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STRUCTURAL APPARATUS

By

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THE first question to ask is "what is structural apparatus?" I spoke to you previously about the body of concepts that formed the schemes of mathematics, and if you think of these relationships and inter-relationships and the concepts themselves as a structure, then structural apparatus is apparatus which is related to one or more of the ideas or relationships within the structure. I want to underline this, because there is a considerable difference in kind between structural apparatus and tactile material that you use for certain activities with children and which has in the past been used to help mathematics learning. It is also different from the sort of game material that is used in learning certain mathematical relationships. Structural apparatus is a cleverly designed, carefully set out, set of materials that must be related to the structure. If it is not related to the structure, then it is not structural material.

The people who design structural apparatus should have in their minds a clear view of the concepts which may be learned through the use of the structural material. Anybody who says that the whole of the structure of concepts and the whole of the set of relationships can be learned by any one set of structural material is obviously making a broad generalisation which would be very difficult to uphold.

The next question to ask is "Why use it?" The answers to this question are really fascinating. Answers that you often get are: "Because it was recommended to me," or "They are using it next door," or "I went to a lecture and it looked good." One should not use it for those reasons. I am going to suggest to you this morning that there is a stage in the education of children where, if you really want them to learn mathematical ideas, you have got to use structural apparatus. There is no question of choice. It just has to be used.

The reason for this bold, brash statement is based upon the established fact that at a certain stage of development, which, incidentally, is known as the stage of concrete operations in thinking, in order to learn ideas, you have to deal with reality. It is not possible to learn ideas without a real model which provides an opportunity to mime the ideas. This stage of concrete operations was defined by Piaget. Subsequent research, much of it in Europe, has established beyond doubt that this stage exists, and that children, as they grow up from five to twelve, pass through it. They pass through it at an individual rate, and all learning is individual because children are individual: you cannot point to a whole class of children and say, "These children are all at the same concrete operational stage." I would say that if you looked at most children between eight and nine, you could say that they were all very likely to be at the stage of concrete operations, but younger than that some of them may not have reached the stage, and as soon as they get towards the end of that year, some of them are ready, and in fact are already moving into the stage of formal operations, which is one in which hypotheses about experience can be set up in the mind without the need of any reality and can be operated upon. In other words, they are at the stage of adult thinking. This concrete operational stage ties up very nicely with the view that you must have reality if ideas are to be learned.

The second thing that I would like to say in answer to the question, "why use it?" is the fact that structural apparatus is mathematically "clean". By this I mean it is not like the environment which is a chaos. You can arrange certain environmental experience; you can structure the environment; you can draw attention to certain elements of environmental experience, so that you can come to see a mathematical idea at work. But unfortunately, it is all very complicated. If you take John Flavell's example of the birds sitting on the cantilever beam, and you watch the deflection of the cantilever beam, there is a bit more in this than meets the eye. It is mathematically complicated. I am not decrying this experience—it is a most excellent experience and it is a very important experience—but it is more complex than it seems. What is more, perhaps the colours of the birds or the squeaks that the birds make might divert attention from the mathematics that is involved. Structural materials are mathematically "clean"; they are almost all mathematics. Perhaps they are not as clean as we would wish. Perhaps colour has been introduced with no pattern to it, or colour such as is in the Cuisenaire Rods or the Colour Factor with some reason in it, or colour with no reason in it, and so on. Perhaps some other things are in it, but in general, these things are not central, they are on the periphery and purposely on the periphery. The attention is

diverted to the mathematical idea which you can see as a result of operations with these actual things.

The other reason why you must use structural materials is that as ideas are abstract, you need a reality in order to see them at work. But the materials are really a symbol system, and instead of squiggles and marks, you have got real things. Young children use real things and set up symbol systems naturally. Have you ever seen girls playing with dolls? These dolls are symbols for real people and they change. One day the doll is one person, one day the doll is another person; one day the doll is sick, one day the doll is fit. It is the same actual thing, the same reality, but all sorts of things are imagined with it. Children grow by using real symbol systems. If we want mathematical ideas to run through their minds, we had better provide some real materials so that they can act out, mime, the mathematical ideas. Structural materials should provide this.

