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Application of lie group analysis and computational methods to the analysis of the flow of a thin non-Newtonian fluid

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Application of Lie Group Analysis and Computational Methods to the Analysis of the Flow of a Thin non-Newtonian Fluid

A THESIS

submitted by

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0413613J

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ABSTRACT

This thesis is primarily concerned with the application of Lie group analysis and numerical methods to the analysis of the flow of a thin non-Newtonian fluid. In fact due to the increasing importance of non-Newtonian fluids in industry and technological applications during the last decades, researchers have written many papers and developed methods to solve the equations resulting from the modelling of the flow of such fluids. It is worth noting here that these equations are highly nonlinear due to the nonlinear dependence of the fluid's viscosity on the velocity gradient, adding more complexity/nonlinearity to the nonlinear Navier-Stokes equations.

We show the importance of combining Lie group analysis and computational methods to describe the flow of a thin non-Newtonian fluid. Lie group analysis provides a systematic way, when well handled, to find exact solutions to certain classes of nonlinear differential equations. When it is impossible to obtain exact solutions, we can therefore make use of approximate methods and numerical schemes.

For the case of the axisymmetric spreading of a power-fluid over a horizontal plane, we use the method of separation of variables combined with the linearization criterion given by Lie to find new exact solutions. We also extend the study of Newtonian fluids to power-law fluids by applying the Lie group method. We determine group-invariant solutions that generalize those of Newtonian fluids and take into account the effects of shear-thinning and shear-thickening. Finally the homotopy analysis method is applied to solve the flow of a generalized second-grade fluid on a moving belt.

DECLARATION

I declare that this thesis is my own unaided work unless otherwise acknowledged. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination to any other institution.

S. N. Neossi Nguetchue

07 March 2009.

DEDICATION

To my Lord and Saviour Jesus-Christ, and to my mother Tchandje Fride.

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Chapter 1

Introduction

1.1 Introduction

Non-Newtonian fluids are characterized by a nonlinear relationship between the shear stress and the shear rate of the flow. These types of fluids are encountered in nature and industrial technologies. The flow analysis of these fluids is very important in theory and practice. From a theoretical point of view, flows of this type are fundamental in fluid mechanics. Practically speaking, these flows have applications in many manufacturing processes in industry. Among them we can cite volcanic lava, mudflow, oil, plastics, oil-based paints, polymer solutions, melts, foodstuffs like tomato sauce and biological fluids like blood. Non-Newtonian fluids exhibit various behaviours: time-independent behaviours (Bingham-plastic, pseudo-plastic and dilatant fluids), time-dependent behaviours (thixotropic and rheopectic fluids) and visco-plastic fluids (e.g egg white). Non-Newtonian fluids are also intensively studied by mathematicians mainly involving the analysis of the resultant differential equations.

In the applied sciences such as rheology or physics of the atmosphere, the approach to fluid mechanics is in an experimental set up leading to the measurement of material coefficients. Due to the great diversity in the physical structure of non-Newtonian fluids, it is not possible to establish a single constitutive equation. Thus, many non-Newtonian fluid models have been proposed of which most are

empirical or semi-empirical.

The non-Newtonian model which is widely used by engineers is the power-law model because it has been found to be successful in describing the behaviour of colloids, suspensions and a variety of polymeric liquids and low molecular weight biological liquids, in the field of glaciology, blood-rheology and geology. The power-law model has limitations. Myers, (2005) compared three non-Newtonian models namely power-law, Carreau and Ellis models. For thin film flow with a constant height free surface, comparisons showed that at low shear rates, the power-law model gave very inaccurate predictions. Comparisons for flow in a channel showed that the power-law model gave inaccurate results due to the high viscosity around the turning point for the velocity. Therefore, when applying the power-law model, great care should be exercised, particularly when the shear rate is low. This model differs from a Newtonian fluid only in that its viscosity is proportional to a power of the symmetric part of the velocity gradient so that, in a simple shear flow, the viscosity depends on the shear rate. The dominant departure of a power-law fluid from the Newtonian behaviour is shear-thinning or shear thickening.

Arguably, the simplest model of non-Newtonian fluids that can predict the normal stress differences is the Rivlin-Ericksen fluid of grade two known as a second-grade fluid (Rivlin and Ericksen, 1955) which is a particular case of fluids of differential type (Dunn and Rajagopal, 1995). A model that combines second-grade and power-law fluids was investigated by Man *et al.*, (1985) and Man, (1992). Gupta and Massoudi, (1993) completed the Man *et al.*, (1995) model by suggesting that the shear viscosity can also depend on temperature.

Flow of liquid film on an inclined plane has drawn the attention of studies in the last five decades due to its various applications in technological development of modern science. These films are susceptible to different types of instabilities due to the influence of various physical factors (e.g. gravity, viscosity, mean surface

tension, Hamaker constant, thermocapillarity). The interfacial waves propagating along the plane show fascinating nonlinear phenomena, such as, solitary waves, transverse secondary instabilities and complex disordered patterns. The earliest studies (Benjamin, 1957 and Yih, 1963) based on linear stability of long waves on a layer of viscous Newtonian fluid flowing down an inclined plane give the critical Reynolds number and show that a vertical uniform film is always unstable, the instability manifesting itself as a gravity-driven surface wave with a wave length much larger than the mean thickness of the film. Using the momentum integral method, Kapitza, (1948), Shkadov, (1967, 1968) and Krylov *et al.*, (1969) performed the analysis to predict the dependencies of the growth rate and the phase velocity of the wave. They predicted the critical Reynolds number and the characteristic of the fastest growing waves. Benny, (1966) applied the lubrication approximation to the Navier-Stokes equations and derived a nonlinear evolution equation for the two dimensional flow of a Newtonian fluid down an inclined plane and then performed a finite amplitude stability analysis. Such finite amplitude stability analysis of the two dimensional flow of a Newtonian fluid down an inclined plane was also performed by Lin, (1969), Gjevik, (1970) and Nakaya, (1975). Benney-type equations are much simpler to study than the full dynamic equations and are often used to study the nonlinear behaviour of film flows.

When the modelling part of a fluid flow or spreading is completed, the next step is to find the solutions that describe the behaviour of the fluid. These solutions could be exact or approximate according to the complexity of the problem. The equations of motion for non-Newtonian fluids are much more complicated than those for Newtonian fluids because they are highly nonlinear due to the complexity of the non-Newtonian rheology. In the case of a thin liquid film for example, when applying the kinematic boundary condition, the resulting equation describing the variation of the film height is of fourth-order when taking into account the effect of surface tension, besides the complexity of the non-Newtonian rheology. Analytic solutions of these equations are not easily obtainable, if at all.

The search for exact analytical solutions of partial differential equations has always been the need of many scientists and engineers. Lie group analysis via Lie symmetry techniques is one of the most systematic ways of finding exact solutions of some partial differential equations, and that is the reason why nowadays, it has gained popularity. Most of the equations governing real life problems are nonlinear, and it is very difficult and sometimes even impossible to find the exact solution of a given nonlinear partial differential equation. One way to circumvent this difficulty is to use numerical or approximate solutions. For the latter case, conventional methods rely on the assumption of infinitesimally small parameters of linearity. The main drawback of linearization is that the resulting solutions may be physically unrealistic.

Various analytical approaches/methods have been proposed by a number of researchers to solve equations arising from the flow of non-Newtonian fluids. For example, the homotopy perturbation method was proposed by He, (1998). The presence of a small parameter in the equation describing the problem is assumed and has been used by Siddiqui *et al.*, (2007) to express the analytical solution of a third-grade fluid and of an Oldroyd 6-constant fluid. Another more generalized method called the homotopy analysis method was proposed earlier by Liao, (1992) without any parameter restriction. The homotopy analysis method is more successful due to the presence of an auxiliary parameter which can be adjusted to provide a convergent analytical solution. See for example Hayat *et al.*, (2004), Sajid *et al.* (2007) and the references cited therein for the application of the homotopy analysis method to non-Newtonian fluids.

Combining Lie group analysis, approximation methods and numerical analysis is a fascinating way of solving real life problems. This approach is applied in our work when solving the partial differential equations arising from the spreading of non-Newtonian thin fluids.

1.2 Objectives and outline of the thesis

In the thesis, an attempt has been made to analyse the dynamics of a non-Newtonian power-law thin film on a horizontal and inclined plane. To this effect we used the combination of Lie group analysis techniques for the systematic search of exact solutions, and numerical methods for the numerical solutions.

In Chapter 2, an outline is given of that part of the theory of group analysis of differential and partial differential equations which is needed for the search of exact solutions. We give some definitions of non-Newtonian fluids and we also recall some numerical methods used through our study.

In Chapter 3, we derive and give in dimensionless form the Navier-Stokes equations of a non-Newtonian fluid and the boundary conditions that will allow us to derive the evolution equations of the height profile in the subsequent chapters.

In Chapter 4, we study the dynamics of a thin power-law fluid. The chapter is divided into two main parts. In the first part, we study the numerical analysis of a thin power-law fluid down a vertical plane and in the second part, we study the spreading of a non soluble surfactant over the surface of the fluid. This latter study has many industrial and biological applications. For instance, the transport of surfactants on thin viscous films and the resulting film deformations are of concern in the treatment of respiratory distress syndrome (RDS), in which the lungs of prematurely born infants are not sufficiently developed to produce sufficient quantities of surfactant in order to reduce the surface tension of the liquid lining of the lungs. In surfactant replacement therapy (SRT), surfactant is instilled into the trachea of a patient who has surfactant-deficient lungs and the surfactant is transported in the large airways primarily by gravity and pressure (Halpern *et al.*, 1998; Espinosa and Kamm, 1999). As the surfactant layer thins to a monolayer, Marangoni flows become the dominant mode of transportation (Halpern *et al.*, 1998; Espinosa and Kamm,

1999). Surface tension driven flows are also important in the clearance of liquid and surfactant from healthy lungs (Davis *et al.*, 1974). We also refer to the work of Jensen and Grotberg, (1992), Matar and Troian, (1999), Matar and Craster, (2001), Dussaud *et al.*, (2005) for their contribution to surfactant spreading.

In Chapter 5, the problem of the axisymmetric spreading under gravity of a thin power-law liquid drop on a horizontal plane is investigated. We use a Lie group analysis approach in the first part of the chapter this leads us to determine the complete set of all the possible symmetries admitted by the governing equations for various values of the associated power-law parameter. Using a linear combination of these symmetries, we study the existence of group-invariant solutions corresponding to different values of the power-law parameter. In the second part of the chapter, we investigate the existence of separable solutions of the equation modelling the axisymmetric spreading under gravity of a thin power-law liquid drop on a horizontal plane. We then apply the linearization criteria to the resulting ordinary differential equation to determine exact solutions for some values of the power-law parameter.

In Chapter 6, we consider the flow of a thin generalized second-grade fluid on a moving belt. A modified model of a second-grade fluid that has shear-dependent viscosity and can predict the normal stress difference is used. The nonlinear equations governing the flow problems are formulated and analyzed and we apply the homotopy analysis method (HAM) to determine explicit analytic solutions. Finally, conclusions are summarised in Chapter 7. A section highlighting possible future work is also presented.

Chapter 2

Mathematical Preliminaries

2.1 Introduction

In this chapter, we present the mathematical tools from the theory of Lie group analysis of differential and partial differential equations that are needed to solve the nonlinear equations which are derived in this thesis. We also describe non-Newtonian fluids by their constitutive equations. Finally, we discuss some numerical methods used in this thesis such as the Newton method for a system of nonlinear equations, and we give an algorithm for its implementation.

2.2 Lie point symmetries

A symmetry of a differential equation is an invertible transformation of the dependent and independent variables that maps the equation to itself. Among symmetries of differential equations, those depending continuously on a small parameter form a local one parameter group of transformation (symmetry group) and they can be computed algorithmically by a procedure due to Sophus Lie. Another useful property of symmetries is that they map solutions to solutions. For partial differential equations, symmetries allow the reduction of the number of independent variables, and they are principally used for obtaining group invariant solutions i.e. solutions that are unaffected under the action of a subgroup of the symmetry group.

We will solve the partial differential equations by deriving their group invariant solutions. The partial differential equations that are derived are second- and fourth-order and contain one dependent variable and two independent variables. To obtain the group invariant solution of a given partial differential equation, its Lie point symmetry generators have first to be derived.

For simplicity and without loss of generality, we consider the second-order partial differential equation of the form

$$F(t, x, h, h_t, h_x, h_{tt}, h_{tx}, h_{xx}) = 0, \tag{2.1}$$

in the two independent variables (t, x).
An infinitesimal transformation

$$\begin{aligned} \overline{t} &\approx t + a\ \xi^1(t, x, h) + \cdots, \\ \overline{x} &\approx x + a\ \xi^2(t, x, h) + \cdots, \\ \overline{h} &\approx h + a\ \eta(t, x, h) + \cdots, \end{aligned}$$

where a is a small parameter, represents a Lie-point symmetry of Eq. (2.1) if

$$X^{[2]} F\big|_{F=0} = 0, \tag{2.2}$$

where

$$X^{[2]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial h} + \zeta_1 \frac{\partial}{\partial h_t} + \zeta_2 \frac{\partial}{\partial h_x} + \zeta_{11} \frac{\partial}{\partial h_{tt}} + \zeta_{12} \frac{\partial}{\partial h_{tx}} + \zeta_{22} \frac{\partial}{\partial h_{xx}}, \tag{2.3}$$

$$\zeta_i = D_i(\eta) - h_k D_i(\xi^k), \quad i = 1, 2 \tag{2.4}$$

$$\zeta_{ij} = D_j(\zeta_i) - h_{ik} D_j(\xi^k), \quad i = 1, 2, \quad j = 1, 2 \tag{2.5}$$

and the repeated index k is summed over the values 1 and 2. The operators D_1 and D_2 are the total differentiation operators with respect to t and x, respectively

$$D_1 = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{xt} \frac{\partial}{\partial h_x} + \cdots, \tag{2.6}$$

$$D_2 = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{tx} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + \cdots. \tag{2.7}$$

The operator defined by

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial h}, \tag{2.8}$$

is called the Lie-point symmetry generator of the partial differential equation (2.1) if Eq. (2.2) is satisfied and this equation, called the determining equation, allows us to determine the coefficients of Eq. (2.8). The operator defined by Eq. (2.3) is called the second prolongation of the Lie-point symmetry generator (2.8). The determining equation is generally separated according to the derivatives of h, and the result should contain constants and possibly arbitrary functions. By setting all the constants and arbitrary functions equal to zero except one, in turn, the Lie-point symmetry generators are obtained.

The group invariant solution for h is obtained by considering a linear combination of the Lie-point symmetry generators

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n, \tag{2.9}$$

where $c_1, c_2, \cdots, c_n$ are constants. The general form of the group invariant solution, $h = \Phi(t, x)$, is obtained by solving the first-order quasi-linear partial differential equation

$$X^{[2]}(h - \Phi(t, x))\big|_{h=\Phi(t,x)} = 0. \tag{2.10}$$

If the group invariant solution is substituted back into the partial differential equation, the partial differential equation will reduce to an ordinary differential equation. The constants in the linear combination (2.9) are generally obtained from the boundary conditions and from auxiliary conditions such as the conservation of certain quantities.

2.3 Conservation laws

In this section, we recall the mathematical tools that will allow us to determine the conservation law of a given partial differential equation and the connection between this

conservation law and a Lie-point symmetry of that same partial differential equation. For more details, the interested reader is referred to Kara and Mahomed, (2000).

$$D_i T^i = 0 \tag{2.11}$$

is a conservation law for Eq. (2.1) if Eq. (2.11) is satisfied for all solutions $h(t,x)$ of Eq. (2.1). In Eq. (2.11), the repeated index i is summed for $i = 1, 2$ and D_i is the operator of total differentiation with respect to x^i defined by Eqs. (2.6) and (2.7), and where $(x^1, x^2) = (t, x)$. The quantities $T^i(x, u, u_{(1)})$, where $u_{(1)}$ denotes the collection of all the first-order derivatives, are the components of the conserved vector $T = (T^i)$.

The following result connects a Lie point symmetry of a differential equation with a conservation law for the differential equation. The two components T^i, $i = 1, 2$ of the conserved vector for the differential equation (2.1), associated with the symmetry X defined by Eq. (2.8), satisfy the two equations

$$X^{[1]}(T^i) + T^i D_k \xi^k - T^k D_k \xi^i = 0, \quad i = 1, 2, \tag{2.12}$$

where X^1 is the first prolongation of X and the repeated index k is the sum for $k = 1, 2$.

2.4 Linearization process

The linearization process consists of writing an ordinary differential equation in terms of a polynomial of its derivatives with the coefficients satisfying a certain compatibility condition. The question is, under which conditions can one linearize a second-order ordinary differential equation via an invertible transformation. In the case of scalar second-order ordinary differential equations, it is well-known that any scalar linear second-order ordinary differential equation is locally equivalent to $y'' = 0$.

A complete solution to the algebraic criteria for linearization for scalar second-order ordinary differential equations was obtained by Lie, (1883). We quote this result in the following theorem:

Theorem 1 *The following statements are equivalent:*

(i) The second-order ordinary differential equation

$$y'' = f(x, y, y') \tag{2.13}$$

is linearizable by a point change of variables

$$t = t(x, y), \quad u = u(x, y); \quad \frac{\partial(t, u)}{\partial(x, y)} \equiv t_x u_y - t_y u_x \neq 0; \tag{2.14}$$

(ii) equation (2.13) admits an 8-dimensional Lie group;

(iii) equation (2.13) has the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0, \tag{2.15}$$

with coefficients F_3, F_2, F_1, F satisfying the compatibility conditions of the auxiliary system

$$\begin{aligned}
\frac{\partial z}{\partial x} &= z^2 - Fw - F_1 z + \frac{\partial F}{\partial y} + FF_2, \\
\frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y}, \\
\frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_2}{\partial x}, \\
\frac{\partial w}{\partial y} &= -w^2 - F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3;
\end{aligned} \tag{2.16}$$

(iv) equation (2.13) admits a two-dimensional Lie algebra with basis
$X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}$, $X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}$, *such that*

$$X_1 \vee X_2 = \xi_1 \eta_2 - \eta_1 \xi_2 = 0. \tag{2.17}$$

Recently Mahomed, (2007) in a survey paper proposed a theorem similar to the above theorem. He states an easier invariant condition compared to the compatibility conditions of the auxiliary system given by Eq. (2.16).

2.5 Non-Newtonian fluids

Newton's law of viscosity states that the shear stress is proportional to the velocity gradient. Fluids that obey Newton's law of viscosity are called Newtonian fluids. Most Newtonian fluids are fluids with simple molecular structures such as water, ethyl alcohol, benzene, hexane. On the other hand there are many categories of fluids that violate Newton's law of viscosity; these fluids are called non-Newtonian fluids. Non-Newtonian fluids are fluids with complex molecular structures, particularly with long-chain molecules. Most of the fluids used in industry are non-Newtonian fluids of complex mixtures such as slurries, pastes, gels, polymer solutions, etc. If we denote by $\mathbf{T}$ the Cauchy stress tensor of a given fluid, it can be written in general as

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{S}, \tag{2.18}$$

where p denotes the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{I}$ is the identity tensor and $\mathbf{S}$ is the extra stress tensor. We recall that for a non-Newtonian fluid, the extra stress tensor is related to the viscosity of the fluid, which is a nonlinear function of the symmetric part of the velocity gradient.

In the following subsections, we will describe the constitutive relations of some non-Newtonian fluids, particularly those that will be used in this work.

2.5.1 Power-law fluids

For a non-Newtonian power-law fluid, the extra stress tensor is given by (see Bird *et al.*, 1977, Wang *et al.*, 2005)

$$\mathbf{S} \;=\; \mu\,[tr(\mathbf{A}_1^2)]^m \mathbf{A}_1, \tag{2.19}$$

where

$$\mathbf{A}_1 \;=\; (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^T, \tag{2.20}$$

is the first Rivlin-Ericksen tensor,

$$\mathbf{V} = (u(x,y,t), v(x,y,t), 0)^T, \tag{2.21}$$

denotes the flow velocity, μ is the apparent coefficient of viscosity (with dimension $[ML^{-1}T^{2m-1}]$) and m is the dimensionless power-law parameter.

For the sake of methodology and its importance for the rest of this thesis, we give the explicit form of the extra stress tensor and the stress tensor in three-dimensional matrix form. The derivations are indeed two-dimensional and can be carried out as such, but we choose the three-dimensional representation due to the two forms of the velocity vector $\mathbf{V}$ given by Eqs. (2.21) and (2.28).

$$\mathbf{A}_1 = \begin{bmatrix} 2u_x & u_y + v_x & 0 \\ u_y + v_x & 2v_y & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{A}_1^2 = \begin{bmatrix} 4u_x^2 + (u_y + v_x)^2 & 0 & 0 \\ 0 & (u_y + v_x)^2 + 4v_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.22}$$

$$\mathrm{tr}(\mathbf{A}_1^2) = 4(u_x^2 + v_y^2) + 2(u_y + v_x)^2, \tag{2.23}$$

$$\mathbf{S} = \begin{bmatrix} 2\mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m u_x & \mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m (u_y + v_x) & 0 \\ \mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m (u_y + v_x) & 2\mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m v_y & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.24}$$

$$\mathbf{T} = \begin{bmatrix} -p + 2\mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m u_x & \mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m (u_y + v_x) & 0 \\ \mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m (u_y + v_x) & -p + 2\mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m v_y & 0 \\ 0 & 0 & -p \end{bmatrix}. \tag{2.25}$$

We conclude from (2.24) that

$$\begin{aligned} &S_{xx} = 2\mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m u_x, \quad S_{xy} = \mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m (u_y + v_x), \\ &S_{yx} = \mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m (u_y + v_x), \quad S_{yy} = 2\mu\ [\mathrm{tr}(\mathbf{A}_1^2)]^m v_y. \end{aligned} \tag{2.26}$$

We mention here that there is another power-law model commonly used in the literature called the Ostwald-de-Waele power-law model and its nonlinear shear-dependent viscosity η is given by (in keeping with the notation used by Miladinova *et al.*, 2004)

$$\eta = \mu_m |\dot{\gamma}|^{n-1}, \tag{2.27}$$

where $\dot{\gamma}$ is the second invariant of the rate of strain tensor, μ_m and n are constants for the particular liquid: μ_m (with units of PaS^n) is a measure of the consistency of the liquid; the larger μ_m, the more 'viscous' the liquid; n (dimensionless) is a measure of the degree of non-Newtonian behaviour, and the greater its departure from unity, the more pronounced are the non-Newtonian properties of the liquid. When $n < 1$ the model describes shear-thinning behaviour, whereas $n > 1$ represents shear-thickening behaviour. For clarity and in order to avoid confusion, the link between the two power-law models dealt with in this thesis is given in Table 2.1 below.

	Generalized Power-Law	Ostwald-de Waele Model
Stress Tensor	$\mathbf{T} = -p\mathbf{I} + \mu[\mathrm{tr}(\mathbf{A}_1^2)]^m\mathbf{A}_1$	$\mathbf{T} = -p\mathbf{I} + \mu_m\lvert\dot{\gamma}\rvert^{n-1}\mathbf{A}_1$
Apparent viscosity	$\eta = \mu[\mathrm{tr}(\mathbf{A}_1^2)]^m$	$\eta = \mu_m\lvert\dot{\gamma}\rvert^{n-1}$
Relationship	$\mu = \mu_m/2^{(n-1)/2},\ m = (n-1)/2$	$\lvert\dot{\gamma}\rvert = \left[\frac{1}{2}\mathrm{tr}(\mathbf{A}_1^2)\right]^{1/2}$

Table 2.1: Table of comparison of two power-law models.

In the case of the Ostwald model, we use the following form of the velocity vector in the Cartesian coordinates x, z, where the coordinates are chosen with x as the streamwise coordinate and z being measured normal to the xy-plane

$$\mathbf{V} = (u(x,z,t), 0, w(x,z,t))^T, \tag{2.28}$$

and the explicit forms of the extra stress tensor and the stress tensor in matrix form are

$$\mathbf{A}_1 = \begin{bmatrix} 2u_x & 0 & u_z + w_x \\ 0 & 0 & 0 \\ u_z + w_x & 0 & 2w_z \end{bmatrix}; \quad \mathbf{A}_1^2 = \begin{bmatrix} 4u_x^2 + (u_z + w_x)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (u_z + w_x)^2 + 4w_z^2 \end{bmatrix}, \tag{2.29}$$

$$\mathrm{tr}(\mathbf{A}_1^2) = 4(u_x^2 + w_z^2) + 2(u_z + w_x)^2, \tag{2.30}$$

$$\mathbf{S} = \eta A_1 = \begin{bmatrix} 2\mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} u_x & 0 & \mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} (u_z + w_x) \\ 0 & 0 & 0 \\ \mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} (u_z + w_x) & 0 & 2\mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} w_z \end{bmatrix}, \tag{2.31}$$

$$\mathbf{T} = \begin{bmatrix} -p + 2\mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} u_x & 0 & \mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} (u_z + w_x) \\ 0 & -p & 0 \\ \mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} (u_z + w_x) & 0 & -p + 2\mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} w_z \end{bmatrix}. \tag{2.32}$$

We conclude from (2.31) that

$$\begin{aligned} &S_{xx} = 2\mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} u_x, \quad S_{xz} = \mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} (u_z + w_x), \\ &S_{zx} = \mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} (u_z + w_x), \quad S_{zz} = 2\mu_m \left[\mathrm{tr}(\mathbf{A}_1^2)\right]^{\frac{n-1}{2}} w_z. \end{aligned} \tag{2.33}$$

Note that in the above computations, and for simplicity, we have used the fact that the fluid under consideration is incompressible.

2.5.2 Fluids of differential types: case of second-grade fluids

For a non-Newtonian second-grade fluid, namely the Rivlin-Ericksen fluid of grade two, the Cauchy stress tensor is related to the motion in the following manner (see Rivlin and Ericksen, 1955)

$$\mathbf{T} = -p\,\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.34}$$

where μ is the coefficient of viscosity and α_1 and α_2 are material moduli, also referred to as normal stress moduli. The kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through Eq. (2.20) and

$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1, \tag{2.35}$$

where

$$\mathbf{L} = \text{grad}\mathbf{V} \quad \text{and} \quad \mathbf{L}^T = (\text{grad}\mathbf{V})^T. \tag{2.36}$$

The thermodynamics and stability of fluids of second-grade have been studied in detail by Dunn and Fosdick, (1994). They show that if the fluid is to be thermodynamically consistent in the sense that all motions of the fluid meet the Clausius-Duhem inequality (an interpretation of the second law of thermodynamics) and that the specific Helmholtz free energy of the fluid be a minimum in equilibrium, then

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{2.37}$$

It is important to note that the second-grade model can capture the normal stress effects (which are manifestations of the stresses that develop orthogonally to planes of shear) but it does not exhibit shear-thinning nor shear-thickening.

2.5.3 Generalized second-grade fluids

In an effort to obtain a model that exhibits both normal stress effects and shear-thinning/thickening, Man and Sun, (1987), and Man, (1992) modified the constitutive equation for a second-grade fluid by allowing the viscosity coefficient to depend upon the rate of deformation. One of the two models proposed is

$$\mathbf{T} = -p\,\mathbf{I} + \mu\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.38}$$

where

$$\Pi = \frac{1}{2}\text{tr}(\mathbf{A}_1^2)$$

is the second invariant of the symmetric part of the velocity gradient, and m is a material parameter (power-law parameter). When $m < 0$, the fluid is shear-thinning,

and when $m > 0$, the fluid is shear-thickening. This constitutive equation is thus capable of predicting shear-thinning and shear-thickening phenomena. A subclass of models given by Eq. (2.38) is the (generalized) power-law model, which can be obtained from Eq. (2.38) by setting $\alpha_1 = \alpha_2 = 0$, i.e.

$$\mathbf{T} = -p\,\mathbf{I} + \mu_0\,[\mathrm{tr}(\mathbf{A}_1^2)]^m \mathbf{A}_1.$$

Gupta and Massoudi, (1991, 1993) generalized the model given by Eq. (2.38), by allowing the (shear) viscosity to be a function of temperature, i.e.

$$\mathbf{T} = -p\,\mathbf{I} + \mu(\theta)\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{2.39}$$

The latter model has applications in many fluids such as lubricants, polymers, and coal slurries where viscous dissipation is substantial. In this case an appropriate form for $\mu(\theta)$ should be used.

2.6 The homotopy analysis method (HAM)

The basic idea of the method is discussed in this section. For more detail we refer the interested reader to Liao, (1992, 1995, 1997, 1999, 2003, 2004, 2005).
Suppose we are given a differential equation of the form

$$N[u(\tau)] = 0, \tag{2.40}$$

where N is a nonlinear operator, τ denotes an independent variable, $u(\tau)$ is an unknown function. For simplicity, we ignore all initial/boundary conditions, which can be treated similarly. By means of generalizing the traditional homotopy method, Liao, (1992, 1995, 1997, 1999, 2003, 2004, 2005) constructed the so-called zeroth-order deformation equation as

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = p\hbar H(\tau)N[\phi(\tau;p)], \tag{2.41}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial

guess of $u(\tau)$ and $\phi(\tau, p)$ is an unknown function. Notice that

$$\phi(\tau; 0) = u_0(\tau) \text{ and } \phi(\tau; 1) = u(\tau).$$

Thus as p increases from 0 to 1, the solution $\phi(\tau; p)$ varies or is deformed from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau; p)$ in a Taylor series with respect to p, we have

$$\phi(\tau; p) = u_0(\tau) + \sum_{k=1}^{+\infty} u_k(\tau) p^k, \tag{2.42}$$

where

$$u_k(\tau) = \frac{1}{k!} \frac{\partial^k \phi(\tau; p)}{\partial p^k}\bigg|_{p=0}. \tag{2.43}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series given by Eq. (2.42) converges at $p = 1$, and we have

$$u(\tau) = u_0(\tau) + \sum_{k=1}^{+\infty} u_k(\tau), \tag{2.44}$$

which must be one of the solutions of the original nonlinear equation. Following Liao, (1992, 1995, 1997, 1999, 2003, 2004, 2005), letting $\hbar = -1$ and $H(\tau) = 1$, Eq. (2.41) reduces to the following equation used in the homotopy perturbation method (HPM)

$$(1 - p)L[\phi(\tau; p) - u_0(\tau)] + pN[\phi(\tau; p)] = 0. \tag{2.45}$$

The original nonlinear equation (2.40) can be deduced from the zeroth-order deformation Eq. (2.41) by letting $p = 1$.

In order to define the higher-order deformation equation, we define a vector as

$$\vec{u}_l = \{u_0(\tau), u_1(\tau), \ldots, u_l(\tau)\}.$$

Differentiating the zeroth-order deformation equation (2.41) k times with respect to p,

dividing by $k!$ and finally setting $p = 0$, we obtain the so-called k-th order deformation equation

$$L[u_k(\tau) - \chi_k u_{k-1}(\tau)] = \hbar H(\tau) R_k(\vec{u}_{k-1}), \tag{2.46}$$

where

$$R_k(\vec{u}_{k-1}) = \frac{1}{(k-1)!} \frac{\partial^{k-1} N[\phi(\tau; p)]}{\partial p^{k-1}}\bigg|_{p=0} \tag{2.47}$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k \geq 2. \end{cases} \tag{2.48}$$

Note that the expression for R_k indicates its dependency on the components $u_0(\tau), \ldots, u_{k-1}(\tau)$. The k-th order deformation equation (2.46) is linear, with linear boundary conditions derived from the original equation, and can thus easily be solved by means of symbolic computation software.

2.7 Conclusion

In this chapter, we have discussed mathematical preliminaries that will be useful to tackle various problem described in this thesis.

Chapter 3

Navier-Stokes equations and evolution equations for the flow of a non-Newtonian Fluid

3.1 Introduction

In this chapter, we give a thin film approximation of the Navier-Stokes equation for the flow of a non-Newtonian fluid and the boundary conditions which we use to derive the evolution equation of the free surface. We consider rectangular cartesian coordinates and cylindrical polar coordinates only for the axisymmetric spreading under gravity of a thin power-law liquid drop on a horizontal plane. In order to derive the evolution equation of the flow of a thin non-Newtonian fluid, we use the thin film approximation of the Navier-Stokes equations and some appropriate boundary conditions. The Navier-Stokes equations are coupled to these boundary conditions, in this regard, we derive the surface kinematic boundary condition, the surface stress conditions and mention the no-slip condition. We then define some dimensionless parameters that will allow us to write our equations and boundary conditions in dimensionless form. The purpose of this procedure is to simplify our equations by only keeping the most important features of the flow and to neglect the less significant ones. Finally, we integrate the corresponding set of partial differential equations to obtain the evolution

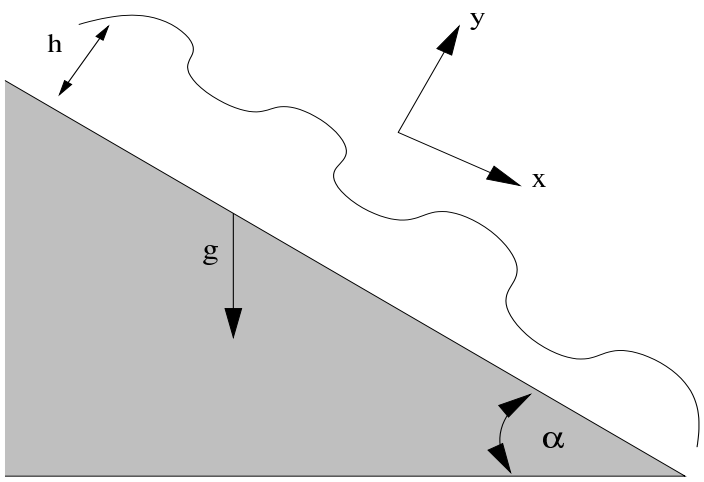

Figure 3.1: Thin liquid film on an inclined plane.

equations for each problem tackled in this work.

3.2 Flow of a power-law fluid over an inclined plane

3.2.1 Statement and mathematical formulation of the problem

We consider a fluid moving on an inclined plane, whose angle of inclination is α (see Fig. 3.1). We denote the components of the velocity of the fluid in the x and y directions as u and v, respectively. Let $y = h \equiv h(x,t)$ be the free surface of the flow, whose characteristic thickness and characteristic length along the plane are denoted by h_0 and L, respectively. The forces assumed to be driving the thin film are surface tension, surface tension gradients, gravity and viscous shear.

3.2.2 Continuity and momentum equations

For the flow of an incompressible fluid, the basic governing equations are the continuity equation and the conservation of linear momentum given by

$$\text{div } \mathbf{V} = 0, \tag{3.1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \text{div } \mathbf{T} + \rho \mathbf{g}, \tag{3.2}$$

where ρ is the fluid density, t is the time, d/dt denotes the material time derivative, $\mathbf{g} = (g \sin\alpha, -g\cos\alpha, 0)$ is the gravity, and $\mathbf{T}$ is the stress tensor defined by Eqs. (2.18)–(2.20). For unsteady two-dimensional flows the velocity vector is given in cartesian coordinates by Eq. (2.21), i.e.

$$\mathbf{V} = [u(x,y,t), v(x,y,t), 0], \tag{3.3}$$

and we find that the balance equations (3.1)-(3.2) yield

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.4}$$

$$\rho \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right) u = -\frac{\partial p(x,y,t)}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \rho g \sin\alpha, \tag{3.5}$$

$$\rho \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right) v = -\frac{\partial p(x,y,t)}{\partial y} + \frac{\partial S_{yx}}{\partial x} + \frac{\partial S_{yy}}{\partial y} - \rho g \cos\alpha, \tag{3.6}$$

where S_{xx}, S_{xy}, S_{yx} and S_{yy} have been defined in Eq. (2.26).

3.2.3 Boundary conditions

In order to solve the Navier-Stokes equations associated with the flow of a thin liquid film, besides the no-slip condition at the substrate (i.e., $u = v = 0$ at $y = 0$), we need to specify some boundary conditions at the fluid-fluid interface which here can be taken to be air-liquid (see Mei, 2003). We assume that air (1) is on top of the liquid (2) as depicted in Fig. 3.2.

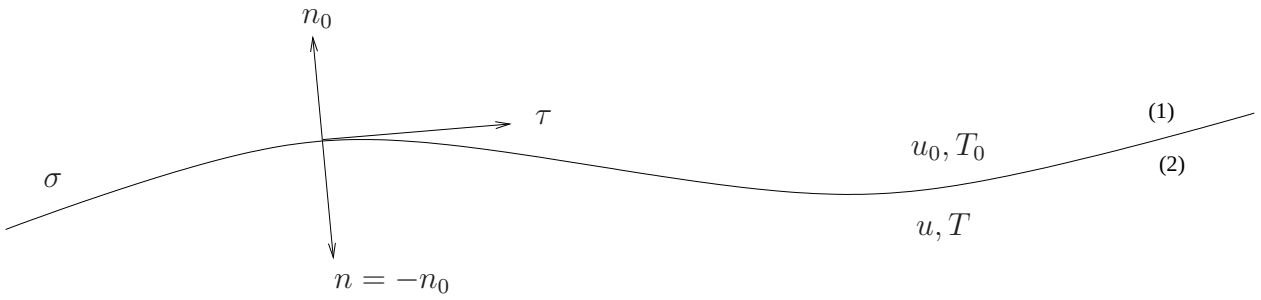

Figure 3.2: Free surface.

I- Surface kinematic boundary condition

The equation of a fluid-fluid interface is given by

$$F(t, x, y) = y - h(t, x) = 0. \tag{3.7}$$

In the special case of an air-liquid interface, we assume there is no air penetration at the free surface, so that

$$\frac{dF}{dt} = 0, \quad \text{at} \quad y = h(t, x). \tag{3.8}$$

The free surface in this case is a material boundary for which a particle initially on the boundary will remain there. Expanding Eq. (3.8), we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\mathbf{V}.\nabla)F = 0, \quad \text{at} \quad y = h(t, x),$$

that is

$$\frac{\partial(y-h)}{\partial t} + u\frac{\partial(y-h)}{\partial x} + v\frac{\partial(y-h)}{\partial y} = 0, \quad \text{at} \quad y = h(t, x).$$

This result leads to the surface boundary condition

$$h_t + uh_x = v, \quad \text{at} \quad y = h(t, x). \tag{3.9}$$

II- Surface tension and stress conditions at the fluid-fluid interface

Molecules in a fluid have a mutual attraction. When this attraction force is overcome by thermal agitation, the molecules pass into a gaseous phase. A liquid molecule, for instance, in the fluid bulk is surrounded by attractive

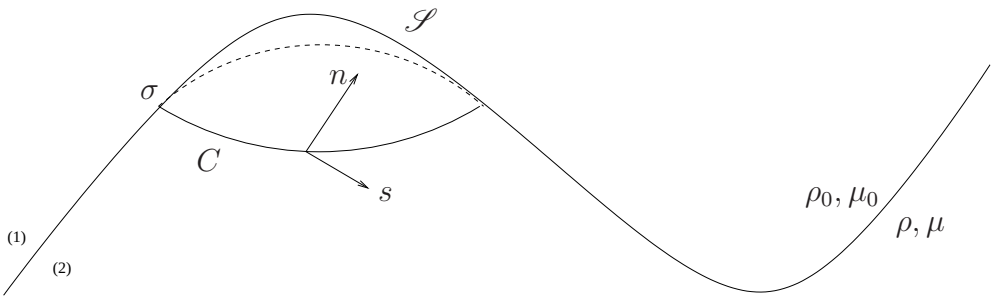

Figure 3.3: Interface between gas and liquid.

neighbours, while a molecule at the interface is attracted by a reduced number of neighbours and is in an energetically unfavourable state. The creation of a new surface is thus energetically costly, and a fluid system will act to minimize surface areas. Thus small fluid bodies tend to evolve into spheres.

Surface tension is a material property of a fluid-fluid interface whose origins lie in the different attractive intermolecular forces that act in the two fluid phases. The result is an interfacial energy per area that acts to resist the creation of a new interface, and that is equivalent to a line tension acting in all directions parallel to the interface. Fluids between which no interfacial tension arises are said to be miscible. Surface tension σ has the units of force/length or equivalently energy/area, and so may be thought of as a negative surface pressure. When an area of liquid covered with a spread substance is separated from a clean area of surface by a mechanical barrier, the force acting on unit length of the barrier is called the surface pressure.

In what follows, we denote by $\mathscr{S}$ a fluid-fluid interface characterized by a surface tension σ, bound by a closed contour C (see Fig. 3.3). One may think of there being a force per unit length of magnitude σ in the s-direction at every point along C that acts to flatten the surface $\mathscr{S}$. Perform a force balance on a volume element $\mathscr{V}$ enclosing the interfacial surface $\mathscr{S}$ defined by the contour C (see Fig. 3.4, and Mei, 2003),

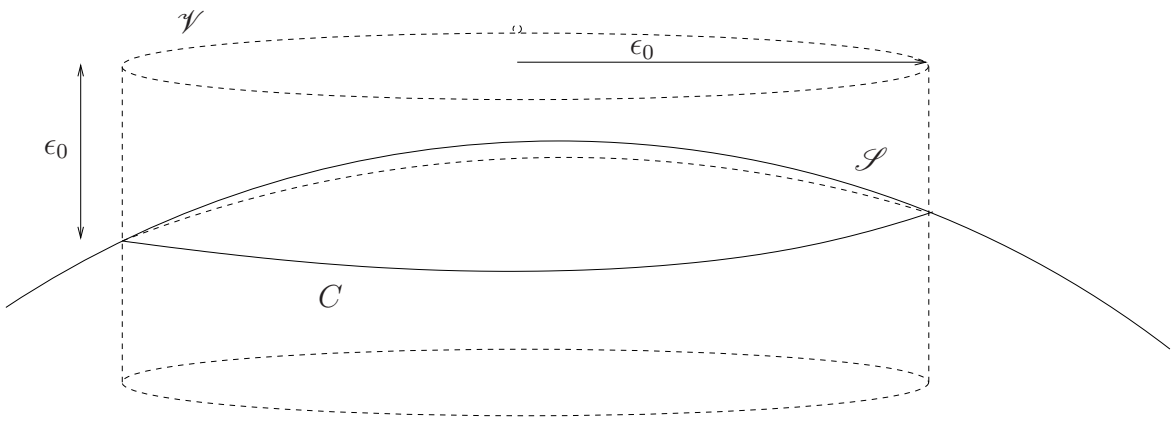

Figure 3.4: A Gaussian fluid pillbox of height ϵ_0 and radius ϵ_0 spanning the interface evolves under the combined influence of volume and surface forces.

$$\int_{\mathscr{V}} \rho \frac{d\mathbf{V}}{dt}\, d\mathscr{V} \;=\; \int_{\mathscr{V}} \mathbf{f}\, d\mathscr{V} \;+\; \int_{\mathscr{S}} [\hat{\mathbf{t}}(\hat{\mathbf{n}}) + \mathbf{t}(\mathbf{n})]\, d\mathscr{S} \;+\; \int_C \sigma \mathbf{s}\, d\ell, \quad (3.10)$$

where ℓ denotes arclength and $d\ell$ a length increment along the curve C.

