Symmetry and Conservation Laws of Recurrence Equations and Classes of Time-Fractional Equations

Nkosingiphile Mnguni (734552@students.wits.ac.za)
School of Mathematics
Supervisors: Dr M. Folly-Gbetoula and Prof. S. Jamal

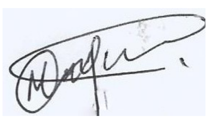
A thesis submitted to the Faculty of Science, University of the Witwatersrand, in requirement for the degree Doctor of Philosophy.

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

Johannesburg, January 2021

Declaration

I hereby declare that the work contained in this thesis is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly. It is submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other institution.



N. Mnguni

Signed at Johannesburg on the 1^{st} day of January 2021.

Abstract

The thesis focuses on the symmetry analysis of two types of equation classes. The first part of the thesis involves the novel study of symmetry analysis on classes of time-fractional partial differential equations. Firstly, a Lie group analysis method is used to obtain the solutions of fractional-order population growth models, where a time component is defined by Riemann-Liouville derivative. Furthermore, the Lie group analysis is extended to the time-fractional axisymmetric spreading of a thin incompressible power-law fluid on a horizontal plane. The solutions and their graphical representation are obtained. The second part of the thesis involves the study of symmetries to find the exact solutions of recurrence equations. We investigate the symmetry analysis on some fourth, fifth and sixth order recurrence equations and obtain their respective solutions for special cases of the recurrence equations. Furthermore this analysis is extended to a general $k+1$-th order recurrence equation.

Acknowledgments

I would like to extend my gratitude to Dr M. Folly-Gbetoula and Prof. S. Jamal for their great supervision of my thesis. The guidance and patience that they have given me throughout my research has been great.

I would also like to give special thanks to my family and friends for the support they have given me throughout this journey.

Contents

List of Figures

Chapter 1

Introduction

The study of symmetry analysis on differential equations (DEs) was first introduced by Marius Sophus Lie (1842-1899) [1]. Lie was able to unify and extend a number of techniques that were required in solving DEs. Lie realised that these techniques were special cases of a general integration procedure based on the invariance of DEs under a continuous group of symmetries. This concept of symmetries is strongly related to the existence of conservation laws and the relationship between them has attracted great interest among researchers following the work of Noether [2].

The extension of this idea to recurrence equations is now well-documented (see [3] and references herein). Recurrence equations have numerous real life applications. Their importance in mathematical biology and other areas of science cannot be ignored. In [3], Hydon developed a symmetry based algorithm enabling one to derive solutions of difference equations without making any special lucky guesses. Hydon emphasized on second-order difference equations, although his algorithm is valid for any order. When it comes to higher-order equations, computations can be cumbersome and extra ansatz may be needed to ease the calculations. Difference equations and recurrence equations are used interchangeably in this thesis.

Outline of the thesis

The goal of this thesis is to study the symmetry analysis on classes of time-fractional partial differential equations and recurrence equations. The work in this thesis produced four publications and two more articles that have been submitted for publication.

In the second chapter, we focus on symmetry analysis for time-fractional partial differential equations and is sub divided into sections as shown below:

- Section 2.1 introduces the notation and theory that is used in the subsequent sections of the chapter.
- In Section 2.2 of the chapter, we consider two categories of fractional-order population growth models, where a time component is defined by Riemann-Liouville derivative. These models are studied under the Lie symmetry approach, and we reduce the fractional partial differential (FDEs) equations to non linear ordinary differential equations (ODEs). Subsequently, solutions of the latter are determined numerically or with the aid of Laplace transform. Graphical representations for integral and trigonometric solutions are presented. A key feature of these models is the connection between spatial patterning of organisms versus competitive coexistence.

  The results for this section have been submitted for publication.
- In Section 2.3, we consider a fractional-order, incompressible power-law fluid on a horizontal plane, where the time component is defined by Riemann-

Liouville derivatives. The model is characterized by a nonlinear second-order partial differential equation comprising of a power-law parameter β. We transform the model into nonlinear ODEs and subsequently, solutions of the latter are determined analytically. In the case of a Newtonian fluid, we show that moving front solutions are obtained irrespective of the presence of fractional derivatives. Graphical representations for the moving front solutions are presented. Lastly, we find a nonclassical solution for the integer-order power-law fluid model.

The results of this section have been published [4].

The third chapter focuses on symmetry analysis for recurrence equations of varying orders and is divided into sections as shown below:

- Section 3.1 introduces the notation and theory that is used in the subsequent sections.
- In Section 3.2, all the Lie point symmetries of difference equations of the form

  $$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}},$$

  where, $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ are sequences of real numbers, are obtained. We perform reduction of order using the invariant of the group of transformations. Furthermore, we obtain their solutions.

  The results of this section have been published [5].

- In Section 3.4, we consider difference equations of the form

  $$u_{n+5} = \frac{u_n}{A_n + B_n \prod_{i=0}^{4} u_{n+i}}$$

  where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are sequences of real numbers. We find non-trivial Lie point symmetry generators of the equations and obtain solutions.

The results of this section have been published [6].

- In Section 3.4, a full Lie point symmetry analysis on

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}}.$$

is performed. Non-trivial symmetries are derived and exact solutions using these symmetries are obtained.

The results of this section have been published [7].

- In Section 3.5, a full Lie point symmetry analysis of the rational difference equations of the form

$$u_{n+k+1} = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}},$$

where A_n and B_n are real sequences, is considered and new symmetries are found. These symmetries are utilized to investigate the existence of solutions. The results in this paper generalize some results in existing literature.

The results for this section have been submitted for publication.

Chapter 2

Classes of Time-Fractional Equations

Partial differential equations (PDEs) with fractional-order are indispensable tools in understanding the dynamics of biological systems, and are at the forefront of theoretical investigations- as opposed to integer-order models that are afflicted with memory effects. Accordingly, the concept of fractional-order DEs (FDEs) have been successfully applied to many, if not all, scientific spheres. To mention just a few examples, we mention studies within biology, fluid dynamics, medicine etc., see [8, 9, 10] and references therein. In the literature, it has often been reported that FDEs are more consistent with natural phenomena than integer-order models. Hence, this has driven much research involving fractional equations as generalizations of classical integer-order DEs. Although fractional models are increasingly complex to analyse and solve, this problem was met with a diverse number of proposals. The techniques available to solve FDEs include, but are not limited to: the homotopy perturbation method [11] and modified homotopy method [12], the

Adomian decomposition method [13], Laplace transform method [14], extrapolation method [15], multistep method [16] and the monotone iterative technique [17]. Another, less common approach is the method of Lie group analysis [18, 19]. This approach is extremely successful in the analysis of integer-order equations and the fractional ones as well [20, 21, 22, 23, 24, 25, 26]. Adaptations to the original theory are necessary when dealing with FDEs and we discuss these modifications below. Nevertheless, the Lie point symmetries of FDEs also lead to group invariant solutions like their integer equation counterparts.

In the previous paragraphs we have discussed our motivation for addressing this problem, and we now turn our attention to the chapter's structure. In the next section we outline the relevant theory on calculus of integral and derivatives of fractional order and, the important properties of the associated Lie point symmetries of FDEs. Section 2.3 is devoted to the Lie symmetry analysis of fractional-order spatio-temporal PDEs, their solutions and graphical analyses. The final section is devoted to the Lie symmetry analysis on moving front solutions of a time fractional, thin power-law fluid under gravity, their solutions and graphical analysis.

2.1 Fractional Calculus and Symmetry Generators

In this section, we present the mathematical framework required in subsequent sections of this chapter. In existence, there are several different definitions of fractional derivatives. Time-fractional derivatives are commonly discussed in terms of Caputo, Grünwald-Letnikov or Riemann-Liouville derivatives [14, 27, 28, 29]. In this work, we limit ourselves to the latter. That is, we shall introduce the linear operators of differentiation in the framework of Riemann-Liouville fractional calculus, followed by the procedure for determining point symmetries of time-fractional PDEs.

The Riemann-Liouville fractional derivative is defined by

$$D_t^\alpha f(t) = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\theta,x)}{(t-\theta)^{\alpha+1-n}} d\theta, & n-1 < \alpha < n, n \in \mathbb{N} \end{cases} \tag{2.1}$$

where $\Gamma(z)$ is the Euler gamma function.

For the purpose of this section, we assume that we have a second-order, time-fractional PDEs with dependent variable u and independent variables x, t:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - f\left(x, t, u, u_x, u_{xx}\right) = 0, \tag{2.2}$$

where $0 < \alpha < 1$ is the parameter describing the order of the fractional time derivative. Now, suppose that Equation (2.2) is invariant under the one-parameter Lie group of point transformations

$$\begin{aligned} t^* &= \quad t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ x^* &= \quad x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ u^* &= \quad u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \quad \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0(x,t,u) + O(\epsilon^2), \\ \frac{\partial \bar{u}}{\partial \bar{x}} &= \quad \frac{\partial u}{\partial x} + \epsilon\eta^x(x,t,u) + O(\epsilon^2), \\ \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \quad \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon^2), \end{aligned} \tag{2.3}$$

where ϵ is an infinitesimal parameter, and the individual terms in the above expression are

$$\begin{aligned}
\eta^{x} &= \eta_x + \eta_u u_x - (\xi_x + \xi_u u_x)\, u_x - (\tau_x + \tau_u u_x)\, u_t, \\
\eta^{xx} &= \eta_{xx} + 2\,\eta_{xu} u_x - 2\,\xi_{xu} {u_x}^2 - \xi_{xx} u_x - 2\,\tau_{ux} u_t u_x - \tau_{xx} u_t - \xi_{uu} {u_x}^3 \\
&\quad -\tau_{uu} {u_x}^2 u_t + \eta_{uu} {u_x}^2 - 3\,\xi_u u_x u_{xx} - \tau_u u_t u_{xx} + \eta_u u_{xx} - 2\,\xi_x u_{xx} \\
&\quad -2\,\tau_u u_x u_{tx} - 2\,\tau_x u_{tx}.
\end{aligned} \tag{2.4}$$

From the generalised Leibniz rule and a generalisation of the chain rule, we have that [19]

$$\begin{aligned}
\eta_\alpha^0 &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta - \alpha D_t(\tau))\, \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n\,(\xi)\, D_t^{\alpha-n}\,(u_x) \\
&\quad + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n}{\partial t^n} \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}.
\end{aligned} \tag{2.5}$$

The D_t^α is the total fractional derivative operator, and

$$\begin{aligned}
\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma\,(n+1-\alpha)} \\
\times (-u)^r \frac{\partial^m}{\partial t^m} \left(u^{k-r}\right) \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial u^k}.
\end{aligned} \tag{2.6}$$

It is important to remark that by convention in the literature, $\eta(x, t, u)$ is taken to be linear in the variable u so that μ vanishes, see for example [20, 30]. Such an ansatz is necessary to considerably simplify the determining system of the symmetry. We adopt this idea in the work below as well.

Let the generator

$$X = \tau(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u} \tag{2.7}$$

span the associated Lie algebra, that is,

$$\tau(x, t, u) = \frac{dt^*}{d\epsilon}\Big|_{\epsilon=0}, \qquad \xi(x, t, u) = \frac{dx^*}{d\epsilon}\Big|_{\epsilon=0}, \qquad \eta(x, t, u) = \frac{du^*}{d\epsilon}\Big|_{\epsilon=0}.$$

The prolongation operator for the vector field is $pr^{(\alpha,2)}$ where

$$\begin{aligned} pr^{(\alpha,2)} &= \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u} \\ &\quad + \eta_{\alpha}^{0}\partial_{\partial_t^{\alpha}u} + \eta^{x}\partial_{u_x} + \eta^{xx}\partial_{u_{xx}}. \end{aligned} \tag{2.8}$$

The infinitesimal invariance condition may be stated as follows. The generator X is a Lie point symmetry of (2.2) if and only if

$$pr^{(\alpha,2)}X\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - f\left(x,t,u,u_x,u_{xx}\right)\right) = 0. \tag{2.9}$$

A solution of the FDE (2.2) $u = \sigma(x,t)$ is an invariant solution of (2.2) if $pr^{(\alpha,2)}X\sigma = 0$. Moreover, it is essential that the transformation (2.3) leaves invariant the lower limit of the fractional derivative $\frac{\partial^{\alpha}u}{\partial t^{\alpha}}$, which translates into the additional constraint condition [18]

$$\tau(x,t,u)\Big|_{t=0} = 0. \tag{2.10}$$

2.2 Fractional-order Spatio-temporal Partial Differential Equations

In this section, the Lie group analysis method is used to obtain solutions of selected time-fractional DEs of the ecological sciences. The FDEs that we consider belong to the family of equations, that are second-order in derivatives, time-fractional PDEs with dependent variable u and independent variables x, t:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = K\left(x,t,u,u_x,u_{xx}\right). \tag{2.11}$$

Here, $0 < \alpha < 1$ is the parameter describing the order of the fractional time derivative.

Generally speaking, PDEs depict a broad class of ecological interactions, where many assume that processes such as birth, death, dispersion etc. are continuous values in space and time. Primarily, spatial PDEs showcase links between competition and habitat geometry, since spatial patterning emerges due to competitive coexistence and dispersal. Solutions for such models lay the foundations for predictability models, data assimilation, and thus, the importance of the aforementioned class of ecological equations.

In particular, we shall study the invariance properties of the time-fractional standard biological model as well as an ecological stimulus equation. The resulting vector fields will be used for reduction purposes to transform the fractional PDEs into ODEs. One of the explicit solution is obtained with Laplace transforms. Additionally, the invariant solutions obtained are explored graphically.

The structure of the section is as follows, The next subsection focuses on generating the Lie symmetries using the concept in Section 2.1. The next two subsections focus on the Lie symmetry analyses, their solutions and graphical analysis of Stimuli Models and Standard BP Model. Conclusion and final remarks are given in the final subsection.

2.2.1 Stimuli Models

In most classical applications of PDEs in population ecology, organisms are assumed to have Brownian random motion which is time and space invariant. Such an assumption characterizes diffusion models having $u(x,t)$ as the density of organisms at a spatial coordinate x and time t, and usually involves a parameter, say b as the the diffusion coefficient that measures the dispersal rate in the units distance^2/time. Simple diffusion models have effectively described the movement of animals in mark-recapture studies (e.g., [31]). As a result of external stimuli, the orientation of organisms is affected and drift or convection terms are added to diffusive equations, resulting in a fractional version of the stimulus model [32]:

$$K\left(x,t,u,u_x,u_{xx}\right) = bu_{xx} - \omega u_x, \tag{2.12}$$

where ω is the drift velocity. In the work that follows, we determine the invariance properties of the time-fractional stimulus equation. Thereafter, we construct an exact solution and explore its behaviour. The numerical part of the study is performed with Maple. According to Lie's theory and Section 2.1, applying the second prolongation $pr^{(\alpha,2)}X$ to Equation (2.12), provides

$$\eta_\alpha^0 - b\eta^{xx} - \omega\eta^x = 0. \tag{2.13}$$

The use of (2.4) and (2.5) into (2.13), yields the determining system:

$$\begin{aligned}
&\frac{\partial^2}{\partial u^2}\eta = 0, \frac{\partial}{\partial u}\tau = 0, \frac{\partial}{\partial x}\tau = 0, \frac{\partial}{\partial u}\xi = 0, \frac{\partial}{\partial t}\xi = 0,\\
&b\left(-\left(\frac{\partial}{\partial t}\tau\right)\alpha + 2\frac{\partial}{\partial x}\xi\right) = 0,\\
&\alpha\left(-\alpha\frac{\partial^2}{\partial t^2}\tau + 2\frac{\partial^2}{\partial u\partial t}\eta + \frac{\partial^2}{\partial t^2}\tau\right) = 0,\\
&-\omega\frac{\partial}{\partial x}\xi - 2\,b\frac{\partial^2}{\partial x\partial u}\eta + b\frac{\partial^2}{\partial x^2}\xi + \alpha\left(\frac{\partial}{\partial t}\tau\right)\omega = 0,\\
&\alpha\left(\alpha - 1\right)\left(-\alpha\frac{\partial^3}{\partial t^3}\tau + 3\frac{\partial^3}{\partial u\partial t^2}\eta + 2\frac{\partial^3}{\partial t^3}\tau\right) = 0.
\end{aligned} \tag{2.14}$$

Solving this system and considering constraint (2.10) provides:

$$\xi(x,t,u) = 2\,c_2\,\alpha\,bx + c_4,\;\; \tau(x,t,u) = 4\,c_2\,bt,$$

$$\eta(x,t,u) = F\left(x,t\right) + 2\,u\alpha\,c_2\,b + u\alpha\omega x c_2 + uc_5 - 2\,uc_2\,b,$$

where c_2, c_4, c_5 are constants. The above symmetry coefficients and $F(x,t)$ must also satisfy the conditions

$$
\begin{aligned}
-b\frac{\partial^2}{\partial x^2}\big(F(x,t) + 2\,u\alpha\,c_2\,b + u\alpha\,\omega\,xc_2 + uc_5 - 2\,uc_2\,b\big) +\\
\omega\,\frac{\partial}{\partial x}\big(F(x,t) + 2\,u\alpha\,c_2\,b + u\alpha\,\omega\,xc_2 + uc_5 - 2\,uc_2\,b\big)\\
+\frac{\partial^\alpha}{\partial t^\alpha}\big(F(x,t) + 2\,u\alpha\,c_2\,b + u\alpha\,\omega\,xc_2 + uc_5 - 2\,uc_2\,b\big)\\
-u\frac{\partial^{\alpha+1}}{\partial t^\alpha \partial u}\big(F(x,t) + 2\,u\alpha\,c_2\,b + u\alpha\,\omega\,xc_2 + uc_5 - 2\,uc_2\,b\big) = 0,\\
\sum_{n=3}^{\infty}\binom{\alpha}{n}\Big(\frac{\partial^{1+n}}{\partial t^n \partial u}\big(F(x,t) + 2\,u\alpha\,c_2\,b + u\alpha\,\omega\,xc_2 + uc_5 - 2\,uc_2\,b\big)\Big)D_{t^{\alpha-n}}(u)\\
-\sum_{n=3}^{\infty}\frac{1}{1+n}\binom{\alpha}{n}\big(D_{t^{\alpha-n}}(u)\big(4\,D_{t^{1+n}}(c_2\,bt) + D_{t^{1+n}}(c_1)\big)\alpha - D_{t^{\alpha-n}}(u)\\
\times\big(4\,D_{t^{1+n}}(c_2\,bt) + D_{t^{1+n}}(c_1)\big)n + D_{t^{\alpha-n}}\big(\frac{\partial}{\partial x}u\big)\big(2\,D_{t^n}(c_2\,\alpha\,bx) + D_{t^n}(c_4)\big)n\\
+D_{t^{\alpha-n}}\big(\frac{\partial}{\partial x}u\big)\big(2\,D_{t^n}(c_2\,\alpha\,bx) + D_{t^n}(c_4)\big)\big) = 0.
\end{aligned}
\tag{2.15}
$$

Taking the simplest case, we let $F(x,t) = 0$, that gives the symmetry vectors,

$$X_1 = \partial_x,\ \ X_2 = \partial_u,$$

$$X_3 = 2\alpha bx\partial_x + 4bt\partial_t + (2u\alpha b + u\alpha\omega x - 2ub)\,\partial_u.$$

Taking X_1, we have the invariant,

$$u(x,t) = w(t).$$

Substituting this invariant into (2.12), we have the time-fractional ODE:

$$\frac{\mathrm{d}^\alpha w(t)}{\mathrm{d}t^\alpha} = 0, \tag{2.16}$$

which has a solution given by the method of Laplace transforms [14]. That is, let

$$L\{w(t)\} = \int_0^\infty e^{-st} w(t) dt = W(s).$$

Then, the Laplace transform of Riemann-Liouville fractional derivatives is

$$L\{D_t^p w(t)\} = s^p W(s) - \sum_{k=0}^{n-1} s^k \left[D_t^{p-k-1} w(t) \right]_{t=0}, \quad p > 0,\ n-1 < p \leq n. \quad (2.17)$$

Assume that the fractional integral of $w(t)$ at $t = 0$ is a constant, say W_0, i.e.

$$\left[D_t^{\alpha-1} w(t) \right]_{t=0} = W_0.$$

Taking Laplace transforms of (2.16), and since $0 < \alpha < 1$, we have that $n = 1$ in (2.17), which implies that

$$L\{D_t^\alpha w(t)\} = s^\alpha W(s) - \left[D_t^{\alpha-1} w(t) \right]_{t=0} = 0.$$

Hence,

$$W(s) = \frac{W_0}{s^\alpha},$$

and inverting the transform finally leads to a solution for $w(t)$ and therefore $u(x,t)$, namely

$$w(t) = \frac{W_0}{\Gamma(\alpha)} t^{\alpha-1}. \quad (2.18)$$

The illustration in Fig. 2.1 depicts the evolution of the solution (2.18).

