\psi}\overleftrightarrow{\partial_t}\psi\right) + \nabla\left(\frac{1}{2mi}\bar{\psi}\overleftrightarrow{\nabla}\psi\right) = 0, \tag{2.32}$$

Comparing this equation to the conservation equation given in (2.25), we can identify the density

$$\rho = \frac{i}{2m}\bar{\psi}\overleftrightarrow{\partial_t}\psi, \tag{2.33}$$

and the current

$$\mathbf{j} = \frac{1}{2mi}\bar{\psi}\overleftrightarrow{\nabla}\psi. \tag{2.34}$$

In fact, we know that $\rho = \bar{\psi}\psi$ is the probability density for the Schrödinger equation. Since the equation (2.31) is not positive definite, it is predictable that the density ρ derived from the Klein-Gordon equation can take negative values. Therefore, it involves negative probabilities because the Klein-Gordon equation is a second order equation and ψ and $\partial_t\psi$ can be choosen arbitrarily at a given time so that ρ may take negative values. The Klein-Gordon equation does not take into account the particle's spin. But as it is expressing the relativistic relation between energy, momentum, and matter, it must hold for particles of any spin. Unfortunately, it holds only for a particle with zero spin, that is a scalar particle, which has only one component denoted ψ. That is why, Dirac postulated that the Schrödinger equation for a particle with high speed can be written as:

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\alpha\cdot\nabla\psi + \beta mc^2\psi, \tag{2.35}$$

where the numbers α_i (which denote the components of α) and β are not simple coefficients, see [8]. Rather, they are square $2n \times 2n$ matrices with complex values acting on a wave function with many components and will be treated as linear operators. Likewise, the fields ψ are not scalars. They are call Dirac spinors. In fact, a spinor is an element of a representation space of the spin group. They have the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{2.36}$$

where N is the dimension of the space that we consider.

The idea here is to write down the Dirac equation in the form of the Klein-Gordon equation. Therefore, taking the square of the operators in (2.35), we get:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \left[\sum_{i,j=1}^{3} \frac{\alpha_i \alpha_j + \alpha_j \alpha_i}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} \right] + \frac{\hbar m c^3}{i} \left[\sum_{i=1}^{3} (\alpha_i \beta + \beta \alpha_i) \frac{\partial \psi}{\partial x^i} \right] + \beta^2 m^2 c^4 \psi. \tag{2.37}$$

Identifying the above equation with the Klein-Gordon equation, we have the following conditions:

$$\begin{aligned} \frac{\alpha_i \alpha_j + \alpha_j \alpha_i}{2} &= \delta_{ij}, \\ \alpha_i \beta + \beta \alpha_i &= 0, \\ \beta^2 &= 1. \end{aligned} \tag{2.38}$$

If $i = j$, then $\alpha_i^2 = 1$. From the relations (2.38), α_i and β have the properties below:

1) They are hermitian because $(-i\hbar c\alpha \cdot \nabla + \beta m c^2)$ is the Hamiltonian of the system and should be Hermitian.

2) Their eigenvalues are ± 1 for their square is the identity matrix.

3) They are traceless and to prove this, we recall that $tr(AB) = tr(BA)$. Here, tr denotes the trace of a matrix and A and B are matrices. When we use $\beta^2 = 1$, we find that $\beta = \beta^{-1}$. It follows:

$$\begin{aligned} \alpha_i \beta + \beta \alpha_i = 0 &\implies \beta(\alpha_i \beta + \beta \alpha_i) = 0 \\ &\implies \beta \alpha_i \beta + \beta \beta \alpha_i) = 0 \\ &\implies \beta \alpha_i \beta = -\alpha_i \\ &\implies tr(\beta \alpha_i \beta) = -tr(\alpha_i) \\ &\implies tr(\alpha_i) = -tr(\alpha_i) \\ &\implies tr(\alpha_i) = 0. \end{aligned} \tag{2.39}$$

Similarly,

$$\begin{aligned} \alpha_i\alpha_i = 1 \quad &\Longrightarrow \quad \beta\alpha_i\alpha_i = \beta \\ &\Longrightarrow \quad -\alpha_i\beta\alpha_i = \beta \\ &\Longrightarrow \quad -tr(\alpha_i\beta\alpha_i) = tr(\beta) \\ &\Longrightarrow \quad tr(\beta) = -tr(\beta) \\ &\Longrightarrow \quad tr(\beta) = 0. \end{aligned} \tag{2.40}$$

4) They are $2n \times 2n$ dimensional matrices from properties 1 and 2.

There is no solution in 2 dimensions (for $n = 1$). This because, when we square respectively α_i and β, we get the identity matrix and in addition, α_i and β anticommute. Therefore, α_i and β are independent and cannot be the identity matrix. We know that there are only three independent Hermitian matrices in 2 dimensions which are different from the identity matrix and are known as Pauli's matrices. Finally, we can easily find a solution in 4 dimensions by using the fact that Pauli's matrices have similar properties so that:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{2.41}$$

and

$$\beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{2.42}$$

where σ_i are Pauli matrices and $\mathbb{I}_2$, the identity matrix. The Pauli matrices are given by

$$\sigma_a = \begin{pmatrix} \delta_{a3} & \delta_{a1} - i\delta_{a2} \\ \delta_{a1} + i\delta_{a2} & -\delta_{a3} \end{pmatrix}, \tag{2.43}$$

with $a = 1, 2, 3$ and $i^2 = -1$. We see that we are obliged to extend the space representing the states in the Schrödinger equation, since α and β are in 4 dimensions. It is not a space of functions with complex values. Rather, it is a space of functions of (at least) 4 complex components. We notice that it is possible to find many other matrices which are solutions of the (2.38). In fact, any set of four matrices constructed from the previous matrices by a transformation $M \longrightarrow UMU^\dagger$, where U is a unitary matrix (that is $U^\dagger U = UU^\dagger = 1$) and M a given matrix, satisfies the system of constraints in (2.38). All these solutions lead to an equivalent representation of the physics.

One introduces the γ^μ matrices to give the more symmetric form between the spatial and temporal coordinates so that $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$. It follows that the Dirac matrices given by:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad . \tag{2.44}$$

By multiplying the initial Dirac equation written in (2.35) to the constant $\frac{\gamma^0}{c}$, we have:

$$i\hbar \sum_{i=0}^{3} \gamma^i \frac{\partial \psi}{\partial x^i} - mc\psi = 0. \tag{2.45}$$

The Dirac matrices satisfy the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}. \tag{2.46}$$

When $\hbar = 1 = c$, we get the Dirac equation given by

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{2.47}$$

One defines the conjugate spinor $\bar{\psi} = \psi^\dagger \gamma^0$, called the adjoint spinor to ψ. Its Dirac equation is given by:

$$\bar{\psi}(i\gamma^\mu \partial_\mu + m) = 0. \tag{2.48}$$

Therefore, we can write down the conserved current, called Dirac current as

$$j^\mu = \bar{\psi}\gamma^\mu \psi, \tag{2.49}$$

where $j^0 = \rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$, is positive.

For convenience in Chapter 3, we shall review in more details and in different notations, the notions of spinors. In fact, the Dirac equation previously discussed can also be written simply as

$$\gamma^\mu P_\mu \Psi = m\Psi, \tag{2.50}$$

where γ^μ the Dirac matrices, P_μ the four-momentum operator, m, the mass and Ψ the Dirac spinor. There exists many type of representation for the Dirac γ^0 and $\overrightarrow{\gamma}$. We choose the representation

$$\gamma_0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix},$$

and

$$\overrightarrow{\gamma} = \begin{pmatrix} 0 & -\overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix}, \tag{2.51}$$

so that

$$\gamma^\mu = (\gamma^0, \overrightarrow{\gamma}). \tag{2.52}$$

By making use of the flat spacetime metric

$$\eta_{\mu\nu} = diag(1, -1, -1. -1), \tag{2.53}$$

the corresponding quantity to γ^μ (which is in a contravariant index) with a covariant index is given by

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\mu = (\gamma^0, -\overrightarrow{\gamma}). \tag{2.54}$$

Now define the 4×4 matrix

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.55}$$

It is defined in arbitrary basis by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5, \tag{2.56}$$

and is called chiral or Weyl representation.

We can see that the Dirac matrices are written in terms of the Pauli matrices. So, let us recall that the Pauli matrices are hermitian, that is

$$(\sigma^i)^\dagger = \sigma^i. \tag{2.57}$$

The product of two Pauli matrices is

$$\sigma^i\sigma^j = \mathbf{1}\delta^{ij} + i\varepsilon^{ijk}\sigma^k, \tag{2.58}$$

where δ^{ij} is the Kronecker delta, and ε^{ijk} is the Levi-Civita tensor. By using (2.58), one can introduce the commutation relation of two Pauli matrices as follows

$$\begin{aligned} [\sigma^i, \sigma^j] &= \sigma^i\sigma^j - \sigma^j\sigma^i \\ &= 2i\varepsilon^{ijk}\sigma^k \ . \end{aligned} \tag{2.59}$$

Similarly, the anticommutation relation is given by

$$\begin{aligned} \{\sigma^i, \sigma^j\} &= \sigma^i\sigma^j + \sigma^j\sigma^i \\ &= 2i\delta^{ij}\mathbf{1} \ . \end{aligned} \tag{2.60}$$

By using the Levi-Civita tensor, the cross-product of two vector $\overrightarrow{U}$ and $\overrightarrow{V}$ on the kth component can be written by

$$\begin{aligned} (\overrightarrow{U} \times \overrightarrow{V})^k &= \varepsilon^{ijk}U^iV^j \\ &= \varepsilon^{kij}U^iV^j \ . \end{aligned} \tag{2.61}$$

Thereby, we can see that given a vector $\overrightarrow{V}$, which has numbers as components, we can calculate an expression of the form $\overrightarrow{V} \cdot \overrightarrow{\sigma}\sigma^j$ which will be useful in computation in the next chapter. The computation gives:

$$\begin{aligned} \overrightarrow{V} \cdot \overrightarrow{\sigma}\sigma^j &= V^i\sigma^i\sigma^j \\ &= V^i\sigma^j\sigma^i + V^i[\sigma^i, \sigma^j] \\ &= V^i\sigma^j\sigma^i + 2i\varepsilon^{ijk}V^i\sigma^k \\ &= V^i\sigma^j\sigma^i + 2i\varepsilon^{kij}V^i\sigma^k \\ &= \sigma^j\overrightarrow{V} \cdot \overrightarrow{\sigma} + 2i(\overrightarrow{\sigma} \times \overrightarrow{V})^j. \end{aligned} \tag{2.62}$$

A four-component Dirac spinor describes Weyl spinor as follows:

$$\Psi = \begin{pmatrix} \eta \\ \chi \end{pmatrix}, \tag{2.63}$$

where η and χ are two-component spinor. One remarks that by setting either η or χ equal to zero in Dirac spinor, yields eigenstates of γ_5, that is

$$\gamma_5 \begin{pmatrix} \eta \\ 0 \end{pmatrix} = + \begin{pmatrix} \eta \\ 0 \end{pmatrix}, \tag{2.64}$$

$$\gamma_5 \begin{pmatrix} 0 \\ \chi \end{pmatrix} = - \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{2.65}$$

A Weyl spinor with positive chirality is called a right chiral spinor, while a Weyl spinor with negative chirality is called a left chiral spinor. Consider a Dirac spinor Ψ. One defines the right and left chiral operator by

$$P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.66}$$

$$P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.67}$$

so that

$$P_R\Psi = P_R \begin{pmatrix} \eta_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} = \Psi_R, \tag{2.68}$$

$$P_L\Psi = P_L \begin{pmatrix} \eta_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} = \Psi_L, \tag{2.69}$$

where Ψ_R and Ψ_L are respectively the right and the left chiral spinors. The two operators satisfy the following relations

$$\begin{aligned} P_R^2 &= P_R \\ P_L^2 &= P_L \\ P_L P_R &= P_R P_L = 0 \\ \gamma_5 P_R &= P_R \gamma_5 = P_R \\ \gamma_5 P_L &= P_L \gamma_5 = -P_L. \end{aligned} \tag{2.70}$$

By substituting (2.63) in the Dirac equation, one gets

$$(E - \overrightarrow{\sigma}.\overrightarrow{p})\eta_R = m\chi_L, \tag{2.71}$$

$$(E - \overrightarrow{\sigma}.\overrightarrow{p})\chi_L = m\eta_R, \tag{2.72}$$

where $P^\mu = (E, \overrightarrow{p})$. Consider the Pauli matrices σ^μ, written on the form

$$\sigma^\mu = (1, \overrightarrow{\sigma}), \tag{2.73}$$

we have

$$\sigma_\mu \equiv \bar{\sigma}^\mu = (1, -\overrightarrow{\sigma}). \tag{2.74}$$

We also have the following relations:

$$\begin{aligned} \sigma^2(\sigma^\mu)^T\sigma^2 &= \bar{\sigma}^\mu, \\ \sigma^2(\bar{\sigma}^\mu)^T\sigma^2 &= \sigma^\mu. \end{aligned} \tag{2.75}$$

The transposed of (2.75) is

$$\begin{aligned} \sigma^2(\sigma^\mu)\sigma^2 &= \bar{\sigma}^{\mu T}, \\ \sigma^2(\bar{\sigma}^\mu)\sigma^2 &= \sigma^{\mu T}, \end{aligned} \tag{2.76}$$

where $(\sigma^2)^T = -\sigma^2$. The relation (2.75) and (2.76) are also valid when one replaces $(\sigma^i)^T$ by σ^{i*}. Hence we have the following properties

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = \eta^{\mu\nu}\mathbf{1}, \tag{2.77}$$

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = \eta^{\mu\nu}\mathbf{1}, \tag{2.78}$$

from which, we deduce

$$tr(\sigma^\mu\sigma^\nu) = 2\eta^{\mu\nu}. \tag{2.79}$$

We also have the Majorana spinor Ψ_M which is given in terms of a single two-dimensional spinor as

$$\Psi_M = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}, \tag{2.80}$$

and we notice that in Chapter 3, all these relations will be used, while the spinors will carry indices.

2.1.3 Equation of motion in quantum field theory

The description of the interaction of elementary particles can be formulated from the principle of least action, which is a simple generalization of the situation in classical mechanics. In classical mechanics, the equations of motion of a system of a point particles are obtained from a functional action which takes functions as argument and returns a number. The functional action is defined by:

$$S[q] = \int_{t_i}^{t_f} dt L(q, \dot{q}, t), \tag{2.81}$$

where L is the Lagrangian describing the system, $q = q(t) = \{q_1(t), \cdots, q_{3N}(t)\}$ are the coordinates of N particles system in three dimensions at a time t and $\dot{q} = \dot{q}(t) = \{\dot{q}_1(t), \cdots, \dot{q}_{3N}(t)\}$, see [7]. The dot denotes the time derivative. When we make use of the variational principle, under the transformation $q_i \longrightarrow q_i + \delta q_i$, it follows:

$$\begin{aligned} \delta S[q] &= \int_{t_i}^{t_f} dt \sum_{i=1}^{3N} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \\ &= \int_{t_i}^{t_f} dt \sum_{i=1}^{3N} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) \\ &= \int_{t_i}^{t_f} dt \sum_{i=1}^{3N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i, \end{aligned} \tag{2.82}$$

where, on the second line, we apply integration by parts and the second term vanishes because $\delta q_i(t_i) = \delta q_i(t_f) = 0$ as the two end points of the trajectory are fixed. Since δq is arbitrary for $t_i < t < t_f$, the stationarity condition, that is, $\delta S = 0$, implies

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \cdots, 3N. \tag{2.83}$$

This is called the Euler-Lagrange equation describing the equations of motion for a discrete system. Likewise, in quantum field theory, the Lagrangian formalism can be extended to describe the dynamics of fields $\varphi(\mathbf{x}, t) = \varphi(t)$. In this case, instead of using the Lagrangian in the continuous case, we will make use of a Lagrangian density $\mathcal{L}(\varphi, \partial_\mu \varphi)$. The Lagrangian density is related to the Lagrangian by the relation

$$L = \int d^3x \mathcal{L}(\varphi, \partial \varphi). \tag{2.84}$$

Therefore, the quantum field theory action can be written as

$$S(\varphi) = \int d^4x \mathcal{L}(\varphi, \partial \varphi). \tag{2.85}$$

We recall that the variational problem involves finding a function that optimizes this action. The condition that guarantees that $\delta S = 0$ is the Euler-Lagrange equation given by

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} = 0. \tag{2.86}$$

We provide a generalization in the case that the action depends on many fields $\varphi^i(\mathbf{x})$, $i = 1, \cdots, n$. The stationarity condition of the action leads to the following Euler-Lagrange equation

$$\frac{\partial \mathcal{L}(\varphi^j, \partial_\mu \varphi^j)}{\partial \varphi^i} - \partial_\mu \frac{\partial \mathcal{L}(\varphi^j, \partial_\mu \varphi^j)}{\partial \partial_\mu \varphi^i} = 0. \tag{2.87}$$

It is worth mentioning that the Lagrangian formulation is important and useful for us, because we often impose symmetry on the Lagrangian. Therefore, if we want equation of motion with a particular symmetry, we can start from a Lagrangian which enjoys that symmetry. The solution may not enjoy the same symmetries, however.

2.2 Overview of Electromagnetism

In this section, we review electromagnetism and the necessary details on the relativistic formulation of electrodynamics. We start by presenting the Maxwell equations in Subsection 2.2.1. In Subsection 2.2.2, we derive the potentials A and ϕ from which we write down the gauge transformations. These transformations help us to reduce the four Maxwell equations to two and finally write down the equations of wave propagation.

2.2.1 Maxwell equations

Maxwell's equations are:

$$\begin{cases} \nabla \cdot \mathbf{B} &= & 0\,, & \text{(a)} \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= & 0\,, & \text{(b)} \\ \nabla \cdot \mathbf{E} &= & 4\pi\rho\,, & \text{(c)} \\ \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= & \frac{4\pi}{c}\mathbf{j}\,, & \text{(d)} \end{cases} \tag{2.88}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, c is the speed of light and ρ and $\mathbf{j}$ are respectively the charge and current densities. The first homogeneous equation (a) says that there is no magnetic charge. The second homogeneous equation (b) is the induction law of Faraday. The two inhomogeneous equations are respectively Coulomb's law and Ampère's law.

2.2.2 Equations of wave propagation

We can obtain the continuity equation, when we combine the two inhomogeneous equations. For this purpose, by taking the divergence of (d), acting the operator $\frac{1}{c}\frac{\partial}{\partial t}$ on (c), and thereafter, combining them, it gives the continuity equation.

Now, when we get back to the homogeneous equations, we can deduce from (2.88), (a) that there exists a magnetic vector potential $\mathbf{A}$ for which

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{2.89}$$

It is obvious to see from vector analysis that

$$\nabla \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0, \tag{2.90}$$

for any vector potential $\mathbf{A}$. The Faraday's equation, when we replace $\mathbf{B}$ as defined in (2.89), becomes:

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{2.91}$$

From the vector analysis, (2.91) implies that there exists some scalar function ϕ (the electric potential) with

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{2.92}$$

Maxwell's equations consist of a set of four equations and contain the two variables $\mathbf{E}$ and $\mathbf{B}$ which we wrote in terms of ϕ and $\mathbf{A}$ from the two homogeneous equations. It is convenient to rewrite the Maxwell equations:

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -4\pi\rho \tag{2.93}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - (\nabla\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}) = -\frac{4\pi}{c}\mathbf{j}. \tag{2.94}$$

At this stage, we can see clearly that we created two objects called respectively scalar potential and vector potential. We made use of them and we reduced the initial four equations into two as it appears in (2.93) and (2.94). However, it is obvious to see that these equations are still coupled.

The potentials ϕ and $\mathbf{A}$ are not fixed only by the conditions given in (2.89) and (2.92). Since we have learnt from vector analysis that for any scalar function f, $\boldsymbol{\nabla} \times (\nabla f) = 0$, then, if we transform $\mathbf{A}$ by

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\chi, \tag{2.95}$$

where χ is a scalar function, the magnetic field $\mathbf{B}$ remains unchanged. If we alter again ϕ by

$$\phi \longrightarrow \phi - \frac{1}{c}\frac{\partial\chi}{\partial t}, \tag{2.96}$$

the electric field $\mathbf{E}$ is also invariant. There is a redundant description of the physical system since the fields remain unchanged.