$\hat{\mathbf{t}}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \hat{\mathbf{T}}$ is the stress tensor, the force/area exerted by the upper (1) fluid on the interface. Similarly, the stress exerted on the interface by the lower (2) fluid is $\mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T}$. Physically, the left-hand side term in Eq. (3.10) represents the inertial force due to the acceleration of fluid within $\mathscr{V}$, the first term of the right-hand side represents the body forces acting on the fluid within $\mathscr{V}$, the second and third term hydrodynamic force exerted at the interface by fluid (1) and (2) respectively. Lastly the fifth term represents the surface tension force exerted along perimeter C. If we now consider ϵ_0 as being the lengthscale of the element $\mathscr{V}$, then the acceleration and body forces will scale as ϵ_0^3, but the surface force will scale as ϵ_0^2. Hence, in the limit $\epsilon_0 \to 0$, we have that the surface forces must balance, that is

$$\int_{\mathscr{S}} [\hat{\mathbf{t}}(\hat{\mathbf{n}}) + \mathbf{t}(\mathbf{n})]\, d\mathscr{S} \;+\; \int_C \sigma \mathbf{s}\, d\ell \;=\; 0.$$

Since $\mathbf{n} = -\hat{\mathbf{n}}$, we now have $\mathbf{t}(\mathbf{n}) = -\hat{\mathbf{n}} \cdot \mathbf{T}$. Moreover, the application of Stokes theorem gives

$$\int_C \sigma \mathbf{s}\, d\ell \;=\; \int_{\mathscr{S}} [\nabla_s \sigma \;-\; \sigma \hat{\mathbf{n}} (\nabla_s \cdot \hat{\mathbf{n}})]\, d\mathscr{S},$$

where the tangential gradient operator, defined by

$$\nabla_s \;=\; [\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}] \cdot \nabla \;=\; \nabla - \hat{\mathbf{n}}\frac{\partial}{\partial n}$$

appears because σ and $\hat{\mathbf{n}}$ are defined only on the surface. For simplicity, we drop the subscript s on ∇. The surface balance thus becomes

$$\int_{\mathscr{S}} [\hat{\mathbf{n}} \cdot \hat{\mathbf{T}} - \hat{\mathbf{n}} \cdot \mathbf{T}]\, d\mathscr{S} \;=\; \int_{\mathscr{S}} [\sigma\hat{\mathbf{n}}(\nabla \cdot \hat{\mathbf{n}}) \;-\; \nabla\sigma]\, d\mathscr{S}.$$

Since the surface element is arbitrary, the integrand must vanish identically and we therefore obtain the interfacial stress balance equation

$$\hat{\mathbf{n}} \cdot (\hat{\mathbf{T}} - \mathbf{T}) \;=\; \sigma\hat{\mathbf{n}}(\nabla \cdot \hat{\mathbf{n}}) \;-\; \nabla\sigma. \tag{3.11}$$

The first term on the right-hand side of Eq. (3.11) represents the normal curvature associated with the local curvature of the interface $\nabla \cdot \mathbf{n}$, and the second term the tangential stress associated with the gradients in the surface tension. Both the normal and the tangential component must balance at the interface, and we shall therefore consider each component in turn.

II-1 Normal stress balance

Taking the scalar product of $\hat{\mathbf{n}}$ with Eq. (3.11) yields the normal stress balance at the interface, that is

$$\hat{\mathbf{n}} \cdot (\hat{\mathbf{T}} - \mathbf{T}) \cdot \hat{\mathbf{n}} \;=\; \sigma(\nabla \cdot \hat{\mathbf{n}}). \tag{3.12}$$

The jump in the normal stress across the interface must balance the curvature force per unit area. We note that a surface with non-zero curvature reflects a jump in the normal stress across the interface.

II-2 Tangential stress balance

Taking the scalar product of $\tilde{\tau}$ with Eq. (3.11), where $\tilde{\tau}$ is any unit vector tangent to the interface, yields the tangential stress balance at the interface, that is

$$\hat{\mathbf{n}} \cdot (\hat{\mathbf{T}} - \mathbf{T}) \cdot \tilde{\tau} \;=\; \nabla\sigma \cdot \tilde{\tau}. \tag{3.13}$$

Note that the left-hand side of Eq. (3.13) represents the jump in tangential components of the hydrodynamic stress at the interface and it contains only velocity gradients (not pressure) and therefore, a non-zero surface tension gradient $\nabla\sigma$ at a fluid interface must always drive motion. On the other hand, the right-hand side of Eq. (3.13) represents the tangential stress associated with gradients in σ, which may result from gradients in temperature or chemical composition at the interface.

• Application

Refer to Fig. 3.2. We assume that fluid (1) is air and is static above the power-law fluid (2). The stress tensor of (1) is thus $\hat{\mathbf{T}} = \mathbf{T}_0 = -p_0\mathbf{I}$, where p_0 is the atmospheric pressure, and the stress tensor of (2) is given by Eqs. (2.18)-(2.19). The normal to the interface in Eq. (3.7) is defined by

$$\hat{\mathbf{n}} = \frac{\nabla F}{\|\nabla F\|} = \frac{(-h_x, 1, 0)}{(1+h_x^2)^{1/2}},$$

the curvature by

$$\kappa = \nabla\cdot\hat{\mathbf{n}} = -\frac{h_{xx}}{(1+h_x^2)^{3/2}},$$

and the tangent to the interface by

$$\tilde{\tau} = \frac{(1, h_x, 0)}{(1+h_x^2)^{1/2}}.$$

Eqs. (3.12) and (3.13) then yield respectively

$$p - p_0 - \frac{2\mu\left(u_x h_x^2 - (u_y+v_x)\,h_x + v_y\right)\left[\mathrm{tr}(A_1^2)\right]^m}{1+h_x^2} = -\frac{\sigma h_{\mathrm{xx}}}{(1+h_x^2)^{3/2}}, \qquad (3.14)$$

and

$$-\frac{\mu\left(\mathrm{tr}A_1^2\right)^m\left(-u_y - v_x + h_x\left[4u_x + h_x\left(u_y+v_x\right)\right]\right)}{1+h_x^2} = \frac{\sigma_x + h_x\sigma_y}{(1+h_x^2)^{1/2}}. \qquad (3.15)$$

Finally, using the continuity equation $v_y = -u_x$ in Eq. (3.14) and after re-arrangement, Eqs. (3.14) and (3.15) become respectively

$$\begin{aligned} p - p_0 &+ 2\mu\frac{(1-h_x^2)}{1+h_x^2}u_x\left[\mathrm{tr}(A_1^2)\right]^m \\ &+ 2\mu\frac{(u_y+v_x)}{1+h_x^2}h_x\left[\mathrm{tr}(A_1^2)\right]^m = -\frac{\sigma h_{xx}}{(1+h_x^2)^{3/2}}, \end{aligned} \qquad (3.16)$$

and

$$\mu\frac{(1-h_x^2)}{(1+h_x^2)^{1/2}}\left(u_y+v_x\right)\left[\mathrm{tr}(A_1^2)\right]^m-\frac{4\mu h_x}{(1+h_x^2)^{1/2}}u_x\left[\mathrm{tr}(A_1^2)\right]^m=\sigma_x+h_x\sigma_y. \quad (3.17)$$

3.3 Scalings

In order to reduce the number of parameters in our equations, and to highlight the particular physical aspects of our model, we need to write the Navier-Stokes equations and the boundary conditions in dimensionless form. This allows us to neglect the less significant terms in our equations.

3.3.1 Derivation using the power-law model

We introduce dimensionless variables (with bars) as follows:

$$\bar{x}=\frac{x}{L},\quad \bar{y}=\frac{y}{h_0},\quad \bar{u}=\frac{u}{U},\quad \bar{v}=\frac{v}{\delta U},\quad \bar{p}=\frac{p-p_0}{(\mu U/\delta^2 L)*(2U^2/h_0^2)^m},\quad \bar{\sigma}=\frac{\sigma-\sigma_0}{\Delta\sigma},$$

where h_0, L, U and $\delta=L/h_0\ll 1$ are the initial film height (characteristic thickness), the characteristic length along the plane, the characteristic velocity in the x-direction and the aspect ratio, respectively. Note here that the scaling and notation are made in order to generalize the derivations given by Myers, (1998) for the case of a Newtonian fluid. The pressure difference and extra stress tensor scaling $(\mu U/\delta^2 L)*(2U^2/h_0^2)^m$ are chosen to balance the pressure gradient with the viscous force. We also assume that the surface tension has the form $\sigma=\sigma_0+\Delta\sigma\ \sigma_1(x)$, where σ_0 is a constant. Surface tension is the dominant force in our study and it therefore determines the choice of the velocity. U is chosen to make the generalized inverse capillary number $C=(\delta^3\sigma_0 a/\mu U)*(h_0^2/2U^2)^m=1$. However, for the present derivation U will remain arbitrary.

The Navier-Stokes equations for a thin power-law fluid on a slope (see Eqs. (3.4), (3.5), (3.6)) are (we have now dropped the bars for the dimensionless variables)

$$\begin{aligned}
\frac{\rho U^2}{L}\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]u &= \rho g\sin\alpha-\frac{\mu U}{\delta^2L^2}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial p}{\partial x}\\
&+\frac{\mu U}{\delta^2L^2}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial S_{xx}}{\partial x} + \frac{\mu U}{\delta^2Lh_0}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial S_{xy}}{\partial y}, \qquad (3.18)\\
\frac{\rho h_0U^2}{L}\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]v &= -\rho g\cos\alpha-\frac{\mu U}{\delta^2Lh_0}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial p}{\partial y}\\
&+\frac{\mu U}{\delta^2L^2}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial S_{yx}}{\partial x} + \frac{\mu U}{\delta^2Lh_0}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial S_{yy}}{\partial y}. \qquad (3.19)
\end{aligned}$$

Multiplying Eqs. (3.18) and (3.19) by $(\delta^2L^2/\mu U)\,(h_0^2/2U^2)^m$ and $(\delta^2Lh_0/\mu U)\,(h_0^2/2U^2)^m$ respectively, we obtain:

$$\begin{aligned}
\frac{\delta^2\rho UL}{\mu}\left(\frac{h_0^2}{2U^2}\right)^m\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]u &= \frac{\rho g\delta^2L^2}{\mu U}\left(\frac{h_0^2}{2U^2}\right)^m\sin\alpha-\frac{\partial p}{\partial x}\\
&+\delta\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}, \qquad (3.20)\\
\frac{\delta^2\rho h_0U}{\mu}\left(\frac{h_0^2}{2U^2}\right)^m\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]v &= -\frac{\rho g\delta^2Lh_0}{\mu U}\left(\frac{h_0^2}{2U^2}\right)^m\cos\alpha\\
&-\frac{\partial p}{\partial y}+\delta^2\frac{\partial S_{yx}}{\partial x}+\delta\frac{\partial S_{yy}}{\partial y}. \qquad (3.21)
\end{aligned}$$

Defining the generalized Reynolds and Bond numbers respectively by $\Re_e=(\rho Uh_0/\mu)*(h_0^2/(2U^2))^m$ and $B=(\rho g\delta^2L^2/(\mu U))*(h_0^2/(2U^2))^m$, the equations (3.20) and (3.21) read:

$$\begin{aligned}
\delta\,\Re_e\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]u &= B\sin\alpha-\frac{\partial p}{\partial x}+\frac{\partial S_{xx}}{\partial x}+\frac{1}{\delta}\frac{\partial S_{xy}}{\partial y}, \qquad (3.22)\\
\delta^3\Re_e\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]v &= -\delta B\cos\alpha-\frac{\partial p}{\partial y}+\frac{1}{\delta}\frac{\partial S_{yx}}{\partial x}+\frac{\partial S_{yy}}{\partial y}. \qquad (3.23)
\end{aligned}$$

Using $\delta\ll1$, and S_{xx}, S_{yy}, S_{xy} as given by Eq. (2.26), in dimensionless form we obtain

$$\begin{aligned}
S_{xx} &= 2\delta^2\frac{\partial u}{\partial x}\left[4\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}\right)^2\right]^m, \qquad (3.24)\\
S_{yy} &= 2\delta^2\frac{\partial v}{\partial y}\left[4\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}\right)^2\right]^m, \qquad (3.25)\\
S_{xy}=S_{yx} &= \delta\left(\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}\right)\left[4\delta^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}\right)^2\right]^m. \qquad (3.26)
\end{aligned}$$

The leading terms of the governing equations (3.4)–(3.6) are as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.27}$$

$$-\frac{\partial p}{\partial x}+B\sin\alpha+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^m\right)=0, \tag{3.28}$$

$$\frac{\partial p}{\partial y}+\delta B\cos\alpha=0, \tag{3.29}$$

where $\delta = h_0/L$. For $m = 0$ we obtain the Myers, (1998) case. If we include the effects of van der Waals forces, as in Ruckenstein and Jain, (1973), that is we assume

$$\phi(x,t)=\frac{\mathcal{A}}{h^3}\ ,$$

where $\mathcal{A}$ is related to the Hamaker constant $A^{'}$ by $\mathcal{A} = A^{'}/6\pi\delta L^2\mu U\,(2U^2/h_0^2)^m$, the above equations (3.27)–(3.29) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.30}$$

$$-\frac{\partial (p+\phi)}{\partial x}+B\sin\alpha+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^m\right)=0, \tag{3.31}$$

$$\frac{\partial (p+\phi)}{\partial y}+\delta B\cos\alpha=0. \tag{3.32}$$

The boundary conditions in dimensionless form become (on dropping the bars)

- the **no-slip condition** at the substrate $y = 0$,

$$u\ =\ v\ =\ 0, \tag{3.33}$$

- the **kinematic boundary condition** at the interface $y = h(t, x)$,

$$h_t+uh_x\ =\ v, \tag{3.34}$$

- the **continuity of normal stress** at the interface $y = h(t, x)$,

$$\begin{aligned} p\ &+\ 2\delta^2\frac{(1-\delta^2h_x^2)}{1+\delta h_x^2}u_x\left[4\delta^2u_x^2+(u_y+\delta^2v_x)^2\right]^m \\ &+\ 2\delta^2\frac{(u_y+\delta^2v_x)}{1+h_x^2}h_x\left[4\delta^2u_x^2+(u_y+\delta^2v_x)^2\right]^m=-\frac{\delta^3\sigma_0}{\mu U}\left(\frac{h_0^2}{2U^2}\right)^m\frac{h_{xx}}{(1+\delta^2h_x^2)^{3/2}}. \end{aligned} \tag{3.35}$$

Since $\delta^2 \ll 1$, Eq. (3.35) simplifies to

$$p = -Ch_{xx}, \tag{3.36}$$

• and the **continuity of the tangential or shear stress** at the interface $y = h(t, x)$,

$$\begin{aligned} \frac{(1-\delta^2 h_x^2)}{(1+\delta^2 h_x^2)^{1/2}} \ \left(u_y + \delta^2 v_x\right) \qquad & \left[4\delta^2 u_x^2 + (u_y + \delta^2 v_x)^2\right]^m \\ - \ 4\delta^2 \frac{h_x}{(1+\delta^2 h_x^2)^{1/2}} \ u_x \qquad & \left[4\delta^2 u_x^2 + (u_y + \delta^2 v_x)^2\right]^m \\ = \qquad & \frac{\delta \Delta\sigma}{\mu U} \left(\frac{h_0^2}{2U^2}\right)^m \sigma_x. \end{aligned} \tag{3.37}$$

Since $\delta^2 \ll 1$ Eq. (3.37) simplifies to

$$u_y \left[u_y^2\right]^m = M\sigma_x, \tag{3.38}$$

where $M = (\delta\Delta\sigma/\mu U) * (h_0^2/2U^2)^m$ is the generalized Marangoni number.

3.3.2 Derivation using the Ostwald model

In thin film flow the coordinate z and the interface position, $z = h(x, t)$ are scaled by the mean thickness of the film h_0, while the longitudinal coordinate x is measured by a length l, proportional to the disturbance wavelength. Thus, the ratio $\epsilon = h_0/l$ is a small parameter.

We choose the scales $\bar{u}_0$ and $\bar{u}_0\epsilon$ for the streamwise velocity u, and the transverse velocity w respectively, where $\bar{u}_0$ is the depthwise average velocity given by $\bar{u}_0 = (1/h_0)\int_0^{h_0} u^0(z)\, dz$, and u^0 is the dimensional streamwise velocity of undisturbed flat film. Time t is scaled by $l/\bar{u}_0$ and the pressure difference $p - p_0$, is scaled by $\rho(\bar{u}_0)^2$. There are three dimensionless groups, namely, the Reynolds number, $Re = \rho(\bar{u}_0)^{2-n} h_0^n/\mu_m$, the Froude number, $Fr = (\bar{u}_0)^2/h_0 g$, and the Weber number, $We = \rho(\bar{u}_0)^2 h_0/\sigma_0$. We assume that $Re = O(1)$, $Fr = O(1)$ and $We = O(\epsilon^2)$. Proceeding in a similar manner, we find that

• the **continuity and momentum equations** written in dimensionless forms are

$$u_x + w_z = 0, \tag{3.39}$$

$$\epsilon Re\,[u_t + uu_x + wu_z] = -\epsilon Rep_x + 2\epsilon^2[\bar{\eta}u_x]_x + [\bar{\eta}(u_z + \epsilon^2 w_x)]z + \frac{Re}{Fr}\sin\beta, \tag{3.40}$$

$$\epsilon^2 Re\,[w_t + uw_x + ww_z] = -Rep_z + \epsilon[\bar{\eta}(u_z + \epsilon^2 w_x)]x + 2\epsilon^2[\bar{\eta}w_z]_z - \frac{Re}{Fr}\cos\beta, \tag{3.41}$$

where the dimensionless shear-dependent viscosity model is given by

$$\bar{\eta} = \left[2\epsilon^2(u_x^2 + w_z^2) + (u_z + \epsilon^2 w_x)^2\right]^{(n-1)/2}. \tag{3.42}$$

When using the continuity equation (3.39), Eq. (3.42) simplifies to

$$\bar{\eta} = \left[4\epsilon^2 u_x^2 + (u_z + \epsilon^2 w_x)^2\right]^{(n-1)/2}.$$

The corresponding dimensionless boundary conditions are given by

• the **no slip condition**

$$u = w = 0 \tag{3.43}$$

at the rigid plate, $z = 0$,

• the **normal stress balance**

$$Re\; p + 2\epsilon\bar{\eta}u_x(1 - \epsilon^2 h_x^2) + 2\epsilon\bar{\eta}(u_z + \epsilon^2 w_x)h_x = -\frac{\epsilon^2 Re}{We}\frac{h_{xx}}{(1 + \epsilon^2 h_x^2)^{1/2}}, \tag{3.44}$$

• the **shear stress balance**

$$\bar{\eta}(u_z + \epsilon^2 w_x)(1 - \epsilon^2 h_x^2) - 4\epsilon^2\bar{\eta}u_x h_x = 0, \tag{3.45}$$

and the **kinematic boundary condition**

$$h_t + uh_x = w, \tag{3.46}$$

at the interface $z = h(x, t)$.

3.4 Mass conservation of surfactant

In the case where an insoluble surfactant is spreading on the liquid film, we need an additional equation. The equation of mass conservation of surfactant is given by

$$\partial_t \Gamma + \nabla_s \cdot (\mathbf{V}_s \Gamma) + (\mathbf{V} \cdot \mathbf{n})(\nabla_s \cdot \mathbf{n})\Gamma = \mathcal{D}\nabla_s^2 \Gamma, \tag{3.47}$$

where Γ is the surfactant surface concentration, $\mathcal{D}$ is the surfactant surface diffusion coefficient assumed constant, and $\mathbf{V}_s$ denotes the velocity vector evaluated at the interface. The second and third terms on the left-hand side of equation (3.47) represent transport by convection and surface dilation respectively, while the term on the right-hand side represents diffusive transport.

The presence of surfactant at the interface between fluid and gas modifies the structure of the interface by modifying the surface tension which now varies with the surfactant concentration. We assume that the surfactant concentration is dilute enough to permit use of a linear equation of state of the surface-tension, that is

$$\sigma(\Gamma) = \sigma_0 - \frac{\mathcal{S}}{\Gamma_s}\Gamma. \tag{3.48}$$

σ_0 is the surface tension of the perfectly clean surfactant-free interface; Γ_s represents the maximal value of surfactant concentration; $\mathcal{S} = \sigma_0 - \sigma_s$ denotes the maximal spreading pressure, and σ_s represents the minimal value of surface tension.

In order to write Eq. (3.47) in dimensionless form, we use the previously defined scaling and

$$\bar{\Gamma} = \Gamma/\Gamma_s, \quad \bar{\sigma} = (\sigma - \sigma_s)/\mathcal{S}.$$

Eq. (3.47) written in dimensionless form (dropping the bars) becomes

$$\partial_t \Gamma + (u_s \Gamma)_x + \delta^2 \left(-h_x u_s + v_s\right) h_{xx} \Gamma = \mathcal{P}_e^{-1} \Gamma_{xx}, \tag{3.49}$$

where $\mathcal{P}_e = LU/\mathcal{D}$ is the surface Peclet number, which represents the ratio of surfactant transport by Marangoni stresses to that by surface diffusion. Since $\delta \ll 1$, Eq. (3.49)

simplifies to

$$\partial_t \Gamma + (u_s \Gamma)_x = \mathcal{P}_e^{-1} \Gamma_{xx}, \tag{3.50}$$

and Eq. (3.48) in dimensionless form reads

$$\sigma(\Gamma) = 1 - \Gamma. \tag{3.51}$$

3.5 Evolution equations for the flow of a thin power-law fluid

3.5.1 Spreading over an incline plane

For the power-law model defined by Eqs. (2.18) and (2.19), we saw that the thin film approximation to the Navier-Stokes equations in this case are defined by Eqs. (3.27)–(3.29) subject to boundary conditions in Eqs. (3.33)–(3.34), (3.36), and (3.38).
Integrating Eq. (3.29) with respect to y and using the boundary condition (3.36), we obtain

$$p = -\delta B \cos\alpha (y - h) - C h_{xx}. \tag{3.52}$$

Equation (3.28) implies

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^m\right) = -C h_{xxx} + \delta B \cos\alpha h_x - B \sin\alpha. \tag{3.53}$$

Letting

$$f(x) = -C h_{xxx} + \delta B \cos\alpha h_x - B \sin\alpha, \tag{3.54}$$

in Eq. (3.53), integrating once with respect to y and imposing the boundary condition (3.38), we obtain

$$\left(\frac{\partial u}{\partial y}\right)^{2m+1} = f(x)(y - h) + M\sigma_x. \tag{3.55}$$

Integrating Eq. (3.55) once with respect to y and imposing the no-slip condition, leads to

$$u = \frac{1}{\beta f(x)} \left([f(x)(y-h) + M\sigma_x]^\beta - [-f(x)h + M\sigma_x]^\beta \right), \tag{3.56}$$

where

$$\beta = (2m+2)/(2m+1).$$

The continuity equation (3.27) defines the y component of the velocity on the free surface

$$v(h) = -\int_0^h u_x dy. \tag{3.57}$$

This expression together with the kinematic condition leads to the governing equation for film height

$$h_t + Q_x = 0, \tag{3.58}$$

where

$$Q = \int_0^h u dy,$$

is the fluid flux. We have used the well-known Leibnitz rule

$$\int_0^{h(x)} \frac{\partial}{\partial x} [g(x,y)]\, dy = \frac{\partial}{\partial x} \int_0^{h(x)} g(x,y) dy - g(x,h) \frac{\partial h}{\partial x},$$

to obtain Eq. (3.58). Integrating Eq. (3.56) once with respect to y, we have

$$Q = \frac{1}{\beta(\beta+1)f(x)^2} \left[(M\sigma_x)^{\beta+1} - (\beta f(x)h + M\sigma_x) (-f(x)h + M\sigma_x)^\beta \right]. \tag{3.59}$$

The evolution equation describing the variation of the thin film height is therefore

$$h_t + \left(\frac{1}{\beta(\beta+1)f(x)^2} \left[(M\sigma_x)^{\beta+1} - (\beta f(x)h + M\sigma_x) (-f(x)h + M\sigma_x)^\beta \right] \right)_x = 0, \tag{3.60}$$

where $\delta = h_0/L$, $\beta = (2m+2)/(2m+1)$ and $f(x)$ is defined in Eq. (3.54). This equation is a generalization of the one obtained by Myers, (1998) for the Newtonian case.

3.5.2 Derivation using Benney-type approximation

The physical system consists of a thin liquid film which is draining down a rigid plate inclined at angle β with the horizontal. The film is bounded above by a motionless gas at ambient pressure p_0. The liquid is considered as non-Newtonian with constant density ρ and surface tension σ_0. The Ostwald de Waele power-law model is used, and the nonlinear shear dependent viscosity η is given by Eq. (2.27). We assume that the liquid film is very thin and the induced gravity-driven flow is relatively slow, so that the flow regime is close to that predicted by lubrication theory. Except for very small angles of inclination, the gravity-driven surface waves with a wavelength much larger than the film thickness cause instability first. To formulate the two-dimensional hydrodynamic problem, Cartesian coordinates x, z are chosen with x as the streamwise coordinate and z being measured normal to the plate (see Fig. 3.5). The Navier-Stokes equations and boundary conditions in dimensionless form are given by Eqs. (3.39)–(3.41) and Eqs. (3.42)–(3.46) respectively.

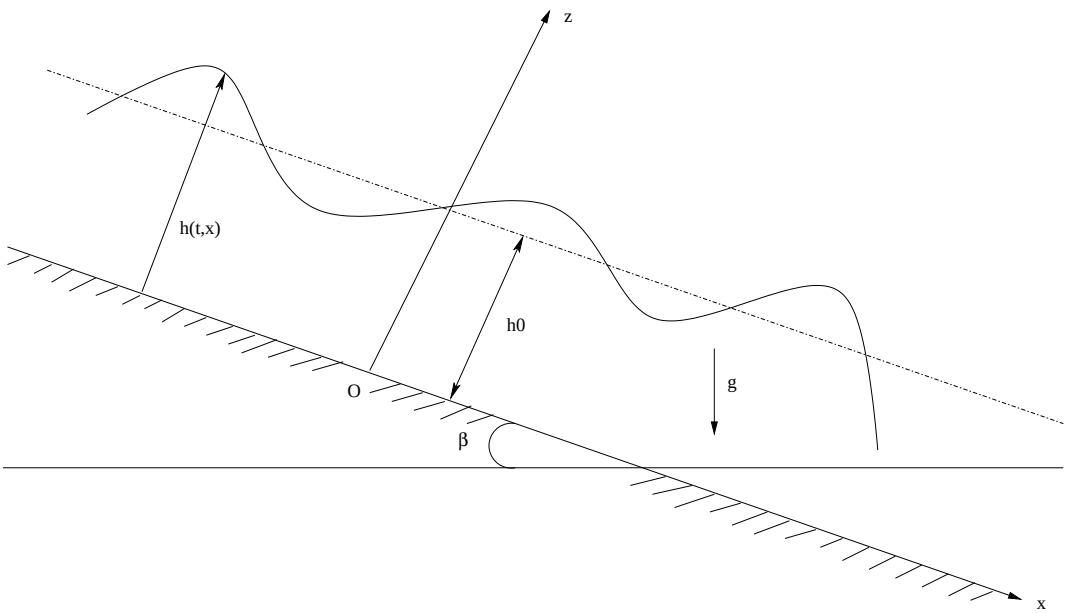

Figure 3.5: Configuration of the physical problem.

In this section, we give a complete derivation of the equation governing the thin film flow of the physical problem. In the paper by Miladinova *et al.*, (2004), only the main steps to the equation have been given.

We consider an approximate solution using a regular perturbation ϵ with the Reynolds number Re, the Weber number We and the Froude number Fr fixed,

$$(u, w, p) = (u_0, w_0, p_0) + \epsilon(u_1, w_1, p_1) + O(\epsilon^2).$$

The film thickness $h(x,t)$ and its derivatives are assumed to be of order of unity. Following Benney's approach (Benney, 1966), the complicated nonlinear system Eqs. (3.39)-(3.46) can be reduced to a single nonlinear evolution equation for the film thickness. We proceed with the asymptotic analysis by substituting the approximate solution into Eqs. (3.39)-(3.46). After equating to zero the coefficients of the same powers of ϵ in each equation and boundary condition, we obtain a sequence of equations. The leading-order terms of the governing dimensionless system are as follows

- **zeroth-order**

$$\begin{aligned}
\eta_0 &= (u_{0z})^{n-1}, \\
u_{0x} + w_{0x} &= 0, && (3.61)\\
\left[(u_{0z})^n\right]_z &= -\frac{Re\sin\beta}{Fr}, && (3.62)\\
p_{0z} &= -\frac{1}{Fr}\cos\beta, && (3.63)\\
z = 0 \ : \quad & u_0 = w_0 = 0, && (3.64)\\
z = h(x,t) \ : \quad & u_{0z} = 0, \quad w_0 = h_t + u_0 h_x, \quad p_0 = -\frac{\epsilon^2}{We} h_{xx}. && (3.65)
\end{aligned}$$

Integrating Eq. (3.62) twice with respect to z and using the boundary conditions given by Eqs. (3.64)-(3.65a) we obtain

$$u_0 = \frac{n}{n+1}\left(\frac{Re}{Fr}\sin\beta\right)^{1/n}\left[h^{(n+1)/n} - (h-z)^{(n+1)/n}\right]. \qquad (3.66)$$

Now, in order to define $\bar{u}_0$ the depthwise average velocity of the flow, we need to determine u^0 the dimensional streamwise velocity of undisturbed flat film. Consider the basic state $[u^0(z), 0, 0]$ in the steady flow down the plane. The momentum equation

becomes (the continuity equation is identically satisfied)

$$0 = -p_x + \rho g \sin\beta + \left(\eta (u^0)'\right)', \tag{3.67}$$

$$0 = -p_z - \rho g \cos\beta, \tag{3.68}$$

subject to boundary conditions

$$u^0(0) = 0, \qquad \text{(no-slip)} \tag{3.69}$$

$$(u^0)'(h_0) = 0, \qquad \text{(zero shear)} \tag{3.70}$$

$$p(x, h_0) = P_0 = \text{constant}, \tag{3.71}$$

where $\eta = \mu_m [(u^0)']^{n-1}$. Integrating Eq. (3.68) with respect to z subject to Eq. (3.71), we obtain that

$$p = P_0 + \rho g \cos\beta (h_0 - z) \tag{3.72}$$

is a function of z only. Substituting Eq. (3.72) into Eq. (3.67) and integrating the resulting equation twice with respect to z subject to Eqs. (3.69) and (3.70), we obtain

$$u^0(z) = \frac{n}{n+1} \left(\frac{\rho g \sin\beta}{\mu_m} \right)^{1/n} \left[h_0^{(n+1)/n} - (h_0 - z)^{(n+1)/n} \right]. \tag{3.73}$$

The depthwise average velocity $\bar{u}_0$ is defined as

$$\bar{u}_0 = \frac{1}{h_0} \int_0^{h_0} u^0(z)\, dz = \frac{n}{2n+1} \left(\frac{\rho g \sin\beta}{\mu_m} \right)^{1/n} h_0^{(n+1)/n}. \tag{3.74}$$

Substituting Eq. (3.74) into the definition of the Reynolds and Froude numbers, we find that

$$\frac{Re}{Fr} \sin\beta = \left(\frac{2n+1}{n} \right)^n. \tag{3.75}$$

Integrating Eq. (3.63) with respect to z subject to Eq. (3.65c), we find

$$p_0 = \frac{\cos\beta}{Fr} (h - z) - \frac{\epsilon^2}{We} h_{xx}. \tag{3.76}$$

Finally using the continuity equation (3.61) subject to Eq. (3.64b), after integration we obtain

$$w_0 = \frac{n}{n+1} \left(\frac{Re}{Fr} \sin\beta \right)^{1/n} h_x \left[h^{(n+1)/n} - (h - z)^{(n+1)/n} - \frac{(n+1)}{n} z\, h^{1/n} \right]. \tag{3.77}$$

- **first-order**

$$u_{1x} + w_{1x} = 0, \tag{3.78}$$

$$Re\,[u_{0t} + w_0 u_{0z} + u_0 u_{0x}] = -Re\; p_{0x} + n\left[u_{0z}^{n-1} u_{1z}\right]_z, \tag{3.79}$$

$$0 = -\,Re\; p_{1z} + \left[(u_{0z})^n\right]_x, \tag{3.80}$$

subject to

$$z = 0: \quad u_1 = w_1 = 0, \tag{3.81}$$

$$z = h(x,t): \quad nu_{0z}^{n-1} u_{1z} = 0, \quad Re\; p_1 + 2(u_{0z})^{n-1} u_{0x} + 2(u_{0z})^n h_x = 0,$$

$$w_1 = u_1 h_x. \tag{3.82}$$

Eq. (3.79) can be rearranged as

$$n\left[u_{0z}^{n-1} u_{1z}\right]_z = Re\,[u_{0t} + w_0 u_{0z} + u_0 u_{0x} + p_{0x}]. \tag{3.83}$$

Substituting u_0 from Eq. (3.66), w_0 from Eq. (3.77) and p_0 from Eq. (3.76) into Eq. (3.83), yields

$$\begin{aligned} n\left[u_{0z}^{n-1} u_{1z}\right]_z = Re\,\Big\{ & K_{\beta n}\; h_t\; \left[h^{1/n} - (h-z)^{1/n}\right] \\ & + \frac{n}{n+1} K_{\beta n}^2\; h_x \left[h^{(n+1)/n} - (h-z)^{(n+1)/n} - \left(\frac{n+1}{n}\right) z h^{1/n}\right](h-z)^{1/n} \\ & + \frac{n}{n+1} K_{\beta n}^2\; h_x \left[h^{(n+1)/n} - (h-z)^{(n+1)/n}\right]\left[h^{1/n} - (h-z)^{1/n}\right] + f(t,x)\Big\}, \end{aligned} \tag{3.84}$$

where $K_{\beta n} = \left(\dfrac{Re}{Fr}\sin\beta\right)^{1/n}$, and $f(t,x) = \dfrac{1}{Fr}\cos\beta\; h_x - \dfrac{\epsilon^2}{We}\; h_{xxx}$.
Let

$$\begin{aligned} K(t,x,z) = \Big\{ & K_{\beta n}\; h_t\; \left[h^{1/n} - (h-z)^{1/n}\right] \\ & + \frac{n}{n+1} K_{\beta n}^2\; h_x \left[h^{(n+1)/n} - (h-z)^{(n+1)/n} - \left(\frac{n+1}{n}\right) z h^{1/n}\right](h-z)^{1/n} \\ & + \frac{n}{n+1} K_{\beta n}^2\; h_x \left[h^{(n+1)/n} - (h-z)^{(n+1)/n}\right]\left[h^{1/n} - (h-z)^{1/n}\right] + f(t,x)\Big\}. \end{aligned} \tag{3.85}$$

Integrating Eq. (3.83) once with respect to z, we obtain

$$\begin{aligned} u_{1z} &= \frac{Re}{n} u_{0z}^{1-n} \int_h^z K(t,x,y)\, dy, \\ &= \frac{Re}{n} K_{\beta n}^{1-n} (h-z)^{(1-n)/n} \int_h^z K(t,x,y)\, dy. \end{aligned} \tag{3.86}$$

In order to compute the integral in Eq. (3.86), we use the following quantities

$$\begin{aligned} I_1 &= \int z(h-z)^{1/n}\, dz = -\frac{n}{(n+1)(2n+1)} \left[(n+1)z + nh\right](h-z)^{(n+1)/n}, \\ I_2 &= \int z(h-z)^{(1-n)/n}\, dz = -\frac{n}{n+1}(z+nh)(h-z)^{1/n}, \\ I_3 &= \int \left[(n+1)z + nh\right](h-z)^{2/n}\, dz = -\frac{n}{2(n+2)} \left[(n+2)z + 3nh\right](h-z)^{(n+2)/n}. \end{aligned}$$

We then obtain the following results

$$\begin{aligned} \int_h^z K(t,x,y)\, dy &= \Bigg\{ K_{\beta n} h_t \left[y h^{1/n} + \frac{n}{n+1}(h-y)^{(n+1)/n} \right] \\ &+ \frac{n}{n+1} K_{\beta n}^2 h_x \Bigg[y h^{(n+2)/n} + \frac{n}{n+1} h^{(n+1)/n}(h-y)^{(n+1)/n} \\ &+ \frac{n}{2n+1} h^{1/n}(h-y)^{(2n+1)/n} - \frac{n}{2n+2}(h-y)^{(2n+2)/n} \Bigg] \\ &+ \frac{n}{n+1} K_{\beta n}^2 h_x \Bigg[-\frac{n}{n+1} h^{(n+1)/n}(h-y)^{(n+1)/n} \\ &+ \frac{n}{2n+2}(h-y)^{(2n+2)/n} \\ &+ \frac{1}{2n+1}\left[(n+1)y + nh\right](h-y)^{(n+1)/n} h^{1/n} \Bigg] + f(t,x) y \Bigg\}_h^z \end{aligned} \tag{3.87}$$

$$\begin{aligned} &= \Bigg\{ K_{\beta n} h_t \left[z h^{1/n} + \frac{n}{n+1}(h-z)^{(n+1)/n} - h^{(n+1)/n} \right] \\ &+ \frac{n}{n+1} K_{\beta n}^2 h_x \Bigg[z h^{(n+2)/n} + \frac{n}{n+1} h^{(n+1)/n}(h-z)^{(n+1)/n} \\ &+ \frac{n}{2n+1} h^{1/n}(h-z)^{(2n+1)/n} - \frac{n}{2n+2}(h-z)^{(2n+2)/n} \\ &+ -h^{(2n+2)/n} \Bigg] + \frac{n}{n+1} K_{\beta n}^2 h_x \Bigg[-\frac{n}{n+1} h^{(n+1)/n}(h-z)^{(n+1)/n} \\ &+ \frac{n}{2n+2}(h-z)^{(2n+2)/n} + \frac{1}{2n+1}\big[(n+1)z \\ &\qquad + nh\big](h-z)^{(n+1)/n} h^{1/n} \Bigg] + f(t,x)(z-h) \Bigg\} \end{aligned} \tag{3.88}$$

$$= \mathscr{K}(t,x,z). \tag{3.89}$$

Letting $L(t,x,z) = (h-z)^{(1-n)/n}\mathscr{K}(t,x,z)$, Eq. (3.86) reads

$$u_{1z} = \frac{Re}{n}K_{\beta n}^{1-n}L(t,x,z), \tag{3.90}$$

and integrating once with respect to z, we obtain

$$u_1 = \frac{Re}{n}K_{\beta n}^{1-n}\int_0^z L(t,x,y)\,dy. \tag{3.91}$$

Computing the integral in Eq. (3.91), we obtain after simplification that

$$\begin{aligned}
u_1 \;=\; & \frac{Re}{n}K_{\beta n}^{1-n}\Bigg\{K_{\beta n}h_t\Bigg[-\frac{n}{n+1}(nh+z)(h-z)^{1/n}h^{1/n} \\
& -\frac{n^2}{(n+1)(n+2)}(h-z)^{(n+2)/n}+n(h-z)^{1/n}h^{(n+1)/n} \\
& -\frac{2n}{(n+1)(n+2)}h^{(n+2)/n}\Bigg]+\frac{n}{n+1}K_{\beta n}^2h_x\Bigg[-\frac{n}{n+1}(nh+z)(h-z)^{1/n}h^{(n+2)/n} \\
& -\frac{n^2}{(2n+1)(2n+2)}(h-z)^{(2n+2)/n}h^{1/n}+n(h-z)^{1/n}h^{(2n+2)/n} \\
& -\frac{n}{2(n+2)(2n+1)}[(n+2)z+3nh](h-z)^{(n+2)/n}h^{1/n} \\
& -\frac{n(5n+4)}{2(n+1)(n+2)(2n+1)}h^{(2n+3)/n}\Bigg]+\frac{n}{n+1}f(t,x)[(h-z)^{(n+1)/n} \\
& -h^{(n+1)/n}]\Bigg\}.
\end{aligned} \tag{3.92}$$

Using the expression for h_t at $z=h$ given by Eq. (3.65b) we have

$$\begin{aligned}
h_t \;&=\; \frac{n}{n+1}K_{\beta n}h_x\left[h^{(n+1)/n}-(h-z)^{(n+1)/n}-\frac{n+1}{n}zh^{1/n}\right]_{z=h} \\
&\quad +\; \frac{n}{n+1}K_{\beta n}h_x\left[(h-z)^{(n+1)/n}-h^{(n+1)/n}\right]_{z=h} \\
&=\; \frac{n}{n+1}K_{\beta n}h_x\left[-\left(\frac{n+1}{n}\right)zh^{1/n}\right]_{z=h} \\
&=\; -K_{\beta n}h_xh^{(n+1)/n}
\end{aligned} \tag{3.93}$$

