Partial Differential Equations in Applied Mathematics 11 (2024) 100749

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

Investigating invariance and conservation laws for the classes of nonlinear parabolic and wave systems

A. Raza [a], A.H. Kara [b,*]

[a] Department of Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, Pakistan
[b] School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa

ARTICLE INFO

Keywords:
Symmetries
Conservation laws
Partial differential equations

ABSTRACT

We study various classes of the nonlinear dynamics of some high order parabolic equations like the Oskolkov–Benjamin–Bona–Mahony–Burgers and Benjamin–Bona–Mahony–Peregrine–Burger equations that arise in the study of some wave phenomena. Also, a broader class of partial differential equations are used in modelling ocean waves that originate from the Ostrovsky equation. We study the invariance properties via the Lie invariance method for the nonlinear systems and further establish classes of conservation laws for models arises in this study. We show how the relationship leads to double reductions of the systems. This relationship is determined by a recent result involving multipliers that lead to a total divergence or closed form of the differential equation under investigation.

1. Introduction

In our analysis, we consider two broad classes of models that have been and are of interest.

I. A large class of equations in mathematical physics are particular cases of the partial differential equation (PDE), viz., the Benjamin–Bona–Mahony–Peregrine–Burgers (BBMPB),[1] are given by

$$u_t + \beta u_{xxx} + \gamma u_x + \theta u u_x = u_{xxt} + \alpha u_{xx}. \tag{1.1}$$

PDE (1.1) exhibits a connection between dispersive and nonlinear effects and incorporates dissipation and the problem arises in, both, bore propagation and the water waves. Furthermore, it describes long waves on water surface in a channel with small-amplitude. A special case is a version of the KdV equation[1,2]

$$u_t + u_{xxt} + u_x + \theta u u_x = 0. \tag{1.2}$$

If $\beta = 0$ and $\alpha = \beta = 0$, we obtain the standard BBM (Benjamin–Bona–Mahoney) and the Oskolkov–Benjamin–Bona–Mahony–Burgers (OBBMB) equations, respectively. With specific choices of the parameters, other cases that arise are the Oskolkov equation describing incompressible viscoelastic Kelvin–Voigt fluid[3] and Camassa–Holm equation which represents a bi-Hamiltonian model for shallow water waves.[4] An extension of this is given by

$$u_t - u_{xxt} + \alpha u_x + 3\theta u^2 u_x - \gamma u_x u_{xx} + 2\beta u u_x - u u_{xxx} = 0, \tag{1.3}$$

leading to, with appropriate choices of parameters, the Degasperis-Procesi[5] and Fornberg-Whitham[6] equations, inter alia.

The above equations may be construed as models for shallow water phenomena and their long term behaviour is similar to that of the well known the Camassa–Holm equation and the KdV equation.

II. Ostrovsky's model for wave phenomena is known for special forms involving 'nonlinear surface and internal waves'.[7–10] In case for rotating ocean, the PDE is given by

$$(q u_{xxx} + p u u_x + c_0 u_x + u_t)_x - \gamma u = 0, \tag{1.4}$$

where c_0 represents the velocity of linear dispersion, γ represents the Coriolis dispersion, which measure the effect of rotation, q represents Boussinesq dispersion and p represents the coefficient of nonlinearity. By choosing suitable parameters, Vakhnenko equation $(u u_x + u_t)_x + u = 0$ transforms to equation, demonstrated in,[11] given by

$$u u_{xxt} + u^2 u_t = u_x u_{xt}. \tag{1.5}$$

Lie invariance method deals with the symmetries admitted by PDE that forms the one parameter Lie groups of transformations. Symmetries obtained then provides invariant solutions which are also called exact invariant solutions, well cited and demonstrated in Refs. 12, 13. Furthermore, the discussion of conservation laws is important for studying physical properties of PDEs such as momentum and energy and for studying others properties of PDEs that predict integrability. Lastly, new conservation laws are constructed from known conservation laws using the symmetry invariance of a given system of PDEs.[14,15]

* Corresponding author.
E-mail address: abdul.kara@wits.ac.za (A.H. Kara).

