Equivalent Lagrangians and Transformation Maps for Differential Equations

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DECLARATION

I declare that the contents of this dissertation are original except where due references have been made. It is being submitted for the degree of Master of Science at the University of the Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

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Nicole Wilson

This ____________ day of ____________, 2012, at ____________.

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Abstract

The *Method of Equivalent Lagrangians* is used to find the solutions of a given differential equation by exploiting the possible existence of an isomorphic Lie point symmetry algebra and, more particularly, an isomorphic Noether point symmetry algebra. Applications include ordinary differential equations such as the *Kummer Equation* and the *Combined Gravity-Inertial-Rossby Wave Equation* and certain classes of partial differential equations related to the $(1+1)$ linear wave equation. We also make generalisations to the $(2+1)$ and $(3+1)$ linear wave equations.

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Introduction

Lie Theory was first formulated by Sophus Lie (1842 - 1899) as a means of unifying the various integration techniques associated with the solving of differential equations, [14]. This lead to the study of continuous groups of symmetries (transformations that leave a given differential equation invariant, see [1]). However, although this field has its origins in differential equations, Lie Theory's astounding applicability to fields such as differential geometry lead to its rapid "abstraction and globalization,"[14], in particular by Schreier in 1925, who discovered the importance of the topology of Lie groups, a concern that was taken much further by Élie Cartan in the 1930's, [16]. Hence the applicability of Lie Theory to differential equations was largely overlooked for decades until the subject was resurrected by mathematicians such as Birkhoff and Ovsiannikov, [14]. Since then, this aspect of Lie Theory has grown considerably and indeed it is in this arena that this study will make its contribution.

The *Method of Equivalent Lagrangians* is used to find the solutions of a given differential equation by exploiting the possible existence of an isomorphic Lie point symmetry algebra and, more particularly, an isomorphic Noether point symmetry algebra. The underlying idea of the method is to construct a regular point transformation which maps the Lagrangian of a 'simpler' differential equation (with known solutions) to the Lagrangian of the differential equation in question. Once determined, this point transformation will then provide a way of mapping the solutions of the simpler differential equation to the solutions of the equation which we seek to solve. This transformation can also be used to find conserved quantities for the equation in question, if the conserved quantities for the simpler differential equation are known. The advantage of this method is that once a transformation map is found, it can be used to find all possible solutions and conserved quantities of the differential equation in question, thereby avoiding the complex integration procedures which are normally required. In this study, the *Method of Equivalent Lagrangians* is described for scalar second-order linear ordinary differential equations (ODEs) and for certain classes of partial differential equations (PDEs) in two, three and four independent variables.

Previous investigations of this method have been limited. In [6], Kara and Mahomed apply the method to two cases of the equation of the form

$$\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2 q^- 1 + f(t)q^n.$$

This study builds on their work through the generalisation of the method to a larger class of scalar second-order ODEs, by applying the method to two new ODEs. Another new contribution of this study is the extension of the method to certain classes of PDEs in two, three and four independent variables. In [5], Kara uses the definition of equivalent Lagrangians for PDEs in two independent variables in order to find an equivalent Lagrangian under a given transformation. In this study we will describe this procedure in detail as it is the "reverse" of the *Method of Equivalent Lagrangians*. Following this, we will apply the method to the same equivalent Lagrangians in order to recover the known transformations. We shall also generalise the method to some classes of PDEs in three and four independent variables.

In all cases, the aim is to demonstrate the method in a comprehensive manner, which naturally comes at the price of generality. However, the design and layout of the method is such that it may be easily imitated for other classes of differential equations.

The structure of the dissertation is as follows. In the first chapter we state the fundamental concepts that are required to describe and apply the *Method of Equivalent Lagrangians*. We also provide an illustrative example of the method and its application in finding solutions and conserved quantities of differential equations. The second chapter deals with the application of the method to two linear second-order ODEs in mathematical physics, namely the *Kummer Equation* and the *Combined Gravity-Inertial-Rossby Wave Equation*. In the third chapter we extend the method to PDEs in two independent variables. Using the standard Lagrangian and previous knowledge of the $(1+1)$ wave equation, we are able to apply the method to certain PDEs such as the canonical form of the wave equation, the wave equation with dissipation and a Klein-Gordon equation. The fourth chapter discusses the extension of the method to certain classes of PDEs in three and four independent variables, using previous knowledge of the standard Lagrangian for the linear wave equations in three and four independent variables, respectively.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we look at the theoretical foundations on which the *Method of Equivalent Lagrangians* is based. The definitions and concepts stated here are essential to an understanding of this study and will be used throughout the dissertation. They are all well known and can be found in [14].

1.2 Fundamentals

Consider an rth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{1.1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives. The total differentiation operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n. \tag{1.2}$$

A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{1.3}$$

along the solutions of (1.1). T is also known as a *conserved quantity.*

The *Euler-Lagrange operator* is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1}, \ldots, D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{1.4}$$

Hence, the *Euler-Lagrange equations* are of the form

$$\frac{\delta L}{\delta u^\alpha} = 0, \qquad \alpha = 1, \ldots, m, \tag{1.5}$$

where L is a *Lagrangian* of some order. The solutions of equation (1.5) are the optimizers of the functional

$$\int L(x, u, u_{(1)}, \ldots)\; dx. \tag{1.6}$$

A vector field X of the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \xi^i,\ \eta^\alpha \in\ \mathcal{A}, \tag{1.7}$$

which leaves (1.6) invariant (up to, possibly, a gauge vector) is known as a *Noether symmetry*, where $\mathcal{A}$ is the space of differential functions. Equivalently, X is a *Noether symmetry* of L if there is a vector $B = (B^1, \ldots, B^n)\ \in \mathcal{A}$ such that

$$X(L) + LD_i(\xi^i) = D_i(B^i), \tag{1.8}$$

where X is prolonged to the degree of L, [14]. If the vector B is identically zero, then X is a *strict* Noether symmetry, [3].