We are trying to build up in the minds of children a set of abstract ideas called concepts. These concepts represent classes of operations. The concept of 7 represents the class of collections of real objects that are always 7 in number, always have 7 discrete, that means separate, elements in the set. It represents not one set, but the whole infinite class of sets that have this set of attributes. Therefore, in order to get a general concept, there are many people who say that the experience that you give children, from which they abstract the ideas leading to the concept, should vary. It should be perceptually varied so that the conceptual constancy is seen to be there. If you are learning an idea, which consists of a set of attributes, it is no good abstracting from one experience, because you will not develop a general notion, but a very strong association between the concept and the experience that led to the concept, and in mathematics, as in all other learning, we do not want associative learning at all. We need a general idea, not a severely limited one. Therefore, we must consider this question very carefully. If we are going to use structural materials, is it appropriate to put all our money on the one horse and have just the one kind? Are we not in some danger of building up an association rather than a general idea? I am not in any way decrying the work of Gattegno, Cuisenaire and Stephen Pollock with his Colour Factor material, but there is no doubt in my mind that if you use just one set of materials such as Cuisenaire or Colour Factor, this does in fact take place.

May I qualify this remark by saying that my present point of view is that perceptual variability can be overdone? You can sometimes vary the experience so much that children get very confused, and clear concepts do not develop. My present point of view is that, on the other hand, I would not have one set of materials, not just alpha one experience, but alpha two. This arises from the fact that development of the concept depends upon ignoring the irrelevent attributes of two different experiences and concentrating attention on the common attributes. If you do not have two different experiences, how can you ignore what is irrelevant and concentrate on what is common? Harold Fletcher would agree with me that if you introduce young children to rods, as he said, "the black seven"—it sounds like a character from *Ivanhoe*—the blackness is for some children over-ridingly powerful and the sevenness is just something that is said, just a mere rote repetition. Let me stress once again that this does not in any way invalidate this essentially and vitally important and first-rate material. There are some people who would argue with me and say, "Look, the ideas are abstract, aren't they? One model is good enough." When children are actually working with Cuisenaire Rods and they have never seen anything else, they in fact use the Cuisenaire Rods rather like a calculating machine. They imagine them. They build up visual percepts in their minds, and by means of visual imagery they often get solutions. We have yet to know, by the way, whether these children are any better at mathematics at a time when it counts. No children that I know anywhere have reached the age of 15 or 16 on a diet of Cuisenaire Rods or any of the other materials that I am talking about.

What kinds of materials would you consider to be structural? Let us start with materials for younger children. One of the most important materials, I think, is called Unifix. You will notice during this talk that most of the good materials are extraordinarily simple. Most of the good ideas are extraordinarily simple anyway and Unifix is an extraordinarily simple idea. Unifix consists of plastic cubes of three-quarter inch edge which interlock. The fact that they are discrete and can be made into a continuous mass suggests that this material is extraordinarily useful in dealing with the original and cardinal value or concept of natural number. Cardinal number describes the amount or the quantity of the set of separate objects. Unifix is the only material which you can push together quite simply and make a tower, for instance, and because these are unit cubes, by bringing these into juxtaposition, you can see that the number has another side to it—it has an ordinal side; you can seriate the numbers and put them in order. You cannot do that with Cuisenaire. A measure of any of the rods depends upon a far more complicated notion. Piaget has done some work on this. The idea of measuring

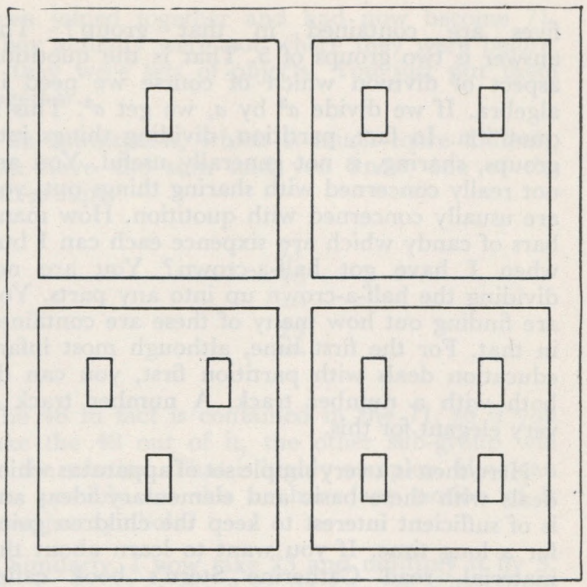
two things with a third independent unit is very difficult and complex. Measurement is quite a sophisticated notion. You may remember the tally sticks—direct correspondence between the notches, and you split the stick down the middle: "That's your bit, that's my bit." There was no independent unit. You actually needed the things to be together.