https://doi.org/10.1016/j.padiff.2024.100749
Received 18 April 2024; Received in revised form 26 May 2024; Accepted 3 June 2024
Available online 6 June 2024
2666-8181/© 2024 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

In a system of PDEs of n independent variables x and dependent variables u, for $\mu = 1, \dots, \tilde{m}$

$$G^{\mu}(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{1.6}$$

where $u_{(1)}, u_{(2)}, \dots, u_{(r)}$ represent respectively all first, second and higher order partial derivatives. A vector T of functions $(T^1, \dots, T^n)$ is called conserved vector if it holds the divergence expression given by

$$D_i T^i = 0 \tag{1.7}$$

along the solutions of PDE (1.6). Conservation law computed from multipliers $Q_\mu(x, u, u_{(1)}, \dots)$ holds the identity given by

$$Q_\mu G^\mu = D_i T^i. \tag{1.8}$$

Conserved vectors are constructed by first finding the multipliers Q_μ which are obtained by annihilating the total divergence using the Euler–Lagrange operator, $\frac{\delta}{\delta u^\alpha}$ represented as $\frac{\delta}{\delta u^\alpha}[Q_\mu G^\mu] = 0$. defined for Q_μ. Conserved flow Φ then can be obtained using a well known 'homotopy' formula.

Furthermore, if $X = \xi\partial_x + \tau\partial_t + \eta\partial_u$ is a Lie point symmetry that leaves a scalar PDE in (1.6), say,

$$G(x, t, u, u_x, u_t, u_{xx} \dots) = 0, \tag{1.9}$$

invariant with

$$XG = RG \tag{1.10}$$

and

$$\lambda = D_t\tau + D_x\xi \tag{1.11}$$

such that

$$XQ + (R + \lambda)Q = 0, \tag{1.12}$$

then X is associated with the corresponding conserved flow $\Phi = (T, \phi)$ and, via X and Φ, double reduction may be obtained.[16,17] Here, $D_t T + D_x\phi = 0|_{(1.9)}$. Moreover, in transformed coordinates (ζ, ρ, U), $\bar{X} = \partial_\rho$ so that $X(\rho) = 1$, $\bar{\Phi} = (\bar{T}, \bar{\phi})$ and $D_\zeta\bar{T} + D_\rho\bar{\phi} = 0$. This procedure leads to a double reduction of the original system since

$$k = \bar{T}|_{\zeta, U, U', \dots}, \tag{1.13}$$

where k is a constant — for details, see Ref. 17.

2. Symmetry generators and conservation laws

In this section, symmetries and conservation laws are studied and their relationship is presented.

2.1. Parabolic equations

2.1.1. BBMBP equations

With $\theta = 1$, (1.2) becomes

$$u_t + u_{xxt} + u_x + uu_x = 0 \tag{2.1}$$

which admits a three dimensional algebra of point symmetry generated by

$$\partial_t, \qquad \partial_x, \qquad t\partial_t - (1 + u)\partial_u \tag{2.2}$$

so that travelling solutions, which have been studied, are generated by $X_1 = \partial_t + c\partial_x$. We obtain two multipliers $Q_1 = 1$ and $Q_2 = u$. For X_1, $\lambda_1 = R_1 = 0$ so that (1.12) is satisfied for Q_1 and Q_2. For the scaling symmetry $X_2 = t\partial_t - (1 + u)\partial_u$, $\lambda_2 = 1$ and $R_2 = 0$ so that (1.12) is not satisfied. In this case, towards obtaining a scaling invariant solution, we will reduce (2.2) in the usual way.