The transformations given in (2.95) and (2.96) are called gauge transformations, where the scalar function $\chi(t, x)$ is completely arbitrary and is called a gauge function. The invariance of the fields under the gauge transformation is called gauge invariance. Under the gauge transformations,

$$\begin{aligned} \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \quad &\longrightarrow \quad \nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} + \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \\ &= \quad \left(\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\chi + \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}. \end{aligned} \tag{2.97}$$

Thus, if we find a scalar function χ for which

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\chi = -\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \right) \tag{2.98}$$

the transformed potentials,

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \quad \text{and} \quad \phi' = \phi - \frac{1}{c}\frac{\partial\chi}{\partial t}, \tag{2.99}$$

satisfy

$$\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial\phi'}{\partial t} = 0, \tag{2.100}$$

which is called the Lorentz gauge condition. Under the Lorentz gauge condition (here, we omit the prime) given in (2.100), Maxwell's equations have a particularly simple form because they are completely uncoupled:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \tag{2.101}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}. \tag{2.102}$$

or simply

$$\Box\phi = 4\pi\rho, \tag{2.103}$$

$$\Box\mathbf{A} = \frac{4\pi}{c}\mathbf{j}. \tag{2.104}$$

These are called the d'Alembert's equations or equations of wave propagation. The equations (2.101) and (2.102) together with the Lorentz gauge condition are equivalent to the Maxwell's equations. The four equations of first order have been transformed into two equations of second order. One can use the Lorentz gauge condition and see that (2.98) becomes

$$\left(\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\chi = 0. \tag{2.105}$$

It is worth mentioning that the d'Alembert's equations tell us on their left-hand sides about quantities related to "radiation" while on the right-hand sides, it is a question of "matter", as sources of the equations of motion.

In electrostatics, $\dot{\mathbf{E}} = \dot{\mathbf{B}} = 0$, where the dot denotes the time derivative. The Faraday's equation implies that $\boldsymbol{\nabla} \times \mathbf{E} = 0$. Therefore, $\mathbf{E} = -\nabla\phi$ so that

$$\nabla^2\phi = -4\pi\rho, \tag{2.106}$$

which is called the Poisson equation. The solution of this equation which decreases for $x \longrightarrow \infty$ if the source is contained within a finite volume is given by:

$$\phi(x) = \int d^3x' \frac{\rho(x')}{|x - x'|}, \tag{2.107}$$

see [12]. In magnetostatics, we can choose $\mathbf{A}$ so that

$$\nabla \cdot \mathbf{A} = 0. \tag{2.108}$$

This choice is called the Coulomb gauge or radiation or transverse gauge. For $\dot{\mathbf{E}} = 0$, we obtain the Ampère law

$$-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \nabla(\boldsymbol{\nabla} \times \mathbf{A}) - \nabla^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j}. \tag{2.109}$$

This is simply

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{j}, \tag{2.110}$$

due to the first Maxwell equation which says that there is no magnetic charge, that is $\nabla \cdot B = \nabla(\boldsymbol{\nabla} \times A) = 0$. The solution of this equation which decreases for $x \longrightarrow \infty$ if the source is contained within a finite volume is given by:

$$\mathbf{A}(x) = \frac{1}{c}\int d^3x' \frac{\mathbf{j}(x')}{|x - x'|}, \tag{2.111}$$

see [12] The equations (2.112) and (2.113) determine the electro and magnetostatics potentials and then the fields **E** and **B** for a static distribution for given currents and charges.

Generally, (2.98) and (2.100) are solved by using the Green function G.The solutions are given by:

$$\phi(x,t) = \int d^3x' dt G(x - x', t - t')\rho(x', t') = \int d^3x' \frac{\rho\left(x', t - \frac{|x-x'|}{c}\right)}{|x - x'|}, \tag{2.112}$$

and

$$\mathbf{A}(x) = \int d^3x' dt G(x - x', t - t')\mathbf{j}(x', t') = \frac{1}{c}\int d^3x' \frac{\mathbf{j}\left(x', t - \frac{|x-x'|}{c}\right)}{|x - x'|}. \tag{2.113}$$

2.3 Lorentz Covariance of Maxwell's Equations

We aim to write down in this section, the compact form of the Maxwell's equations. For this purpose, we shall review the gauge symmetry. We will make use of $A(x) = A^\mu = (\phi, \mathbf{A})$, where $\mu = 0, \cdots, 3$. We recall that the electric and magnetic fields are not part of A^μ. Another four-vector which we will make use of is $j(x) = j^\mu = (c\rho, \mathbf{j})$. From the definition of A^μ, we can rewrite the Lorentz condition that we gave in (2.100), by

$$\partial_\mu A^\mu = \partial_0 A^0 + \nabla \cdot \mathbf{A} = 0. \tag{2.114}$$

similarly, the combination of the two gauge transformations given in equations (2.95) and (2.96) gives in terms of A^μ

$$A^\mu(x) \longrightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu \chi. \tag{2.115}$$

The equations (2.101) and (2.102) are combined as

$$\Box A^\mu = \frac{4\pi}{c} j^\mu. \tag{2.116}$$

We end this section by writing the Maxwell equations in terms of the field strength $F_{\mu\nu}$.

2.3.1 Symmetries

There exists two kinds of symmetries: a local symmetry where the symmetry group depends on spacetime, and the global symmetry which does not depend on spacetime. A global symmetry applies at all points in spacetime. A local symmetry varies from point to point; it is a function of the position in spacetime. By gauge transformations, we are referring to local, not global, symmetries. QED, Quantum Chromodynamics (QCD) and the electroweak theory are some examples. Sometimes, those theories can be applied to theories that are invariant under transformations, that depend on several parameters which may or may not commute. When the parameters do not commute, we say that we have a non-abelian gauge theory.

1. Gauge symmetries

A gauge transformation arises from local symmetries. Using a gauge transformation, one sees that the electric and magnetic fields remain unchanged. In this way, one has not changed the state of the system. Therefore it is not a symmetry. Rather, it is a redundant description of a physical system. The group of transformations leading to this redundancy can be abelian or non-abelian. There are ways of changing the field that is not physical. Thus local gauge symmetry is needed for physics to make sense.

Gauges theories are in the description of electromagnetism, the weak interaction, and the strong interaction. Physically, QED is explained by a $U(1)$ abelian gauge theory. The Yang-Mills theory has appeared to generalize the abelian $U(1)$ gauge theory to the non-abelian gauge theory case. The gauge transformation leaves the action and the classical equations of motion invariant. Therefore, it is clear that symmetries leave the solutions to the equations of motion unaffected. The existence of self interaction in the non-abelian case shows one of the major differences between abelian and non-abelian gauge theories.

2. Global symmetries

A symmetry is defined to be any transformation that leaves the action S unchanged. Consider global symmetries. Noether's theorem states that for any continuous symmetry of a physical system, there is an associated conserved current, and so there is a corresponding conserved charge. Since charges generate symmetries and lead to quantization, Noether's theorem becomes relevant. To see how the Noether's theorem manifests, we consider an infinitesimal transformation about a classical scalar:

$$\phi \longrightarrow \phi' = \phi + \delta\phi, \tag{2.117}$$

so that

$$\partial_\mu\phi \longrightarrow \partial_\mu\phi' = \partial_\mu\phi + \delta\partial_\mu\phi, \tag{2.118}$$

see [9]. Taking the derivative of (2.117) and comparing it to (2.118), it is clear that

$$\delta\partial_\mu\phi = \partial_\mu\delta\phi. \tag{2.119}$$

We now consider the action given by

$$\mathcal{S} = \int d^4x\mathcal{L}. \tag{2.120}$$

Considering $\delta\mathcal{L} = \partial^\mu F_\mu$, we show by using the Hamiltonian principle that $\delta\mathcal{S}$ vanishes and

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \\
&= \partial_\mu F^\mu.
\end{aligned} \tag{2.121}
$$

Therefore, we can write

$$
\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\ - F^\mu\right) &= 0 \\
\partial_\mu j_n^\mu &= 0,
\end{aligned} \tag{2.122}
$$

where $j_n^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu$ is the Noether's current. This equation is the local conservation law and can be written in the following way,

$$
\frac{\partial j_n^0}{\partial t} + \nabla\cdot\mathbf{j}_n = 0. \tag{2.123}
$$

Evaluating the integral of (2.123), we derive the conserved charge given by

$$
Q = \int d^3x j_n^0. \tag{2.124}
$$

We see that the Noether theorem implies the existence of a conserved current for a continuous symmetry of the action. Noether's theorem is also applied for symmetries acting on spacetime, and then to the invariance under Poincaré's transformations. The conserved current in this case is different from the one we derived previously for a continuous symmetry. We are now about to construct the conserved current associated to the invariance under translations. These currents play an important role since they express the conservations of energy and momentum. They also provide the Hamiltonian of a system. In fact, when we consider translations in spacetime given by the transformation

$$
x^\mu \longrightarrow x^\mu + a^\mu, \tag{2.125}
$$

with a^μ, a small arbitrary change in spacetime, the field can change as

$$
\phi(x^\mu) \longrightarrow \phi(x^\mu + a^\mu) = \phi(x^\mu) + a^\mu\partial_\mu\phi(x^\mu), \tag{2.126}
$$

where we used the Taylor expansion. However, under a small variation, a change in field given in (2.117) comparing to (2.126) allows us to write the change in the field explicitly by

$$
\delta\phi = a^\mu\partial_\mu\phi. \tag{2.127}
$$

Considering the variation of the Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$, in the classical field theory, we have

$$
\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi. \tag{2.128}
$$

Now, when we make use of the Euler-Lagrange equation derived in (2.86), the equation (2.128) becomes:

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}a^\nu\partial_\nu\phi\right). \end{aligned} \tag{2.129}$$

Like in (2.127), a small variation in Lagrangian can be written as

$$\delta\mathcal{L} = a^\mu\partial_\mu\mathcal{L} = \partial_\mu\left(a^\mu\mathcal{L}\right). \tag{2.130}$$

Comparing (2.129) to (2.130), we have

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}a^\nu\partial_\nu\phi\right) = \partial_\mu\left(a^\mu\mathcal{L}\right). \tag{2.131}$$

We can end the computation by moving both the terms to the same hand side of the above equation and it follows:

$$\begin{aligned} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}a^\nu\partial_\nu\phi\right) - \partial_\mu\left(a^\mu\mathcal{L}\right) &= 0 \\ \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}a^\nu\partial_\nu\phi\right) - \partial_\mu\left(\delta^\mu_\nu a^\nu\mathcal{L}\right) &= 0 \\ a^\nu\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}\right) &= 0 \\ a^\nu\partial_\mu T^\mu_\nu &= 0. \end{aligned} \tag{2.132}$$

The quantity T^μ_ν is called the energy-momentum tensor. It is convenient to mention that for $\mu = \nu = 0$, T^0_0 corresponds to the Hamiltonian density $\mathcal{H}$.

The current density and the energy momentum tensor are related as we can see as follows:

$$j^\mu = a^\nu T^\mu_\nu. \tag{2.133}$$

Therefore

$$j^0 = a^0 T^0_0 = a\mathcal{H}. \tag{2.134}$$

We can integrate the quantity j^0 as follows:

$$\int dx^3 j^0 = aH \equiv Q, \tag{2.135}$$

where H is the Hamiltonian and Q the conserved charge. Although the conserved charge is unique, the conserved current is not. However, it is worth to mention that the conserved charge generates the transformation. To see how the conserved charge generates the transformation, we consider an element of Lie group U defined as

$$U = e^{iQ}. \tag{2.136}$$

The change in field is given by

$$\phi \longrightarrow U\phi U^{\dagger}. \tag{2.137}$$

Now we compute $U\phi U^{\dagger}$ as follows:

$$\begin{aligned}
U\phi U^{\dagger} &= e^{iQ}\phi e^{-iQ} \\
&= e^{iaH}\phi e^{-iaH} \\
&= (1+iaH)\phi(1-iaH) \\
&= \phi + ia[H,\phi] \\
&= \phi + ia\left(-i\frac{\partial\phi}{\partial t}\right) \\
&= \phi + a\frac{\partial\phi}{\partial t} \\
&= \phi + \delta\phi.
\end{aligned} \tag{2.138}$$

Therefore, it is clear that

$$\text{Lie group element} = e^{i(\text{generator of Lie algebra})}. \tag{2.139}$$

2.3.2 Compact form of Maxwell equations

The theory of electrodynamics rests on Maxwell tensor $F^{\mu\nu}$. The tensor is antisymmetric and covariant and is defined in terms of the four-potential A^{μ} by

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}. \tag{2.140}$$

For two states that are related by a gauge symmetry to be identified, they should have the same physical state. This because of the insufficiency of Maxwell's equations to describe the evolution of A^{μ}. We shall formulate the theory purely in terms of the electric and magnetic fields. Therefore, one creates the electromagnetic field strength $F^{\mu\nu}$ [9, 13]. The invariance of $F^{\mu\nu}$ under the gauge transformation given in (2.115) follows:

$$\begin{aligned}
F^{\mu\nu} \longrightarrow F'^{\mu\nu} &= \partial^{\mu}A'^{\nu} - \partial^{\nu}A'^{\mu} \\
&= \partial^{\mu}(A^{\nu} + \partial^{\nu}\chi) - \partial^{\nu}(A^{\mu} + \partial^{\mu}\chi) \\
&= F^{\mu\nu} + \partial^{\mu}\partial^{\nu}\chi - \partial^{\nu}\partial^{\mu}\chi \\
&= F^{\mu\nu},
\end{aligned} \tag{2.141}$$

where we used in the last step the fact that partial derivatives commute. Under a Lorentz transformation Λ, $F^{\mu\nu}$ is transformed as

$$F^{\mu\nu} \longrightarrow F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}. \tag{2.142}$$

Now, we could write the electric and magnetic fields in terms of the four-potential $A^{\mu} = (\phi, \mathbf{A})$. When we consider (2.92), the electric field is given as

$$\begin{aligned}
E^{i} &= -\frac{1}{c}\frac{\partial A^{i}}{\partial t} - \nabla\phi \\
&= -\partial_0 A^{i} - \partial_i A^{0} \\
&= -\partial^0 A^{i} + \partial^{i} A^{0} \\
&= \partial^{i}A^{0} - \partial^{0}A^{i} \\
&= F^{i0}
\end{aligned} \tag{2.143}$$

where, we use $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ such that $\partial^0 = \partial_0$ and $\partial^i = -\partial_i$. Similarly, for the magnetic field, we refer to (2.89) and we have

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= \mathbf{e}_1(\partial_2 A^3 - \partial_3 A^2) + \mathbf{e}_2(\partial_3 A^1 - \partial_1 A^3) + \mathbf{e}_3(\partial_1 A^2 - \partial_2 A^1) \\
&= \mathbf{e}_1 B^1 + \mathbf{e}_2 B^2 + \mathbf{e}_3 B^3.
\end{aligned} \tag{2.144}$$

A quick calculation shows that

$$\begin{aligned}
B^1 &= \partial^3 A^2 - \partial^2 A^3 = F^{32}, \\
B^2 &= \partial^1 A^3 - \partial^3 A^1 = F^{13}, \\
B^3 &= \partial^2 A^1 - \partial^1 A^2 = F^{21}.
\end{aligned} \tag{2.145}$$

Using the totally antisymmetric Levi-Civita tensor $\epsilon^{ijk} = \epsilon_{ijk}$, we can write

$$\begin{aligned}
F^{ij} &= \partial^i A^j - \partial^j A^i \\
&= -\epsilon^{ijk} B^k.
\end{aligned} \tag{2.146}$$

Finally, we can write down the Maxwell or electromagnetic field strength by combining (2.143) and (2.146), and it follows:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{2.147}$$

To summarize, we consider the inhomogeneous equations. It follows:

$$\begin{aligned}
\nabla \cdot \mathbf{E} = 4\pi\rho \Longrightarrow \partial_i E^i &= \frac{4\pi}{c} j^0, \\
\partial_i F^{i0} &= \frac{4\pi}{c} j^0.
\end{aligned}$$

Given $F^{\mu\nu}$ is antisymmetric, that is $F^{ii} = 0$, we have

$$\partial_\mu F^{\mu 0} = \frac{4\pi}{c} j^0. \tag{2.148}$$

Similarly,

$$\begin{aligned}
\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{j} \\
(\nabla \times B)^i - \frac{1}{c}\frac{\partial E^i}{\partial t} &= \frac{4\pi}{c} j^i \\
\partial_j \left(\epsilon^{ij}_k B^k\right) - \frac{1}{c}\frac{\partial E^i}{\partial t} &= \frac{4\pi}{c} j^i \\
\partial_j F^{ji} + \partial_0 F^{0i} &= \frac{4\pi}{c} j^i,
\end{aligned} \tag{2.149}$$

which in more general form is written as

$$\partial_i F^{\mu i} = \frac{4\pi}{c} j^\mu. \tag{2.150}$$

Finally, (2.151) and (2.155) combined, give

$$\partial_\nu F^{\mu\nu} \;=\; \frac{4\pi}{c} j^\mu. \tag{2.151}$$

The existence of $\mathbf{B}$ may thus be a pure consequence of the electrodynamics covariance; this is a clear example which shows the power of symmetries in physics. The requirement for which equations of motion should be the same in all reference frame, that is they should be covariant, brings us from electrodynamics to dynamical Maxwell's equations and implies the existence of $\mathbf{B}$.

To generalize the two homogeneous equations, we consider

$$(\boldsymbol{\nabla} \times E)^i + \frac{\partial B^i}{\partial t} \;=\; 0. \tag{2.152}$$

Now, we can expand the above equation as it follows:

$$\begin{cases} \partial_0 B^1 - \partial_2 F^{03} + \partial_3 F^{02} &= 0, \\ \partial_0 B^2 + \partial_1 F^{03} - \partial_3 F^{01} &= 0, \\ \partial_0 B^3 - \partial_1 F^{02} + \partial_2 F^{01} &= 0. \end{cases} \tag{2.153}$$

When we replace B^i as defined in (2.145), a quick calculation gives

$$\begin{cases} \partial_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20} &= 0, \\ \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} &= 0, \\ \partial_0 F_{21} + \partial_1 F_{02} + \partial_2 F_{10} &= 0. \end{cases} \tag{2.154}$$

We can finally write, from (2.154), the following equation:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \tag{2.155}$$

In conclusion, the Maxwell's equations are consistent with special relativity and their covariant forms are given explicitly in (2.151) and (2.155). When we observe (2.151), we can see that it is the Gauss's law written in terms of the field strength.

2.4 The Aharonov-Bohm and Electromagnetism Memory Effects

In this section, we review what are the Aharonov-Bohm and electromagnetism memory effects. In Subsection 2.4.1, we follow [14] and explain how the Aharonov-Bohm effect is considered as a quantum effect. In Subsection 2.4.2, we shall introduce the memory effect and prove that it can be expressed as a local conservation law and also prove by using gauge transformations, that the relative phase associated can be eliminated.

2.4.1 The Aharonov-Bohm effect

Aharonov and Bohm, in 1959, published a paper titled "Significance of Electromagnetic Potentials in the Quantum Theory". Through their work, the electromagnetic potential ϕ and $\mathbf{A}$ were predicted to be real physical quantities. This could be shown according to them through the electron interference experiment, [14]. In quantum field theory, mathematical objects representing fields are described as objects deriving from a potential. It follows that a field derives from a potential. Aharonov and Bohm proposed an experiment in which a particle moving in the absence of electromagnetic field could be influenced by a potential from which derive the corresponding electric and magnetic fields. The experimental setting suggested is made of a solenoid as shown below:

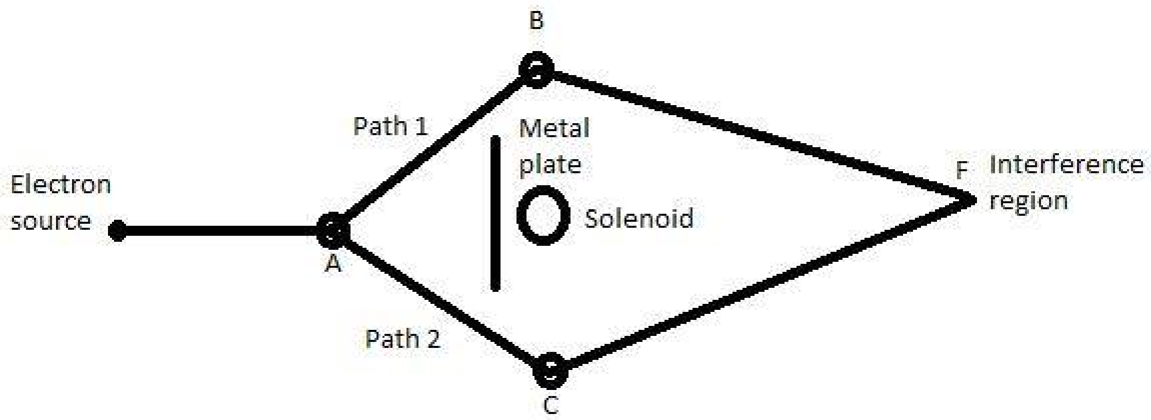

Figure 2.1: Schematic experiment showing the Aharonov-Bohm effect. A, B, C are suitable devices to separate and divert beams. F is the interference region.

The electrons are emitted by a source and they are split coherently at A. The solenoid considered is assumed infinite so that the magnetic field outside the solenoid can be negligible. The magnetic field inside the solenoid is strong but a metal plate is placed at the end of the solenoid facing an incoming electron beam to prevent electrons of entering the solenoid. The fact that the magnetic field is zero outside and is confined inside implies that the vector potential cannot be negligible everywhere. This shows that the fields are not sufficient to describe what is happening. Therefore, the corresponding potential can also affect the passage of the electron.