Two linear algebras (e.g. Lie symmetry or Noether symmetry algebras) are said to be *isomorphic* if they each possess the same structure constants in their respective bases, [2].

It is well known that if the Noether symmetry algebras for two Lagrangians, L and $\overline{L}$, are isomorphic, the Lagrangians can be mapped from one to the other. In the light of this, we define the notion of equivalent Lagrangians.

Definition 1. Let $x = (x^1, x^2, \ldots, x^n)$ be a vector of n independent variables and $u = (u^1, u^2, \ldots, u^m)$ a vector of m dependent variables, where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, $\ldots$, rth-order partial derivatives of u. Let X and U be similarly defined. Two Lagrangians,

$$L = L(x, u, u_{(1)}, \ldots, u_{(r)})$$

and

$$\overline{L} = \overline{L}(X, U, U_{(1)}, \ldots, U_{(r)}),$$

are said to be *equivalent* up to guage if and only if there exists a transformation, $X = X(x, u)$ and $U = U(x, u)$, such that

$$L(x, u, u_{(1)}, \ldots) = \overline{L}(X, U, U_{(1)}, \ldots)\ J(x, u, u_{(1)}) + div\ \tilde{f}, \tag{1.9}$$

where J is the Jacobian, [5], and $\tilde{f} = (f_1, f_2, \ldots, f_n)$ is a guage vector such that $f_i = f_i(x, u)$ for all $i = 1, 2, \ldots, n$.

For ordinary differential equations, in which $u = u(x)$, i.e. x is scalar, the definition of equivalence up to gauge becomes

*Definition 1**. Two Lagrangians, L and $\overline{L}$, are said to be equivalent up to guage if and only if there exists a transformation, $X = X(x,u)$ and $U = U(x,u)$, such that

$$L(x,u,u') = \overline{L}(X,U,U')\frac{dU}{dx} + f_x + u'f_u, \tag{1.10}$$

where the guage function, f, is an arbitrary function of x and u, [6].

For functions of a single variable, we shall use the *prime* notation to denote total differentiation. For example, if $u = u(x)$, then $\frac{du}{dx} = u'$. Similarly, $\frac{d^2u}{dx^2} = u''$, and so on. For functions of more than one variable, partial differentiation will be denoted by subscript notation. For example, if $u = u(x,t)$, then $\frac{\partial u}{\partial x} = u_x$ and $\frac{\partial u}{\partial t} = u_t$. Similarly, $\frac{\partial^2 u}{\partial x^2} = u_{xx}$, and so on.

Remark: The definitions of equivalent Lagrangians imply that given a variational differential equation with corresponding Lagragian L, we can find a regular point transformation $X = X(x,u)$ and $U = U(x,u)$ which maps another (equivalent) Lagrangian $\overline{L}$ to L. This regular point transformation also maps the solutions of the differential equation associated with $\overline{L}$ to the solutions of the original differential equation.

In addition to this, once we have found the regular point transformation $X = X(x,u)$ and $U = U(x,u)$ mentioned above, it is possible to use this transformation to map the (known) conserved quantities of the differential equation associated with $\overline{L}$ to the conserved quantities of the equation in question. This is demonstrated in the following section by means of an illustrative example.

1.3 Illustrative Example

Consider the well-known harmonic oscillator ODE

$$y'' + y = 0, \tag{1.11}$$

with Lagrangian

$$L = \frac{1}{2}y'^{\,2} - \frac{1}{2}y^2. \tag{1.12}$$

We wish to solve and find conserved quantities for the ODE (1.11) using the *Method of Equivalent Lagrangians*. In order to do so, we must find the regular point transformation $X = X(x, y)$ and $Y = Y(x, y)$ that maps the Lagrangian

$$\overline{L} = \frac{1}{2}Y'^{\,2}, \tag{1.13}$$

associated with the free particle ODE,

$$Y'' = 0, \tag{1.14}$$

to the Lagrangian (1.12) associated with the ODE (1.11). This can be done because equations (1.11) and (1.14) are equivalent to each other, and hence can be transformed from one to the other by means of invertible maps, [10]. The corollory of this is that these equations have isomorphic eight-dimensional Lie algebras ([8], [9]) and isomorphic five-dimensional Noether algebras, [11]. Hence L and $\overline{L}$ are equivalent Lagrangians and therefore we can apply Definition 1* in order to find the required transformation map. Once the transformation map has been found, we will be able use it to transform the known solutions and conserved quantities of equation (1.14) to those of equation (1.11).

From Definition 1* we have that

$$\frac{1}{2}y'^{\,2} - \frac{1}{2}y^2 = \frac{1}{2}Y'^{\,2}\frac{dX}{dx} + f_x + y'f_y. \tag{1.15}$$

Now, by the chain rule,

$$Y' = \frac{\frac{dY}{dx}}{\frac{dX}{dx}},$$

and by the total differentiation operator (see (1.2)),

$$\frac{dY}{dx} = \frac{\partial Y}{\partial x} + y'\frac{\partial Y}{\partial y}.$$

Similarly, $\frac{dX}{dx} = \frac{\partial X}{\partial x} + y'\frac{\partial X}{\partial y}$. Hence equation (1.15) becomes

$$\begin{aligned}
\frac{1}{2}y'^{\,2} - \frac{1}{2}y^2 &= \frac{1}{2}\left(\frac{\frac{dY}{dx}}{\frac{dX}{dx}}\right)^2 \frac{dX}{dx} + \frac{df}{dx} \\
&= \frac{1}{2}\frac{\left(\frac{dY}{dx}\right)^2}{\frac{dX}{dx}} + \frac{df}{dx} \\
&= \frac{1}{2}\frac{(Y_x + y'Y_y)^2}{(X_x + y'X_y)} + f_x + y'f_y.
\end{aligned}$$