We, firstly, perform two double reductions of (2.2). Note that

$$\frac{\partial T}{\partial t} + \frac{\partial\phi}{\partial x} = \frac{\partial T}{\partial\zeta}\frac{\partial\zeta}{\partial t} + \frac{\partial T}{\partial\rho}\frac{\partial\rho}{\partial t} + \frac{\partial\phi}{\partial\zeta}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial\rho}\frac{\partial\rho}{\partial x}. \tag{2.3}$$

For X_1, $\zeta = x - ct$ (c being the speed of the travelling wave) and $U(\zeta) = u$. Thus,

$$\bar{T} = -cT + \phi, \qquad \bar{\phi} = \frac{1}{c}\phi. \tag{2.4}$$

The multiplier Q_1 and Q_2 leads to the conserved flows

$$\begin{aligned} T_1 &= \tfrac{1}{3}u_{xx} + u,\\ \phi_1 &= \tfrac{2}{3}u_{xt} + \tfrac{1}{2}u^2 + u,\\ T_2 &= \tfrac{1}{3}uu_{xx} - \tfrac{1}{6}u_x^2 + \tfrac{1}{2}u^2,\\ \phi_2 &= \tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 - \tfrac{2}{3}uu_{xt} - \tfrac{1}{3}u_xu_t \end{aligned} \tag{2.5}$$

Double reductions based on these can only be performed with X_1 which from (2.4)(a), respectively,

$$\begin{aligned} k_1 &= U'' + \tfrac{c-1}{c}U - \tfrac{1}{2c}U^2,\\ k_2 &= -cUU'' + \tfrac{c}{2}U'^2 + \tfrac{1}{2}(1-c)U^2 + \tfrac{1}{3}U^3. \end{aligned} \tag{2.6}$$

By letting $U = V$ and $U' = W$, these reduce to first-order odes. In (2.6)(a), if $k_1 = 0$, the solution of the ode is

$$3cW^2 = V^3 + 3(1-c)V^2 + k. \tag{2.7}$$

With $k_2 = 0$ in (2.6)(b), we obtain a solution

$$W = -\frac{1-c}{3c}V^{\frac{5}{2}} - \frac{2}{15c}V^{\frac{7}{2}} + k. \tag{2.8}$$

Even though there is no association of a conservation law with X_2, a direct reduction is possible; we obtain invariants $\zeta = x$ and $U = t(1+u)$ so that another reduction leading to an exact solution of (2.2) is obtained by

$$U'' + UU' - U = 0. \tag{2.9}$$

By letting $U = V$ and $U' = W$, (2.9) reduces to a ODE with solution

$$-2W - \ln(1-W)^2 = V^2 + k. \tag{2.10}$$

2.1.2. Benjamin–Bona–Mahony–Peregrine-Burgers (BBMPB)

We now seek (double) reductions of (1.1) with a similar procedure. The Lie point symmetry algebra is generated by

$$\partial_t, \quad \partial_x, \quad (\text{for} \quad \alpha = 0) \quad X_2 = -\beta t\partial_x + t\partial_x - (\frac{\beta+\gamma}{\theta} + u)\partial_u \tag{2.11}$$

Travelling solutions are obtained by $X_1 = \partial_t + c\partial_x$. We obtain two multipliers $Q_1 = 1$ and $Q_2 = u$. For X_1, $\lambda_1 = R_1 = 0$ so that (1.12) is satisfied for Q_1 and Q_2. For the symmetry X_2, $\lambda_2 = 1$ and $R_2 = 0$ so that (1.12) is not satisfied. The multiplier Q_1 and Q_2 leads to the conserved flows

$$\begin{aligned} T_1 &= \tfrac{1}{3}u_{xx} - u,\\ \phi_1 &= \tfrac{2}{3}u_{xt} - \beta u_{xx} + \alpha u_x - \tfrac{\theta}{2}u^2 - \gamma u,\\ T_2 &= -\tfrac{1}{3}uu_{xx} + \tfrac{1}{6}u_x^2 + \tfrac{1}{2}u^2,\\ \phi_2 &= \tfrac{\theta}{3}u^3 + \tfrac{\gamma}{2}u^2 + \beta uu_{xx} - \tfrac{\beta}{2}u_x^2 - \tfrac{2}{3}uu_{xt} + \tfrac{1}{3}u_xu_t. \end{aligned} \tag{2.12}$$

For X_1, $\zeta = x - ct$ and $U(\zeta) = u$, double reductions can be performed with X_1 which from (2.4)(a), respectively, is

$$\begin{aligned} k_1 &= -(\beta + c)U'' + (c-\gamma)U + \alpha U' - \tfrac{\theta}{2}U^2,\\ k_2 &= (\beta + c)UU'' - \tfrac{1}{2}(c+\beta)U'^2 + \tfrac{1}{2}(1+\gamma)U^2 + \tfrac{\theta}{2}U^3. \end{aligned} \tag{2.13}$$

Using X_2 with invariants $\zeta = x + \beta t$ and $U = t(u + \frac{\beta+\gamma}{\theta})$, a direct reduction of (1.1) for $\alpha = 0$ is

$$U'' + \theta UU' - U = 0. \tag{2.14}$$

The three second-order odes are all easily reducible to first-order odes.