Substituting Eq. (3.93) in Eq. (3.92), we obtain after simplification that

$$\begin{aligned} u_1 &= \frac{Re}{n}K_{\beta n}^{1-n}\Bigg\{K_{\beta n}^2 h_x\left[\frac{n}{(n+1)^2}(nh+z)(h-z)^{1/n}h^{(n+2)/n}\right. \\ &\quad -\frac{n}{n+1}(h-z)^{1/n}h^{(2n+2)/n}+\frac{n(3n+2)}{2(n+1)^2(2n+1)}h^{(2n+3)/n} \\ &\quad +\frac{n^2}{(n+1)(n+2)}(h-z)^{(n+2)/n}h^{(n+1)/n}-\frac{n^3}{2(n+1)^2(2n+1)}(h-z)^{(2n+2)/n}h^{1/n} \\ &\quad \left.-\frac{n^2}{2(n+1)(n+2)(2n+1)}\left[(n+2)z+3nh\right](h-z)^{(n+2)/n}h^{1/n}\right] \\ &\quad -\frac{n}{n+1}\left[h^{(n+1)/n}-(h-z)^{(n+1)/n}\right]f(t,x)\Bigg\}. \end{aligned} \tag{3.94}$$

We have $u = u_0 + u_1 + O(\epsilon^2)$, $\quad w = w_0 + w_1 + O(\epsilon^2)$. Only $w_1 = -\int_0^z u_{1z}\, dz$ is not yet computed. We use the formula $h_t + \dfrac{\partial}{\partial x}\left\{\displaystyle\int_0^{h(t,x)} u(t,x,z)\, dz\right\} + O(\epsilon^2) = 0$ obtained from the continuity equation and the kinematic boundary condition.

$$\begin{aligned} \int_0^h u_0\, dz &= K_{\beta n}\frac{n}{n+1}\int_0^h \left[h^{(n+1)/n}-(h-z)^{(n+1)/n}\right]dz \\ &= K_{\beta n}\frac{n}{n+1}\left[zh^{(n+1)/n}+\frac{n}{2n+1}(h-z)^{(2n+1)/n}\right]_0^h \\ &= \frac{n}{2n+1}K_{\beta n}h^{(2n+1)/n}, \end{aligned} \tag{3.95}$$

so that

$$\frac{\partial}{\partial x}\left(\int_0^h u_0 dz\right) = \left(h^{(2n+1)/n}\right)_x. \tag{3.96}$$

Integrating u_1 with respect to z, we have

$$\begin{aligned} \int_0^h u_1\, dz &= \frac{Re}{n}K_{\beta n}^{1-n}\Bigg\{K_{\beta n}^2 h_x\left[-\frac{n^2}{(n+1)^2(2n+1)}(2nh+z)(h-z)^{(n+1)/n}h^{(n+2)/n}\right. \\ &\quad +\frac{n^2}{(n+1)^2}(h-z)^{(n+1)/n}h^{(2n+2)/n}+\frac{n(3n+2)}{2(n+1)^2(2n+1)}zh^{(2n+3)/n} \\ &\quad -\frac{n^3}{2(n+1)^2(n+2)}(h-z)^{(2n+2)/n}h^{(n+1)/n} \\ &\quad +\frac{n^4}{2(n+1)^2(2n+1)(3n+2)}(h-z)^{(3n+2)/n}h^{1/n} \\ &\quad +\frac{n^3}{2(n+1)^2(n+2)(2n+1)(3n+2)}\big[(n+1)(n+2)z \\ &\qquad\qquad +n(5n+4)h\big](h-z)^{(2n+2)/n}h^{1/n}\big] \\ &\quad \left.-\frac{n}{n+1}\left[zh^{(n+1)/n}+\frac{n}{2n+1}(h-z)^{(2n+1)/n}\right]f(t,x)\right\}_{z=0}^{z=h}. \end{aligned}$$

After substituting of $z = 0$ and $z = h$ into this expression, we obtain

$$\begin{aligned}\int_0^h u_1\, dz &= \frac{Re}{n}K_{\beta n}^{1-n}\Bigg\{K_{\beta n}^2 h_x\Bigg[\frac{n(3n+2)}{2(n+1)^2(2n+1)}+\frac{2n^3}{(n+1)^2(2n+1)}\\ &\quad -\frac{n^2}{(n+1)^2}+\frac{n^3}{2(n+1)^2(n+2)}-\frac{n^4}{2(n+1)^2(2n+1)(3n+2)}\\ &\quad -\frac{n^4(5n+4)}{2(n+1)^2(n+2)(2n+1)(3n+2)}\Bigg]h^{(3n+3)/n}\\ &\quad -\frac{n}{2n+1}h^{(2n+1)/n}f(t,x)\Bigg\}. \end{aligned} \tag{3.97}$$

Rearranging terms, we obtain after simplification

$$\begin{aligned}\int_0^h u_1\, dz &= \frac{Re}{n}K_{\beta n}^{1-n}\Bigg\{K_{\beta n}^2 h_x\Bigg[\frac{2n}{(2n+1)(3n+2)}\Bigg]h^{(3n+3)/n}\\ &\quad -\frac{n}{2n+1}\,h^{(2n+1)/n}f(t,x)\Bigg\}\\ &= \frac{Re}{n}\left(\frac{n}{2n+1}\right)K_{\beta n}^{1-n}\left\{\frac{2K_{\beta n}^2 h_x}{3n+2}h^{(3n+3)/n}-h^{(2n+1)/n}f(t,x)\right\}\\ &= \frac{1}{n}ReK_{\beta n}^{-n}\left\{\frac{2h_x}{3n+2}K_{\beta n}^2 h^{(n+2)/n}-f(t,x)\right\}h^{(2n+1)/n}. \end{aligned} \tag{3.98}$$

Since

$$\begin{aligned} f(t,x) &= \frac{1}{Fr}\cos\beta h_x-\frac{\epsilon^2}{We}h_{xxx}\\ &= -\frac{\sin\beta}{Fr}\left(-\cos\beta h_x+\frac{\epsilon^2 Fr}{We\sin\beta}h_{xxx}\right)\\ &= -\frac{\sin\beta}{Fr}\left(-\cos\beta h_x+Sh_{xxx}\right)\\ &= -\frac{K_{\beta n}^n}{Re}\left(-\cos\beta h_x+Sh_{xxx}\right), \end{aligned} \tag{3.99}$$

where $S=\epsilon^2 Fr/(We\sin\beta)=O(1)$ and $K_{\beta n}=(2n+1)/n$, Eq. (3.98) becomes

$$\int_0^h u_1\, dz=\frac{1}{n}\left\{\frac{2Reh_x}{3n+2}\left(\frac{2n+1}{n}\right)^{2-n}h^{(n+2)/n}-\cot\beta h_x+Sh_{xxx}\right\}h^{(2n+1)/n}. \tag{3.100}$$

Hence the governing equation is given by

$$\begin{aligned} h_t+\left(h^{(2n+1)/n}\right)_x &+ \frac{\epsilon}{n}\Bigg\{\Bigg[\frac{2Re\ h_x}{3n+2}\left(\frac{2n+1}{n}\right)^{2-n}h^{(n+2)/n}\\ &\quad -\cot\beta\ h_x+Sh_{xxx}\Bigg]h^{(2n+1)/n}\Bigg\}_x+O(\epsilon^2)=0, \end{aligned} \tag{3.101}$$

where h is the film height, n is the power-law index, $S = \epsilon^2 Fr/We \sin\beta = O(1)$ is the surface tension parameter, Re is the Reynolds number, Fr is the Froude number and $\epsilon = h_0/l \ll 1$ is a small parameter. We assume that $Re = O(1)$ and $Fr = O(1)$.

Eq. (3.101) can be written in a simpler form as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(b_1(h)h\right) + \epsilon\frac{\partial}{\partial x}\left\{\frac{1}{n}b_2(h)h_x + \frac{1}{n}b_3(h)h_{xxx}\right\} + O(\epsilon^2) = 0, \tag{3.102}$$

where

$b_1(h) = h^m$, $b_2(h) = a_n h^{3m} + bh^{m+1}$, $b_3(h) = Sh^{m+1}$, $a_n = \dfrac{2Re}{3n+2}\left(\dfrac{2n+1}{n}\right)^{2-n}$, $b = -\cot\beta$ and $m = (n+1)/n$.

3.5.3 Axisymmetric spreading of a liquid drop over a horizontal plane

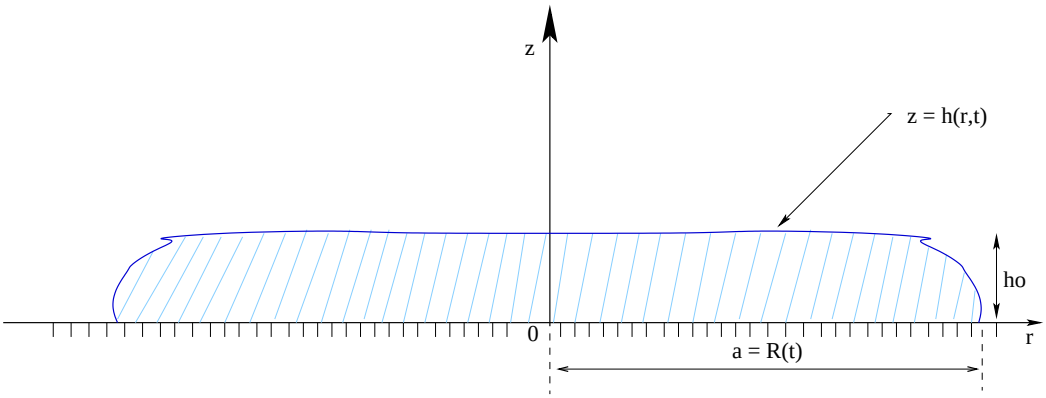

Figure 3.6: Fluid over a horizontal plane.

The stress tensor is defined in Eqs. (2.18)–(2.20). We use cylindrical polar coordinates (r, θ, z) and the velocity vector is defined by

$$\mathbf{V} = (v_r, v_\theta, v_z). \tag{3.103}$$

The flow is axisymmetric flow ($\partial/\partial\theta = 0$) and there is no flow in the θ direction so that $v_\theta = 0$. The initial radius of the film is "a" and the initial height of the film at $r = 0$ is "h_0". The fluid is viscous and incompressible.

The fluid variables are:

$$v_r = v_r(r, z, t), \quad v_\theta = 0, \quad v_z = v_z(r, z, t), \quad p = p(r, z, t).$$

The motion of the fluid is governed by the conservation of mass and momentum

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0, \tag{3.104}$$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right) = \rho F_r - \frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\ \tau_{rr}) + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r}, \tag{3.105}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right) = \rho F_\theta - \frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\ \tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}. \tag{3.106}$$

The components of the extra stress tensor are given by

$$\tau_{rr} = S_{rr} = \mu\left[\mathrm{tr}\left(\mathbf{A}_1^2\right)\right]^m E_{rr}, \tag{3.107}$$

$$\tau_{rz} = \tau_{zr} = S_{rz} = S_{zr} = \mu\left[\mathrm{tr}\left(\mathbf{A}_1^2\right)\right]^m E_{rz}, \tag{3.108}$$

$$\tau_{zz} = S_{zz} = \mu\left[\mathrm{tr}\left(\mathbf{A}_1^2\right)\right]^m E_{zz}, \tag{3.109}$$

$$\tau_{\theta\theta} = S_{\theta\theta} = \mu\left[\mathrm{tr}\left(\mathbf{A}_1^2\right)\right]^m E_{\theta\theta}, \tag{3.110}$$

where

$$E_{rr} = 2e_{rr} = 2\frac{\partial v_r}{\partial r}, \tag{3.111}$$

$$E_{rz} = E_{rz} = 2e_{rz} = 2e_{zr} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}, \tag{3.112}$$

$$E_{\theta\theta} = 2e_{\theta\theta} = \frac{\partial v_r}{r}, \tag{3.113}$$

$$E_{zz} = 2e_{zz} = \frac{\partial v_z}{\partial z}, \tag{3.114}$$

$$\mathrm{tr}\left(\mathbf{A}_1^2\right) = 2e_{ij}e_{ij} = 2\left[2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{\partial v_r}{r}\right)^2 + 2\left(\frac{\partial v_z}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2\right]. \tag{3.115}$$

Scaling

The characteristic quantities are:

Characteristic length in r-direction: a

Characteristic length in z-direction: h_0

Characteristic velocity in r-direction: U

Characteristic velocity in z-direction: $(h_0/a)U$

Characteristic time: a/U

Characteristic pressure: $\mathcal{P} = \mu(aU/h_0^2)\,(2U^2/h_0^2)^m$.

For the derivation of the characteristic velocity in z-direction, we use the continuity equation with $v_\theta = 0$ and $\partial/\partial\theta = 0$, that is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0,$$

$$O\left(\frac{U}{a}\right) + O\left(\frac{U}{a}\right) + O\left(\frac{v_z}{h_0}\right) = 0,$$

and therefore

$$v_z = O\left(\frac{h_0}{a}U\right).$$

For the derivation of the characteristic pressure, we use the r-component of the Navier-Stokes equation. Balancing the pressure gradient in the r-direction with the viscous force in the r-direction, we obtain

$$\begin{aligned}
\frac{\partial p}{\partial r} &\sim \frac{\tau_{rr}}{r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} \\
&\sim \mu\left[2\left\{\mathrm{tr}(\mathbf{A}_1^2)\right\}^m \frac{\partial v_r}{\partial r} + 2\frac{\partial}{\partial r}\left(\left\{\mathrm{tr}(\mathbf{A}_1^2)\right\}^m \frac{\partial v_r}{\partial r}\right) + \frac{\partial}{\partial z}\left(\left\{\mathrm{tr}(\mathbf{A}_1^2)\right\}^m \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\right)\right], \\
\frac{p}{a} &\sim \mu\left[2\left(\frac{2U^2}{h_0^2}\right)\frac{U}{a} + 2\left(\frac{2U^2}{h_0^2}\right)^m \frac{U}{a^2} + \left(\frac{2U^2}{h_0^2}\right)^m \left(\frac{U}{h_0^2} + \frac{U}{a^2}\right)\right].
\end{aligned}$$

In the previous evaluation, we assumed $\mathrm{tr}(\mathbf{A}_1^2) \sim 2U^2/h_0^2$. (The proof will be shown later.)

$$\begin{aligned}
\frac{p}{a} &\sim \mu\left[2\left(\frac{2U^2}{h_0^2}\right)\frac{U}{a} + 2\left(\frac{2U^2}{h_0^2}\right)^m \frac{U}{a^2} + \left(\frac{2U^2}{h_0^2}\right)^m \left(\frac{U}{h_0^2} + \frac{U}{a^2}\right)\right] \\
&\sim \mu\left(\frac{2U^2}{h_0^2}\right)^m \left[\frac{2U}{a} + \frac{2U}{a^2} + \frac{U}{h_0^2} + \frac{U}{a^2}\right] \\
&\sim \mu\frac{U}{h_0^2}\left(\frac{2U^2}{h_0^2}\right)^m \left[1 + \frac{2h_0}{a} + \frac{3h_0}{a^2}\right],
\end{aligned}$$

and therefore

$$p \sim \mu\frac{aU}{h_0^2}\left(\frac{2U^2}{h_0^2}\right)^m \left[1 + O\left(\frac{h_0}{a}\right)\right].$$

Proposition 1

$$\mathrm{tr}(\boldsymbol{A}_1^2) \sim \frac{2U^2}{h_0^2}.$$

Proof 1

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{A}_1^2) &\sim 2\left[2\ O\left(\frac{U^2}{a^2}\right) + 2\ O\left(\frac{U^2}{a^2}\right) + 2\ O\left(\frac{U^2}{a^2}\right) + O\left(\left(\frac{U}{h_0} + \frac{h_0 U}{a^2}\right)^2\right)\right] \\
&\sim 2\left[O\left(\frac{U^2}{a^2}\right) + O\left(\left(\frac{U}{h_0} + \frac{h_0 U}{a^2}\right)^2\right)\right] \\
&\sim 2\left(\frac{U^2}{h_0^2}\right)\left[O\left(\frac{h_0^2}{a^2}\right) + 1 + O\left(\left(\frac{h_0^2}{a^2}\right)^2\right)\right] \\
&\sim \left(\frac{2U^2}{h_0^2}\right)\left[1 + O\left(\frac{h_0^2}{a^2}\right)\right]. \qquad \square
\end{aligned}$$

We introduce the dimensionless variables

$$\begin{aligned}
&\overline{t} = \frac{U}{a}t, \quad \overline{r} = \frac{r}{a}, \quad \overline{z} = \frac{z}{h_0}, \quad \overline{v}_r = \frac{v_r}{U}, \quad \overline{v}_z = \frac{a}{h_0}\frac{v_z}{U}, \\
&\overline{p} = \frac{h_0^2}{\mu a U}\left(\frac{h_0^2}{2U^2}\right)^m p\ , \quad \overline{\underline{F}} = \frac{\underline{F}}{F_0}, \quad \overline{\tau} = \frac{h_0^2}{\mu a U}\left(\frac{h_0^2}{2U^2}\right)^m \tau.
\end{aligned}$$

Let

$$T_r^m = \left\{\mathrm{tr}(\mathbf{A}_1^2)\right\}^m,$$

$$\begin{aligned}
T_r^m &= 2^m\left[2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{v_r}{r}\right)^2 + 2\left(\frac{\partial v_z}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2\right]^m \\
&= 2^m\left[\frac{2U^2}{a^2}\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}}\right)^2 + \frac{2U^2}{a^2}\left(\frac{\overline{v}_{\overline{r}}}{\overline{r}}\right)^2 + \frac{2\delta^2 U^2}{h_0^2}\left(\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}\right)^2 + \left(\frac{U}{h_0}\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}} + \frac{\delta U}{a}\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right)^2\right]^m \\
&= 2^m\left[\frac{2U^2}{a^2}\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}}\right)^2 + \frac{2U^2}{a^2}\left(\frac{\overline{v}_{\overline{r}}}{\overline{r}}\right)^2 + \frac{2U^2}{a^2}\left(\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}\right)^2 + \frac{U^2}{h_0^2}\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}} + \delta^2\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right)^2\right]^m \\
&= \left(\frac{2U^2}{h_0^2}\right)^m\left[2\delta^2\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}}\right)^2 + 2\delta^2\left(\frac{\overline{v}_{\overline{r}}}{\overline{r}}\right)^2 + 2\delta^2\left(\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}\right)^2 + \left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}} + \delta^2\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right)^2\right]^m \\
&= \left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m,
\end{aligned}$$

where

$$\overline{T}_{\overline{r}}^m = \left[2\delta^2\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}}\right)^2 + 2\delta^2\left(\frac{\overline{v}_{\overline{r}}}{\overline{r}}\right)^2 + 2\delta^2\left(\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}\right)^2 + \left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}} + \delta^2\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right)^2\right]^m.$$

Writing the components of τ in terms of dimensionless variables, we obtain

$$\begin{aligned}
\tau_{rr} &= \frac{2\mu U}{a}\left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m \frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}} \\
\tau_{rz} = \tau_{zr} &= \frac{\mu U}{h_0}\left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m \left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}} + \delta^2 \frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right) \\
\tau_{\theta\theta} &= \frac{2\mu U}{a}\left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m \frac{\overline{v}_{\overline{r}}}{\overline{r}} \\
\tau_{zz} &= \frac{2\mu U}{a}\left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m \frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}.
\end{aligned}$$

Remark 1 *We can find the value of the components of $\overline{\tau}$ just by using the relation $\overline{\tau} = \mathcal{P}^{-1}\tau$, but it is not necessary to do this since we can readily substitute the previous values of the components of τ written in dimensionless form into the Navier-Stokes equations.*

We write the Navier-Stokes equations (3.104)–(3.106) with $\partial/\partial\theta = 0$ and $v_\theta = 0$ in dimensionless form.

r-component:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right) = \rho F_r - \frac{\partial p}{\partial r} + \frac{\tau_{rr}}{r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r}.$$

To write this equation in a dimensionless form, we proceed as follows:

$$\begin{aligned}
\rho\frac{U^2}{a}\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{t}} + \overline{v}_{\overline{r}}\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}} + \overline{v}_{\overline{z}}\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}}\right) = {} & \rho F_0 \overline{F}_r - \frac{\mu U}{h_0^2}\left(\frac{2U^2}{h_0^2}\right)^m \frac{\partial \overline{p}}{\partial \overline{r}} + \frac{2\mu U}{a^2}\left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m \frac{1}{\overline{r}}\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}} \\
& + \frac{2\mu U}{a^2}\left(\frac{2U^2}{h_0^2}\right)^m \frac{\partial}{\partial \overline{r}}\left[\overline{T}_{\overline{r}}^m \frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{r}}\right] \\
& + \frac{\mu U}{a^2}\left(\frac{2U^2}{h_0^2}\right)^m \frac{\partial}{\partial \overline{z}}\left[\overline{T}_{\overline{r}}^m \left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}} + \delta^2 \frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right)\right] \\
& - \frac{2\mu U}{a^2}\left(\frac{2U^2}{h_0^2}\right)^m \overline{T}_{\overline{r}}^m \frac{\overline{v}_{\overline{r}}}{\overline{r}^2}.
\end{aligned}$$

Multiplying both sides of the latter equation by $(h_0^2/\mu U) * (h_0^2/2U^2)^m$ and letting $\nu = \mu/\rho$, we obtain

$$\frac{Ua}{\nu}\left(\frac{h_0^2}{2U^2}\right)^m\left(\frac{h_0}{a}\right)^2\left[\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{t}}+\overline{v}_{\overline{r}}\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{r}}+\overline{v}_{\overline{z}}\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{z}}\right]= \frac{F_0h_0^2}{\nu U}\left(\frac{h_0^2}{2U^2}\right)^m\overline{F}_r-\frac{\partial\overline{p}}{\partial\overline{r}} + 2\left(\frac{h_0}{a}\right)^2\overline{T}_{\overline{r}}^m\frac{1}{\overline{r}}\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{r}} + 2\left(\frac{h_0}{a}\right)^2\frac{\partial}{\partial\overline{r}}\left[\overline{T}_{\overline{r}}^m\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{r}}\right] + \frac{\partial}{\partial\overline{z}}\left[\overline{T}_{\overline{r}}^m\left(\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{z}}+\delta^2\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{r}}\right)\right] - 2\left(\frac{h_0}{a}\right)^2\overline{T}_{\overline{r}}^m\frac{\overline{v}_{\overline{r}}}{\overline{r}^2}.$$

Letting $\Re e=(Ua/\nu)\left(h_0^2/2U^2\right)^m$ be the generalized Reynolds number, the latter equation becomes

$$\Re e\left(\frac{h_0}{a}\right)^2\left[\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{t}}+\overline{v}_{\overline{r}}\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{r}}+\overline{v}_{\overline{z}}\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{z}}\right]= \frac{F_0h_0^2}{\nu U}\left(\frac{h_0^2}{2U^2}\right)^m\overline{F}_r-\frac{\partial\overline{p}}{\partial\overline{r}}+2\left(\frac{h_0}{a}\right)^2\overline{T}_{\overline{r}}^m\frac{1}{\overline{r}}\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{r}} + 2\left(\frac{h_0}{a}\right)^2\frac{\partial}{\partial\overline{r}}\left[\overline{T}_{\overline{r}}^m\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{r}}\right]+\frac{\partial}{\partial\overline{z}}\left[\overline{T}_{\overline{r}}^m\left(\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{z}}+\delta^2\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{r}}\right)\right] - 2\left(\frac{h_0}{a}\right)^2\overline{T}_{\overline{r}}^m\frac{\overline{v}_{\overline{r}}}{\overline{r}^2}. \tag{3.116}$$

θ- component:

Since $v_\theta=0$ and $\partial/\partial\theta=0$, it reduces to

$$F_\theta=\ 0. \tag{3.117}$$

z-component:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)= \rho F_z-\frac{\partial p}{\partial z}+\frac{\tau_{rz}}{r}+\frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\tau_{zz}}{\partial z}.$$

To write this equation in a dimensionless form, we proceed as follows:

$$\rho U^2\frac{h_0}{a^2}\left(\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{t}}+\overline{v}_{\overline{r}}\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{r}}+\overline{v}_{\overline{z}}\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{z}}\right)= \rho F_0\overline{F}_z-\frac{\mu Ua}{h_0^3}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial\overline{p}}{\partial\overline{r}} + \frac{\mu U}{ah_0}\left(\frac{2U^2}{h_0^2}\right)^m\overline{T}_{\overline{r}}^m\frac{1}{\overline{r}}\left(\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{z}}+\delta^2\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{r}}\right) + \frac{\mu U}{h_0a}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial}{\partial\overline{r}}\left[\overline{T}_{\overline{r}}^m\left(\frac{\partial\overline{v}_{\overline{r}}}{\partial\overline{z}}+\delta^2\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{r}}\right)\right] + \frac{2\mu U}{ah_0}\left(\frac{2U^2}{h_0^2}\right)^m\frac{\partial}{\partial\overline{z}}\left[\overline{T}_{\overline{r}}^m\frac{\partial\overline{v}_{\overline{z}}}{\partial\overline{z}}\right].$$

Multiplying both sides of the latter equation by $(h_0^3/\mu U a) * (h_0^2/2U^2)^m$, we obtain

$$\begin{aligned}\Re e\left(\frac{h_0}{a}\right)^4\left[\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{t}}+\overline{v}_{\overline{r}}\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}+\overline{v}_{\overline{z}}\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}\right] = \quad & \frac{F_0 h_0^3}{\nu U a}\left(\frac{h_0^2}{2U^2}\right)^m \overline{F}_z-\frac{\partial \overline{p}}{\partial \overline{z}} \\ & +\left(\frac{h_0}{a}\right)^2 \overline{T}_{\overline{r}}^m \frac{1}{\overline{r}}\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}}+\delta^2\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right) \\ & +\left(\frac{h_0}{a}\right)^2 \frac{\partial}{\partial \overline{r}}\left[\overline{T}_{\overline{r}}^m\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}}+\delta^2\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{r}}\right)\right] \\ & +2\left(\frac{h_0}{a}\right)^2 \frac{\partial}{\partial \overline{z}}\left[\overline{T}_{\overline{r}}^m\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}\right]. \end{aligned} \tag{3.118}$$

Continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)+\frac{\partial v_z}{\partial z}=0.$$

For the dimensionless form, we write

$$\frac{1}{a\overline{r}}\frac{\partial}{\partial(a\overline{r})}\left(a\overline{r}U\overline{v}_{\overline{r}}\right)+\frac{\partial(\delta U\overline{v}_{\overline{z}})}{\partial(h_0\overline{z})}=0,$$

$$\frac{U}{a}\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\ \overline{v}_{\overline{r}}\right)+\frac{U}{a}\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}=0.$$

Therefore

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\ \overline{v}_{\overline{r}}\right)+\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}=0. \tag{3.119}$$

Imposing the thin film approximation

$$\frac{h_0}{a}\ll 1,$$

$$\Re e\left(\frac{h_0}{a}\right)^2\ll 1,$$

Eqs. (3.116) to (3.119) reduce to

$$0=-\frac{\partial \overline{p}}{\partial \overline{r}}+\frac{F_0 h_0^2}{\mu U}\left(\frac{h_0^2}{2U^2}\right)^m\overline{F}_r+\frac{\partial}{\partial \overline{z}}\left[\left(\frac{\partial \overline{v}_{\overline{r}}}{\partial \overline{z}}\right)^{2m+1}\right], \tag{3.120}$$

$$0=\overline{F}_\theta, \tag{3.121}$$

$$0=-\frac{\partial \overline{p}}{\partial \overline{z}}+\frac{F_0 h_0^3}{\mu U a}\left(\frac{h_0^2}{2U^2}\right)^m\overline{F}_z, \tag{3.122}$$

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\ \overline{v}_{\overline{r}}\right)+\frac{\partial \overline{v}_{\overline{z}}}{\partial \overline{z}}=0. \tag{3.123}$$

Balancing the pressure gradient in the z-direction with the force of gravity in the z-direction, since $F = (0, 0, -g)$, we obtain:

$$
\begin{aligned}
-\frac{\partial p}{\partial z} &\sim \rho F_z = -\rho g, \\
\frac{\partial p}{\partial z} &\sim \rho g, \\
\frac{\mathcal{P}}{h_0} &= \rho g, \\
\mathcal{P} &= \rho g h_0.
\end{aligned}
$$

Equating the two expressions of the characteristic pressure we obtain

$$
\begin{aligned}
\mu \frac{aU}{h_0^2} \left(\frac{2U^2}{h_0^2} \right)^m &= \rho g h_0, \\
U &= \left(\frac{g h_0^{2m+3}}{\nu a 2^m} \right)^{1/(2m+1)}.
\end{aligned}
$$

We take $F_0 = 1$ and suppress the bars to keep the notation simple. The Navier-Stokes and continuity equations (3.120)–(3.123) then reduce to

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial z} \left[\left(\frac{\partial v_r}{\partial z} \right)^{2m+1} \right], \tag{3.124}$$

$$\frac{\partial p}{\partial z} = -1, \tag{3.125}$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r \; v_r) + \frac{\partial v_z}{\partial z} = 0. \tag{3.126}$$

The boundary conditions are:

$$\text{at} \quad z = 0 \qquad : \quad v_r = v_z = 0, \tag{3.127}$$

$$\text{at} \quad z = h(r,t) : \quad p(r,h,t) = p_0, \quad \text{(the surface tension is neglected)} \tag{3.128}$$

$$\text{at} \quad z = h(r,t) : \quad \tau_{zr} = \frac{h_0}{a} \left(\frac{\partial v_r}{\partial z} \right)^{2m+1} (r,h,t) = 0, \tag{3.129}$$

$$\text{at} \quad z = h(r,t) : \quad v_z(r,h,t) = \frac{\partial h}{\partial t}(r,t) + v_r(r,h,t) \frac{\partial h}{\partial r}(r,t). \tag{3.130}$$

Eq. (3.127) represents the no-slip condition, Eq. (3.128) represents the normal stress balance, Eq. (3.129) represents the tangential stress balance and Eq. (3.130)

represents the kinematic boundary condition.

The continuity equation (3.126) is integrated with respect to z from 0 to $h(r,t)$ and the boundary condition (3.127) is imposed

$$\int_0^h \frac{1}{r}\frac{\partial}{\partial r}(rv_r)dz + \int_0^h \frac{\partial v_z}{\partial z}dz = 0, \tag{3.131}$$

$$\frac{1}{r}\int_0^h \frac{\partial}{\partial r}(rv_r)dz + v_z(r,h,t) = v_z(r,0,t), \tag{3.132}$$

$$\frac{1}{r}\int_0^h \frac{\partial}{\partial r}(rv_r)dz + \frac{\partial h}{\partial t} + v_r(r,h,t)\frac{\partial h}{\partial r} = 0, \tag{3.133}$$

$$\frac{\partial h}{\partial t} + \frac{1}{r}\int_0^h \frac{\partial}{\partial r}(rv_r)dz + v_r(r,h,t)\frac{\partial h}{\partial r} = 0. \tag{3.134}$$

Leibnitz rule gives

$$\frac{1}{r}\frac{\partial}{\partial r}\int_0^h rv_r dz = \frac{1}{r}\int_0^h \frac{\partial}{\partial r}(rv_r)dz + v_r(r,h,t)\frac{\partial h}{\partial r}, \tag{3.135}$$

so that

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\int_0^h rv_r dz = 0, \tag{3.136}$$

that is

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\int_0^h v_r(r,z,t)dz\right) = 0. \tag{3.137}$$

Eq. (3.125) is readily integrated subject to the boundary condition (3.128) to give

$$p(r,z,t) = p_0 + h(r,t) - z. \tag{3.138}$$

Substituting Eq. (3.138) into Eq. (3.124), we obtain

$$\frac{\partial h}{\partial r} = \frac{\partial}{\partial z}\left[\left(\frac{\partial v_r}{\partial z}\right)^{2m+1}\right]. \tag{3.139}$$

Integrating Eq. (3.139) with respect to z

$$z\frac{\partial h}{\partial r} = \left(\frac{\partial v_r}{\partial z}\right)^{2m+1} + c,$$

and the condition (3.129) implies that

$$c = h\frac{\partial h}{\partial r}.$$

Thus

$$\left(\frac{\partial v_r}{\partial z}\right)^{2m+1} = (z-h)\,\frac{\partial h}{\partial r} \qquad (m \neq -1/2),$$

so that

$$\frac{\partial v_r}{\partial z} = \left[(z-h)\,\frac{\partial h}{\partial r}\right]^{1/(2m+1)}. \tag{3.140}$$

If $m = -3/4$, Eq. (3.140) reads

$$\frac{\partial v_r}{\partial z} = \left[(z-h)\,\frac{\partial h}{\partial r}\right]^{-2}.$$

Integrating the latter equation with respect to z and applying the boundary condition (3.127), we obtain

$$v_r = \left[\frac{1}{h-z} - \frac{1}{h}\right]\left(\frac{\partial h}{\partial r}\right)^{-2}. \tag{3.141}$$

The integral $\int_0^h v_r dz$, diverges since $\int_0^h (1/(h-z) - 1/h)\,dz$ does not converge on the interval $[0, h]$ and the behaviour of $h_r \to 0$ as $r \to 0$ cannot remove the singularity since h_r in Eq. (3.141) is raised to the power of -2. Therefore unless otherwise specified, we assume that $m \neq -3/4$. Integrating Eq. (3.140) with respect to z we obtain

$$v_r(r,z,t) = \frac{1}{\beta}\,(z-h)^{\beta}\left(\frac{\partial h}{\partial r}\right)^{1/(2m+1)} + c',$$

where $\beta = (2m+2)/(2m+1)$, and c' is an integration constant.

Applying the boundary condition (3.127) we obtain

$$c' = -\frac{1}{\beta}h^{\beta}\left(\frac{\partial h}{\partial r}\right)^{\beta-1},$$

and hence

$$v_r(r,z,t) = \frac{1}{\beta}\left[(z-h)^{\beta} - h^{\beta}\right]\left(\frac{\partial h}{\partial r}\right)^{\beta-1}. \tag{3.142}$$

Finally, substituting Eq. (3.142) into Eq. (3.137), we obtain

$$\frac{\partial h}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\int_0^h\frac{1}{\beta}\left[(z-h)^\beta-h^\beta\right]\left(\frac{\partial h}{\partial r}\right)^{\beta-1}dz\right)=0,$$

$$\int_0^h\left[(z-h)^\beta-h^\beta\right]dz=-\frac{\beta}{\beta+1}h^{\beta+1}.$$

Thus the equation modelling the height of a thin power-law film on a horizontal plane in the presence of gravity is defined by

$$\frac{\partial h}{\partial t}=\frac{1}{(\beta+1)r}\frac{\partial}{\partial r}\left(rh^{\beta+1}\left(\frac{\partial h}{\partial r}\right)^{\beta-1}\right). \tag{3.143}$$

3.6 Governing equation of the thin film flow of a generalized second-grade fluid on a moving belt

The fundamental equations governing the motion of an incompressible fluid neglecting thermal effects, are given by the continuity and linear momentum equations

$$\text{div }\mathbf{V}\ =\ 0, \tag{3.144}$$

$$\rho\frac{d\mathbf{V}}{dt}\ =\ \text{div }\mathbf{T}\ +\ \rho\mathbf{b}, \tag{3.145}$$

where $\mathbf{V}$ is the velocity vector, ρ is the fluid density, t is the time, $\mathbf{b}$ is the body force and d/dt is the material time derivative. The Cauchy stress tensor $\mathbf{T}$ for a generalized second-grade fluid is given by Eq. (2.38), i.e.

$$\mathbf{T}\ =\ -p\,\mathbf{I}\ +\ \mu\Pi^{m/2}\mathbf{A}_1\ +\ \alpha_1\mathbf{A}_2\ +\ \alpha_2\mathbf{A}_1^2,$$

where

$$\Pi=\frac{1}{2}\text{tr}(\mathbf{A}_1^2),$$

is the second invariant of the symmetric part of the velocity gradient, m is the material parameter, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$, $\mathbf{A}_2$, are the kinematical tensors defined in Eqs. (2.20) and (2.35) respectively. When $m < 0$, the fluid is shear-thinning, and when $m > 0$, the fluid is shear-thickening. This constitutive equation is capable of predicting shear-thinning and shear-thickening phenomena.

It is worth mentioning here that a power-law fluid has a shear-dependent viscosity but exhibits no normal stress differences. The generalized second-grade fluid, which has been used successfully in modelling icy mesh flow (Man *et al.*, 1985), exhibits both shear thinning and normal stress differences. If $\alpha_1 = \alpha_2 = 0$, the power-law model (Bird *et al.*, 1977) is recovered from Eq. (2.38). Moreover, if $\alpha_1 = \alpha_2 = 0$ and $m = 0$ we are left with the classical model of Navier and Stokes.

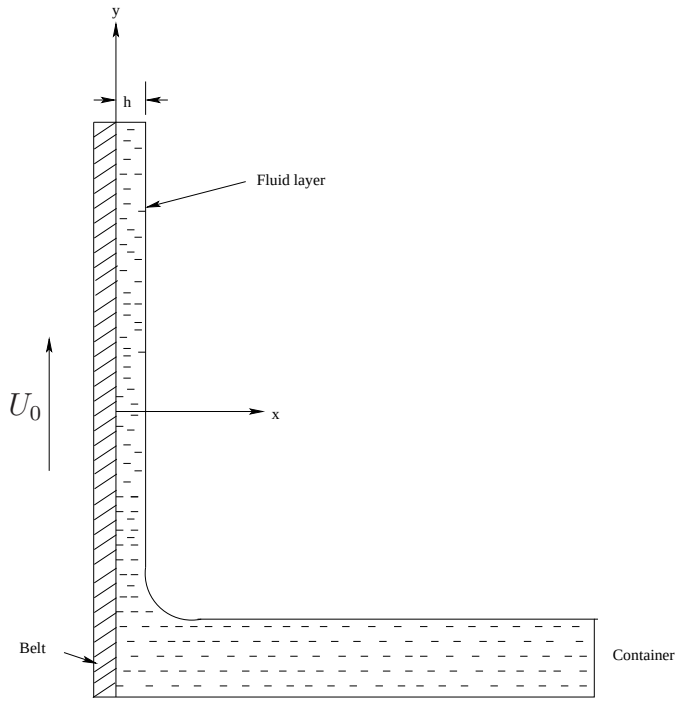

Figure 3.7: Geometry of the flow of moving belt through a non-Newtonian fluid.

Consider a container having a generalized second-grade fluid in it. A wide moving

belt, which moves vertically upward with constant velocity U_0, passes through this container as shown in Fig. 3.7. Since the belt moves upward and passes through the fluid, it picks up a film fluid of thickness $h(y,t)$. Due to gravity, the fluid film tends to drain down the belt. For simplicity, some assumptions are made

(i) the flow is in steady state,

(ii) the flow is laminar and uniform,

(iii) the film fluid thickness h is uniform, thus does not depend on y.

We choose an xy-coordinate system and position the x-axis parallel to the fluid and normal to the belt, the y-axis upward along the belt and the z-axis normal to the plane. The only velocity component is in the y-direction, therefore

$$\mathbf{V} \;=\; (0, v(x), 0), \tag{3.146}$$

and also the extra stress tensor is a function of x only, that is

$$\mathbf{S} \;=\; \mu\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \;=\; \mathbf{S}(x). \tag{3.147}$$

Eq. (3.146) satisfies the continuity equation (3.144) identically. Using Eq. (2.38) in the momentum equation (3.145), we obtain

x-momentum

$$2(2\alpha_1 + \alpha_2)\left(\frac{dv}{dx}\right)\left(\frac{d^2v}{dx^2}\right) \;=\; \frac{\partial p}{\partial x}, \tag{3.148}$$

y-momentum

$$\mu(m+1)\left(\left|\frac{dv}{dx}\right|\right)^m\left(\frac{d^2v}{dx^2}\right) - \rho g \;=\; \frac{\partial p}{\partial y}, \tag{3.149}$$

z-momentum

$$0 \;=\; \frac{\partial p}{\partial z}. \tag{3.150}$$

Eq. (3.148) can be rewritten as

$$(2\alpha_1 + \alpha_2)\frac{d}{dx}\left[\left(\frac{dv}{dx}\right)^2\right] \;=\; \frac{\partial p}{\partial x}, \tag{3.151}$$

and defining the modified pressure by $\hat{p}$

$$\hat{p} \;=\; p-(2\alpha_1+\alpha_2)\left(\frac{dv}{dx}\right)^2, \tag{3.152}$$

Eqs. (3.149)–(3.152) imply that

$$\mu(m+1)\left(\left|\frac{dv}{dx}\right|\right)^m\left(\frac{d^2v}{dx^2}\right)-\rho g \;=\; \frac{\partial\hat{p}}{\partial y}, \tag{3.153a}$$

$$\frac{\partial\hat{p}}{\partial x} \;=\; \frac{\partial\hat{p}}{\partial z} \;=\; 0. \tag{3.153b, c}$$

Eqs. $(3.153b, c)$ imply that $\hat{p} \;=\; \hat{p}(y)$ and Eq. $(3.153a)$ implies that $\partial\hat{p}/\partial y$ is at most a constant. As there is no pressure gradient along the y-direction, this constant can be taken to be zero and thus the fluid is moving upwards. We note that at this level $\hat{p} \;=\;$ constant throughout the flow, and p is affected by $v(x)$ through the normal stress moduli α_1 and α_2. Eq. $(3.153a)$ therefore becomes

$$\mu(m+1)\left(\left|\frac{dv}{dx}\right|\right)^m\left(\frac{d^2v}{dx^2}\right)-\rho g=0. \tag{3.154}$$

The boundary conditions will be

$$v \;=\; U_0, \quad \text{at} \quad x \;=\; 0 \quad \text{(at the belt)}, \tag{3.155a}$$

$$S_{xy} \;=\; 0, \qquad \text{at} \quad x \;=\; h \quad \text{(shear stress at the free surface)}, \tag{3.155b}$$

where

$$S_{xy} \;=\; \mu\frac{dv}{dx}\left|\frac{dv}{dx}\right|^m. \tag{3.156}$$

Substituting Eq. (3.156) in Eq. $(3.155b)$, we obtain

$$\frac{dv}{dx} \;=\; 0, \quad \text{at} \quad x \;=\; h. \tag{3.157}$$

Thus the flow of a generalized second-grade fluid on a vertically moving belt is governed by

$$\mu(m+1)\left(\left|\frac{dv}{dx}\right|\right)^m\left(\frac{d^2v}{dx^2}\right)-\rho g=0, \tag{3.158}$$

subject to the boundary conditions

$$\begin{aligned} v &= U_0, \quad \text{at } x = 0 \\ \frac{dv}{dx} &= 0, \quad \text{at } x = h. \end{aligned} \tag{3.159}$$

In order to write Eqs. (3.158) and (3.159) in dimensionless form, we introduce the dimensionless variables

$$x^* = \frac{x}{h}, \quad v^* = \frac{v}{U_0}, \quad k^* = \frac{\rho g h^2}{\mu U_0}\left(\frac{h}{U_0}\right)^m. \tag{3.160}$$

Recall that for a generalized second grade fluid, $[\mu] = ML^{-1}T^{m-1}$. Dropping the stars, the dimensionless forms of Eqs. (3.158) and (3.159) are given respectively by

$$(m+1)\left|\frac{dv}{dx}\right|^m \frac{d^2v}{dx^2} - k = 0, \tag{3.161}$$

$$\begin{aligned} v &= 1, \quad \text{at } x = 0, \\ \frac{dv}{dx} &= 0, \quad \text{at } x = 1. \end{aligned} \tag{3.162}$$

We remark here that $k \neq 0$ in Eq. (3.160).