Unifix obviously is useful with very young children and it is quite a good exercise to ask yourself what it is you are supposed to be teaching —not in terms of techniques and processes, but in terms of basic concepts. The fact that you can make these into discrete sets or make them into continuous ones is a unique feature of this material.

One of the first books on structural arithmetic was written by Dr. Catherine Stern. She is an American who worked in a private school in New York, and she decided that the teaching of arithmetic there was just terrible. Without meaning to offend you, the syllabus that she had to work to looked very much like the syllabus I have seen here. She revolted against this and decided that the only thing she could do was to invent material which would help children really to learn some mathematics. So she invented what was called the Stern material. This is an example, by the way, of the independent school being progressive in the United States, rather than the State school. However, the work of Catherine Stern has largely been unrecognised in the United States, but is now widely recognised in Britain and also in Europe.

Catherine Stern again used the idea of a unit cube. When she first made them, the cubes had a 1″ edge, but in Britain they now have a $\frac{3}{4}$″ edge. She has a whole set of bars welded together up to 10, and she has cubes; she is introducing a new kind of structure here—instead of the structure just being in the cube, she is introducing things into which you put other things.

Catherine Stern also had to do some "pattern boards". You probably remember the patterns relating to numbers that were prevalent when you were very small. We had a frieze round the room, and the teacher pointed to them and we all counted. There was structure in this, and when Stern decided to make pattern boards, which were actually boards, she decided that she ought to arrange these in a rather special way, so she realised that the set of numbers could be roughly divided into odd and even numbers, and so she arranged them in sets like this:



She also had a counting board, which was a large board into which the bars would fit. There were numerals fitting at the top and she had a fascinating set of number trays. These trays are simply pieces of hardboard with a ridge round: if you took the six tray, then it would be six units in length and six units in breadth. From this tray, if you put a three bar and a three bar in, they would just fit in, and by means of this tray, the composition of six in terms of four plus two, three plus three, five plus one and so on, was demonstrated.

Stern also had a number track. This was a groove, into which bars representing numbers would fit and she marked a cross on it and put numerals on the edge. The number track is a fascinating piece of apparatus because it gives one an extraordinarily clear view of what is a very complex arithmetical operation, the operation of division. In division, you can phrase the same question by writing it differently. Instead of writing it like this—

$$\frac{10}{5} = 2$$

you could write it like this—

$$5\overline{)10}\ 2$$

It is obvious that these are completely different. The first one is a partition example. In this you have a group of 10; you divide it into five equal parts and there are 2 in each part. But the second one says that if you have a group of 10, how many

fives are contained in that group? The answer is two groups of 5. That is the quotition aspect of division which of course we need in algebra. If we divide a^3 by a, we get a^2. This is quotition. In fact, partition, dividing things into groups, sharing, is not generally useful. You are not really concerned with sharing things out, you are usually concerned with quotition. How many bars of candy which are sixpence each can I buy when I have got half-a-crown? You are not dividing the half-a-crown up into any parts. You are finding out how many of these are contained in that. For the first time, although most infant education deals with partition first, you can do both with a number track. A number track is very elegant for this.

Here then is a very simple set of apparatus which deals with these basic and elementary ideas and is of sufficient interest to keep the children going for a long time. If you want to learn about this material, read Catherine Stern's book called *Children Discover Arithmetic*, which is a fascinating book on structural materials and in fact was the first one to be written.

The Cuisenaire Rods consist of colour families which to me are not very relevant. It is useful to identify the rods, but no more. The reason why Cuisenaire Rods have fallen into disrepute is that people keep on applying numbers to them. The great beauty of this material is that it is algebraic and not arithmetical. The flexibility of this material is its great strength. It is rather nice to have colour, but I get very worried when I see children writing equations such as:

$$r + y = b$$

I accept this—

$$N(r) + N(y) = N(b)$$

which says that the number which we associate with r plus the number which we associate with y is the same number as the number we associate with black, but there are books in which you have this strange $(r + y = b)$. Here is a warning: if you are using structural material that is mathematically clean, do not use dirty hands yourself. You can muck it up completely and utterly. You can ruin the great value of the material by not understanding it, not knowing what it is supposed to be doing. Half the teachers in the world using Cuisenaire call these things by numbers far too quickly—this was never intended.