2.2. II. Nonlinear Ostrovsky equations

For Eq. (1.5), the symmetry generators are represented by $Y_1 = \partial_x$, $Y_2 = F(t)\partial_t$, $Y_3 = x\partial_x - 2u\partial_u$. For reduction, we consider $Y = \partial_t + c\partial_x$ and $Z = x\partial_x + \mu t\partial_t - 2u\partial_u$ and the multiplier $Q = 1$ yield the conserved flow

$$\begin{aligned} T &= \tfrac{1}{3}u^3 + \tfrac{7}{12}uu_{xx} - \tfrac{5}{12}u_x^2 \\ \phi &= \tfrac{5}{12}uu_{xt} - \tfrac{7}{12}u_x u_t. \end{aligned} \tag{2.15}$$

The condition for association (1.12) is satisfied for Y but not for scaling Z. In the first case, we use the conserved flow (2.15) to obtain the reduced equation

$$k = UU'' + \tfrac{1}{3}U^3 - U'^2. \tag{2.16}$$

Via Z, a direct reduction of (1.5) is obtainable with invariants $\zeta = tx^{\mu}$ and $t^{\frac{2}{\mu}}U = u$. All the multipliers including $Q = 1$ and the corresponding flows are represented in the form of conserved vectors.

a. $Q_1 = \frac{1}{6}u^{-4}(3u^2u_t a'(t) + a(t)(6u^2u_{tt} + 3u_{xt}{}^2))$:

For $a(t) = 1$, we get the density

$$\begin{aligned} T_1 \;&= \int_0^1 \tfrac{1}{12\lambda u^4}(3\lambda u^3(4tu_{xt}{}^2 + u_x(u_{xt} + 2tu_{xtt}) + u_t u_{xx} + 2tu_{tt}u_{xx} \\ &+6tu_t u_{xxt}) - 15tu_x{}^2u_{xt}{}^2 \\ &+ tuu_{xt}(14u_x u_{xxt} + 5u_{xt}u_{xx}) + 4\lambda u^4(3t\lambda u_t{}^2 - u_{xxt} - 2tu_{xxtt}) \\ &-u^2(3\lambda u_t u_x(u_x + 6tu_{xt}) \\ &+ 2t(3\lambda u_{tt}u_x{}^2 + u_{xxt}{}^2 + u_{xt}u_{xxxt})))d\lambda \end{aligned}$$

b. $Q_2 = \frac{1}{u^4}(6u^3u_{xx} - 3u^2u_x^2 + 4u^5)$:

$$\begin{aligned} \phi_2 \;&= [-3u_t u_x{}^3 - 3uu_x{}^2u_{xt} + u^2(6u_{xt}u_{xx} + 6u_x u_{xxt} - 4u_t u_{xxx}) \\ &-4u^3(2u_t u_x + u_{xxxt})]/4u^2, \\ T_2 \;&= u^4 + \tfrac{3u_x{}^4}{4u^2} + 4u^2u_{xx} - \tfrac{9u_x{}^2u_{xx}}{4u} + \tfrac{1}{2}(3u_{xx}{}^2 - u_x u_{xxx}) + u(u_{xxxx} - u_x{}^2) \end{aligned}$$

c. $Q_3 = \frac{1}{u^2}$:

$$\phi_3 = \int_0^1 \frac{u_t u_x - uu_{xt}}{2\lambda u^2}\,d\lambda, \qquad T_3 = \int_0^1 \frac{2\lambda u^3 - u_x{}^2 + uu_{xx}}{2\lambda u^2}\,d\lambda$$