3.7 Conclusion

In this chapter we considered two variants of the power-law model of a non-Newtonian fluid. We used lubrication theory for the first variant to obtain a simplified form of the corresponding Navier-Stokes equations and gave the relationship between the two models. For the boundary conditions, we derived the surface kinematic boundary condition, the surface stress conditions and mentioned the no-slip condition. We defined some dimensionless parameters in order to write the Navier-Stokes equations and the boundary conditions in dimensionless form. We also discussed the equation of the spreading of an insoluble surfactant over a thin film and its dimensionless form. Finally we derived the evolution equations of the various flows studied in the subsequent chapters.

Chapter 4

Numerical Analysis of the Dynamics of a Thin Power-Law Fluid

4.1 Introduction

In this chapter, we study the dynamics of a thin power-law fluid. The chapter is divided in two main parts. In the first part, we study the numerical solution of the fourth-order ode[1] resulting from the spreading of a thin power-law fluid down a vertical plane and in the second part, we study the spreading of a non-soluble surfactant over the surface of the fluid.

[1] ode in this chapter stands for ordinary differential equation

4.2 Numerical solution of a fourth-order ode modelling thin power-law fluids

4.2.1 Introduction

The applications of thin film flow down a vertical plane has been the subject of study for the last five decades due to its various applications in the technological development of modern science (see e.g. the reviews of Oron *et al.*, 1997 and Myers, 1998). The dynamics of waves propagating on the free surface of a thin film has received much attention from various industries due to its dramatic effect on the transport rate of mass (Steinthrop and Wild, 1967), heat (Fedotkin and Firisyuk, 1969 and Williams *et al.*, 1968) and momentum (Dukler, 1972), in designing distillation and absorption columns, evaporators, condensers, nuclear reactor emergency cooling systems, etc. Understanding how waves propagate on the free surface of a thin film is important in modern precision coating of photographic emulsions, magnetic material, protective paints, flow of molten metal/lava, etc.

Study of the motion of a wave on the free surface of a falling film started with the pioneering experiments by Kapitza, (1948) and Kapitza and Kapitza, (1949). Many other researchers have worked on this fascinating problem (see for example, the reviews by Fulford, 1964, Lin and Wang, 1985 and Chang, 1994). Moriarty and Schwartz, (1991) have studied the spreading of a thin liquid drop draining down a vertical wall under gravity and small surface tension with and without slip at the contact line. Goodwin and Homsy, (1991) studied viscous flow down a slope in the vicinity of a contact line, introducing a contact angle boundary condition at the leading edge of the flow. The above mentioned studies assume that the fluid is Newtonian.

Since most of the fluids used in industry are basically non-Newtonian, it was then necessary to investigate these fluids. Some studies on linear stability of non-

Newtonian liquid film flow were made by Gupta, (1967) considering a second-order fluid; by Liu and Mei, (1989) for a Bingham fluid; by Lai, (1967) for an Oldroyd-B fluid; by Hwang *et al.*, (1994) and Berezin *et al.*, (1998) for a power-law fluid. Weidner and Schwartz, (1994) studied the contact-line motion of shear-thinning liquids using a three-constant Ellis viscosity model. This model allows for power-law shear-thinning behaviour at high shear stress and the authors used this property to mitigate the moving contact line paradox at the leading edge of the flow. More recently, Dandapat and Mukhopadhyay, (2003) investigated waves that occur at the surface of a falling film of a thin power-law fluid on a vertical plane. They derived an evolution equation representing two wave equations under long wave approximations. Based on the different ranges of the physical parameters, they showed that different types of waves are possible on the surface of the film: kinematic and dynamic waves. They concluded that these waves are a result of nonlinear interaction between kinematic and dynamic waves and these wave characteristics strongly depend on the power-law index m.

In this part of the chapter, we consider the evolution of a wave on the free surface of a non-Newtonian thin film with a contact angle boundary condition imposed at both the leading and trailing contact lines under gravity and surface tension. Our main contribution is in deriving and analyzing the fourth-order ordinary differential equation that models the evolution of waves on the free surface of a non-Newtonian fluid. We approximate the solution of the fourth-order ode numerically by imposing contact angle boundary condition at the leading contact line. The results obtained here offer new insights into the relationship between the contact angle and power law index m. We also show a dependence of the capillary ridge height on m.

4.2.2 Governing equation

We consider a thin film on an inclined plane whose angle of inclination is α to the horizontal (see Fig. 3.1). We denote the components of the velocity of the fluid in the x and y directions by u and v respectively. Let $y = h(x,t)$ be the free surface of

the flow, whose characteristic thickness and characteristic length along the plane are denoted by h_0 and L respectively. The forces assumed to be driving the thin film are surface tension, surface tension gradients, gravity and viscous shear. The evolution equation is given by Eq. (3.60).

4.2.3 Travelling wave solutions

In this section we consider only capillary flow ($C \neq 0$) for the case $M = 0$ and $\alpha = \pi/2$, i.e. we investigate the flow of a thin power-law fluid down a vertical wall under the effects of gravity and surface tension. Equation (3.60) then reduces to the fourth-order nonlinear partial differential equation

$$\frac{\partial h}{\partial t} = -\frac{1}{\beta+1}\frac{\partial}{\partial x}\left[h^{\beta+1}\left(C\frac{\partial^3 h}{\partial x^3} + B\right)^{\beta-1}\right]. \tag{4.1}$$

Travelling-wave solutions admitted by Eq. (4.1) are found by writing the film height in the form

$$h(t,x) = y(\overline{x}), \qquad \overline{x} = x - vt, \tag{4.2}$$

where v is the constant wave velocity. Substituting Eq. (4.2) into Eq. (4.1) we obtain the fourth-order ode

$$-vy' = -\frac{1}{\beta+1}\frac{d}{dx}\left[y^{\beta+1}\left(Cy''' + B\right)^{\beta-1}\right]. \tag{4.3}$$

We have dropped the bars and $' = d/d\overline{x}$. We can integrate Eq. (4.3) once with respect to x to obtain

$$y''' = \left(ay^{-k} + by^{1-k}\right)\frac{1}{k-2} \quad - c, \tag{4.4}$$

where $a = kA/C^{\beta-2}$, $b = kv/C^{\beta-2}$, $c = B/C$, $k = \beta + 1$, and A is a constant of integration. We note that for $k = 3$ Eq. (4.4) reduces to the well-known third-order ode investigated by Moriarty and Schwartz, (1992) and Tuck and Schwartz, (1990).

We investigate numerical solutions admitted by the fourth-order ode (4.3) subject to the following conditions:

$$x < 0, \quad y \to 1 \quad \text{as} \quad x \to -\infty \tag{4.5}$$

$$x = \bar{x}^*, \quad y(\bar{x}^*) = h_p, \quad y'(\bar{x}^*) = -\tan\phi. \tag{4.6}$$

ϕ is the contact angle at the leading edge of the film where the contact line occurs at $x = \bar{x}^*$. We cannot impose zero boundary conditions on the contact line because of the singular nature of Eq. (4.3) as $y \to 0$. This condition is replaced by a precursor film $h_p \ll 1$. In this work we choose $h_p = 10^{-2}$. Note that the contact angle should satisfy the condition $0 < \tan\phi \ll 1$. For our calculations we choose $\phi \in \{-3, -2.5, 0.25, 0.5\}$. The wave velocity v is chosen to be unity. We rescale Eq. (4.3) to the canonical form

$$y' = \frac{d}{dx}\left[y^{\beta+1}\left(y''' + 1\right)^{\beta-1}\right]. \tag{4.7}$$

4.2.4 Numerical results and discussion

Consider Eq. (4.7). This fourth-order ode can be integrated once with respect to x to obtain

$$y^{\beta+1}\left(\frac{d^3y}{dx^3} + 1\right)^{\beta-1} + y = c_0, \tag{4.8}$$

where c_0 is a constant of integration. Note that Eq. (4.8) has a constant solution $y = 1$, i.e. $c_0 = 2$. We follow the same procedure as described by Tuck and Schwartz, (1990) and Momoniat *et al.*, (2005) to find an approximate solution which will provide a starting point for a marching scheme. To this end, we perturb the constant solution

$$y = 1 + \varepsilon g(x), \qquad \varepsilon << 1, \tag{4.9}$$

where $g(x)$ is an unknown function of x, to obtain the third-order linear ordinary differential equation

$$(\beta - 1)\frac{d^3g(x)}{dx^3} + (\beta + 1)g(x) = 0. \tag{4.10}$$

The general solution of the above third-order linear ordinary differential equation is given by

$$\begin{aligned} g(x) &= c_0 \exp\left(-\left(\frac{\beta+1}{\beta-1}\right)^{1/3} x\right) + \exp\left(\frac{1}{2}\left(\frac{\beta+1}{\beta-1}\right)^{1/3} x\right)\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\left(\frac{\beta+1}{\beta-1}\right)^{1/3} x\right)\right. \\ &+ \left. c_2 \sin\left(\frac{\sqrt{3}}{2}\left(\frac{\beta+1}{\beta-1}\right)^{1/3} x\right)\right]. \end{aligned} \tag{4.11}$$

There are three possible solutions for the description of the film height but only one is necessary. We choose $c_0 = 0$ since the solution will be unbounded which results in a physically irrelevant case. Given that the equation is autonomous and can consequently be shifted arbitrarily in x, we choose $c_1 = 1$ and $c_2 = 0$. This results in a solution of the form

$$y = 1 + \varepsilon \exp\left(\frac{1}{2}\left(\frac{\beta+1}{\beta-1}\right)^{1/3} x\right) \cos\left(\frac{\sqrt{3}}{2}\left(\frac{\beta+1}{\beta-1}\right)^{1/3} x\right), \tag{4.12}$$

which can be used as starting point for the application of the Runge-Kutta method to approximate the solution of Eq. (4.7) numerically. Note that to achieve the constant height as $x \to -\infty$, we impose $(\beta+1)/(\beta-1) > 0$ in Eq. (4.12), that is, $\beta \notin [-1, 1]$.

A computer code was written for the fourth-order Runge-Kutta method which we used for the inner solution near the moving contact line combined with `bvp4c` in Matlab® to solve Eq. (4.7) subject to Eqs. (4.5) and (4.6). We impose a matching condition at $x_m \approx 5$ where the minimum occurs in the inner solution. Our numerical results are obtained using Matlab® and the graphical results are shown in Figures 4.1–4.3.

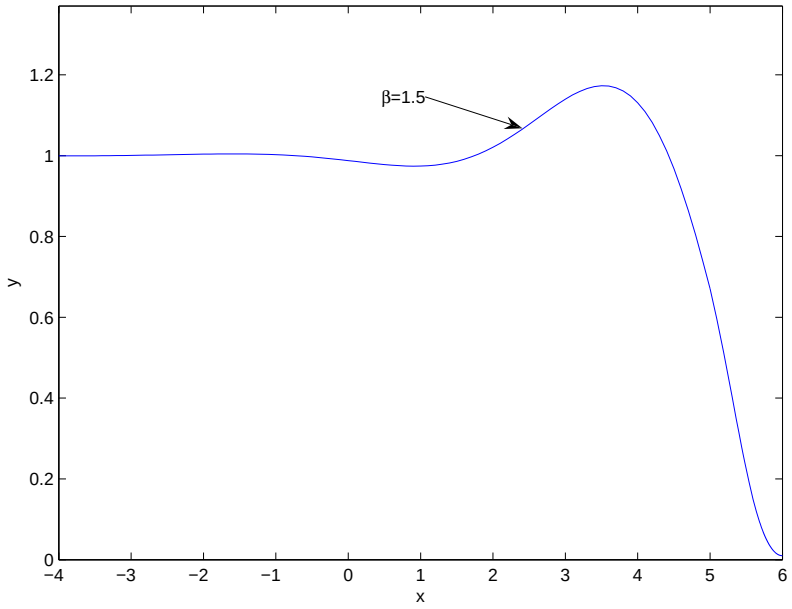

Figure 4.1: Solution to Eq. (4.7) with constant contact angle $\phi = -2.5$ (rad) for $\beta = 1.5$ (shear thinning).

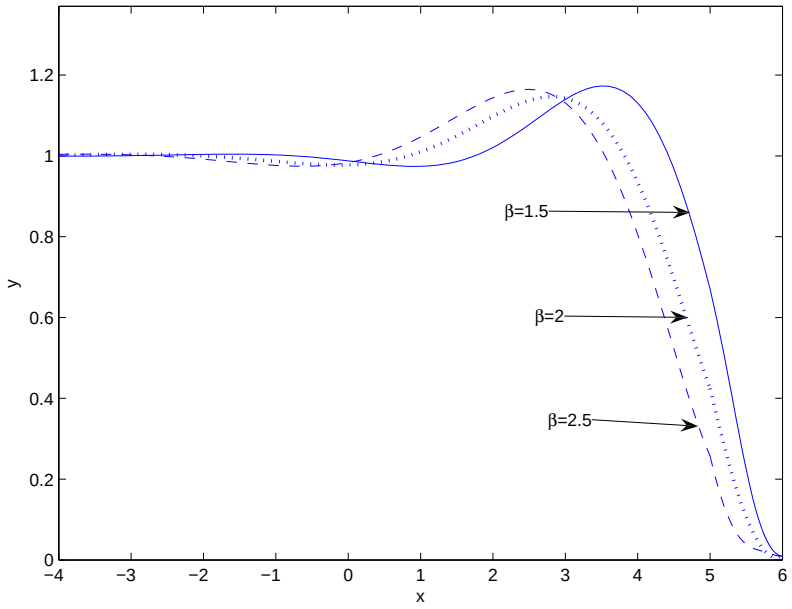

Figure 4.2: Film profile for the variation of β with constant angle $\phi = -2.5$ (rad)

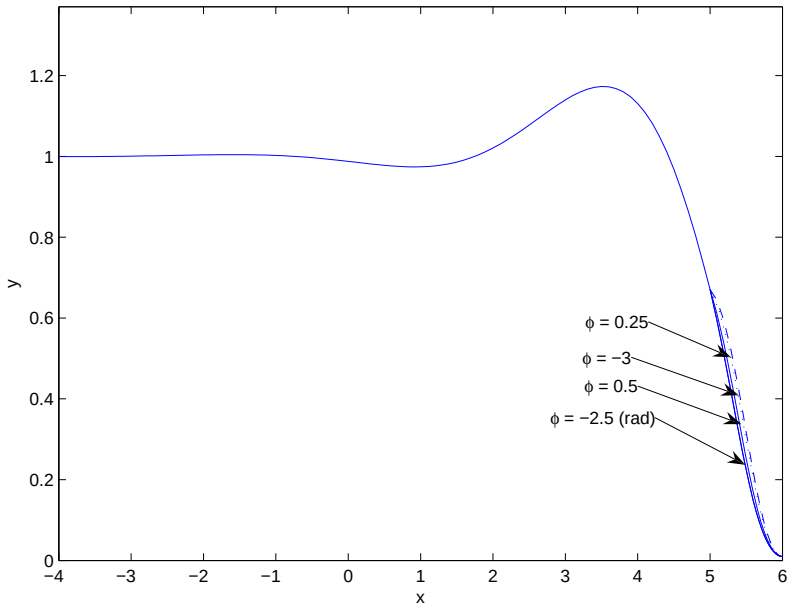

Figure 4.3: Film profile for the variation of ϕ with $\beta = 1.5$

From Figures 4.1–4.3 we observe the development of capillary ridges at the leading contact line. In Figure 4.2 we notice the influence of the power-law parameter $\beta = (2m + 2)/(2m + 1)$ on the film profile. There is a neat departure from $\beta = 2$ (Newtonian fluid) for $\beta = 1.5$ and $\beta = 2.5$. This is not surprising since we know that we are dealing with a power-law model which exhibits shear thickening behaviour ($\beta > 2$) and shear thinning behaviour ($\beta < 2$). We also realize from Figure 4.3 that the contact angle has a slight effect on the film profile near the contact angle but does not influence the capillary ridge and the film profile behind the ridge. We refer to Smith, (1995) for a discussion on these types of flows. We have not done a comparison of our results with those of Smith.

4.2.5 Conclusion

We make the following concluding remarks:

- We solved a fourth-order ode describing travelling wave solutions instead of a third-order ode as carried out in the literature (see for example Moriarty and Schwartz, 1992 and Tuck and Schwartz, 1990).
- We imposed a contact angle at the leading contact line.
- We studied the influence of both shear-thickening and shear-thinning by the variation of the power-law index m via $\beta = (2m + 2)/(2m + 1)$.

We have provided a deeper insight into the effects of varying power-law index β, and leading contact angle ϕ. Our results show that the film profile is considerably influenced by the power-law index β and that the variation of the contact angle ϕ in a prescribed domain has a negligible effect on the overall film profile.

4.3 Insoluble surfactant over a thin power-law fluid: non-Newtonian effect

In this section, we are concerned with the spreading of an insoluble surfactant over a thin power-law fluid. We study the effect of both the surfactant and the power-law index over the film height.

4.3.1 Introduction

Surfactants or surface active agents are molecules that preferentially accumulate at the fluid interface. It is energetically favourable for them to occupy the surface rather than the bulk fluid so they reduce surface energy or surface tension. In other terms, surface tension decreases as the surfactant concentration increases. The study of the dynamics of a surfactant over a thin liquid film has many applications in science, industry and medicine. For instance, the transport of surfactants on thin viscous films and the resulting film deformations are of concern in the treatment of respiratory distress syndrome (RDS), in which the lungs of prematurely born infants are not developed enough to produce sufficient quantities of surfactant to reduce the surface tension of the lungs liquid lining. In surfactant replacement therapy (SRT), surfactant is instilled into the trachea of a patient with surfactant-deficient lungs and the surfactant is transported in the large airways primarily by gravity and pressure (Halpern *et al.*, 1998; Espinosa and Kamm, 1999). As the surfactant layer thins to a monolayer, Marangoni flows become the dominant mode of transportation (Halpern *et al.*, 1998; Espinosa and Kamm, 1999). Surface tension–driven flows are also important in the clearance of liquid and surfactant from healthy lungs (Davis *et al.*, 1974).

In all of the studies mentioned, Newtonian rheology was used. Some authors have studied non-Newtonian rheology and among them Zhang *et al.*, (2003) investigated the rupture mechanism of a precorneal thin mucus coating sandwiched between the aqueous tear film and the corneal epithelial surface with a monolayer of surfactant

overlying the aqueous layer in the eye. They used the Ostwald power-law model to model mucus and a linear equation of state to describe the relationship between surface tension and surfactant concentration. Their results reveal that the influence of rheological properties, aqueous-mucus thickness ratio, and the interfacial tension on the time required for the film rupture can be significant and varies considerably depending on the magnitude of the Hamaker constants governing the strength of the van der Waals forces.

4.3.2 Governing equations

A thin power-law fluid is flowing over an inclined plane and over the fluid a monolayer of surfactant has been deposited.

The flow is governed by a coupled system of partial differential equations defined by Eqs. (3.60) and (3.50) where $u_s = u(t, x, y = h(t, x))$ with u defined in Eq. (3.56) and σ defined in Eq. (3.51). The resulting system (after substitution and simplification) becomes:

$$h_t - \left(\frac{1}{\beta(\beta+1)f(x)^2} \left[(M\Gamma_x)^{\beta+1} + (\beta f(x)h - M\Gamma_x)\left(f(x)h + M\Gamma_x\right)^{\beta} \right] \right)_x = 0, \tag{4.13}$$

$$\Gamma_t + \left(\frac{1}{\beta f(x)} \left([M\Gamma_x]^{\beta} - [f(x)h + M\Gamma_x]^{\beta} \right) \Gamma \right)_x = \mathcal{P}e^{-1}\Gamma_{xx}, \tag{4.14}$$

where $f(x)$ is defined in Eq. (3.54).

When $\alpha \neq 0$, $\delta B h_x << B\sin\alpha$, and in Eq. (3.48) we take the limit case $\sigma_0 = 0$ which in this case implies $C = 0$, thus $f(x,t) \simeq -B\sin\alpha$.

The system reduces to

$$h_t \quad + \quad c_0\Big((M\Gamma_x)^{\beta+1} - (\beta B\sin\alpha\ h + M\Gamma_x)(B\sin\alpha - M\Gamma_x)^{\beta}\Big)_x = 0, \qquad (4.15)$$

$$\Gamma_t \quad + \quad c_1\Big(\left[(M\Gamma_x)^{\beta} - (B\sin\alpha\ h - M\Gamma_x)^{\beta}\right]\Gamma\Big)_x - \mathcal{P}_e^{-1}\Gamma_{xx} = 0, \qquad (4.16)$$

where $c_0 = -1/[\beta(\beta+1)B^2\sin^2\alpha]$, and $c_1 = -1/(\beta B\sin\alpha)$ are constants. The system (4.15) and (4.16) is subject to the initial conditions:

$$h(x,0) \ = \ 1 \ + \ \exp(-x^2), \quad \Gamma(x,0) \ = \ \Gamma_0\exp(-(x/L)^2),$$

and to the boundary conditions

$$h(0,t) \ = \ 2, \quad h(\infty,t) \ = \ 1, \quad \frac{\partial\Gamma}{\partial x}(0,t) \ = \ 0, \quad \Gamma(\infty,t) \ = \ 0.$$

4.3.3 Numerical results and discussion

In order to solve the system defined by Eqs. (4.15) and (4.16), we used PDECOL a Fortran 77 Package (see Madsen and Sincovec, 1979), which uses the finite element collocation method based on piecewise polynomials for the spatial discretization. We obtained the curves shown in figures 4.4 – 4.10. For the data, except otherwise mentioned, the angle $\beta = \pi/4$, the Bond number $B = 10^{-1}$ and the Marangoni number $M = 10$.

We observe that in the absence of surface tension, the profile of the film height is not smooth. In general, this is true because it is known that surface tension plays a role in the smoothness of the film height because it opposes the surface deformation due to pressure and other forces. Besides, in the absence of surface tension the spreading of a surfactant over a fluid propagates with the development of shocks. In figure 4.8 for $\beta = 2$ (Newtonian fluid) and in figure 4.10 for $\beta = 2.5$, we observe that there is an appearance of shocks for $t = 10$, and this situation corresponds in both cases to the surface concentration $\Gamma \approx 0$ (i.e. too small).

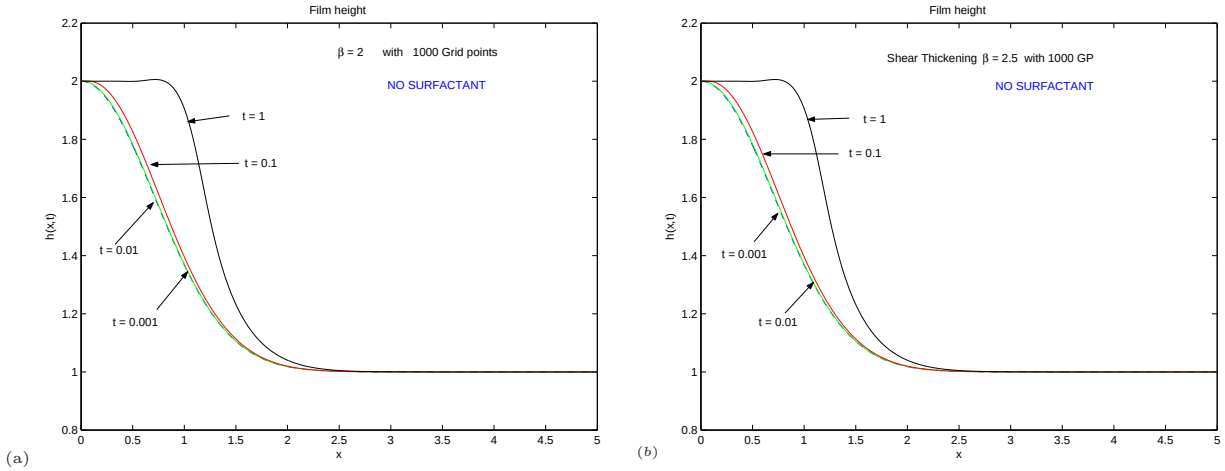

Figure 4.4: Plot of solution (4.15) – (4.16) without surfactant. (a): Film profile for $\beta = 2$; (b): Film profile for $\beta = 2.5$.

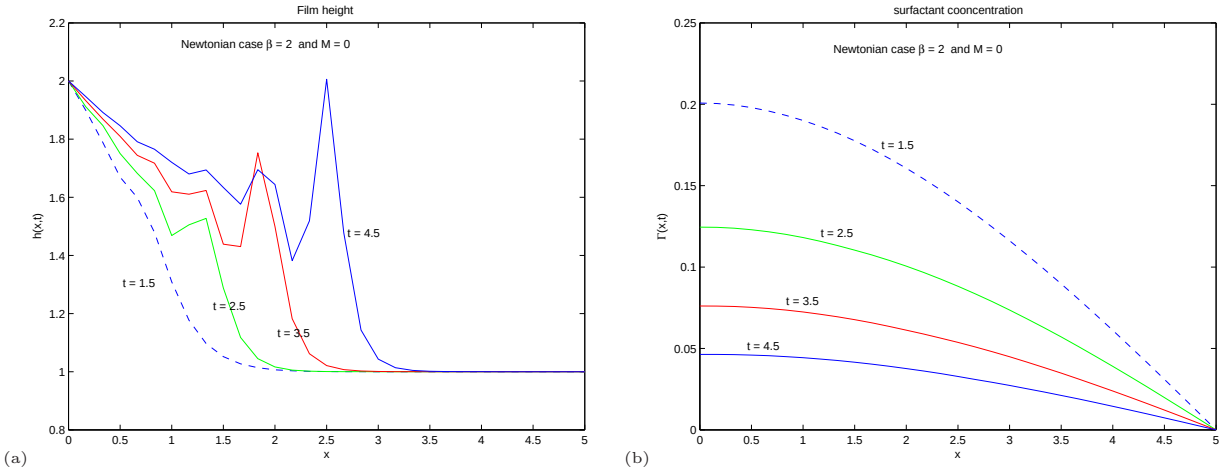

Figure 4.5: Plot of solution (4.15) – (4.16). (a): Film profile ; (b): Concentration profile.

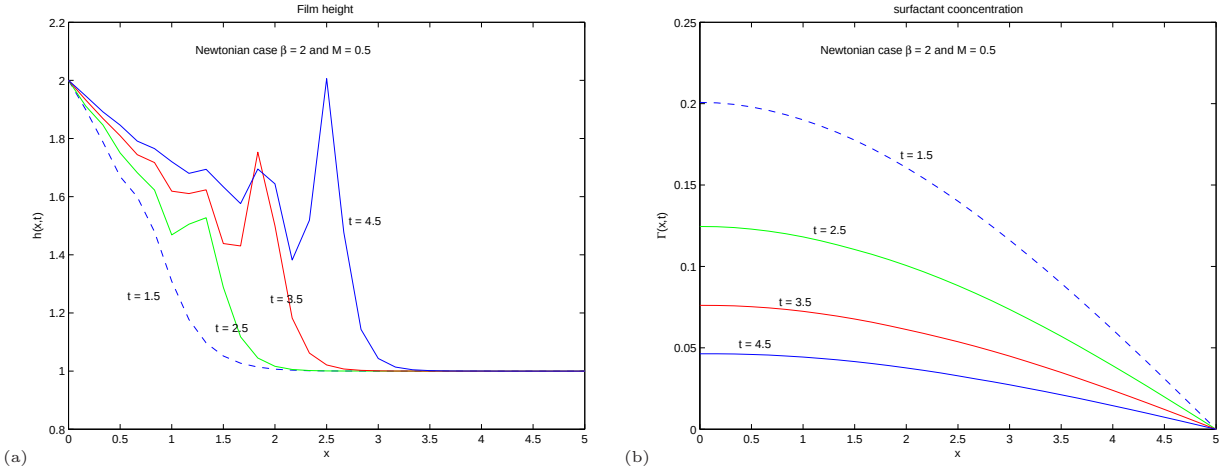

Figure 4.6: Plot of solution (4.15)−(4.16). (a): Film profile; (b): Concentration profile.

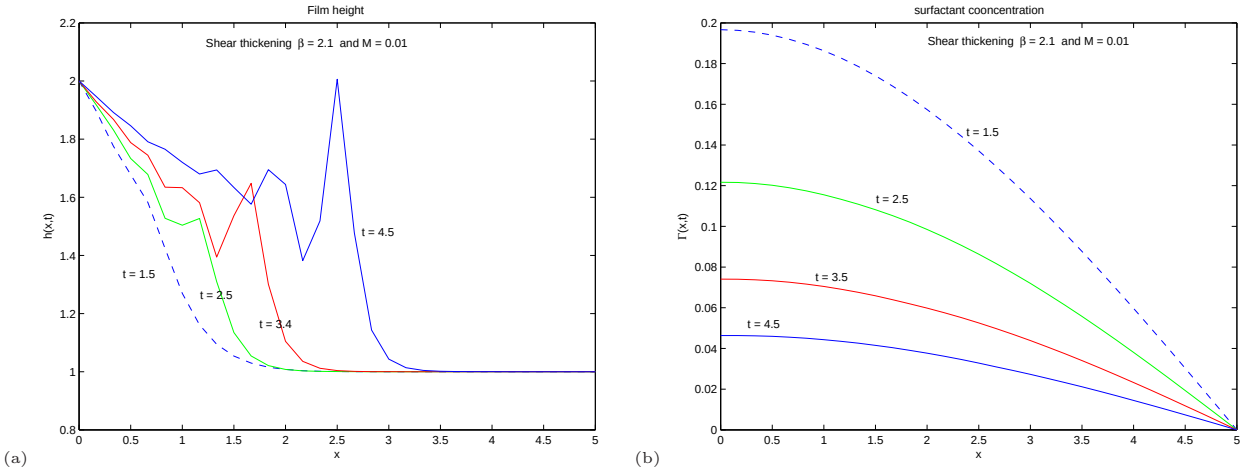

Figure 4.7: Plot of solution (4.15)−(4.16). (a): Film profile; (b): Concentration profile.

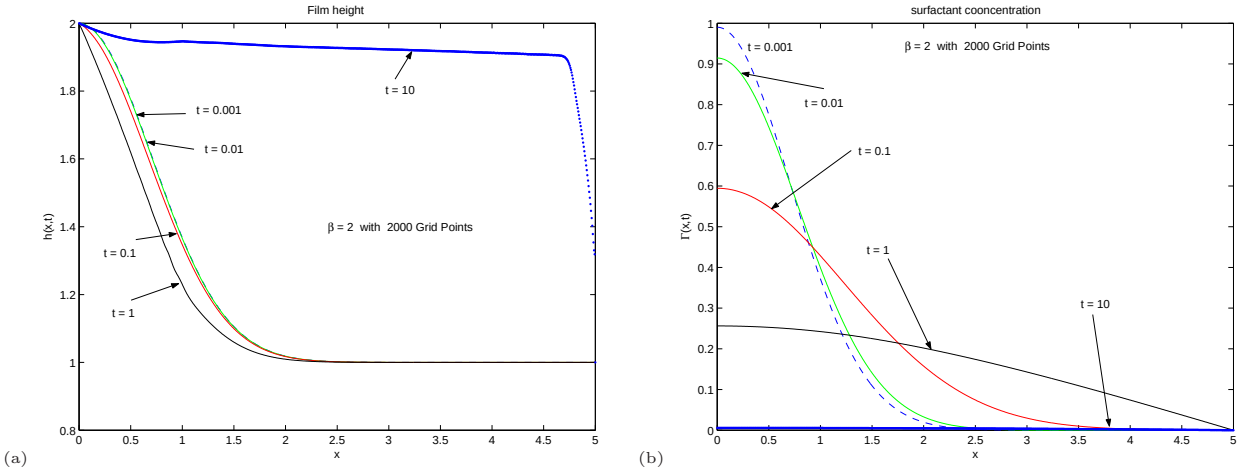

Figure 4.8: Plot of solution (4.15)−(4.16). (a): Film profile; (b): Concentration profile.

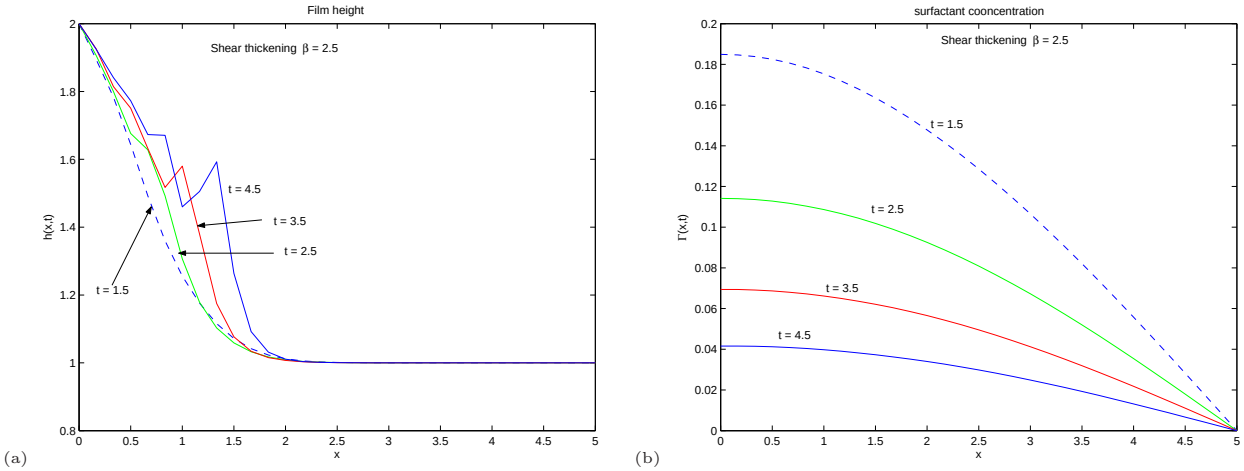

Figure 4.9: Plot of solution (4.15)−(4.16). (a): Film profile; (b): Concentration profile.

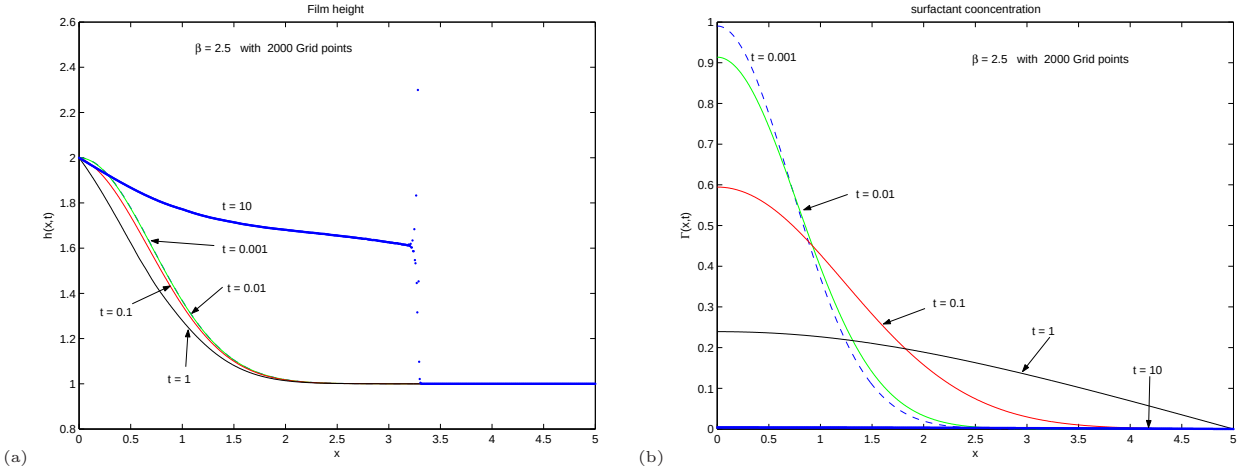

Figure 4.10: Plot of solution (4.15) − (4.16). (a): Film profile; (b): Concentration profile.

4.3.4 Conclusion

In this second part of the chapter, we have solved a coupled system of second-order partial differential equations describing the flow of a thin power-law fluid over an inclined plane and over the fluid a monolayer of surfactant has been deposited. The numerical results revealed that when considering Marangoni effects in the absence of surface tension, the profile of the film height is not smooth, and for some values of the power-law parameter, shocks developed after a long period of time. Surface tension has a smoothing effect on the film profile, that is the reason why in its absence we observed that the film profile is not smooth. We also noticed that the surfactant concentration profile was not influenced by the absence of surface tension and remained smooth in all the figures.

Chapter 5

Axisymmetric Spreading Under Gravity of a Thin Power-Law Liquid Drop on a Horizontal Plane

5.1 Introduction

The theory of axisymmetric spreading has many applications in science and technology. In geophysics for instance, the problem has important applications in the spreading of molten lava on the surface of the earth. Since in industry and even in nature, most of the fluids are well described by non-Newtonian models, we have chosen to investigate this class of power-law fluids. For Newtonian fluids, a similarity solution was first obtained by Huppert, (1982) and his work has been outlined by Middleman, (1995). Sherman, (1990) gave a slightly different derivation of a similarity solution. Greenspan, (1978) used lubrication theory to construct a model that describes the spreading of a drop over a horizontal surface and used the dynamic contact angle to describe the forces that act on the fluid at the contact line. For non-Newtonian fluids, we cite the pioneering work of Biswas and Gupta, (1987) who investigated the flow of a thin free layer of a non-Newtonian power-law fluid over a horizontal plane assuming the existence of a similarity solution. Betelu and Fontelos, (2003) investigated the

spreading of a non-Newtonian power-law drop under the effect of surface tension only and showed that for the power-law parameter greater than unity, the governing dimensionless equation admits compact-supported source-type similarity solutions with bounded energy dissipation rate. Gratton *et al.*, (1999) derived the governing equation for unidimensional and axisymmetric creeping currents of a non-Newtonian liquid with a power-law rheology, generalising the usual lubrification theory. Among their results, they obtained similarity solutions for currents whose volume varies as a power of time and they also found analytical solutions for the spread of constant volume that are in good agreement with experiment. In a more detailed work Myers, (2005) compared various non-Newtonian rheologies namely power-law, Carreau and Ellis rheologies. He compared the three models for free surface flow and flow within a channel and showed that for some choices of parameters, the Ellis model can closely approximate the Carreau model. The three models were then compared for thin film flow with a constant height free surface. For low shear rates, the power-law model gave very inaccurate predictions. Comparisons for flow in a channel showed that the power-law model gave inaccurate results due to the high viscosity around the turning point for the velocity. Therefore, great care should be exercised when applying the power-law model, particularly when the shear rate is low. It should only be used in situations where the majority of the flow is subject to a shear rate significantly above the transition range.

Momoniat *et al.*, (2001) obtained the group-invariant solution of the problem by considering a linear combination of the Lie points symmetries of the nonlinear diffusion equation for the surface profile of the liquid drop and, in later work, Mason and Momoniat, (2004) obtained the general form of the group invariant-solution of the same problem but this time with suction and blowing of fluid at the base. For the latter case, they obtained two particular solutions, each depending essentially on only one parameter which can be varied to yield a range of models.

For other studies of axisymmetric spreading of a viscous drop on a smooth

horizontal surface, we refer to the paper by Ehrhard and Davis, (1991) where lubrication theory is used and the forces acting on the drop are capillary, thermocapillary and gravity forces. Various contact-angle-versus-speed conditions are also imposed. Predictions of their model compare well with the experimental data, and the heat transfer has a sensitive influence on the spreading.

In this chapter we are particularly concerned with the analytical solution of the problem of axisymmetric spreading of a thin power-law liquid of an incompressible fluid on a horizontal plane. The effects of surface tension are neglected. Our main contribution is in analyzing (in a separable form) the solution of the second-order partial differential equation modelling the evolution of surface of a non-Newtonian fluid and, analysing the resulting second-order ordinary differential equation using Lie group methods.