The effect is merely a quantum mechanics's phenomenon described by the Hamiltonian formalism. The first experiment to demonstrate the Aharonov-Bohm's statement was focused only on the magnetic Aharonov-Bohm effect and were performed in 1960 by Chambers, [15].The effect is observed when an electromagnetic field event influence a particle passing through a region of space while in this region, the electric field and magnetic field were zero, see [16]. And it has been shown in the beam experiment that if one makes a change in the magnetic field in the region made of solenoid, a change in phase difference between beams is observed. This phase shift is the so-called the Aharonov-Bohm effect and is similar to the memory effect described by Susskind, see [4].

As the effect appears as a quantum effect, we shall consider the Hamiltonian for charged particle in electromagnetic field given by

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}(x))^2 + e\phi(x) + V(x), \tag{2.156}$$

where $V(x)$ is a possible non-electric potential, see [17]. We shall recall that the momentum operator is given by

$$\mathbf{p} = -i\hbar\nabla. \tag{2.157}$$

By replacing (2.157) in (2.156) and by putting the Hamiltonian in the Schrödinger equation, one gets

$$\frac{1}{2m}[(i\hbar\nabla - e\mathbf{A}(x))^2 + e\phi(x) + V(x)]\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{2.158}$$

The above equation describes the dynamics of charged particle in electromagnetic field. We can now perform a gauge transformation of the potentials as we defined them previously and see what happens to the general Schrödinger equation in (2.158). A wave function ψ which solves (2.158) has the form

$$\psi(x) = e^{i\frac{S(x)}{\hbar}}\psi', \tag{2.159}$$

where

$$S(x) = \frac{e}{\hbar}\int A(x)dx, \tag{2.160}$$

see [?]. Plugging (2.159) into (2.158) gives after we cancel out the factor $e^{i\frac{S(x)}{\hbar}}$:

$$-\frac{\hbar^2}{2m}\nabla^2\psi' + V(x)\psi' = i\hbar\frac{\partial\psi'}{\partial t}, \tag{2.161}$$

which shows that ψ' is solution of the Schrödinger equation given in (2.158) in absence of the potential vector. So, the solution of (2.161) in presence of vector field is the same wave function multiplied by a phase factor $e^{i\frac{S(x)}{\hbar}}$.

2.4.2 The memory effect

The memory effect exists in electromagnetism and has been recently studied by Susskind [4]. The effect is observable after charged particles have been ejected from a large sphere which surrounds an explosion. Before the explosion, there is no charge density, no current density, no vector potential and no electromagnetic field in the sphere. The explosion here, is the divergence of an ensemble of equally many positive and negative charges. Thereafter, a gauge field A has formed in the sphere after all the charged particles have passed through it. This gauge field has the form $A = \nabla\lambda$ and can be eliminated by a gauge transformation. We therefore have an electromagnetic memory effect as a relative phase. The equation of motion in this process is just the Schrödinger equation which remains unchanged under the gauge transformation, leading to the memory effect. The memory effect can be expressed as a local (instantaneous) conservation law.

To see how the memory effect is expressed as a local conservation law, let us consider an explosion which ejects charged particles from a large sphere Ω of radius r centered on the charge Q. We recall that the left hand side of Gauss's law expresses a divergence, therefore all the particles must pass through the sphere. It has been assumed that particles are moving lightlike along the radial component due to the fact that the explosion is near the center of the sphere. The charge density, current density and electromagnetic field were zero before the explosion. We know that Gauss's law is valid in all dimension and is true everywhere at all times and has the form:

$$\nabla\cdot\mathbf{E} = \rho. \tag{2.162}$$

In the temporal gauge, that is $\nabla\phi = 0$, we have $\mathbf{E} = -\dot{\mathbf{A}}$ from (2.92). The equation (2.162) becomes

$$\nabla\cdot\dot{\mathbf{A}} = -\rho. \tag{2.163}$$

We use $c\rho$ with $c = 1$ as the radial component of the current noted j_r and it follows:

$$\nabla \cdot \dot{\mathbf{A}} = -j_r. \tag{2.164}$$

Then, when we integrate (2.164) over time as all charges have passed through the sphere, we have:

$$\nabla \cdot \mathbf{A} = -Q. \tag{2.165}$$

This obviously allows us to write

$$j_r = \dot{Q}. \tag{2.166}$$

When we get back to (2.164) and make use of (2.166), we finally have

$$\frac{d}{dt}(\nabla \cdot \mathbf{A} + Q) = 0, \tag{2.167}$$

which is a conservation law. It should satisfy (2.165) since when integrated over time, change in $\mathbf{A}$ is not zero, hence not trivial. Therefore, this is a conservation law that is valid at every point on Ω. The associated conserved charge generates BMS transformation.

We can show how to eliminate the gauge field using the gauge transformation on the sphere by creating a relative phase defined by

$$\lambda = e^{\frac{ie\chi}{\hbar c}}. \tag{2.168}$$

For this purpose, we can check to see how the magnetic field influences the dynamics of particles. Therefore, it is helpful to evaluate the relative weight of the A-dependent contributions to the quantum Hamiltonian of a particle with charge e and mass m moving in an electromagnetic field. The classical Hamiltonian in Gaussian units is given by

$$\mathbf{H} = \frac{1}{2m}\left(p - \frac{eA}{c}\right)^2 + e\phi, \tag{2.169}$$

where p is the canonical momentum of the particle so that $[x, p] = i\hbar$. It is related to the mechanical momentum by

$$p_{mech} = p - \frac{eA}{c}. \tag{2.170}$$

The corresponding Schrödinger equation is given by

$$\frac{1}{2m}\left[-i\hbar\nabla - \frac{eA}{c}\right]^2 \psi + e\phi\psi = i\hbar\frac{\partial}{\partial t}\psi, \tag{2.171}$$

where ψ is a wave function in this case. In classical electromagnetism, the gauge transformation leads to a redundant description of physics. In particular, the electric and magnetic fields remain unchanged. And under the gauge transformation, the Schrödinger equation becomes

$$\frac{1}{2m}\left[-i\hbar\nabla - \frac{eA}{c} - \frac{e}{c}\nabla\chi\right]^2 \psi' + e\left(\phi - \frac{1}{c}\frac{\partial\chi}{\partial t}\right)\psi' = i\hbar\frac{\partial}{\partial t}\psi', \tag{2.172}$$

where we take $\psi' = \lambda\psi$ and we use

$$i\hbar\frac{\partial\psi'}{\partial t} + \frac{e}{c}\frac{\partial\chi}{\partial t}\psi' = e^{\frac{ie\chi}{\hbar c}} i\hbar\frac{\partial\psi}{\partial t}. \tag{2.173}$$

The original equation given in (2.171) is restored, see [18]. Thus, under the gauge transformation, the wave function is transformed as

$$\psi'(x,t) = \lambda\psi(x,t). \tag{2.174}$$

We conclude by saying that the electromagnetism is invariant under gauge transformations. The λ factor can be viewed as a 1×1 unitary matrix. Therefore, we have a group symmetry, where the transformation group is $U(1)$. This corresponds to a local phase change, which, changes from point to point. It is not a global phase factor. This phase, which defines the memory effect does not visibly affect the physics of the system since the same physical equation, that is the Schrödinger equation, is obtained. This means that no observable change occurs when we change the phase of the wave function. Thus the gauge transformation here, can be regarded as the change in phase of a wave function. This is explained by the fact that Schrödinger equation is rewritten in the same way exactly before transformation by making use of transformed wavefunction, scalar and vector potentials. The potential just connects the phase at one spacetime to the phase at another spacetime location.

2.5 Quantum Electrodynamics

Quantum electrodynamics is the theory of light interacting with charged matter. In this section, we shall write down the action of the model and review some aspects of the theory, see [9]. The Lagrangian of the theory is made of the Dirac Lagrangian density and the Yang-Mills Lagrangian density. In Subsection 2.5.1, we present the Lagrangian density in Dirac field and derive the associated current density. Subsection 2.5.2 presents the massless Lagrangian density. We finally write down the electrodynamics Lagrangian density and its conserved current in Subsection 2.5.3.

2.5.1 The Dirac Lagrangian density

The Dirac Lagrangian density has the form

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi, \tag{2.175}$$

where ψ and $\bar{\psi}$ should be considered as independent fields. We can see that the derivative of $\mathcal{L}_D$ with respect to $\bar{\psi}$ leads to (2.35). While the derivative with respect to ψ gives (2.48). The Dirac equation is consistent with the principle of special relativity. The Lagrangian density is invariant under a global continuous symmetry $U(1)$. The transformations are

$$\psi \longrightarrow \psi' = e^{i\alpha}\psi \quad \text{and} \quad \bar{\psi} \longrightarrow \bar{\psi}' = e^{-i\alpha}\bar{\psi}, \tag{2.176}$$

where α is a constant phase factor. Noether's theorem, then, implies the current

$$-\alpha\mathbf{j}^\mu_D = i\alpha\frac{\partial\mathcal{L}_D}{\partial\partial_\mu\psi}\psi = i\alpha\frac{\partial\mathcal{L}'}{\partial\partial_\mu\psi}\psi - i\alpha\bar{\psi}\frac{\partial\mathcal{L}'}{\partial\partial_\mu\bar{\psi}} = -\alpha\bar{\psi}\gamma^\mu\psi, \tag{2.177}$$

is conserved as

$$\partial_\mu\mathbf{j}^\mu_D = 0\ , \tag{2.178}$$

see [7]. Here $\mathcal{L}'$ is a modified form of $\mathcal{L}_D$ given by

$$\mathcal{L}' = \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi. \tag{2.179}$$

It treats more symmetrically ψ and $\bar{\psi}$ and is different from $\mathcal{L}_D$ by a derivative $\partial_\mu(\cdots)$ which does not contribute to the equations of motion. This Lagrangian density leads to the canonical momentum field $\pi(x)$ as given by

$$\pi(x) = \frac{\partial\mathcal{L}'}{\partial\dot{\psi}(x)} = i\psi^\dagger(x). \tag{2.180}$$

We can see by computing the derivative of ψ and $\bar{\psi}$, that the transformations become

$$\partial_\mu\psi \longrightarrow \partial_\mu\psi' = e^{i\alpha}\partial_\mu\psi \quad \text{and} \quad \partial_\mu\bar{\psi} \longrightarrow \partial_\mu\bar{\psi}' = e^{-i\alpha}\partial_\mu\bar{\psi} \tag{2.181}$$

and the invariance of the Lagrangian is shown as

$$\begin{aligned} \mathcal{L}_D \longrightarrow \mathcal{L}'_D &= \bar{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' \\ &= e^{-i\alpha}\bar{\psi}(i\gamma^\mu\partial_\mu - m)e^{i\alpha}\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &= \mathcal{L}_D. \end{aligned}$$

However, the symmetry is spontaneously broken, that is the Lagrangian does not enjoy the symmetry, when we promote the global symmetry transformation to local symmetry. We shall introduce an electromagnetic field. The local symmetry is given by

$$\psi \longrightarrow \psi' = e^{i\chi(x)}\psi, \tag{2.182}$$

and

$$\bar{\psi} \longrightarrow \bar{\psi}' = e^{-i\chi(x)}\bar{\psi}. \tag{2.183}$$

Likewise, in local symmetry, we can compute the derivative of the fields given as

$$\partial_\mu\psi \longrightarrow \partial_\mu\psi' = e^{i\chi(x)}\left(\partial_\mu + i\partial_\mu\chi(x)\right)\psi, \tag{2.184}$$

$$\partial_\mu\bar{\psi} \longrightarrow \partial_\mu\bar{\psi}' = e^{-i\chi(x)}\left(\partial_\mu - i\partial_\mu\chi(x)\right)\bar{\psi}. \tag{2.185}$$

Obviously, the first term in the Dirac Lagrangian density is not invariant. A potential vector A_μ should be introduced to maintain this Lagrangian invariant. Therefore, we define a new derivative operator called gauge covariant derivative which is given by

$$D_\mu = \partial_\mu - iA_\mu(x), \tag{2.186}$$

and is transformed as

$$D_\mu \longrightarrow D'_\mu = e^{i\chi(x)}D_\mu e^{-i\chi(x)}. \tag{2.187}$$

The interaction is possible due to the conserved current to ensure that the Lagrangian is invariant. The interaction Lagrangian is given by

$$\mathcal{L}_{int} = j^\mu A_\mu. \tag{2.188}$$

The Lagrangian becomes invariant under $U(1)$ gauge symmetry. We imposed to A_μ the transformation given in (2.115). Finally the Lagrangian has the form

$$\mathcal{L} = \mathcal{L}_D + j^\mu A_\mu, \tag{2.189}$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$, and we replace ∂_μ in $\mathcal{L}_D$ by the covariant derivative D_μ defined previously. The field strength $F_{\mu\nu}$, whose invariance has been proved early can be written in terms of the covariant derivative as follows:

$$F_{\mu\nu} = i\left[D_\mu, D_\nu\right]. \tag{2.190}$$

2.5.2 Yang-Mills Lagrangian density

The Lagrangian of massless free particle charges of electromagnetic field is defined by

$$\mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.191}$$

It is invariant under the gauge transformation since $F_{\mu\nu}$ is invariant. It describes the propagation and interactions of gauge fields. The corresponding action is the following

$$\begin{aligned} S &= \int d^4x\mathcal{L}_{YM} \\ &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}. \end{aligned} \tag{2.192}$$

and produces the equations of motion given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu F^{\mu\nu} = 0. \tag{2.193}$$

We can write (2.193) in terms of components of A^μ as follows:

For $\nu = 0$,

$$\partial_i F^{i0} = 0 \Longrightarrow \partial_i\partial^i A^0 - \partial_i\partial^0 A^i = 0. \tag{2.194}$$

For $\nu = j$,

$$\partial_\mu F^{\mu j} = 0 \Longrightarrow \partial_0\partial^0 A^j + \partial_i\partial^i A^j - \partial_0\partial^j A^0 - \partial_i\partial^j A^i = 0. \tag{2.195}$$

Now, we shall decompose the vector field A^j into two parts: a transverse part and a longitudinal part as $A^j = \partial^j\phi + A_t^j$, where A_t^j is the transverse part. The equation (2.194) becomes

$$-\nabla^2 A^0 + \partial_0\nabla^2\phi = 0, \tag{2.196}$$

$$\partial_i A_t^i = 0, \tag{2.197}$$

and (2.195) is

$$\left\{\begin{array}{rcl} \partial_0\partial^0 A_t^j + \nabla^2 A_t^j &=& 0\,, \\ \partial_0\partial^0\partial^j\phi - \nabla^2\partial^j\phi - \partial_0\partial^j A^0 + \partial^j\nabla^2\phi &=& 0. \end{array}\right. \tag{2.198}$$

Hence

$$A_0 = \partial_0\phi, \tag{2.199}$$

$$\partial_0\partial^0 A_t^j + \nabla^2 A_t^j = 0, \tag{2.200}$$

$$\partial_0\partial^0\partial^j\phi + \partial^j\partial_0 A^0 = 0. \tag{2.201}$$

We can see that (2.199) and (2.201) are not independent. Thus $F^{0i} = \partial^0 A_t^i$ and $F^{ij} = \partial^i A_t^j - \partial^j A_t^i$. The electric and magnetic fields in particular do not depend on the longitudinal component ϕ. Therefore, we choose $\phi = 0$ and it implies that $A_0 = 0$. This shows that the massless vector field is a pure transverse field.

We have shown that $F_{\mu\nu}$ is invariant under the transformation

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x). \tag{2.202}$$

Consequently, the corresponding action and Lagrangian are also invariant. This invariance is known as gauge symmetry. The longitudinal part can be modified by using the above transformation. Therefore, we can adjust $\chi(x)$ so that this longitudinal part vanishes everywhere. This choice is called Coulomb gauge condition. Under this condition, the conjugate momenta are the components of the electric field given as

$$\pi^i(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i(x))} = F^{i0} = E^i. \tag{2.203}$$

A_0 is not a dynamical field. It results $\pi^0 = 0$. We can see that for the moment, the momentum conjugate π^i is the electric field as shown in (2.203).

Likewise, under Lorentz gauge condition, it is obvious to see that the equations of motion given in (2.193) becomes

$$\partial_\mu\partial^\mu A^\nu = 0. \tag{2.204}$$

Therefore, for us to recover these equations of motion from the Lagrangian, we should add an extra term to the original Lagrangian given in (2.191) so that it may become

$$\mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2, \tag{2.205}$$

where the second term is called gauge fixing term. It may be written in more general form as

$$\mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \tag{2.206}$$

where α is finite and arbitrary. In the (2.205), the condition $\alpha = 1$, is called Feynman gauge. While in the case $\alpha = 0$, it is called Landau gauge [9]. The components A^0 and A^i are dynamical fields in Lorentz gauge. The momentum π^μ conjugate to A^μ is written as follows:

$$\pi^0(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0(x))} = -\partial_\mu A^\mu, \tag{2.207}$$

$$\pi^i(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i(x))} = F^{i0} = E^i. \tag{2.208}$$

2.5.3 Charged fermion Lagrangian density

To write down the Lagrangian density of the electrodynamics, we consider here the simplest gauge theory which describes a fermion carrying an electric charge e and a photon. The gauge group $U(1)$ is abelian and has a single parameter. We recall that the theory contains a Dirac fermion, which transforms as

$$\psi(x) \longrightarrow \psi'(x) = e^{i\chi(x)}\psi(x). \tag{2.209}$$

The infinitesimal transformations of the gauge field and the field strength are

$$\delta A_\mu = \frac{1}{e}\partial_\mu \chi(x) \qquad \text{and} \quad \delta F_{\mu\nu} = 0, \tag{2.210}$$

where e is the gauge coupling constant. Setting $e = 1$, the invariant form of the Lagrangian density is given by

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.211}$$

This theory provides a quantum electrodynamics and contains a fermion-photon interaction of the form $eA_\mu\bar{\psi}\gamma^\mu\psi$, where the gauge field A_μ representing a photon field is coupled with the Noether's current $\mathbf{j}^\mu = e\bar{\psi}\gamma^\mu\psi$.

Chapter 3

$\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

3.1 Introduction

The importance of internal symmetries was observed in particle physics in the 1960s. Coleman and Mandula [19] show that the Lie algebra of symmetries of the scattering matrix in quantum field theory contains uniquely the Poincaré algebra and a finite set of internal symmetry generators T_A. In fact, the scattering matrix is defined as the unitary matrix connecting asymptotic particle states in the Hilbert space of physical states. The generators of Lorentz transformations and translations are respectively $M_{\mu\nu}$ and P_μ. The generators $M_{\mu\nu}$ contain three rotations J_i and three boosts K_i, with $i = 1, 2, 3$. They satisfy the commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \quad [K_i, K_j] = -i\varepsilon_{ijk} J_k, \quad \text{and} \quad [K_i, J_j] = i\varepsilon_{ijk} K_k, \tag{3.1}$$

where ε_{ijk} is the Levi-Civita tensor. T_A are Lorentz invariant generators of a Lie algebra $\mathcal{A}$. A Lie algebra $\mathcal{A}$ is called semisimple if the only commutative ideal of $\mathcal{A}$ is $\{0\}$. The generators of the Lie algebra $\mathcal{A}$ acting on multiparticle states are hermitian matrices and the algebra $\mathcal{A}$ is the direct sum of a semisimple algebra $\mathcal{A}_1$ and an abelian algebra $\mathcal{A}_2$ as $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. The algebra is:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i(g_{\nu\rho} M_{\mu\sigma} - g_{\nu\sigma} M_{\mu\rho} + g_{\mu\sigma} M_{\nu\rho} - g_{\mu\rho} M_{\nu\rho}), \\
[M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu), \\
[P_\mu, P_\nu] &= 0, \\
[T^A, T^B] &= iC^{ABC} T^C, \\
[M_{\mu\nu}, T^A] &= [P_\mu, T^A] = 0.
\end{aligned} \tag{3.2}$$

Golfand and Likhtman [20] have shown that one can evade the Coleman-Mandula theorem by adopting the generalized symmetry superalgebra. Therefore, we must consider a graded Lie algebra, which is the extension of the Poincaré algebra and the internal symmetry as follows [21]:

$$\begin{cases}
[\mathcal{B}, \mathcal{B}] &= \mathcal{B}, \\
[\mathcal{F}, \mathcal{B}] &= \mathcal{F}, \\
\{\mathcal{F}, \mathcal{F}\} &= \mathcal{B}.
\end{cases} \tag{3.3}$$

The generators $\mathcal{B}$ are bosonic and $\mathcal{F}$ are fermionic. The relations (3.3) are the superalgebras and they show that bosonic generators form a Lie algebra whose representations are fermionic generators. The generators $\mathcal{B}$ and $\mathcal{F}$ obey the Jacobi identities below:

$$\begin{aligned} [\mathcal{B}_1,[\mathcal{B}_2,\mathcal{B}_3]]+[\mathcal{B}_2,[\mathcal{B}_3,\mathcal{B}_1]]+[\mathcal{B}_3,[\mathcal{B}_1,\mathcal{B}_2]]&=0\\ [\mathcal{B}_1,[\mathcal{B}_2,\mathcal{F}_3]]+[\mathcal{B}_2,[\mathcal{F}_3,\mathcal{B}_1]]+[\mathcal{F}_3,[\mathcal{B}_1,\mathcal{B}_2]]&=0\\ [\mathcal{B}_1,\{\mathcal{F}_2,\mathcal{F}_3\}]+\{\mathcal{F}_2,[\mathcal{F}_3,\mathcal{B}_1]\}+\{\mathcal{F}_3,[\mathcal{B}_1,\mathcal{F}_2]\}&=0\\ [\mathcal{F}_1,\{\mathcal{F}_2,\mathcal{F}_3\}]+[\mathcal{F}_2,\{\mathcal{F}_3,\mathcal{F}_1\}]+[\mathcal{F}_3,\{\mathcal{F}_1,\mathcal{F}_2\}]&=0. \end{aligned} \tag{3.4}$$

These Jacobi identities are called super Jacobi identities.