The conserved density corresponding to multiplier $Q_* = u^{-4}(6ku^4 + 3mu^2)$ is given by

$$T_* = \int_0^1 (6k\lambda^2u^3 - 5k\lambda u_x{}^2 - \frac{3mu_x{}^2}{2\lambda u^2} + \frac{3mu_{xx}}{2\lambda u} + u(3m + 7k\lambda u_{xx}))\,d\lambda$$

The invariance analysis of the conserved vectors presented above can be followed by the action of the symmetry on the multipliers. Firstly, we can show that

$$\begin{aligned} &Y_1 Q_i = 0, && i = 1,2,3,* \\ &Y_2 Q_j = 0, && j = 2,3,* \\ &Y_2 Q_1 = F(t)\partial_t Q_1 = \bar{Q}_1. \end{aligned}$$

Therefore, the conserved vectors in above list are strictly invariant under symmetry Y_1 where (b), (c) and combined one are invariant strictly under symmetry Y_2. The conserved vectors in (a), (b) and (c) are ray invariant under the symmetry generator Y_3; the calculations demonstrated that

$$Y_3 Q_1 = 2Q_1, \quad Y_3 Q_2 = -2Q_2, \quad Y_3 Q_3 = 4Q_3.$$

In general, the ray invariance condition corresponding to Y_3 is represented by

$$Y_3 Q_i = (\lambda + 5)Q_i, \quad i = 1,2,3,$$

so that $\lambda = -3, -7, -1$, respectively.

Table 1
Table of adjoint representation.

$Ad(\exp(\epsilon Y_i))Y_j$	Y_1	Y_2	Y_3
Y_1	Y_1	Y_2	Y_3
Y_2	Y_1	Y_2	$Y_3 - \epsilon Y_3$
Y_3	Y_1	$Y_2 e^{\epsilon}$	Y_3

3. Optimal system of subalgebras admitted by parabolic equations and reduction

In this section, we now construct optimal system admitted by (BBMPB) equations. The adjoint action[18,19] is given by

$$Ad(\exp(\epsilon Y_i))(Y_j) = Y_j - \epsilon[Y_i, Y_j] + \frac{\epsilon^2}{2!}[Y_i,[Y_i,Y_j]] - \cdots, \tag{3.1}$$

where ϵ is real and the Lie product $[Y_i, Y_j]$ is defined by

$$[Y_i, Y_j] = Y_iY_j - Y_jY_i. \tag{3.2}$$

3.1. BBMBP equations

First we construct the optimal system for BBMBP equation discussed in Section 2 given by

$$u_t + u_{xxt} + u_x + uu_x = 0,$$

which admits a three dimensional algebra of point symmetry generated by

$$Y_1 = \partial_t, \qquad Y_2 = \partial_x, \qquad Y_3 = t\partial_t - (1+u)\partial_u, \tag{3.3}$$

with nonzero commutator $[Y_2, Y_3] = Y_2$. The adjoint representation corresponding to above algebra is presented in Table 1 that help us in constructing the optimal system of subalgebra for Eq. (2.1).

Consider $Y \in \mathcal{L}_3$, we have

$$Y = a_1Y_1 + a_2Y_2 + a_3Y_3. \tag{3.4}$$

Now executing the adjoint operator (3.1) on Y and using the adjoint Table 1, we obtain an optimal system which is as follows. Non-similar symmetry generators are given by

$$\begin{aligned} X^1 = Ad(e^{aY_2})Y &= Y_1 + Y_3 \quad &;\quad a_3 \neq 0, a_1 \neq 0 \\ X^2 = Ad(e^{aY_2})Y &= Y_3 \quad &;\quad a_3 \neq 0, a_1 = 0 \\ X^3 = Ad(e^{aY_3})Y &= c_1Y_1 \pm Y_2 \quad &;\quad a_3 = 0, a_2 \neq 0 \\ X^4 = Ad(e^{aY_3})Y &= Y_1 \quad &;\quad a_3 = 0, a_2 = 0 \end{aligned}$$