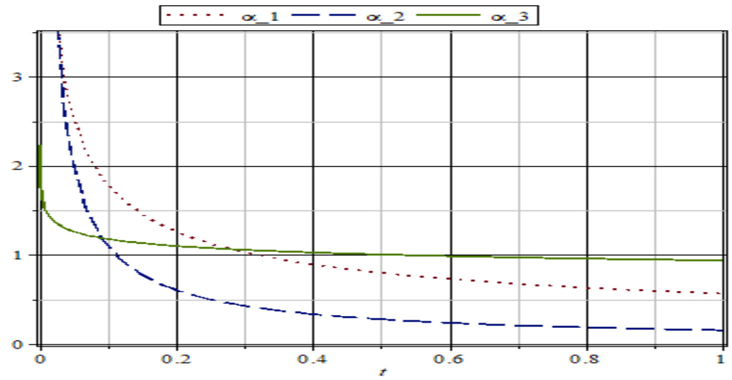


Fig. 2.1: Evolution of the analytical solution of (2.18) is depicted, parameter values are $W_0 = 1$, while fractional values are taken as $\alpha_1 = \frac{1}{2},\ \alpha_2 = \frac{1}{7},\ \alpha_3 = \frac{9}{10}$. Clearly, over time, the population density decreases, independent of any spatial variable, the rate of decline is impacted by the size of the time-fractional parameter.

2.2.2 Standard BP Model

Diffusion models, like the one mentioned above are easily altered to encompass interactions between con-specifics. Gurney and Nisbet [33] proposed that a simple diffusion model can be replaced by a biased random motion model to study the attraction or repulsion between organisms. The fractional-order equivalent of this model is

$$K\left(x, t, u, u_x, u_{xx}\right) = bu_{xx} + c\left(u_x^2 + uu_{xx}\right), \tag{2.19}$$

where again $u(x, t)$ is population density, and c is a measure of the tendency to move away from or towards con-specifics. The scenario of $c > 0$ depicts repulsion and $c < 0$ indicates attraction between con-specifics. In fact, there are many permutations of simple diffusion models to describe spatial processes. In order to investigate more complex behavioural patterns, scientists opted to study reaction-diffusion models - where the PDE characterizes population dispersal and multispecies interactions.

A typical time-fractional reaction-diffusion model, also referred to as a standard biological population (BP) model, may be expressed as the time-fractional PDE

$$K\left(x,t,u,u_x,u_{xx}\right)=\left(u^2\right)_{xx}+h\left(u\right). \tag{2.20}$$

The first term represents diffusive movement and the last term, $h(u)$, is the reaction term that describes population growth dynamics. This model also assumes that population dispersal occurs through Brownian random motion, but is still relevant even if the diffusion is modified to depict less simple behaviour [34]. Several laws exist for the choice of the constitutive function $h(u)$, such as Malthusian law [35], Verhulst law [36] and a porous media law [37]. Equation (2.20) together with its various modifications have been studied extensively via numerous methods [38, 39, 40, 41, 42].

For Equation (2.20), the invariance condition (2.9) is equivalent to the following equation:

$$\eta_\alpha^0-4u_x\eta^x-2\left(u\eta^{xx}+u_{xx}\eta\right)-\eta\frac{\mathrm{d}}{\mathrm{d}u}h\left(u\right)=0. \tag{2.21}$$

Substituting (2.4) and (2.5) into (2.21), and equating the coefficients of the monomials, we obtain the determining equations for the symmetry group of Equation (2.20), viz.

$$\begin{aligned}
\frac{\partial^2}{\partial u^2}\eta=0,\ \frac{\partial}{\partial u}\tau=0,\ \frac{\partial}{\partial x}\tau=0,\ \frac{\partial}{\partial u}\xi=0,\ \frac{\partial}{\partial t}\xi=0,\\
-2\,\eta+4\,u\frac{\partial}{\partial x}\xi-2\,\alpha\,u\frac{\partial}{\partial t}\tau=0,-4\,\frac{\partial}{\partial x}\eta-4\,u\frac{\partial^2}{\partial x\partial u}\eta+2\,u\frac{\partial^2}{\partial x^2}\xi=0,\\
\alpha\,\left(-\alpha\,\frac{\partial^2}{\partial t^2}\tau+2\,\frac{\partial^2}{\partial u\partial t}\eta+\frac{\partial^2}{\partial t^2}\tau\right)=0,\\
4\,\frac{\partial}{\partial x}\xi-2\,\frac{\partial}{\partial u}\eta-2\,\alpha\,\frac{\partial}{\partial t}\tau=0,\\
\alpha\,\left(\alpha-1\right)\left(-\alpha\,\frac{\partial^3}{\partial t^3}\tau+3\,\frac{\partial^3}{\partial u\partial t^2}\eta+2\,\frac{\partial^3}{\partial t^3}\tau\right)=0,
\end{aligned} \tag{2.22}$$

that satisfy condition (2.10) and the following auxiliary conditions that involve the unspecified constitutive function $h(u)$:

$$\begin{aligned}
&\left(\frac{\partial}{\partial u}\left(c_4\,\alpha\,u+2\,c_2\,u\right)\right)h\left(u\right)-\left(c_4\,\alpha\,u+2\,c_2\,u\right)\frac{\mathrm{d}}{\mathrm{d}u}h\left(u\right)\\
&-2\,u\frac{\partial^2}{\partial x^2}\left(c_4\,\alpha\,u+2\,c_2\,u\right)-\alpha\,h\left(u\right)\frac{\partial}{\partial t}\left(-c_4\,t+c_3\right)+\frac{\partial^\alpha}{\partial t^\alpha}\left(c_4\,\alpha\,u+2\,c_2\,u\right)\\
&-u\frac{\partial^{\alpha+1}}{\partial t^\alpha\partial u}\left(c_4\,\alpha\,u+2\,c_2\,u\right)=0,\\
&\sum_{n=3}^{\infty}\binom{\alpha}{n}\left(\frac{\partial^{1+n}}{\partial t^n\partial u}\left(c_4\,\alpha\,u+2\,c_2\,u\right)\right)D_{t^{\alpha-n}}\left(u\right)\\
&-\sum_{n=3}^{\infty}\frac{1}{1+n}\binom{\alpha}{n}\Big(D_{t^{\alpha-n}}\left(u\right)\\
&\times\big(-D_{t^{1+n}}\left(c_4\,t\right)+D_{t^{1+n}}\left(c_3\right)\big)\alpha-D_{t^{\alpha-n}}\left(u\right)\big(-D_{t^{1+n}}\left(c_4\,t\right)+D_{t^{1+n}}\left(c_3\right)\big)n\\
&+D_{t^{\alpha-n}}\big(\frac{\partial}{\partial x}u\big)\big(D_{t^n}\left(c_2\,x\right)+D_{t^n}\left(c_1\right)\big)n+D_{t^{\alpha-n}}\Big(\frac{\partial}{\partial x}u\Big)\\
&\times\big(D_{t^n}\left(c_2\,x\right)+D_{t^n}\left(c_1\right)\big)\Big)=0.
\end{aligned} \tag{2.23}$$

The auxiliary conditions are simply to ensure that all the determining equations are all identically satisfied. Next, we define the constitutive function in order to exemplify some particular solutions. Suppose $h(u) = h_0u^2$. Solving the system of equations (2.22) yields the symmetry coefficients

$$\xi(x,t,u) = c_2\,x + c_1,\;\; \tau(x,t,u) = -c_4\,t,\;\; \eta(x,t,u) = c_4\,\alpha\,u + 2\,uc_2,$$

which we list in vector form

$$X_1,\;\; X_4 = x\partial_x + 2u\partial_u,$$

$$X_5 = -t\partial_t + \alpha u\partial_u.$$

For the operator X_5, we obtain the invariant

$$u(x,t) = w(x)t^{-\alpha}. \tag{2.24}$$

At this stage, we recall the formula [43]

$$\frac{d^{\alpha}t^{\beta}}{dt^{\alpha}} = \frac{t^{-\alpha+\beta}\Gamma\left(1+\beta\right)}{\Gamma\left(1-\alpha+\beta\right)}, \qquad \beta > -1, \tag{2.25}$$

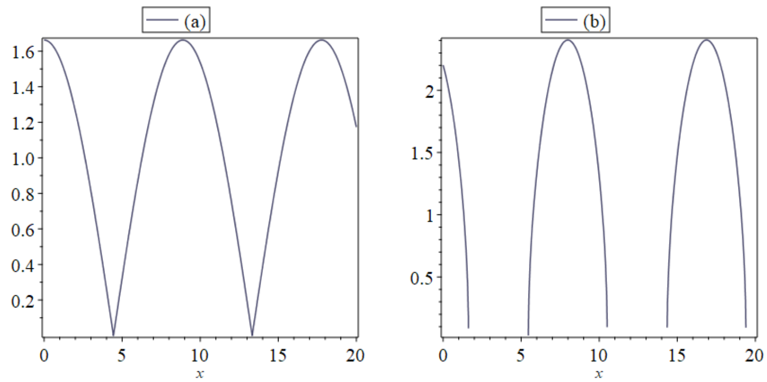
which is useful in the reduction of the FDE. Consequently, with the application of (2.24) and (2.25) we transform Equation (2.20) into an ODE:

$$\begin{aligned} -\Gamma\left(\frac{1}{2}-\alpha\right)\left(w\left(x\right)\right)^{2}h + 4^{\alpha}\sqrt{\pi}w\left(x\right) - 2\,\Gamma\left(\frac{1}{2}-\alpha\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}w\left(x\right)\right)^{2} \\ -2\left(\frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}w\left(x\right)\right)\Gamma\left(\frac{1}{2}-\alpha\right)w\left(x\right) = 0. \end{aligned} \tag{2.26}$$

We find the solution of this ODE to be ($\alpha \neq \frac{1}{2}$):

$$\begin{aligned} \int^{w(x)} -2\,\frac{\Gamma\left(\frac{1}{2}-\alpha\right)\sqrt{3}\_a}{\sqrt{\Gamma\left(\frac{1}{2}-\alpha\right)\left(-3\,\_a^{4}h_{0}\Gamma\left(\frac{1}{2}-\alpha\right)+2^{2+2\,\alpha}\sqrt{\pi}\_a^{3}+12\,c_{1}\,\Gamma\left(\frac{1}{2}-\alpha\right)\right)}}d\_a \\ -x-c_{2}=0, \end{aligned} \tag{2.27}$$

and one can recover the solution for the population density $u(x,t)$ from (2.24). Numerically, the behaviour of $w(x)$ can be seen in Fig. 2.2, as expected we observe the oscillatory nature of the solution.



(a) (b)

Fig. 2.2: The population density Equation (2.24), we take the positive sign for the square root. The habitat size has an oscillatory effect on population growth. Parameter values are : (a) $h_0 = \frac{1}{2}, \alpha = \frac{1}{4},\ c_1 = c_2 = 0$; and, (b) $h_0 = \frac{1}{2}, \alpha = \frac{1}{3},\ c_1 = c_2 = 1$.

Secondly, consider a simpler model with zero reaction, viz $h(u) = 0$, which inherits the same generators $X_1,\ X_4,\ X_5$. Subsequently, to obtain an exact solution, we choose the same invariant as before, Equation (2.24) and utilize (2.25). This time, the fractional-order reaction-diffusion equation reduces to the ODE:

$$2\,\Gamma\left(\frac{1}{2}-\alpha\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}w\left(x\right)\right)^{2}+\left(\frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}w\left(x\right)\right)\Gamma\left(\frac{1}{2}-\alpha\right)w\left(x\right)-4^{\alpha}\sqrt{\pi}w\left(x\right)=0.$$

This nonlinear ODE has the implicit solution ($\alpha \neq \frac{1}{2}$):

$$\pm\int^{w(x)}\frac{\Gamma\left(\frac{1}{2}-\alpha\right)\sqrt{5}\_a^{2}}{\sqrt{\Gamma\left(\frac{1}{2}-\alpha\right)\left(2\,\_a^{5}4^{\alpha}\sqrt{\pi}+c_1\right)}}d\_a-x-c_2=0. \tag{2.28}$$

A graphical analysis of the solution (2.28) in conjunction with the population density Equation (2.24) is probed in Fig. 2.3. Note that it is straightforward to solve the integral for particular values of the free parameters, but not as easy for arbitrary values.

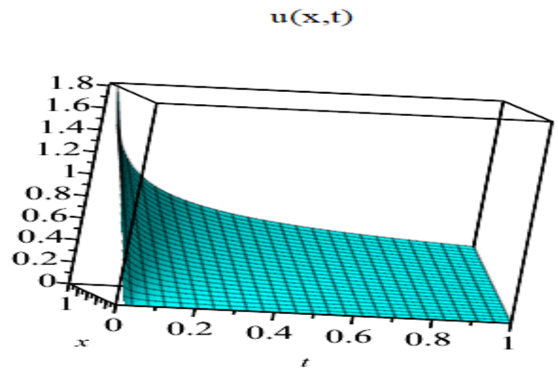


Fig. 2.3: The population density Equation (2.24) with parameter values $\alpha = \frac{1}{3}$, $c_1 = c_2 = 0$, and we take the positive sign for the integral.

2.2.3 Conclusion

In this section, we investigated a family of fractional-order models that describe population dynamics. The fractional derivative was with respect to time in order to gain insight into rates of population growth, dispersal, habitat patch size and reactions to external stimuli. Lie symmetries, which formulate a geometric method, were employed to perform group-invariant reductions. In this regard, we determined ODEs which were further analysed. One of the resultant exact solutions was implicit and comprised of an integral solution, while another involved trigonometric functions. The stimuli model had a solution determined by Laplace transforms. Lastly, we also looked at the graphical behaviour of the solutions. The results may further serve as a guide for computer simulations in conservation biology and for comparisons with other numerical data.

2.3 Moving Front Solutions of a Time-fractional, Thin Power-law Fluid under Gravity

The axisymmetric spreading of fluids is an important subject in all sciences and industrial processes. For instance, the spread of molten lava on the earth's surface, polymeric fluids, paint, lubricants, blood, mucus, melted chocolate, etc. Similarity solutions have been explored for integer-order fluid equations, both Newtonian and non-Newtonian, see [44, 45, 46, 47]. We are particularly interested in the model investigated by [48], which studied the problem of a power-law fluid on a horizontal plane ignoring surface tension. This study assumed the existence of a separable solution and applied linearization criteria to obtain analytical solutions. The same model, recast in fractional time is:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{(\beta+1)\,x}\left(xu^{\beta+1}\left(u_x\right)^{\beta-1}\right)_x = 0. \tag{2.29}$$

The function $u(x,t)$ is the film height, x is a radial coordinate, t is time, β is the power-law parameter and $0 < \alpha < 1$ is the parameter describing the order of the fractional time derivative. All quantities are nondimensional. Equation (2.29) models the fractional time evolution of the height of a thin power-law film on a horizontal plane in the presence of gravity. That is, $0 < \alpha < 1$ allows us to examine the intermediate evolutionary behaviour of the fluid at fractional time. In particular, the next state of the fluid height might depend not only on its current state but also on historical states. This is referred to as the non-local property which can be successfully modelled by the introduction of the fractional derivative [30]. The above reasons serve as a strong motivation for the study of the fluid model Equation (2.29). Moreover, the numerous applications of axisymmetric spreading [49] of the integer equation are shared with the fractional equation.

In this section, we determine the symmetries and solutions admitted by the time-fractional equation for certain values of the power-law parameter β. We compare the solutions of the fractional-order equation to the solutions of the integer-order equation and a nonclassical solution. We show that the solutions describe moving front solutions by considering the evolution of the film heights.

In Section 2.1, we presented the relevant theory on fractional calculus. The important properties of the associated Lie point symmetries of FDEs were also discussed. In Subsection 2.3.1, we study the film equation using Lie group analysis and describe the special cases of the power-law coefficient. Subsection 2.3.2 contains concluding remarks.