There are several reasons why supersymmetry is considered in theoretical physics as well as in high energy physics. The supersymmetric field theory is generally more constrained by the higher degree of symmetry. Supersymmetric models often are easier to solve analytically than non-supersymmetric ones. A typical example is the $\mathcal{N}=4$ super Yang-Mills theory [22]. Supersymmetry is also interesting to study because it solves the hierarchy problem in the standard model and offers the proton stability. Supersymmetry unifies the coupling constants and offers a candidate for the dark matter. The supersymmetric theory proposes a first step to unify gauge theories and gravity and is necessary in string theory.

In this chapter, following the spirit of [19, 20, 21, 23, 22, 24, 25, 26, 27, 28], we review the super Poincaré algebra. Thereafter, we present the massless and massive state representations of this algebra. By using the same method as in the state representations, we review the extension on the representations of fields. This allows us to review the field theory of the supersymmetry. Starting from the abelian gauge theory, we end up with the supersymmetry nonabelian gauge theory which is called $\mathcal{N}=4$ super Yang-Mills theory and is analytically solvable. We aim to provide later a brief statement for extending the memory effect discussed in the Chapter 4 to non-Abelian gauge theory

3.2 Supersymmetry

3.2.1 Supersymmetry algebra

We consider a superalgebra given in (3.3) whose bosonic generators are generators of the Poincaré algebra (P_μ and $M_{\mu\nu}$) as well as generators of internal symmetry algebra T^A, satisfying the relations (3.2). This is due to the fact that $\mathcal{B}$ must satisfy the Coleman-Mandula theorem. So, $\mathcal{B}$ belongs to $(P_\mu, M_{\mu\nu}, T^A)$. $\mathcal{F}$ are made of generators Q^I_α as well as their adjoints $(Q^I_\alpha)^\dagger=\bar{Q}^I_{\dot\alpha}$, where the index I labels all the different 2-spinors Q_α that are present and runs from 1 to some integer N. The index α is the spin index. This kind of superalgebra is the so-called supersymmetry algebra. In the case where $N>1$, we talk about an extended supersymmetry. By using the relation (3.3), we must have the relation

$$[Q^I_\alpha,\mathcal{B}]=b_\alpha^\beta Q^I_\beta \tag{3.5}$$

for a bosonic generator $\mathcal{B}$, where b_α^β are matrices and represent the Lie algebra of bosonic generators. When we apply the super Jacobi identity using two bosonic operators $\mathcal{B}_1$ and $\mathcal{B}_2$ with Q^I_α, we get

$$
\begin{aligned}
[Q^I_\alpha, [\mathcal{B}_1, \mathcal{B}_2]] &= [[Q^I_\alpha, \mathcal{B}_1], \mathcal{B}_2] - [[Q^I_\alpha, \mathcal{B}_2], \mathcal{B}_1] \\
&= [(b_1)^\beta_\alpha Q^I_\beta, \mathcal{B}_2] - [(b_2)^\beta_\alpha Q^I_\beta, \mathcal{B}_1] \\
&= (b_1 b_2)^\beta_\alpha Q^I_\beta - (b_2 b_1)^\beta_\alpha Q^I_\beta \\
&= (b_1 b_2 - b_2 b_1)^\beta_\alpha Q^I_\beta \\
&= [b_1, b_2]^\beta_\alpha Q^I_\beta. \qquad (3.6)
\end{aligned}
$$

We can identify three cases:

1) Consider $\mathcal{B} = \mathcal{M}_{\mu\nu}$. Then (3.5) becomes

$$[Q^I_\alpha, M_{\mu\nu}] = (b_{\mu\nu})^\beta_\alpha Q^I_\beta, \qquad (3.7)$$

with

$$[b_{\mu\nu}, b_{\rho\sigma}] = i(g_{\nu\rho} b_{\mu\sigma} - g_{\nu\sigma} b_{\mu\rho} + g_{\mu\sigma} b_{\nu\rho} - g_{\mu\rho} b_{\nu\rho}). \qquad (3.8)$$

The matrices $(b_{\mu\nu})^\beta_\alpha$ form a representation of Lorentz algebra; that is, the generators Q^I_α carry a representation of the Lorentz group. We shall assume that Q^I_α and its conjugate denoted $\bar{Q}^I_{\dot{\alpha}}$ belong to the representation of Lorentz group, $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, where $\oplus$ denotes the union, and hence $b_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$.

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \qquad (3.9)$$

where σ^μ is the Pauli matrix. If Q^I_α belongs to $(\frac{1}{2}, 0)$ then $\bar{Q}^I_{\dot{\alpha}}$ belongs to $(0, \frac{1}{2})$. It follows that the anticommutator $\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\}$ belongs to $(\frac{1}{2}, \frac{1}{2})$. Moreover, P_μ is the only bosonic object in the representation $(\frac{1}{2}, \frac{1}{2})$ since it is a generator of translation. Therefore, the anticommutator $\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\}$ must be proportional to P_μ times a matrix U^{IJ}, and we have

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = \sigma^\mu_{\alpha\dot{\beta}} P_\mu U^{IJ}, \qquad (3.10)$$

where the matrix U is a Lorentz invariant. We notice that a vector is decomposed as spinors because two symmetry transformations $Q^I_\alpha \bar{Q}^I_{\dot{\beta}}$ have the effect of a translation. The adjoint of (3.10) is

$$\{Q^J_\beta, \bar{Q}^I_{\dot{\alpha}}\} = \sigma^\mu_{\alpha\dot{\beta}} P_\mu (U^{IJ})^\dagger, \qquad (3.11)$$

where we used the fact that P_μ and σ^μ are hermitian. Comparing (3.10) and (3.11), we find that U is hermitian and then, can be diagonalized. One can redefine Q^I_α and its adjoint and fixes the normalization constant to 2. In this case, U^{IJ} is proportional to the Krönecker δ^{IJ} and we have

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ}. \qquad (3.12)$$

2) Consider $\mathcal{B} = P_\mu$. Then (3.5) becomes

$$[Q^I_\alpha, P_\mu] = [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0. \qquad (3.13)$$

To prove (3.12), because of its importance, we refer to (3.5) and write it as

$$[Q^I_\alpha, P_\mu] = Z^{IJ} \sigma^\mu_{\alpha\dot{\beta}} \bar{Q}^J_{\dot{\beta}}. \qquad (3.14)$$

The adjoint of the above equation is

$$[\bar{Q}^I_{\dot{\alpha}}, P_\mu] = Z^{*IJ}\bar{\sigma}^\mu_{\alpha\dot{\beta}}\bar{Q}^J_{\dot{\beta}}. \tag{3.15}$$

Now, we can write the super Jacobi identity when we consider two bosonic operators P_μ and P_ν and one fermionic operator Q^I_α as

$$[[Q^I_\alpha, P_\mu], P_\nu] - [[Q^I_\alpha, P_\nu], P_\mu] = [Q^I_\alpha, [P_\mu, P_\nu]] = 0. \tag{3.16}$$

This is equal to zero due to the fact that $[P_\mu, P_\nu] = 0$. We can work out the left hand side of (3.16) as

$$[[Q^I_\alpha, P_\mu], P_\nu] - [[Q^I_\alpha, P_\nu], P_\mu] = (ZZ^*)^{IJ}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)^\beta_\alpha Q^I_\alpha. \tag{3.17}$$

By comparing (3.16) and (3.17), we see that $ZZ^* = 0$. Since Q^I_α belongs to the representation $(\frac{1}{2}, 0)$ of the algebra of the Lorentz group, it is clear that the anticommutator $\{Q^I_\alpha, Q^J_\beta\}$ belongs to $(1, 0)$. This anticommutator can be decomposed in symmetric and antisymmetric parts in index as

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}X^{[IJ]} + \sigma^{\mu\nu}_{\alpha\beta}M_{\mu\nu}Y^{[IJ]}, \tag{3.18}$$

where $X^{[IJ]}$ and $Y^{[IJ]}$ are respectively antisymmetric and symmetric matrices in the indices I, J and $\varepsilon_{\alpha\beta}$ is a tensor. The symmetric part in the spinorial index has a spin 1, and only the Lorentz generator $M_{\mu\nu}$ is consistent according to the Coleman-Mandula theorem. Writing the super Jacobi identity of one bosonic operator P_μ and two fermionic operators Q^I_α and Q^J_β, it can be contracted with the tensor $\varepsilon^{\alpha\beta}$. The super Jacobi identity chooses the antisymmetric part in α and β of (3.18). The matrix X is a Lorentz invariant and commutes with P_μ. Therefore, the term $\varepsilon^{\alpha\beta}[P_\mu, \{Q^I_\alpha, Q^J_\beta\}]$ vanishes and we have

$$\begin{aligned}
\varepsilon^{\alpha\beta}\{Q^J_\beta, [Q^I_\alpha, P_\mu]\} + \varepsilon^{\alpha\beta}\{Q^I_\alpha, [Q^J_\beta, P_\mu]\} &= 0 \\
\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma_{\mu\beta\dot{\beta}}\left(Z^{IK}\{Q^J_\alpha, \bar{Q}^K_{\dot{\alpha}}\} - Z^{JK}\{Q^I_\alpha, \bar{Q}^K_{\dot{\alpha}}\}\right) &= 0 \\
2\bar{\sigma}^{\dot{\beta}\beta}_\mu\sigma^\nu_{\beta\dot{\beta}}(Z^{IJ} - Z^{JI})P_\nu &= 0,
\end{aligned} \tag{3.19}$$

where, we used (3.14) and (3.15). The above equation implies that $Z^{IJ} - Z^{JI} = 0$. This means that the matrix Z is symmetric. In addition, we have seen that $ZZ^* = 0$. Therefore, $Z = 0$ and hence

$$[Q^I_\alpha, P_\mu] = [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0. \tag{3.20}$$

The super Jacobi identity of P_μ, Q^I_α and Q^J_β implies that $[P_\mu, \{Q^I_\alpha, Q^J_\beta\}] = 0$ and we have

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}X^{[IJ]}. \tag{3.21}$$

3) It is worth mentioning that the supersymmetry genarators Q^I_α and its adjoint $\bar{Q}^I_{\dot{\alpha}}$ carry a representation of the internal symmetry whose generators is T^A. We recall that T^A is hermitian and using (3.5), it follows:

$$\begin{aligned}
[Q^I_\alpha, T_A] &= (B_A)^{IJ}Q^J_\alpha\,, \\
[\bar{Q}^I_{\dot{\alpha}}, T_A] &= (B_A)^{IJ}\bar{Q}^I_{\dot{\alpha}}.
\end{aligned} \tag{3.22}$$

The super Jacobi identity of T_A, Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ takes the form

$$\{\bar{Q}^J_{\dot{\beta}}, [Q^I_\alpha, T_A]\} + \{Q^I_\alpha, [\bar{Q}^J_{\dot{\beta}}, T_A\} = [\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\}, T_A] \tag{3.23}$$

and becomes

$$(B_A)^{IK}\{Q^K_\alpha, \bar{Q}^J_{\dot{\beta}}\} - (B^*_A)^{JK}\{Q^I_\alpha, \bar{Q}^K_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}\delta^{IJ}[P_\mu, T_A] = 0. \tag{3.24}$$

Using (3.12), we have

$$2\sigma^\mu_{\alpha\dot{\beta}}P_\mu((B_A)^{IJ} - (B^*_A)^{JI}) = 0. \tag{3.25}$$

This shows that B_A is hermitian. Then, the T_A generators act by a unitary representation on a N-dimension space of the supercharges. The largest internal symmetry group which can non-trivially act on the supercharges is $U(N)$, and is known as the R-symmetry. This physically means that the supersymmetry allows for a global symmetry which does not commute with supersymmetry.

3.2.2 Representations of supersymmetry algebra

We consider finite dimensional representations of the supersymmetry algebra. We recall that the Poincaré algebra is a subalgebra of the supersymmetry algebra. Therefore, an irreducible representation of the supersymmetry algebra will also be a representation of the Poincaré algebra, but reducible in general, see [22]. Thereby, as each irreducible representation of the Poincaré algebra is a particle, an irreducible representation of the supersymmetry algebra will correspond to a collection of many type of particles, which are connected to one another by the supercharges Q^I_α. Given that supercharges are fermionic variables, they connect bosonic and fermionic states. The set of irreducible representation states of finite dimension of the supersymmetry algebra is called a supermultiplet. The supersymmetry algebra has two Casimir operators. The Casimir operators are the elements of the enclosing algebra that commute with any element of the algebra's generators. We have seen in (3.13) that P_μ commutes with the supercharges. Therefore, the mass operator $P^2 = P^\mu P_\mu$ must also commute with them. It follows that P^2 is a Casimir operator. The second Casimir operator is

$$W^2 = W^\mu W_\mu, \tag{3.26}$$

where

$$W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} \tag{3.27}$$

is the Pauli-Ljubanski operator, see [23]. It is obvious that all particles or all states which belong to a same supermultiplet have the same mass. This is because P^2 remains a Casimir operator. To see this clearly, we consider two states (one bosonic state $|\mathcal{B}\rangle$ and one fermionic state $|\mathcal{F}\rangle$) of respective mass m_b and m_f. They are related by supersymmetry:

$$Q^I_\alpha|\mathcal{B}\rangle = |\mathcal{F}\rangle. \tag{3.28}$$

The eigenvalues of the operator P^2 is m^2 and it follows:

$$P^\mu P_\mu|\mathcal{B}\rangle \;\; = \;\; m_b^2|\mathcal{B}\rangle \;,$$

$$P^\mu P_\mu|\mathcal{F}\rangle = m_f^2|\mathcal{F}\rangle. \tag{3.29}$$

By using the above relations with (3.28), we have

$$P^\mu P_\mu Q_\alpha^I|\mathcal{B}\rangle = P^\mu P_\mu|\mathcal{F}\rangle = m_f^2|\mathcal{F}\rangle. \tag{3.30}$$

Similarly, using the fact that P^2 and Q_α^I commute, we also have

$$P^\mu P_\mu Q_\alpha^I|\mathcal{B}\rangle = Q_\alpha^I P^\mu P_\mu|\mathcal{B}\rangle = m_b^2 Q_\alpha^I|\mathcal{B}\rangle = m_b^2|\mathcal{F}\rangle. \tag{3.31}$$

Comparing (3.30) and (3.31), it is clear that $m_b = m_f$. We remark that in the spectrum of the known-particles, bosons and fermions do not have the same masses. Therefore, if supersymmetry is realized in nature, it is necessarily broken.

Finite dimensional representations of the supersymmetry algebra contain an equal number of bosonic and fermionic states. It is interesting to introduce the fermion number operator defined by [23]

$$N_F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(1-\sigma^3), \tag{3.32}$$

where σ^3 is a Pauli matrix. The fermion number operator relates bosonic and fermionic states by

$$(-1)^{N_F}|\mathcal{B}\rangle = |\mathcal{B}\rangle \quad \text{and} \quad (-1)^{N_F}|\mathcal{F}\rangle = -|\mathcal{F}\rangle. \tag{3.33}$$

Since supersymmetry generators Q_α^I interchange bosonic and fermionic states as we have seen in (3.28), it follows

$$(-1)^{N_F} Q_\alpha^I|\mathcal{B}\rangle = (-1)^{N_F}|\mathcal{F}\rangle = -|\mathcal{F}\rangle, \tag{3.34}$$

$$Q_\alpha^I(-1)^{N_F}|\mathcal{B}\rangle = Q_\alpha^I|\mathcal{B}\rangle = |\mathcal{F}\rangle. \tag{3.35}$$

We can see that Q_α^I and $(-1)^{N_F}$ anticommute. In finite dimensional representations, the trace is defined over I and J indices and we have

$$tr\left((-1)^{N_F}\{Q_\alpha^I, \bar{Q}_{\dot\beta}^J\}\right) = 0. \tag{3.36}$$

By using

$$\{Q_\alpha^I, \bar{Q}_{\dot\beta}^J\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ}, \tag{3.37}$$

this can be shown as it follows:

$$\begin{aligned}
2\delta^{IJ}\sigma^\mu_{\alpha\dot\beta} tr\left((-1)^{N_F}P_\mu\right) &= tr\left((-1)^{N_F}\{Q_\alpha^I, \bar{Q}_{\dot\beta}^J\}\right) \\
&= -tr\left((-1)^{N_F}(Q_\alpha^I\bar{Q}_{\dot\beta}^J - \bar{Q}_{\dot\beta}^J Q_\alpha^I)\right) \\
&= -tr\left(-Q_\alpha^I(-1)^{N_F}\bar{Q}_{\dot\beta}^J - \bar{Q}_{\dot\beta}^J(-1)^{N_F}Q_\alpha^I\right) \\
&= -tr\left(-Q_\alpha^I(-1)^{N_F}\bar{Q}_{\dot\beta}^J - Q_\alpha^I(-1)^{N_F}\bar{Q}_{\dot\beta}^J\right) \\
&= 0,
\end{aligned} \tag{3.38}$$

where we have used the trace cyclicity. Now we can use (3.12) and write

$$tr\left((-1)^{N_F}\{Q_\alpha^I, \bar{Q}_{\dot\beta}^J\}\right) = 2\delta^{IJ}\sigma^\mu_{\alpha\dot\beta} tr\left((-1)^{N_F}P_\mu\right) = 0. \tag{3.39}$$

Therefore, we can see that

$$tr\left((-1)^{N_F}\right) = 0, \tag{3.40}$$

for all P_μ not equal to zero. Using the relation above, we have

$$\begin{aligned}
tr\left((-1)^{N_F}\right) &= 0,\\
\sum_{\text{bosons}} \langle\mathcal{B}|(-1)^{N_F}|\mathcal{B}\rangle + \sum_{\text{fermions}} \langle\mathcal{F}|(-1)^{N_F}|\mathcal{F}\rangle &= 0,\\
\sum_{\text{bosons}} \langle\mathcal{B}|\mathcal{B}\rangle - \sum_{\text{fermions}} \langle\mathcal{F}|\mathcal{F}\rangle &= 0,\\
n_b - n_f &= 0. \qquad (3.41)
\end{aligned}$$

n_b and n_f are respectively the number of bosons and fermions. However, by considering the positivity of the Hilbert space and a supermultiplet state $|\psi\rangle$, we can write

$$\begin{aligned}
\|Q^I_\alpha|\psi\rangle\|^2 + \|(Q^I_\alpha)^\dagger|\psi\rangle\|^2 = \langle\psi|Q^I_\alpha(Q^I_\alpha)^\dagger + (Q^I_\alpha)^\dagger Q^I_\alpha|\psi\rangle &\geq 0,\\
\langle\psi|\{Q^I_\alpha, (Q^I_\alpha)^\dagger\}|\psi\rangle &\geq 0,\\
\langle\psi|2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu|\psi\rangle &\geq 0. \qquad (3.42)
\end{aligned}$$

By contracting the above equation in spinor index and by using the trace of Pauli's matrices, we obtain

$$4\langle\psi|P_0|\psi\rangle \;\geq\; 0. \tag{3.43}$$

The energy P_0 is always positive in a supersymmetric theory. Usually we can set the zero of energy where we like. This is not true because the supersymmetry is spontaneously broken (i.e. it is not a symmetry of the vacuum) if and only if

$$\langle\psi|P_0|\psi\rangle \;\neq\; 0, \tag{3.44}$$

where $|\psi\rangle$ is the state of the vacuum of the theory. There exists two representations corresponding to massless and massive representation.

1) Massless states representations.