For the sake of simplicity we shall now assume that X is a function of x only (i.e. $X = X(x)$). Hence $X_y = 0$. From this, the above equation simplifies to

$$\begin{aligned}
\frac{1}{2}y'^{\,2} - \frac{1}{2}y^2 &= \frac{(Y_x + y'Y_y)^2}{2X_x} + f_x + y'f_y \\
&= \frac{1}{2}\frac{Y_x^2}{X_x} + \frac{Y_xY_y}{X_x}y' + \frac{1}{2}\frac{Y_y^2}{X_x}y'^{\,2} + f_x + y'f_y.
\end{aligned}$$

Since X and Y are dependent on x and y only, and not on their derivatives, it follows that we can separate the above equation by powers of y'. From this we get the following system of three PDEs.

$$\begin{aligned}
y'^2 &: \frac{1}{2} = \frac{1}{2}\frac{Y_y^2}{X_x}, \\
y' &: 0 = \frac{Y_xY_y}{X_x} + f_y, \\
1 &: -\frac{1}{2}y'^{\,2} = \frac{1}{2}\frac{Y_x^2}{X_x} + f_x.
\end{aligned}$$

From the first equation we have that

$$Y_y^2 = X_x.$$

Since we have already assumed that $X = X(x)$, we can now make the additional assumption that X is of the form

$$X = \int a^2 \, dx,$$

where a is a function of x. Hence

$$Y_y^2 = a^2(x),$$

which implies that

$$Y_y = a(x),$$

and so

$$Y = \int a(x) \, dy,$$

from which it follows that

$$Y = a(x)y + b(x),$$

where we can assume, without loss of generality, that $b(x) = 0$. Differentiating Y partially with respect to x, we obtain the expression

$$Y_x = a'y.$$

Solving for f_y in the second equation in the system above, and then substituting our newly-found expressions for X_x, Y_x and Y_y, we deduce that

$$f_y = -\frac{a'}{a}y.$$

Integrating the above equation with respect to y gives us the expression

$$f = -\frac{1}{2}\frac{a'}{a}y^2 + c(x),$$

where, without loss of generality, we can take $c(x) = 0$. Now that we have an expression for f, we can differentiate this partially with respect to x and substitute f_x into the third equation in our system. From this it follows that

$$-\frac{1}{2}y^2 = \frac{1}{2}\frac{(a'y)^2}{a} + \frac{1}{2}\left(\frac{a'}{a}\right)' y^2. \tag{1.16}$$

Equating the coefficients of y^2, we arrive at the equation

$$-\frac{1}{2} = \frac{1}{2}\frac{a'^{\,2}}{a} + \frac{1}{2}\left(\frac{a'}{a}\right)'. \tag{1.17}$$

Now we can make the substitution $A = \frac{a'}{a}$, and multiply through by 2, thus simplifying the above equation to

$$-1 = A^2 - A'. \tag{1.18}$$

Rewriting this equation in the form

$$\frac{dA}{dx} = 1 + A^2, \tag{1.19}$$

we realise that it is a separable ODE, the solution of which is

$$\arctan(A) = x + k,$$

where k is an arbitrary constant which we can assume, for our purposes, to be equal to 0. Hence,

$$A = \frac{a'}{a} = \tan x.$$

Integrating both sides of this equation yields the result

$$\ln a = \ln(\sec x) + \ln d,$$

where d is an arbitrary constant. From this we arrive at the solution

$$a = d \sec x. \tag{1.20}$$

Thus, the transformation in question is given by the equations

$$X = d^2 \tan x, \quad Y = dy \sec x, \tag{1.21}$$

and the guage function is given by

$$f = -\frac{1}{2} y^2 \tan x.$$

This transformation in turn maps the solutions of (1.14) to the solutions of (1.11). The solutions of (1.14) are easy to obtain by integrating both sides of the equation twice. The general solution is therefore of the form

$$Y = \alpha X + \beta, \tag{1.22}$$

where α and β are arbitrary constants. By substituting our transformation $X = X(x, y)$ and $Y = Y(x, y)$, given by (1.21), into equation (1.22) and solving for y, we obtain the expression

$$y = \alpha \sin x + \beta \cos x, \tag{1.23}$$

which is the general solution for equation (1.11). In order to verify this, we note that

$$y' = \alpha \cos x - \beta \sin x,$$

which implies that

$$y'' = -\alpha \sin x - \beta \cos x.$$

Hence

$$y'' + y = -\alpha \sin x - \beta \cos x + \alpha \sin x + \beta \cos x = 0, \tag{1.24}$$

and so equation (1.11) is satisfied.

In addition to finding solutions, we can use our transformation to find the conserved quantities of (1.11), by applying our transformation map to the conserved quantities of equation (1.14).

Consider, for example, the known conserved quantity

$$\bar{I} = XY' - Y \tag{1.25}$$

of equation (1.14). Using transformation $X = X(x, y)$ and $Y = Y(x, y)$, given by (1.21), it follows that a conserved quantity for equation (1.11) is

$$\begin{aligned}
I &= X\left(\frac{dY}{dX}\right) - Y \\
&= X\left(\frac{\frac{dY}{dx}}{\frac{dX}{dx}}\right) - Y \\
&= \frac{\tan x \sec x(y' + y \tan x)}{\sec^2 x} - y \sec x \\
&= \sin x(y' + y \tan x) - y \sec x \\
&= y' \sin x + y \sin x \tan x - y \sec x.
\end{aligned}$$

This is verified by the fact that

$$\begin{aligned}
\frac{dI}{dx} &= y'' \sin x + y \sin x \\
&= \sin x(y'' + y) \\
&= 0.
\end{aligned}$$

Hence we have demonstrated how to find transformation maps using the definition of equivalent Lagrangians. Furthermore, once the transformation has been found, we have shown how it can be used to find solutions and conserved quantities of a given differential equation. In the next chapter, the *Method of Equivalent Lagrangians* will be applied to two scalar second-order linear ODEs, namely the *Kummer Equation* and the *Combined Gravity-Inertial-Rossby Wave Equation*.