2.3.1 The Power-law Film Model

The definitions and formulae discussed in Section 2.1 will be used to investigate the invariance properties of the fluid model. The resulting vector fields will be used for reduction purposes to transform the FDEs into ODEs. We demonstrate how one of the explicit solutions below is obtained with the aid of Laplace transforms. Additionally, the invariant solutions are explored graphically.

For the integer-order version of Equation (2.29), the following exact solutions for (2.29) are known [48]: a) For $\beta = 0$,

$$u(x,t) = (At + B)\, F\,(x)\,, \tag{2.30}$$

where $y = F(x)$ is a solution of the implicit equation $\left(y - \frac{1}{A}\right) e^{Ay} = AC_2x^2 - \frac{AC_1}{2}$, A, B, C_1, C_2 are constants, and b) For $\beta = 2$,

$$u(x,t) = \frac{3A^{\frac{1}{3}}x^{\frac{2}{3}}}{2\,(3At - B)^{\frac{1}{3}}}. \tag{2.31}$$

Furthermore, we find that the integer-order equation admits two nonclassical symmetries

$$\begin{aligned} Z_1 &= -u\partial u + \partial t + \frac{x\left(-2\beta+1\right)\partial x}{\beta}, \\ Z_2 &= -\frac{u\partial u}{t} + \partial t + \frac{2x\left(-\beta+1\right)\partial x}{t\beta}. \end{aligned} \tag{2.32}$$

Such symmetries arise by imposing an extra condition on the infinite criterion for invariance, known as the invariant surface condition. A reduction of the integer version of Equation (2.29) involving Z_2 leads to the nonclassical solution, viz.

$$u(x,t) = x^{\frac{\beta}{2\beta-1}} t^{-2\frac{\beta}{2\beta-1}} \left|\left(2-\beta^{-1}\right)^{\frac{\beta}{2\beta-1}}\right| \left|\left(-\frac{\beta\left(\beta+1\right)}{\left(2\beta-1\right)\left(5\beta-2\right)}\right)^{\frac{1}{(2\beta-1)}}\right|, \tag{2.33}$$

where $\beta \neq 0, \frac{1}{2}, \frac{2}{5}$. The solutions (2.30) and (2.31) are classical solutions. In Figure 2.4, we plot the height function (2.33) for a Newtonian fluid.

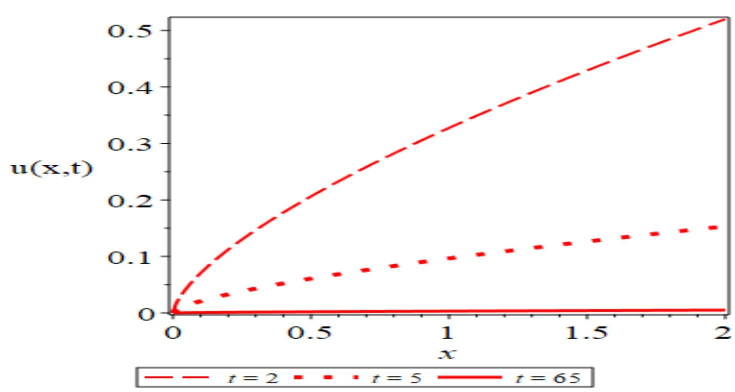


Fig. 2.4: Progression of the analytical solution (2.33), where $\beta = 2$.

Now we move on to the fractional study. According to Lie's theory and section 2.1, applying the second prolongation $pr^{(\alpha,2)}X$ to Equation (2.29), provides

$$
\begin{aligned}
&\eta_\alpha^0 + \frac{1}{(\beta+1)\,x^2}u^{\beta+1}u_x^{\beta-1}\xi - \left(\frac{1}{x}u^\beta u_x^{\beta-1} + \beta u^{\beta-1}u_x^\beta + (\beta-1)\,u^\beta u_x^{\beta-2}u_{xx}\right)\eta \\
&-\left(\frac{\beta-1}{(\beta+1)\,x}u^{\beta+1}u_x^{\beta-2} + \beta u^\beta u_x^{\beta-1} + \frac{(\beta-2)\,(\beta-1)}{(\beta+1)}u^{\beta+1}u_x^{\beta-3}u_{xx}\right)\eta^x \\
&-\frac{\beta-1}{\beta+1}u^{\beta+1}u_x^{\beta-2}\eta^{xx} = 0.
\end{aligned}
\tag{2.34}
$$

The use of (2.4) and (2.5) into (2.34), yields the determining system for X:

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2}\eta = 0, \frac{\partial}{\partial t}\xi = 0, \frac{\partial}{\partial u}\xi = 0, \frac{\partial}{\partial u}\tau = 0,\\
\frac{\partial}{\partial x}\tau = 0, \left(-\alpha\,\frac{\partial^2}{\partial t^2}\tau + 2\,\frac{\partial^2}{\partial u \partial t}\eta + \frac{\partial^2}{\partial t^2}\tau\right)\alpha = 0,\\
\frac{(\beta-1)\left(2\,\frac{\partial^2}{\partial x \partial u}\eta - \frac{\partial^2}{\partial x^2}\xi\right)}{\beta+1} = 0,\\
\frac{(\beta-1)\left(-u^{\beta+1}\frac{\partial}{\partial t}\tau\alpha - u^\beta\eta\beta + 2\,u^{\beta+1}\frac{\partial}{\partial x}\xi - u^\beta\eta\right)}{\beta+1} = 0,\\
(\alpha-1)\left(-\alpha\,\frac{\partial^3}{\partial t^3}\tau + 3\,\frac{\partial^3}{\partial u \partial t^2}\eta + 2\,\frac{\partial^3}{\partial t^3}\tau\right)\alpha = 0.
\end{aligned}
\tag{2.35}
$$

In solving this system, we consider two special cases, viz. $\beta = 2$ and $\beta = 0$. The physical relevance of the case $\beta = 2$ is that it corresponds to a Newtonian fluid. Newtonian fluids obey Newton's law of viscosity which states that there is a proportional relationship between shear stress and velocity gradient. Such fluids are the basis for classical fluid mechanics and have simple molecular structures, for example water, alcohol, ethyl, etc. The second case $\beta = 0$ is also of interest in the literature we have cited as it sets the power-law parameter to zero. Recall that the integer-order solution, Equation (2.30), is established through linearization when $\beta = 0$.

2.3.1.1 Case A: $\beta = 2$

For the Newtonian fluid we obtain the symmetries,

$$X_1 = 3t\partial_t - \alpha u\partial_u,\ X_2 = 2u\partial_u + 3x\partial_x,$$

$$X_3 = -2u\partial_u + 13x\partial_x,$$

upon solving the determining system (2.35) in conjunction with (2.10). Using Lie's method for X_1, the invariant function is obtained as,

$$u(x,t) = w\left(x\right)t^{-\frac{\alpha}{3}}, \tag{2.36}$$

so that (2.29) reduces to the nonlinear fractional ODE:

$$-\frac{1}{3x}w^2\left(ww_{xx}x + 3\,xw_x^2 + ww_x\right) + \frac{\partial^\alpha t^{-\frac{\alpha}{3}}}{\partial t^\alpha}w = 0. \tag{2.37}$$

At this stage, we recall the formula [43]:

$$\frac{d^\alpha t^\beta}{dt^\alpha} = \frac{t^{-\alpha+\beta}\Gamma\left(1+\beta\right)}{\Gamma\left(1-\alpha+\beta\right)}, \qquad \beta > -1, \tag{2.38}$$

which is useful in the reduction of FDEs. Therefore (2.37) can be recast as a nonlinear ODE

$$-\frac{1}{3x}w^2\left(ww_{xx}x + 3\,xw_x^2 + ww_x\right) + \frac{\Gamma\left(-\frac{1}{3}\alpha\right)}{4\Gamma\left(-\frac{4}{3}\,\alpha\right)}w = 0. \tag{2.39}$$

Equation (2.39) is an integer-order equation and we find that it admits the Lie point symmetry $w\partial_w + \frac{3}{2}\,x\partial_x$. The use of this symmetry yields the solution of (2.39) as:

$$w\left(x\right) = \frac{3}{4}\sqrt[3]{\frac{\Gamma\left(-\frac{1}{3}\alpha\right)}{\Gamma\left(-\frac{4}{3}\,\alpha\right)}}x^{\frac{2}{3}}. \tag{2.40}$$

Thus, the height of the Newtonian fluid is given by the solution

$$u\left(x,t\right) = \frac{3}{4}\sqrt[3]{\frac{\Gamma\left(-\frac{1}{3}\alpha\right)}{\Gamma\left(-\frac{4}{3}\,\alpha\right)}}x^{\frac{2}{3}}t^{-\frac{\alpha}{3}}. \tag{2.41}$$

Fig. 2.5 shows the moving front solution given by (2.41). We observe the influence of the fractional parameter α on the height of the power-law film: as $\alpha \to 0$, the film tends to a greater height for a given t value. Moreover, as for $\frac{3}{4} < \alpha \leq 1$, $u(x,t)$ becomes negative and is therefore non-physical.

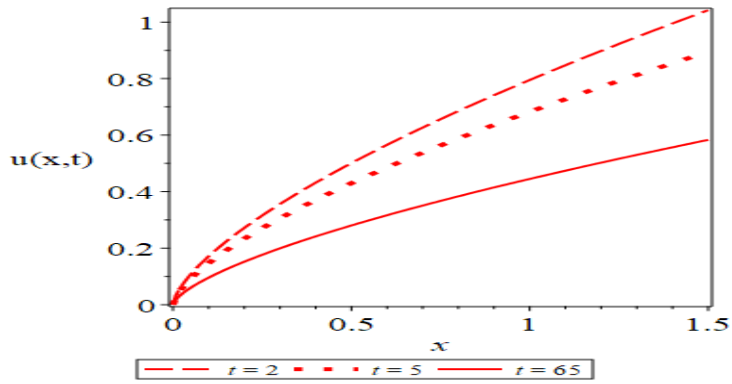


Fig. 2.5: Evolution of the analytical solution of (2.41) is depicted for various time values and $\alpha = \frac{1}{2}$.

2.3.1.2 Case B: $\beta = 0$

We solve the system (2.35) with constraint (2.10) and find that the symmetries of Equation (2.29) are

$$Y_1 = t\partial_t + \alpha u \partial_u \qquad \text{and} \qquad Y_2 = \partial_x.$$

Firstly, suppose we apply Y_1 to reduce the FDE. Then, the invariants are $u(x,t) = w(x)\, t^{\alpha}$, so that Equation (2.29), in a similar process as above, reduces to the nonlinear ODE:

$$w\Gamma(\alpha)\,\alpha - \frac{w}{xw_x} - 1 + ww_{xx}w_x^2 = 0. \tag{2.42}$$

Equation (2.42) admits a single Lie point symmetry $2\,w\partial_w + x\partial_x$, which is used to obtain the invariant solution $w\,(x) = \frac{1}{\Gamma(\alpha)\alpha}$. Consequently, the height of a thin power-law film on a horizontal plane in the presence of gravity is time dependent, viz.

$$u(x,t) = \frac{1}{\Gamma\,(\alpha)\,\alpha}t^{\alpha}. \tag{2.43}$$

This solution is a moving front solution, see Fig. 2.6.

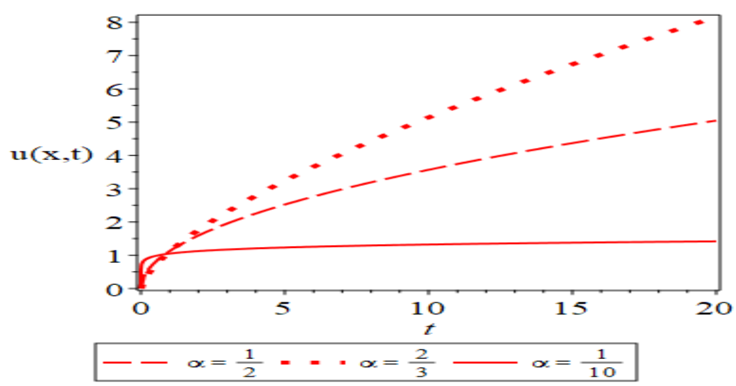


Fig. 2.6: Evolution of the analytical solution of (2.43) is depicted for various fractional values. Clearly, over time, the height of the power-law film is influenced by the size of the fractional parameter.

On the other hand, taking the Lie point symmetry Y_2, we have the invariant,

$$u(x,t) = w(t).$$

Substituting this invariant into (2.29), provides us with a time-fractional ODE:

$$\frac{\mathrm{d}^{\alpha} w(t)}{\mathrm{d}t^{\alpha}} = 0. \tag{2.44}$$

The solution for equation (2.44) is similar to the solution obtained in section 2.3.2 above, which is,

$$u(x,t) = \frac{W_0}{\Gamma\,(\alpha)}t^{\alpha-1}. \tag{2.45}$$

2.3.2 Conclusion

In Section 2.3, we have investigated the time-fractional problem of axisymmetric spreading of a thin incompressible power-law fluid on a horizontal plane. The surface tension effects were ignored. Our main result was to obtain the moving front solutions admitted by the FDE modelling the evolution of the surface of a fluid using Lie group methods. For two cases of the power-law parameter, we used quadrature to determined ODEs which were then further analysed. One of the solutions was determined with the aid of Laplace transforms. We also looked at the graphical behaviour of the solutions. Solutions of the integer equation, namely Equation (2.30), (2.31) represent moving fronts. We have shown that solutions of the fractional-order equation, i.e. (2.41) and (2.43) also represent moving front solutions. For case A, the film solution is negative for $\frac{3}{4} < \alpha \leq 1$ due to the Gamma functions. However for $\alpha \to \frac{3}{4}$, the two solutions (2.41) and (2.31) are similar. Each solution has the identical radial term, but differs with respect to the time component, so that the fractional solution (2.41) produces a greater film height at every time value. In case B, we observe that as the limiting case is approached $\alpha \to 1$, Equation (2.43) is equivalent to Equation (2.30) with $C_1 = C_2 = B = 0$. The second solution, Equation (2.45) tends to a constant as $\alpha \to 1$. The results found may further serve as a guide for computer simulations in fluid mechanics and for comparisons with other numerical data.

Chapter 3

Symmetry of Recurrence Equations

Recently, rational difference equations have become a centre of interest for many authors, see [50, 51, 52]. Many methods have been developed to solve difference equations in closed form, that is, when every solution can be written in terms of the initial values and the indexing variable index n only. Among others, is the Lie symmetry approach used for DEs. The DEs method for difference equations was studied by P. Hydon [3] and others (see [53, 54, 55, 56, 57, 58, 59]). In [3], the author introduced an algorithm for obtaining symmetries and conservation laws of second-order difference equations. Now, it is known that these tools can be used to lower the order, via the invariants of the Lie group of transformations, as it was established for DEs.

In the next section we introduce the definitions and notations that will be used in the succeeding sections when determining the solutions for recurrence equations under study. Sections 3.2, 3.3 and 3.4 of the chapter are devoted to the symmetry analysis on the fifth, fourth and sixth order recurrence equations, respectively. The final section focuses on the symmetry analysis on a general $k+1$ th-order recurrence equation.

3.1 Definitions and Notation

The definitions are taken from Hydon [3], and most of the notation follows from the same book.

Definition 1. *[60] Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \text{for all} \qquad x \in M,$$

and every infinitesimal generator X of G.

Definition 2. *A parameterized set of point transformations,*

$$\Gamma_\epsilon \to \hat{x}\left(x; \epsilon\right), \tag{3.1}$$

where $x = x_i$ and $i = 1, ..., p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. *Γ_0 is the identity map if $\hat{x} = x$ where $\epsilon = 0$*
2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0*
3. *Each $\hat{x}_i$ can be represented as a Taylor series (in a neighbourhood of $\epsilon = 0$ that is determined by x), and therefore*

$$\hat{x}_i\left(x; \epsilon\right) = x_i + \epsilon \xi_i(x) + O\left(\epsilon^2\right), \qquad i = 1, ..., p.$$

Consider the p^{th}-order difference equation

$$u_{n+p} = \Omega\left(n, u_n, ..., u_{n+p+1}\right), \tag{3.2}$$

for some function Ω. Assume the point transformations are of the form

$$\begin{aligned} \hat{n} &= n, \\ \hat{u}_n &= u_n + \epsilon Q\left(n, u_n\right) + O\left(\epsilon^2\right), \end{aligned} \tag{3.3}$$

with the corresponding infinitesimal symmetry generator

$$\begin{aligned} X = \ & Q(n, u_n)\frac{\partial}{\partial u_n} + SQ(n, u_n)\frac{\partial}{\partial u_{n+1}} + ... \\ & + S^{p-1}Q(n, u_n)\frac{\partial}{\partial u_{n+p-1}}, \end{aligned} \tag{3.4}$$

where S is the shift operator, i.e. $S\colon n \to n+1$. The symmetry condition is defined as

$$\hat{u}_{n+p} = \Omega\left(n, \hat{u}_n, \hat{u}_{n+1}, ..., \hat{u}_{n+p-1}\right), \tag{3.5}$$

whenever (3.2) is true. Substituting the Lie point symmetries (3.3) into the symmetry condition (3.5) leads to a linearized symmetry condition

$$S^{(p)}Q - X\Omega = 0, \tag{3.6}$$

whenever (3.2) holds.

One can solve for the characteristic $Q(n, u_n)$ using the method of elimination and thereafter lower the order of the difference equation (3.2) via the canonical coordinate [61]

$$S_n = \int \frac{du_n}{Q(n, u_n)}. \tag{3.7}$$

Definition 3. *V_n is invariant under the Lie group of transformations (3.3) if and only if $XV_n = 0$.*

We define the functions

$$\Theta^n\left(\theta_j, s, l\right) = \prod_{j=s}^{l} \theta_{2j+n}, \qquad n = 0, 1, \tag{3.8}$$

and we adopt the standard conventions

$$\begin{aligned} &\prod_{j=s}^{l} \theta_j = 1, \text{when } s > l \\ &\prod_{j=n}^{l_0} \theta_j = 0, \text{when } n > l_0. \end{aligned} \tag{3.9}$$

We refer the reader to [3] for a deeper understanding of the concept of symmetry analysis of recurrence equations. In general, the constraints on the constants in the characteristics give one a clear idea (without any lucky guess) about a perfect change of variables.