   For massless states, the two Casimir operators are zero, that is $P^\mu P_\mu = W^\mu W_\mu = 0$, where $W_\mu = \lambda P_\mu$ and the eigenvalue of P_μ is $p_\mu = (E, 0, 0, E)$. The λ factor is called helicity and is the projection of the angular momentum onto the direction of motion. However,

   $$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu\delta^{IJ} = \begin{pmatrix} 0 & 0\\ 0 & 4E\end{pmatrix}_{\alpha\dot{\beta}}\delta^{IJ}. \tag{3.45}$$

   Observing this anticommutator, we see in particular that $\{Q^I_1, \bar{Q}^J_{\dot{1}}\} = 0, \quad \forall \;\; I, J$. Therefore, $Q^I_1 = \bar{Q}^J_{\dot{1}} = 0$, $\forall\; I, J$ in the representation of a positive definite Hilbert space. Hence, we are left with only Q^I_2 and $\bar{Q}^J_{\dot{2}}$ in the representation. Consequently, the central charges $X^{[IJ]}$ vanish, that is

   $$\{Q^I_1, \bar{Q}^J_{\dot{2}}\}|E, \lambda\rangle = 0 = \varepsilon_{12}X^{IJ}|E, \lambda\rangle, \tag{3.46}$$

   where $|E, \lambda\rangle$ denotes the state in the representation. In massless representations, the helicity is just M_{12}, where $M_{\mu\nu}$, the Lorentz generator is

   $$L = (L^1, L^2, L^3) = (M_{23}, M_{31}, M_{12}). \tag{3.47}$$

The supersymmetry algebra commutation relations show that

$$[L_3, Q_1^I] = -\frac{1}{2} Q_1^I \quad \text{and} \quad [L_3, Q_2^I] = \frac{1}{2} Q_2^I. \tag{3.48}$$

The conjugate equations are

$$[L_3, \bar{Q}_{\dot{1}}^I] = \frac{1}{2} \bar{Q}_{\dot{1}}^I \quad \text{and} \quad [L_3, \bar{Q}_{\dot{2}}^I] = -\frac{1}{2} \bar{Q}_{\dot{2}}^I. \tag{3.49}$$

We use the language of Clifford algebras to discuss the representation in N-dimensions. A Clifford algebra is a set of matrices (we called them gamma matrices) satisfying the anticommutation relations:

$$\{\Gamma_p, \Gamma_q\} = 2\delta_{pq}, \tag{3.50}$$

where $p, q = 0, 1, \cdots (N-1)$. The supercharges $\bar{Q}_{\dot{1}}^I$ and Q_2^I raise the helicity by $\frac{1}{2}$, while Q_1^I and $\bar{Q}_{\dot{2}}^I$ lower the helicity by $\frac{1}{2}$. One defines

$$q^I = \frac{Q_2^I}{2\sqrt{E}} \quad \text{and} \quad q^{I\dagger} = \frac{\bar{Q}_{\dot{2}}^I}{2\sqrt{E}}, \tag{3.51}$$

the normalized operators which satisfy the anticommutation relations of a Clifford algebra in N-dimensions:

$$\{q^I, q^{J\dagger}\} = \delta^{IJ} \quad \text{and} \quad \{q^I, q^J\} = \{q^{I\dagger}, q^{J\dagger}\} = 0. \tag{3.52}$$

The operators q^I and $q^{I\dagger}$ are known as anticommuting annihilation and creation operators. Any irreducible representation of the Clifford algebra is characterized by a Clifford ground state $|E, \lambda_0\rangle$, with

$$q^I |E, \lambda_0\rangle = 0. \tag{3.53}$$

The other states are generated by applying successively the creation operator $q^{I\dagger}$. It follows:

$$\begin{aligned} q^I |E, \lambda_0\rangle &= 0, \\ q^{I\dagger} |E, \lambda_0\rangle &= |E, \lambda_0 + \frac{1}{2}\rangle_I, \\ q^{I\dagger} q^{J\dagger} |E, \lambda_0\rangle &= |E, \lambda_0 + 1\rangle_{IJ}, \\ &\vdots \\ q^{1\dagger} q^{2\dagger} \cdots q^{N\dagger} |E, \lambda_0\rangle &= |E, \lambda_0 + \frac{N}{2}\rangle_{1,\cdots,N}. \end{aligned} \tag{3.54}$$

Due to the antisymmetry in indices, we have $\binom{N}{k}$ states with helicity $\lambda = \lambda_0 + \frac{k}{2}, k = 0, 1, \ldots, N$. The total number of states is $\sum_{k=0}^{N} \binom{N}{k} = 2^N$. The fermionic and bosonic states numbers are equal to 2^{N-1}. For $N = 4$, the theory is called $\mathcal{N} = 4$ super Yang-Mills theory, with $\lambda_0 = -1$. The associated spectrum is given by helicities $(-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$ and the respective corresponding number of states by (1, 4, 6, 4, 1).

2) Massive states representations

Massive states have $P^2 > 0$, with eigenvalue $p_\mu = (m, 0, 0, 0)$, and a priori arbitrary central charges X^{IJ}. The second Casimir operator is $W^2 = -m^2 L^2$. The supersymmetry algebra takes the form

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} &= 2m\delta_{\alpha\dot\beta}\delta^{IJ}, \\ \{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta}X^{IJ}, \\ \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} &= \varepsilon_{\dot\alpha\dot\beta}X^{*IJ}. \end{aligned} \tag{3.55}$$

The first line in (3.55) can be written as

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ}. \tag{3.56}$$

Contrary to the massless case, here the central charges can be non-zero. Therefore, we distinguish two cases. If the central charges are zero, the equation (3.56) defines $2N$ annihilation and creation operators:

$$q^I_\alpha = \frac{Q^I_\alpha}{\sqrt{2m}} \quad \text{and} \quad q^{I\dagger}_\alpha = \frac{\bar{Q}^I_{\dot\alpha}}{\sqrt{2m}}, \tag{3.57}$$

leading to 2^{2N} states. In $\mathcal{N} = 4$ super Yang-Mills theory, we have 256 states in total. In the case where $X^{IJ} \neq 0$, the central charges can be brought to block diagonal form made by 2×2 antisymmetric matrices as

$$X = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \tag{3.58}$$

where D has the form

$$D = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_{\frac{N}{2}} \end{pmatrix}. \tag{3.59}$$

All eigenvalues of D are real and positive and can be generalized by $z_r, r = 1, \ldots, \frac{N}{2}$. One introduces a particular linear combination of supercharges

$$A^\pm_{\alpha r} = \frac{1}{\sqrt{2}}(Q_{\alpha 1r} \pm Q^{\dot\alpha 2r}), \tag{3.60}$$

and finds that the only non-vanishing structures are

$$\{A^\pm_{\alpha r}, A^\pm_{\beta s}\} = \delta_{\alpha\beta}\delta_{rs}(2m \mp z_r). \tag{3.61}$$

For the anticommutation relation to remain positive definite, it requires

$$z_r \leq 2m.$$

We still have 2^{2N} states as in the case where $X^{IJ} = 0$.

3.2.3 Fields

We have discussed supersymmetry representations on states. However, fields are the basis of any quantum field theory. Therefore, we will write representations of the supersymmetry algebra on fields. In fact, states of the Hilbert space in quantum field theory are generated by acting a field-valued operator $\phi(x)$ on a vacuum state

$$|x\rangle = \phi(x)|0\rangle. \tag{3.62}$$

For convenience, we shall introduce superspace and superfields. Superspace is a generalization of ordinary Minkowski space, and superfields are simply fields defined on superspace. We will follow the strategy of states representations and start from a complex scalar field $\phi(x)$ which is the analogous of the Clifford vacuum state that we saw previously, the ground state of the representation. We recall that the action of P_μ on a field $\phi(x)$ is given by the relation

$$[\phi(x), P_\mu] = i\partial_\mu \phi(x). \tag{3.63}$$

For simplicity, we will review a representation on fields for $\mathcal{N} = 1$. For this matter, one imposes the commutation relation

$$[\bar{Q}_{\dot{\alpha}}, \phi(x)] = 0. \tag{3.64}$$

The generalized Jacobi identity for $\phi(x), Q_\alpha, \bar{Q}_{\dot{\beta}}$ is

$$\begin{aligned} [\phi(x), \{Q_\alpha, \bar{Q}_{\dot{\beta}}\}] &= \{[\phi(x), Q_\alpha], \bar{Q}_{\dot{\beta}}\} + \{[\phi(x), \bar{Q}_{\dot{\beta}}], Q_\alpha\} \\ &= 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\phi. \end{aligned} \tag{3.65}$$

The above equation implies that the field $\phi(x)$ is a complex field, otherwise we would have

$$[Q_\alpha, \phi(x)] = 0 \tag{3.66}$$

and

$$[\bar{Q}_{\dot{\alpha}}, \phi(x)] = 0. \tag{3.67}$$

This would mean that the field $\phi(x)$ is a constant, that is

$$\partial_\mu \phi(x) = 0. \tag{3.68}$$

One introduces new fields $\psi_\alpha(x)$, $F_{\alpha\beta}(x)$ and $X_{\alpha\dot{\beta}}(x)$, see [24]:

$$[Q_\alpha, \phi(x)] = 2i\psi_\alpha(x), \tag{3.69}$$

$$\{Q_\alpha, \psi_\beta(x)\} = iF_{\alpha\beta}(x), \tag{3.70}$$

$$\{\bar{Q}_{\dot{\alpha}}, \psi_\beta(x)\} = X_{\dot{\alpha}\beta}(x). \tag{3.71}$$

When we use (3.64) and (3.71), the right hand side of (3.65) becomes

$$2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\phi(x) = -2i\psi_\alpha, \bar{Q}_{\dot{\beta}} = 2iX_{\alpha\dot{\beta}}(x). \tag{3.72}$$

That is

$$X_{\alpha\dot{\beta}}(x) = \sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\phi(x). \tag{3.73}$$

We can see that $X_{\dot{\alpha}\beta}(x)$ is not in reality a new field. Rather, it is proportional to the spacetime derivative of the scalar field $\phi(x)$ and (3.71) becomes

$$\{\bar{Q}_{\dot{\alpha}}, \psi_\beta(x)\} = \sigma^\mu_{\dot{\alpha}\beta}\partial_\mu\phi(x). \tag{3.74}$$

In addition, the generalized Jacobi identity for $\phi(x), Q_\alpha, Q_\beta$ is

$$\{Q_\alpha, [Q_\beta, \phi(x)]\} - \{Q_\beta, [\phi(x), Q_\alpha]\} = 0. \tag{3.75}$$

Introducing (3.69) in (3.75), we have

$$\begin{aligned} \{Q_\alpha, \psi_\beta\} + \{Q_\beta, \psi_\alpha\} &= 0, \\ F_{\alpha\beta}(x) + F_{\beta\alpha}(x) &= 0, \end{aligned} \tag{3.76}$$

which says that $F_{\alpha\beta}(x)$ is antisymmetric and can be written as

$$F_{\alpha\beta}(x) = \varepsilon_{\alpha\beta}F(x). \tag{3.77}$$

$F(x)$ is a new complex scalar field and (3.70) becomes

$$\{Q_\alpha, \psi_\beta(x)\} = i\varepsilon_{\alpha\beta}F(x). \tag{3.78}$$

We use the super Jacobi identity for $(\psi, Q_\alpha, Q_\gamma)$ and get the action of the supercharge Q_α on the field $F(x)$.

$$[Q_\alpha, \{Q_\gamma, \psi_\beta\}] + [Q_\gamma, \{Q_\alpha, \psi_\beta\}] = [\{Q_\alpha, Q_\gamma\}, \psi_\beta] = 0. \tag{3.79}$$

The above relation, when we use (3.78), becomes

$$\varepsilon_{\gamma\beta}[Q_\alpha, F(x)] + \varepsilon_{\alpha\beta}[Q_\gamma, F(x)] = 0. \tag{3.80}$$

Contracting (3.80) with $\varepsilon_{\beta\gamma}$, we obtain

$$[Q_\alpha, F(x)] = 0. \tag{3.81}$$

We similarly proceed by using the generalized Jacobi identity for $(\psi_\gamma, Q_\alpha, \bar{Q}_{\dot{\beta}})$

$$[\bar{Q}_{\dot{\beta}}, \{Q_\alpha, \psi_\gamma\}] + [Q_\alpha, \{\bar{Q}_{\dot{\beta}}, \psi_\gamma\}] = [\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}, \psi_\gamma] = 0. \tag{3.82}$$

That is

$$i\varepsilon_{\alpha\gamma}[\bar{Q}_{\dot{\beta}}, F(x)] + \sigma^\mu_{\dot{\beta}\gamma}[Q_\alpha, \partial_\mu\phi] = [2\sigma^\mu_{\alpha\dot{\beta}}P_\mu, \psi_\gamma]. \tag{3.83}$$

A quick calculation leads to

$$[\bar{Q}_{\dot{\alpha}}, F(x)] = -2\sigma^\mu_{\dot{\alpha}\beta}\partial_\mu\psi^\beta(x). \tag{3.84}$$

We verify the remaining super Jacobi identity and it follows:

$$[\psi_\gamma, \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}] = [F(x), \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}] = [F(x), \{Q_\alpha, Q_\beta,\}] = 0, \tag{3.85}$$

$$[F(x), \{Q_\alpha, \bar{Q}_{\dot{\beta}}\}] = 2\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu F(x). \tag{3.86}$$

This says that there is no new field in addition. Hence the multiplet of field (ϕ, ψ, F) forms a representation of the supersymmetry. The field ψ has two components and is known as the Weyl spinor. Two Weyl spinors are described by a four-component Dirac spinor. The multiplet (ϕ, ψ, F) is a matter multiplet and is called chiral or Wess-Zumino multiplet. It contains two complex scalar fields where each of which has two degrees of freedom and one Weyl spinor which has four degrees of freedom. There are a total of four bosonic degrees of freedom and four fermionic degrees of freedom. The multiplet finally has eight degrees of freedom. This is the smallest possible number in four spacetime dimensions. It means that the multiplet is irreducible.

One extends the spacetime coordinates by introducing anticommuting complex variables or spinor parameters ξ^α and $\bar{\xi}^{\dot{\alpha}} = (\xi^\alpha)^*$:

$$\{\xi^\alpha, \xi^\beta\} = \{\bar{\xi}^{\dot{\alpha}}, \bar{\xi}^{\dot{\beta}}\} = \{\xi^\alpha, \bar{\xi}^{\dot{\beta}}\} = 0. \tag{3.87}$$

These spinor parameters together with the identity element form the Grassmann algebra, whose variables known as Grassmann variables are θ^α and $\bar{\theta}_{\dot{\alpha}}$. We write:

$$\theta\theta = \theta^\alpha\theta_\alpha, \quad \bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}, \quad \theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta, \quad \theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta. \tag{3.88}$$

The Grassmann variable and the spinor parameter are related by the supersymmetric transformations below:

$$\theta'^\alpha = \theta^\alpha + \xi^\alpha, \quad \bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \tag{3.89}$$

One can use the spinor parameters and define the infinitesimal supersymmetry variation of (ϕ, ψ, F) by

$$\delta(\phi, \psi, F) = -i[(\phi, \psi, F), \xi Q + \bar{Q}\bar{\xi}], \tag{3.90}$$

where

$$\begin{aligned} \delta\phi &= 2\xi\psi, \\ \delta\psi &= -\xi F - i\partial_\mu\phi\sigma^\mu\bar{\xi}, \\ \delta F &= -2i\partial_\mu\psi\sigma^\mu\bar{\xi}. \end{aligned} \tag{3.91}$$

The whole algebra has the form

$$[\delta_1\ , \delta_2](\phi, \psi, F) = 2i(\xi_1\sigma^\mu\bar{\xi}_2 - \xi_2\sigma^\mu\bar{\xi}_1)(\phi, \psi, F). \tag{3.92}$$

The Hermitian conjugation of (ϕ, ψ, F) is $(\phi^\dagger, \psi^\dagger, F^\dagger)$ with

$$\begin{aligned} \delta\phi^\dagger &= 2\bar{\psi}\bar{\xi}, \\ \delta\psi^\dagger &= -F^\dagger\bar{\xi} + i\xi\sigma^\mu\partial_\mu\bar{\phi}^\dagger, \\ \delta F^\dagger &= 2i\xi\sigma^\mu\partial_\mu\bar{\psi}. \end{aligned} \tag{3.93}$$

In superfields, the generator $P_\mu = i\partial_\mu$ remains the same, while the supercharges become

$$\begin{aligned} \mathcal{Q}_\alpha &= i\left[\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right], \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= -i\left[\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right], \\ \bar{\mathcal{Q}}^{\dot{\alpha}} &= i\left[\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\bar{\sigma}^{\mu\alpha\dot{\alpha}}\theta_\alpha\partial_\mu\right]. \end{aligned} \tag{3.94}$$

One introduces spinor covariant derivatives:

$$\begin{aligned}
\mathcal{D}_\mu &= -i\partial_\mu,\\
\mathcal{D}_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu,\\
\bar{\mathcal{D}}_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu,\\
\bar{\mathcal{D}}^{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\bar\sigma^{\mu\alpha\dot\alpha}\theta_\alpha\partial_\mu.
\end{aligned} \tag{3.95}$$

They satisfy the relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\mathcal{D}_{\dot\alpha}\mathcal{D}_{\dot\beta}\} = 0,$$

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{Q}}_{\dot\beta}\} = 0.$$

A superfield is given in terms of Grassmann variables by [23]

$$\begin{aligned}
\mathcal{F}(x,\theta^\alpha,\bar\theta_{\dot\alpha}) \;=\;& f(x) + \theta^\alpha\phi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) + \theta^\alpha\theta_\alpha m(x) + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}n(x) + (\theta^\alpha\sigma^\mu\bar\theta_{\dot\alpha})A_\mu(x)\\
&+\theta^\alpha\theta_\alpha\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\theta^\alpha\psi_\alpha(x) + \theta^\alpha\theta_\alpha\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}d(x).
\end{aligned} \tag{3.96}$$

The fields $f(x), m(x), n(x)$ and $d(x)$ are complex scalar fields, $\phi_\alpha(x), \psi_\alpha(x), \bar\chi^{\dot\alpha}(x)$ and $\bar\lambda^{\dot\alpha}(x)$ are right and left Weyl spinors and A_μ is a complex vector field. They are the components of the superfield. We note that a general superfield has 16 bosonic degrees of freedom and 16 fermionic degrees of freedom. The action of the spinor parameters on $\mathcal{F}$ is

$$\mathcal{D}_\alpha[\delta_s\ ,\mathcal{F}] = \delta_s[\mathcal{D}_\alpha\ ,\mathcal{F}]. \tag{3.97}$$

Consider a superfield $\Phi(x,\theta,\bar\theta)$. It is said to be chiral if it satisfies the relation

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0. \tag{3.98}$$

Solving (3.98) for θ^α and $y^\mu = x^\mu - \theta\sigma^\mu\bar\theta$, we have

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^2F(y), \tag{3.99}$$

see [23]. One rewrites the above chiral superfield in terms of Grassmann variables and x^μ as

$$\begin{aligned}
\Phi(x^\mu,\theta,\bar\theta) \;=\;& \phi(x) - i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x)\\
&+\sqrt{2}\ \theta^\alpha\psi_\alpha(x) + \frac{i}{\sqrt{2}}\ \theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta^2F(x).
\end{aligned} \tag{3.100}$$

Similarly, the superfield $\Phi(x,\theta,\bar\theta)$ is said to be antichiral if

$$\bar{\mathcal{D}}_\alpha\Phi^\dagger(x,\theta,\bar\theta) = 0, \tag{3.101}$$

where

$$\Phi^\dagger(y^\dagger,\bar\theta) = \phi^*(y^\dagger) + \sqrt{2}\bar\theta^\alpha\bar\psi_\alpha(y^\dagger) + \bar\theta^2F(y^\dagger), \tag{3.102}$$

which in terms of Grassmann variables and x^μ takes the form

$$\Phi^\dagger(x^\mu,\theta,\bar\theta) \;=\; \phi^*(x)+i\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x)-\frac{1}{4}\theta^2\bar\theta^2\Box\phi^*(x)$$
$$+\sqrt{2}\;\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x)-\frac{i}{\sqrt{2}}\;\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\psi(x)+\bar\theta^2F^*(x). \tag{3.103}$$

A product of two chiral superfields $\Phi_i(y,\theta)$ and $\Phi_j(y,\theta)$ is also a chiral superfield. We have

$$\Phi_i\Phi_j(y,\theta)=\phi(y)+\sqrt{2}\theta^\alpha\psi_\alpha(y)+\theta^2F(y), \tag{3.104}$$

where

$$\begin{aligned}\phi &= \phi_i\phi_j,\\ \psi &= \psi_i\phi_j+\phi_i\psi_j,\\ F &= F_i\phi_j+\phi_iF_j-\psi_i\psi_j.\end{aligned} \tag{3.105}$$

The double action of the covariant derivatives on $\mathcal{F}$ is as follows:

$$\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{F}=\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\mathcal{F}=-4(m(x)+\theta\psi(x)+\theta^2d(x)), \tag{3.106}$$

and

$$\mathcal{D}\mathcal{D}\mathcal{F}=\mathcal{D}^\alpha\mathcal{D}_\alpha\mathcal{F}=-4(m(x)+\bar\theta\bar\psi(x)+\bar\theta^2d(x)). \tag{3.107}$$

We deduce that $\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{F}$ and $\mathcal{D}\mathcal{D}\mathcal{F}$ are chiral and antichiral superfields respectively.