Chapter 2

The *Kummer* and *Combined Gravity-Inertial-Rossby Wave* ODEs

2.1 Introduction

Second-order ODEs can be divided into equivalence classes based on their Lie symmetries, [11]. Two equations belong to the same equivalence class if there exists a diffeomorphism that transforms one of the equations to the other, [11]. If a second-order ODE admits eight Lie symmetries (the maximum number of Lie symmetries of a scalar second-order ODE, by Lie's 'Counting Theorem,' [7]), it belongs to the equivalence class of the equation $Y'' = 0$, [11]. Hence it can be mapped to this equation by means of a regular point transformation. All linear second-order ODEs belong to the equivalence class of the equation $Y'' = 0$ (see [10] and [15]). It is an important result from the work of Lie ([8], [9]) that such equations possess the maximum eight-dimensional Lie symmetry algebra, [10].

In [11], Mahomed *et al.* prove that the maximum dimension of the Noether symmetry algebra for a scalar second-order ODE is five, and that the equation (1.14) with standard Lagrangian (1.13) attains this maximum. This five-dimensional Noether algebra is therefore isomorphic to any other five-dimensional Noether algebra for which the associated Euler-Lagrange equation is of the same equivalence class as (1.14), [11]. Therefore it follows that for any scalar second-order linear ODE with Lagrangian, L, generating a five-dimensional Noether algebra, $\overline{L}$ can be mapped to L by means of a regular point transformation $X = X(x, y)$ and $Y = Y(x, y)$ (this transformation evidently also transforms the corresponding Euler-Lagrange equations, for L and $\overline{L}$ respectively, from one to the other, [11]).

We use the *Method of Equivalent Lagrangians* detailed above to find the transformation maps that can be used to find solutions and conserved quantities for two scalar second-order linear ODEs, namely the *Kummer Equation* and the *Combined Gravity-Inertial-Rossby Wave Equation*.

2.2 The *Kummer Equation*

The *Kummer Equation* has several applications in theoretical physics. Inter alia, it models the velocity distribution of electrons in a high frequency gas discharge. Using the solutions of this equation, which are called the *Confluent Hypergeometric Functions*, [12], together with kinetic theory, it is thus possible to predict the high frequency breakdown electric field for gases, [12]. The differential equation is given by

$$xy'' + (2k - x)y' - ky = 0, \tag{2.1}$$

where k is an arbitrary constant. By rearranging this equation and multiplying by an integrating factor, we discover that a Lagrangian for this equation is

$$L = \frac{1}{2}x^2ke^{-x}(y' + \frac{k}{x}y^2). \tag{2.2}$$

Equation (2.1) has eight Lie symmetries (this is because it is a second-order linear ODE, see [10]). Therefore there is a point transformation, $X = X(x, y)$ and $Y = Y(x, y)$, which maps equation (1.14), with Lagrangian (1.13), to equation (2.1). The Lagrangian for equation (1.14) is known to have five Noether symmetries, [11]. From the existence of the point transformation above, it follows that the Lagrangian for the *Kummer Equation* (2.1), given by (2.2), also has five Noether symmetries, [10]. Therefore Lagrangians (2.2) and (1.13) are equivalent. Invoking Definition 1* and substituting L and $\overline{L}$ into equation (1.10), we can find the point transformation $X = X(x, y)$ and $Y = Y(x, y)$ that maps (1.13) to (2.2), and hence equation (1.14) to equation (2.1).

Equation (1.10) gives us

$$\begin{aligned}
\frac{1}{2}x^{2k}e^{-x}(y'^{\,2} + \frac{k}{x}y^2) &= \frac{1}{2}Y'^{\,2}\frac{dX}{dx} + f_x + y'f_y \\
&= \frac{1}{2}\left(\frac{dY}{dX}\right)^2\frac{dX}{dx} + f_x + y'f_y \\
&= \frac{1}{2}\left(\frac{\frac{dY}{dx}}{\frac{dX}{dx}}\right)^2\frac{dX}{dx} + f_x + y'f_y \\
&= \frac{1}{2}\left(\frac{Y_x + 2Y_xY_yy' + y'^{\,2}Y_y^2}{X_x + y'X_y}\right) + f_x + y'f_y.
\end{aligned}$$

In order to simplify the above equation, we assume that X is a function of x only and is of the form

$$X = \int a^2 \, dx,$$

where a is a function of x. That is, $X_y = 0$, and the above equation becomes

$$\frac{1}{2}x^{2k}e^{-x}(y'^{\,2} + \frac{k}{x}y^2) = \frac{1}{2}\left(\frac{Y_x^2 + 2Y_xY_yy' + y'^{\,2}Y_y^2}{a^2}\right) + f_x + y'f_y. \tag{2.3}$$

Now, since X and Y are dependent on x and y only, and not on their derivatives, we can separate equation (2.3) by powers of y', after which we obtain a system of three equations,

$$\begin{aligned} y'^{\,2} \quad &: \quad \frac{1}{2}x^{2k}e^{-x} = \frac{1}{2}\left(\frac{Y_y^2}{a^2}\right), \\ y' \quad &: \quad 0 = \frac{1}{2}\left(\frac{2Y_xY_y}{a^2}\right) + f_y, \\ 1 \quad &: \quad \frac{k}{2x}x^{2k}e^{-x}y^2 = \frac{1}{2}\left(\frac{Y_x^2}{a^2}\right) + f_x. \end{aligned}$$

From the first equation we determine that

$$Y_y = ax^ke^{-\frac{1}{2}x}.$$

Integrating with respect to y results in the expression

$$Y = ax^ke^{-\frac{1}{2}x}y + b(x), \tag{2.4}$$

for which we can assume, without loss of generality, that $b(x)$=0. We can differentiate equation (2.4) partially with respect to x, and substitute expressions for Y_x and Y_y (given above) into the second equation in the system, in order to obtain the expression

$$f_y = -x^{2k-1}e^{-x}y\left(\frac{a'}{a}x + k - \frac{1}{2}x\right),$$

from which we conclude that

$$f = -\frac{1}{2}x^{2k-1}e^{-x}y^2\left(\frac{a'}{a}x + k - \frac{1}{2}x\right) + c(x), \tag{2.5}$$

where we again assume, without loss of generality, that $c(x)$=0.