3.2 Symmetry Lie Algebra and Exact Solutions of Some Fourth-order Difference Equations

In this section we extend on the work done by Elsayed [62] where the author investigated the dynamics and solutions of

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}. \tag{3.10}$$

For related work, see [55, 56]. One can notice that (3.10) is special case of a more general form

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1}x_{n-3}} \tag{3.11}$$

where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are arbitrary sequences. we will use a symmetry-based method to solve (3.11). Equivalently, we study:

$$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}}$$

instead, where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are arbitrary sequences. This means we can only compare x_i with u_{i+3}.

3.2.1 Main results

We apply the symmetry condition (3.6) to

$$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}} \tag{3.12}$$

to solve for the characteristic Q. This yields:

$$\begin{aligned} &Q\left(n+4, u_{n+4}\right) + \frac{B_n u_n^2}{\left(B_n u_n u_{n+2} + A_n\right)^2} Q\left(n+2, u_{n+2}\right) - \\ &\frac{A_n}{\left(B_n u_n u_{n+2} + A_n\right)^2} Q\left(n, u_n\right) = 0. \end{aligned} \tag{3.13}$$

In order to get an equation that involves u_n only, we first differentiate implicitly (3.13) with respect to u_n (keeping u_{n+4} fixed). This results in,

$$\frac{\partial}{\partial u_{n+2}} Q\left(n+2, u_{n+2}\right) - \frac{\partial}{\partial u_n} Q\left(n, u_n\right) + \frac{2}{u_n} Q\left(n, u_n\right) = 0. \tag{3.14}$$

Secondly, we differentiate (3.14) with respect to u_n. This yields,

$$\frac{d}{du_n}\left(-\frac{\partial}{\partial u_n} Q(n, u_n) + \frac{2}{u_n} Q(n, u_n)\right) = 0.$$

So,

$$-\frac{\partial}{\partial u_n} Q(n, u_n) + \frac{2}{u_n} Q(n, u_n) = f_2(n).$$

Thus Q takes the form,

$$Q(n, u_n) = f_1(n)u_n^2 + f_2(n)u_n, \tag{3.15}$$

where f_1 and f_2 are functions of n. Finally, we substitute (3.15) into (3.13) and do the separation by powers of shifts of u_n. This yields a system of equations that simplifies to:

$$\begin{cases} f_1(n) = 0, \\ f_2(n) + f_2\,(n+2) = 0 \end{cases} \tag{3.16}$$

Hence,

$$f_1(n) = 0, \qquad f_2(n) = \alpha^n, \qquad f_2(n) = \bar{\alpha}^n,$$

where $\alpha = \exp\{i\pi/2\}$ and $\bar{\alpha}$ is the complex conjugate. Hence, we have two generators given as follows:

$$X_0 = \alpha^n u_n \partial_{u_n} + \alpha^{n+1} u_{n+1} \partial_{u_{n+1}} + \alpha^{n+2} u_{n+2} \partial_{u_{n+2}} + \alpha^{n+3} u_{n+3} \partial_{u_{n+3}},$$

$$X_1 = \bar{\alpha}^n u_n \partial_{u_n} + \bar{\alpha}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\alpha}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\alpha}^{n+3} u_{n+3} \partial_{u_{n+3}}.$$

Let

$$S_n = \int \frac{du_n}{\alpha^n u_n} = \frac{1}{\alpha^n} \ln |u_n|$$

We perform a change of variable, thanks to (3.16):

$$\tilde{V_n} = S_n \alpha^n + S_{n+2} \alpha^{n+2}.$$

We have

$$X_0 \tilde{V_n} = X_1 \tilde{V_n} = 0$$

so that $\tilde{V_n}$ is an invariant of the group of transformations (3.3). Considering the fact that the equation being studied is rational, it is convenient to utilise,

$$|V_n| = \exp\{-\tilde{V_n}\}.$$

Using V_n with (3.12), we obtain

$$V_{n+2} = A_n V_n \pm B_n. \tag{3.17}$$

Now we make the choice of using the plus sign and thus write the solution of (3.17) as:

$$V_{2n+i} = V_i \left(\prod_{k_1=0}^{n-1} A_{2k_1+i} \right) + \sum_{l=0}^{n-1} \left(B_{2l+i} \prod_{k_2=l+1}^{n-1} A_{2k_2+i} \right), \tag{3.18}$$

where $i = 0, 1$. Thus the solution of (3.12) can be obtained by reversing the changes of variables. We have:

$$\begin{aligned}
|u_n| = \exp\left(\alpha_n S_n\right) &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \frac{1}{2}\sum_{k_1=0}^{n-1} \alpha^n \bar{\alpha}^{k_1} \ln|V_{k_1}| + \frac{1}{2}\sum_{k_2=0}^{n-1} \bar{\alpha}^n \alpha^{k_2} \ln|V_{k_2}| \right) \\
&= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \frac{1}{2}\sum_{k=0}^{n-1} \left(\alpha^{n-k} + \bar{\alpha}^{n-k}\right) \ln|V_k| \right) \\
&= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \sum_{k=0}^{n-1} Re\left(\alpha^{n-k}\right) \ln|V_k| \right) \\
&= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \sum_{k=0}^{n-1} \cos\left(\frac{(n-k)\pi}{2}\right) \ln|V_k| \right).
\end{aligned}$$

Setting $n = 4n + j$ and $H_j = \alpha^j c_1 + \bar{\alpha}^j c_2$, we obtain:

$$|u_{4n+j}| = \exp\left(H_j + \sum_{k=0}^{4n+j-1} \left(\cos\left(\frac{(j-k)\pi}{2}\right)\right) \ln|V_k| \right). \tag{3.19}$$

For $j = 0$, (3.19) becomes,

$$|u_{4n}| = \exp\left(H_0\right) \exp\left(\ln|V_0| - \ln|V_2| + \ln|V_4| - \ln|V_6| + \cdots + \ln|V_{4n-4}| - \ln|V_{4n-2}|\right).$$

However, using (3.19) with $j = 0, n = 0$, $|u_0| = \exp(H_0)$. It can be shown that we do not need the absolute values, thus:

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{V_{4s}}{V_{4s+2}}.$$

Using (3.18), where $n := 2s$ and $i = 0$ for V_{4s}, and $n := 2s+1$ and $i = 0$ for V_{4s+2}, we have:

$$\begin{aligned}
V_{4s} &= V_0 \prod_{k_1=0}^{2s-1} A_{2k_1} + \sum_{l=0}^{2s-1} B_{2l} \prod_{k_2=l+1}^{2s-1} A_{2k_2} \\
&= V_0 \left(\prod_{k_1=0}^{2s-1} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{2s-1} B_{2l} \prod_{k_2=l+1}^{2s-1} A_{2k_2} \right) \\
&= \frac{1}{u_0 u_2} \left(\prod_{k_1=0}^{2s-1} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s-1} B_{2l} \prod_{k_2=l+1}^{2s-1} A_{2k_2} \right),
\end{aligned}$$

and,

$$\begin{aligned}
V_{4s+2} &= V_0 \prod_{k_1=0}^{2s} A_{3k_1} + \sum_{l=0}^{2s} B_{3l} \prod_{k_2=l+1}^{2s} A_{3k_2} \\
&= V_0 \left(\prod_{k_1=0}^{2s} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \right) \\
&= \frac{1}{u_0 u_2} \left(\prod_{k_1=0}^{2s} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \right).
\end{aligned}$$

Thus:

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s-1} B_{2l} \prod_{k_2=l+1}^{2s-1} A_{2k_2}}{\prod_{k_1=0}^{2s} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2}},$$

which implies that:

$$x_{4n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{2k_1} + x_{-3}x_{-1} \sum_{l=0}^{2s-1} b_{2l} \prod_{k_2=l+1}^{2s-1} a_{2k_2}}{\prod_{k_1=0}^{2s} a_{2k_1} + x_{-3}x_{-1} \sum_{l=0}^{2s} b_{2l} \prod_{k_2=l+1}^{2s} a_{2k_2}}. \tag{3.20}$$

For $j=1$, (3.19) becomes:

$$|u_{4n+1}| = \exp(H_1)\exp\left(\ln|V_1| - \ln|V_3| + \ln|V_5| - \ln|V_7| + \cdots + \ln|V_{4n-3}| - \ln|V_{4n-1}|\right).$$

However, using (3.19) with $j=1, n=0$, $|u_1| = \exp(H_1)$. It can be shown that we do not need the absolute values, thus:

$$u_{4n+1} = u_1 \prod_{s=0}^{n-1} \frac{V_{4s+1}}{V_{4s+3}}.$$

By (3.18), we have:

$$\begin{aligned} V_{4s+1} &= V_1 \prod_{k_1=0}^{2s-1} A_{2k_1+1} + \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1} \\ &= V_1 \left(\prod_{k_1=0}^{2s-1} A_{2k_1+1} + \frac{1}{V_1} \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1} \right) \\ &= \frac{1}{u_1 u_3} \left(\prod_{k_1=0}^{2s-1} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1} \right), \end{aligned}$$

and similarly,

$$V_{4s+3} = \frac{1}{u_1 u_3} \left(\prod_{k_1=0}^{2s} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s} B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1} \right).$$

Now,

$$u_{4n+1} = u_1 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1}}{\prod_{k_1=0}^{2s} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s} B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}},$$

which implies that:

$$x_{4n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{2k_1+1} + x_{-2}x_0 \sum_{l=0}^{2s-1} b_{2l+1} \prod_{k_2=l+1}^{2s-1} a_{2k_2+1}}{\prod_{k_1=0}^{2s} a_{2k_1+1} + x_{-2}x_0 \sum_{l=0}^{2s} b_{2l+1} \prod_{k_2=l+1}^{2s} a_{2k_2+1}}. \tag{3.21}$$

For $j=2$, we find that (3.19) becomes:

$$|u_{4n+2}| = \exp(H_2)\exp\left(-\ln|V_0| + \ln|V_2| - \ln|V_4| + \cdots + \ln|V_{4n-2}| - \ln|V_{4n}|\right).$$

Similar to the earlier cases, setting $n = 0$ and $j = 2$ yields the equation $|u_2| = \frac{1}{|V_0|}\exp(H_2)$. So we have:

$$u_{4n+2} = u_2 \prod_{s=0}^{n-1} \frac{V_{4s+2}}{V_{4s+4}}.$$

The expressions for V_{4s+2} and V_{4s+4} are obtained from (3.18), by setting $n = 2s+1$, $i = 0$ and $n = 2s+2$, $i = 0$, respectively. They are as follows:

$$\begin{aligned} V_{4s+2} &= V_0 \prod_{k_1=0}^{2s} A_{2k_1} + \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \\ &= V_0 \left(\prod_{k_1=0}^{2s} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \right) \\ &= \frac{1}{u_0 u_2} \left(\prod_{k_1=0}^{2s} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \right), \end{aligned}$$

and

$$\begin{aligned} V_{4s+4} &= V_0 \prod_{k_1=0}^{2s+1} A_{2k_1} + \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2} \\ &= V_0 \left(\prod_{k_1=0}^{2s+1} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2} \right) \\ &= \frac{1}{u_0 u_2} \left(\prod_{k_1=0}^{2s+1} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2} \right), \end{aligned}$$

Hence,

$$u_{4n+2} = u_2 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2}}{\prod_{k_1=0}^{2s+1} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2}},$$

which gives:

$$x_{4n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{2k_1} + x_{-3} x_{-1} \sum_{l=0}^{2s} b_{2l} \prod_{k_2=l+1}^{2s} a_{2k_2}}{\prod_{k_1=0}^{2s+1} a_{2k_1} + x_{-3} x_{-1} \sum_{l=0}^{2s+1} b_{2l} \prod_{k_2=l+1}^{2s+1} a_{2k_2}}. \tag{3.22}$$

For $j=3$, (3.19) becomes:

$$|u_{4n+3}| = \exp\left(H_3\right)\exp\left(-\ln|V_1| + \ln|V_3| - \ln|V_5| + \cdots + \ln|V_{4n-1}| - \ln|V_{4n+1}|\right).$$

Setting $n=0$ and $j=3$, we find that $|u_3| = \frac{1}{|V_1|}\exp\left(H_3\right)$. Hence,

$$u_{4n+3} = u_3 \prod_{s=1}^{n-1} \frac{V_{4s+3}}{V_{4s+5}}.$$

Following a similar approach as was done in the earlier cases ($j=0,1,2$), the reader can verify that

$$u_{4n+3} = u_3 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s} B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}}{\prod_{k_1=0}^{2s+1} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s+1} B_{2l+1} \prod_{k_2=l+1}^{2s+1} A_{2k_2+1}}.$$

Thus,

$$x_{4n} = x_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{2k_1+1} + x_{-2} x_0 \sum_{l=0}^{2s} b_{2l+1} \prod_{k_2=l+1}^{2s} a_{2k_2+1}}{\prod_{k_1=0}^{2s+1} a_{2k_1+1} + x_{-2} x_0 \sum_{l=0}^{2s+1} b_{2l+1} \prod_{k_2=l+1}^{2s+1} a_{2k_2+1}}. \tag{3.23}$$

Therefore, the solution x_n to the equation:

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1} x_{n-3}}$$

satisfies equations (3.20), (3.21), (3.22), and (3.23), as long as the denominators do not vanish.

3.2.1.1 The Case when a_j and b_j are 2-periodic Sequences

We assume that $\{a_j\}_{j\geq 0} = \{a_0, a_1, a_0, a_1, ...\}$ where $a_0 \neq a_1$, and $\{b_j\}_{j\geq 0} = \{b_0, b_1, b_0, b_1, ...\}$ where $b_0 \neq b_1$.

a) $a_0 \neq 1$**,** $a_1 \neq 1$

Then after substitution, (3.20), (3.21), (3.22), and (3.23) become:

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{a_0^{2s} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{a_1^{2s} + x_{-2}x_0b_1\frac{1-a_1^{2s}}{1-a_1}}{a_1^{2s+1} + x_{-2}x_0b_1\frac{1-a_1^{2s+1}}{1-a_1}},$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+2}}{1-a_0}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{a_1^{2s+1} + x_{-2}x_0b_1\frac{1-a_1^{2s+1}}{1-a_1}}{a_1^{2s+2} + x_{-2}x_0b_1\frac{1-a_1^{2s+2}}{1-a_1}},$$

where

$$\prod_{i=1}^{2}\left(\frac{b_0\left(1-a_0^{2s+i}\right)}{1-a_0}x_{-3}x_{-1} + a_0^{2s+i}\right)\left(\frac{b_1\left(1-a_1^{2s+i}\right)}{1-a_1}x_{-2}x_0 + a_1^{2s+i}\right) \neq 0$$

for all $s = 1, 2, ..., n-1$.

3.2.1.2 The Case when a_j and b_j are 1-periodic Sequences

In this case, replace a_1 and b_1, in the above section, by a_0 and b_0, respectively.

a) $a_0 \neq 1$

In this case, the solution equations are given by:

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{a_0^{2s} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{a_0^{2s} + x_{-2}x_0b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1} + x_{-2}x_0b_0\frac{1-a_0^{2s+1}}{1-a_0}},$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+2}}{1-a_0}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{a_0^{2s+1} + x_{-2}x_0b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2} + x_{-2}x_0b_0\frac{1-a_0^{2s+2}}{1-a_0}},$$

where

$$\prod_{i=1}^{2}\left(\frac{b_0\left(1-a_0^{2s+i}\right)}{1-a_0}x_{-3}x_{-1} + a_0^{2s+i}\right)\left(\frac{b_0\left(1-a_0^{2s+i}\right)}{1-a_0}x_{-2}x_0 + a_0^{2s+i}\right) \neq 0$$

for all $s = 1, 2, ..., n-1$.

<u>$a_0 = -1$, $b_0 = 1$</u>

The solution is given by the following equations, and it appears in [62] Theorem 6.

$$x_{4n-3} = x_{-3}\left(-1 + x_{-3}x_{-1}\right)^{-n}, \qquad x_{4n-2} = x_{-2}\left(-1 + x_{-2}x_0\right)^{-n},$$

$$x_{4n-1} = x_{-1}\left(-1 + x_{-3}x_{-1}\right)^{n}, \qquad x_{4n} = x_0\left(-1 + x_{-2}x_0\right)^{n},$$

where $x_{-2}x_0 \neq 1$ and $x_{-3}x_{-1} \neq 1$.

<u>$a_0 = -1$, $b_0 = -1$</u>

The solution is given by the following equations, and it appears in [62] Theorem 9.

$$x_{4n-3} = (-1)^n x_{-3}\left(1 + x_{-3}x_{-1}\right)^{-n}, \qquad x_{4n-2} = (-1)^n x_{-2}\left(1 + x_{-2}x_0\right)^{-n},$$

$$x_{4n-1} = (-1)^n x_{-1}\left(1 + x_{-3}x_{-1}\right)^{n}, \qquad x_{4n} = (-1)^n x_0\left(1 + x_{-2}x_0\right)^{n},$$

where $x_{-2}x_0 \neq -1$ and $x_{-3}x_{-1} \neq -1$.

b) $a_0 = 1$

The solution is given by the following equations:

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1 + 2sb_0x_{-3}x_{-1}}{1 + (2s+1)\,b_0x_{-3}x_{-1}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1 + 2sb_0x_{-2}x_0}{1 + (2s+1)\,b_0x_{-2}x_0},$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,b_0x_{-3}x_{-1}}{1 + (2s+2)\,b_0x_{-3}x_{-1}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1 + (2s+1)\,b_0x_{-2}x_0}{1 + (2s+2)\,b_0x_{-2}x_0},$$

where $jb_0x_{-3}x_{-1} \neq -1$ and $jb_0x_{-2}x_0 \neq -1$ for all $j = 1, 2, ..., 2n$.

$\underline{a_0 = 1,\ b_0 = 1}$

The solution is given by the following equations, and it appears in [62] Theorem 1.

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1+2sx_{-3}x_{-1}}{1+(2s+1)\,x_{-3}x_{-1}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1+2sx_{-2}x_0}{1+(2s+1)\,x_{-2}x_0},$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1+(2s+1)\,x_{-3}x_{-1}}{1+(2s+2)\,x_{-3}x_{-1}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1+(2s+1)\,x_{-2}x_0}{1+(2s+2)\,x_{-2}x_0},$$

where $jx_{-3}x_{-1} \neq -1$ and $jx_{-2}x_0 \neq -1$ for all $j = 1, 2, ..., 2n$.