We have received four elements X^1, X^2, X^3 and X^4 in optimal set of non-equivalent symmetry generators admitted by Eq. (2.1). Reduction of X^2 and X^4 has already performed in Section 2. At the conclusion of this section, reduction under optimal system is performed using the direct method. To follow the reduction of new symmetries obtained in optimal system, first consider the symmetry generator $X^1 = Y_1 + Y_3$ has invariant solution $u = F(x)$ which is the invariant solution of (2.1) and satisfies the ODE $F'(x) = 0$. Invariant solution corresponding to symmetry generator $X^3 = cY_1 + Y_2$ holds the form

$u(x,t) = F(-cx + t)$, where $\alpha = -cx + t$

and satisfy the ODE

$$F'(\alpha) + F'''(\alpha)c^2 - F'(\alpha)c - F(\alpha)F'(\alpha)c = 0 \tag{3.5}$$

3.2. Benjamin–Bona–Mahony–Peregrine-Burgers (BBMPB)

The Lie point symmetry algebra is generated by

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad (\text{for} \quad \alpha = 0) \quad Y_3 = -\beta t\partial_x + t\partial_t - (\frac{\beta+\gamma}{\theta} + u)\partial_u$$

Table 2
Table of adjoint representation.

$Ad(\exp(\epsilon Y_i))Y_j$	Y_1	Y_2	Y_3
Y_1	Y_1	Y_2	Y_3
Y_2	Y_1	Y_2	$Y_3 - \epsilon(Y_2 - \beta Y_1)$
Y_3	Y_1	$e^{\epsilon} Y_2 + (1 - e^{\epsilon})\beta Y_1$	Y_3

with the only nonzero commutator $[Y_2, Y_3] = Y_2 - \beta Y_1$. The adjoint representation corresponding to above algebra is presented in Table 2 that help us in constructing the optimal system for Eq. (2.11).

Consider $Y \in \mathcal{L}_3$, we have

$$Y = a_1 Y_1 + a_2 Y_2 + a_3 Y_3. \tag{3.6}$$

Now executing the adjoint operator (3.1) on Y and using the adjoint Table 1, we obtain an optimal system which is as follows. Non-similar symmetry generators are given by

$$X^1 = Ad(e^{aY_2})Y = (1+\beta)Y_1 + Y_3 \quad ; \quad a_3 \neq 0, a_1 \neq 0$$
$$X^2 = Ad(e^{aY_2})Y = Y_3 \quad ; \quad a_3 \neq 0, a_1 = 0$$
$$X^3 = Ad(e^{aY_3})Y = Y_2 \quad ; \quad a_3 = 0, a_2 \neq 0$$
$$X^4 = Ad(e^{aY_3})Y = Y_1 \quad ; \quad a_3 = 0, a_2 = 0$$

For Benjamin–Bona–Mahony–Peregrine–Burgers, we again have received four symmetries X^1, X^2, X^3 and X^4 in optimal system which are non-equivalent. Reduction under optimal system for the symmetry generator

$$X^1 = (1+\beta)Y_1 + Y_3,$$

has invariant solution of the form

$$u = F\left(\ln(\beta + 1 + t)\beta^2 - \beta t + \ln(\beta + 1 + t)\beta - x\right),$$

and satisfies the ODEs

$$-\left(F'''(\alpha)\beta + \alpha F''(\alpha) + F'(\alpha)(\theta F(\alpha) + \gamma)\right)(1+\beta) = 0, \tag{3.7}$$

$$-\alpha F''(\alpha) - F'(\alpha)(\theta F(\alpha) + \beta + \gamma) = 0. \tag{3.8}$$

The corresponding invariant variables are given by

$\alpha = \ln(\beta + 1 + t)\beta^2 - \beta t + \ln(\beta + 1 + t)\beta - x$ and $u = F(\alpha)$.

4. Conclusion

Complex nonlinear dynamical systems are usually dealt with very well by invariance methods in conjunction with conservation laws. This approach has not been previously use, to the best of our knowledge, on the classes of equations studied here. We have shown how a study of the relationship between symmetries and conservation laws are attained by the notion of multipliers; once established, we can obtain double reduction of the original equations which is a double step in determining solutions. A large of class of parabolic equations were studied in this way, viz., the Oskolkov–Benjamin–Bona–Mahony–Burgers equations, Benjamin–Bona–Mahony–Peregrine–Burger equation and the Ostrovsky equation.