5.2 Governing equation

The equation modelling the height of a thin power-law film on a horizontal plane in the presence of gravity has been derived in chapter 3 and is given by Eq. (3.143) as

$$\frac{\partial h}{\partial t} = \frac{1}{(\beta+1)r}\frac{\partial}{\partial r}\left(rh^{\beta+1}\left(\frac{\partial h}{\partial r}\right)^{\beta-1}\right).$$

5.3 Symmetry analysis

In this section, we look for all the symetries X of Eq. (3.143). Expanding (3.143), we obtain

$$\begin{aligned} h_t &= \frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1} + h^{\beta}h_r^{\beta} + \frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr} \\ &= F(t, r, h, h_r, h_{rr}). \end{aligned} \tag{5.1}$$

The second prolongation of X is given by

$$\begin{aligned} X^{[2]} &= \xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial r}+\eta\frac{\partial}{\partial h}+\zeta_1\frac{\partial}{\partial h_t}+\zeta_2\frac{\partial}{\partial h_r}+\zeta_{11}\frac{\partial}{\partial h_{tt}} \\ &\quad +\zeta_{12}\frac{\partial}{\partial h_{\mathrm{tr}}}+\zeta_{22}\frac{\partial}{\partial h_{rr}}, \end{aligned} \tag{5.2}$$

where

$$\zeta_1 = \eta_t+h_t\eta_h-h_t\xi^1_t-(h_t)^2\xi^1_h-h_r\xi^2_t-h_th_r\xi^2_h, \tag{5.3}$$

$$\zeta_2 = \eta_r+h_r\eta_h-h_t\xi^1_r-h_th_r\xi^1_h-h_r\xi^2_r-(h_r)^2\xi^2_h, \tag{5.4}$$

$$\begin{aligned} \zeta_{22} &= -h_r^3\xi^2_{hh}-2h_{rt}\xi^1_r+h_{rr}\left(\eta_h-h_t\xi^1_h-3h_r\xi^2_h-2\xi^2_r\right) \\ &\quad +h_r^2\left(\eta_{hh}-h_t\xi^1_{hh}-2\xi^2_{rh}\right)+\eta_{rr}-h_t\xi^1_{rr} \\ &\quad -h_r\left(2h_{rt}\xi^1_h-2\eta_{rh}+2h_t\xi^1_{rh}+\xi^2_{rr}\right). \end{aligned} \tag{5.5}$$

The determining equation of Eq. (3.143) given by

$$X^{[2]}\left(h_t-F(t,r,h,h_r,h_{rr})\right)\big|_{(3.143)}=0, \tag{5.6}$$

is

$$-\xi^1F_t-\xi^2F_r-\eta F_h+\zeta_1-\zeta_2F_{h_r}-\zeta_{22}F_{h_{rr}}=0. \tag{5.7}$$

Substituting Eqs. (5.3)–(5.5) into Eq. (5.7) and replacing h_t in Eq. (5.7) using Eq. (5.1)

we have

$$
\begin{aligned}
&\xi^2\left(\frac{1}{(\beta+1)r^2}h^{\beta+1}h_r^{\beta-1}\right)-\eta\left(\frac{1}{r}h^{\beta}h_r^{\beta-1}+\beta h^{\beta-1}h_r^{\beta}+(\beta-1)h^{\beta}h_r^{\beta-2}h_{rr}\right)\\
&+\eta_t+\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right)\eta_h\\
&-\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right)\\
&\quad\times\left(\left[\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right]\xi_h^1+\xi_t^1\right)\\
&-h_r\left(\left[\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right]\xi_h^2+\xi_t^2\right)\\
&-\left[h_r\eta_h+\eta_r-\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right)\right.\\
&\quad\left.\times\left(h_r\xi_h^1+\xi_r^1\right)-h_r\left(h_r\xi_h^2+\xi_r^2\right)\right]\left(\frac{\beta-1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-2}+\beta h^{\beta}h_r^{\beta-1}\right.\\
&\left.+\frac{(\beta-2)(\beta-1)}{\beta+1}h^{\beta+1}h_r^{\beta-3}h_{rr}\right)-\left\{-h_r^3\xi_{hh}^2-2h_{rt}\xi_r^1\right.\\
&+h_{rr}\left(\eta_h-\left[\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right]\xi_h^1\right.\\
&\left.-3h_r\xi_h^2-2\xi_r^2\right)+h_r^2\left(\eta_{hh}-\left[\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}\right.\right.\\
&\left.\left.+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right]\xi_{hh}^1-2\xi_{rh}^2\right)+\eta_{rr}-\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^{\beta}h_r^{\beta}\right.\\
&\left.+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right)\xi_{rr}^1-h_r\left(2h_{rt}\xi_h^1-2\eta_{rh}+2\left[\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}\right.\right.\\
&\left.\left.\left.+h^{\beta}h_r^{\beta}+\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right]\xi_{rh}^1+\xi_{rr}^2\right)\right\}\left(\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}\right)=0. \qquad (5.8)
\end{aligned}
$$

Since ξ^1, ξ^2 and η do not depend on the derivatives of h, the determining equation (5.8) can be separated by these derivatives and their power. Therefore, separating (5.8) by $h_r^{\beta-1}h_{rt}$ and $h_r^{\beta-2}h_{rt}$, we obtain

$$h_r^{\beta-1}h_{rt}: \quad 2\frac{\beta-1}{\beta+1}h^{\beta+1}\xi_h^1=0, \qquad (5.9)$$

$$h_r^{\beta-2}h_{rt}: \quad 2\frac{\beta-1}{\beta+1}h^{\beta+1}\xi_r^1=0. \qquad (5.10)$$

Since $\beta \neq 1$, we conclude that $\xi^1 = \xi^1(t)$. The determining equation (5.8) reduces to

$$\begin{aligned}
&\xi^2\left(\frac{1}{(\beta+1)r^2}h^{\beta+1}h_r^{\beta-1}\right) - \eta\left(\frac{1}{r}h^\beta h_r^{\beta-1} + \beta h^{\beta-1}h_r^\beta + (\beta-1)h^\beta h_r^{\beta-2}h_{rr}\right)\\
&+\eta_t + \left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1} + h^\beta h_r^\beta + \frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right)\eta_h\\
&-\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1} + h^\beta h_r^\beta + \frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right)\xi_t^1\\
&-h_r\left(\left[\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1} + h^\beta h_r^\beta + \frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}h_{rr}\right]\xi_h^2 + \xi_t^2\right)\\
&-\left[h_r\eta_h + \eta_r - h_r\left(h_r\xi_h^2 + \xi_r^2\right)\right]\times\left(\frac{\beta-1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-2}\right.\\
&\qquad\left.+\beta h^\beta h_r^{\beta-1} + \frac{(\beta-2)(\beta-1)}{\beta+1}h^{\beta+1}h_r^{\beta-3}h_{rr}\right)\\
&-\left\{-h_r^3\xi_{hh}^2 + h_{rr}\left(\eta_h - 3h_r\xi_h^2 - 2\xi_r^2\right) + h_r^2\left(\eta_{hh} - 2\xi_{rh}^2\right)\right.\\
&\qquad\left.+\eta_{rr} - h_r\left(-2\eta_{rh} + \xi_{rr}^2\right)\right\}\left(\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-2}\right) = 0.
\end{aligned} \tag{5.11}$$

Separating (5.11) by $h_r^{\beta-1}h_{rr}$, we obtain

$$h_r^{\beta-1}h_{rr}: \qquad \frac{(\beta-1)\beta}{\beta+1}h^{\beta+1}\xi_h^2 = 0. \tag{5.12}$$

Since $\beta \neq 1$, we have two cases: $\beta = 0$ and $\beta \neq 0$.

- **Case** $\beta = 0$

From Eq. (5.12), ξ_h^2 is not necessarily zero. Substituting $\beta = 0$ back into the determining equation (5.11) and noting that the coefficients of $h_r^{-1}h_{rr}$ cancel, we obtain

$$\begin{aligned}
&\xi^2\left(\frac{1}{r^2}hh_r^{-1}\right) - \eta\left(\frac{1}{r}h_r^{-1} - h_r^{-2}h_{rr}\right) + \eta_t\\
&+\left(\frac{1}{r}hh_r^{-1} + 1 - hh_r^{-2}h_{rr}\right)\eta_h - \left(\frac{1}{r}hh_r^{-1} + 1 - hh_r^{-2}h_{rr}\right)\xi_t^1\\
&-\left(\frac{1}{r}h + h_r\right)\xi_h^2 - h_r\xi_t^2 - \left[h_r\eta_h + \eta_r - h_r(h_r\xi_h^2 + \xi_r^2)\right]\left(-\frac{1}{r}hh_r^{-2} + 2hh_r^{-3}h_{rr}\right)\\
&-\left\{-h_r^3\xi_{hh}^2 + h_{rr}\left(\eta_h - 3h_r\xi_h^2 - 2\xi_r^2\right) + h_r^2\left(\eta_{hh} - 2\xi_{rh}^2\right)\right.\\
&\qquad\left.+\eta_{rr} - h_r\left(-2\eta_{rh} + \xi_{rr}^2\right)\right\}\left(-hh_r^{-2}\right) = 0.
\end{aligned} \tag{5.13}$$

Separating Eq. (5.13) by h_r, we obtain

$$h_r: \qquad \xi_t^2 + \xi_h^2 + h\xi_{hh}^2 = 0. \tag{5.14}$$

The determining equation (5.13) reduces to

$$\begin{aligned}
&\xi^2\left(\frac{1}{r^2}hh_r^{-1}\right)-\eta\left(\frac{1}{r}h_r^{-1}-h_r^{-2}h_{rr}\right)+\eta_t\\
&+\left(\frac{1}{r}hh_r^{-1}+1-hh_r^{-2}h_{rr}\right)\eta_h-\left(\frac{1}{r}hh_r^{-1}+1-hh_r^{-2}h_{rr}\right)\xi_t^1\\
&-\frac{1}{r}h\xi_h^2-\left[h_r\eta_h+\eta_r-h_r(h_r\xi_h^2+\xi_r^2)\right]\left(-\frac{1}{r}hh_r^{-2}+2hh_r^{-3}h_{rr}\right)\\
&-\left\{h_{rr}\left(\eta_h-3h_r\xi_h^2-2\xi_r^2\right)+h_r^2\left(\eta_{hh}-2\xi_{rh}^2\right)\right.\\
&\qquad\left.+\eta_{rr}-h_r\left(-2\eta_{rh}+\xi_{rr}^2\right)\right\}\left(-hh_r^{-2}\right)=0.
\end{aligned}\tag{5.15}$$

Separating Eq. (5.15) by $h_r^{-2}h_{rr}$, we find that

$$h_r^{-2}h_{rr}:\qquad \eta-2h\eta_h+h\xi_t^1=0.\tag{5.16}$$

The determining equation (5.15) reduces to

$$\begin{aligned}
&\xi^2\left(\frac{1}{r^2}hh_r^{-1}\right)-\eta\left(\frac{1}{r}h_r^{-1}\right)+\eta_t+\left(\frac{1}{r}hh_r^{-1}+1\right)\eta_h-\left(\frac{1}{r}hh_r^{-1}+1\right)\xi_t^1\\
&-\frac{1}{r}h\xi_h^2-\left(-\frac{1}{r}hh_r^{-2}+2hh_r^{-3}h_{rr}\right)(\eta_r-h_r^2\xi_h^2)+\frac{1}{r}\left(\eta_h-\xi_r^2\right)hh_r^{-1}\\
&+\left\{-3h_rh_{rr}\xi_h^2+h_r^2\left(\eta_{hh}-2\xi_{rh}^2\right)+\eta_{rr}-h_r\left(-2\eta_{rh}+\xi_{rr}^2\right)\right\}hh_r^{-2}=0.
\end{aligned}\tag{5.17}$$

Separating Eq. (5.17) by h_r^{-2}, we obtain

$$h_r^{-2}:\qquad r\eta_{rr}+\eta_r=0.\tag{5.18}$$

The determining equation (5.17) reduces to

$$\begin{aligned}
&\xi^2\left(\frac{1}{r^2}hh_r^{-1}\right)-\eta\left(\frac{1}{r}h_r^{-1}\right)+\eta_t+\left(\frac{1}{r}hh_r^{-1}+1\right)\eta_h-\left(\frac{1}{r}hh_r^{-1}+1\right)\xi_t^1\\
&-\frac{1}{r}h\xi_h^2-\left(2hh_r^{-3}h_{rr}\right)\eta_r+\frac{1}{r}\left(\eta_h-\xi_r^2\right)hh_r^{-1}+\left(-\frac{1}{r}h+2hh_r^{-1}h_{rr}\right)\xi_h^2\\
&+\left\{-3h_rh_{rr}\xi_h^2+h_r^2\left(\eta_{hh}-2\xi_{rh}^2\right)-h_r\left(-2\eta_{rh}+\xi_{rr}^2\right)\right\}hh_r^{-2}=0.
\end{aligned}\tag{5.19}$$

Finally, separating the determining equation (5.19) by the remaining powers $h_r^{-3}h_{rr}$, $h_r^{-1}h_{rr}$, h_r^{-1} and 1, and after simplification, we obtain respectively

$$h_r^{-3}h_{rr}:\qquad \eta_r=0,\tag{5.20}$$

$$h_r^{-1}h_{rr}:\qquad \xi_h^2=0,\tag{5.21}$$

$$h_r^{-1}:\qquad r^2\xi_{rr}^2+r\xi_r^2-\xi^2=0,\tag{5.22}$$

$$1:\qquad h\eta_{hh}+\eta_h+\eta_t-\xi_t^1=0.\tag{5.23}$$

Eqs. (5.14) and (5.21) imply that $\xi^2 = \xi^2(r)$. Eq. (5.22) is easily integrated with respect to r and we obtain

$$\xi^2(r) \;=\; C_1\, r + C_2\, \frac{1}{r}, \tag{5.24}$$

where C_1 and C_2 are arbitrary constants.

Eq. (5.20) implies that $\eta = \eta(t, h)$. Integrating Eq. (5.16), we obtain

$$\eta(t, h) \;=\; h\xi^1_t + h^{1/2} C_3(t). \tag{5.25}$$

Substituting Eq. (5.25) into Eq. (5.23), we obtain after simplification

$$h\; \ddot{\xi}^1(t) + h^{1/2}\; \dot{C}_3(t) + \frac{1}{4} h^{-1/2}\; C_3(t) = 0. \tag{5.26}$$

Separating Eq. (5.26) by powers of h, we obtain

$$\ddot{\xi}^1(t) = 0, \tag{5.27}$$

and

$$C_3(t) = 0. \tag{5.28}$$

Thus

$$\xi^1(t) = a\; t + b, \tag{5.29}$$

where a and b are arbitrary constants, and

$$\eta(h) = a\; h. \tag{5.30}$$

In conclusion,

$$\xi^1(t) = a\; t + b, \quad \xi^2(r) = C_1\; r + C_2\; \frac{1}{r}, \quad \eta(h) = a\; h.$$

We obtain four symmetry generators in this case and they are given by

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + h\frac{\partial}{\partial h},$$

$$X_3 = r\frac{\partial}{\partial r}, \qquad X_4 = \frac{1}{r}\frac{\partial}{\partial r}.$$

- **Case** $\beta \neq 0$

Eq. (5.12) implies that $\xi_h^2 = 0$ and therefore $\xi^2 = \xi^2(t, r)$. The determining equation (5.11) reduces to

$$\begin{aligned}
&\xi^2 \left(\frac{1}{(\beta+1)r^2} h^{\beta+1} h_r^{\beta-1} \right) - \eta \left(\frac{1}{r} h^\beta h_r^{\beta-1} + \beta h^{\beta-1} h_r^\beta + (\beta-1) h^\beta h_r^{\beta-2} h_{rr} \right) \\
&+\eta_t + \left(\frac{1}{(\beta+1)r} h^{\beta+1} h_r^{\beta-1} + h^\beta h_r^\beta + \frac{\beta-1}{\beta+1} h^{\beta+1} h_r^{\beta-2} h_{rr} \right) \eta_h \\
&- \left(\frac{1}{(\beta+1)r} h^{\beta+1} h_r^{\beta-1} + h^\beta h_r^\beta + \frac{\beta-1}{\beta+1} h^{\beta+1} h_r^{\beta-2} h_{rr} \right) \xi_t^1 - h_r \xi_t^2 \\
&- \left(h_r \eta_h + \eta_r - h_r \xi_r^2 \right) \\
&\qquad \times \left(\frac{\beta-1}{(\beta+1)r} h^{\beta+1} h_r^{\beta-2} + \beta h^\beta h_r^{\beta-1} + \frac{(\beta-2)(\beta-1)}{\beta+1} h^{\beta+1} h_r^{\beta-3} h_{rr} \right) \\
&- \left\{ h_{rr} \left(\eta_h - 2\xi_r^2 \right) + h_r^2 \eta_{hh} + \eta_{rr} - h_r \left(-2\eta_{rh} + \xi_{rr}^2 \right) \right\} \\
&\qquad \times \left(\frac{\beta-1}{\beta+1} h^{\beta+1} h_r^{\beta-2} \right) = 0. \qquad (5.31)
\end{aligned}$$

Separating Eq. (5.31) by $h_r^{\beta-2} h_{rr}$ and after simplification, we obtain

$$h_r^{\beta-2} h_{rr} : \qquad -\frac{\eta}{h} - \frac{1}{\beta+1} \xi_t^1 - \frac{\beta-2}{\beta+1} \eta_h + \frac{\beta}{\beta+1} \xi_r^2 = 0. \qquad (5.32)$$

Differentiating Eq. (5.32) with respect to h, we obtain

$$\frac{\partial}{\partial h} \left(\frac{\eta}{h} + \frac{\beta-2}{\beta+1} \eta_h \right) = 0. \qquad (5.33)$$

Thus

$$\frac{\eta}{h} + \frac{\beta-2}{\beta+1} \eta_h = C_1(t, r), \qquad (5.34)$$

where C_1 is an arbitrary function of t and r. Multiplying Eq. (5.34) by h, we obtain

$$\eta + \left(\frac{\beta-2}{\beta+1} \right) h \eta_h = h C_1(t, r). \qquad (5.35)$$

From Eq. (5.35), we can distinguish two cases: $\beta = 2$ and $\beta \neq 2$.

- **Case $\beta = 2$**

  This case has been studied in detail by Momoniat, (1999) in his PhD thesis. However our general result for $\beta \neq 0$ will include this case. We recall here that the three symmetry generators are given by

$$X_1 = t\frac{\partial}{\partial t}; \qquad X_2 = t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}; \qquad X_3 = \frac{3r}{2}\frac{\partial}{\partial r} + h\frac{\partial}{\partial h}. \tag{5.36}$$

- **Case $\beta \neq 2$**

  Separating the determining equation (5.31) with respect to $h_r^{\beta-3}h_{rr}$, we obtain

$$h_r^{\beta-3}h_{rr}: \qquad \eta_r = 0. \tag{5.37}$$

  1) If $\beta = 1/2$

  Integrating Eq. (5.35), we obtain

$$\eta = -h\log(h)\ C_1(t,r) + h\ C_7(t,r), \tag{5.38}$$

  where C_7 is an arbitrary function of t and r. Substituting Eq. (5.38) into Eq. (5.32), we obtain after simplification

$$-\ C_1 - \frac{2}{3}\ \xi_t^1 + \frac{1}{3}\ \xi_r^2 = 0. \tag{5.39}$$

Separating the determining equation (5.31) by $h_r^{\beta-3}h_{rr}$, i.e $h_r^{5/3}h_{rr}$ for $\beta = 1/2$, we obtain

$$h_r^{5/3}h_{rr}: \qquad \eta_r = 0. \tag{5.40}$$

Thus

$$\eta = -h\log(h)\ C_1(t) + h\ C_7(t), \tag{5.41}$$

The determining equation (5.31) reduces to

$$\begin{aligned}
&\xi^2\left(\frac{2}{3r^2}h^{3/2}h_r^{-1/2}\right)-(-h\log(h)C_1+hC_7)\left(\frac{1}{r}h^{1/2}h_r^{-1/2}+\frac{1}{2}h^{-1/2}h_r^{1/2}\right)\\
&-h\log(h)\dot{C}_1+h\dot{C}_7+\left(\frac{2}{3r}h^{3/2}h_r^{-1/2}+h^{1/2}h_r^{1/2}\right)[-(1+\log(h))C_1+C_7]\\
&-\left(\frac{2}{3r}h^{3/2}h_r^{-1/2}+h^{1/2}h_r^{1/2}\right)\xi_t^1-h_r\xi_t^2\\
&-\left[-(1+\log(h))C_1+C_7-\xi_r^2\right]\left(-\frac{1}{3r}h^{3/2}h_r^{-1/2}+\frac{1}{2}h^{1/2}h_r^{1/2}\right)\\
&+\left\{-C_1h^{-1}h_r^2-\xi_{rr}^2h_r\right\}\left(\frac{1}{3}h^{3/2}h_r^{-3/2}\right)=0. \qquad (5.42)
\end{aligned}$$

Separating the determining equation (5.42) by $(h_r)^0$, we obtain

$$(h_r)^0: \qquad -h\log(h)\dot{C}_1(t)+h\dot{C}_7(t)=0, \qquad (5.43)$$

thus

$$-\log(h)\dot{C}_1(t)+\dot{C}_7(t)=0. \qquad (5.44)$$

Separating by coefficients of $\log(h)$, we obtain

$$\dot{C}_1(t)=\dot{C}_7(t)=0. \qquad (5.45)$$

Therefore, C_1 and C_7 are arbitrary constants.

Since ξ^1 and ξ^2 no longer depend on h, we can still separate the determining equation (5.42) by powers of h and h_r. We obtain after simplification

$$h^{1/2}h_r^{1/2}: \qquad -\frac{5}{6}C_1-\xi_t^1+\frac{1}{2}\xi_r^2=0, \qquad (5.46)$$

$$h_r: \qquad \xi_t^2=0, \qquad (5.47)$$

$$h^{3/2}h_r^{-1/2}: \qquad \frac{2}{3r^2}\xi^2-\frac{1}{3r}\xi_r^2-\frac{1}{3}\xi_{rr}^2-\frac{2}{3r}\xi_t^1-\frac{C_1}{r}=0. \qquad (5.48)$$

Eq. (5.47) implies that

$$\xi^2=\xi^2(r). \qquad (5.49)$$

Equating C_1 from Eqs. (5.39) and (5.46), we obtain

$$\xi_r^2=2\xi_t^1, \qquad (5.50)$$

that is

$$\xi^2(r)' = 2\dot{\xi}^1(t) = C_0, \tag{5.51}$$

where C_0 is an arbitrary constant. Integrating Eq. (5.51), we find

$$\xi^2(r) = C_0 r + D_0, \tag{5.52}$$

$$\xi^1(t) = \frac{1}{2} C_0 t + E_0, \tag{5.53}$$

where D_0 and E_0 are arbitrary constants. Substituting Eq. (5.52) and Eq. (5.53) into Eq. (5.48), we obtain after simplification

$$C_1 = D_0 = 0. \tag{5.54}$$

In conclusion, for $\beta = 1/2$ we have

$$\xi^1(t) = \frac{1}{2} C_0 \; t + E_0; \quad \xi^2(r) = C_0 \; r; \quad \eta(h) = C_7 \; h, \tag{5.55}$$

and we therefore obtain the following three symmetry generators

$$X_1 = \frac{\partial}{\partial t}; \quad X_2 = t\frac{\partial}{\partial t} + 2r\frac{\partial}{\partial r}; \quad X_3 = h\frac{\partial}{\partial h}. \tag{5.56}$$

2) If $\beta \neq 1/2$

Separating the determining equation (5.31) by $(h_r)^0$, we obtain

$$(h_r)^0 : \qquad \eta_t = 0. \tag{5.57}$$

Eqs. (5.37) and (5.57) therefore imply that

$$\eta = \eta(h). \tag{5.58}$$

Integrating Eq. (5.35), we obtain

$$\eta = \left(\frac{\beta + 1}{2\beta - 1}\right) h \; C_1 + h^{-(\beta+1)/(\beta-2)} \; C_2, \tag{5.59}$$

where C_2 is an arbitrary constant and C_1 is also an arbitrary constant. Substituting Eq. (5.59) into Eq. (5.32), we obtain after simplification

$$-C_1 - \frac{1}{\beta + 1} \; \xi^1_t + \frac{\beta}{\beta + 1} \; \xi^2_r = 0. \tag{5.60}$$

The determining equation (5.31) reduces to

$$\begin{aligned}
&\xi^2\left(\frac{1}{(\beta+1)r^2}h^{\beta+1}h_r^{\beta-1}\right)-\eta\left(\frac{1}{r}h^\beta h_r^{\beta-1}+\beta h^{\beta-1}h_r^\beta\right)\\
&+\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^\beta h_r^\beta\right)\eta_h-\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^\beta h_r^\beta\right)\xi_t^1\\
&-\xi_t^2-\left(\eta_h-\xi_r^2\right)\times\left(\frac{\beta-1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+\beta h^\beta h_r^\beta\right)\\
&-\left(h_r\eta_{hh}-\xi_{rr}^2\right)\times\left(\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-1}\right)=0,
\end{aligned}\tag{5.61}$$

that is

$$\begin{aligned}
&\xi^2\left(\frac{1}{(\beta+1)r^2}h^{\beta+1}h_r^{\beta-1}\right)\\
&-\left(\frac{\beta+1}{2\beta-1}hC_1+h^{-(\beta+1)/(\beta-2)}C_2\right)\left(\frac{1}{r}h^\beta h_r^{\beta-1}+\beta h^{\beta-1}h_r^\beta\right)\\
&+\left(\frac{\beta+1}{2\beta-1}C_1-\frac{\beta+1}{\beta-2}h^{-(2\beta-1)/(\beta-2)}C_2\right)\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^\beta h_r^\beta\right)\\
&-\left(\frac{1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+h^\beta h_r^\beta\right)\xi_t^1-h_r\xi_t^2\\
&-\left(\frac{\beta+1}{2\beta-1}C_1-\frac{\beta+1}{\beta-2}h^{-(2\beta-1)/(\beta-2)}C_2-\xi_r^2\right)\\
&\qquad\times\left(\frac{\beta-1}{(\beta+1)r}h^{\beta+1}h_r^{\beta-1}+\beta h^\beta h_r^\beta\right)\\
&-\left(\frac{(\beta+1)(2\beta-1)}{(\beta-2)^2}h^{-3(\beta-1)/(\beta-2)}h_rC_2-\xi_{rr}^2\right)\times\left(\frac{\beta-1}{\beta+1}h^{\beta+1}h_r^{\beta-1}\right)=0.
\end{aligned}\tag{5.62}$$

Separating the determining equation (5.62) by the powers of h_r, $h_r^{\beta-1}$ and h_r^β respectively, we obtain after simplification

$$h_r: \quad \xi_t^2=0, \tag{5.63}$$

$$h_r^{\beta-1}: \quad \xi^2-r\xi_r^2+(\beta-1)r^2\xi_{rr}^2=0, \tag{5.64}$$

$$h_r^\beta: \quad -(\beta+1)C_1+\frac{2\beta-1}{(\beta-1)(\beta-2)}h^{(2\beta-1)/(\beta-2)}C_2+\beta\xi_x^2-\xi_t^1=0. \tag{5.65}$$

Eq. (5.63) implies that $\xi^2=\xi^2(r)$ and therefore Eq. (5.64) becomes

$$\xi^2(r)-r\ (\xi^2)'(r)+(\beta-1)\ r^2\ (\xi^2)''(r)=0. \tag{5.66}$$

Integrating Eq. (5.66), we obtain

$$\xi^2(r)=C_3\ r+C_4\ r^{1/(\beta-1)}, \tag{5.67}$$

where C_3 and C_4 are arbitrary constants. From Eq. (5.60) we have that

$$\frac{\beta}{\beta+1}\ \xi_r^2 = C_1 + \frac{1}{\beta+1}\ \xi_t^1. \tag{5.68}$$

Since $\xi^1 = \xi^1(t)$ and $\xi^2 = \xi^2(r)$, the equality in Eq. (5.68) is an arbitrary constant say C_5, that is

$$\frac{\beta}{\beta+1}\ \xi_r^2 = C_1 + \frac{1}{\beta+1}\ \xi_t^1 = C_5. \tag{5.69}$$

From Eq. (5.69) integrating $(\beta/(\beta+1))\ \xi_r^2 = C_5$, we obtain

$$\xi^2(r) = \left(\frac{\beta+1}{\beta}\right) C_5\ r + C_6, \tag{5.70}$$

where C_6 is an arbitrary constant. Equating Eqs. (5.67) and (5.70), we obtain

$$\left[\left(\frac{\beta+1}{\beta}\right) C_5 - C_3\right]\ r + C_6 - C_4\ r^{1/(\beta-1)} = 0. \tag{5.71}$$

Separating Eq. (5.71) in powers of r, we obtain

$$C_3 = \left(\frac{\beta+1}{\beta}\right) C_5, \quad C_6 = 0, \quad \text{and} \quad C_4 = 0. \tag{5.72}$$

Thus

$$\xi^2(r) = \left(\frac{\beta+1}{\beta}\right) C_5\ r. \tag{5.73}$$

In the same manner, from Eq. (5.69), we have

$$\xi_t^1 = (\beta+1)(C_5 - C_1) = \alpha_\beta. \tag{5.74}$$

Integrating Eq. (5.74) with respect to t, we obtain

$$\xi^1(t) = \alpha_\beta\ t + \gamma_\beta, \tag{5.75}$$

where α_β and γ_β are arbitrary constants depending only on β. Substituting η , ξ^1 and ξ^2 in Eq. (5.65), we obtain after simplification

$$-\frac{2\beta-1}{(\beta-2)^2} h^{(2\beta-1)/(\beta-2)} C_2 = 0. \tag{5.76}$$

Since $\beta \neq 1/2$, we have $C_2 = 0$, and therefore

$$\eta(h) = \left(\frac{\beta+1}{2\beta-1}\right) h\ C_1. \tag{5.77}$$

From Eq. (5.74), we obtain $C_5 = (1/(\beta+1))\,\alpha_\beta + C_1$.

In conclusion, for $\beta \neq 1/2$, we have

$$\begin{aligned}
\xi^1(t) &= \alpha_\beta\ t + \gamma_\beta, \\
\xi^2(r) &= \frac{1}{\beta}\left[\alpha_\beta + (\beta+1)C_1\right]\ r, \\
\eta(h) &= \left(\frac{\beta+1}{2\beta-1}\right) h\ C_1,
\end{aligned}$$

and we therefore obtain the three symmetry generators

$$X_1 = \frac{\partial}{\partial t}; \quad X_2 = t\frac{\partial}{\partial t} + \frac{1}{\beta}\ r\frac{\partial}{\partial r}; \quad X_3 = \left(\frac{2\beta-1}{\beta}\right)\ r\frac{\partial}{\partial r} + h\ \frac{\partial}{\partial h}. \tag{5.78}$$

Finally, we can summarize all the results of the symmetry generators admitted by Eq. (3.143) for the various values of β in the following table:

Values of β	Symmetry generators
$\beta = 0$	$X_1 = \frac{\partial}{\partial t},\ X_2 = t\frac{\partial}{\partial t} + h\frac{\partial}{\partial h},\ X_3 = r\frac{\partial}{\partial r},\ X_4 = \frac{1}{r}\frac{\partial}{\partial r}$
$\beta = 1/2$	$X_1 = \frac{\partial}{\partial t}; \quad X_2 = t\frac{\partial}{\partial t} + 2r\frac{\partial}{\partial r}; \quad X_3 = h\ \frac{\partial}{\partial h}$
$\beta \neq 1/2\ (\beta \neq \pm 1)$	$X_1 = \frac{\partial}{\partial t}; \quad X_2 = t\frac{\partial}{\partial t} + \frac{1}{\beta}\ r\frac{\partial}{\partial r}; \quad X_3 = \left(\frac{2\beta-1}{\beta}\right)\ r\frac{\partial}{\partial r} + h\ \frac{\partial}{\partial h}$

Table 5.1: Table of the symmetry generators admitted by Eq. (3.143) for different values of β.

5.4 Group-invariant solution

Here we include all the procedure to find the group invariant solution for various values of β. For $\beta \neq 0$, Eq. (3.143) admits three symmetries

$$X_1 = \frac{\partial}{\partial t}; \quad X_2 = t\frac{\partial}{\partial t} + \frac{r}{\beta}\frac{\partial}{\partial r}; \quad X_3 = \left(\frac{2\beta - 1}{\beta}\right) r\frac{\partial}{\partial r} + h\frac{\partial}{\partial h}. \tag{5.79}$$

$h = \Phi(t, r)$ is a group-invariant solution of Eq. (3.143) provided

$$X\left(h - \Phi(t, r)\right)\big|_{h=\Phi(t,r)} = 0, \tag{5.80}$$

where

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3. \tag{5.81}$$

Thus

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_2 + (2\beta - 1)c_3)\frac{r}{\beta}\frac{\partial}{\partial r} + c_3 h\frac{\partial}{\partial h}.$$

We assume that c_1, c_2 and c_3 are all non-zero. Therefore there are two independent non-zero ratios of constants which are determined by two conditions

$$V = \int_0^{R(t)} rh(r, t)dr = \text{constant}, \tag{5.82}$$

and

$$h(R(t), t) = 0 \quad \text{at} \quad r = R(t). \tag{5.83}$$

We will work out the ratios c_1/c_2 and c_3/c_2. Dividing X by c_2, we obtain

$$Y = (\frac{c_1}{c_2} + t)\frac{\partial}{\partial t} + \left(1 + (2\beta - 1)\frac{c_3}{c_2}\right)\frac{r}{\beta}\frac{\partial}{\partial r} + \frac{c_3}{c_2}h\frac{\partial}{\partial h}.$$

Now $h = \Phi(t, r)$ is a group-invariant solution of Eq. (3.143) provided that

$$\left[\left(\frac{c_1}{c_2} + t\right)\frac{\partial}{\partial t} + \left(1 + (2\beta - 1)\frac{c_3}{c_2}\right)\frac{r}{\beta}\frac{\partial}{\partial r} + \frac{c_3}{c_2}h\frac{\partial}{\partial h}\right]\left(h - \Phi(t, r)\right)\Big|_{h=\Phi} = 0, \tag{5.84}$$

that is

$$\left(\frac{c_1}{c_2} + t\right)\frac{\partial \Phi}{\partial t} + \left(1 + (2\beta - 1)\frac{c_3}{c_2}\right)\frac{r}{\beta}\frac{\partial \Phi}{\partial r} = \frac{c_3}{c_2}\Phi. \tag{5.85}$$

The differential equations of the characteristic curves of Eq. (5.85) are

$$\frac{dt}{\frac{c_1}{c_2}+t} \quad = \quad \frac{dr}{\left[1+(2\beta-1)\frac{c_3}{c_2}\right]\frac{r}{\beta}} \quad = \quad \frac{d\Phi}{\frac{c_3}{c_2}\Phi}. \tag{5.86}$$

We can split the above equations into two as follows

$$(a) \;:\; \frac{dt}{\frac{c_1}{c_2}+t} \quad = \quad \frac{dr}{\left[1+(2\beta-1)\frac{c_3}{c_2}\right]\frac{r}{\beta}},$$

and

$$(b) \;:\; \frac{dt}{\frac{c_1}{c_2}+t} \quad = \quad \frac{d\Phi}{\frac{c_3}{c_2}\Phi}.$$

Integrating (a) and (b) respectively, we have

$$\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]\log\left(\frac{c_1}{c_2}+t\right) = \log r + \text{const},$$

and

$$\log\left(\frac{c_1}{c_2}+t\right) = \frac{c_2}{c_3}\log\Phi + \text{const},$$

which implies

$$\log\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]} = \log r + \text{const},$$

and

$$\log\left(\frac{c_1}{c_2}+t\right)^{\frac{c_3}{c_2}} = \log\Phi + \text{const}.$$

Thus

$$\frac{r}{\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}} = \bar{\alpha}_1, \tag{5.87}$$

and

$$\frac{\Phi}{\left(\frac{c_1}{c_2}+t\right)^{\frac{c_3}{c_2}}} = \bar{\alpha}_2, \tag{5.88}$$

where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are arbitrary constants. Writing $\bar{\alpha}_2 = F(\bar{\alpha}_1)$ where F is an arbitrary function, we obtain

$$\frac{\Phi}{\left(\frac{c_1}{c_2}+t\right)^{\frac{c_3}{c_2}}} = F\left(\frac{r}{\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}}\right),$$

and therefore

$$\Phi(t,r) = \left(\frac{c_1}{c_2}+t\right)^{\frac{c_3}{c_2}} F\left(\frac{r}{\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}}\right).$$

Since $h = \Phi(t,r)$, the group-invariant solution of Eq. (3.143) is of the form

$$h(t,r) = \left(\frac{c_1}{c_2}+t\right)^{\frac{c_3}{c_2}} F(\xi), \tag{5.89}$$

with

$$\xi = \frac{r}{\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}}. \tag{5.90}$$

For a given time t, make a change of variable from r to ξ in Eq. (5.82)

$$r = \xi\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]},$$

so that

$$dr = \left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]} d\xi.$$

$$0 \le r \le R(t) \quad \text{implies} \quad 0 \le \xi \le \frac{R(t)}{\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}}.$$

We also obtain

$$V = \left(\frac{c_1}{c_2}+t\right)^{\frac{2}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]+\frac{c_3}{c_2}} \int_0^{R(t)/\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}} \xi F(\xi) d\xi. \tag{5.91}$$

For V to remain constant in time, we impose the two conditions

$$\left(\frac{c_1}{c_2}+t\right)^{\frac{2}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]+\frac{c_3}{c_2}} = \bar{\gamma}_1, \tag{5.92}$$

and

$$\frac{R(t)}{\left(\frac{c_1}{c_2}+t\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}} = \bar{\gamma}_2. \tag{5.93}$$

Eq. (5.92) is valid if and only if

$$\frac{2}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]+\frac{c_3}{c_2}=0,$$

that is

$$\frac{c_3}{c_2} = -\frac{2}{5\beta-2}, \quad (\beta \neq 2/5)$$

and therefore $\bar{\gamma}_1 = 1$. From the initial condition $R(0) = 1$, Eq. (5.93) implies that

$$\bar{\gamma}_2 = \left(\frac{c_2}{c_1}\right)^{\frac{1}{\beta}\left[1+(2\beta-1)\frac{c_3}{c_2}\right]}.$$

Consequently, Eq. (5.93) gives

$$R(t) = \left(1 + \frac{c_2}{c_1} t\right)^{\frac{1}{5\beta - 2}}, \tag{5.94}$$

and Eq. (5.91) simply reads

$$V = \int_0^{\left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}}} \xi F(\xi) d\xi. \tag{5.95}$$

Eqs. (5.89) and (5.90) now reduce to

$$h(t, r) = \frac{F(\xi)}{\left(\frac{c_1}{c_2} + t\right)^{2/(5\beta-2)}}, \tag{5.96}$$

and

$$\xi = \frac{r}{\left(\frac{c_1}{c_2} + t\right)^{1/(5\beta-2)}} \equiv \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}} \frac{r}{R(t)}. \tag{5.97}$$

The boundary condition (5.83) applied with h defined by Eqs. (5.96)–(5.97), and $\xi = (c_2/c_1)^{\frac{1}{5\beta-2}}$, at $r = R(t)$ gives

$$F\left((c_2/c_1)^{\frac{1}{5\beta-2}}\right) = 0. \tag{5.98}$$

In order to substitute Eqs. (5.96) and (5.97) into Eq. (3.143), we perform the following calculations

$$h(t, r) = \frac{F(\xi)}{(c_2/c_1)^{-2/(5\beta-2)} R(t)^2},$$

$$\begin{aligned} \frac{\partial h}{\partial t} &= (c_2/c_1)^{2/(5\beta-2)} \frac{\partial}{\partial t}\left(\frac{F(\xi)}{R(t)^2}\right), \\ &= (c_2/c_1)^{2/(5\beta-2)} \left[\frac{\partial F(\xi)}{\partial t} R(t)^2 - 2R(t)F(\xi)R'(t)\right] \frac{1}{R(t)^4}, \end{aligned}$$

and

$$\frac{\partial h}{\partial r} = \left[\frac{\partial}{\partial r}F(\xi)\right]\frac{(c_2/c_1)^{2/(5\beta-2)}}{R(t)^2}.$$

The chain rule gives

$$\frac{\partial F}{\partial t} = \frac{\partial \xi}{\partial t}\frac{dF}{d\xi},$$

and

$$\frac{\partial F}{\partial r} = \frac{\partial \xi}{\partial r}\frac{dF}{d\xi}.$$

However,

$$\begin{aligned}\frac{\partial \xi}{\partial t} &= \frac{\partial}{\partial t}\left[\left(\frac{c_2}{c_1}\right)^{1/(5\beta-2)}\frac{r}{R(t)}\right],\\ &= -\left(\frac{c_2}{c_1}\right)^{1/(5\beta-2)}R'(t)\frac{r}{R(t)^2},\\ &= -\frac{1}{5\beta-2}\left(\frac{c_2}{c_1}\right)^{(5\beta-1)/(5\beta-2)}\frac{r}{R(t)^{5\beta-1}},\end{aligned}$$

since $R'(t) = \frac{1}{5\beta-2}\frac{c_2}{c_1}R(t)^{-5\beta+3}$

and

$$\frac{\partial \xi}{\partial r} = \left(\frac{c_2}{c_1}\right)^{1/(5\beta-2)}\frac{1}{R(t)}. \tag{5.99}$$

Thus

$$\frac{\partial F}{\partial t} = -\frac{1}{5\beta-2}\left(\frac{c_2}{c_1}\right)^{(5\beta-1)/(5\beta-2)}\frac{r}{R(t)^{5\beta-1}}\frac{dF}{d\xi},$$

and

$$\frac{\partial F}{\partial r} = \left(\frac{c_2}{c_1}\right)^{1/(5\beta-2)}\frac{1}{R(t)}\frac{dF}{d\xi}.$$

Hence

$$\begin{aligned}
\frac{\partial h}{\partial t} &= (c_2/c_1)^{2/(5\beta-2)}\left[\frac{\partial F(\xi)}{\partial t}R(t)^2 - 2R(t)F(\xi)R'(t)\right]\frac{1}{R(t)^4}, \\
&= \frac{(c_2/c_1)^{\frac{2}{(5\beta-2)}}}{R(t)^4}\left[-\frac{1}{5\beta-2}\left(\frac{c_2}{c_1}\right)^{\frac{(5\beta-1)}{(5\beta-2)}}\frac{r}{R(t)^{5\beta-3}}\left(\frac{dF}{d\xi}\right) - \frac{2(c_2/c_1)}{(5\beta-2)R(t)^{5\beta-4}}F(\xi)\right], \\
&= -\left[\frac{(c_2/c_1)^{\frac{(5\beta-1)}{(5\beta-2)}}}{5\beta-2}\frac{r}{R(t)^{5\beta+1}}\left(\frac{dF}{d\xi}\right) + \frac{2(c_2/c_1)}{(5\beta-2)R(t)^{5\beta}}F(\xi)\right]\left(\frac{c_2}{c_1}\right)^{\frac{2}{(5\beta-2)}}, \\
&= -\frac{(c_2/c_1)^{\frac{5\beta}{(5\beta-2)}}}{(5\beta-2)R(t)^{5\beta}}\left[\xi\frac{dF}{d\xi}(\xi) + 2F(\xi)\right], \qquad (5.100)
\end{aligned}$$

and

$$\frac{\partial h}{\partial r} = \frac{(c_2/c_1)^{\frac{3}{(5\beta-2)}}}{R(t)^3}\left(\frac{dF}{d\xi}\right).$$

From the previous results we have

$$rh^{\beta+1}\left(\frac{\partial h}{\partial r}\right)^{\beta-1} = \frac{(c_2/c_1)\xi}{R(t)^{5\beta-2}}F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}\right)^{\beta-1},$$

so that

$$\frac{\partial}{\partial r}\left[rh^{\beta+1}\left(\frac{\partial h}{\partial r}\right)^{\beta-1}\right] = \frac{(c_2/c_1)^{\frac{5\beta-1}{5\beta-2}}}{R(t)^{5\beta-1}}\frac{d}{d\xi}\left[\xi F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}\right].$$

It follows that

$$\frac{1}{(\beta+1)r}\frac{\partial}{\partial r}\left[rh^{\beta+1}\left(\frac{\partial h}{\partial r}\right)^{\beta-1}\right] = \frac{(c_2/c_1)^{\frac{5\beta}{5\beta-2}}}{(\beta+1)R(t)^{5\beta}\xi}\frac{d}{d\xi}\left[\xi F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}\right]. \qquad (5.101)$$

Now equating Eqs. (5.100) and (5.101) and after simplification, Eq. (3.143) reads

$$\frac{d}{d\xi}\left[\xi F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}\right] + \frac{(\beta+1)\xi}{5\beta-2}\left[\xi\frac{dF}{d\xi}(\xi) + 2F(\xi)\right] = 0. \qquad (5.102)$$

Finally, F satisfies the ordinary differential equation

$$\frac{d}{d\xi}\left[\xi F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}\right]+\frac{(\beta+1)}{5\beta-2}\frac{d}{d\xi}\left[\xi^2F(\xi)\right]=0. \tag{5.103}$$

Integrating Eq. (5.103) once with respect to ξ, we obtain

$$\xi F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}+\frac{(\beta+1)}{5\beta-2}\xi^2F(\xi)=A=\text{constant}. \tag{5.104}$$

The boundary condition (5.98) implies that $A=0$ provided that

$$F^{\beta+1}\left(\left(\frac{c_2}{c_1}\right)^{1/(5\beta-2)}\right)\frac{dF}{d\xi}\left(\left(\frac{c_2}{c_1}\right)^{1/(5\beta-2)}\right)=0. \tag{5.105}$$

We take $A=0$ and verify that the solution obtained satisfies Eq. (5.105). Hence, Eq. (5.104) reads

$$\xi F(\xi)^{\beta+1}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}+\frac{(\beta+1)}{5\beta-2}\xi^2F(\xi)=0,$$

that is

$$F(\xi)^{\beta}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}+\frac{(\beta+1)}{5\beta-2}\xi=0. \tag{5.106}$$

Eq. (5.106) is separable for some particular values of β. For example if β is even or a rational number of the form $\beta=p/q,\ q\neq 0$, where: i) p and q are such that $p-q$ is odd i.e p and q are both odd or ii) q is even and p is a given integer. We discuss the solution of Eq. (5.106) in these three cases subject to Eq. (5.105).