A superfield $\mathcal{V}$ is said to be vector superfield if it obeys the relation

$$\mathcal{V}(x,\theta,\bar\theta)=\mathcal{V}^\dagger(x,\theta,\bar\theta), \tag{3.108}$$

and can be written as:

$$\begin{aligned}\mathcal{V}(x,\theta,\bar\theta) \;=\;& C(x)+i\theta\chi(x)-i\bar\theta\bar\chi(x)+\frac{i}{2}\theta^2[M(x)+iN(x)]-\frac{i}{2}\bar\theta^2[M(x)-iN(x)]\\ &+\theta\sigma^\mu\bar\theta A_\mu(x)+\theta^2\bar\theta_{\dot\alpha}\left[\bar\lambda^{\dot\alpha}(x)+\frac{1}{2}(\bar\sigma^\mu\partial_\mu\chi(x))^{\dot\alpha}\right]\\ &+\bar\theta^2\theta^\alpha\left[\lambda_\alpha(x)-\frac{1}{2}(\sigma^\mu\partial_\mu\bar\chi(x))_\alpha\right]-\frac{1}{2}\theta^2\bar\theta^2\left[D(x)+\frac{1}{2}\Box C(x)\right],\end{aligned} \tag{3.109}$$

where C, D, M and N are real scalar fields, A_μ is a real vector field, χ and λ are spinorial fields, see [23]. It is worth mentioning that a vector superfield contains eight bosonic components and eight fermionic components and does not describe a supermultiplet yet. A supersymmetric infinitesimal transformation of parameter ξ of $\mathcal{V}$ is

$$\delta_\xi\mathcal{V}=-i(\xi\mathcal{Q}+\bar\xi\bar{\mathcal{Q}}), \tag{3.110}$$

and by identification, we obtain transformations on the vector superfield components as

$$\begin{aligned}\delta_\xi C &= i(\chi\xi-\bar\chi\bar\xi),\\ \delta_\xi\chi &= 2\xi M+\sigma^\mu\bar\xi(A_\mu-i\partial_\mu C),\\ \delta_\xi(M+iN) &= \bar\xi\bar\lambda,\\ \delta_\xi A_\mu &= \xi\sigma_\mu\bar\lambda-\bar\xi\bar\sigma_\mu\lambda+\xi\partial_\mu\chi+\bar\xi\partial_\mu\bar\chi,\\ \delta_\xi\lambda &= 2\xi D-\frac{1}{2}\sigma^{\mu\nu}\xi F_{\mu\nu},\\ \delta_\xi D &= i(\xi\sigma^\mu\partial_\mu\bar\lambda+\bar\xi\bar\sigma^\mu\partial_\mu\lambda).\end{aligned} \tag{3.111}$$

All the fields $\lambda, F_{\mu\nu}$ and D mix and form a submultiplet of the vector multiplet. Then, the submultiplet contains four bosonic degrees of freedom and also four fermionic degrees of freedom and is irreducible. Therefore, the vector multiplet is reducible but not completely. This is a particular characteristic of the supersymmetry theory. A vector superfield can be constructed by combining a chiral superfield Φ and an antichiral superfield $\Phi^\dagger$ as

$$\begin{aligned}\Phi+\Phi^\dagger &= \phi+\phi^*+\sqrt{2}\,\theta^\alpha\psi_\alpha+\sqrt{2}\,\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}+\theta^2F+\bar\theta^2F^*-i\theta\sigma^\mu\bar\theta\partial_\mu(\phi-\phi^*)\\ &\quad+\frac{i}{\sqrt{2}}\,\theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta-\frac{i}{\sqrt{2}}\,\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\psi-\frac{1}{4}\theta^2\bar\theta^2\Box(\phi+\phi^*),\end{aligned}\tag{3.112}$$

$$\begin{aligned}i(\Phi-\Phi^\dagger) &= i(\phi-\phi^*)+i\sqrt{2}\,(\theta^\alpha\psi_\alpha-\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha})+i\theta^2F-i\bar\theta^2F^*+\theta\sigma^\mu\bar\theta\partial_\mu(\phi+\phi^*)\\ &\quad-\frac{1}{\sqrt{2}}\,\theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta+\frac{1}{\sqrt{2}}\,\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\psi-\frac{i}{4}\theta^2\bar\theta^2\Box(\phi-\phi^*).\end{aligned}\tag{3.113}$$

A comparison in coefficient of the term $\theta\sigma^\mu\bar\theta$ in $\mathcal{V}$ and $i(\Phi-\Phi^\dagger)$ leads to a generalization of supersymmetry gauge transformation of the vector superfield by

$$\mathcal{V}\longrightarrow\mathcal{V}'=\mathcal{V}+i(\Phi-\Phi^\dagger),\tag{3.114}$$

where

$$\begin{aligned}C&\to C+i(\phi-\phi^*),\\ \chi&\to\chi+\sqrt{2}\psi,\\ M+iN&\to M+iN+2F,\\ A_\mu&\to A_\mu+\partial_\mu(\phi-\phi^*),\\ \lambda&\to\lambda,\\ D&\to D.\end{aligned}\tag{3.115}$$

These equations show that through a choice of the chiral superfield Φ, we can set the components C, M, N and χ in the superfield to zero. The component fields λ and D are invariant. This gauge choice is the so-called Wess-Zumino gauge, where the vector superfield is

$$\mathcal{V}_{WZ}(x,\theta,\bar\theta)=\theta\sigma^\mu\bar\theta A_\mu(x)+i\theta^2\bar\theta\bar\lambda(x)-i\bar\theta^2\theta\lambda(x)+\frac{1}{2}\theta^2\bar\theta^2D(x),\tag{3.116}$$

see [22]. The degrees of freedom of a vector superfield contains the Maxwell tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, the spinor $\lambda(x)$ called gaugino and the auxiliary field $D(x)$. The successive powers of the vector superfield in the Wess-Zumino gauge are

$$\mathcal{V}_{WZ}^2=(\theta\sigma^\mu\bar\theta A_\mu)(\theta\sigma^\nu\bar\theta A_\nu)=\frac{1}{2}\theta^2\bar\theta^2A^\mu A_\nu,\quad\text{and}\quad\mathcal{V}_{WZ}^n=0\ ,\forall n\geq 3.\tag{3.117}$$

However, it is possible to build by using the vector superfield $\mathcal{V}$, two superfields W_α and $\overline{W}^{\dot\alpha}$. Their spinorial fields λ_α and $\bar\lambda^{\dot\alpha}$ form the fields component of minimum dimension:

$$\begin{aligned}W_\alpha&=-\frac{1}{4}\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha\mathcal{V},\\ \overline{W}^{\dot\alpha}&=-\frac{1}{4}\mathcal{D}^\alpha\mathcal{D}_\alpha\overline{\mathcal{D}}^{\dot\alpha}\mathcal{V}.\end{aligned}\tag{3.118}$$

These superfields are respectively chiral and antichiral, that is

$$\begin{aligned}
\overline{\mathcal{D}}^{\dot\beta} W_\alpha &= -\frac{1}{4}\overline{\mathcal{D}}^{\dot\beta}\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha\mathcal{V} = 0,\\
\mathcal{D}_\beta\overline{W}^{\dot\alpha} &= -\frac{1}{4}\mathcal{D}_\beta\mathcal{D}^\alpha\mathcal{D}_\alpha\overline{\mathcal{D}}^{\dot\alpha}\mathcal{V} = 0.
\end{aligned}\tag{3.119}$$

We used the fact that $\overline{\mathcal{D}}_{\dot\alpha}$ anticommute among themselves and $\dot\alpha$ can take only two values $\dot 1$, $\dot 2$, and then $\overline{\mathcal{D}}^3 = \mathcal{D}^3 = 0$. The superfields W_α and $\overline{W}^{\dot\alpha}$ are also gauge invariant. For the superfield W_α for example, this can be shown by using a supersymmetry gauge transformation in (3.114):

$$\begin{aligned}
W_\alpha \longrightarrow W'_\alpha &= W_\alpha - \frac{i}{4}\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha(\Phi - \Phi^\dagger)\\
&= W_\alpha - \frac{i}{4}\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha\Phi\\
&= W_\alpha + \frac{i}{4}\overline{\mathcal{D}}^{\dot\alpha}(\{\overline{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_\alpha\}\Phi\\
&= W_\alpha - \frac{1}{2}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\partial_\mu\Phi\\
&= W_\alpha.
\end{aligned}\tag{3.120}$$

3.3 Supersymmetric Field Theory

In this section, we are interested in the dynamic of the fields. For this purpose, we shall construct the supersymmetric action. However, we recall that the invariance refers to the property of being left unchanged by symmetry operations and the covariance refers to equations whose form is preserved by a change of coordinate system. For the equations of motion to be covariant in a supersymmetry transformation, the action must be invariant. This is true unless the action is independent of Grassmann variables:

$$\delta_{\xi,\bar\xi}\int dx\ \mathcal{L}(x,\theta,\bar\theta) = -i\int dx\ (\xi\mathcal{Q} + \overline{\xi\mathcal{Q}})\mathcal{L}(x,\theta,\bar\theta) = 0. \tag{3.121}$$

The differentiation and the integration are equivalent operations for the Grassmann variables. One gets the invariant actions by integrating the superfields on the superspace coordinates. Consider the chiral superfield Φ and the antichiral superfield $\Phi^\dagger$, we have:

$$\begin{aligned}
-\frac{1}{4}\mathcal{D}^2\Phi|_0 &= -\frac{1}{4}\mathcal{D}^\alpha\mathcal{D}_\alpha\Phi|_{\theta^2} = \int d^2\theta\ \Phi = F(x),\\
-\frac{1}{4}\overline{\mathcal{D}}^2\Phi^\dagger|_0 &= -\frac{1}{4}\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\Phi^\dagger|_{\bar\theta^2} = \int d^2\bar\theta\ \Phi^\dagger = F^\dagger(x).
\end{aligned}\tag{3.122}$$

In a similar way, consider a vector superfield $\mathcal{V}$. We have

$$-\frac{1}{16}\mathcal{D}^2\overline{\mathcal{D}}^2\mathcal{V}|_0 = -\frac{1}{16}\mathcal{D}^\alpha\mathcal{D}_\alpha\overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}\mathcal{V}|_{\theta^2\bar\theta^2} = \int d^2\theta d^2\bar\theta\ \mathcal{V} = D(x). \tag{3.123}$$

It turns out that the Lagrangian density of the supersymmetry field theory has two types of terms:

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_F, \tag{3.124}$$

where the D-term is

$$\mathcal{L}_D = \int d^2\theta d^2\bar{\theta}\ \mathcal{V}(x,\theta,\bar{\theta}), \tag{3.125}$$

and the F-term is

$$\mathcal{L}_F = \int d^2\theta\ \Phi(x,\theta,\bar{\theta}) + \int d^2\bar{\theta}\ \Phi^\dagger(x,\theta,\bar{\theta}). \tag{3.126}$$

The most general form for a chiral superfield Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_K + \mathcal{L}_I \\ &= \int d^2\theta d^2\bar{\theta}\ K(\Phi,\Phi^\dagger) + \int d^2\theta\ W(\Phi) + \int d^2\bar{\theta}\ W^\dagger(\Phi).\end{aligned} \tag{3.127}$$

Here $\mathcal{L}_K$ represents the D-term and is given by the first integral in (3.127). $\mathcal{L}_I$ represents the F-term and is given by the last two integrals in (3.127). In the D-term, $K(\Phi,\Phi^\dagger) = \Phi^\dagger\Phi$ is called the Kähler potential and is given by

$$\begin{aligned}\Phi^\dagger\Phi &= -\frac{1}{4}((\partial^2\phi_i^\dagger)\phi_i + \phi_i^\dagger\partial^2\phi_i) + \frac{1}{2}\partial_\mu\phi_i^\dagger\partial^\mu\phi_i + \frac{i}{2}\psi_i\sigma^\mu\partial_\mu\overline{\psi_i} - \frac{i}{2}\partial_\mu\psi_i\sigma^\mu\overline{\psi_i} + F_i^\dagger F_i \\ &= \partial_\mu\phi_i^\dagger\partial^\mu\phi_i + \frac{i}{2}(\psi_i\sigma^\mu\partial_\mu\overline{\psi_i} - \partial_\mu\psi_i\sigma^\mu\overline{\psi_i}) + F_i^\dagger F_i + \text{total derivative}\end{aligned} \tag{3.128}$$

In the F-term, $W(\Phi)$ is called the superpotential. The superpotential is a holomorphic function in Φ, since it does not depend on its complex conjugate. Otherwise, it will not be a chiral field. Its general form which is renormalizable writes

$$W(\Phi) = f_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k, \tag{3.129}$$

where f_i, m_{ij} and g_{ijk} are some parameters. We note that m_{ij} and g_{ijk} are symmetric in the indices.

3.3.1 Abelian gauge theory in supersymmetry

Here, we are about to write the supersymmetric quantum electrodynamics Lagrangian density. We recall that the fields $F(x)$ and $D(x)$ are called auxiliary fields. This is because their equations of motion do not describe propagation in spacetime. However, we note that the field equations for $F(x)$ and $D(x)$ are purely algebraic. These fields can be eliminated from the Lagrangian and then from the equations of motion by use of their own field equations. We impose the Wess-Zumino gauge. The superfields W_α and $\overline{W}^{\dot{\alpha}}$ are:

$$W_\alpha(y,\theta,\bar{\theta}) = -i\lambda_\alpha(y) + \theta_\alpha D(y) + \sigma_\alpha^{\mu\nu\beta}\theta_\beta F_{\mu\nu}(y) - \theta^2\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y) \tag{3.130}$$

$$\overline{W}^{\dot{\alpha}}(y^\dagger,\theta,\bar{\theta}) = i\bar{\lambda}^{\dot{\alpha}}(y^\dagger) - \bar{\theta}^{\dot{\alpha}}D(y^\dagger) + \bar{\sigma}_{\dot{\beta}}^{\mu\nu\dot{\alpha}}\bar{\theta}^{\dot{\beta}}F_{\mu\nu}(y^\dagger) + \bar{\theta}^2\bar{\sigma}^{\mu\alpha\dot{\alpha}}\partial_\mu\lambda_\alpha(y^\dagger), \tag{3.131}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is an abelian field strength and V_μ is a vector superfield, see [22]. Similarly, the vector superfield in Wess-Zumino gauge is

$$\mathcal{V}_{WZ}(y,\theta,\bar{\theta}) = \theta\sigma^\mu\bar{\theta}A_\mu(y) + i\theta^2\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}^2\theta\lambda(y) + \frac{1}{2}\theta^2\bar{\theta}^2(D(y) - \partial_\mu A^\mu(y)). \tag{3.132}$$

To write down the Lagrangian density of the superfields, we recall that the superfields W_α and $\overline{W}^{\dot\alpha}$ are chiral and antichiral respectively. The Lagrangian density of a free vector field which is gauge invariant is given by:

$$\begin{aligned}\mathcal{L}_V &= \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}|_{\bar\theta\bar\theta}) \\ &= \frac{1}{4}\int d^2\theta\ W^\alpha W_\alpha + \frac{1}{4}\int d^2\bar\theta \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}. \end{aligned} \tag{3.133}$$

Due to the fact that W_α is chiral, $W^\alpha W_\alpha$ is also chiral and $\mathcal{L}_V$ is invariant under supersymmetry. The Lagrangian in (3.133) is given in terms of the field strength as

$$\mathcal{L}_V = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}(\lambda\sigma^\mu\partial_\mu\bar\lambda + \bar\lambda\bar\sigma^\mu\partial_\mu\lambda) + \frac{1}{2}D^2. \tag{3.134}$$

The superfields W_α and $\overline{W}^{\dot\alpha}$ contain spinorial fields λ and its conjugate $\bar\lambda$. They are supersymmetry partners of the gauge field V_μ. By using Majorana spinor given by

$$\Psi_M = \begin{pmatrix}\lambda_\alpha \\ \bar\lambda^{\dot\alpha}\end{pmatrix}, \tag{3.135}$$

the relation (3.133) becomes

$$\mathcal{L}_V = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\Psi}_M\gamma^\mu\partial_\mu\Psi_M + \frac{1}{2}D^2, \tag{3.136}$$

where γ^μ is the Dirac spinor.

The theory can be coupled. For this, we couple the gauge field to matter fields described by superfields Φ_i and $\Phi_i^\dagger$ which under the group $U(1)$ transform as

$$\Phi_j \longrightarrow e^{i\Lambda}\Phi_j \quad \text{and} \quad \Phi_j^\dagger \longrightarrow \Phi_j^\dagger e^{i\Lambda^\dagger},$$

where Λ is the angle of rotation. The matter Lagrangian density is:

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar\theta \Phi^\dagger e^{\mathcal{V}}\Phi + \int d^2\theta\ W(\Phi) + \int d^2\bar\theta\ W^\dagger(\Phi), \tag{3.137}$$

see [22]. However, in Wess-Zumino gauge, we have

$$e^{\mathcal{V}} = 1 + \mathcal{V} + \frac{1}{2}\mathcal{V}^2. \tag{3.138}$$

Therefore,

$$\begin{aligned}\mathcal{L}_{matter} &= \int d^2\theta d^2\bar\theta(\Phi^\dagger\Phi + \Phi^\dagger\mathcal{V}\Phi + \frac{1}{2}\Phi^\dagger\mathcal{V}^2\Phi) \\ &\quad + \int d^2\theta W(\Phi) + \int d^2\bar\theta W^\dagger(\Phi). \end{aligned} \tag{3.139}$$

The first term in the first integral is given in (3.128). When we use Φ and $\mathcal{V}$ defined previously in Wess-Zumino gauge, the first integral in (3.137) becomes

$$\int d^2\theta d^2\bar{\theta}\ \Phi^\dagger e^{\mathcal{V}}\Phi = ((\mathcal{D}^\mu\phi_i)^\dagger \mathcal{D}_\mu\phi_i + \frac{i}{2}(\psi_i\sigma^\mu\mathcal{D}_\mu\overline{\psi}_i - \mathcal{D}_\mu\psi_i\sigma^\mu\overline{\psi}_i) + F_iF_i^\dagger$$
$$-\frac{i}{2}\sqrt{2}(\phi_i^\dagger\lambda\psi_i - \overline{\psi}_i\bar{\lambda}\phi_i) - \frac{1}{2}\phi_i^\dagger D\phi_i, \quad (3.140)$$

where $\mathcal{D}_\mu$ is the usual covariant derivative.

It is now possible to write the complete form of a supersymmetric quantum electrodynamics Lagrangian density. If one wants to couple a chiral supermultiplet with a Majorana spinor to an abelian gauge supermultiplet, one faces the problem that a gauge transformation is incompatible with the reality imposed by the Majorana condition given by $\psi^* = C\bar{\psi}^T$, where C is the charge conjugation matrix and T denotes the transpose. We must thus introduce a complex Majorana spinor $\psi_\pm$. The supersymmetric quantum electrodynamics Lagrangian density in terms of two superfields $\Phi_\pm$ is:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}(\Phi_+^\dagger e^{e\mathcal{V}}\Phi_+ + \Phi_-^\dagger e^{-e\mathcal{V}}\Phi_-)\int d^2\theta\ (m\Phi_+\Phi_- + \frac{1}{4}W^\alpha W_\alpha)$$
$$+\int d^2\bar{\theta}\ (m\Phi_+^\dagger\Phi_-^\dagger + \frac{1}{4}\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}). \quad (3.141)$$

Starting with two chiral supermultiplets $(\phi_\pm, \psi_\pm, F_\pm)$ of charge $\pm e$ and by using the covariant derivative

$$\mathcal{D}_\mu(\bullet)_\pm = \partial_\mu \pm ieA_\mu(\bullet)_\pm, \quad (3.142)$$

the Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2 + (\mathcal{D}^\mu\phi_+)^\dagger\mathcal{D}_\mu\phi_+ + (\mathcal{D}^\mu\phi_-)^\dagger\mathcal{D}_\mu\phi_- + F_+^\dagger F_+$$
$$+F_-^\dagger F_- + i(\psi_+\sigma^\mu\mathcal{D}_\mu\overline{\psi}_+ + \psi_-\sigma^\mu\mathcal{D}_\mu\overline{\psi}_-) + \frac{1}{2}ie\sqrt{2}(\phi_+^\dagger\lambda\psi_+ - \overline{\psi}_+\bar{\lambda}\phi_+)$$
$$-\frac{1}{2}ie\sqrt{2}(\phi_-^\dagger\lambda\psi_- - \overline{\psi}_-\bar{\lambda}\phi_-)\frac{1}{2}eD(\phi_+^\dagger\phi_+ - \phi_-^\dagger\phi_-)$$
$$+m(F_+\phi_- + F_-\phi_+ - \psi_+\psi_-) + m(F_+^\dagger\phi_-^\dagger + F_-^\dagger\phi_+^\dagger - \overline{\psi}_+\overline{\psi}_-)\ . \quad (3.143)$$

The equations of motion of the auxiliary fields $F_\pm$ and D are respectively

$$F_\pm = -m\phi_\mp^\dagger,$$
$$D = \frac{1}{2}e(\phi_+^\dagger\phi_+ - \phi_-^\dagger\phi_-). \quad (3.144)$$

Thus, when we substitute (3.144) into (3.143), we obtain

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2 + (\mathcal{D}^\mu\phi_+)^\dagger\mathcal{D}_\mu\phi_+ + (\mathcal{D}^\mu\phi_-)^\dagger\mathcal{D}_\mu\phi_- + i(\psi_+\sigma^\mu\mathcal{D}_\mu\overline{\psi}_+$$
$$+\psi_-\sigma^\mu\mathcal{D}_\mu\overline{\psi}_-) + \frac{1}{2}ie\sqrt{2}(\phi_+^\dagger\lambda\psi_+ - \overline{\psi}_+\bar{\lambda}\phi_+) - \frac{1}{2}ie\sqrt{2}(\phi_-^\dagger\lambda\psi_- - \overline{\psi}_-\bar{\lambda}\phi_-)$$
$$-m(\psi_+\psi_- + \overline{\psi}_+\overline{\psi}_-)\ + V(\phi_+, \phi_-). \quad (3.145)$$

The potential $V(\phi_+, \phi_-)$ is given by

$$V(\phi_+, \phi_-) = m^2(\phi_+^\dagger\phi_+ + \phi_-^\dagger\phi_-) + \frac{1}{8}e^2(\phi_+^\dagger\phi_+ - \phi_-^\dagger\phi_-)^2. \quad (3.146)$$

3.3.2 Nonabelian gauge theory in supersymmetry

From the abelian theory, we move to the nonabelian gauge theory. We consider a nonabelian group G, with a coupling constant g, and we denote the generators of the group by T^a $(a = 1, \cdots, \dim G)$. The group generators T^a are hermitian and satisfy the commutation relation

$$[T^a, T^b] = C^{abc} T^c, \tag{3.147}$$

where C^{abc} is the structure constant and is antisymmetric.