The great virtue of Cuisenaire is that it can show you with a great degree of elegance, for example, the commutative identity. Using the Madison Project notation—

$$\square \times \triangle = \triangle \times \square$$

If I substitute or put into this box $\square$ on the left-hand side and that box $\triangle$ on the left-hand side any of the infinite set of numbers, provided I put the same one in the right-hand $\square$ and in the right-hand $\triangle$ this is known as the commutative identity or commutative law for multiplication, which is usually written—

$$\forall\, a, b, — a \times b = b \times a$$

$a \times b$ is always the same number as $b \times a$.

This material is useful for fractions. You can get children doing the most difficult examples with understanding.

With materials of this kind, if children really understand what they are doing, they can get on fast and go far, using the material as a mime. You can do a lot of very important things with it, but unfortunately, addition and subtraction are a bit more complicated.

Cuisenaire and Gattegno realised that this material was rather limiting—you cannot go any further with calculation within a system of numeration which depends, as ours does, upon place value, upon base, with this kind of material, because, strange as it may seem, although these are all cuboids, the over-riding view that the child has of it is of its linearity. If I were using the Cuisenaire material seriously, I would have simple balances made, soda straw balances with little pans, so that you could see the relationship between the masses. I do not like just this linear idea; I am more for varying the percept.

When it comes to dealing with the four operations within a place value system, none of these materials that I have described to you is very good. Here are some materials that happen to be mine. These are centimetre units; they are made in plastic because children like plastic for some reason. These are cuboids, but the over-riding percept is that of superficiality or area.

You can, of course, carry out operations with these, but the most important kind of percept comes from material which is called the Dienes Multi-Base Arithmetical Blocks. Dienes and I developed this material in 1958 when we found that the idea of place value was not really growing in our children. This is not an original idea—nothing is original; this is similar to the Montessori material. We took a unit cube and constructed models of the first three powers. With the first power, the over-riding percept is one-dimensional; with the second power, the over-riding percept is two dimensional; with the third power, the over-riding percept is three-dimensional.

I am very disturbed by some of the falsification of structural materials that goes on to make the mathematics fit the materials. The materials have got to fit the mathematics.

The Dienes Multi-Base Arithmetical Blocks are used to help children with the operations of addition, subtraction, multiplication and division within a system of numeration based upon these notions.

Many people have said that I am opposed to calculation or to doing sums. I am opposed to doing sums for their own sake, but it is quite obvious that primary education *has* to help children have competence in calculation. It also has to give children opportunities to see that calculation is useful. But the priority seems to me to be wrong. If you write a very simple calculation on the board or in a text-book, the ideas of which this calculation is an expression are very complicated. In order to carry out the operation of addition, for example, it has to be realised that the law of closure for addition enables you to find a number which is the sum. Secondly, if you do not want to add 8 to 3, but 3 to 8, then you are invoking the commutative identity and the commutative nature of addition. These ideas are built up from experiences with Cuisenaire Rods.

It is most important that the algebra is learned before we apply the actual numbers, so you want algebraic material like Cuisenaire Rods and other things. Then you can start letting the numbers go to work in accordance with the rules of the game, that is all. You do not generalise the algebraic ideas from arithmetical examples, in my view. You go straight to the algebra and then let the algebra control the numbers. This is the axiomatic view and here is an axiomatic view, not with symbols, but with materials. Can you see the similarity between what I am doing with materials and what Bob Davis and others are doing with words and symbols? I think I shall eventually compromise and see that both symbols and materials have a place. But this is my present position—I see that for some children, symbols are better, for others, materials are better.

The actual operation itself of addition is highly complicated. For example, if we resolve this—

$$\begin{array}{r} 23 \\ +\ 48 \\ \hline 71 \\ \hline\hline \end{array}$$

and find that the sum is 71, very few children that I knew some ten years ago realised (and how could they?) that the 23 and the 48 had simply been joined together and had now become 71. They actually were not where they were before, if they were sets of objects. You just put them together.