For the third equation, we can substitute our expression for Y_x to obtain

$$xk = \left(\frac{a'}{a}\right)x^2 + 2\left(\frac{a'}{a}\right)x(k - \frac{1}{2}x) + (k - \frac{1}{2}x)^2 + f_x.$$

Making the substitution

$$A = \frac{a'}{a}$$

simplifies the above equation to

$$y^2 e^{-x} x^{2k-1} k = (A^2x^2 + 2Ax(k - \frac{1}{2}x) + (k - \frac{1}{2}x)^2)x^{2k}e^{-x}y^2 + f_x.$$

We then differentiate (2.5) partially with respect to x, and obtain an expression for f_x, which we can substitute into the above equation. This simplifies to

$$0 = A^2 - A' - \frac{k}{x}\left(\frac{k}{x} - 1\right) - \frac{1}{4}. \tag{2.6}$$

Integrating the equation $A = \frac{a'}{a}$ gives us the expression

$$a = Be^{\int A\ dx},$$

where A satisfies equation (2.6) and B is an arbitrary constant not equal to zero. Hence we have that

$$X = \int B^2 e^{2\int A\ dx}\ dx, \tag{2.7}$$

and

$$Y = Be^{\int A\ dx} x^k e^{-\frac{1}{2}x} y. \tag{2.8}$$

Equations (2.7) and (2.8), where A satisfies (2.6), define our regular point transformation $X = X(x, y)$ and $Y = Y(x, y)$, which transforms equation (1.14) to equation (2.1).

We know that the general solution of equation (1.14) is given by

$$Y = \alpha X + \beta,$$

where α and β are arbitrary constants. Therefore, we can substitute expressions (2.7) and (2.8), for X and Y respectively, to obtain an expression for y which is the general solution of the *Kummer Equation* (2.1). The regular point transformation found above can also be used to find the conserved quantities of the *Kummer Equation* in the same way in which we found the conserved quantities for the harmonic oscillator equation in the previous section.

2.3 The *Combined Gravity-Inertial-Rossby Wave Equation*

The *Combined Gravity-Inertial-Rossby Wave Equation* is given by

$$y'' + g(x)y = 0, \tag{2.9}$$

where $g(x)$ is an arbitrary function of x. The derivation of this equation is outlined in [13]. Very briefly, the governing equations for the Combined Gravity-Inertial-Rossby Waves on a β-plane reduce to a partial differential equation, which, with Fourier plane wave analysis, becomes a second-order ODE describing the latitudinal structure of the perturbations. In equation (2.9), x and y are local Cartesian coordinates and $g(x)$ is the wave number, see [13]. By inspection, we find that

$$L = \frac{1}{2}y'^{\,2} - \frac{1}{2}g(x)y^2 \tag{2.10}$$

is a Lagrangian for equation (2.9). As for the *Kummer Equation* and its corresponding Lagrangian, it can be shown that equation (2.9) with Lagrangian (2.10) has an eight-dimensional Lie symmetry algebra (this is because it is a linear second-order ODE for both arbitrary and specific $g(x)$, see [10]) and a five-dimensional Noether algebra (which follows from the equivalence of (2.9) and (1.14), see [10]). Therefore equation

(1.14) can be mapped to this equation using the *Method of Equivalent Lagrangians*. We follow the same procedure as for the *Kummer Equation* in the previous section, with our aim being to find the regular point transformation $X = X(x,y)$ and $Y = Y(x,y)$ that maps equation (1.14) to equation (2.9).

As before, we begin by substituting expressions for L and $\overline{L}$ (given by (2.10) and (1.13) respectively) into equation (1.10). This gives the equation

$$\frac{1}{2}y'^{\,2} - \frac{1}{2}g(x)y^2 = \frac{1}{2}\left(\frac{Y_x + 2Y_xY_yy' + y'^{\,2}Y_y^2}{X_x + y'X_y}\right) + f_x + y'f_y. \tag{2.11}$$

Again we assume that X is of the form

$$X = \int a^2 \; dx,$$

where $a = a(x)$, which simplifies the above equation to

$$\frac{1}{2}y'^{\,2} - \frac{1}{2}g(x)y^2 = \frac{1}{2}\left(\frac{Y_x^2 + 2Y_xY_yy' + y'^{\,2}Y_y^2}{a^2}\right) + f_x + y'f_y. \tag{2.12}$$

Separating by powers of y', we obtain the following system of three equations,

$$\begin{aligned} y'^{\,2} &: \; \frac{1}{2} = \frac{1}{2}\frac{Y_y^2}{a^2}, \\ y' &: \; 0 = \frac{Y_xY_y}{a^2} + f_y, \\ 1 &: \; -\frac{1}{2}g(x)y^2 = \frac{1}{2}\frac{Y_x^2}{a^2} + f_x. \end{aligned}$$

From the first equation we deduce that

$$Y = ay + b(x), \tag{2.13}$$

where we can assume, without loss of generality, that $b(x) = 0$. Substituting expressions for Y_y and Y_x into the second equation in the system, and then integrating with respect to y, we have that

$$f = -\frac{1}{2}\frac{a'}{a}y^2 + c(x), \tag{2.14}$$

for which we again assume, without loss of generality, that $c(x) = 0$. Finally, after substituting expressions for Y_x and f_x into the third equation and making the substitution $A = \frac{a'}{a}$, we obtain the equation

$$-\frac{1}{2}g(x)y^2 = \frac{1}{2}A^2y^2 - \frac{1}{2}A'y^2, \tag{2.15}$$

which simplifies to

$$0 = A' - A^2 - g(x). \tag{2.16}$$

Thus, as before, $a = e^{\int A\ dx}$, where A satisfies equation (2.16). Hence we have that our regular point transformation, $X = X(x, y)$ and $Y = Y(x, y)$, which transforms equation (1.14) to equation (2.9), is given by

$$X = \int e^{2\int A\ dx}\ dx, \qquad Y = e^{\int A\ dx}y,$$

where A satisfies (2.16). This enables us to find the solutions of the *Combined Gravity-Inertial-Rossby Wave Equation* in the same way that we found the solutions of the harmonic oscillator equation in the illustrative example. This regular point transformation can in turn be used to find the conserved quantities of the *Combined Gravity-Inertial-Rossby Wave Equation.*