Rewrite Eq. (5.106) as

$$F(\xi)^{\beta}\left(\frac{dF}{d\xi}(\xi)\right)^{\beta-1}=-\frac{(\beta+1)}{5\beta-2}\xi,$$

so that

$$F(\xi)^{\frac{\beta}{\beta-1}}\frac{dF}{d\xi}(\xi)=\left[\frac{(\beta+1)}{5\beta-2}\right]^{\frac{1}{\beta-1}}(-\xi)^{\frac{1}{\beta-1}}.$$

Letting $C_\beta^0 = \left[\frac{(\beta+1)}{5\beta-2}\right]^{\frac{1}{\beta-1}}$ and separating the equation, we have

$$F^{\frac{\beta}{\beta-1}} dF = C_\beta^0 (-\xi)^{\frac{1}{\beta-1}} d\xi.$$

Integrating with respect to ξ, we obtain

$$\frac{F^{\frac{2\beta-1}{\beta-1}}}{\frac{2\beta-1}{\beta-1}} = -C_\beta^0 \frac{(-\xi)^{\frac{\beta}{\beta-1}}}{\frac{\beta}{\beta-1}} + C_0.$$

Applying the boundary condition (5.98) we have

$$C_0 = C_\beta^0 \left(\frac{\beta-1}{\beta}\right) \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}}.$$

Thus

$$\frac{F^{\frac{2\beta-1}{\beta-1}}}{\frac{2\beta-1}{\beta-1}} = C_\beta^0 \left(\frac{\beta-1}{\beta}\right) \left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - (-\xi)^{\frac{\beta}{\beta-1}}\right].$$

Since $\beta = 2(m+1)/(m+1)$, and $(-\xi)^{\frac{\beta}{\beta-1}} = \xi^{\frac{\beta}{\beta-1}}$, we obtain

$$F^{\frac{2\beta-1}{\beta-1}} = C_\beta^0 \left(\frac{2\beta-1}{\beta}\right) \left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}}\right].$$

Hence

$$F(\xi) = \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}^{\frac{1}{2\beta-1}} \left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}}. \tag{5.107}$$

$r \leq R(t)$, implies via Eq. (5.97) that $\xi \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}}$.

If $\frac{\beta}{\beta-1} \geq 0$, then $\xi^{\frac{\beta}{\beta-1}} \leq \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}}$ and therefore

$$\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}} \geq 0.$$

Otherwise

$$\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}} \leq 0.$$

We have

$$F(\xi)\frac{dF}{d\xi}(\xi) = -\left(\frac{\beta}{2\beta-1}\right)\left\{\frac{\beta+1}{2-5\beta}\left(\frac{1-2\beta}{\beta}\right)\right\}^{\frac{1}{2\beta-1}} F(\xi).$$

Multiplying by $F(\xi)^\beta$ we obtain

$$F(\xi)^{\beta+1}\frac{dF}{d\xi}(\xi) = -\left(\frac{\beta}{2\beta-1}\right)\left\{\frac{\beta+1}{2-5\beta}\left(\frac{1-2\beta}{\beta}\right)\right\}^{\frac{1}{2\beta-1}} F(\xi)^{\beta+1}.$$

Eq. (5.98) implies that Eq. (5.105) is satisfied. From Eqs. (5.96), (5.97) and (5.107)

$$\begin{aligned} h(t,r) &= \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}} \frac{F(\xi)}{R(t)^2}, \\ &= \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}} \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}^{\frac{1}{2\beta-1}} \frac{1}{R(t)^2} \times \\ &\quad \left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} \frac{r^{\frac{\beta}{\beta-1}}}{R(t)^{\frac{\beta}{\beta-1}}}\right]^{\frac{\beta-1}{2\beta-1}}, \end{aligned}$$

$$h(t,r) = \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}} \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(2\beta-1)(5\beta-2)}} \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}^{\frac{1}{2\beta-1}} \frac{1}{R(t)^2} \times \left[1-\frac{r^{\frac{\beta}{\beta-1}}}{R(t)^{\frac{\beta}{\beta-1}}}\right]^{\frac{\beta-1}{2\beta-1}},$$

$$= \left(\frac{c_2}{c_1}\right)^{\frac{1}{2\beta-1}} \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}^{\frac{1}{2\beta-1}} \frac{1}{R(t)^2} \left[1-\frac{r^{\frac{\beta}{\beta-1}}}{R(t)^{\frac{\beta}{\beta-1}}}\right]^{\frac{\beta-1}{2\beta-1}}.$$

Hence

$$h(t,r) = \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\frac{c_2}{c_1}\right\}^{\frac{1}{2\beta-1}} \frac{1}{R(t)^2}\left[1-\frac{r^{\frac{\beta}{\beta-1}}}{R(t)^{\frac{\beta}{\beta-1}}}\right]^{\frac{\beta-1}{2\beta-1}} \qquad (5.108)$$

where $R(t)$ is given by Eq. (5.94). Both $h(t,r)$ and $R(t)$ depend on the ratio c_2/c_1 which is obtained by substituting Eq. (5.107) in Eq. (5.95), that is

$$V = \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}^{\frac{1}{2\beta-1}} \int_0 \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}} \xi\left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}} d\xi.$$

Let

$$I = \int_0 \left(\frac{c_2}{c_1}\right)^{\frac{1}{5\beta-2}} \xi\left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}} d\xi,$$

$$\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}} = \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}}\left[1-\frac{\xi^{\frac{\beta}{\beta-1}}}{(c_2/c_1)^{\beta/(\beta-1)(5\beta-2)}}\right],$$

$$= \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}}\left[1-\left\{\frac{\xi}{(c_2/c_1)^{1/(5\beta-2)}}\right\}^{\frac{\beta}{\beta-1}}\right].$$

It follows that

$$\left[\left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(\beta-1)(5\beta-2)}} - \xi^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}} = \left(\frac{c_2}{c_1}\right)^{\frac{\beta}{(2\beta-1)(5\beta-2)}} \times \left[1 - \left\{\frac{\xi}{(c_2/c_1)^{1/(5\beta-2)}}\right\}^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}}.$$

Making the change of variable in I from ξ to η by letting

$$\eta = \frac{\xi}{(c_2/c_1)^{1/(5\beta-2)}},$$

we have $\xi = (c_2/c_1)^{1/(5\beta-2)}\eta$ and $d\xi = (c_2/c_1)^{1/(5\beta-2)}d\eta$.

Now $0 \le \xi \le (c_2/c_1)^{1/(5\beta-2)}$ implies $0 \le \eta \le 1$,

and hence

$$I = (c_2/c_1)^{1/(2\beta-1)} \int_0^1 \eta \left[1 - \eta^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}} d\eta.$$

Therefore

$$V = \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\frac{c_2}{c_1}\right\}^{\frac{1}{2\beta-1}} \int_0^1 \eta \left[1 - \eta^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}} d\eta,$$

$$= \left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\frac{c_2}{c_1}\right\}^{\frac{1}{2\beta-1}} J(\beta),$$

where

$$J(\beta) = \int_0^1 \eta \left[1 - \eta^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{2\beta-1}} d\eta.$$

Writing the change of variable $t = \eta^{\dfrac{\beta}{\beta-1}}$, we have $\eta = t^{(\beta-1)/\beta} = t^{1/\alpha}$, where $\alpha = \dfrac{\beta}{\beta-1}$.

Furthermore

$$\begin{aligned} \eta d\eta &= \frac{1}{\alpha} t^{2/\alpha - 1} dt \\ &= \frac{1}{\alpha} t^{\gamma' - 1} dt, \end{aligned}$$

where

$$\gamma' = \frac{2}{\alpha} = \frac{2(\beta-1)}{\beta}.$$

Now $0 \leq \eta \leq 1$ implies for $\dfrac{\beta}{\beta-1} \geq 0$, that $0 \leq t \leq 1$, and hence

$$J(\beta) = \frac{1}{\alpha} \int_0^1 t^{\gamma'-1} (1-t)^{\gamma-1} dt.$$

If $\gamma' > 0$ and $\gamma > 0$ then

$$\begin{aligned} J(\beta) &= \frac{1}{\alpha} B(\gamma', \gamma), \\ &= \frac{1}{\alpha} \frac{\Gamma(\gamma')\Gamma(\gamma)}{\Gamma(\gamma'+\gamma)}, \\ &= \frac{1}{\alpha\gamma'} \frac{\Gamma(\gamma'+1)\Gamma(\gamma)}{\Gamma(\gamma'+\gamma)}, \end{aligned}$$

where B and Γ are respectively the Beta and Gamma functions (see Abramowitz and Stegun, 1965). In the last equality, we have used $\Gamma(\gamma') = \Gamma(\gamma'+1)/\gamma'$.

If $\dfrac{\beta}{\beta-1} > 0$ (which implies $\gamma' > 0$) and $\dfrac{3\beta-2}{2\beta-1} > 0$ (equivalent to $\gamma' > 0$), then

$$J(\beta) = \frac{\Gamma\left(3 - \dfrac{2}{\beta}\right) \Gamma\left(\dfrac{3\beta-2}{2\beta-1}\right)}{2\,\Gamma\left(\dfrac{2+\beta(7\beta-8)}{\beta(2\beta-1)}\right)}.$$

For other values of β, $J(\beta)$ can be calculated by numerical integration. We finally obtain the ratio

$$\frac{c_2}{c_1} = \frac{V^{2\beta-1}}{\left[\dfrac{\beta+1}{5\beta-2}\left(\dfrac{2\beta-1}{\beta}\right)^{\beta-1}\right] J(\beta)^{2\beta-1}}. \tag{5.109}$$

Substituting Eq. (5.109) into Eq. (5.108), we obtain

$$h(t,r) = \frac{V}{J(\beta)R(t)^2}\left(1-\frac{r^{\frac{\beta}{\beta-1}}}{R(t)^{\frac{\beta}{\beta-1}}}\right)^{\frac{\beta-1}{2\beta-1}}, \tag{5.110}$$

where

$$R(t) = \left[1+\frac{V^{2\beta-1}}{\left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}J(\beta)^{2\beta-1}}\,t\right]^{\frac{1}{5\beta-2}}. \tag{5.111}$$

The Lie point symmetry which generates the group-invariant solution is proportional to

$$Y = \left(\frac{\left\{\frac{\beta+1}{5\beta-2}\left(\frac{2\beta-1}{\beta}\right)^{\beta-1}\right\}J(\beta)^{2\beta-1}}{V^{2\beta-1}}+t\right)\frac{\partial}{\partial t}+\frac{1}{5\beta-2}r\frac{\partial}{\partial r}-\frac{2}{5\beta-2}h\frac{\partial}{\partial h}. \tag{5.112}$$

From the expression for h given by Eq. (5.110), we realize that for the non-Newtonian power-law case, the initial surface profile cannot be specified arbitrarily. Since $R(0)=1$, the initial surface profile is given by

$$h(0,r) = \frac{V}{J(\beta)}\left(1-r^{\frac{\beta}{\beta-1}}\right)^{\frac{\beta-1}{2\beta-1}}. \tag{5.113}$$

Differentiating h with respect to r

$$\frac{\partial h}{\partial r} = -\left(\frac{\beta}{2\beta-1}\right)\frac{V\,r^{\frac{1}{\beta-1}}}{J(\beta)R(t)^{(3\beta-2)/(\beta-1)}}\left(1-\frac{r^{\beta/(\beta-1)}}{R(t)^{\beta/(\beta-1)}}\right)^{-\left(\frac{\beta}{2\beta-1}\right)}. \tag{5.114}$$

• If $\dfrac{\beta}{2\beta-1} > 0, \quad \dfrac{\partial h}{\partial r} \to -\infty \quad \text{as} \quad r \to R(t).$

For these values of β, the solution near the outer rim is incorrect, because the thin film approximation is not satisfied there. So the surface tension, which was neglected, has to be included for a better description of the spreading, because surface tension becomes important near the outer rim $r = R(t)$.

• If $\dfrac{\beta}{2\beta-1} \leq 0, \quad \dfrac{\partial h}{\partial r} \to 0 \quad \text{as} \quad r \to R(t).$

For these values of β, the above result physically reveals the presence of an extremum and Eq. (5.83) implies that it is a minimum, and the tangent at $r = R(t)$ is in the horizontal plane. In this case surface tension can be neglected.

For $\beta = 0$, Eq. (3.143) admits four symmetries

$$X_1 = \frac{\partial}{\partial t}; \quad X_2 = t\frac{\partial}{\partial t} + h\frac{\partial}{\partial h}; \quad X_3 = r\frac{\partial}{\partial r}; \quad X_4 = \frac{1}{r}\frac{\partial}{\partial r}. \tag{5.115}$$

We investigate the existence of group-invariant solutions.

$h = \Phi(t, r)$ is a group-invariant solution of Eq. (3.143) provided

$$X\left(h - \Phi(t,r)\right)\big|_{h=\Phi(t,r)} = 0, \tag{5.116}$$

where

$$X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4. \tag{5.117}$$

We assume that the constants c_1, c_2, c_3, and c_4 are not all zero, say $c_1 \neq 0$, so that dividing X by c_1, we obtain

$$Y = \left(1 + \frac{c_2}{c_1}t\right)\frac{\partial}{\partial t} + \left(\frac{c_3}{c_1}r + \frac{c_4}{c_1 r}\right)\frac{\partial}{\partial r} + \frac{c_2}{c_1}h\frac{\partial}{\partial h}.$$

Now $h = \Phi(t, r)$ is a group-invariant solution of Eq. (3.143) provided that

$$Y\left(h - \Phi(t,r)\right)\big|_{h=\Phi(t,r)} = 0, \tag{5.118}$$

that is

$$(1 + \frac{c_2}{c_1}t)\frac{\partial\Phi}{\partial t} + \left(\frac{c_3}{c_1}r + \frac{c_4}{c_1 r}\right)\frac{\partial\Phi}{\partial r} = \frac{c_2}{c_1}\Phi. \tag{5.119}$$

Proceeding in a manner similar to the previous case where $\beta \neq 0$, we obtain the following results:

$$\Phi(t,r) = \left(1 + \frac{c_2}{c_1}t\right) F\left(\frac{c_4/c_1 + (c_3/c_1)r^2}{(1+(c_2/c_1)t)^{2c_3/c_2}}\right), \tag{5.120}$$

and the group-invariant solution is of the form

$$h(t,r) = \left(1 + \frac{c_2}{c_1}t\right) F(\xi), \tag{5.121}$$

where

$$\xi = \frac{c_4/c_1 + (c_3/c_1)r^2}{(1+(c_2/c_1)t)^{2c_3/c_2}}. \tag{5.122}$$

Fixing t and making the change of variable from r to ξ in Eq. (5.82), assuming $c_3/c_1 > 0$, we obtain

$$V = \frac{c_1}{2c_3}\left(1 + \frac{c_2}{c_1}t\right)^{2c_3/c_2+1} \int_a^b F(\xi)\, d\xi, \tag{5.123}$$

where

$$a = \frac{c_4/c_1}{\left(1 + \frac{c_2}{c_1}t\right)^{2c_3/c_2}} \quad \text{and} \quad b = a + \frac{c_4/c_1 + (c_3/c_1)R^2(t)}{\left(1 + \frac{c_2}{c_1}t\right)^{2c_3/c_2}}. \tag{5.124}$$

For V to remain constant in time, we impose three conditions

$$\frac{c_1}{2c_3}\left(1 + \frac{c_2}{c_1}t\right)^{2c_3/c_2+1} = \bar{\gamma}_1, \tag{5.125}$$

$$a = \frac{c_4/c_1}{\left(1 + \frac{c_2}{c_1}t\right)^{2c_3/c_2}} = \bar{\gamma}_2, \tag{5.126}$$

$$b = a + \frac{c_4/c_1 + (c_3/c_1)R^2(t)}{\left(1 + \frac{c_2}{c_1}t\right)^{2c_3/c_2}} = \bar{\gamma}_3. \tag{5.127}$$

Eq. (5.125) is valid if and only if

$$\frac{c_3}{c_2} = -\frac{1}{2}, \tag{5.128}$$

and therefore

$$\bar{\gamma}_1 = \frac{c_1}{2c_3}. \tag{5.129}$$

From Eq. (5.128), Eq. (5.126) is valid if and only if $c_4 = 0$, therefore

$$\bar{\gamma}_2 = 0. \tag{5.130}$$

From the initial condition $R(0) = 1$, Eq. (5.127) implies that

$$\bar{\gamma}_3 = \frac{c_3}{c_1} > 0. \tag{5.131}$$

Consequently, Eq. (5.127) gives

$$R(t) = \left(1 + \frac{c_2}{c_1}t\right)^{-1/2} = \left(1 - 2\frac{c_3}{c_1}t\right)^{-1/2}, \tag{5.132}$$

and Eq. (5.123) becomes

$$V = \frac{c_1}{2c_3}\int_0^{c_3/c_1} F(\xi)\, d\xi. \tag{5.133}$$

We notice from Eq. (5.132) that if the group-invariant solution exists, there will be a blow up in the solution at $t = c_1/(2c_3)$, since $c_3/c_1 > 0$. Furthermore, Eq. (3.143) in terms of t and ξ reads

$$\frac{d}{d\xi}\left[F(\xi)\ \left(\frac{d\ F}{d\xi}\right)^{-1}\right] + 2\left(\frac{c_3}{c_1}\right)^{1/2}\ \xi\ \frac{d\ F}{d\xi} + 2\left(\frac{c_3}{c_1}\right)^{1/2}\ F = 0, \tag{5.134}$$

subject to the boundary conditions

$$F(c_3/c_1) = 0 \tag{5.135}$$

and

$$\frac{d\ F}{d\xi}(c_3/c_1) = 0. \tag{5.136}$$

The boundary condition (5.135) is obtained from Eq. (5.124) and Eq. (5.136) by substituting Eq. (5.135) into Eq. (5.134).

Eq. (5.134) subject to Eqs. (5.135) and (5.136) does not admit a physical solution, therefore Eq. (3.143) does not possess a group-invariant solution for $\beta = 0$.

5.4.1 Discussion

In this section, we briefly discuss the influence of the power-law parameter on the film profile. We consider three cases: $1 < \beta < 2$ for shear-thickening, $\beta = 2$ for Newtonian, and finally $\beta < 1$ and $\beta > 2$ for shear-thinning. In figures 5.1–5.3, graphs of $h(r, t)$ are plotted against r for scaled times t and values of β chosen to illustrate the different type of behaviour of the flow. The general remark is that $h(0, t)$ decreases and the radius $R(t)$ increases with time for the shear-thickening, Newtonian and shear-thinning cases. Figure 5.1 shows that a shear-thickening drop flows faster than the Newtonian and shear-thinning drops, and figure 5.3 shows that the shear-thinning drop flows slower than the Newtonian and shear-thickening drops, with the Newtonian drop playing an intermediate role. The shape of the drop differs depending on the nature of the fluid, be it shear-thickening, Newtonian or shear-thinning. We notice the strong influence of the power-law coefficient of the spreading or profile of the liquid drop. This behaviour of the fluid drop could be predicted due to the form of the analytical solution.

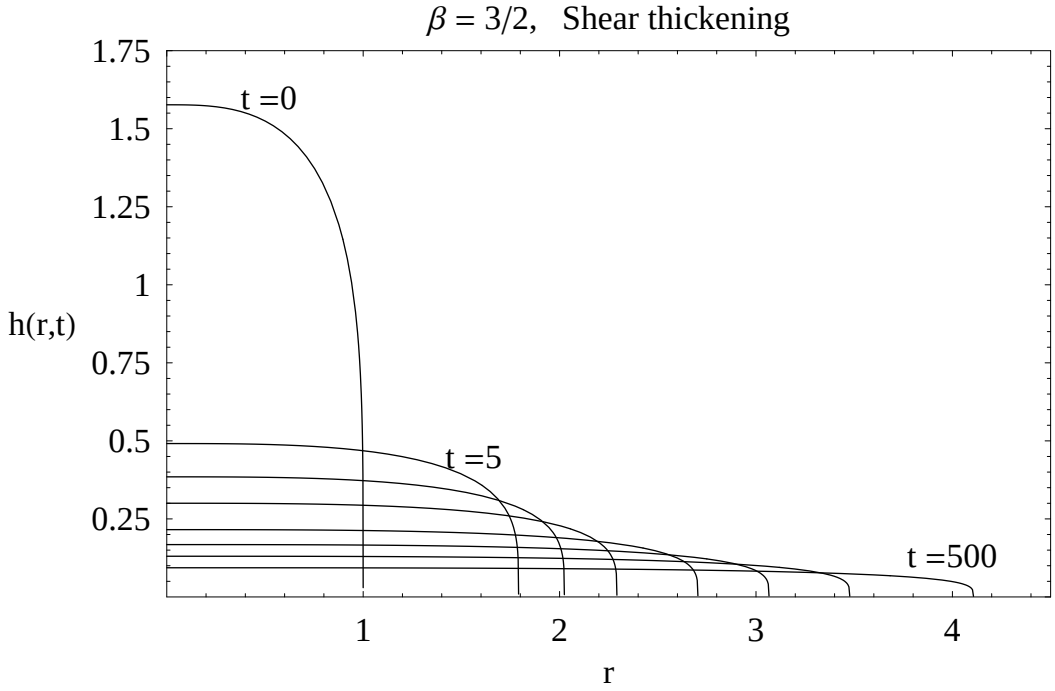

Figure 5.1: Free surface profile for a shear-thickening fluid, $\beta = 3/2$.

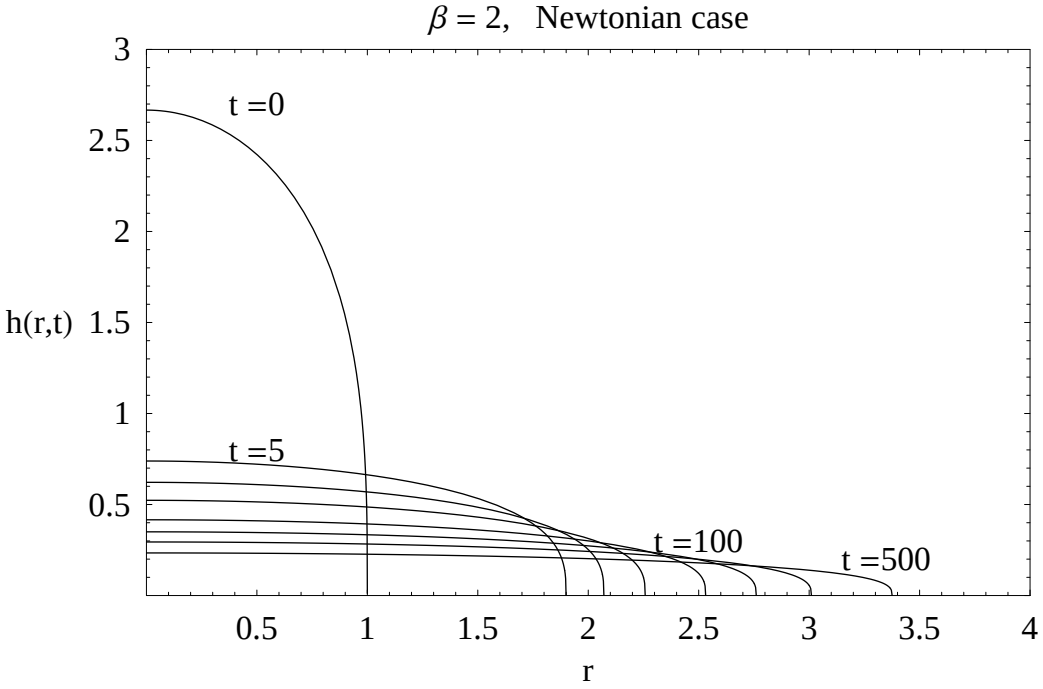

Figure 5.2: Free surface profile for a Newtonian fluid, $\beta = 2$.

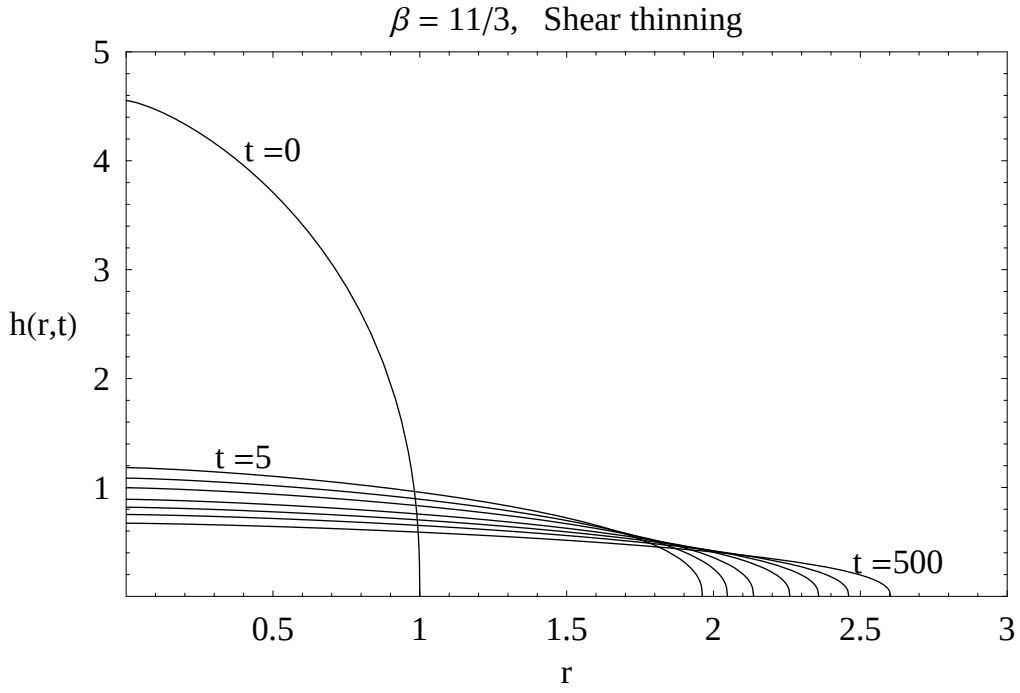

Figure 5.3: Free surface profile for a shear-thinning fluid, $\beta = 11/3$.

5.4.2 Conclusion

We derived and solved Eq. (3.143) which is a highly nonlinear partial differential equation modelling the axisymmetric spreading under gravity of a thin power-law fluid on a horizontal plane. We then used Lie group analysis (via invariant solution) to reduce this partial differential equation to a highly nonlinear second-order ordinary differential equation defined by Eq. (5.103). Through our investigations, we show that this ordinary differential equation is integrable. After reduction of this ordinary differential equation, we show that the resulting equation is integrable for particular values of β, for instance for β even or of the form $\beta = p/q,\ q \neq 0$, where p and q are both odd or q is even and p a given integer. We then study the influence of the power-law parameter β on the evolution of the profile of the free surface. We notice that this parameter has a great influence on the profile of the free surface. For $\beta = 0$, we showed that Eq. (3.143) does not possess a group-invariant solution. Our results for the Newtonian case are in full agreement with similar work done by Momoniat *et al.*, (2001) and appear to be a good generalization of non-Newtonian power-law fluids. These results are important since non-Newtonian fluids are used in many industrial and technological applications, and power-law constitutes a simple and widely used model.

5.5 Conservation Laws

In this section, we want to establish a connection between conservation of total volume V of the liquid drop and an elementary conservation law for the partial differential equation Eq. (3.143).

$$D_1 T^1 + D_2 T^2 = 0 \tag{5.137}$$

is a conservation law for Eq. (3.143) if Eq. (5.137) is satisfied for all solutions $h(t, r)$ of Eq. (3.143). The quantities $T^i(t, r, h, h_t, h_r)$ are the components of the conserved vector $T = (T^1, T^2)$. Eq. (3.143) implies

$$r\frac{\partial h}{\partial t} - \frac{\partial}{\partial r}\left[\frac{1}{\beta+1} r h^{\beta+1} (h_r)^{\beta-1}\right] = 0,$$

that is

$$\frac{\partial}{\partial t}(rh) + \frac{\partial}{\partial r}\left(-\frac{1}{\beta+1} r h^{\beta+1} (h_r)^{\beta-1}\right) = 0.$$

Compute

$$\begin{aligned}
D_1(rh) + D_2\left(-\frac{1}{\beta+1} r h^{\beta+1} (h_r)^{\beta-1}\right) &= r h_t - \frac{1}{\beta+1}\left[h^{\beta+1} (h_r)^{\beta-1} + \right. \\
r(\beta+1) h^{\beta} (h_r)^{\beta} &+ \left. r(\beta-1) h^{\beta+1} (h_r)^{\beta-2} h_{rr}\right] \\
&= \frac{1}{(\beta+1) r}\frac{\partial}{\partial r}\left(r h^{\beta+1}\left(\frac{\partial h}{\partial r}\right)^{\beta-1}\right) \\
-\frac{1}{\beta+1}\left[h^{\beta+1} (h_r)^{\beta-1} + r(\beta+1) h^{\beta} (h_r)^{\beta}\right. &+ \left. r(\beta-1) h^{\beta+1} (h_r)^{\beta-2} h_{rr}\right] \\
&= \frac{1}{\beta+1}\left[h^{\beta+1} (h_r)^{\beta-1} + r(\beta+1) h^{\beta} (h_r)^{\beta}\right. \\
r(\beta-1) h^{\beta+1} (h_r)^{\beta-2} h_{rr}\Big] &- \frac{1}{\beta+1}\left[h^{\beta+1} (h_r)^{\beta-1} + r(\beta+1) h^{\beta} (h_r)^{\beta}\right. \\
r(\beta-1) h^{\beta+1} (h_r)^{\beta-2} h_{rr}\Big] &= 0. \qquad (5.138)
\end{aligned}$$

The components of the conserved vector are

$$T^1 = rh, \qquad (5.139a)$$

$$T^2 = -\frac{1}{\beta+1} r h^{\beta+1} (h_r)^{\beta-1}. \qquad (5.139b)$$

The result linking X to T, is given by

$$X^{[1]}\left(T^i\right) + T^i D_k \xi^k - T^k D_k \xi^i = 0, \quad \text{i} = 1,2 \qquad (5.140)$$

that is

$$\text{for i=1}: \quad X^{[1]}\left(T^1\right) + T^1 D_2 \xi^2 - T^2 D_2 \xi^1 = 0, \qquad (5.141)$$

$$\text{for i=2}: \quad X^{[1]}\left(T^2\right) + T^2 D_1 \xi^1 - T^1 D_1 \xi^2 = 0. \qquad (5.142)$$

The first prolongation $X^{[1]}$ is given by

$$X^{[1]} = X + \zeta_t \frac{\partial}{\partial h_t} + \zeta_r \frac{\partial}{\partial h_r},$$

where

$$\zeta_t = D_1(\eta) - h_t D_1(\xi^1) - h_r D_1(\xi^2),$$

and

$$\zeta_r = D_2(\eta) - h_t D_2(\xi^1) - h_r D_2(\xi^2).$$

The calculations give

$$\begin{aligned}\zeta_t &= (c_3 - c_2)\, h_t,\\ \zeta_r &= -\frac{1}{\beta}\left[c_2 + (\beta-1)c_3\right] h_r.\end{aligned}$$

Therefore

$$\begin{aligned}X^{[1]} &= (c_1 + c_2 t)\frac{\partial}{\partial t} + \left[c_2 + (2\beta-1)c_3\right]\frac{r}{\beta}\frac{\partial}{\partial r} + c_3 h\frac{\partial}{\partial h},\\ &\quad + (c_3 - c_2)\, h_t\frac{\partial}{\partial h_t} - \frac{1}{\beta}\left[c_2 + (\beta-1)c_3\right] h_r\frac{\partial}{\partial h_r}.\end{aligned} \tag{5.143}$$

Substituting both Eqs. (5.139a)–(5.139b) and Eq. (5.143) into Eq. (5.141) and Eq. (5.142) respectively, we obtain

$$\frac{1}{\beta}\left[c_2 + (2\beta-1)c_3\right] rh + c_3 rh + (rh)\frac{1}{\beta}\left[c_2 + (2\beta-1)c_3\right] = 0,$$

and

$$\begin{aligned}-\frac{1}{\beta(\beta+1)}\left[c_2 + (2\beta-1)c_3\right] rh^{\beta+1}(h_r)^{\beta-1} + \frac{\beta-1}{\beta+1}\left[c_2 + (\beta-1)c_3\right] rh^{\beta+1}(h_r)^{\beta-1}\\ -c_3 rh^{\beta+1}(h_r)^{\beta-1} - \frac{1}{\beta+1}c_2 rh^{\beta+1}(h_r)^{\beta-1} = 0.\end{aligned}$$

After simplification, we obtain

$$\frac{2}{\beta}\left[c_2 + \left(\frac{5}{2}\beta - 1\right)c_3\right] rh = 0,$$

and

$$-\frac{2}{\beta(\beta+1)}\left[c_2 + \left(\frac{5}{2}\beta - 1\right)c_3\right] rh^{\beta+1}(h_r)^{\beta-1} = 0.$$

Finally, we obtain

$$\left[c_2 + \left(\frac{5}{2}\beta - 1\right) c_3\right] rh = 0,$$

and

$$\left[c_2 + \left(\frac{5}{2}\beta - 1\right) c_3\right] rh^{\beta+1} (h_r)^{\beta-1} = 0.$$

Hence, Eqs. (5.141) and (5.142) are satisfied provided

$$\frac{c_3}{c_2} = -\frac{2}{5\beta - 2}. \tag{5.144}$$

The ratio c_3/c_2 of the constants in the linear combination Eq. (5.81) must satisfy Eq. (5.144) for Eq. (5.81) to be the symmetry generator associated with the conserved vectors Eq. (5.139a) and Eq. (5.139b). This is the same condition on the ratio c_3/c_2 that was derived when imposing (on the group-invariant solution) the condition of conservation of total volume of the liquid drop. We therefore realise that the same conditions for the Newtonian case have been extended to power-law fluids. We have applied the Kara-Mahomed Theorem, (2000) in the reduction of the constants. The symmetries of this conserved quantity give the physical invariant solution.

5.6 Method of the separation of variables

In view of the separable solution and the resulting ordinary differential equation, we rewrite Eq. (3.143) in the form

$$\frac{\partial h}{\partial t} = \frac{1}{(\beta+1)x} \frac{\partial}{\partial x} \left[xh^{\beta+1} \left(\frac{\partial h}{\partial x}\right)^{\beta-1} \right]. \tag{5.145}$$

We investigate separable solutions of the form (see Polyanin and Zaitshev, (2004))

$$h(x,t) = y(x)\psi(t), \tag{5.146}$$

admitted by Eq. (5.145). Substituting Eq. (5.146) into Eq. (5.145) we obtain

$$\frac{\dot{\psi}(t)}{\psi^{\beta'}(t)} = y^{\beta'-1}(x) F\left(x, \frac{y'(x)}{y(x)}, \frac{y''(x)}{y(x)}\right), \tag{5.147}$$

which we can separate as

$$\frac{\dot{\psi}(t)}{\psi^{\beta'}(t)} = A, \tag{5.148}$$

$$y^{\beta'-1}(x)F\left(x, \frac{y'(x)}{y(x)}, \frac{y''(x)}{y(x)}\right) = A, \tag{5.149}$$

where A is constant. Integrating Eq. (5.148) with respect to t we obtain

$$\psi(t) = [(1-\beta')At + B]^{\frac{1}{1-\beta'}}, \tag{5.150}$$

where B is a constant. Thus from Eq. (5.146)

$$h(x,t) = [(1-\beta')At + B]^{\frac{1}{1-\beta'}}\, y(x), \tag{5.151}$$

where y satisfies the ordinary differential equation

$$y^{\beta'-1}F\left(r, \frac{y'}{y}, \frac{y''}{y}\right) = A, \tag{5.152}$$

and where $\beta' = 2\beta$. The separable solution is

$$h(x,t) = [(1-2\beta)At + B]^{\frac{1}{1-2\beta}}\, y(x), \tag{5.153}$$

where y satisfies the following ordinary differential equation

$$\frac{y(x)^{2\beta-1}}{\beta+1}\left[\frac{1}{x}\left(\frac{y'(x)}{y(x)}\right)^{\beta-1} + (\beta+1)\left(\frac{y'(x)}{y(x)}\right)^{\beta} + (\beta-1)\left(\frac{y'(x)}{y(x)}\right)^{\beta-2}\frac{y''(x)}{y(x)}\right] = A, \tag{5.154}$$

which can be written as

$$y'' = \frac{(\beta+1)A}{(\beta-1)y^{\beta}y'^{\beta-2}} - \frac{1}{x}\frac{y'}{\beta-1} + \frac{\beta+1}{\beta-1}\frac{y'^2}{y}. \tag{5.155}$$

In order to write Eq. (5.155) in a linearizable form, we need $\beta \in [-1,2] \cap \mathbb{Z}$, $\beta \neq \pm 1$; consequently, $\beta \in \{0,2\}$.

5.6.1 Lie group method and linearization

The Lie group method provides a systematic way of determining infinitesimal point transformations of the form

$$\overline{x} \approx x + a\xi(x, y), \qquad \overline{y} \approx y + a\eta(x, y), \tag{5.156}$$

that leave the equation under consideration invariant. The transformations (5.156) form a group where a is the group parameter and the usual group properties hold. The generator of the group is given by

$$X = \xi\partial_x + \eta\partial_y, \tag{5.157}$$

where $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$ and

$$\xi = \frac{d\overline{x}}{da}, \qquad \eta = \frac{d\overline{y}}{da}. \tag{5.158}$$

Writing the equation under consideration as

$$F(x, y, y', y'') = 0, \tag{5.159}$$

where $' = d/dx$ we obtain the coefficients ξ and η in (5.157) by solving a determining equation

$$\left(X^{[2]}F\right)\big|_{F=0} = 0, \tag{5.160}$$

where $X^{[2]}$ is a second prolongation of X given by

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''},$$

and where

$$\zeta_1 = \eta_x + (\eta_y - \xi_x)\, y' - y'^2\xi_y,$$

$$\zeta_2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\xi_{xy})\, y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)\, y''.$$

For more information on the application of Lie group methods to ordinary differential equations, the interested reader is referred to Bluman and Kumei, (1989) and

Ibragimov, (1994, 1996).