In subtraction, which is much more difficult, you have the sum and you know one of the sub-groups:

$$\begin{array}{r} 71 \\ -\ 48 \\ \hline 23 \\ \hline\hline \end{array}$$

The 48 in fact is contained in the 71, so if you take the 48 out of it, the other sub-group will be the solution. That is highly complex. How can you possibly see this or have any notion of it from just symbols?

Similarly, I now take 23 and multiply it by 3. We know what the 3 in 23 was: it represented 3 units. But what is it now when used for multiplying? In the nature of things, you cannot just tell from that example.

Very few children understand that if you are dividing 14 into 7 equal parts, you have actually got 7 equal parts, but all that you are interested in is the size of one of them—that in fact the answer is really seven twos. But you are not interested because they are all equal, you are just concentrating on one particular quality, that is the size of each group.

So calculations and operations are very complicated and you just cannot get away with it with systems of symbols and squiggles on boards. Let us see if materials can help us to understand these four simple things.

One of the advantages of working in number scales other than 10 is that your number bonds are very simple. If you are using the decimal scale, you can have $9 + 8$. But if you are using the binary scale, the most you can have is $1 + 1$. Why should the fact that children stumble over number bonds hold up their opportunities to learn about the operations? Working in base 3, the most difficult combination you can have at any one moment of time is $2 + 2$. The sequence is very simple and this is the way that these structural materials are used. First of all, children play with the materials and see the relationships between them. I do mean play—random play—build, construct—play is a respectable word in the modern educational scene. The play is one of adjustment—mental and emotional adjustment to get the right set for what you have to do, but play is much more important than that. You cannot

discover the relationships unless you do play. Then you inject into the play situation some structured games which enable children to see the relationships between the various pieces. I am talking now about the M.A.B. blocks. Then you provide children with a series of programme cards or instructions which tell them what to do. But you give them the very minimum of guidance because you want them to feel involved and you deliberately say "do this", and leave them free to do it. The beauty of this material is that the framework is provided by the material itself. Usually we provided children with boards; we asked them what they were called and they said, "That's a long one", so we accepted that it was called a "long"; they said, "That's a flat one", so we accepted that it was called "a flat" and we asked them what they thought that was and they said it was "a block", so we called it a "block". Then we said, "Suppose you had three of these, these would make one of the next kind, what would you call that?" They replied, "It's a long block", and the next power is a "flat block" and the next power is a "block block". Names do not matter, so we accepted these and we used this nomenclature. In fact we should have base to the zeroth power, base to the first power, base to the second power, base to the third power, but we are a long way from understanding base. Concepts take a long time to grow. In my innocence and youth, I thought you could learn concepts very quickly, but you cannot; there is always another layer to put on top of a concept; in my view, a growth stage is followed by a sort of plateau which is operational to a certain extent, and then you suddenly find something else out and you go on. Then it has a wider sphere of operations and so on, and for the whole of our lives, our concepts grow more general. To take a very simple example, the concept of any of us older than thirty or forty of a car would have been very limited in the past. Look at what we now accept as belonging to the class of things which we symbolise by the term "car". It is much wider.

Children put these things out on the boards. They first of all work with the real situation and then record what they have done so that their "sums" are really records of what they have done. They do things with materials, they record faithfully what they have done. Two children work together.

The difficulty is that they think these symbols are associated with bits of wood. That is all right, but it is not good enough because you have to be much more flexible.

Let me tell you one or two things that we have discovered about this work. I did say at the beginning that we do not yet know what is the result of this experience. I have a subjective view based on a number of children who have now gone to high school, and generally speaking, because we made some gross errors in the early stages, I cannot honestly say that they are very much better mathematically. At that time we were bumbling along and making the most hideous mistakes, but now we are getting rather better. The only way in which you can really find out what is happening to children is to sit with them individually. The application of group tests is highly suspect.

Finally, I would like to quote from a research project of a friend of mine who enquired into the development of number concepts of children between seven and ten years of age. She said: "Despite the development of new mathematical materials based on sound mathematical and psychological principles, unless the teachers using these materials have themselves acquired a depth of understanding about how young children learn and about the stages through which they move in this process, then the position in connection with the learning of mathematics is little better than before these materials were developed."