$\underline{a_0 = 1,\ b_0 = -1}$

The solution is given by the following equations, and it appears in [62] Theorem 4.

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1-2sx_{-3}x_{-1}}{1-(2s+1)\,x_{-3}x_{-1}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1-2sx_{-2}x_0}{1-(2s+1)\,x_{-2}x_0},$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1-(2s+1)\,x_{-3}x_{-1}}{1-(2s+2)\,x_{-3}x_{-1}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1-(2s+1)\,x_{-2}x_0}{1-(2s+2)\,x_{-2}x_0},$$

where $jx_{-3}x_{-1} \neq 1$ and $jx_{-2}x_0 \neq 1$ for all $j = 1, 2, ..., 2n$.

3.2.2 Conclusion

We obtained two non-trivial symmetry generators of shifted equation and used one of these generators to obtain exact solutions to difference equations of the form

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1} x_{n-3}}$$

where (a_n) and (b_n) are arbitrary sequences of real numbers. In particular, our work generalised a result of Elsayed [62].

3.3 On Solutions of Some Fifth-order Recurrence equation

In [63], khalaf Allah studied the asymptotic behaviour of solutions to the rational difference equations:

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}. \tag{3.31}$$

Abo-Zeid discussed global asymptotic stability of all solutions of difference equations of the form:

$$x_{n+1} = \frac{A x_{n-2}}{B + C x_n x_{n-1} x_{n-2}} \tag{3.32}$$

where A, B and C are positive real numbers [64]. Our goal is to give exact solutions of the following difference equations via the invariant of their group of transformations:

$$x_{n+1} = \frac{x_{n-4}}{a_n + b_n x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4}}, \tag{3.33}$$

where (a_n) and (b_n) are arbitrary sequences of real numbers. Without loss of generality, we rather study the equation:

$$u_{n+5} = \frac{u_n}{A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4}}.$$

3.3.1 Main results

Consider the fifth-order difference equation:

$$u_{n+5} = \frac{u_n}{A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4}}. \tag{3.34}$$

To get the symmetry algebra of equation (3.34), we impose the invariance criterion (3.6) to (3.34) to get:

$$
\begin{aligned}
&S^5Q + \frac{B_n u_n^2 u_{n+1} u_{n+2} u_{n+3} S^4 Q}{\left(B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n\right)^2} + \frac{B_n u_n^2 u_{n+1} u_{n+2} u_{n+4} S^3 Q}{\left(B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n\right)^2} + \\
&\frac{B_n u_n^2 u_{n+1} u_{n+3} u_{n+4} S^2 Q}{\left(B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n\right)^2} + \frac{B_n u_n^2 u_{n+2} u_{n+3} u_{n+4} S Q}{\left(B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n\right)^2} - \\
&\frac{A_n Q}{\left(B_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n\right)^2} = 0. \qquad (3.35)
\end{aligned}
$$

To deal away with u_{n+5}, we utilize implicit differentiation with respect to u_n (treating u_{n+1} as a function of u_n, u_{n+2}, u_{n+3}, u_{n+4} and u_{n+5}) by applying the differential operator:

$$L = \frac{\partial}{\partial u_n} - \frac{\omega_{u_n}}{\omega_{u_{n+1}}} \frac{\partial}{\partial u_{n+1}}.$$

Consequently, we have:

$$
\begin{aligned}
&u_n u_{n+2} u_{n+3} S^4 Q + u_n u_{n+2} u_{n+4} S^3 Q + u_n u_{n+3} u_{n+4} S^2 Q + u_n u_{n+2} u_{n+3} u_{n+4} (SQ)' \\
&+ 2u_{n+2} u_{n+3} u_{n+4} Q - u_n u_{n+2} u_{n+3} u_{n+4} Q' = 0. \qquad (3.36)
\end{aligned}
$$

The notation $'$ stands for the derivative with respect to the continuous variable. Differentiating (3.36) with respect to u_n, while u_{n+1}, u_{n+2}, u_{n+3} and u_{n+4} are fixed, results in the differential equation:

$$Q''' \left(n, u_n\right) = 0 \qquad (3.37)$$

whose solution is as follows:

$$Q\left(n, u_n\right) = \alpha(n) u_n^2 + \beta(n) u_n + \gamma(n), \qquad (3.38)$$

for some functions α, β and γ which are arbitrary. Performing a substitution of (3.38) and its shifts in (3.35), and thereafter, replacing the expression of u_{n+5} given in (3.34) in the obtained equation gives a lengthy equation that we decide to omit.

After simplifying the resulting equation and separation with respect to powers of shifts of u_n, we get the following system:

$$
\begin{aligned}
&1 && : A_n^2\gamma(n+5) = 0\\
&u_n && : -A_n\beta(n) + A_n\beta(n+5) = 0\\
&u_n^2 && : -A_n\alpha(n) + \alpha(n+5) = 0\\
&u_n^2 u_{n+1}u_{n+2}u_{n+4} && : B_n\gamma(n+3) = 0\\
&u_n^2 u_{n+1}u_{n+2}u_{n+3} && : B_n\gamma(n+4) = 0\\
&u_n^2 u_{n+2}u_{n+3}u_{n+4} && : B_n\gamma(n+1) = 0\\
&u_n^2 u_{n+1}u_{n+3}u_{n+4} && : B_n\gamma(n+2) = 0\\
&u_n u_{n+1}u_{n+2}u_{n+3}u_{n+4} && : 2A_nB_n\gamma(n+5) = 0\\
&u_n^2 u_{n+1}u_{n+2}^2u_{n+3}u_{n+4} && : B_n\alpha(n+2) = 0\\
&u_n^2 u_{n+1}^2u_{n+2}u_{n+3}u_{n+4} && : B_n\alpha(n+1) = 0\\
&u_n^2 u_{n+1}u_{n+2}u_{n+3}u_{n+4}^2 && : B_n\alpha(n+4) = 0\\
&u_n^2 u_{n+1}u_{n+2}u_{n+3}^2u_{n+4} && : B_n\alpha(n+3) = 0\\
&u_n^2 u_{n+1}^2u_{n+3}^2u_{n+4}^2 && : B_n^2\gamma(n+5) = 0\\
&u_n^2 u_{n+1}u_{n+2}u_{n+3}u_{n+4} && : B_n\big[\beta(n+1) + \beta(n+2) + \beta(n+3) + \beta(n+4)\\
& && \quad + \beta(n+5)\big] = 0
\end{aligned}
$$

which reduces to

$$
\begin{cases}
\alpha(n) = 0,\\
\beta(n) + \beta(n+1) + \beta(n+2) + \beta(n+3) + \beta(n+4) = 0.
\end{cases}
$$

The four independent solutions of the linear fourth-order difference equation above are given by

$$
\theta^n, \quad \bar{\theta}^n, \quad \beta^n \quad \text{and} \quad \bar{\beta}^n, \tag{3.39}
$$

where $\beta = \exp\{2i\pi/5\}$ and $\theta = \exp\{4i\pi/5\}$. Therefore, the difference equation (3.34) admits a four-dimensional symmetry algebra spanned by the Lie point symmetry generators:

$$\begin{aligned} X_1 =& \beta^n u_n \partial_{u_n} + \beta^{n+1} u_{n+1} \partial_{u_{n+1}} + \beta^{n+2} u_{n+2} \partial_{u_{n+2}} + \beta^{n+3} u_{n+3} \partial_{u_{n+3}} \\ &+ \beta^{n+4} u_{n+4} \partial_{u_{n+4}}, \end{aligned} \tag{3.40}$$

$$\begin{aligned} X_2 =& \bar{\beta}^n u_n \partial_{u_n} + \bar{\beta}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\beta}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\beta}^{n+3} u_{n+3} \partial_{u_{n+3}} \\ &+ \bar{\beta}^{n+4} u_{n+4} \partial_{u_{n+4}}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} X_3 =& \theta^n u_n \partial_{u_n} + \theta^{n+1} u_{n+1} \partial_{u_{n+1}} + \beta^{n+2} u_{n+2} \partial_{u_{n+2}} + \theta^{n+3} u_{n+3} \partial_{u_{n+3}} \\ &+ \theta^{n+4} u_{n+4} \partial_{u_{n+4}}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} X_4 =& \bar{\theta}^n u_n \partial_{u_n} + \bar{\theta}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\theta}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\theta}^{n+3} u_{n+3} \partial_{u_{n+3}} \\ &+ \bar{\theta}^{n+4} u_{n+4} \partial_{u_{n+4}}. \end{aligned} \tag{3.43}$$

We use the canonical coordinate,

$$S_n = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n| \tag{3.44}$$

and the relation (3.39) to derive the function,

$$\tilde{V}_n = S_n \beta^n + S_{n+1} \beta^{n+1} + S_{n+2} \beta^{n+2} + S_{n+3} \beta^{n+3} + S_{n+4} \beta^{n+4}. \tag{3.45}$$

We have proven that:

$$X_1 \tilde{V}_n = \beta^n + \beta^{n+1} + \beta^{n+2} + \beta^{n+3} + \beta^{n+4} = 0, \tag{3.46}$$

$$X_2 \tilde{V}_n = \bar{\beta}^n + \bar{\beta}^{n+1} + \bar{\beta}^{n+2} + \bar{\beta}^{n+3} + \bar{\beta}^{n+4} = 0, \tag{3.47}$$

$$X_3\tilde{V}_n = \theta^n + \theta^{n+1} + \theta^{n+2} + \theta^{n+3} + \theta^{n+4} = 0, \tag{3.48}$$

and

$$X_4\tilde{V}_n = \bar{\theta}^n + \bar{\theta}^{n+1} + \bar{\theta}^{n+2} + \bar{\theta}^{n+3} + \bar{\theta}^{n+4} = 0. \tag{3.49}$$

Therefore, $\tilde{V}_n$ is an invariant function. For convenience, we introduce the variable

$$|V_n| = \exp\{-\tilde{V}_n\}. \tag{3.50}$$

We express (3.34) in terms of V_n and find that

$$V_{n+1} = A_n V_n \pm B_n. \tag{3.51}$$

Note. The fifth-order difference equation (3.34) has been reduced to a first-order linear difference equation (3.51).

Using the plus sign, we can write the solution of (3.51) in closed form as:

$$V_j = V_0 \left(\prod_{k_1=0}^{j-1} A_{k_1} \right) + \sum_{l=0}^{j-1} \left(B_l \prod_{k_2=l+1}^{j-1} A_{k_2} \right). \tag{3.52}$$

By combining (3.44), (3.45) and (3.50), we have:

$$u_n = u_{n-5s} \left(\prod_{j=0}^{s-1} \frac{V_{n-5j}}{V_{n-5j+1}} \right). \tag{3.53}$$

Another interesting thing about this method is that the angle obtained while computing the symmetry generators explains how one can further present the solutions. For instance, given that $\beta^5 = 1$, we can write solutions (3.53) in a manner that is popularly considered in literature.

Let $n := 5n + i$ and $s = n$. Then the equation above becomes:

$$u_{5n+i} = u_i \left(\prod_{j=1}^{n} \frac{V_{5j+i}}{V_{5j+i+1}} \right),$$

so that

$$u_{5n+i} = u_i \prod_{j=1}^{n} \frac{V_0 \left(\prod_{k_1=0}^{5j+i-1} A_{k_1}\right) + \sum_{l=0}^{5j+i-1} \left(B_l \prod_{k_2=l+1}^{5j+i-1} A_{k_2}\right)}{V_0 \left(\prod_{k_1=0}^{5j+i} A_{k_1}\right) + \sum_{l=0}^{5j+i} \left(B_l \prod_{k_2=l+1}^{5j+i} A_{k_2}\right)}$$

$$= u_i \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j+i-1} A_{k_1} + \frac{1}{V_0} \sum_{l=0}^{5j+i-1} \left(B_l \prod_{k_2=l+1}^{5j+i-1} A_{k_2}\right)}{\prod_{k_1=0}^{5j+i} A_{k_1} + \frac{1}{V_0} \sum_{l=0}^{5j+i} \left(B_l \prod_{k_2=l+1}^{5j+i} A_{k_2}\right)}.$$

Since

$$V_i = \frac{1}{(u_i u_{i+1} u_{i+2} u_{i+3} u_{i+4})},$$

upon substituting for $\frac{1}{V_0}$, u_{5n+i} becomes:

$$u_{5n+i} = u_i \left(\prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j+i-1} A_{k_1} + u_0 u_1 u_2 u_3 u_4 \sum_{l=0}^{5j+i-1} \left(B_l \prod_{k_2=l+1}^{5j+i-1} A_{k_2}\right)}{\prod_{k_1=0}^{5j+i} A_{k_1} + u_0 u_1 u_2 u_3 u_4 \sum_{l=0}^{5j+i} \left(B_l \prod_{k_2=l+1}^{5j+i} A_{k_2}\right)} \right).$$

Hence, the solution to (3.33) is given by:

$$x_{5n+i-4} = x_{i-4} \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j+i-1} a_{k_1} + x_{-4} x_{-3} x_{-2} x_{-1} x_0 \sum_{l=0}^{5j+i-1} \left(b_l \prod_{k_2=l+1}^{5j+i-1} a_{k_2}\right)}{\prod_{k_1=0}^{5j+i} a_{k_1} + x_{-4} x_{-3} x_{-2} x_{-1} x_0 \sum_{l=0}^{5j+i} \left(b_l \prod_{k_2=l+1}^{5j+i} a_{k_2}\right)}$$

where $i = 0, 1, 2, 3, 4$.

More explicitly, we have the following:

$$x_{5n-4} = x_{-4} \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j-1} a_{k_1} + x_{-4} x_{-3} x_{-2} x_{-1} x_0 \sum_{l=0}^{5j-1} \left(b_l \prod_{k_2=l+1}^{5j-1} a_{k_2}\right)}{\prod_{k_1=0}^{5j} a_{k_1} + x_{-4} x_{-3} x_{-2} x_{-1} x_0 \sum_{l=0}^{5j} \left(b_l \prod_{k_2=l+1}^{5j} a_{k_2}\right)}, \tag{3.54}$$

$$x_{5n-3} = x_{-3} \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j} a_{k_1} + x_{-4} x_{-3} x_{-2} x_{-1} x_0 \sum_{l=0}^{5j} \left(b_l \prod_{k_2=l+1}^{5j} a_{k_2}\right)}{\prod_{k_1=0}^{5j+1} a_{k_1} + x_{-4} x_{-3} x_{-2} x_{-1} x_0 \sum_{l=0}^{5j+1} \left(b_l \prod_{k_2=l+1}^{5j+1} a_{k_2}\right)}, \tag{3.55}$$

$$x_{5n-2} = x_{-2} \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j+1} a_{k_1} + x_{-4}x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{5j+1} \left(b_l \prod_{k_2=l+1}^{5j+1} a_{k_2} \right)}{\prod_{k_1=0}^{5j+2} a_{k_1} + x_{-4}x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{5j+2} \left(b_l \prod_{k_2=l+1}^{5j+2} a_{k_2} \right)}, \tag{3.56}$$

$$x_{5n-1} = x_{-1} \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j+2} a_{k_1} + x_{-4}x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{5j+2} \left(b_l \prod_{k_2=l+1}^{5j+2} a_{k_2} \right)}{\prod_{k_1=0}^{5j+3} a_{k_1} + x_{-4}x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{5j+3} \left(b_l \prod_{k_2=l+1}^{5j+3} a_{k_2} \right)}, \tag{3.57}$$

$$x_{5n} = x_0 \prod_{j=1}^{n} \frac{\prod_{k_1=0}^{5j+3} a_{k_1} + x_{-4}x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{5j+3} \left(b_l \prod_{k_2=l+1}^{5j+3} a_{k_2} \right)}{\prod_{k_1=0}^{5j+4} a_{k_1} + x_{-4}x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{5j+4} \left(b_l \prod_{k_2=l+1}^{5j+4} a_{k_2} \right)}, \tag{3.58}$$

as long as the denominators do not vanish.

3.3.1.1 The Case a_j and b_j are Constant

In this case set $a_j = a$ and $b_j = b$ for all j. In all cases explored below, we set $f = x_{-4}x_{-3}x_{-2}x_{-1}x_0$.

a) The Case $a \neq 1$

$$x_{5n-4} = x_{-4} \prod_{j=1}^{n} \frac{a^{5j} + bf\frac{1-a^{5j}}{1-a}}{a^{5j+1} + bf\frac{1-a^{5j+1}}{1-a}},$$

$$x_{5n-3} = x_{-3} \prod_{j=1}^{n} \frac{a^{5j+1} + bf\frac{1-a^{5j+1}}{1-a}}{a^{5j+2} + bf\frac{1-a^{5j+2}}{1-a}},$$

$$x_{5n-2} = x_{-2} \prod_{j=1}^{n} \frac{a^{5j+2} + bf\frac{1-a^{5j+2}}{1-a}}{a^{5j+3} + bf\frac{1-a^{5j+3}}{1-a}},$$

$$x_{5n-1} = x_{-1} \prod_{j=1}^{n} \frac{a^{5j+3} + bf \frac{1-a^{5j+3}}{1-a}}{a^{5j+4} + bf \frac{1-a^{5j+4}}{1-a}},$$

$$x_{5n} = x_0 \prod_{j=1}^{n} \frac{a^{5j+4} + bf \frac{1-a^{5j+4}}{1-a}}{a^{5j+5} + bf \frac{1-a^{5j+5}}{1-a}},$$

where

$$\prod_{i=1}^{5} \left(a^{5j+i} + bf \frac{1-a^{5j+i}}{1-a} \right) \neq 0$$

for all $j = 1, 2, \ldots, n$.

b) The Case $a = -1$

In this case, we obtain the following:

$$x_{5n-4} = \begin{cases} x_{-4}(-1+bf)^{n-2-2\lfloor \frac{n-1}{2} \rfloor} & \text{if } n \text{ is even,} \\ x_{-4}(-1+bf)^{2\lfloor \frac{n-1}{2} \rfloor - n+2} & \text{if } n \text{ is odd,} \end{cases}$$

$$x_{5n-3} = \begin{cases} x_{-3}(-1+bf)^{2\lfloor \frac{n-1}{2} \rfloor - n+2} & \text{if } n \text{ is even,} \\ x_{-3}(-1+bf)^{n-2-2\lfloor \frac{n-1}{2} \rfloor} & \text{if } n \text{ is odd,} \end{cases}$$

$$x_{5n-2} = \begin{cases} x_{-2}(-1+bf)^{n-2-2\lfloor \frac{n-1}{2} \rfloor} & \text{if } n \text{ is even,} \\ x_{-2}(-1+bf)^{2\lfloor \frac{n-1}{2} \rfloor - n+2} & \text{if } n \text{ is odd,} \end{cases}$$

$$x_{5n-1} = \begin{cases} x_{-1}(-1+bf)^{2\lfloor \frac{n-1}{2} \rfloor - n+2} & \text{if } n \text{ is even,} \\ x_{-1}(-1+bf)^{n-2-2\lfloor \frac{n-1}{2} \rfloor} & \text{if } n \text{ is odd,} \end{cases}$$

$$x_{5n} = \begin{cases} x_0(-1+bf)^{n-2-2\lfloor \frac{n-1}{2} \rfloor} & \text{if } n \text{ is even,} \\ x_0(-1+bf)^{2\lfloor \frac{n-1}{2} \rfloor - n+2} & \text{if } n \text{ is odd,} \end{cases}$$

where $bf \neq 1$.