In nonabelian gauge field, a vector superfield is $\mathcal{V}_i^j = \mathcal{V}^a (T^a)_i^j$. We shall consider $e^{\mathcal{V}}$ instead of $\mathcal{V}$. The generalized supersymmetry gauge transformation in (3.114) for $e^{\mathcal{V}}$ is:

$$e^{\mathcal{V}} \longrightarrow e^{\mathcal{V}'} = e^{-i\Lambda^\dagger} e^{\mathcal{V}} e^{i\Lambda}, \tag{3.148}$$

where $\Lambda_i^j = \Lambda^a (T^a)_i^j$ and the parameters Λ^a of the transformation are a set of chiral superfields. The relation in (3.148), expanded in first order is

$$e^{\mathcal{V}} \longrightarrow e^{\mathcal{V}'} = 1 + \mathcal{V} + i(\Lambda - \Lambda^\dagger) + \cdots . \tag{3.149}$$

Therefore, like in the $U(1)$ case, we use the Wess-Zumino gauge in which $\mathcal{V}^n = 0$, $\forall\, n \geq 3$. The generalization of the chiral superfields containing the Maxwell tensor is

$$\begin{aligned} W_\alpha &= -\frac{1}{4} \overline{\mathcal{D}}_{\dot\alpha} \overline{\mathcal{D}}^{\dot\alpha} e^{-\mathcal{V}} \mathcal{D}_\alpha e^{\mathcal{V}}, \\ \overline{W}^{\dot\alpha} &= \frac{1}{4} \mathcal{D}^\alpha \mathcal{D}_\alpha e^{\mathcal{V}} \overline{\mathcal{D}}^{\dot\alpha} e^{-\mathcal{V}}. \end{aligned} \tag{3.150}$$

However, these superfields are not invariant. They transform as it follows:

$$\begin{aligned} W_\alpha &\longrightarrow W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda}, \\ \overline{W}^{\dot\alpha} &\longrightarrow \overline{W}'^{\dot\alpha} = e^{-i\Lambda^\dagger} \overline{W}^{\dot\alpha} e^{i\Lambda^\dagger}, \end{aligned} \tag{3.151}$$

that is

$$\begin{aligned} W'_\alpha &= -\frac{1}{4} e^{-i\Lambda} \left(\overline{\mathcal{D}}_{\dot\alpha} \overline{\mathcal{D}}^{\dot\alpha} (e^{-\mathcal{V}} \mathcal{D}_\alpha e^{\mathcal{V}}) \right) e^{i\Lambda}, \\ \overline{W}'^{\dot\alpha} &= -\frac{1}{4} e^{-i\Lambda} \left(\mathcal{D}^\alpha \mathcal{D}_\alpha (e^{\mathcal{V}} \overline{\mathcal{D}}^{\dot\alpha} e^{-\mathcal{V}}) \right) e^{i\Lambda}. \end{aligned} \tag{3.152}$$

The superfields W_α and $\overline{W}^{\dot\alpha}$ are given explicitly in Wess-Zumino gauge by

$$\begin{aligned} W_\alpha &= -\frac{1}{4} \overline{\mathcal{D}}^2 \left((1 - \mathcal{V} + \frac{1}{2}\mathcal{V}^2) \mathcal{D}_\alpha (1 + \mathcal{V} + \frac{1}{2}\mathcal{V}^2) \right) \\ &= -\frac{1}{4} \overline{\mathcal{D}}^2 \mathcal{D}_\alpha \mathcal{V} + \frac{1}{8} \overline{\mathcal{D}}^2 [\mathcal{V}, \mathcal{D}_\alpha \mathcal{V}], \\ \overline{W}^{\dot\alpha} &= -\frac{1}{4} \mathcal{D}^2 \left((1 + \mathcal{V} + \frac{1}{2}\mathcal{V}^2) \overline{\mathcal{D}}^{\dot\alpha} (1 - \mathcal{V} + \frac{1}{2}\mathcal{V}^2) \right) \\ &= -\frac{1}{4} \mathcal{D}^2 \overline{\mathcal{D}}^{\dot\alpha} \mathcal{V} - \frac{1}{8} \mathcal{D}^2 [\mathcal{V}, \overline{\mathcal{D}}^{\dot\alpha} \mathcal{V}], \end{aligned} \tag{3.153}$$

where

$$[\mathcal{V}, \mathcal{D}_\alpha \mathcal{V}] = i\bar\theta^2 \sigma_\alpha^{\mu\nu\beta} \theta_\beta [A_\mu, A_\nu] + i\theta^2 \bar\theta^2 \sigma^\mu_{\alpha\dot\alpha} [A_\mu, \bar\lambda^{\dot\alpha}]. \tag{3.154}$$

By using the fact that $\overline{\mathcal{D}\mathcal{D}}\bar{\theta}\bar{\theta} = -4$, we end up with

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + \sigma_\alpha^{\mu\nu\beta}\theta_\beta(F_{\mu\nu} - \frac{i}{2}[A_\mu, A_\nu]) - \theta^2\sigma^\mu_{\alpha\dot{\alpha}}(\partial_\mu\bar{\lambda}^{\dot{\alpha}} + \frac{i}{2}[A_\mu, \bar{\lambda}^{\dot{\alpha}}]). \qquad (3.155)$$

The invariant kinetic term of the Lagrangian density is obtained by contracting the spinor indices and by using their trace as

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= \frac{1}{4}\int d^2\theta\ tr(W^\alpha W_\alpha) + \frac{1}{4}\int d^2\bar{\theta}\ tr(\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}) \\ &= -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}\bar{\lambda}^a i\gamma^\mu\mathcal{D}_\mu\lambda^a + \frac{1}{2}D^aD^a, \end{aligned} \qquad (3.156)$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - igC^{abc}A^b_\mu A^c_\nu \\ \mathcal{D}_\mu\bar{\lambda}^a &= \partial_\mu\bar{\lambda}^a + igC^{abc}A^b_\mu\lambda^c. \end{aligned} \qquad (3.157)$$

For matter, the gauge group generators T^a belong to a representation $\mathcal{R}$. The chiral superfield Φ_i, with $(i = 1, \cdots, dim\mathcal{R})$ transform as

$$\begin{aligned} \Phi_i &\longrightarrow \Phi'_i = (e^{-i\Lambda})^j_i\Phi_j, \\ \Phi^\dagger_i &\longrightarrow \Phi'^\dagger_i = \Phi'^\dagger_j(e^{i\Lambda^\dagger})^j_i. \end{aligned} \qquad (3.158)$$

Similarly, instead of using $\mathcal{V}$, we use $e^{\mathcal{V}}$ and the kinetic terms is given by

$$\mathcal{L}_{\text{kinetic}} = \int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{\mathcal{V}}\Phi. \qquad (3.159)$$

In this case, the gauge field is $A_\mu = A^a_\mu T^a$ and one introduces the coupling constant by redefining $\mathcal{V} \to g\mathcal{V}$. The Lagrangian density of the matter field is

$$\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\bar{\theta}\ \Phi^\dagger e^{2g\mathcal{V}}\Phi + \int d^2\theta\ W(\Phi) + \int d^2\bar{\theta}\ W(\Phi^\dagger). \qquad (3.160)$$

The Lagrangian describing the coupled gauge field to the chiral superfields is:

$$\begin{aligned} \mathcal{L} &= \frac{1}{4g^2}\left(\int d^2\theta\ tr(W^\alpha W_\alpha) + \int d^2\bar{\theta}\ tr(\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}})\right) \\ &\quad + \int d^2\theta d^2\bar{\theta}\ \Phi^\dagger e^{g\mathcal{V}}\Phi + \int d^2\theta\ W(\Phi) + \int d^2\bar{\theta}\ W(\Phi^\dagger). \end{aligned} \qquad (3.161)$$

To assure the consistency of the gauge symmetry, the superpotential $W(\Phi)$ shall be a gauge invariant. In terms of components, we have

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}\bar{\lambda}^a i\gamma^\mu\mathcal{D}_\mu\lambda^a + \frac{1}{2}D^aD^a + (\mathcal{D}^\mu\phi_i)^\dagger\mathcal{D}_\mu\phi_i \\ &\quad + \frac{i}{2}(\psi_i\sigma^\mu\mathcal{D}_\mu\overline{\psi_i} - \mathcal{D}_\mu\psi_i\sigma^\mu\overline{\psi_i}) + F^\dagger_iF_i + ig\sqrt{2}(\phi^\dagger_i(T^a)^{ij}\psi_j\lambda^a - \bar{\lambda}^a\overline{\psi_i}(T^a)^{ij}\phi_j) \\ &\quad + gD^a\phi^\dagger_i(T^a)^{ij}\phi_j + F_i\frac{\partial W}{\partial\phi_i} + F^\dagger_i\frac{\partial W^\dagger}{\partial\phi^\dagger_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} - \frac{1}{2}\overline{\psi_i\psi_j}\frac{\partial^2 W^\dagger}{\partial\phi^\dagger_i\phi^\dagger_j}, \end{aligned} \qquad (3.162)$$

where

$$\mathcal{D}\phi_i = \partial_\mu\phi_i + igA^a_\mu(T^a)^{ij}\phi_j,$$

$$\mathcal{D}\psi_i \;=\; \partial_\mu\psi_i + igA^a_\mu(T^a)^{ij}\psi_j, \tag{3.163}$$

see [23]. The equation of motion of the auxiliary fields are

$$F_i^\dagger = -\frac{\partial W}{\partial\phi_i}, \tag{3.164}$$

$$F_i = -\frac{\partial W^\dagger}{\partial\phi_i^\dagger}, \tag{3.165}$$

$$D^a = -g\phi_i^\dagger(T^a)^{ij}\psi_j. \tag{3.166}$$

Finally, we deduce after substituting the auxiliary fields into (3.162), the Lagrangian:

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}\bar{\lambda}^a i\gamma^\mu\mathcal{D}_\mu\lambda^a + (\mathcal{D}^\mu\phi_i)^\dagger\mathcal{D}_\mu\phi_i + \frac{i}{2}(\psi_i\sigma^\mu\mathcal{D}_\mu\overline{\psi_i} - \mathcal{D}_\mu\psi_i\sigma^\mu\overline{\psi_i}) \\ & + ig\sqrt{2}(\phi_i^\dagger(T^a)^{ij}\psi_j\lambda^a - \bar{\lambda}^a\overline{\psi_i}(T^a)^{ij}\phi_j) - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} \\ & - \frac{1}{2}\overline{\psi_i\psi_j}\frac{\partial^2 W^\dagger}{\partial\phi_i^\dagger\phi_j^\dagger} - V(\phi_i,\phi_i^\dagger),\end{aligned} \tag{3.167}$$

where

$$V(\phi_i,\phi_i^\dagger) = F_i^\dagger F_i + \frac{1}{2}D^aD^a = \sum_i \left|\frac{\partial W}{\partial\Phi_i}\right|^2_{\Phi_i=\phi_i} + \frac{1}{2}g^2\sum_a(\phi_i^\dagger(T^a)^{ij}\phi_j)^2. \tag{3.168}$$

3.3.3 The $\mathcal{N}=4$ Super Yang-Mills theory from dimensional reduction

We introduce in this subsection the four dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We impose the Majorana-Weyl condition which will be given to maintain the symmetry between the number of fermion and boson degrees of freedom in the theory coupled to fermion in ten-dimensions. We shall apply the method of dimensional reduction to the $\mathcal{N}=1$ super Yang-Mills theory to obtain the $\mathcal{N}=4$ theory. The four dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory is a very special quantum field theory. It is a theory in Minkowski space and comes from dimensional reduction of a Lagrangian in 10 dimensions. For $\mathcal{N}\leq 4$, there exists a gauge multiplet which transforms under the adjoint representation of the gauge group. The gauge multiplet is unique for $\mathcal{N}=4$. It is given by

$$(A_\mu, \lambda^a_\alpha, \phi^i), \tag{3.169}$$

where A_μ is the usual spin-1 gauge field, λ^a_α $(a=1,2,3,4)$ are Weyl spinors and are equivalent to two Dirac fermions. The fields ϕ^i are real scalars and are equivalent to three complex scalars, see [29]. The theory presents many interesting properties. One of these properties exhibited is the exact electric-magnetic duality. In other words, the theory is invariant under the interchange of electric and magnetic quantum numbers. The theory is also invariant under the replacement of a weak gauge coupling g with a strong gauge coupling $4\pi/g$.

We consider the $\mathcal{N}=1$ super Yang-Mills Lagrangian. It contains a Yang-Mills term and a fermion term and has the form

$$\mathcal{L}_{10} = tr(-\frac{1}{2}F_{MN}F^{MN} + i\bar{\Psi}\Gamma^M D_M\Psi), \tag{3.170}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M + ig_{YM}[A_M, A_N]$, Γ^M are the Dirac (gamma) matrices in 10 dimensions, see [29]. The spinor Ψ satisfies the Majorana and Weyl conditions, that is

$$\Psi = C_{10}\bar{\Psi}^T, \tag{3.171}$$

$$\Psi = \pm\Gamma_{11}\bar{\Psi}, \tag{3.172}$$

with C, the conjugation matrix operator which satisfies

$$C^{-1}\Gamma^M C = -(\Gamma^M)^T. \tag{3.173}$$

The spinor Ψ will be replaced by the gaugino λ with its eight degrees of freedom corresponding to those of the ten-dimensional gauge field A_M. In fact, the gaugino has eight degrees of freedom because the Dirac spinor in ten dimensions has 32 degrees of freedom which is reduced to 16 by imposing the Weyl condition, and thereafter imposing the Majorana condition on this Weyl spinor, we end up with eight degrees of freedom. The covariant derivative D_M is given by

$$D_M = \partial_M(\bullet) + ig_{YM}[A_M, \bullet]. \tag{3.174}$$

The constant g_{YM} is the gauge coupling. The action which corresponds to the Lagrangian in (3.170) is invariant under the supersymmetry transformations

$$\delta_\xi A_M = i\bar{\xi}\Gamma_M\Psi, \tag{3.175}$$

$$\delta_\xi\Psi = \Gamma_{MN}F^{MN}\xi. \tag{3.176}$$

The Lagrangian in (3.170) is supersymmetric. Its dimensional reduction to the Minkowski space gives the so-called $\mathcal{N}=4$ Super Yang-Mills Lagrangian. This dimensional reduction implies that the Lorentz group $SO(1,9)$ splits according to

$$SO(1,9) \longrightarrow SO(1,3)\otimes SO(6). \tag{3.177}$$

This $SO(6) \simeq SU(4)$, which is R-symmetry. The gauge field A_M reduces to A_μ, $\mu = 0,1,2,3$, transforming under $SO(1,3)$ symmetry. The reduced components of the gauge field A_i, $i = 1,2,3,4,5,6$, become six real scalars which we denoted ϕ^i. The Yang-Mills term in (3.170) decomposes as

$$\begin{aligned} -\frac{1}{2}F_{MN}F^{MN} &= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{ij}F^{ij} - F_{\mu i}F^{\mu i} \\ &= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g_{YM}^2[\phi^i,\phi^j]^2 - D_\mu\phi^i D^\mu\phi^i. \end{aligned} \tag{3.178}$$

To squeeze the second term in (3.170) into four dimensions, we choose the representation in which the Clifford algebra splits as follows:

$$\Gamma^\mu = \mathbf{1}_8\otimes\gamma^\mu, \quad \mu = 0,1,2,3, \tag{3.179}$$

$$\Gamma^i \approx \gamma^{pq} = \begin{pmatrix} 0 & \rho^{pq} \\ \rho_{pq} & 0 \end{pmatrix} \otimes \gamma_5 \quad p,q = 1,2,3,4, \tag{3.180}$$

where γ^μ are the ordinary 4×4 Dirac gamma matrices, the matrices γ^{pq} are antisymmetric in indices and ρ^{pq} are 4×4 matrices. The matrix elements of ρ^{pq} are defined by

$$(\rho^{pq})_{kl} = \delta_{pk}\delta_{ql} - \delta_{qk}\delta_{pl}, \tag{3.181}$$

$$(\rho^{pq})_{kl} = \frac{1}{2}\epsilon_{pqmn}(\rho^{mn})_{kl} = \epsilon_{pqkl}. \tag{3.182}$$

In our chosen representation, the chirality matrix Γ_{11} in terms of γ^5 is

$$\Gamma_{11} = \Gamma_0\Gamma_1\Gamma_2\cdots\Gamma_9 = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & \mathbf{1}_4 \end{pmatrix} \otimes \gamma_5, \tag{3.183}$$

and the charge conjugation operator is

$$C_{10} = C \otimes \begin{pmatrix} 0 & \mathbf{1}_4 \\ \mathbf{1}_4 & 0 \end{pmatrix}, \tag{3.184}$$

where C is the charge conjugation operator in four dimensions. The fermion terms are

$$\begin{aligned} i\bar{\lambda}\Gamma^M D_M\lambda &= \frac{1}{2}(i\bar{\lambda}\Gamma^\mu D_\mu\lambda + i\bar{\lambda}\Gamma^i D_i\lambda) \\ &= \frac{1}{2}(i\bar{\lambda}\Gamma^\mu D_\mu\lambda + i(ig_{YM})\bar{\lambda}\Gamma^i[\phi^i,\lambda]) \\ &= \frac{1}{2}(i\bar{\lambda}\Gamma^\mu D_\mu\lambda - g_{YM}\bar{\lambda}\Gamma^i[\phi^i,\lambda]). \end{aligned} \tag{3.185}$$

Thus the ten-dimensionally reduced Lagrangian of (3.170) is

$$\begin{aligned} \mathcal{L} &= tr(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g_{YM}^2[\phi^i,\phi^j]^2 - D_\mu\phi^i D^\mu\phi^i + \frac{1}{2}(i\bar{\lambda}\Gamma^\mu D_\mu\lambda \\ &\quad -g_{YM}\bar{\lambda}\Gamma^i[\phi^i,\lambda])). \end{aligned} \tag{3.186}$$

The supersymmetry transformation laws in (3.175) and (3.176) take the following forms after dimensional reduction

$$\delta_\xi A_\mu = -i\bar{\xi}\Gamma_\mu\lambda, \quad \delta_\xi\phi^i = -i\bar{\xi}\Gamma_i\lambda, \tag{3.187}$$

$$\delta_\xi\lambda = (\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu} + D_\mu\phi^j\Gamma^{\mu j} + \frac{i}{2}[\phi^i,\phi^j]\Gamma^{ij})\xi. \tag{3.188}$$

The general form of (3.186) including the coupling constant g_{YM} and the so-called instanton angle is given by

$$\begin{aligned} \mathcal{L} &= tr(-\frac{1}{2g_{YM}}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\sum_a \bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu\phi^i D_\mu\phi^i \\ &\quad +g_{YM}\sum_{i,a,b} C_i^{ab}\lambda_a[\phi^i,\lambda_b] + g_{YM}\sum_{i,a,b}\bar{C}_{iab}\bar{\lambda}^a[\phi^i,\bar{\lambda}^b] + \frac{g_{YM}^2}{2}\sum_{ij}[\phi^i,\phi^j]^2), \end{aligned} \tag{3.189}$$

where θ_I is the instanton angle, C_i^{ab} is the constant structure for $SU(4)$ R-symmetry and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{3.190}$$

is called the Poincaré dual.