2.4 Discussion and Conclusion

In this chapter, we have expanded on the work of Kara and Mahomed in [6] by applying the *Method of Equivalent Lagrangians* to the second-order *Kummer Equation* and the *Combined Gravity-Inertial-Rossby Wave Equation.* Because both of these equations possess Lagrangians whose Noether algebras are isomorphic to that of the Lagrangian (1.13), we were able to find transformations that map (1.13) to the Lagrangians of these two equations respectively. Once a transformation is known, it can be used to retrieve all solutions and conserved quantities of the equation in question, since all solutions and conserved quantities for equation (1.14), associated with Lagrangian (1.13), are known. This is extremely useful as it means that the lengthy calculations usually associated with the integration techniques used to solve these equations can be avoided. A limitation of this method is that the equation in question must possess a Lagrangian, which is not always the case. In addition, this Lagrangian must be equivalent to another Lagrangian whose associated ODE has known solutions (and conserved quantities), which in our case was (1.13). If it is not known whether or not two Lagrangians are equivalent, their Noether symmetries must be computed. If their Noether symmetries are of the same dimension, then it follows that their Noether symmetry algebras are isomorphic, [2], and hence the Lagrangians are equivalent. Under these conditions, the *Method of Equivalent Lagrangians* can be applied to other classes of ODEs using the method demonstrated above. In the next chapter we shall turn our attention to equivalent Lagrangians of PDEs in two independent variables.

Chapter 3

The $(1+1)$ Wave Equation and Related PDEs

3.1 Introduction

We now study the application of the method to some classes of PDEs in two independent variables. We first demonstrate, through the use of two illustrative examples, that given a Lagrangian, $\overline{L}$, and a known transformation, one can construct an equivalent Lagrangian, L, using Definition 1 given in the first chapter. Following this, we turn our attention to the construction of a standard form for the Lagrangian equivalent to the usual Lagrangian of the one-dimensional linear wave equation. This will enable us to apply the *Method of Equivalent Lagrangians* to PDEs whose Lagrangians are known to be equivalent to that of the one-dimensional linear wave equation. In this latter situation, the aim of the method is to construct a transformation that maps one Lagrangian, $\overline{L}$, to its equivalent, L. As before, this will enable us to find solutions and conserved quantities of the PDE associated with L, as long as those of the PDE associated with $\overline{L}$ are known. In the final section of this chapter, we apply the *Method of Equivalent Lagrangians* to map the one-dimensional linear wave equation with dissipation to a particular Klein-Gordon equation which we seek to solve. This is the main application of this chapter.

3.2 Illustrative Examples

Example 1. In the first example, we use a given Lagrangian, $\overline{L}$, and a given transformation, $X = X(x,t,u)$, $T = T(x,t,u)$ and $U = U(x,t,u)$, in order to construct an equivalent Lagrangian, L.

Consider the $(1+1)$ linear wave equation with unit wave speed,

$$U_{TT} - U_{XX} = 0. \tag{3.1}$$

In this equation, X and T are independent variables, and U is the dependent variable. In other words, $U = U(X,T)$. Equation (3.1) is known to have the Lagrangian

$$\overline{L} = \frac{1}{2}(U_T^2 - U_X^2), \tag{3.2}$$

see [1]. Suppose we are given the transformation

$$X = t + x, \qquad T = t - x, \qquad U = u,$$

which is the standard transformation to canonical form, [5]. By making the correct substitutions into equation (1.9), we can calculate L.

Firstly, the Jacobian, J, is given by

$$J = \begin{vmatrix} \frac{dX}{dx} & \frac{dT}{dx} \\ \frac{dX}{dt} & \frac{dT}{dt} \end{vmatrix}, \tag{3.3}$$

for two independent variables x and t, [5].

The Lagrangian L is a function of the variables x, t and u, where $u = u(x,t)$. Hence, using our canonical transformation above, we have that

$$J = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

From equation (1.9), assuming zero guage, it follows that

$$L = \frac{1}{2}(U_T^2 - U_X^2)2 \tag{3.4}$$

In order to find U_T and U_X, we use the two equations

$$\frac{dU}{dt} = \frac{\partial T}{\partial t}\frac{\partial U}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial U}{\partial X} \tag{3.5}$$

and

$$\frac{dU}{dx} = \frac{\partial T}{\partial x}\frac{\partial U}{\partial T} + \frac{\partial X}{\partial x}\frac{\partial U}{\partial X}, \tag{3.6}$$

which are derived from the multivariable case of the chain rule, and solve for $\frac{\partial U}{\partial T}$ and $\frac{\partial U}{\partial X}$.

Using our standard transformation to canonical form, (3.5) and (3.6) become

$$u_t = U_T + U_X, \qquad u_x = -U_T + U_X.$$

Solving these simultaneously, we obtain

$$U_T = \frac{1}{2}(u_t - u_x), \qquad U_X = \frac{1}{2}(u_t + u_x).$$

We now substitute these expressions into equation (3.4), from which we have

$$\begin{aligned} L &= \frac{1}{4}(u_t^2 - 2u_x u_t + u_x^2) - \frac{1}{4}(u_t^2 + 2u_x u_t + u_x^2) \\ &= -u_x u_t. \end{aligned}$$

Hence L is equivalent to $\overline{L}$ in the sense of Definition 1. The *Euler-Lagrange Equation* associated with L is

$$u_{xt} = 0, \tag{3.7}$$

which is the canonical form of the wave equation given in (3.1), see [1].