- **Case** $\beta = 0$

Setting $\beta = 0$ Eq. (5.155) reads

$$y'' = -\left(A + \frac{1}{y}\right) y'^2 + \frac{1}{x} y'. \tag{5.161}$$

The determining equation

$$X^{[2]} \left[y'' + \left(A + \frac{1}{y}\right) y'^2 + \frac{1}{x} y' \right] \Big|_{(5.161)} = 0, \tag{5.162}$$

leads to

$$\begin{aligned} \eta_{xx} + (2\eta_{xy} - \xi - xx)\, y' + (\eta_{yy} - 2\xi - xy)\, y'^2 - \xi_{yy} y'^3 + \\ (\eta_y - 2\xi_x - 3y'\xi_y) \left[-\left(A + \frac{1}{y}\right) y'^2 + \frac{1}{x} y' \right] - \\ \left[\eta_x + (\eta_y - \xi_x)\, y' - y'^2 \xi_y\right] \left(-2 \left(A + \frac{1}{y}\right) y' + \frac{1}{x} \right) = 0. \end{aligned} \tag{5.163}$$

Therefore the determining equation decomposes into the following four equations, obtained by setting the coefficients of the various powers of y' equal to zero

$$(y')^3 \;:\; \xi_{yy} - \left(A + \frac{1}{y}\right) \xi_y = 0, \tag{5.164}$$

$$(y')^2 \;:\; \eta_{yy} - 2\xi_{xy} + \left(A + \frac{1}{y}\right) \eta_y - \frac{2}{x} \xi_y - \frac{1}{y^2} \eta = 0, \tag{5.165}$$

$$\begin{aligned} y' \;:\; & 2\eta_{xy} - \xi_{xx} + \frac{1}{x} (\eta_y - 2\xi_x) \\ & - \left[-2 \left(A + \frac{1}{y}\right) \eta_x + \frac{1}{x} (\eta_y - \xi_x) \right] + \frac{1}{x^2} \xi = 0, \end{aligned} \tag{5.166}$$

$$(y')^0 \;:\; \eta_{xx} - \frac{1}{x} \eta_x = 0. \tag{5.167}$$

Simplifying, we obtain

$$\xi_{yy} - \left(A + \frac{1}{y}\right) \xi_y = 0, \tag{5.168}$$

$$\eta_{yy} - 2\xi_{xy} + \left(A + \frac{1}{y}\right) \eta_y - \frac{2}{x} \xi_y - \frac{1}{y^2} \eta = 0, \tag{5.169}$$

$$2\eta_{xy} - \xi_{xx} + 2 \left(A + \frac{1}{y}\right) \eta_x - \frac{1}{x} \xi_x + \frac{1}{x^2} \xi = 0, \tag{5.170}$$

$$\eta_{xx} - \frac{1}{x} \eta_x = 0. \tag{5.171}$$

Eq. (5.168) implies

$$\xi(x,y) = a(x)(Ay-1)\frac{e^{Ay}}{A^2} + b(x), \tag{5.172}$$

and Eq. (5.171) implies

$$\eta(x,y) = c(y)\frac{x^2}{2} + d(y). \tag{5.173}$$

Substituting Eq. (5.172) and Eq. (5.173) into Eq. (5.169) and Eq. (5.170) and solving the resulting equations, we obtain

$$\xi(x,y) = a_2 x(Ay-1)\frac{e^{Ay}}{A^2} + \frac{b_1}{x} + b_2 x + \frac{c_1}{4}x^3, \tag{5.174}$$

$$\eta(x,y) = \left[c_1\frac{(Ay-1)}{A^2 y} + c_2\frac{e^{-Ay}}{y}\right]\frac{x^2}{2} + d_1\frac{(Ay-1)}{A^2 y} + d_2\frac{e^{-Ay}}{y} + 2a_2\frac{(Ay-1)^2}{A^4 y}e^{Ay}. \tag{5.175}$$

We obtain all the eight possible symmetries of the ordinary differential equation:

$$X_1 = \frac{1}{x}\frac{\partial}{\partial x} \quad ; \quad X_2 = x\frac{\partial}{\partial x},$$
$$X_3 = x^3\frac{\partial}{\partial x} + 2x^2\left(\frac{Ay-1}{A^2 y}\right)\frac{\partial}{\partial y} \quad ; \quad X_4 = x^2\frac{e^{-Ay}}{y}\frac{\partial}{\partial y},$$
$$X_5 = \left(\frac{Ay-1}{A^2 y}\right)\frac{\partial}{\partial y} \quad ; \quad X_6 = \frac{e^{-Ay}}{y}\frac{\partial}{\partial y},$$
$$X_7 = \frac{1}{x}(Ay-1)\frac{e^{Ay}}{A^2}\frac{\partial}{\partial x} \quad ; \quad X_8 = x(Ay-1)\frac{e^{Ay}}{A^2}\frac{\partial}{\partial x} + 2\frac{(Ay-1)^2}{A^4 y}e^{Ay}\frac{\partial}{\partial y}.$$

Their commutators are given in Table 5.2. From Table 5.2 we note that

$$[X_4, X_5] = X_4,$$

implies that the vector space with basis X_4 and X_5 spans a two-dimensional Lie algebra L_2. And we note that $X_4 \vee X_5 = 0$ where the pseudo-scalar product $\vee$ is defined by $X_i \vee X_j = \xi_i\eta_j - \eta_i\xi_j$ for any operator $X_k = \xi_k\frac{\partial}{\partial x} + \eta_k\frac{\partial}{\partial y}$ with $k \in \{i, j\}$. We can now solve Eq. (5.161) by using the two previous symmetries. (Or since $[X_1, X_2] = 2X_1$, we can also use X_1 and X_2 because of its simple form.)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$2X_1$	$4X_2 + 2X_5$	$2X_6$	0	0	0	$2X_7$
X_2	$-2X_1$	0	$2X_3$	$2X_4$	0	0	$-2X_7$	0
X_3	$-4X_2 - 2X_5$	$-2X_3$	0	0	0	$-2X_4$	$-2X_8$	0
X_4	$-2X_6$	$-2X_4$	0	0	X_4	0	$X_2 - 2X_5$	X_3
X_5	0	0	0	$-X_4$	0	$-X_6$	X_7	X_8
X_6	0	0	$2X_4$	0	X_6	0	X_1	$X_2 + 4X_5$
X_7	0	$2X_7$	$2X_8$	$-X_2 + 2X_5$	$-X_7$	$-X_1$	0	0
X_8	$-2X_7$	0	0	$-X_3$	$-X_8$	$-X_2 - 4X_5$	0	0

Table 5.2: Table of commutators.

- **Linearization process**

Now we apply the linearization process as described in chapter 2.

Equation (5.161) is linearizable as it satisfies the conditions indicated in Theorem 1. The basis of L_2 in canonical variables

$$t = x \quad \text{and} \quad u = \frac{1}{Ax^2}\left(y - \frac{1}{A}\right)e^{Ay}, \tag{5.176}$$

can be written as

$$\overline{X}_4 = \frac{\partial}{\partial u}, \quad \overline{X}_5 = u\frac{\partial}{\partial u}. \tag{5.177}$$

Eq. (5.161) is then transformed into the linear second-order ordinary differential equation

$$t\frac{d^2u}{dt^2} + 3\frac{du}{dt} = 0. \tag{5.178}$$

Integrating twice with respect to t, we obtain

$$u = -\frac{C_1}{2t^2} + C_2, \tag{5.179}$$

where C_1 and C_2 are arbitrary constants. Substituting the canonical variable, we obtain the general solution of the original equation Eq. (5.161)

$$\left(y - \frac{1}{A}\right)e^{Ay} = AC_2\ x^2 - \frac{AC_1}{2}. \tag{5.180}$$

We conclude that

$$h(x,t) = (At + B)\,F(x), \tag{5.181}$$

where $y = F(x)$ is a solution of the implicit equation (5.180). We note that the implicit solution is symmetric about the y-axis. We plot the implicit solution (5.180) in figure 5.4 for $x \geq 0$. For the implicit solution to touch the contact line $y = 0$ at $x = 0$ we find that

$$A = \pm\sqrt{\frac{2}{c_1}}. \tag{5.182}$$

We plot the implicit solution for A (with a positive sign) given by Eq. (5.182) in figure 5.5. For $A < 0$ the solution is below the axis $y = 0$ and is hence not physically viable.

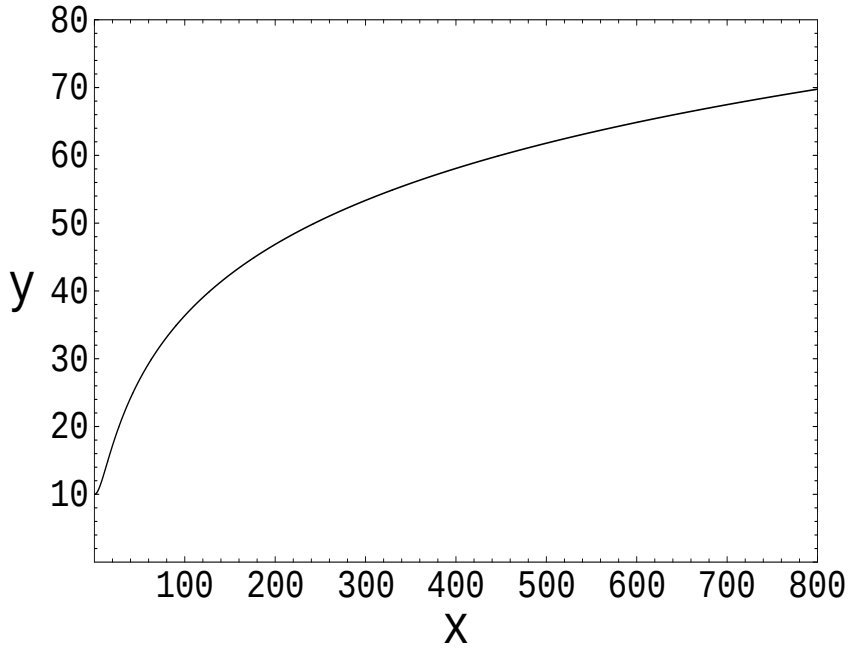

Figure 5.4: Plot of the implicit solution (5.180) for $A = 0.1$, $C_1 = 1$ and $C_2 = 1$.

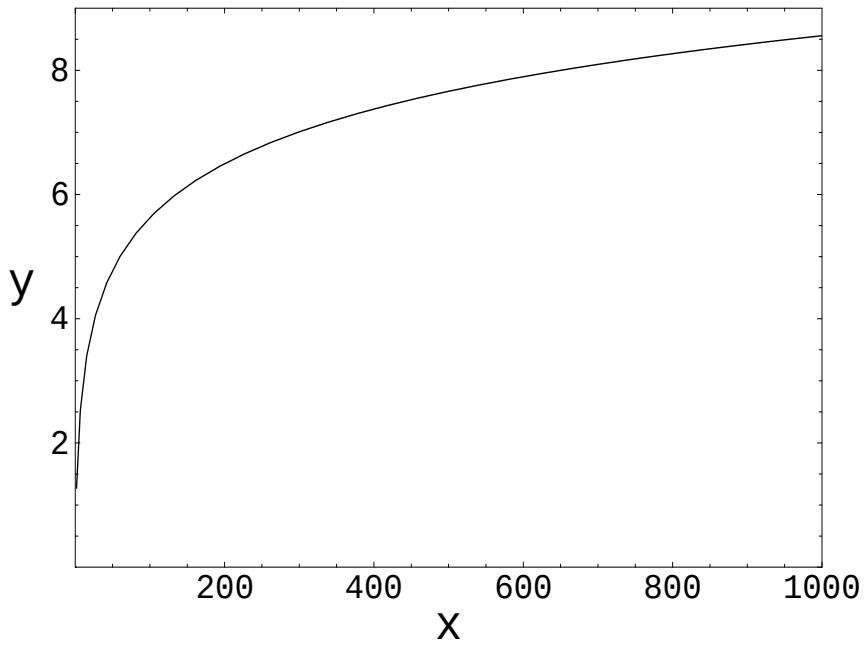

Figure 5.5: Plot of the implicit solution (5.180) for A given by (5.182), $C_1 = 1$ and $C_2 = 1$.

Both the solutions plotted in figures 5.4 and 5.5 represent a moving front of a thin film. The solution (5.181) implies that the profiles indicated in figures 5.4 and 5.5 either increase or decrease in height as the sign of B changes from positive to negative.

- **Case** $\beta = 2$

Eq. (5.155) reads

$$y'' = \frac{3}{y}y'^2 - \frac{1}{x}y' + \frac{3A}{y^2}, \tag{5.183}$$

$$X = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \tag{5.184}$$

We will look for invariant solutions under operator (5.184). $y = \phi(x)$ is an invariant solution of Eq. (5.183) under the infinitesimal operator (5.184) if and only if $y = \phi(x)$ satisfies

$$(i) \qquad \xi(x,\phi)\phi' = \eta(x,\phi), \tag{5.185}$$

$$(ii) \qquad \phi'' = \frac{3}{\phi}\phi'^2 - \frac{1}{x}\phi' + \frac{3A}{\phi^2}. \tag{5.186}$$

Solving (i) implies: $\phi(x) = C_3\ x^{2/3}$, where C_3 is a constant. Substituting ϕ into (ii) we obtain $C_3 = -(3/2)A^{1/3}$. Therefore

$$\phi(x) = -\frac{3}{2}A^{1/3}x^{2/3}. \tag{5.187}$$

In conclusion the exact solution is given by

$$h(x,t) = \frac{3A^{1/3}x^{2/3}}{2(3At - B)^{1/3}}. \tag{5.188}$$

In figure 5.6, we have plotted the graph of the solution (5.188) for $A = 1$ and $B = -1$. We notice that we cannot impose the initial profile. This solution represents the decreasing front height of the film. The case $\beta = 2$ corresponds to a Newtonian fluid. The solution recovered is the well-known moving front solution for a Newtonian fluid.

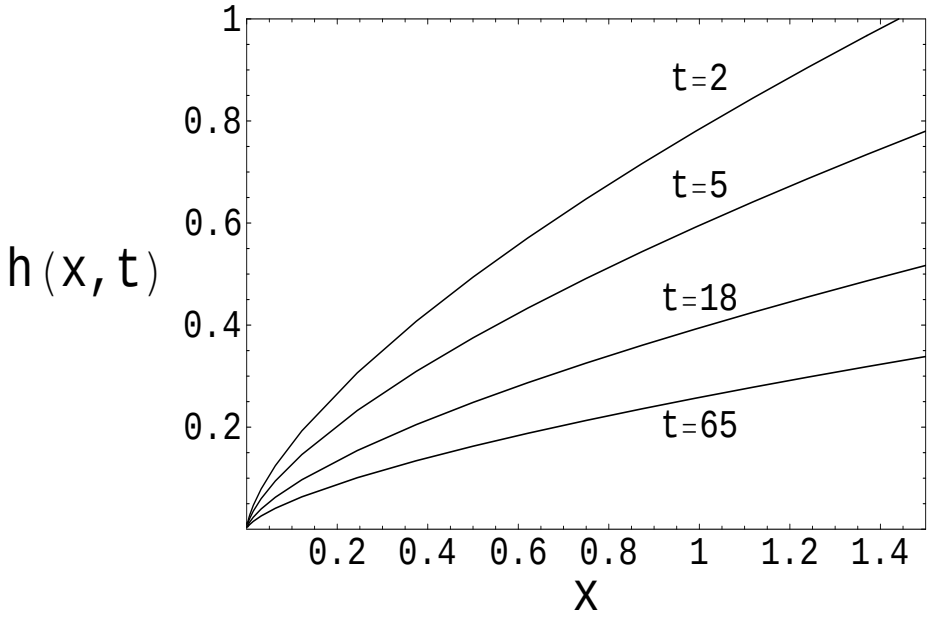

Figure 5.6: Plot of solution (5.188) for $A = 1$ and $B = -1$.

5.6.2 Conclusion

The separable solution (5.153) can be obtained from the conditional symmetry

$$X = [(1-\beta)At + B]\,\frac{\partial h}{\partial t} + Ah\frac{\partial}{\partial h}, \tag{5.189}$$

admitted by Eq. (5.145). We solved Eq. (5.145) modelling the axisymmetric spreading under gravity of a thin power-law fluid on a horizontal plane assuming that the solution is separable. We obtained Eq. (5.155) which is a highly nonlinear second-order ordinary differential equation. The new result obtained is the linearization of the case $\beta = 0$ using the theorem of Lie, (1883). A new moving front solution is obtained. For the case $\beta = 2$ we recovered the standard moving front solution.

Chapter 6

Thin Film Flow of a Generalized Second-Grade Fluid on a Moving Belt by Homotopy Analysis Method

6.1 Introduction

Most real world problems are modelled by nonlinear equations. The most difficult problem faced by mathematicians and engineers is the lack of a sound theory to determine the analytic solution of a given nonlinear equation. Therefore, we have to rely on numerical and semi-exact or approximate methods. Most scientists believe that the combination of numerical and semi-exact analytical methods can provide useful results. The homotopy analysis method (HAM) is one of the well-known methods to solve nonlinear equations. (HAM) has been successfully applied to various nonlinear problems. The readers are referred to the investigations [Liao, (1992, 1995, 1997, 1999, 2003, 2004, 2005), Hayat *et al.*, (2004), Sajid *et al.*, (2006)] and several studies therein.

In this chapter we apply the HAM to the partial differential equation resulting

from the flow of a thin generalized second-grade fluid on a vertically moving belt. The results obtained exhibit the effectiveness and reliability of the homotopy analysis method. This problem is also known as the Landau-Levich or dragout problem, which refers to the withdrawal of a plate or fiber from a liquid bath. The seminal paper on this problem was given by Landau and Levich, (1942). For details on this type of problem, the interested reader is referred to Afanasiev *et al.*, (2007) and the references therein.

In section 6.2, we provide the HAM analysis. This section is divided into two subsections. In the first subsection we solve the equation modelling the flow for positive values of the power-law parameter m and related convergence analysis. In the second subsection, we solve the equation modelling the flow for negative m and we study the convergence of the solution. Section 6.3 deals with discussion of the results and finally in section 6.4 the concluding remarks are included.

6.2 Solution by homotopy analysis method (HAM)

The governing equation modelling the flow of a generalized second-grade fluid on a vertically moving belt is derived in chapter 3 (Eqs. (3.161) and (3.162)). These equations are restated

$$(m+1)\left|\frac{dv}{dx}\right|^m \frac{d^2v}{dx^2} - k = 0,$$

$$\begin{aligned} v &= 1, \quad \text{at} \quad x = 0, \\ \frac{dv}{dx} &= 0, \quad \text{at} \quad x = 1. \end{aligned}$$

The above equation can be solved only by using HAM when m is positive integers. However, we have the advantage that rewriting the above equation, it can also be solved for m negative integers. For the latter case, we rewrite Eq. (3.161) as

$$\frac{d^2v}{dx^2} - \frac{k}{(m+1)}\left|\frac{dv}{dx}\right|^{-m} = 0, \tag{6.1}$$

with $m \neq -1$.

Eqs. (3.161) and (6.1) subject to Eq. (3.162) have the analytical solution

$$v(x) = 1 + \frac{k^{\frac{1}{m+1}}(m+1)}{m+2}\left[(x-1)^{\frac{m+2}{m+1}} - (-1)^{\frac{m+2}{m+1}}\right]. \tag{6.2}$$

Due to gravity the fluid film tends to drain down the belt. Therefore we only consider the case where

- $\left|\frac{dv}{dx}\right| = -\frac{dv}{dx}$.

Hence Eqs. (3.161) and (6.1) become respectively

$$(m+1)(-1)^m \left(\frac{dv}{dx}\right)^m \frac{d^2v}{dx^2} - k = 0, \tag{6.3}$$

and

$$\frac{d^2v}{dx^2} - (-1)^{-m}\frac{k}{(m+1)}\left(\frac{dv}{dx}\right)^{-m} = 0. \tag{6.4}$$

For HAM solution we select the initial guess u_0 and auxiliary linear operator L as follows

$$u_0(x) = \frac{k}{2}(x^2 - 2x) + 1, \tag{6.5}$$

$$L = \frac{d^2}{dx^2}, \tag{6.6}$$

with the following property

$$L[c_1 x + c_2] = 0, \tag{6.7}$$

where c_1 and c_2 are arbitrary constants.

We mention here that for $b = 0$ in Sajid *et al.*, (2007) and $m = 0$ in the present analysis, the results are identical.

6.2.1 Case 1: m positive integer

Consider Eq. (6.3) i.e. the case where m is a positive integer. For simplicity we consider the values $m = 1, 2$ and 3. For other values of m the same procedure can be followed using Molabahrami and Khani's Theorem (see Molabahrami and Khani, (2009)).

For Eq. (6.3), the k-th order deformation equation (2.46) becomes

$$L[u_k(\tau)-\chi_k u_{k-1}(\tau)] = \hbar \left[(m+1)(-1)^m \sum_{i=0}^{k-1} D_i \left[\left(\frac{\partial \phi}{\partial \tau} \right)^m \right] u''_{k-1-i}(\tau) - k^*(1-\chi_k) \right], \tag{6.8}$$

where the homotopy series ϕ is defined by Eq. (6.36) below. Moreover the homotopy derivative D_i is defined in Molabahrami and Khani's Theorem.

For $m = 1$, the k-th order deformation equation (6.8) becomes

$$L[u_k(\tau) - \chi_k u_{k-1}(\tau)] = \hbar \left[-2 \sum_{i=0}^{k-1} u'_i(\tau) u''_{k-i-1}(\tau) - k^*(1-\chi_k) \right]. \tag{6.9}$$

When $m = 2$, the k-th order deformation equation (6.8) is of the form

$$L[u_k(\tau)-\chi_k u_{k-1}(\tau)] = \hbar \left[3 \sum_{i=0}^{k-1} u''_{k-i-1}(\tau) \left(\sum_{j=0}^{i} u'_{i-j}(\tau) u'_j(\tau) \right) - k^*(1-\chi_k) \right], \tag{6.10}$$

and similarly for $m = 3$, the k-th order deformation equation (6.8) is

$$\begin{aligned} &L[u_k(\tau) - \chi_k u_{k-1}(\tau)] = \\ &\hbar \left[-4 \sum_{i=0}^{k-1} u''_{k-i-1}(\tau) \left(\sum_{j=0}^{i} u'_{i-j}(\tau) \sum_{l=0}^{j} u'_{j-l}(\tau) u'_l(\tau) \right) - k^*(1-\chi_k) \right], \end{aligned} \tag{6.11}$$

with

$$u_k(0) \;=\; u'_k(1) \;=\; 0. \tag{6.12}$$

The above equations are subjected with the boundary conditions given in Eq. (6.12). Note that in the above equations, we have replaced the material parameter k by k^* to avoid confusion with the subscript.

Using Mathematica® one can solve the linear differential equations given by Eqs. (6.9)–(6.11) subject to conditions (6.12), up to a few orders of approximations. It is found that $u_k(x)$ is

$$u_k(x) = \sum_{n=0}^{mk+2} a_{k,n} x^n, \quad k \geq 0. \tag{6.13}$$

Substituting Eq. (6.13) into Eqs. (6.9)–(6.11), we obtain the following recurrence formulae for the coefficients $a_{k,n}$ of $u_k(x)$ for $k \geq 1, \;\; 0 \leq n \leq mk+2$:

- For $m = 1$

$$a_{k,1} = \chi_k \chi_{k+2} a_{k-1,1} + \hbar k^* (1-\chi_k) - \sum_{s=0}^{k} \frac{\Delta_{k,s}}{s+1}, \tag{6.14}$$

$$a_{k,2} = \chi_k \chi_{k+1} a_{k-1,2} - \hbar \frac{k^*}{2}(1-\chi_k) + \frac{\Delta_{k,0}}{2}, \tag{6.15}$$

$$a_{k,s} = \chi_k \chi_{k-s+3} a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq k+2, \tag{6.16}$$

$$\Delta_{k,s} = -2\hbar \delta_{k,s}, \tag{6.17}$$

$$\delta_{k,s} = \sum_{i=0}^{k-1} \sum_{q=max\{0,s-i-1\}}^{min\{s,k-i-1\}} b_{i,s-q} c_{k-1-i,q}. \tag{6.18}$$

- For $m = 2$

$$a_{k,1} = \chi_k \chi_{2k+1} a_{k-1,1} + \hbar k^* (1-\chi_k) - \sum_{s=0}^{2k} \frac{\Delta_{k,s}}{s+1}, \tag{6.19}$$

$$a_{k,2} = \chi_k \chi_{2k} a_{k-1,2} - \hbar \frac{k^*}{2}(1-\chi_k) + \frac{\Delta_{k,0}}{2}, \tag{6.20}$$

$$a_{k,s} = \chi_k \chi_{2k-s+2} a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq 2k+2, \tag{6.21}$$

$$\Delta_{k,s} = 3\hbar \delta_{k,s}, \tag{6.22}$$

$$\delta_{k,s} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \sum_{p=max\{0,s-2k+2i+2\}}^{min\{s,2i+2\}} \sum_{q=max\{0,p-2i+2j-1\}}^{min\{p,2j+1\}} b_{j,q} b_{i-j,p-q} c_{k-1-i,s-p}. \tag{6.23}$$

- For $m = 3$

$$a_{k,1} = \chi_k \chi_{3k} a_{k-1,1} + \hbar k^* (1 - \chi_k) - \sum_{s=0}^{3k} \frac{\Delta_{k,s}}{s+1}, \tag{6.24}$$

$$a_{k,2} = \chi_k \chi_{3k-1} a_{k-1,2} - \hbar \frac{k^*}{2} (1 - \chi_k) + \frac{\Delta_{k,0}}{2}, \tag{6.25}$$

$$a_{k,s} = \chi_k \chi_{3k-s+1} a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq 3k+2, \tag{6.26}$$

$$\Delta_{k,s} = -4\hbar \delta_{k,s}, \tag{6.27}$$

$$\delta_{k,s} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{r=max\{0,s-3k+3i+3\}}^{min\{s,3i+3\}} \sum_{p=max\{0,r-3i+3j-1\}}^{min\{r,3j+2\}} \times \sum_{q=max\{0,p-3j+3l-1\}}^{min\{p,3l+1\}} b_{l,q} b_{j-l,p-q} b_{i-j,r-p} c_{k-i-1,s-r}. \tag{6.28}$$

The coefficients $b_{k,s}$ and $c_{k,s}$ are

$$b_{k,s} = (s+1) a_{k,s+1}, \tag{6.29}$$

$$c_{k,s} = (s+1)(s+2) a_{k,s+2}. \tag{6.30}$$

We can generalize the above recurrence formulae for a given value of the power-law parameter m as follows

$$a_{k,1} = \chi_k \chi_{(-m)k+m+3} a_{k-1,1} + \hbar k^* (1 - \chi_k) - \sum_{s=0}^{mk} \frac{\Delta_{k,s}}{s+1}, \tag{6.31}$$

$$a_{k,2} = \chi_k \chi_{mk-m+2} a_{k-1,2} - \hbar \frac{k^*}{2} (1 - \chi_k) + \frac{\Delta_{k,0}}{2}, \tag{6.32}$$

$$a_{k,s} = \chi_k \chi_{mk-m-s+4} a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq mk+2, \tag{6.33}$$

in which

$$\Delta_{k,s} = (-1)^m (m+1) \hbar \delta_{k,s}, \tag{6.34}$$

and $\delta_{k,s}$ are the coefficients of the expansion of

$$\tilde{N}_{k-1}[x] := D_{k-1} \left[\left(\frac{d\phi}{dx} \right)^m \frac{d^2\phi}{dx^2} \right] = \sum_{i=0}^{k-1} D_i (\phi'^{\,m}) u''_{k-1-i}(x), \tag{6.35}$$

as a polynomial in x, where

$$\phi(x,p) = \sum_{k=0}^{+\infty} u_k(x) p^k, \tag{6.36}$$

is the homotopy series,

$$\phi' = \frac{d\phi}{dx}, \tag{6.37}$$

and $u_k(x)$ is defined by Eq. (6.13). The homotopy derivative $D_i(\phi'^{\,m})$ in Eq. (6.35) is calculated through the use of Molabahrami and Khani's Theorem

Theorem 2 *For homotopy series*
$\phi = \sum_{i=0}^{+\infty} u_i q^i,$
it holds
$$D_n(\phi^k) = \sum_{r_1=0}^{n} u_{n-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}},$$
where $n \geq 0$ *and* $k \geq 1$ *are positive integers and* $D_n(\phi) := \left.\frac{1}{n!}\frac{\partial^n \phi(p)}{\partial p^n}\right|_{p=0}$ *is the nth-order homotopy derivative of* ϕ.

For more details about the properties of the homotopy derivatives, the interested reader is referred to Liao, (2008).

The power of the above relations is that the recurrence formulae can be used to calculate all the coefficients $a_{k,s}$ of Eq. (6.13) by using

$$a_{0,0} = 1, \quad a_{0,1} = -k^*, \quad a_{0,2} = \frac{k^*}{2}, \tag{6.38}$$

which are given by the initial guess defined in Eq. (6.5). We can therefore calculate $u_k(x)$ successively for $k = 1, 2, 3, ...$ The M th order of approximation of Eqs. (6.3) and (3.162) is given by

$$u(x) \approx \sum_{k=0}^{M} \left(\sum_{n=0}^{mk+2} a_{k,n} x^n \right). \tag{6.39}$$

Since the coefficients of Eq. (6.39) depend upon the auxiliary parameter $\hbar$, we assume that $\hbar$ is properly chosen so that the series in Eq. (6.39) converges. Therefore we have

the explicit analytical solution

$$u(x) = \lim_{M \to +\infty} \sum_{k=0}^{M} \left(\sum_{n=0}^{mk+2} a_{k,n} x^n \right). \tag{6.40}$$

Analysis of the convergence

One of the advantages of the homotopy analysis method over other methods, such as the perturbation and homotopy perturbation methods, is the great freedom that we have to choose the auxiliary parameter $\hbar$ contained in Eq. (6.40) (see, Liao, 2003a). This is achieved by plotting $\hbar$-curves of the M th order approximate solutions which in this case are the curves of $u''(0)$ as functions of $\hbar$. The range of admissible values of $\hbar$ is the set of values for which the $\hbar$-curves are horizontal. The $\hbar$-curves for 10th- and 20th-order of approximations are displayed in Figure 6.1. From this figure and as we can check from the table given, it is clear that for the 10th-order approximations, the range of the admissible values for $\hbar$ is $-0.8 \leq \hbar \leq -0.2$ and this range is larger for the 20th-order approximation. $u''(0)$ converges to 0.5 for all the admissible values of $\hbar$.

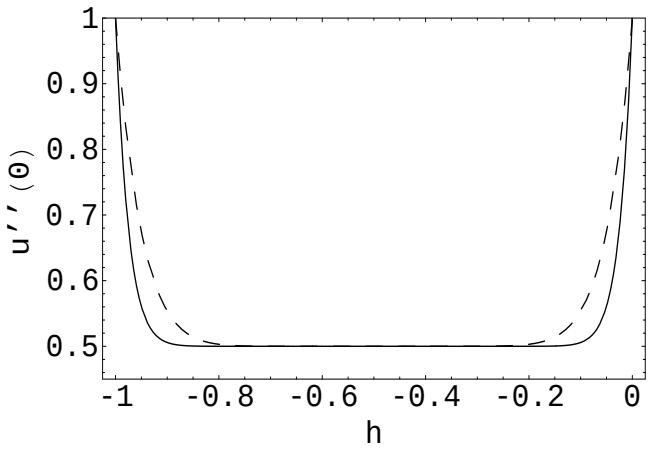

Figure 6.1: $\hbar$-curves for the 10th order of approximation (dash line) and for the 20th order approximation (solid line) when $m = 1$, $k^* = 1$.

Order of approximation	$\hbar = -0.8$	$\hbar = -0.7$	$\hbar = -0.6$	$\hbar = -0.5$	$\hbar = -0.25$
2	0.68	0.58	0.52	0.5	0.625
4	0.5648	0.5128	0.5008	0.5	0.53125
6	0.523328	0.502048	0.500032	0.5	0.507813
8	0.508398	0.500328	0.500001	0.5	0.501953
10	0.503023	0.500052	0.5	0.5	0.500488
12	0.501088	0.500008	0.5	0.5	0.500122
14	0.500392	0.500001	0.5	0.5	0.500031
16	0.500141	0.5	0.5	0.5	0.500008
18	0.500051	0.5	0.5	0.5	0.500002
20	0.500018	0.5	0.5	0.5	0.5

For the 10th-order approximation, we compute the residual and find the optimal value $\hbar = -0.67542$ for which the HAM converges in the whole region of x. We prove the following convergence theorem

Theorem 3 *If the series in Eq. (6.40) converges, its sum is an exact solution of Eq. (6.3) subject to the boundary conditions Eq. (3.162).*

We first state and prove the following Lemma that will be used to prove the above theorem.

Lemma 1 *For any positive integer m,*
if $u(\tau) = \sum_{k=0}^{+\infty} u_k(\tau)$ and $\phi(\tau, p) = \sum_{i=0}^{+\infty} u_i(\tau) p^i$, then
$$\sum_{i=0}^{+\infty} D_i \left[\left(\frac{\partial \phi}{\partial \tau} \right)^m \right] = [u'(\tau)]^m.$$

Proof 2 *We use induction to prove this Lemma and some properties of the homotopy derivative (see Liao, 2008).*

if $m = 1$,

$$\begin{aligned}
\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^m\right] &= \sum_{i=0}^{+\infty} D_i\left(\frac{\partial\phi}{\partial\tau}\right)\\
&= \sum_{i=0}^{+\infty}\frac{\partial}{\partial\tau}\left[D_i\left(\phi\right)\right]\\
&= \sum_{i=0}^{+\infty}\frac{\partial}{\partial\tau}\left[u_i\left(\tau\right)\right]\\
&= \sum_{i=0}^{+\infty} u_i'(\tau)\\
&= [u'(\tau)]^1.
\end{aligned}$$

For $m = 2$,

$$\begin{aligned}
\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2\right] &= \sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)\left(\frac{\partial\phi}{\partial\tau}\right)\right]\\
&= \sum_{i=0}^{+\infty}\left(\sum_{j=0}^{i} D_j(\frac{\partial\phi}{\partial\tau})D_{i-j}(\frac{\partial\phi}{\partial\tau})\right)\\
&= \sum_{i=0}^{+\infty}\left(\sum_{j=0}^{i}\frac{\partial D_j(\phi)}{\partial\tau}\frac{\partial D_{i-j}(\phi)}{\partial\tau}\right)\\
&= \sum_{i=0}^{+\infty}\sum_{j=0}^{i} u_j'(\tau)u_{i-j}'(\tau)\\
&= \sum_{j=0}^{+\infty}\sum_{i=j}^{+\infty} u_j'(\tau)u_{i-j}'(\tau)\\
&= \left(\sum_{j=0}^{+\infty} u_j'(\tau)\right)\left(\sum_{k=0}^{+\infty} u_k'(\tau)\right)\\
&= \left(\sum_{j=0}^{+\infty} u_j'(\tau)\right)^2\\
&= [u'(\tau)]^2.
\end{aligned}$$

Given an arbitrary positive integer n, *assume that* $\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^n\right] = [u'(\tau)]^n$ *and show that* $\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n+1}\right] = [u'(\tau)]^{n+1}$.

We have

$$\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n+1}\right] = \sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^n\frac{\partial\phi}{\partial\tau}\right].$$

By a property of the homotopy derivative to the right-hand side of the above expression we obtain

$$\begin{aligned}
\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n+1}\right] &= \sum_{i=0}^{+\infty}\left(\sum_{j=0}^{i} D_j\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n}\right] D_{i-j}\left(\frac{\partial\phi}{\partial\tau}\right)\right) \\
&= \sum_{j=0}^{+\infty}\sum_{i=j}^{+\infty} D_j\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n}\right] D_{i-j}\left(\frac{\partial\phi}{\partial\tau}\right) \\
&= \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} D_j\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n}\right] D_{k}\left(\frac{\partial\phi}{\partial\tau}\right) \\
&= \left(\sum_{j=0}^{+\infty} D_j\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n}\right]\right)\left(\sum_{k=0}^{+\infty} D_k\left(\frac{\partial\phi}{\partial\tau}\right)\right) \\
&= \left(\sum_{j=0}^{+\infty} D_j\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n}\right]\right)\left(\sum_{k=0}^{+\infty} D_k\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{1}\right]\right).
\end{aligned}$$

Finally using the induction hypothesis and the fact that the Lemma is true for $m = 1$, the above expression gives

$$\begin{aligned}
\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^{n+1}\right] &= [u'(\tau)]^n[u'(\tau)]^1 \\
&= [u'(\tau)]^{n+1},
\end{aligned}$$

and this ends the proof of the Lemma. □

Proof 3 *If the series in Eq. (6.40) converges, its sum $u(x)$ can be written as*

$$u(x) = \sum_{k=0}^{+\infty} u_k(x), \tag{6.41}$$

where $u_k(x)$ is defined by Eq. (6.13), and as a consequence of the convergence of the series we have

$$\lim_{k\to+\infty} u_k(x) = 0. \tag{6.42}$$

From the definition (2.48) of χ_k,

$$\sum_{k=1}^{n}[u_k(x) - \chi_k u_{k-1}(x)] = u_n(x), \tag{6.43}$$

and passing to the limit, we obtain according to Eq. (6.42)

$$\sum_{k=1}^{+\infty}[u_k(x)-\chi_k u_{k-1}(x)] = \lim_{n\to+\infty} u_n(x) = 0. \tag{6.44}$$

Applying the linear operator L defined by Eq. (6.6) to the above expression, we have

$$\sum_{k=1}^{+\infty} L[u_k(x)-\chi_k u_{k-1}(x)] = L\sum_{k=1}^{+\infty}[u_k(x)-\chi_k u_{k-1}(x)] = 0. \tag{6.45}$$

Using the above expression and the k-th order deformation equation (2.46), we obtain

$$\sum_{k=1}^{+\infty} L[u_k(x)-\chi_k u_{k-1}(x)] = \hbar H(x)\sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}) = 0. \tag{6.46}$$

Since $\hbar \neq 0$ and $H(x) = 1$, we obtain

$$\sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}) = 0. \tag{6.47}$$

From Eq. (6.8), we have

$$R_k(\vec{u}_{k-1}) = (m+1)(-1)^m \sum_{i=0}^{k-1} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^m\right] u''_{k-1-i}(\tau) - k^*(1-\chi_k), \tag{6.48}$$

and from the above expression

$$\begin{aligned}
\sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}) &= \sum_{k=1}^{+\infty}\left[(m+1)(-1)^m\sum_{i=0}^{k-1} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^m\right] u''_{k-1-i}(\tau) - k^*(1-\chi_k)\right] \\
&= (m+1)(-1)^m\sum_{k=1}^{+\infty}\sum_{i=0}^{k-1} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^m\right] u''_{k-1-i}(\tau) - k^* \\
&= (m+1)(-1)^m\sum_{i=0}^{+\infty}\sum_{k=i+1}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^m\right] u''_{k-1-i}(\tau) - k^* \\
&= (m+1)(-1)^m\sum_{i=0}^{+\infty} D_i\left[\left(\frac{\partial\phi}{\partial\tau}\right)^m\right]\sum_{j=0}^{+\infty} u''_j(\tau) - k^*.
\end{aligned}$$

Using the above Lemma and the properties of the derivatives of the sum of a convergent series, the above expression finally yields

$$\sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}) = (m+1)(-1)^m[u'(\tau)]^m u''(\tau) - k^*. \tag{6.49}$$

From Eqs. (6.47) and (6.49), we have

$$(m+1)(-1)^m[u'(\tau)]^m u''(\tau) - k^* = 0, \quad 0 < \tau < 1, \tag{6.50}$$

and through Eqs. (6.5) and (6.12), we obtain

$$u(0) = \sum_{k=0}^{+\infty} u_k(0) = u_0(0) + \sum_{k=1}^{+\infty} u_k(0) = 1, \tag{6.51}$$

$$u'(1) = \sum_{k=0}^{+\infty} u'_k(1) = u'_0(1) + \sum_{k=1}^{+\infty} u'_k(1) = 0. \tag{6.52}$$

From Eqs. (6.50)–(6.52), we conclude that $u(x)$ is an exact solution of Eqs. (6.3) and (3.162). This ends the proof of the theorem. □

6.2.2 Case 2: m negative integer

We shall consider for this study three values of the power-law parameter $m = -2, -3$ and -4.