3.3.1.2 The Case $a = 1$

$$x_{5n-4} = x_{-4} \prod_{j=1}^{n} \frac{1 + (5j)bf}{1 + (5j+1)bf},$$

$$x_{5n-3} = x_{-3} \prod_{j=1}^{n} \frac{1 + (5j+1)bf}{1 + (5j+2)bf},$$

$$x_{5n-2} = x_{-2} \prod_{j=1}^{n} \frac{1 + (5j+2)bf}{1 + (5j+3)bf},$$

$$x_{5n-1} = x_{-1} \prod_{j=1}^{n} \frac{1 + (5j+3)bf}{1 + (5j+4)bf},$$

$$x_{5n} = x_{0} \prod_{j=1}^{n} \frac{1 + (5j+4)bf}{1 + (5j+5)bf}$$

where $(5j + i)bf \neq -1$, $i = 1, \ldots, 5$, for all $j = 1, \ldots, n$.

3.3.2 Conclusion

We obtained four non-trivial symmetries of the difference equations (3.33). Consequently, solutions to the equations were obtained. Our approach in this section involved Lie point symmetry analysis.

3.4 A Group Theory Approach Towards Some Rational Difference Equations

We aim to extend the work by Elsayed [51] where the author investigated the dynamics and solutions of:

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}}, \tag{3.59}$$

where the initial conditions x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary non-zero real numbers. One can notice that equations (3.59) are just special cases of a more general form:

$$x_{n+1} = \frac{x_{n-5}}{a_n + b_n x_{n-1}x_{n-3}x_{n-5}}, \tag{3.60}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences. We will use a symmetry based method to solve (3.60). Equivalently, we study:

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}} \tag{3.61}$$

instead, where $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are arbitrary sequences. This means we can only compare x_i with u_{i+5}.

3.4.1 Main Results

Consider the sixth-order difference equation of the form (3.61), that is:

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4}}.$$

We impose the symmetry condition (3.6) and simplify the resulting equation to get:

$$S^6 Q + \frac{B_n u_n^2 u_{n+2} S^4 Q + B_n u_n^2 u_{n+4} S^2 Q - A_n Q}{\left(B_n u_n u_{n+2} u_{n+4} + A_n\right)^2} = 0. \tag{3.62}$$

To solve for Q, we first differentiate (3.62) with respect to u_n, (keeping Ω fixed and viewing u_{n+2} as a function of u_n, u_{n+4} and Ω). After simplification, this leads to:

$$S^4 Q + u_{n+4}\left(S^2 Q\right)' - u_{n+4} Q' + \frac{2u_{n+4}}{u_n} Q = 0, \tag{3.63}$$

where $'$ denotes the derivative with respect to the independent variable. We then differentiate (3.63) with respect to u_n to get:

$$Q''\left(n, u_n\right) - \frac{2}{u_n} Q'\left(n, u_n\right) + \frac{2}{u_n^2} Q\left(n, u_n\right) = 0. \tag{3.64}$$

The solution of (3.64) is given by:

$$Q\left(n, u_n\right) = \alpha_n u_n^2 + \beta_n u_n, \tag{3.65}$$

for some functions α_n and β_n of n. To obtain more information on α_n and β_n, we substitute (3.65) in (3.62) and split the resulting equation to get the following:

$$u_n u_{n+2} u_{n+4}^2 \quad : B_n \alpha_{n+4} = 0 \tag{3.66a}$$
$$u_n u_{n+2}^2 u_{n+4} \quad : B_n \alpha_{n+2} = 0 \tag{3.66b}$$
$$u_n u_{n+2} u_{n+4} \quad : B_n (\beta_{n+2} + \beta_{n+4} + \beta_{n+6}) = 0 \tag{3.66c}$$
$$u_n \quad : -A_n \alpha_n + \alpha_{n+6} = 0 \tag{3.66d}$$
$$1 \quad : A_n\left(\beta_n - \beta_{n+6}\right) = 0. \tag{3.66e}$$

These equations (3.66) reduce to,

$$0\alpha_n = 0,\ \beta_{n+4} + \beta_{n+2} + \beta_n = 0. \tag{3.67}$$

The expression of β_n in (3.67) is merely obtained by solving the corresponding characteristic equation $r^4 + r^2 + 1 = 0$ for r. Assuming that $r_i,\ i = 1, 2, 3, 4$, are the solutions of this characteristic equation, then β_n is a linear combination of the r_i^n's. In other words, the solutions of (3.67) are:

$$\alpha_n = 0 \qquad \text{and} \qquad \beta_n = (-1)^n \beta^n c_1 + (-1)^n \bar{\beta}^n c_2 + \bar{\beta}^n c_3 + \beta^n c_4 \tag{3.68}$$

for some arbitrary constants c_i, $i = 1, \ldots, 4$, and where $\beta = \exp(\pi i/3)$. Thus, we obtain four characteristics given by:

$$Q_1 = (-1)^n \beta^n u_n, \quad Q_2 = (-1)^n \bar{\beta}^n u_n \quad Q_3 = \bar{\beta}^n u_n, \quad Q_4 = \beta^n u_n. \tag{3.69}$$

The four corresponding symmetry generators X_1, X_2, X_3 and X_4 are given by:

$$\begin{aligned} X_1 =& (-1)^n \beta^n u_n \partial u_n - (-1)^n \beta^{n+1} u_{n+1} \partial_{u_{n+1}} + (-1)^n \beta^{n+2} u_{n+2} \partial_{u_{n+2}} - \\ & (-1)^n \beta^{n+3} u_{n+3} \partial_{u_{n+3}} + (-1)^n \beta^{n+4} u_{n+4} \partial_{u_{n+4}} - (-1)^n \beta^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{3.70a}$$

$$\begin{aligned} X_2 =& (-1)^n \bar{\beta}^n u_n \partial u_n - (-1)^n \bar{\beta}^{n+1} u_{n+1} \partial_{u_{n+1}} + (-1)^n \bar{\beta}^{n+2} u_{n+2} \partial_{u_{n+2}} - \\ & (-1)^n \bar{\beta}^{n+3} u_{n+3} \partial_{u_{n+3}} + (-1)^n \bar{\beta}^{n+4} u_{n+4} \partial_{u_{n+4}} - (-1)^n \bar{\beta}^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{3.70b}$$

$$\begin{aligned} X_3 =& \bar{\beta}^n u_n \partial u_n + \bar{\beta}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\beta}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\beta}^{n+3} u_{n+3} \partial_{u_{n+3}} + \\ & \bar{\beta}^{n+4} u_{n+4} \partial_{u_{n+4}} + \bar{\beta}^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{3.70c}$$

$$\begin{aligned} X_4 =& \beta^n u_n \partial u_n + \beta^{n+1} u_{n+1} \partial_{u_{n+1}} + \beta^{n+2} u_{n+2} \partial_{u_{n+2}} + \beta^{n+3} u_{n+3} \partial_{u_{n+3}} + \\ & \beta^{n+4} u_{n+4} \partial_{u_{n+4}} + \beta^{n+5} u_{n+5} \partial_{u_{n+5}}. \end{aligned} \tag{3.70d}$$

Here, using Q_4, we introduce the canonical coordinate [61]:

$$s_n = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n|. \tag{3.71}$$

In view of (3.67), we introduce the variable:

$$r_n = \beta^{n+4} s_{n+4} + \beta^{n+2} s_{n+2} + \beta^n s_n. \tag{3.72}$$

It is easy to check that,

$$X_i \, r_n = \beta^{n+4} + \beta^{n+2} + \beta^n = 0, \quad i = 1, \ldots, 4. \tag{3.73}$$

Therefore r_n is invariant under X_4. It is advantageous to use,

$$|\tilde{r}_n| = \exp\left(-r_n\right), \tag{3.74}$$

that is, $\tilde{r}_n = \pm 1/(u_n u_{n+2} u_{n+4})$. Here, we choose to use the plus sign and we have shown, using (3.61), that:

$$\tilde{r}_{n+2} = A_n \tilde{r}_n + B_n. \tag{3.75}$$

Therefore,

$$\tilde{r}_{2n+k} = \tilde{r}_k \left(\prod_{k_1=0}^{n-1} A_{2k_1+k}\right) + \sum_{l=0}^{n-1}\left(B_{2l+k}\prod_{k_2=l+1}^{n-1} A_{2k_2+k}\right), \quad k = 0, 1. \tag{3.76}$$

It is worthwhile to mention that equations in (3.76) give the solution of (3.75) for all n. By reversing all the change of variables, we have:

$$\begin{aligned}|u_n| =& \exp\Bigg[(-\beta)^n c_5 + (\beta)^n c_6 + \left(\bar{\beta}\right)^n c_7 + \left(-\bar{\beta}\right)^n c_8 + \\ & (-\beta)^{n+1}\sum_{k_1=0}^{n-1}\frac{-i\sqrt{3}}{6}\left(-\bar{\beta}\right)^{k_1} r_{k_1} + (\beta)^{n+1}\sum_{k_2=0}^{n-1}\frac{i\sqrt{3}}{6}\left(\bar{\beta}\right)^{k_2} r_{k_2} + \\ & (\bar{\beta})^{n+1}\sum_{k_3=0}^{n-1}\frac{-i\sqrt{3}}{6}\beta^{k_3} r_{k_3} + (-\bar{\beta})^{n+1}\sum_{k_4=0}^{n-1}\frac{i\sqrt{3}}{6}(-\beta)^{k_4} r_{k_4}\Bigg] \\ =& \Gamma_n \exp\left(\sum_{k=0}^{n-1}\frac{\sqrt{3}}{3}\left[(-1)^{n+k}+1\right]\mathrm{Im}\left[\gamma(n+1,k)\right]\ln|\tilde{r}_k|\right),\end{aligned} \tag{3.77}$$

where $\tilde{r}_k$ is given in (3.76), $\Gamma_n = \exp\{(-\beta)^n c_5 + \beta^n c_6 + \bar{\beta}^n c_7 + \left(-\bar{\beta}\right)^n c_8\}$ and $\gamma(l,k) = \beta^l \bar{\beta}^k$.

Note. Equation (3.77) gives the solution of (3.61) in a unified manner.

We can simplify (3.77) further by splitting it into six categories. We have:

$$|u_{6n}| = \Gamma_{6n}\exp\left(\sum_{k=0}^{6n-1}\frac{\sqrt{3}}{3}\left[(-1)^{6n+k}+1\right]\mathrm{Im}\left[\gamma(6n+1,k)\right]\ln|\tilde{r}_k|\right) \tag{3.78}$$

$$u_{6n} = u_0\prod_{s=0}^{n-1}\frac{\tilde{r}_{6s}}{\tilde{r}_{6s+2}}. \tag{3.79}$$

Similarly, after a straightforward but lengthy computation, we get:

$$u_{6n+i} = u_i \prod_{s=0}^{n-1} \frac{\tilde{r}_{6s+i}}{\tilde{r}_{6s+2+i}}, i = 0, \dots, 5. \tag{3.80}$$

Note. We can obtain (3.80) using (3.74) which need not the use of absolute values.

Now, using (3.76) and (3.80), we obtain the solutions of (3.61) as follows:

$$u_{6n} = u_0 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s-1} A_{2k_1}\right) + u_0u_2u_4 \sum_{l=0}^{3s-1} \left(B_{2l} \prod_{k_2=l+1}^{3s-1} A_{2k_2}\right)}{\left(\prod_{k_1=0}^{3s} A_{2k_1}\right) + u_0u_2u_4 \sum_{l=0}^{3s} \left(B_{2l} \prod_{k_2=l+1}^{3s} A_{2k_2}\right)}, \tag{3.81a}$$

$$u_{6n+1} = u_1 \frac{\left(\prod_{k_1=0}^{3s-1} A_{2k_1+1}\right) + u_1u_3u_5 \sum_{l=0}^{3s-1} \left(B_{2l+1} \prod_{k_2=l+1}^{3s-1} A_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{3s} A_{2k_1+1}\right) + u_1u_3u_5 \sum_{l=0}^{3s} \left(B_{2l+1} \prod_{k_2=l+1}^{3s} A_{2k_2+1}\right)}, \tag{3.81b}$$

$$u_{6n+2} = u_2 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s} A_{2k_1}\right) + u_0u_2u_4 \sum_{l=0}^{3s} \left(B_{2l} \prod_{k_2=l+1}^{3s} A_{2k_2}\right)}{\left(\prod_{k_1=0}^{3s+1} A_{2k_1}\right) + u_0u_2u_4 \sum_{l=0}^{3s+1} \left(B_{2l} \prod_{k_2=l+1}^{3s+1} A_{2k_2}\right)}, \tag{3.81c}$$

$$u_{6n+3} = u_3 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s} A_{2k_1+1}\right) + u_1u_3u_5 \sum_{l=0}^{3s} \left(B_{2l+1} \prod_{k_2=l+1}^{3s} A_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{3s+1} A_{2k_1+1}\right) + u_1u_3u_5 \sum_{l=0}^{3s+1} \left(B_{2l+1} \prod_{k_2=l+1}^{3s+1} A_{2k_2+1}\right)}, \tag{3.81d}$$

$$u_{6n+4} = u_4 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s+1} A_{2k_1}\right) + u_0u_2u_4 \sum_{l=0}^{3s+1} \left(B_{2l} \prod_{k_2=l+1}^{3s+1} A_{2k_2}\right)}{\left(\prod_{k_1=0}^{3s+2} A_{2k_1}\right) + u_0u_2u_4 \sum_{l=0}^{3s+2} \left(B_{2l} \prod_{k_2=l+1}^{3s+2} A_{2k_2}\right)}, \tag{3.81e}$$

$$u_{6n+5} = u_5 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s+1} A_{2k_1+1}\right) + u_1u_3u_5 \sum_{l=0}^{3s+1} \left(B_{2l+1} \prod_{k_2=l+1}^{3s+1} A_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{3s+2} A_{2k_1+1}\right) + u_1u_3u_5 \sum_{l=0}^{3s+2} \left(B_{2l+1} \prod_{k_2=l+1}^{3s+2} A_{2k_2+1}\right)}, \tag{3.81f}$$

whenever the denominators do not vanish, i.e.,

$$u_0u_2u_4 \sum_{l=0}^{s} B_{2l}\Theta^0(A_{k_2}, l+1, s) \neq -\Theta^0(A_{k_1}, 0, s)$$

and

$$u_1u_3u_5 \sum_{l=0}^{s} B_{2l+1}\Theta^1(A_{k_2}, l+1, s) \neq -\Theta^1(A_{k_1}, 0, s).$$

The implication is that solutions of (3.60) are given by:

$$x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s-1} a_{2k_1}\right) + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{3s-1} \left(b_{2l} \prod_{k_2=l+1}^{3s-1} a_{2k_2}\right)}{\left(\prod_{k_1=0}^{3s} a_{2k_1}\right) + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{3s} \left(b_{2l} \prod_{k_2=l+1}^{3s} a_{2k_2}\right)}, \tag{3.82a}$$

$$x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s-1} a_{2k_1+1}\right) + x_{-4}x_{-2}x_0 \sum_{l=0}^{3s-1} \left(b_{2l+1} \prod_{k_2=l+1}^{3s-1} a_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{3s} a_{2k_1+1}\right) + x_{-4}x_{-2}x_0 \sum_{l=0}^{3s} \left(b_{2l+1} \prod_{k_2=l+1}^{3s} a_{2k_2+1}\right)}, \tag{3.82b}$$

$$x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s} a_{2k_1}\right) + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{3s} \left(b_{2l} \prod_{k_2=l+1}^{3s} a_{2k_2}\right)}{\left(\prod_{k_1=0}^{3s+1} a_{2k_1}\right) + x_{-5}x_{-3}x_{-1} \sum_{l=0}^{3s+1} \left(b_{2l} \prod_{k_2=l+1}^{3s+1} a_{2k_2}\right)}, \tag{3.82c}$$

$$x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{3s} a_{2k_1+1}\right) + x_{-4}x_{-2}x_0 \sum_{l=0}^{3s} \left(b_{2l+1} \prod_{k_2=l+1}^{3s} a_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{3s+1} a_{2k_1+1}\right) + x_{-4}x_{-2}x_0 \sum_{l=0}^{3s+1} \left(b_{2l+1} \prod_{k_2=l+1}^{3s+1} a_{2k_2+1}\right)}, \tag{3.82d}$$

$$x_{6n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{3s+1} a_{2k_1}\right) + x_{-5}x_{-3}x_{-1} \sum\limits_{l=0}^{3s+1} \left(b_{2l} \prod\limits_{k_2=l+1}^{3s+1} a_{2k_2}\right)}{\left(\prod\limits_{k_1=0}^{3s+2} a_{2k_1}\right) + x_{-5}x_{-3}x_{-1} \sum\limits_{l=0}^{3s+2} \left(b_{2l} \prod\limits_{k_2=l+1}^{3s+2} a_{2k_2}\right)}, \tag{3.82e}$$

$$x_{6n} = x_0 \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{3s+1} a_{2k_1+1}\right) + x_{-4}x_{-2}x_0 \sum\limits_{l=0}^{3s+1} \left(b_{2l+1} \prod\limits_{k_2=l+1}^{3s+1} a_{2k_2+1}\right)}{\left(\prod\limits_{k_1=0}^{3s+2} a_{2k_1+1}\right) + x_{-4}x_{-2}x_0 \sum\limits_{l=0}^{3s+2} \left(b_{2l+1} \prod\limits_{k_2=l+1}^{3s+2} a_{2k_2+1}\right)}, \tag{3.82f}$$

whenever the denominators do not vanish, i.e.,

$$x_{-5}x_{-3}x_{-1} \sum_{l=0}^{s} b_{2l}\Theta^0(a_{k_2}, l+1, s) \neq -\Theta^0(a_{k_1}, 0, s)$$

and

$$x_{-4}x_{-2}x_{-0} \sum_{l=0}^{s} b_{2l+1}\Theta^1(a_{k_2}, l+1, s) \neq -\Theta^1(a_{k_1}, 0, s).$$