Example 2. In the previous example, we made use of a canonical transformation in order to find a Lagrangian equivalent to $\overline{L}$. In this example, we follow the procedure detailed in [5], in which an equivalent Lagrangian is constructed for the wave equation (3.1), but where the transformed variables are a consequence of the underlying symmetry structure concluded from a generator in the Noether algebra.

It can be verified that

$$\overline{G}(X, T, U) = \frac{\partial}{\partial T} \tag{3.8}$$

is a Noether point symmetry generator for the Lagrangian $\overline{L}$ given by (3.2), [1]. Suppose we wish to map $\overline{G}$ to the dilation symmetry generator

$$G(x, t, u) = x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u}. \tag{3.9}$$

Once this mapping is found, it can be used in formula (1.9) to determine L. The formula for change of variables is given by

$$\overline{G}(X, T, U) = G(X)\frac{\partial}{\partial X} + G(T)\frac{\partial}{\partial T} + G(U)\frac{\partial}{\partial U}. \tag{3.10}$$

Substituting (3.8) into equation (3.10), and equating coefficients, we obtain the three equations

$$G(X) = 0, \qquad G(T) = 1, \qquad G(U) = 0.$$

We solve these equations using the method of invariants, from which we conclude that

$$X = \mathcal{F}(t, \frac{u^2}{x}), \qquad T = \ln\ x + \mathcal{G}(t, \frac{u^2}{x}), \qquad U = \mathcal{H}(t, \frac{u^2}{x}),$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are arbitrary functions. For particular choices of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, we arrive at

$$X = t, \qquad T = \ln\ x, \qquad U = \frac{u^2}{x}.$$

This gives us our transformation. Now, using the expression given in (3.3) for J, we have that

$$J = \begin{vmatrix} 0 & \frac{1}{x} \\ 1 & 0 \end{vmatrix} = -\frac{1}{x}.$$

As in the first example, we find U_T and U_X by solving equations (3.5) and (3.6) simultaneously. From this we obtain

$$U_X = \frac{2uu_t}{x}$$

and

$$U_T = 2uu_x - \frac{u^2}{x}.$$

We now substitute the expressions for J, U_X and U_T into equation (1.9), again assuming zero guage, which gives us

$$\begin{aligned} L &= \frac{1}{2}(U_T^2 - U_X^2).\left(-\frac{1}{x}\right) \\ &= \frac{-1}{2x}\left(4u^2u_x^2 - \frac{4u^3u_x}{x} + \frac{u^4}{x^2} - \frac{4u^2u_t^2}{x^2}\right) \\ &= \frac{-u^2}{2x^3}(4u_x^2x^2 - 4uu_xx + u^2 - 4u_t^2). \end{aligned}$$

Hence we have shown how, for a given PDE in two independent variables, given a Lagrangian, $\overline{L}$, and a transformation (which may be constructed using a symmetry generator in the Noether algebra, as is demonstrated in this example), one can find a Lagrangian L, which is equivalent to $\overline{L}$.

In the applications that follow, we show that if we have two equivalent Lagrangians of PDEs in two independent variables, then we can find the mapping which transforms one Lagrangian to the other. This mapping can be used to find solutions of the PDEs associated with the Lagrangians, as well as to find conserved quantities.

3.3 Form of Equivalent Lagrangian for the Linear Wave Equation in $(1 + 1)$ Dimensions

We now find an expression for the form of a Lagrangian, L, which is equivalent to the usual Lagrangian, $\overline{L}$, of the one-dimensional linear wave equation. Once we have this form, given any L equivalent to $\overline{L}$, we can find the transformation that maps $\overline{L}$ to L, and hence the solutions and conserved quantities of the $(1 + 1)$ linear wave equation to those of the differential equation associated with L.

Solving equations (3.5) and (3.6) simultaneously for U_T and U_X, we obtain

$$U_T = \frac{\frac{dU}{dt}X_x - X_t\frac{dU}{dx}}{T_tX_x - X_tT_x} \tag{3.11}$$

and

$$U_X = \frac{T_t\frac{dU}{dx} - T_x\frac{dU}{dt}}{T_tX_x - X_tT_x}. \tag{3.12}$$

Therefore,

$$\frac{1}{2}(U_T^2 - U_X^2) = \frac{1}{2}\left(\frac{\frac{dU}{dt}^2(X_x^2 - T_x^2) + \frac{dU}{dx}^2(X_t^2 - T_t^2) + 2\frac{dU}{dx}\frac{dU}{dt}(T_tT_x - X_tX_x)}{T_t^2X_x^2 - 2T_tT_xX_tX_x + X_t^2T_x^2}\right),$$

from equation (1.9), assuming that guage is zero. Then

$$J = \begin{vmatrix} \frac{dX}{dx} & \frac{dT}{dx} \\ \frac{dX}{dt} & \frac{dT}{dt} \end{vmatrix}$$

$$= \quad \frac{dX}{dx}\frac{dT}{dt} - \frac{dX}{dt}\frac{dT}{dx}.$$

By the chain rule, $\frac{dX}{dx} = X_x + u_x X_u$, and similarly for $\frac{dT}{dt}$, $\frac{dX}{dt}$ and $\frac{dT}{dx}$. Hence

$$\begin{aligned} J &= (X_x + u_x X_u)(T_t + u_t T_u) - (X_t + u_t X_u)(T_x + u_x T_u) \\ &= X_x(T_t + u_t T_u) - X_t(T_x + u_x T_u) + X_u(u_x T_t - u_t T_x), \end{aligned}$$

and so

$$\begin{aligned} L &= \frac{1}{2}(U_T^2 - U_X^2)J \\ &= \frac{1}{2}\left[\frac{\frac{dU}{dt}^2(X_x^2 - T_x^2) + \frac{dU}{dx}^2(X_t^2 - T_t^2) + 2\frac{dU}{dx}\frac{dU}{dt}(T_t T_x - X_t X_x)}{T_t^2 X_x^2 - 2T_t T_x X_t X_x + X_t^2 T_x^2}\right][X_x(T_t \\ &\quad + u_t T_u) - X_t(T_x + u_x T_u) + X_u(u_x T_t - u_t T_x)]. \end{aligned}$$