When $m = -2$, Eq. (2.46) becomes

$$L[u_k(\tau) - \chi_k u_{k-1}(\tau)] = \hbar \left[u''_{k-1}(\tau) + k^* \sum_{i=0}^{k-1} u'_{k-i-1}(\tau) u'_i(\tau) \right]. \tag{6.53}$$

For $m = -3$, Eq. (2.46) is

$$L[u_k(\tau) - \chi_k u_{k-1}(\tau)] = \hbar \left[u''_{k-1}(\tau) - \frac{k^*}{2} \sum_{i=0}^{k-1} u'_{k-i-1}(\tau) \left(\sum_{j=0}^{i} u'_{i-j}(\tau) u'_j(\tau) \right) \right], \tag{6.54}$$

and $m = -4$, we have

$$L[u_k(\tau) - \chi_k u_{k-1}(\tau)] = \hbar \left[u''_{k-1}(\tau) + \frac{k^*}{3} \sum_{i=0}^{k-1} u'_{k-i-1}(\tau) \left(\sum_{j=0}^{i} u'_{i-j}(\tau) \sum_{l=0}^{j} u'_{j-l}(\tau) u'_l(\tau) \right) \right], \tag{6.55}$$

with

$$u_k(0) = u'_k(1) = 0. \tag{6.56}$$

As mentioned previously, we have replaced the material parameter k by k^* to avoid confusion with the subscript. Using Mathematica® $u_k(x)$ is expressed by

$$u_k(x) = \sum_{n=0}^{(-m)k+2} a_{k,n}x^n, \quad k \geq 0. \tag{6.57}$$

Substituting Eq. (6.57) into Eqs. (6.53)–(6.56), we obtain the following recurrence formulae for the coefficients $a_{k,n}$ of $u_k(x)$ for $k \geq 1, \quad 0 \leq n \leq (-m)k+2$:

- For $m = -2$

$$a_{k,1} = \chi_k\chi_{2k+1}a_{k-1,1} - \sum_{s=0}^{2k}\frac{\Delta_{k,s}}{s+1}, \tag{6.58}$$

$$a_{k,2} = \chi_k\chi_{2k}a_{k-1,2} + \frac{\Delta_{k,0}}{2}, \tag{6.59}$$

$$a_{k,s} = \chi_k\chi_{2k+2-s}a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq 2k+2, \tag{6.60}$$

$$\Delta_{k,s} = \hbar\left[\chi_{2k-s}c_{k-1,s} + k^*\delta_{k,s}\right], \tag{6.61}$$

$$\delta_{k,s} = \sum_{i=0}^{k-1}\sum_{q=max\{0,s-2i-1\}}^{min\{s,2k-2i-1\}} b_{i,s-q}b_{k-1-i,q}. \tag{6.62}$$

- For $m = -3$ we have

$$a_{k,1} = \chi_k\chi_{3k}a_{k-1,1} - \sum_{s=0}^{3k}\frac{\Delta_{k,s}}{s+1}, \tag{6.63}$$

$$a_{k,2} = \chi_k\chi_{3k-1}a_{k-1,2} + \frac{\Delta_{k,0}}{2}, \tag{6.64}$$

$$a_{k,s} = \chi_k\chi_{3k+1-s}a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq 3k+2, \tag{6.65}$$

$$\Delta_{k,s} = \hbar\left[\chi_{3k-1-s}c_{k-1,s} - \frac{k^*}{2}\delta_{k,s}\right], \tag{6.66}$$

$$\delta_{k,s} = \sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{p=max\{0,s-3k+3i+2\}}^{min\{s,3i+2\}}\sum_{q=max\{0,p-3i+3j-1\}}^{min\{p,3j+1\}} b_{j,q}b_{i-j,p-q}b_{k-1-i,s-p}. \tag{6.67}$$

- For $m = -4$ we obtain

$$a_{k,1} = \chi_k \chi_{4k-1} a_{k-1,1} - \sum_{s=0}^{4k} \frac{\Delta_{k,s}}{s+1}, \tag{6.68}$$

$$a_{k,2} = \chi_k \chi_{4k-2} a_{k-1,2} + \frac{\Delta_{k,0}}{2}, \tag{6.69}$$

$$a_{k,s} = \chi_k \chi_{4k-s} a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq 4k+2, \tag{6.70}$$

$$\Delta_{k,s} = \hbar \left[\chi_{4k-2-s} c_{k-1,s} + \frac{k^*}{3} \delta_{k,s} \right], \tag{6.71}$$

$$\delta_{k,s} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{r=max\{0,s-4k+4i+3\}}^{min\{s,4i+3\}} \sum_{p=max\{0,r-4i+4j-1\}}^{min\{r,4j+2\}} \times \sum_{q=max\{0,p-4j+4l-1\}}^{min\{p,4l+1\}} b_{l,q} b_{j-l,p-q} b_{i-j,r-p} b_{k-i-1,s-r}. \tag{6.72}$$

The coefficients $b_{k,s}$ and $c_{k,s}$ are

$$b_{k,s} = (s+1) a_{k,s+1}, \tag{6.73}$$

$$c_{k,s} = (s+1)(s+2) a_{k,s+2}. \tag{6.74}$$

We can generalize the above recurrence formulae for a given value of the power-law parameter m as follows

$$a_{k,1} = \chi_k \chi_{(-m)k+m+3} a_{k-1,1} - \sum_{s=0}^{(-m)k} \frac{\Delta_{k,s}}{s+1}, \tag{6.75}$$

$$a_{k,2} = \chi_k \chi_{(-m)k+m+2} a_{k-1,2} + \frac{\Delta_{k,0}}{2}, \tag{6.76}$$

$$a_{k,s} = \chi_k \chi_{(-m)k+m-s+4} a_{k-1,s} + \frac{\Delta_{k,s-2}}{(s-1)s}, \quad 3 \leq s \leq (-m)k+2, \tag{6.77}$$

where

$$\Delta_{k,s} = \hbar \left[\chi_{(-m)k-s+m+2} c_{k-1,s} + \frac{(-1)^{1-m} k^*}{(m+1)} \delta_{k,s} \right], \tag{6.78}$$

and $\delta_{k,s}$ in this case, and as in the previous section is calculated through $\left(\frac{d\phi}{dx} \right)^{-m}$ using the Molabahrami and Khani's Theorem.

The power of the above relations is that the recurrence formulae can be used to calculate all the coefficients $a_{k,s}$ of Eq. (6.57) by employing

$$a_{0,0} = 1, \quad a_{0,1} = -k^*, \quad a_{0,2} = \frac{k^*}{2}, \tag{6.79}$$

which are given by the initial guess defined in Eq. (6.5). We can therefore calculate $u_k(x)$ successively for $k = 1, 2, 3, ...$ The M th order of approximation of Eqs. (6.4) subject to Eq. (3.162) is

$$u(x) \approx \sum_{k=0}^{M} \left(\sum_{n=0}^{(-m)k+2} a_{k,n} x^n \right). \tag{6.80}$$

Since the coefficients of Eq. (6.80) depend on the auxiliary parameter $\hbar$, we assume that $\hbar$ is properly chosen so that the series in Eq. (6.80) converges. Therefore the explicit analytic solution is of the form

$$u(x) = \lim_{M \to +\infty} \sum_{k=0}^{M} \left(\sum_{n=0}^{(-m)k+2} a_{k,n} x^n \right). \tag{6.81}$$

Analysis of the convergence

The $\hbar$-curves for 20 th and 30 th order of approximations are displayed in Figure 6.2. This figure and table below show that for the 20 th order approximation, the range of the admissible values of $\hbar$ is $-0.9 \leq \hbar \leq -0.5$. This range is larger for the 30 th order approximation; $u''(0)$ converges to 0 for all the admissible values of $\hbar$.

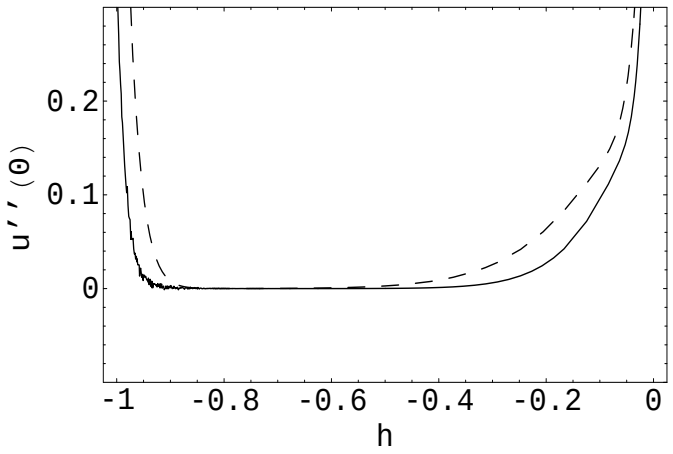

Figure 6.2: $\hbar$-curves for the 20 th order of approximation (dash line) and for the 30th order approximation (solid line) when $m = -2$, $k^* = 1$.

Order of approximation	$\hbar = -0.8$	$\hbar = -0.75$	$\hbar = -0.7$	$\hbar = -0.6$	$\hbar = -0.5$
5	-0.114584	-0.007511	0.046329	0.086727	0.110049
10	0.014869	0.009193	0.011164	0.020668	0.036179
15	0.000068	0.000848	0.001510	0.004193	0.010500
20	0.000055	0.000088	0.000188	0.000782	0.002832
23	$6.152\text{x}10^{-6}$	0.000021	0.000052	0.000278	0.001260
25	$1.193\text{x}10^{-6}$	$8.837\text{x}10^{-6}$	0.000022	0.000139	0.000729
27	$2.785\text{x}10^{-6}$	$6.932\text{x}10^{-6}$	$9.688\text{x}10^{-6}$	0.000069	0.000420
29	$2.265\text{x}10^{-7}$	$2.114\text{x}10^{-7}$	$-5.833\text{x}10^{-6}$	0.000034	0.000240
30	$-1.379\text{x}10^{-4}$	$2.530\text{x}10^{-5}$	$1.200\text{x}10^{-5}$	0.000026	0.000182

For the 20 th order approximation, we compute the residual and find the optimal value $\hbar = -0.88$ for which the HAM converges in the whole region of x.

As discussed previously, we have the following convergence theorem

Theorem 4 *If the series in Eq. (6.81) converges, its sum is an exact solution of Eq. (6.4) subject to the boundary conditions (3.162).*

6.3 Results and discussion

In this section, we study the effect of various fluid parameters on the velocity field.

Figure 6.3 shows the variation of the velocity for $m = 1, 2, 3$ with $k^* = 1$. We notice that the velocity decreases with increasing m. In Figure 6.4 we notice a different behaviour of the velocities for odd and even negative values of m. The velocity appears to decrease slower for odd negative values of m and to decrease more rapidly for even negative values of m. Figures 6.5–6.6 show the effect of k^* on the velocity profile for both positive and negative m. It is observed that the velocity decreases with increasing k^* for positive and negative m.

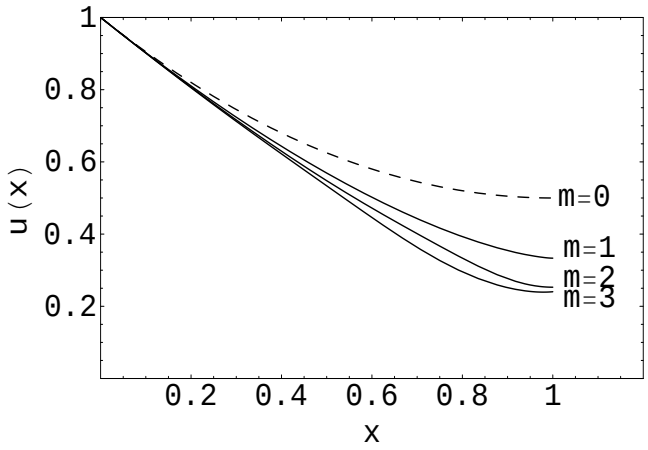

Figure 6.3: Variation of the velocity for positive values of m, for $k^* = 1$.

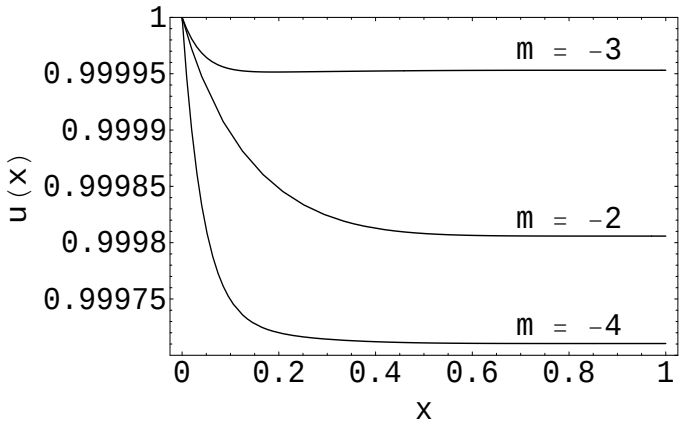

Figure 6.4: Variation of the velocity for negative values of m, for $k^* = 1$.

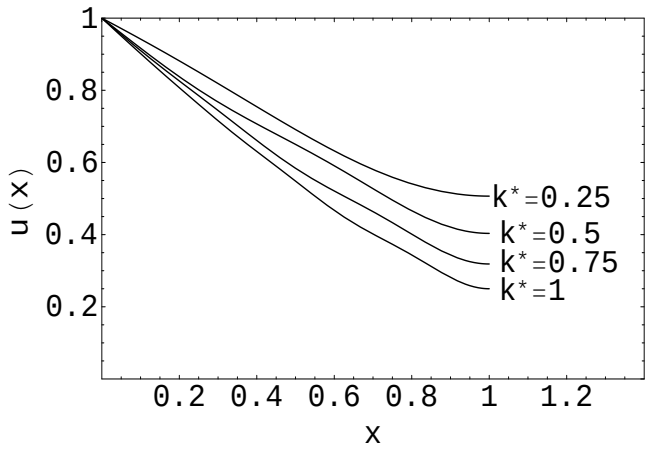

Figure 6.5: Variation of the velocity for $m = 2$ and varying k^*.

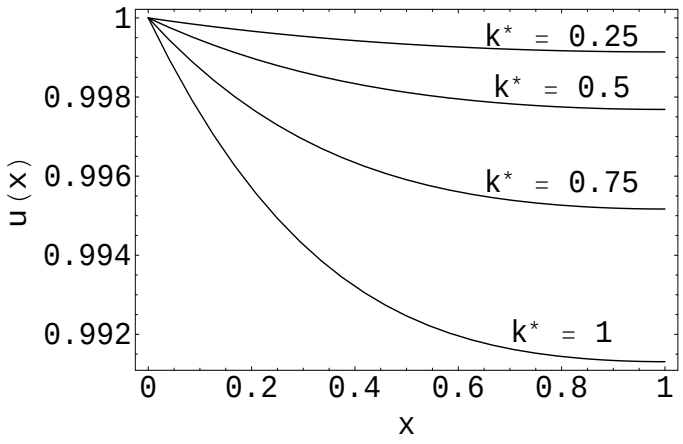

Figure 6.6: Variation of the velocity for $m = -2$, and varying k^*.

6.4 Concluding remarks

In the present work, we consider the flow of a generalized second-grade fluid on a vertically moving belt. We apply the homotopy analysis method to find the solution of an equation modelling the thin film flow of a generalized second-grade fluid on a moving belt. Series solutions are developed for different values of the material parameter m. Convergence in each case is explicitly shown. The influence of emerging flow parameters on the velocity is seen. These results demonstrate that the homotopy analysis method can be used with confidence to compute the solutions of a given nonlinear partial differential equation when we do not have the exact solution at our disposal.

Chapter 7

Conclusion and future work

Numerical analysis has evolved over time and new methods of determining approximate solutions have been developed, for example the Homotopy Analysis Method (HAM) and the Adomian Decomposition Method (ADM). Lie group analysis remains one of the most powerful analytical tools that provides a systematical way to attempt to find exact solutions for certain, if not all, classes of differential equations.

This thesis has illustrated how the combination of Lie group analysis and computational methods can constitute a powerful tool to study the flow of a thin non-Newtonian fluid. To be more precise, when Lie group analysis cannot provide exact solutions, and when semi-analytical solutions cannot be found or do not constitute realistic/physical solutions, the use of numerical schemes (for example the central differences/upwind and Newton methods) can be applied to study the dynamics of the spreading of a thin non-Newtonian fluid. This final chapter serves as a summary of the work presented and suggests possible future avenues of study.

We began this thesis by giving a description and importance of the study of non-Newtonian fluids. We gave a motivation of the use of power-law fluids by engineers and mathematicians compared to other types of non-Newtonian fluids. Chapter 1 gave the outlines and main objectives of this thesis.

In Chapter 2, we presented the mathematical preliminaries, gave a mathematical description of the various constitutive equations of the non-Newtonian fluids used and we derived the evolution equations of the problems tackled in the subsequent chapters. In the mathematical preliminaries, we recalled some important features of the Lie group analysis such as the definition of a symmetry of a differential equation and its associated determining equation. We also introduced the concept of conservation laws and gave the theorem that describes the Lie linearization process of a second-order ordinary differential equation.

In order to describe the flow of fluid, we need to describe the conservation of some physical quantities which are expressed by the Navier-Stokes equations. For the case of non-Newtonian fluids, such equations are required for the particular fluid at hand. Chapter 3 is dedicated to the Navier-Stokes equations of non-Newtonian fluids. These equations with the associated boundary conditions were used in the subsequent chapters to derive the highly nonlinear fourth-order partial differential equation describing the spreading of a non-Newtonian fluid.

In Chapter 4, we studied the dynamics of a thin power-law fluid. The chapter was divided in two main parts. In the first part, we studied the numerical analysis of a thin power-law fluid down a vertical plane and in the second part, we studied the spreading of an insoluble surfactant over the surface of a power-law fluid due to its many industrial and biological applications.

In Chapter 5, we studied the axisymmetry spreading under gravity of a thin power-law liquid drop on a horizontal plane. In the first part of the chapter, we derived the governing equation of the flow and then proceeded to a detailed study of the symmetries admitted by the equation. We found symmetries that depend on the power-law parameter β. We realized that there is symmetry breaking for $\beta = 0$, which corres-ponds to four symmetries admitted by the governing equation. For the other values of β, the governing equation admits only three symmetries which represent a

generalization of the Newtonian case. These symmetries were then used to obtain new group invariant solutions generalizing the solutions obtained in the literature for the Newtonian case. In the second part of the chapter, we used the linearization criterion for second-order nonlinear ordinary differential equations to obtain a new solution of the governing equation.

In Chapter 6, we applied the Homotopy Analysis Method (HAM) to the flow of a thin generalized second-grade fluid over a vertically moving belt. Our analysis revealed that for the problem at hand, solutions were possible for both positive and negative values of the power-law parameter m. This is not always the case when applying the HAM. We studied the convergence and showed that the homotopy solutions converge for selected values of the auxiliary parameter $\hbar$. These results indicate once more that one can use the homotopy methods to compute reliable solutions of a nonlinear partial differential equation when it does not appear to possess exact solution(s).

Future work

We now suggest possible areas of further research.

The most immediate area of interest is the search for a reliable solution technique for solving the coupled system of nonlinear equations (4.13)–(4.14) where surface tension is taken into account. Another area of interest will be to develop a numerical technique that will allow us to determine a numerical solution of the moving boundary problem defined by Eq. (3.143), subject to Eqs. (5.82) and (5.83) and compare this numerical solution with the group-invariant solution given by Eq. (5.110).

List of publications based on this research work

From our research work, a number of papers have been submitted to International Journals for possible publication. The list of the research papers is:

1. Neossi Nguetchue, S. N., and Momoniat, E., Axisymmetric spreading of a thin power-law fluid under gravity on a horizontal plane, *Nonlinear Dyn* (2008) **52**: 361–366.

2. Neossi Nguetchue, S. N., Abelman, S., and Momoniat, E., Symmetries and similarity solutions for the axisymmetric spreading under gravity of a thin power-law liquid drop on a horizontal plane – to appear in *Appl. Math. Mod.*

3. Neossi Nguetchue, S. N., and Abelman, S., A computational algorithm for solving nearly penta-diagonal linear systems, *Appl. Math. Comput.* (2008) **203**: 629–634. (See, Appendix).

4. Neossi Nguetchue, S. N., and Abelman, S., Thin film flow of a generalized second-grade fluid on a moving belt by Homotopy Analysis Method (HAM)– in preparation.

Appendix A

Algorithm NPENTA

A.1 Introduction

Penta-diagonal linear systems arise when solving third- or/and fourth-order partial differential equations numerically using finite differences. When the equations are linear, the associated matrices have constant coefficients and variable coefficients otherwise. In the latter case, the original equations have to be linearized in order to obtain linear systems. In some cases when solving partial differential equations, periodic boundary conditions are imposed depending on the nature or geometry of the problem. The latter situation leads to penta-diagonal linear systems with extra terms at the top right-hand and the bottom left-hand corners of the defining coefficient matrix which give the associated matrices a nearly penta-diagonal structure. Many problems in the field of thin films are solved by imposing periodic boundary conditions. For some examples, we refer the interested reader to Miladinova *et al.*, (2004) and to the references therein.

In this appendix, our main objective is to develop an efficient algorithm for solving general nearly penta-diagonal linear systems of the form

$$Ax = y, \tag{A.1}$$

where

$$A = \begin{pmatrix} d_1 & c_1 & f_1 & 0 & \cdots & 0 & 0 & p_1 & q_1 \\ a_2 & d_2 & c_2 & f_2 & 0 & \cdots & 0 & 0 & r_2 \\ e_3 & a_3 & d_3 & c_3 & f_3 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & e_{n-2} & a_{n-2} & d_{n-2} & c_{n-2} & f_{n-2} \\ \alpha_{n-1} & 0 & \cdots & \cdots & 0 & e_{n-1} & a_{n-1} & d_{n-1} & c_{n-1} \\ \beta_n & \gamma_n & 0 & \cdots & \cdots & 0 & e_n & a_n & d_n \end{pmatrix} \tag{A.2}$$

$x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$ and $n \geq 6$.

From the form of matrix A, each row contains exactly five nonzero terms; therefore this matrix can be stored in exactly $5n$ memory locations by using five vectors $\mathbf{f} = (f_1, \ldots, f_{n-2}, p_1, q_1)^T$, $\mathbf{c} = (c_1, \ldots, c_{n-1}, r_2)^T$, $\mathbf{d} = (d_1, \ldots, d_n)^T$, $\mathbf{a} = (\alpha_{n-1}, a_2, \ldots, a_n)^T$, and $\mathbf{e} = (\beta_n, \gamma_n, e_3, \ldots, e_n)^T$. Here $a_1 = \alpha_{n-1}$, $c_n = r_2$, $e_1 = \beta_n$, $e_2 = \gamma_n$, $f_{n-1} = p_1$ and $f_n = q_1$.

The rest of this appendix is organised as follows. In section A.2, we describe the algorithm for solving a general nearly penta-diagonal system where the associated matrix is given by (A.2). In section A.3, we present an illustrative example arising from the spreading of a non-Newtonian power-law fluid, subject to periodic boundary conditions, down an inclined plane . Finally in section A.4 we present concluding remarks.

A.2 Computational algorithm

In this section, we formulate a computational algorithm for solving the nearly penta-diagonal system (A.1) with the coefficient matrix A given by (A.2). In order to obtain

an efficient LU decomposition of A, we introduce seven additional vectors $\mathbf{b}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{k}$, $\mathbf{l}$, $\mathbf{v}$, $\mathbf{w}$ associated with $\mathbf{a}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$. After decomposing $A = LU$, we find that

$$\begin{pmatrix}
d_1 & c_1 & f_1 & 0 & \cdots & 0 & 0 & p_1 & q_1 \\
a_2 & d_2 & c_2 & f_2 & 0 & \cdots & 0 & 0 & r_2 \\
e_3 & a_3 & d_3 & c_3 & f_3 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & e_{n-2} & a_{n-2} & d_{n-2} & c_{n-2} & f_{n-2} \\
\alpha_{n-1} & 0 & \cdots & \cdots & 0 & e_{n-1} & a_{n-1} & d_{n-1} & c_{n-1} \\
\beta_n & \gamma_n & 0 & \cdots & \cdots & 0 & e_n & a_n & d_n
\end{pmatrix} =$$

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
b_2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
e_3/g_1 & b_3 & 1 & \ddots & \ddots & & & & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & & 0 & e_{n-2}/g_{n-4} & b_{n-2} & 1 & 0 & 0 \\
k_1 & k_2 & \cdots & \cdots & k_{n-4} & k_{n-3} & k_{n-2} & 1 & 0 \\
l_1 & l_2 & \cdots & \cdots & l_{n-4} & l_{n-3} & l_{n-2} & l_{n-1} & 1
\end{pmatrix} \times$$

$$\begin{pmatrix}
g_1 & h_1 & f_1 & 0 & \cdots & \cdots & 0 & v_1 & w_1 \\
0 & g_2 & h_2 & f_2 & 0 & \cdots & 0 & v_2 & w_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & \ddots & g_{n-5} & h_{n-5} & f_{n-5} & 0 & v_{n-5} & w_{n-5} \\
0 & \cdots & \cdots & 0 & g_{n-4} & h_{n-4} & f_{n-4} & v_{n-4} & w_{n-4} \\
0 & \cdots & \cdots & 0 & 0 & g_{n-3} & h_{n-3} & v_{n-3} & w_{n-3} \\
0 & \cdots & \cdots & 0 & 0 & 0 & g_{n-2} & v_{n-2} & w_{n-2} \\
0 & \cdots & \cdots & 0 & 0 & 0 & 0 & g_{n-1} & w_{n-1} \\
0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & g_n
\end{pmatrix} \tag{A.3}$$

This form differs from the results obtained by Karawia (2006) in that we have three top right and bottom left corner elements in A instead of one. The immediate consequence is that there is one extra row vector namely $\mathbf{k}$ in the lower triangular matrix L and another extra column vector namely $\mathbf{v}$ in the upper triangular matrix U of the LU decomposition of A. Note that for this LU decomposition to exist, the following condition must be satisfied

$$g_i \neq 0, \quad 1 \leq i \leq n-1. \tag{A.4}$$

In the construction of both the lower triangular matrix L and the upper triangular matrix U, the $g_i,\ i = 1, 2, \ldots, n-1$, are divisors and thus cannot equal zero.

Equating coefficients in $A = L\,U$, we obtain

$$g_i = \begin{cases}
d_1 & \text{if } i = 1 \\
d_2 - b_2 h_1 & \text{if } i = 2 \\
d_i - (e_i/g_{i-2}) f_{i-2} - b_i h_{i-1} & \text{if } 3 \leq i \leq n-2,
\end{cases} \tag{A.5}$$

$$k_i = \begin{cases} \alpha_{n-1}/g_1 & \text{if } i = 1 \\ -k_1 h_1/g_2 & \text{if } i = 2 \\ -(k_{i-2} f_{i-2} + k_{i-1} h_{i-1})/g_i & \text{if } 3 \le i \le n-4, \\ (e_{n-1} - k_{n-5} f_{n-5} - k_{n-4} h_{n-4})/g_{n-3} & \text{if } i = n-3, \\ (a_{n-1} - k_{n-4} f_{n-4} - k_{n-3} h_{n-3})/g_{n-2} & \text{if } i = n-2, \end{cases} \tag{A.6}$$

$$w_i = \begin{cases} q_1 & \text{if } i = 1 \\ r_2 - b_2 w_1 & \text{if } i = 2 \\ -(e_i/g_{i-2}) w_{i-2} - b_i w_{i-1} & \text{if } 3 \le i \le n-3, \\ f_{n-2} - (e_{n-2}/g_{n-4}) w_{n-4} - b_{n-2} w_{n-3} & \text{if } i = n-2, \\ c_{n-1} - \sum_{j=1}^{n-2} k_j \; w_j & \text{if } i = n-1, \end{cases} \tag{A.7}$$

$$v_i = \begin{cases} p_1 & \text{if } i = 1 \\ -b_2 v_1 & \text{if } i = 2 \\ -(e_i/g_{i-2}) v_{i-2} - b_i v_{i-1} & \text{if } 3 \le i \le n-4, \\ f_{n-3} - (e_{n-3}/g_{n-5}) v_{n-5} - b_{n-3} v_{n-4} & \text{if } i = n-3, \\ c_{n-2} - (e_{n-2}/g_{n-4}) v_{n-4} - b_{n-2} v_{n-3} & \text{if } i = n-2, \end{cases} \tag{A.8}$$

$$l_i = \begin{cases} \beta_n/g_1 & \text{if } i = 1 \\ (\gamma_n - l_1 h_1)/g_2 & \text{if } i = 2 \\ -(l_{i-2} f_{i-2} + l_{i-1} h_{i-1})/g_i & \text{if } 3 \le i \le n-3, \\ (e_n - l_{n-4} f_{n-4} - l_{n-3} h_{n-3})/g_{n-2} & \text{if } i = n-2, \\ \big(a_n - \sum_{j=1}^{n-2} l_j \; v_j\big)/g_{n-1} & \text{if } i = n-1, \end{cases} \tag{A.9}$$

and

$$\begin{aligned} g_{n-1} &= d_{n-1} - \sum_{j=1}^{n-2} k_j v_j \\ g_n &= d_n - \sum_{j=1}^{n-1} l_j w_j. \end{aligned} \tag{A.10}$$

The components of the vectors **b** and **h** are defined in the steps of the proposed algorithm. Before formulating the main algorithm, we remark that, for the linear system (A.1) to possess a unique solution, $g_i \neq 0$, for $1 \leq i \leq n$. This implies that

$$\det(A) = \prod_{i=1}^{n} g_i \neq 0. \tag{A.11}$$

Thus the vector **g** plays a very significant role.

• **Algorithm** In order to solve the linear system (A.1) with coefficient matrix given by (A.2), we proceed as follows

step 1: Set $g_1 = d_1,\ b_2 = a_2/g_1,\ h_1 = c_1,\ k_1 = \alpha_{n-1}/g_1,\ w_1 = q_1,\ v_1 = p_1,$
$l_1 = \beta_n/g_1,\ g_2 = d_2 - b_2 h_1,\ k_2 = -k_1 h_1/g_2,\ w_2 = r_2 - b_2 w_1,\ v_2 = -b_2 v_1,$
$l_2 = (\gamma_n - l_1 h_1)/g_2,\ h_2 = c_2 - b_2 f_1.$

step 2: For $i = 3, 4, \ldots, n-3$, compute
$b_i = \big(a_i - (e_i/g_{i-2})h_{i-2}\big)/g_{i-1},$
$h_i = c_i - b_i f_{i-1},$
$g_i = d_i - (e_i/g_{i-2})f_{i-2} - b_i h_{i-1}.$

step 3: Set $b_{n-2} = \big(a_{n-2} - (e_{n-2}/g_{n-4})h_{n-4}\big)/g_{n-3},$
$g_{n-2} = d_{n-2} - (e_{n-2}/g_{n-4})f_{n-4} - b_{n-2}h_{n-3},$

step 4: For $i = 3, 4, \ldots, n-4$, compute

$$k_i = -(k_{i-2}f_{i-2} + k_{i-1}h_{i-1})/g_i,$$
$$v_i = -(e_i/g_{i-2})v_{i-2} - b_i v_{i-1}.$$

step 5: Set $k_{n-3} = (e_{n-1} - k_{n-5}f_{n-5} - k_{n-4}h_{n-4})/g_{n-3}$,

$$k_{n-2} = (a_{n-1} - k_{n-4}f_{n-4} - k_{n-3}h_{n-3})/g_{n-2},$$
$$v_{n-3} = f_{n-3} - (e_{n-3}/g_{n-5})v_{n-5} - b_{n-3}v_{n-4},$$
$$v_{n-2} = c_{n-2} - (e_{n-2}/g_{n-4})v_{n-4} - b_{n-2}v_{n-3},$$
$$g_{n-1} = d_{n-1} - \textstyle\sum_{i=1}^{n-2} k_i v_i.$$

step 6: For $i = 3, 4, \ldots, n-3$, compute

$$w_i = -(e_i/g_{i-2})w_{i-2} - b_i w_{i-1},$$
$$l_i = -(l_{i-2}f_{i-2} + l_{i-1}h_{i-1})/g_i.$$

step 7: Set $w_{n-2} = f_{n-2} - (e_{n-2}/g_{n-4})w_{n-4} - b_{n-2}w_{n-3}$,

$$w_{n-1} = c_{n-1} - \textstyle\sum_{j=1}^{n-2} k_j\ w_j,$$
$$l_{n-2} = (e_n - l_{n-4}f_{n-4} - l_{n-3}h_{n-3})/g_{n-2},$$
$$l_{n-1} = \big(a_n - \textstyle\sum_{j=1}^{n-2} l_j\ v_j\big)/g_{n-1},$$
$$g_n = d_n - \textstyle\sum_{j=1}^{n-1} l_j w_j.$$

step 8: Set $z_1 = y_1$, $z_2 = y_2 - b_2 z_1$.

step 9: For $i = 3, 4, \ldots, n-2$, compute

$$z_i = y_i - b_i z_{i-1} - (e_i/g_{i-2})z_{i-2}.$$

step 10: Compute $z_{n-1} = y_{n-1} - \sum_{j=1}^{n-2} k_j z_j$,

$$z_n = y_n - \textstyle\sum_{j=1}^{n-1} l_j z_j.$$

step 11: Compute the solution vector $\mathbf{x}$ using

$$x_n = z_n/g_n,\ x_{n-1} = (z_{n-1} - w_{n-1}x_n)/g_{n-1},$$
$$x_{n-2} = (z_{n-2} - v_{n-2}x_{n-1} - w_{n-2}x_n)/g_{n-2},$$
$$x_{n-3} = (z_{n-3} - h_{n-3}x_{n-2} - v_{n-3}x_{n-1} - w_{n-3}x_n)/g_{n-3},$$
$$\text{For } i = n-4, n-5, \ldots, 1, \text{ compute}$$
$$x_i = (z_i - h_i x_{i+1} - f_i x_{i+2} - v_i x_{n-1} - w_i x_n)/g_i.$$

This new algorithm will be referred to as the **NPENTA** algorithm. For the choice $p_1 = r_2 = \alpha_{n-1} = \gamma_n = 0$, we obtain the **KPENTA** algorithm (see Karawia, 2006). The **NPENTA** algorithm for solving the general nearly penta-diagonal system (A.1) is generally preferable because the conditions $g_i \neq 0$, for $1 \leq i \leq n$, are sufficient for its validity.

A.3 Illustrative examples

In order to illustrate the **NPENTA** algorithm, two examples are given. The first example is a general nearly penta-diagonal system of size ten and the second example arises from the numerical solution of the spreading of a thin power-law fluid down an inclined plane when imposing periodic boundary conditions.

Example A.3.1

We solve the following system

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 2 & 3 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\ -2 & 2 & 1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & -5 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 & 4 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 7 & -8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & -4 & 2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & -5 & 1 \\ -2 & -3 & 0 & 0 & 0 & 0 & 0 & 7 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 6 \\ 0 \\ 3 \\ 4 \\ 3 \\ 4 \\ 1 \\ 1 \end{pmatrix} \tag{A.12}$$

by using algorithm **NPENTA**.

Following the algorithm, we obtain using Mathematica®

- **Step 1**

$g_1 = 1;\ b_2 = 2;\ h_1 = 2;\ k_1 = -1;\ w_1 = 3;\ v_1 = 1;\ l_1 = -2;$

$g_2 = -1;\ k_2 = -2;\ w_2 = -4;\ v_2 = -2;\ l_2 = -1;\ h_2 = -2.$

- **Step 2**

$[b_3, b_4, b_5, b_6, b_7] = \left[-6, -\frac{11}{5}, \frac{28}{83}, \frac{266}{477}, \frac{246}{1559}\right],$

$[h_3, h_4, h_5, h_6, h_7] = \left[21, \frac{17}{5}, \frac{82}{83}, \frac{55}{477}, -\frac{12964}{1559}\right],$

$[g_3, g_4, g_5, g_6, g_7] = \left[-5, \frac{166}{5}, -\frac{477}{83}, \frac{1559}{477}, \frac{9257}{1559}\right].$

- **Step 3**

$b_8 = \frac{3063}{9257}, \quad g_8 = \frac{47577}{9257}.$

- **Step 4**

$[v_3, v_4, v_5, v_6] = \left[-10, -12, \frac{170}{83}, -\frac{200}{477}\right],$

$[k_3, k_4, k_5, k_6] = \left[\frac{1}{5}, \frac{9}{166}, \frac{97}{954}, -\frac{251}{3118}\right].$

- **Step 5**

$$[v_7, v_8] = \left[\tfrac{3329}{1559}, -\tfrac{42381}{9257}\right],$$
$$[k_7, k_8] = \left[\tfrac{13135}{18514}, \tfrac{74617}{47577}\right],$$
$$g_9 = \tfrac{2345}{31718}.$$

- **Step 6**

$$[w_3, w_4, w_5, w_6, w_7] = \left[-18, -\tfrac{98}{5}, \tfrac{250}{83}, -\tfrac{238}{477}, \tfrac{2574}{15579}\right],$$
$$[l_3, l_4, l_5, l_6, l_7] = \left[-\tfrac{4}{5}, \tfrac{99}{166}, \tfrac{71}{954}, -\tfrac{1777}{3118}, \tfrac{669}{18514}\right].$$

- **Step 7**

$$[w_8, w_9] = \left[\tfrac{14870}{9257}, -\tfrac{160546}{47577}\right],$$
$$[l_8, l_9] = \left[\tfrac{26044}{15859}, \tfrac{101707}{2345}\right],$$
$$g_{10} = \tfrac{1016128}{7035}.$$

- **Step 8**

$$[z_1, z_2] = [10, -6].$$

- **Step 9**

$$[z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}] = \left[-10, 8, -\tfrac{141}{83}, \tfrac{710}{159}, \tfrac{2196}{1559}, \tfrac{20066}{9257}\right].$$

- **Step 10**

$$[z_9, z_{10}] = \left[\tfrac{14870}{9257}, -\tfrac{160546}{47577}\right].$$

- **Step 11**

$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}] = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

The determinant of the matrix associated with this system of equations is $\det(A) = -1016128$.

Example A.3.2

In this example, we give the procedure to follow in order to obtain a nearly penta-diagonal matrix from an ordinary/partial differential equation subject to periodic boundary conditions.

We consider the following equation derived in Miladinova *et al.*, (2004) and written here in a simpler form as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(b_1(h)h\right) + \epsilon\frac{\partial}{\partial x}\left\{\frac{1}{n}b_2(h)h_x + \frac{1}{n}b_3(h)h_{xxx}\right\} + O(\epsilon^2) = 0, \tag{A.13}$$

where
$b_1(h) = h^m$, $b_2(h) = a_n h^{3m} + bh^{m+1}$, $b_3(h) = Sh^{m+1}$, $a_n = \frac{2Re}{3n+2}\left(\frac{2n+1}{n}\right)^{2-n}$, $b = -\cot\beta$ and $m = (n+1)/n$. (See section 3.5.2 for a detail discussion.)

For simplicity, we consider a steady-state solution ($\frac{\partial}{\partial t} = 0$), and we assume that $b_2(h) = 0$. Integrating Eq. (A.13) with respect to x and choosing the constant of integration to be unity, we obtain,

$$Ch^{m+1}h''' + h^{m+1} - 1 = 0 \quad \text{for} \quad x \in [-1, 1] \quad \text{(say)} \tag{A.14}$$

where $C = (\epsilon S)/n$, subject to periodic boundary conditions
$h^{(s)}(-1) = h^{(s)}(1)$, with $s = 0, 1, 2$.

Eq. (A.14) is solved by a centered finite difference method as follows. Given a positive integer N, we let $\Delta x = 2/N$, $x_i = -1 + (i-1)\Delta x$, $i = 1, \ldots, (N+1)$. For the vector $\mathbf{h} = (h_1, \ldots, h_{N+1})$, the given periodic boundary conditions imply $h_i = h_{N+i}$, $\forall\, i$. Thus Eq. (A.14) can now be written with $\mathbf{h} = (h_1, \ldots, h_N)$ as

$$\mathbf{F}(\mathbf{h}) = \mathbf{0}, \tag{A.15}$$

where

$$\mathbf{F}(\mathbf{h}) = (F_1(\mathbf{h}), \ldots, F_N(\mathbf{h})),$$

and

$$
\begin{aligned}
F_1(\mathbf{h}) &= Ch_1^{m+1}(h_3 - 2h_2 + 2h_N - h_{N-1}) + \delta(h_1^{m+1} - 1), \\
F_2(\mathbf{h}) &= Ch_2^{m+1}(h_4 - 2h_3 + 2h_1 - h_N) + \delta(h_2^{m+1} - 1), \\
&\dots \\
F_i(\mathbf{h}) &= Ch_i^{m+1}(h_{i+2} - 2h_{i+1} + 2h_{i-1} - h_{i-2}) + \delta(h_i^{m+1} - 1), \\
&\dots \\
F_{N-1}(\mathbf{h}) &= Ch_{N-1}^{m+1}(h_1 - 2h_N + 2h_{N-2} - h_{N-3}) + \delta(h_{N-1}^{m+1} - 1), \\
F_N(\mathbf{h}) &= Ch_N^{m+1}(h_2 - 2h_1 + 2h_{N-1} - h_{N-2}) + \delta(h_N^{m+1} - 1),
\end{aligned}
$$

with $\delta = 2(\Delta x)^3$. In order to solve the nonlinear system of equations defined by Eq. (A.15) using the Newton method, we need only solve the linear system

$$
\begin{aligned}
D\mathbf{F}(\mathbf{h}^{(k)})\ \mathbf{s} &= -\ \mathbf{F}(\mathbf{h}^{(k)}), \quad k = 0, 1, \dots \\
\mathbf{h}^{(k+1)} &= \mathbf{s}\ +\ \mathbf{h}^{(k)}
\end{aligned}
\tag{A.16}
$$

where $D\mathbf{F}$ is the Jacobian matrix of $\mathbf{F}$ and $\mathbf{h}^{(k)}$ is the iteration vector which should converge to $\mathbf{h}$ the solution of Eq. (A.15). Computing the components of the Jacobian matrix, it is easy to show that its form is similar to Eq. (A.2).

A.4 Concluding remarks

The method described is an extension of the one obtained by Karawia, (2006). It is a more general method that can be applied in many other fields where a matrix similar to one used here arises. This type of matrix arises naturally in the numerical solution of ordinary/partial differential equations of third- and/or fourth-order using finite differences when periodic boundary conditions are used/imposed. Our method is very efficient and competitive with other methods used to solve nearly penta-diagonal systems and it can easily be implemented with any computer algebra system. Mathematica® was used to implement the **NPENTA** algorithm.

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