3.4.1.1 The Case where (a_n) and (b_n) are 2-periodic Sequences

Let $a_n = (\lambda, \mu, \lambda, \mu, \lambda, \dots)$, $b_n = (\eta, \zeta, \eta, \zeta, \dots)$, $\Phi = x_{-5}x_{-3}x_{-1}$ and $\Psi = x_{-4}x_{-2}x_0$. The solution in this case is given by the equations:

$$x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{\lambda^{3s} + \eta\Phi \sum\limits_{l=0}^{3s-1} \lambda^l}{\lambda^{3s+1} + \eta\Phi \sum\limits_{l=0}^{3s} \lambda^l}, \qquad x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{\mu^{3s} + \zeta\Psi \sum\limits_{l=0}^{3s-1} \mu^l}{\mu^{3s+1} + \zeta\Psi \sum\limits_{l=0}^{3s} \mu^l}, \tag{3.83a}$$

$$x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{\lambda^{3s+1} + \eta\Phi \sum\limits_{l=0}^{3s} \lambda^l}{\lambda^{3s+2} + \eta\Phi \sum\limits_{l=0}^{3s+1} \lambda^l}, \qquad x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{\mu^{3s+1} + \zeta\Psi \sum\limits_{l=0}^{3s} \mu^l}{\mu^{3s+2} + \zeta\Psi \sum\limits_{l=0}^{3s+1} \mu^l}, \tag{3.83b}$$

$$x_{6n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{\lambda^{3s+2}+\eta\Phi\sum\limits_{l=0}^{3s+1}\lambda^{l}}{\lambda^{3s+3}+\eta\Phi\sum\limits_{l=0}^{3s+2}\lambda^{l}}, \qquad x_{6n} = x_0\prod_{s=0}^{n-1}\frac{\mu^{3s+2}+\zeta\Psi\sum\limits_{l=0}^{3s+1}\mu^{l}}{\mu^{3s+3}+\zeta\Psi\sum\limits_{l=0}^{3s+2}\mu^{l}}, \tag{3.83c}$$

where $\left(\sum\limits_{l=0}^{s-1}\lambda^l\right)\eta\Phi \neq -\lambda^s$ and $\left(\sum\limits_{l=0}^{s-1}\mu^l\right)\zeta\Psi \neq -\mu^s, \quad s \leq 3n.$

3.4.1.2 The Case where (a_n) and (b_n) are Constants

Here, $a_n = \mu = \lambda$, $b_n = \eta = \zeta$, $\Phi = x_{-5}x_{-3}x_{-1}$ and $\Psi = x_{-4}x_{-2}x_0$. Equation (3.60) becomes:

$$x_{n+1} = \frac{x_{n-5}}{(\lambda + \eta x_{n-1}x_{n-3}x_{n-5})}$$

.

a) The Case where $\lambda = 1$

The solution given in (3.82) simplifies to:

$$x_{6n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{1+(3s)\eta\Phi}{1+(3s+1)\eta\Phi}, \qquad x_{6n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1+(3s)\eta\Psi}{1+(3s+1)\eta\Psi}, \tag{3.84a}$$

$$x_{6n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1+(3s+1)\eta\Phi}{1+(3s+2)\eta\Phi}, \qquad x_{6n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1+(3s+1)\eta\Psi}{1+(3s+2)\eta\Psi}, \tag{3.84b}$$

$$x_{6n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1+(3s+2)\eta\Phi}{1+(3s+3)\eta\Phi}, \qquad x_{6n} = x_0\prod_{s=0}^{n-1}\frac{1+(3s+2)\eta\Psi}{1+(3s+3)\eta\Psi}, \tag{3.84c}$$

where $s\eta\Phi \neq -1$ and $s\eta\Psi \neq -1, \quad s \leq 3n.$

- If $\eta = 1$, then equations in (3.84) are exactly the ones obtained in Theorem 2.1 in [51] for:

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}x_{n-5}} \tag{3.85}$$

and their restriction (the initial conditions are arbitrary nonzero positive real numbers) is a special case of our restriction (the initial conditions are arbitrary nonzero real numbers and $s\Phi \neq -1$, $s\Psi \neq -1$, $0 \leq s \leq 3n$).

- If $\eta = -1$, then equations in (3.84) are exactly the ones obtained in Theorem 3.1 in [51] for:

$$x_{n+1} = \frac{x_{n-5}}{1 - x_{n-1}x_{n-3}x_{n-5}} \tag{3.86}$$

and their restriction (the initial conditions are arbitrary nonzero real numbers and $jbdf \neq 1$, $jace \neq 1$) coincides with our restriction (the initial conditions are arbitrary nonzero real numbers and $s\Phi \neq 1$, $s\Psi \neq 1$, $0 \leq s \leq 3n$).

b) The Case where $\lambda \neq 1$

In this case, the solution given in (3.82) simplifies to:

$$\begin{aligned}
x_{6n-5} &= x_{-5}\prod_{s=0}^{n-1}\frac{\lambda^{3s} + \eta\Phi\left(\frac{1-\lambda^{3s}}{1-\lambda}\right)}{\lambda^{3s+1} + \eta\Phi\left(\frac{1-\lambda^{3s+1}}{1-\lambda}\right)}, &\quad x_{6n-4} &= x_{-4}\prod_{s=0}^{n-1}\frac{\lambda^{3s} + \zeta\Psi\left(\frac{1-\lambda^{3s}}{1-\lambda}\right)}{\lambda^{3s+1} + \zeta\Psi\left(\frac{1-\lambda^{3s+1}}{1-\lambda}\right)},\\
x_{6n-3} &= x_{-3}\prod_{s=0}^{n-1}\frac{\lambda^{3s+1} + \eta\Phi\left(\frac{1-\lambda^{3s+1}}{1-\lambda}\right)}{\lambda^{3s+2} + \eta\Phi\left(\frac{1-\lambda^{3s+2}}{1-\lambda}\right)}, &\quad x_{6n-2} &= x_{-2}\prod_{s=0}^{n-1}\frac{\lambda^{3s+1} + \zeta\Psi\left(\frac{1-\lambda^{3s+1}}{1-\lambda}\right)}{\lambda^{3s+2} + \zeta\Psi\left(\frac{1-\lambda^{3s+2}}{1-\lambda}\right)},\\
x_{6n-1} &= x_{-1}\prod_{s=0}^{n-1}\frac{\lambda^{3s+2} + \eta\Phi\left(\frac{1-\lambda^{3s+2}}{1-\lambda}\right)}{\lambda^{3s+3} + \eta\Phi\left(\frac{1-\lambda^{3s+3}}{1-\lambda}\right)}, &\quad x_{6n} &= x_{0}\prod_{s=0}^{n-1}\frac{\lambda^{3s+2} + \zeta\Psi\left(\frac{1-\lambda^{3s+2}}{1-\lambda}\right)}{\lambda^{3s+3} + \zeta\Psi\left(\frac{1-\lambda^{3s+3}}{1-\lambda}\right)},
\end{aligned} \tag{3.87}$$

where $(1-\lambda^s)\eta\Phi \neq -\lambda^s(1-\lambda)$ and $(1-\lambda^s)\zeta\Psi \neq -\lambda^s(1-\lambda), \quad s \leq 3n.$

Note. If $\lambda = -1$, then the solution given in (3.87) simplifies to:

$$\begin{aligned}
&x_{12n-5} = x_{6(2n)-5} = x_{-5}, \qquad x_{12n-4} = x_{-4}, \qquad x_{12n-3} = x_{-3},\\
&x_{12n-2} = x_{-2}, \qquad x_{12n-1} = x_{-1}, \qquad x_{12n} = x_0,\\
&x_{12n+1} = \frac{x_{-5}}{-1+\eta\Phi}, \qquad x_{12n+2} = \frac{x_{-4}}{-1+\zeta\Psi}, \qquad x_{12n+3} = x_{-3}(-1+\eta\Phi),\\
&x_{12n+4} = x_{-2}(-1+\zeta\Psi), \qquad x_{12n+5} = \frac{x_{-1}}{(-1+\eta\Phi)}, \qquad x_{12n+6} = \frac{x_0}{-1+\zeta\Psi},\\
&x_{12n+7} = x_{6(2n+2)-5} = x_{-5}.
\end{aligned} \tag{3.88}$$

- When setting $\eta = 1$ in (3.88), we get the result obtained in Theorem 4.1 in [51] for:

$$x_{n+1} = \frac{x_{n-5}}{-1+x_{n-1}x_{n-3}x_{n-5}} \tag{3.89}$$

  and their restriction coincides with our restriction (the initial conditions are arbitrary nonzero real numbers, $\Phi \neq 1$ and $\Psi \neq 1$).

- When setting $\eta = -1$ in (3.88), we get the result obtained in Theorem 5.1 in [51] for:

$$x_{n+1} = \frac{x_{n-5}}{-1-x_{n-1}x_{n-3}x_{n-5}} \tag{3.90}$$

  and their restriction coincides with our restriction (the initial conditions are arbitrary nonzero real numbers, $\Phi \neq -1$ and $\Psi \neq -1$).

3.4.2 Conclusion

We have obtained nontrivial symmetries of some rational ordinary difference equations and their exact solutions were obtained. Most importantly, we note that (3.67) gives a clear idea, without making any lucky guesses, for the most convenient choice of the invariant of (3.61).

3.5 On a Class of $k+1$ th-order Difference Equations with Variable Coefficients

In this section, we aim to use the Lie symmetry approach to solve the following difference equations:

$$x_{n+1} = \frac{x_{n-k}}{\beta_n + \gamma_n \prod_{i=0}^{k} x_{n+i}}, \tag{3.91}$$

where β_n and γ_n are real sequences. The definitions and notation introduced in the preceding section will be used. Therefore, we will have to shift the equation k times and study:

$$u_{n+k+1} = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}}, \tag{3.92}$$

instead.

Our work is a natural generalization of the results by Elabbasy, et. al. [65]. These authors used induction methods to give solutions of

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n+i}}, \tag{3.93}$$

where the parameters α, β and γ are non-negative real numbers and the initial values are positive numbers.

3.5.1 Symmetries

Consider the $k+1$ th-order difference equations of the form (3.92), i.e.,

$$u_{n+k+1} = \Omega = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}}.$$

We impose the symmetry condition (3.6) to get:

$$Q(n+k+1, u_{n+k+1}) - \sum_{i=0}^{k} \Omega_{,u_{n+i}} Q(n+i, u_{n+i}) = 0, \tag{3.94}$$

where $\Omega_{,y}$ denotes the partial derivative of Ω with respect to y.

The characteristic in (3.94) takes different arguments and one can eliminate the undesirable variable by implicit differentiation. In this optic, we differentiate (3.94) with respect to u_{n+1} (keeping Ω fixed) and viewing u_{n+2} as a function of $u_n, u_{n+1}, \ldots, u_{n+k}$ and Ω, that is, we act the operator:

$$L = \frac{\partial}{\partial u_{n+1}} + \frac{\partial u_{n+2}}{\partial u_{n+1}} \frac{\partial}{\partial u_{n+2}} = \frac{\partial}{\partial u_{n+1}} - \frac{\Omega_{,u_{n+1}}}{\Omega_{,u_{n+2}}} \frac{\partial}{\partial u_{n+2}} \tag{3.95}$$

on (3.94). This yields,

$$\begin{aligned} &- \Omega_{,u_{n+1}} Q'(n+1, u_{n+1}) + \Omega_{,u_{n+1}} Q'(n+2, u_{n+2}) \\ &- \sum_{i=0}^{k} \left[\Omega_{,u_{n+i}u_{n+1}} - \frac{\Omega_{,u_{n+1}}}{\Omega_{,u_{n+2}}} \Omega_{,u_{n+i}u_{n+2}} \right] Q(n+i, u_{n+i}) = 0 \end{aligned} \tag{3.96}$$

which simplifies to,

$$\begin{aligned} &- u_{n+1}u_{n+2} Q'(n+2, u_{n+2}) + u_{n+1}u_{n+2} Q'(n+1, u_{n+1}) - u_{n+2} Q(n+1, u_{n+1}) \\ &+ u_{n+1} Q(n+2, u_{n+2}) = 0, \end{aligned} \tag{3.97}$$

after a set of rather long calculations. Note that $'$ stands for the derivative with respect to the continuous variable. The differentiation of (3.97) with respect to u_{n+1}

twice (keeping u_{n+2} fixed) leads to:

$$[u_{n+1}Q'(n+1,u_{n+1}) - Q(n+1,u_{n+1})]'' = 0, \tag{3.98}$$

after simplification. The solution of (3.98) is given by,

$$Q\left(n,u_n\right) = a_n u_n + b_n u_n \ln u_n + c_n \tag{3.99}$$

for some functions $a_n,\ b_n$ and c_n of n. These functions are obtained by substituting (3.99) in (3.94) and by splitting the resulting equations with respect to the product of shifts of u_n, since they are functions of n only. It turns out that $b_n = c_n = 0$ and we are left with the following reduced system:

$$1 \quad : \quad a_{n+k+1} - a_n = 0 \tag{3.100a}$$

$$u_n \ldots u_{n+k} \quad : \quad a_{n+1} + a_{n+2} + \cdots + a_{n+k} + a_{n+k+1} = 0, \tag{3.100b}$$

or equivalently,

$$a_n + a_{n+1} + a_{n+2} + \cdots + a_{n+k} = 0. \tag{3.101}$$

We have found that:

$$a_n = \exp\left(\frac{2\pi ns}{k+1}i\right), \quad 1 \le s \le k. \tag{3.102}$$

Thus, the k infinitesimal generators are given by:

$$X_s = \exp\left(\frac{2\pi ns}{k+1}i\right) u_n \frac{\partial}{\partial u_n}, \quad 1 \le s \le k. \tag{3.103}$$

3.5.2 Reduction and Exact Solutions

Let

$$\theta_s = \exp\left(\frac{2\pi s}{k+1}i\right) \quad \text{and} \quad Q_s(n,u_n) = (\theta_s)^n u_n. \tag{3.104}$$

To lower the order of (3.92), we introduce the canonical coordinate defined in (3.7). We have:

$$S_n = \int \frac{du_n}{Q_s(n, u_n)} = \frac{1}{(\theta_s)^n} \ln |u_n|. \tag{3.105}$$

Thanks to (3.101), we have proved that:

$$X_s \left[(\theta_s)^n S_n + (\theta_s)^{n+1} S_{n+1} + \cdots + (\theta_s)^{n+k} S_{n+k} \right] = 0, \quad 1 \leq s \leq k. \tag{3.106}$$

So,

$$r_n = (\theta_s)^n S_n + (\theta_s)^{n+1} S_{n+1} + \cdots + (\theta_s)^{n+k} S_{n+k}, \tag{3.107}$$

is an invariant function of $X_s,\ s = 0, 1, 2, \ldots, k$. For convenience, we consider:

$$|\tilde{r_n}| = \exp\{-r_n\} = \pm \frac{1}{\prod_{i=0}^{k} u_{n+i}}, \tag{3.108}$$

instead. We choose $\tilde{r_n} = 1/\prod_{i=0}^{k} u_{n+i}$ and the reader can readily check that $\tilde{r}_n$ satisfies:

$$\tilde{r}_{n+1} = B_n \tilde{r}_n + A_n, \tag{3.109}$$

and that,

$$\tilde{r}_n = \tilde{r}_0 \left(\prod_{k_1=0}^{n-1} B_{k_1} \right) + \sum_{l=0}^{n-1} \left(A_l \prod_{k_2=l+1}^{n-1} B_{k_2} \right). \tag{3.110}$$

Thanks to (3.108) and (3.92), we have that:

$$u_{n+k+1} = \frac{\tilde{r}_n}{\tilde{r}_{n+1}} u_n, \tag{3.111}$$

and thus,

$$u_{(k+1)n+j} = u_j \prod_{s=0}^{n-1} \frac{\tilde{r}_{(k+1)s+j}}{\tilde{r}_{(k+1)s+j+1}}, \qquad j = 0, 1, \ldots, k. \tag{3.112}$$

We have:

$$
\begin{aligned}
u_{(k+1)n+j} =& u_j \prod_{s=0}^{n-1} \frac{\tilde{r}_0 \left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1} \right) + \sum\limits_{l=0}^{(k+1)s+j-1} \left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2} \right)}{\tilde{r}_0 \left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1} \right) + \sum\limits_{l=0}^{(k+1)s+j} \left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2} \right)} \\
=& u_j \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1} \right) + \frac{1}{\tilde{r}_0} \sum\limits_{l=0}^{(k+1)s+j-1} \left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2} \right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1} \right) + \frac{1}{\tilde{r}_0} \sum\limits_{l=0}^{(k+1)s+j} \left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2} \right)} \\
=& u_j \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1} \right) + \left(\prod\limits_{i=0}^{k} u_i \right) \sum\limits_{l=0}^{(k+1)s+j-1} \left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2} \right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1} \right) + \left(\prod\limits_{i=0}^{k} u_i \right) \sum\limits_{l=0}^{(k+1)s+j} \left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2} \right)}
\end{aligned}
$$

for $j = 0, 1, \ldots, k$. The solution to the sequence $\{x_n\}$ is then given by:

$$
x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} \beta_{k_1} \right) + \mathcal{P} \sum\limits_{l=0}^{(k+1)s+j-1} \left(\gamma_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} \beta_{k_2} \right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} \beta_{k_1} \right) + \mathcal{P} \sum\limits_{l=0}^{(k+1)s+j} \left(\gamma_l \prod\limits_{k_2=l+1}^{(k+1)s+j} \beta_{k_2} \right)}
$$

where $j = 0, 1, 2, \ldots, k$ and $\mathcal{P} = \prod\limits_{i=0}^{k} x_{-i}$. In the subsequent sections, we investigate solutions to special cases of the difference equations.