The following classes may also be studied, in a similar way, viz., Calogero–Bogoyavlenskii–Schiff equation,[20] magneto-electro-elastic equations,[21] higher-dimensional KP equations.[22]

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

1. Peregrine DH. Calculations of the development of an undular bore. *J Fluid Mech*. 1996;25(2). 321-300.
2. Benjamin TB, Bona JL, Mahony JJ. Model equations for long waves in nonlinear dispersive systems. *Philos Trans R Soc Lond: Ser A, Math Phys Sci*. 1972;272:47–78.
3. Oskolkov AP. Nonlocal problems for one class of nonlinear operator equations that arise in the theory of Sobolev type equations. *J Sov Math*. 1993;64(1):724–736.
4. Johnson RS, Camassa-Holm. Korteweg–de Vries and related models for water waves. *J Fluid Mech*. 2002;455:63–82.
5. Wazwaz AM. New solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa–Holm equations. *Appl Math Comput*. 2007;186(1):130–141.
6. Whitham GB. Variational methods and applications to water waves. *Proc R Soc A: Math, Phys Eng Sci*. 1967;299(1456):6–25.
7. Ostrovsky L. Nonlinear internal waves in a rotating ocean. *Oceanology*. 1978;18:119–125.
8. Ostrovsky L. Evolution equations for strongly nonlinear waves. *Phys Fluids*. 2003;15(10):2934–2948.
9. Karöglu C, Özis T. A novel travelling wave solution for Ostrovsky equation using Exp-function method. *Comput Math Appl*. 2009;58:2142–2146.
10. Gao A, Tian L. The global attractor of the viscous damped forced Ostrovsky equation. *Nonlinear Anal: Real World Appl*. 2009;10:2894–2904.
11. Gillman OA, Grimshaw R, Stepanyants Yu A. Approximate analytical and numerical solutions of the stationary Ostrovsky equation. *Stud Appl Math*. 1995;95:115–126.
12. Ibragimov NH. *Transformation Groups Applied to Mathematical Physics*. Dordrecht: Reidel Publishing. Co. 1985.
13. Olver P. *Application of Lie Groups to Differential Equations*. New York: Springer; 1993.
14. Kara AH, Mahomed FM. Relationship between symmetries and conservation laws. *Int J Theoret Phys*. 2000;39(1):23–40.
15. Kara AH, Mahomed FM. A basis of Conservation Laws for partial differential equations. *J Nonlinear Math Phys*. 2002;9:60–72.
16. Anco SC, Kara AH. Symmetry-invariant conservation laws of partial differential equations. *European J Appl Math*. 2018;29(1):78–117.
17. Anco SC, Gandarias ML. Symmetry multi-reduction method for partial differential equations with conservation laws. *Commun Nonlinear Sci Numer Simul*. 2020;91:105349.
18. Olver PJ. *Application of Lie Groups to Differential Equations*. New York, NY, USA: Springer; 1986.
19. Ibragimov NH. *Selected Works*. Karlskrona, Sweden: ALGA Publications; 2006 Volume II; ISBN 9172959916.
20. Jarads F, Jhangeer A, Awrejcewicz J, Bilal Riaz M, Junaid-U-Rehman M. Investigation of wave solutions and conservation laws of generalized Calogero–Bogoyavlenskii–Schiff equation by group theoretic method. *Results Phys*. 2022;37:105479.
21. Khater MMA, Jhangeer A, Rezazadeh H, Akinyemi L, Ali Akbar M, Inc M. Propagation of new dynamics of longitudinal bud equation among a magneto-electro-elastic round rod. *Modern Phys Lett B*. 2021;35:2150381.
22. Rafiq MH, Jhangeer A, Raza N. Symmetry and complexity: A Lie symmetry approach to bifurcation, chaos, stability and travelling wave solutions of the (3+1)-dimensional Kadomtsev–Petviashvili equation. *Phys Scripta*. 2023;98:115239.