Chapter 4

Nonabelian Yang-Mills Theories

Photons are the mediators in the interaction theory in electromagnetism under the gauge group $U(1)$. The strong and weak interactions are described by using nonabelian vectorial gauge fields. To understand what makes the nonabelian gauge field theory different from the abelian one, we study in this chapter the nonabelian Yang-Mills theory and review the memory effect in the nonabelian's case. Following the spirit of [23, 30, 31], we stay in the nonabelian gauge theory and present the field theory in $SU(2)$. Thereafter, we write down the Aharonov-Bohm phase factor. We finally calculate the memory in the $SU(2)$ gauge group.

4.1 The Gauge Group SU(2)

In this section, we work out the generators of the gauge group $SU(2)$. Thereafter, we write down the corresponding gauge field. Finally, we derive the Euler-Lagrange equations corresponding to the Lagrangian.

4.1.1 Isospin symmetry

The isospin is a quantity introduced by Heisenberg to describe multiplet of particles which have identical properties. In fact, in strong interaction, proton and neutron have approximatively identical mass. A proton-neutron doublet called nucleon writes:

$$\psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \tag{4.1}$$

where ψ_p and ψ_n are respectively the wave functions of proton and neutron. The isospin can be constructed as a vector $I = (I_1, I_2, I_3)$ in the abstract space of isospin. The operator I thus defined, satisfies exactly the same relations as the angular momentum commutation relations:

$$[I_i, I_j] = i\varepsilon_{ijk} I_k, \tag{4.2}$$

$$[I^2, I_3] = 0. \tag{4.3}$$

The relation (4.3) shows the existence of eigenvalues in such a way that for a state $|I, I_3\rangle$, we have

$$I^2|I, I_3\rangle = I(I+1)|I, I_3\rangle, \tag{4.4}$$

$$I_3|I, I_3\rangle = I_3|I, I_3\rangle, \tag{4.5}$$

thus a multiplet of isospin has $(2I+1)$ degenerate eigenvalues. Whence particles such as proton and neutron can be written by the values of I and I_3 as

$$|p\rangle = |I = \frac{1}{2}, I_3 = +\frac{1}{2}\rangle, \quad |n\rangle = |I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle. \tag{4.6}$$

The physics should be invariant like in any other theory. The global transformation is given by

$$\psi_N \longrightarrow \psi'_N = U\psi_N, \tag{4.7}$$

where U is a unitary 2×2 matrix of determinant unity. The matrix U is member of the Lie group $SU(2)$ and form a rotation group of isospin. The matrix writes:

$$U(\theta^a) = e^{-i\theta^a \frac{\sigma^a}{2}} = \cos\frac{|\theta|}{2}\mathbf{1} - i\sin\frac{|\theta|}{2}\frac{\theta^a}{|\theta|}\sigma^a, \tag{4.8}$$

where σ^a with a= 1,2,3 are Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.9}$$

and

$$\theta \equiv \sqrt{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2}. \tag{4.10}$$

The generators of the $SU(2)$ gauge group are $t^a = \frac{\sigma^a}{2}$. They satisfy the trace formula

$$tr(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{4.11}$$

They also satisfy the commutation relation

$$[t^a, t^b] = i\varepsilon^{abc}t^c, \tag{4.12}$$

and the anticommutation relation

$$\{t^a, t^b\} = \frac{1}{2}\delta^{ab}. \tag{4.13}$$

4.1.2 The gauge field of SU(2)

Yang and Mills have found the necessity to introduce a new isospin potential whose local gauge group is the symmetry group $SU(2)$. This is in analogy with the group $U(1)$ in electromagnetism. Therefore, it is important to recall local gauge invariance in electromagnetism. The transformation is

$$\psi \longrightarrow \psi' = e^{i\alpha(x)}\psi, \tag{4.14}$$

where $\alpha(x)$ is the parameter of the transformation. This invariance is assured by the gauge field A_μ and the covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu, \tag{4.15}$$

where e is the electron charge. In a similar way, the local invariance of isospin is assured by the following covariant derivative

$$D_\mu = \partial_\mu - igA^a_\mu t^a, \tag{4.16}$$

where g is the charge of isospin. The analogy of (4.17) in $SU(2)$ writes

$$\psi \longrightarrow \psi' = e^{i\theta^a(x)t^a}\psi. \tag{4.17}$$

Here, the matrice of transformation $U(\theta^a) = e^{i\theta^a(x)t^a}$ are time-dependent. The gauge field in agreement with the gauge transformation is given by

$$A^a_\mu t^a \longrightarrow A^a_\mu t^a = U(\theta)A^a_\mu t^a U(\theta)^{-1} - \frac{i}{g}[\partial_\mu U(\theta)]U(\theta)^{-1}. \tag{4.18}$$

One can make use of the Pauli matrices and define the $SU(2)$ gauge field as

$$A_\mu \equiv A^a_\mu t^a = \begin{pmatrix} \frac{A^3_\mu}{2} & \frac{A^1_\mu - iA^2_\mu}{2} \\ \frac{A^1_\mu + iA^2_\mu}{2} & -\frac{A^3_\mu}{2} \end{pmatrix}. \tag{4.19}$$

4.1.3 Yang-Mills theory

The Yang Mills field is constructed by defining a field strength tensor similar to the one used in general relativity and electromagnetism. This field strength tensor is defined as:

$$F^{\mu\nu} = F^{\mu\nu}_a t_a = -\frac{i}{g}[D^\mu, D^\nu]. \tag{4.20}$$

Here, the field strength tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu], \tag{4.21}$$

and transforms covariantly under a gauge transformation in a similar way as the gauge field $A_\mu = A^a_\mu t^a$ as:

$$F^{\mu\nu} \longrightarrow F'^{\mu\nu} = U^{-1}F^{\mu\nu}U. \tag{4.22}$$

The Yang-Mills Lagrangian which is invariant for the gauge field writes:

$$\mathcal{L} = -\frac{1}{2}tr(F_{\mu\nu}F^{\mu\nu}). \tag{4.23}$$

Normalizing the $SU(2)$ generator $t_a = \frac{\sigma_a}{2}$ as given in (4.11), we have

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{4.24}$$

One important problem raised by the theory is due to the mass. We know that the mass of proton and neutron are non-zero. However, the local gauge invariance requires that the mass of the gauge field should be null. The fundamental reason is that the mass in the potential part of the Lagrangian is in second order

$$m^2 A_\mu A^\mu. \tag{4.25}$$

Due to this term, the theory is not gauge invariant. As a result, this means that the mass in the gauge theory should vanish in order for the structure to be conserved. In addition, it leaves out the electromagnetic interaction which would make sure to break the symmetry and would restore the identity of each particle.

The later developments formulated an invariant Lagrangian under a gauge transformation given by

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{4.26}$$

The mass term in (4.25) is not allowed for A_μ because it breaks the gauge symmetry while the mass term in (4.26) is invariant. Its invariance is shown by

$$m\overline{\psi_i}\psi_i \longrightarrow m\overline{\psi'_i}\psi'_i = m\overline{\psi_j}(U^j_i)^\dagger(U^j_i)\psi_j = m\overline{\psi_i}\psi_i, \tag{4.27}$$

where we used

$$\psi_i \longrightarrow \psi'_i = U^j_i\psi_j \quad \text{and} \quad \overline{\psi_i} \longrightarrow \overline{\psi'_i} = \overline{\psi_j}(U^\dagger)^j_i. \tag{4.28}$$

The Lagrangian in (4.26) represents a spinor field interacting with a gauge field such as formulated by Yang-Mills theory. The first term in (4.26) represents the Dirac field Lagrangian and the second term term represents the gauge field formulated previously. It is important to mention that the mass term appears here, but this time, it is in first order and remains invariant. This Lagrangian so formulated leads directly to the fields equations of the gauge theory. The corresponding Euler-Lagrange equations are

$$\partial^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\nu A^\mu_a)}\right) = \frac{\partial\mathcal{L}}{\partial A^\mu_a}, \tag{4.29}$$

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\overline{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial\overline{\psi}}. \tag{4.30}$$

The equation of motion given by the Lagrangian in (4.26) is

$$\partial^\mu F^a_{\mu\nu} + gC^{abc}A^\mu_b F_{\mu\nu c} = g\overline{\psi}\gamma_\nu t^a\psi. \tag{4.31}$$

The above equation of motion can be written by using the covariant derivative. For this purpose, we consider the covariant derivative of the gauge field:

$$D^\mu A^\nu_a = \partial^\mu A^\nu_a - gC_{abc}A^\mu_a A^\nu_b, \tag{4.32}$$

and we make the replacement below in the adjoint representation

$$(T_a)_{bc} = -iC_{abc}, \tag{4.33}$$

where

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Making use of the infinitesimal transformation of A^a_μ given by

$$\delta A^a_\mu = -\frac{1}{g}[\partial_\mu\theta^a - igA^c_\mu(T^c)^{ab}] \equiv -\frac{1}{g}D_\mu\theta^a \tag{4.34}$$

and (4.32), we can write the infinitesimal transformation of $F^a_{\mu\nu}$ as

$$\begin{aligned}\delta F^a_{\mu\nu} &= \partial_\mu \delta A^a_\nu - \partial_\nu \delta A^a_\mu + gC^{abc}(\delta A^b_\mu A^c_\nu + A^b_\mu \delta A^c_\nu) \\ &= (D_\mu \delta A_\nu)^a - (D_\nu \delta A_\mu)^a. \end{aligned} \tag{4.35}$$

It follows the variation of the Lagrangian as given below

$$\delta \mathcal{L} = -g\overline{\psi}\gamma^\mu t^a \psi \delta A^a_\mu - \frac{1}{2}\delta F^a_{\mu\nu} F^{a\mu\nu}, \tag{4.36}$$

which leads to

$$D^\mu F^a_{\mu\nu} = g\overline{\psi}\gamma_\nu t^a \psi. \tag{4.37}$$

The relations (4.37) is the gauge field equation when this field is interacting with Dirac spinors.

The introduction of the $SU(2)$ symmetry group based on isospin invariance has generalized the phase invariance concept in quantum mechanics. This has contributed to construct a new interpretation of symmetries. Therefore, particle physicists have considered a new broader symmetry such as $SU(3)$ in quantum chromodynamics. For convenience we recall the definitions of the electric and magnetic fields as seen in the abelian gauge theory in the Chapter 2 as follows:

$$E_i := F_{0i}, \quad B_k := -\frac{1}{2}\epsilon_{kij}F^{ij}, \tag{4.38}$$

where $i, j, k = 1, 2, 3$. Using the definitions of E_i and B_k in (4.38), we replace $F_{i0} = -F_{0i}$ and F_{ij} respectively by F^a_{i0} and F^a_{ij}. The covariant nonabelian gauge field is written as $A^a_\mu = (A^a_0, -\mathbf{A}^a)$. It follows the electric and magnetic fields of $SU(3)$ as written bellow

$$E^a_i := F^a_{i0} = \partial_i A^a_0 - \partial_0 A^a_i + gC^{abc}A^b_i A^c_0, \tag{4.39}$$

$$B^{aj} := -\frac{1}{2}\epsilon^{jik}F^a_{ik} = -\frac{1}{2}\epsilon^{jik}(\partial_i A^a_k - \partial_k A^a_i + gC^{abc}A^b_i A^c_k). \tag{4.40}$$

In vectorial notation, the electric and magnetic color fields have the form

$$\mathbf{E}^a = -\nabla A^a_0 - \partial_t \mathbf{A}^a + gC^{abc}A^b_0 \mathbf{A}^c, \tag{4.41}$$

$$\mathbf{B}^a = \boldsymbol{\nabla} \times \mathbf{A}^a - \frac{1}{2}gC^{abc}(\mathbf{A}^b \times \mathbf{A}^c). \tag{4.42}$$

4.2 The Nonabelian Time-Dependent Aharonov-Bohm effect

We discussed the Aharonov-Bohm effect in the context of abelian gauge theory. For recall, the effect is a quantum-mechanical phenomenon in which the wave function of a charge particle travelling aroung an extremely long solenoid undergoes a phase shift depending on the magnetic field between the paths even if the magnetic field vanishes along the paths themselves. The effect can be observed in two ways: the type I Aharonov-Bohm effect and the type II Aharonov-Bohm effect. In fact, the type I Aharonov-Bohm effect is observed in the case when a particle is passing through a region in the space in the absence of electric and magnetic fields. The type II Aharonov-Bohm effect refers to the case where the particle is travelling a region in the presence of fields. The effect can be studied in both abelian and nonabelian gauge fields.

In this section, we are interesting in the study of the nonabelian Aharonov-Bohm effect with time-dependent magnetic field. Consider the discussion on the effect in Chapter 2. The abelian gauge group is $U(1)$. The corresponding gauge field writes $A_\mu = (A_0, -\mathbf{A})$ and transforms as

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \chi(x), \tag{4.43}$$

where χ is a real scalar function. The relativistic form of the Aharonov-Bohm effect writes

$$\beta = e^{S(x)}, \tag{4.44}$$

where $S(x)$ is a function defined by

$$S(x) = \frac{e}{\hbar} \oint A_\mu dx^\mu = \frac{e}{\hbar} \oint \left(A_0 dt - \mathbf{A} dx\right), \tag{4.45}$$

e is the charge of the particle and $\hbar$ the reduced Planck constant or Dirac constant. One can make use of the Stokes' theorem to rewrite the function $S(x)$ by introducing the field strength $F_{\mu\nu}$. It follows:

$$S(x) = -\frac{e}{2\hbar} \int F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{4.46}$$

Similarly in (4.43), the infinitesimal local gauge transformation of the nonabelian gauge field can be written in the form

$$A^a_\mu(x) \longrightarrow A'^a_\mu(x) = A^a_\mu(x) + C^{abc}\alpha^b(x) A^c_\mu - \frac{1}{g}\partial^\mu \alpha^a(x), \tag{4.47}$$

where g is the gauge coupling constant and α is an arbitrary real function. The generalized phase factor of the nonabelian Aharonov-Bohm effect is then given by

$$\beta = P e^{S(x)}, \tag{4.48}$$

where P is the path-ordering operator and A_μ is replaced by $A^a_\mu T^a$ in the function $S(x)$, see [32]. In the past five years, people have considerably worked on the time-dependent nonabelian Aharonov-Bohm effect. They found that the time-dependent part of the phase shift vanishes for the first order term in its Taylor expansion, see [31].

4.3 The Memory of SU(2)

The memory as reminder, is defined as a frozen in phase.This phase in the $SU(2)$ gauge group from the later developments has the form:

$$\beta = P\left[\exp\left[\frac{g}{\hbar}\oint A^a_\mu t^a dx^\mu\right]\right]. \tag{4.49}$$

We follow the method in [30] to compute the above phase factor. We make use of the Taylor expansion of (4.49) up to the second order as follows:

$$\beta \simeq 1 + \frac{g}{\hbar} t^a \oint A^a_\mu dx^\mu + P\left(\frac{g}{\hbar}\right)^2 \oint\oint A^a_\mu dx^\mu \cdot A^b_\nu dx^\nu t^a t^b + \cdots. \tag{4.50}$$

Now, we shall evaluate β by focusing our interest on the first order term named β' in (4.50)

$$\beta' = \frac{g}{\hbar} t^a \oint A^a_\mu dx^\mu. \tag{4.51}$$

Making use of the Stokes' theorem, we have

$$\beta' = \frac{g}{\hbar} t^a \int (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) dx^\mu \wedge dx^\nu. \tag{4.52}$$

From the definition of $F^a_{\mu\nu}$ given below

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc} A^b_\mu A^c_\nu, \tag{4.53}$$

we derive the relation

$$\partial_\mu A^a_\nu - \partial_\nu A^a_\mu = F^a_{\mu\nu} - gC^{abc} A^b_\mu A^c_\nu. \tag{4.54}$$

Plugging (4.54) into (4.52), one gets (4.51) in terms of the field strength $F^a_{\mu\nu}$ as given as follows:

$$\beta' = \frac{g}{2\hbar} t^a \int F^a_{\mu\nu} dx^\mu \wedge dx^\nu - \frac{g^2}{\hbar} C^{abc} t^a \int A^a_\mu A^b_\nu dx^\mu \wedge dx^\nu. \tag{4.55}$$

where

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \tag{4.56}$$

is the wedge product and the factor $\frac{1}{2}$ comes from the antisymmetric property of $F^a_{\mu\nu}$ and the wedge product $dx^\mu \wedge dx^\nu$. The field strength in terms of the electric and magnetic color fields writes:

$$F^a_{\mu\nu} = \begin{pmatrix} 0 & E^a_1 & E^a_2 & E^a_3 \\ -E^a_1 & 0 & -B^a_3 & B^a_2 \\ -E^a_2 & B^a_3 & 0 & -B^a_1 \\ E^a_3 & -B^a_2 & B^a_1 & 0 \end{pmatrix}. \tag{4.57}$$

This allows us to rewrite (4.55) as

$$\begin{aligned} \beta' &= \frac{g}{\hbar} t^a \int ((E^a_1 dx + E^a_2 dy + E^a_3 dz) \wedge dt + B^a_1 dy \wedge dz + B^a_2 dz \wedge dx + B^a_3 dx \wedge dy) \\ &\quad - \frac{g^2}{\hbar} C^{abc} t^a \int A^b_\mu A^c_\nu dx^\mu \wedge dx^\nu. \end{aligned} \tag{4.58}$$

One can split the above equation into three terms and obviously identifies the first term to the electric phase, the second term to the magnetic phase. The last term comes from the nonabelian term appearing in the field strength. Therefore, we have

$$\beta' = \beta_1 + \beta_2 + \beta_3, \tag{4.59}$$

where

$$\beta_1 = \frac{g}{\hbar} t^a \int (E^a_1 dx + E^a_2 dy + E^a_3 dz) \wedge dt \tag{4.60}$$

is the electric phase,

$$\beta_2 = \frac{g}{\hbar} t^a \int (B^a_1 dy \wedge dz + B^a_2 dz \wedge dx + B^a_3 dx \wedge dy) \tag{4.61}$$

is the magnetic phase, and

$$\beta_3 \quad = \quad -\frac{g^2}{\hbar} C^{abc} t^a \int A^b_\mu A^c_\nu dx^\mu \wedge dx^\nu \tag{4.62}$$

is the nonabelian term. Staying in the temporal gauge where $E_a = -\dot{A}_a$, the electric phase in (4.60) becomes

$$\beta_1 \quad = \quad -\frac{g}{\hbar} t^a \oint A^a \cdot dl. \tag{4.63}$$

The magnetic phase difference β_1 is given by:

$$\beta_2 \quad = \quad \frac{g}{\hbar} t^a \int B^a \cdot dS. \tag{4.64}$$

We now replace the definition of B^a in (4.42) into (4.64). Hence

$$\beta_2 \quad = \quad \frac{g}{\hbar} t^a \int (\mathbf{\nabla} \times \mathbf{A}^a + \frac{1}{2} g C^{abc} (\mathbf{A}^b \times \mathbf{A}^c)) \cdot dS. \tag{4.65}$$

Making use of the Stokes' theorem, we end up with

$$\beta_2 \quad = \quad \frac{g}{\hbar} t^a \oint A^a \cdot dl + \frac{g^2}{2\hbar} C^{abc} t^a \int (A^b \times A^c) \cdot dS. \tag{4.66}$$

Still in the temporal gauge, we use the antisymmetry property of the wedge product and β_3 gives:

$$\beta_3 \quad = \quad -\frac{g^2}{2\hbar} C^{abc} t^a \int (A^b \times A^c) \cdot dS. \tag{4.67}$$

Finally, we can see that the first order term of the memory vanishes. In [31], It has been conjectured that all the higher order term of the gauge field also vanish.

Chapter 5

Conclusion

In this dissertation, we have reviewed the quantum electrodynamics, the supersymmetry and the nonabelian Yang-Mills theory. Staying in quantum electrodynamics, we have shown the construction of the Lagrangian density of the theory. We have also discussed the memory effect in classical electromagnetism and we have found that this memory can be shifted by performing a gauge transformation where the resulting phase factor is a 1×1 unitary matrix. To understand the memory in non-confining theory, we have extended our study to the structure of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, which is a nonabelian gauge theory. In this framework, we have shown the construction of the Lagrangian density of the theory by using dimensional reduction of the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. Moreover, we have investigated the simplest action that is consistent when one includes a fermionic variable in the theory. As happens in electrodynamics, we have looked at symmetries under which the system is invariant and we have derived the conserved charge (supercharge).

Finally, we have reviewed the gauge theory of the $SU(2)$ and the Yang-Mills theory in nonabelian gauge. By using the nonabelian gauge field, we have written down the electric and magnetic fields. This has led to the equation of motion in terms of the nonabelian gauge field strength. we have repeated the construction of the memory effect from QED in $SU(2)$. This has led to the memory in the $\mathcal{N} = 4$ super-Yang-Mills theory since the group generators do not affect the calculation.

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