3.5.3 The Case when β_n and γ_n are 1-periodic

In this case, we assume that $\beta_0 = \beta_j$ for all $j \geq 1$ and $\gamma_0 = \gamma_j$ for all $j \geq 1$.

3.5.3.1 The Case when $\beta_0 \neq 1$

The solution becomes:

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{\beta_0^{(k+1)s+j} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j}}{1-\beta_0}\gamma_0}{\beta_0^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j+1}}{1-\beta_0}\gamma_0}$$

where

$$\beta_0^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j+1}}{1-\beta_0}\gamma_0 \neq 0$$

for all $s = 0, 2, \ldots, n-1$ and all $j = 0, 1, 2, \ldots, k$.

Set $\beta_0 = \gamma_0 = \frac{1}{a}$ where a is a constant. Then the solution reduces to:

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{(a^{-1})^{(k+1)s+j} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-(a^{-1})^{(k+1)s+j}}{1-a^{-1}}a^{-1}}{(a^{-1})^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-(a^{-1})^{(k+1)s+j+1}}{1-a^{-1}}a^{-1}},$$

which is equivalent to,

$$x_{(k+1)n+j-k} = x_{j-k}a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j} a^l},$$

where,

$$1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j} a^l \neq 0$$

for all $s = 0, 1, 2, \ldots, n-1$.

More explicitly, we have:

$$x_{(k+1)n-k} = x_{-k}a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s} a^l},$$

$$x_{(k+1)n+1-k} = x_{1-k}a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+1} a^l},$$

$$x_{(k+1)n+2-k} = x_{2-k}a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+1} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+2} a^l},$$

$$\vdots$$

$$\vdots$$

$$x_{(k+1)n-1} = x_{-1}a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k-2} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k-1} a^l}$$

and

$$x_{(k+1)n} = x_0 a^n \prod_{s=0}^{n-1} \frac{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k-1} a^l}{1+\left(\prod_{i=0}^{k} x_{-i}\right)\sum_{l=0}^{(k+1)s+k} a^l}.$$

This solution has appeared in [65].

3.5.3.2 The Special Case $\beta = -1$ and k is Odd

The solution simplifies to:

$$x_{(k+1)n-k} = x_{-k}(-1+(x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$x_{(k+1)n+1-k} = x_{1-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^n,$$

$$x_{(k+1)n+2-k} = x_{2-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$\vdots$$

$$\vdots$$

$$x_{(k+1)n-1} = x_{-1}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$x_{(k+1)n} = x_{j-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^n,$$

as long as the denominator does not vanish.

However, the solution above can be written in compact form as:

$$x_{(k+1)n+j-k} = x_{j-k}\left(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0\right)^{(-1)^{j+1}n},$$

for $j = 0, 1, \dots, k$ and $\gamma_0 \prod_{i=0}^{k} x_{-i} \neq 1$.

This solution has appeared in [65] (See Theorem 9).

Remark 3.5.1. Note that if $\gamma_0 \prod_{i=0}^{k} x_{-i} = 2$, the solution is periodic with period $k+1$.

3.5.3.3 The Special Case $\beta = -1$ and k is Even

In this case, we have:

$$\begin{aligned}
x_{(k+1)n+j-k} &= x_{j-k} \prod_{s=0}^{n-1} \frac{(-1)^{s+j} + \left(\prod_{i=0}^{k} x_{-i}\right)^{\frac{1-(-1)^{s+j}}{2}} \gamma_0}{(-1)^{s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right)^{\frac{1-(-1)^{s+j+1}}{2}} \gamma_0} \\
&= x_{j-k} \prod_{s\geq 0, s-j \text{ is even}}^{n-1} \frac{1}{-1 + \left(\prod_{i=0}^{k} x_{-i}\right) \gamma_0} \\
&\quad \times \prod_{s\geq 0, s-j \text{ is odd}}^{n-1} \left(-1 + \left(\prod_{i=0}^{k} x_{-i}\right) \gamma_0\right).
\end{aligned}$$

If j is even and n is odd,

$$\begin{aligned}
x_{(k+1)n+j-k} &= x_{j-k} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{-\lfloor \frac{n-1}{2} \rfloor - 1} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{\lfloor \frac{n-1}{2} \rfloor} \\
&= x_{j-k} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{-1}.
\end{aligned}$$

If j is odd and n is odd,

$$\begin{aligned}
x_{(k+1)n+j-k} &= x_{j-k} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{-\lfloor \frac{n-1}{2} \rfloor} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1} \\
&= x_{j-k} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right).
\end{aligned}$$

If j is even and n is even,

$$\begin{aligned}
x_{(k+1)n+j-k} &= x_{j-k} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{-\lfloor \frac{n-1}{2} \rfloor - 1} \left(-1 + \gamma_0 \prod_{i=0}^{k} x_{-i}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1} \\
&= x_{j-k}.
\end{aligned}$$

If j is odd and n is even,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}. \end{aligned}$$

In summary, and more compactly, the solution is

$$x_{(k+1)n+j-k} = \begin{cases} x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{(-1)^{j+1}}, & \text{if } n \text{ is odd} \\ x_{j-k}, & \text{if } n \text{ is even.} \end{cases}$$

where $\gamma_0\prod_{i=0}^{k}x_{-i} \neq 1$. This solution has appeared in [65] (See Theorem 8).

3.5.3.4 The Case when $\beta_0 = 1$

The solution is given by:

$$x_{(k+1)n+j-k} = x_{j-k}\prod_{s=0}^{n-1}\frac{1+\left(\prod_{i=0}^{k}x_{-i}\right)((k+1)s+j)\gamma_0}{1+\left(\prod_{i=0}^{k}x_{-i}\right)((k+1)s+j+1)\gamma_0}$$

where,

$$\left(\prod_{i=0}^{k}x_{-i}\right)((k+1)s+j+1)\gamma_0 \neq -1$$

for all $s = 0, 1, 2, 3, \ldots, n-1$ and for all $j = 0, 1, 2, \ldots, k$.

3.5.4 Conclusion

We utilized symmetry analysis to find point symmetries for certain $(k+1)$ th-order difference equations. We performed the group reduction of the equations using one of these symmetries and solutions were given in a unified manner. Our results generalise those in [65] in the sense that (a) α, β and γ need not necessarily be non-negative integers and (b) the constants can be replaced with sequences (variable constants).

Chapter 4

Conclusion

In this thesis we have applied the symmetry analysis a number of problems that falls into two classes of equations. The nature of the equations under study were time-fractional PDEs and recurrence equations.

In chapter two, we studied PDEs that had a time-fractional component. This work produced two manuscripts of which one was published. In this chapter we investigated a family of fractional-order models that describes population dynamics. Lie symmetries, which formulate a geometric method, were employed to perform group-invariant reductions. In this regard, we determined ODEs which were further analysed. One of the resultant exact solutions was implicit and comprised of an integral solution, while another involved trigonometric functions. We further investigated the time-fractional problem of axisymmetric spreading of a thin incompressible power-law fluid on a horizontal plane. We obtained the moving front solutions admitted by the FDE modelling of the evolution of the surface fluid using Lie group methods. The graphical representation of the solutions are also included.

In chapter three, we obtained nontrivial symmetries and exact solutions of the recurrence equation. We first studied some fourth and fifth order recurrence equation, this was the extension to the work done by Elsayed [62] and Abo-Zeid [64], respectively. However, the method we used is completely different to theirs. We then studied some sixth order recurrence equation where nontrivial symmetries and their exact solutions were obtained. Furthermore, we utilized symmetry analysis to find point symmetries for certain $(k + 1)$ th-order difference equations with variable coefficients. We performed the group reduction of the equations using one of these symmetries and solutions were given in a unified manner. Our results generalise those in [65] in the sense that (a) α, β and γ need not necessarily be non-negative integers and (b) the constants can be replaced with sequences (variable constants).

Bibliography

[1] S. Lie, Klassification und Integration von gewohnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestetten. I, Math. Ann. **22** (1888), 213-253.

[2] E. Noether, Invariante variationsprobleme, Mathematisch- Physikalische Klasse **2** (1918), 235-257.

[3] P.E. Hydon, Difference Equations by Differential Equation Method, Cambridge University Press, UK 2014

[4] S. Jamal, N. Mnguni, Moving front solutions of a time-fractional powerlaw fluid under gravity, Quaestiones Mathematicae, (2020), 1-10.

[5] N. Mnguni, D. Nyirenda, M. Folly-Gbetoula, Symmetry Lie algebra and exact solutions of some fourth-order difference equations, Journal of Nonlinear Sciences and Applications, **11** (2018), 1262-1270.

[6] N. Mnguni, D. Nyirenda and M. Folly-Gbetoula, On Solutions of Some Fifth-order Difference Equations, Far East Journal of Mathematical Sciences, **102** (2017), 3053-3065.

[7] M. Folly-Gbetoula , N. Mnguni, and A.H. Kara, A Group Theory Approach towards Some Rational Difference Equations, Hindawi Journal of Mathematics, **2019** (2019), 1505619.

[8] A.H. Bhrawy and M.A. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation, Nonl. Dyn. **80** (2015), 101-116.

[9] K. Moaddy, S. Momani and I. Hashim, The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics, Comput. Math. Appl. **61(4)** (2011), 1209-1216.

[10] E. Ahmed, A. Hashish and F.A. Rihan, On fractional order cancer model, J. Fract. Calc. Appl. Anal. **3(2)** (2012), 16.

[11] S. Momani and Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A **365** (2007), 345-350.

[12] Z. Odibat and S. Momani, Modifed homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order, Chaos, Sol. Frac. **36(1)** (2006), 167-174.

[13] A.M.A. El-Sayed and M. Gaber, The Adomian decomposition method for solving partial differential equations of fractal order in finite domains, Phys. Lett. A **359(3)** (2006), 175-182.

[14] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Academic press, New York, 1998.

[15] K. Diethelm and G. Walz, Numerical solution of fractional order differential equations by extrapolation, Num. Algor. **16(3)** (1997), 231-253.

[16] L. Galeone and R. Garrappa, On multistep methods for differential equations of fractional order, Medit. J. Math. **3(3)** (2006), 565-580.

[17] G. Wang, R. P. Agarwal and A. Cabada, Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations, Appl. Math. Lett. **25(6)** (2012), 1019-1024.

[18] R.K. Gazizov, A.A. Kasatkin and S.Y. Lukashchuk, Symmetry properties of fractional diffusion equations Phys. Scr. **T136** (2009), 014016.

[19] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Continuous transformation groups of fractional differential equations, Vestn. USATU **9** (2007) 125-135.

[20] G.W. Wang, X.Q. Liu and Y.Y. Zhang, Lie symmetry analysis to the time fractional generalized fifth-order KdV equation. Commun. Nonlinear Sci. Numer. Simul. **18** (2013), 2321-2326.

[21] H.Z. Liu, Complete group classifications and symmetry reductions of the fractional fifth-order KdV types of equations. Stud. Appl. Math. **131(4)** (2013), 317-330.

[22] Q. Huang and R. Zhdanov, Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann- Liouville derivative. Phys. A **409** (2014), 110-118.

[23] N. Dimakis, A. Giacomini, S. Jamal, G. Leon and A. Paliathanasis, Noether symmetries and stability of ideal gas solutions in Galileon cosmology, Phys. Rev. D **95** (2017), 064031.

[24] S. Jamal, nth-Order Approximate Lagrangians Induced by Perturbative Geometries, Math. Phys. Anal. Geom. **21(25)** (2018), 1-9.

[25] S. Jamal,A group theoretical application of SO(4,1) in the de Sitter universe, Gen. Relativ. Grav. **49(88)** (2017), 1-14.

[26] S. Jamal and A. Paliathanasis, Group invariant transformations for the Klein-Gordon equation in three dimensional flat spaces, J. Geom. Phys. **117** (2017), 50-59.

[27] M. Caputo, Linear Models of Dissipation whose Q is almost Frequency IndependentII, Geophys. J. R. Astr. Soc. **13** (1967), 529.

[28] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential equations, Wiley, New York, 1993.

[29] V. Kiryakova, Generalised Fractional Calculus and Applications, Pitman Res. Notes in Math. **301**, (1994).

[30] R. Sahadevan and T. Bakkyaraj, Invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations, J. Math. Anal. Appl. **393** (2012) 341-347.

[31] A. Okubo, Diffusion and Ecological Problems: Mathematical Models, Springer, Berlin, 1980.

[32] I.S. Helland, J.M. Hoff and G. Anderbrant, Attraction of bark beetles (Coleoptera: Scolytidae) to a pheromone trap: experiment and mathematical models, J. Chem. Ecol. **10** (1984), 723-752.

[33] W.S.C. Gurney and R.M. Nisbet, The regulation of inhomogeneous populations, J. Theor. Biol. **52** (1975), 441-457.

[34] E.E. Holmes, M.A. Lewis, J.E. Banks and R.R. Veit, Partial differential equations in ecology: spatial interactions and population dynamics. Ecology **75(1)** (1994), 17-29.

[35] M. E. Gurtin and R. C. MacCamy, On the diffusion of biological populations, Math. Biosci. **33** (1977), 35-49.

[36] Y.G. Lu, Hölder estimates of solutions of biological population equations, Appl. Math. Lett. **13** (2000), 123-126.

[37] A. Okubo, Diffusion and ecological problems: mathematical models, An extended version of the Japanese edition, Ecology and diffusion, Translated by G. N. Parker. Biomathematics, Springer-Verlag, Berlin-New York, 1980.

[38] A.M.A. El-Sayed, S.Z. Rida and A.A.M. Arafa, Exact solutions of fractional-order biological population model, Commun. Theor. Phys. **52** (2009), 992-996.

[39] Y.Q. Liu, Z.L. Li and Y.Y. Zhang, Homotopy perturbation method to fractional biological population equation, Fract. Differ. Calc. **1** (2011), 117-124.

[40] J. Singh, D. Kumar and A. Kiliman, Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological population-s, Abstr. Appl. Anal. **2014** (2014), 1-12.

[41] S. Sarwar, M. A. Zahid and S. Iqbal, Mathematical study of fractional-order biological population model using optimal homotopy asymptotic method, Int. J. Biomath. **9** (2016), 1-17.

[42] O. Acan, M.M Al Qurashi and D. Baleanu, New exact solution of generalized biological population model, J. Nonlinear Sci. Appl. **10** (2017), 3916-3929.

[43] A. Atangana and A. Secer, A note on fractional order derivatives and table of fractional derivatives of some special functions, Abstr. Appl. Anal. **2013** (2013), 279681.

[44] S.I Betelu and M.A. Fontelos, Capillarity driven spreading of power-law fluids, Appl. Math. Lett. **16** (2003), 1315-1320

[45] G. Biswas and A.S. Gupta, Spreading of non-Newtonian fluid drops on a horizontal plane, Mech. Res. Commun. **14**(5/6) (1987), 361-370

[46] H.E. Huppert, The propagation of two-dimensional and axisymmetric viscous currents over a rigid horizontal surface, J. Fluid Mech. **121** (1982), 43-58

[47] S. Middleman, Modeling Axistmmetric Flows, Academic Press, New York, 1995.

[48] S.N. Neossi Nguetchue and E. Momoniat, Axisymmetric spreading of a thin power-law fluid under gravity on a horizontal plane, Nonl. Dyn, **52** (2008), 361-366

[49] S.H. Davis and L.M. Hocking, Spreading and imbiition of viscous liquid on a porous base, Phys. Fluids **11** (1999), 48-57

[50] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = ax_{n-1}/(1+bx_nx_{n-1})$,Appl. Math. Comput. **156** (2004), 587-590.

[51] E.M. El-Sayed, Dynamics of a rational recursive sequence, International journal of Differential Equations **4** (2009), 185-200.

[52] T.F. Ibrahim, On the third order rational difference equation $x_{n+1} = x_nx_{n-2}/x_{n-1}(a+bx_nx_{n-2})$, Int. J. Contemp. Math. Sciences **4:27** (2009), 1321-1334.

[53] P. E. Hydon, Symmetries and first integrals of ordinary difference equations, Proc. Roy. Soc. Lond. A **456** (2000), 2835-2855.

[54] G. R. W. Quispel and R. Sahadevan, Lie symmetries and the integration of difference equations, Phys. Lett. A **184** (1993), 64-70.

[55] M. Folly-Gbetoula, Symmetry, reductions and exact solutions of difference equations $u_{n+2} = \frac{au_n}{1+bu_n u_{n+1}}$, Journal of Difference Equations and Applications, vol. 23 **6** (2017), 1017-1024.

[56] M. Folly-Gbetoula and A. H Kara, Symmetries, conservation laws, and integrability of difference equtions, Advances in Difference Equations, **1** (2014), 224.

[57] D. Levi, L. Vinet and P. Winternitz, Lie group formalism for difference equations, J. Phys. A: Math. Gen. **30** (1997), 633-649.

[58] S. Maeda, The similarity method for difference equations, IMA J. Appl. Math. **38** (1987), 129-134.

[59] D. Nyirenda and M. Folly-Gbetoula, Invariance analysis and exact solutions of some sixth-order difference equations, J. Nonlinear Sci. Appl. **10** (2017) 6262-6273.

[60] P. J. Olver, Applications of Lie Groups to Differential Equations, Second Edition, Springer, New York, 1993.

[61] N. Joshi and P. Vassiliou, The existence of Lie Symmetries for First-Order Analytic Discrete Dynamical Systems, J. Math. Anal. Appl. **195** (1995), 872-887.

[62] E. M. Elsayed, On the Solution of Some Difference Equations, Eur. J. Pure Appl. Math. **3** (2011), 287303.

[63] R. Khalaf-Allah, Asymptotic behavior and periodic nature of two difference equations, Ukr. Math. J. **6** (2009), 988-993.

[64] R. Abo-Zeid, On the oscillation of a third order rational difference equation, Journal of the Egyptian Mathematical Society, **23** (2015), 62-66.

[65] E.M. Elsayed, E.M Elabbasy and H. El-Metwally, On the difference equation $x_{n+1} = \alpha x_{n-k}/(\beta + \gamma \prod_{i=0}^{k} x_{n-i})$, J. Concrete and Applicable Mathematics, **5(2)** (2007), 101-113.