If we assume that $X_u = 0$ and $T_u = 0$ (i.e. we assume that $X = X(x,t)$ and $T = T(x,t)$), then the above expression for the Lagrangian reduces to

$$\begin{aligned} L &= \frac{1}{2}\left[\frac{\frac{dU}{dt}^2(X_x^2 - T_x^2) + \frac{dU}{dx}^2(X_t^2 - T_t^2) + 2\frac{dU}{dx}\frac{dU}{dt}(T_t T_x - X_t X_x)}{(X_x T_t - X_t T_x)^2}\right](X_x T_t - X_t T_x) \\ &= \frac{1}{2}\left[\frac{\frac{dU}{dt}^2(X_x^2 - T_x^2) + \frac{dU}{dx}^2(X_t^2 - T_t^2) + 2\frac{dU}{dx}\frac{dU}{dt}(T_t T_x - X_t X_x)}{X_x T_t - X_t T_x}\right]. \end{aligned}$$

Now, by total differentiation (see (1.2)), we have that

$$\frac{dU}{dt} = U_t + u_t U_u$$

and

$$\frac{dU}{dx} = U_x + u_x U_u.$$

Therefore we conclude that L takes the following form.

Form of the Equivalent Lagrangian. The general form for a Lagrangian equivalent to the usual Lagrangian of the one-dimensional linear wave equation is

$$\begin{aligned}
L &= \tfrac{1}{2(X_xT_t - X_tT_x)}[(U_t + u_tU_u)^2(X_x^2 - T_x^2) \\
&+ (U_x + u_xU_u)^2(X_t^2 - T_t^2) + 2(U_x + u_xU_u)(U_t + u_tU_u)(T_tT_x - X_tX_x)] \\
&= \left[\tfrac{1}{2(X_xT_t - X_tT_x)}\right]\{u_xu_t[2U_u^2(T_tT_x - X_tX_x)] \\
&+ u_x^2[U_u^2(X_t^2 - T_t^2)] + u_t^2[U_u^2(X_x^2 - T_x^2)] \\
&+ u_x[2U_uU_x(X_t^2 - T_t^2) + 2U_tU_u(T_tT_x - X_tX_x)] \\
&+ u_t[2U_tU_u(X_x^2 - T_x^2) + 2U_uU_x(T_tT_x - X_tX_x)] \\
&+ [U_t^2(X_x^2 - T_x^2) + U_x^2(X_t^2 - T_t^2) + 2U_tU_x(T_tT_x - X_tX_x)]\},
\end{aligned} \tag{3.13}$$

where $U = U(x,t,u)$, $X = X(x,t)$ and $T = T(x,t)$.

Once we have a Lagrangian which we know to be equivalent to the Lagrangian $\overline{L}$ given by (3.2), we can reverse the process of the two examples above, and use the form of the Lagrangian given in the above equation in order to find the transformation that maps the solutions of the one-dimensional wave linear equation (3.1) to the solutions of the differential equation associated with the equivalent Lagrangian. As a means of demonstrating and validating this method, we use the Lagrangians found in the previous two examples (as we know these to be equivalent to (3.2) by construction) and attempt to recover the required transformations.

3.4 Finding Transformations - Illustrative Examples

Example 1. Consider equation (3.7) with its Lagrangian

$$L = -u_xu_t, \tag{3.14}$$

which we found to be equivalent to (3.2) in Example 1 of the previous section. We use the *Form of the Equivalent Lagrangian* found in the previous section in order to find the transformation that maps the Lagrangian (3.2) to this Lagrangian.

Substituting (3.14) for L in equation (3.13), we obtain the equation

$$\begin{aligned}
-u_x u_t &= \frac{1}{2(X_x T_t - X_t T_x)}[(U_t + u_t U_u)^2 (X_x^2 - T_x^2) \\
&+ (U_x + u_x U_u)^2 (X_t^2 - T_t^2) + 2(U_x + u_x U_u)(U_t + u_t U_u)(T_t T_x - X_t X_x)] \\
&= \left[\frac{1}{2(X_x T_t - X_t T_x)}\right] \{u_x u_t [2U_u^2 (T_t T_x - X_t X_x)] \\
&+ u_x^2 [U_u^2 (X_t^2 - T_t^2)] + u_t^2 [U_u^2 (X_x^2 - T_x^2)] \\
&+ u_x [2U_u U_x (X_t^2 - T_t^2) + 2U_t U_u (T_t T_x - X_t X_x)] \\
&+ u_t [2U_t U_u (X_x^2 - T_x^2) + 2U_u U_x (T_t T_x - X_t X_x)] \\
&+ [U_t^2 (X_x^2 - T_x^2) + U_x^2 (X_t^2 - T_t^2) + 2U_t U_x (T_t T_x - X_t X_x)]\}.
\end{aligned}$$

Since the transformation X, T and U is dependent on x, t and u alone, and not on their derivatives, it follows that we can separate the above equation into six different equations by the derivative terms of u. We arrive at the equations

$$\begin{aligned}
u_t u_x \ &: \ -1 = \frac{2U_u^2 (T_t T_x - X_t X_x)}{2(X_x T_t - X_t T_x)}, \\
u_x^2 \ &: \ 0 = \frac{U_u^2 (X_t^2 - T_t^2)}{2(X_x T_t - X_t T_x)}, \\
u_t^2 \ &: \ 0 = \frac{U_u^2 (X_x^2 - T_x^2)}{2(X_x T_t - X_t T_x)}, \\
u_x \ &: \ 0 = \frac{2U_u U_x (X_t^2 - T_t^2) + 2U_t U_u (T_t T_x - X_t X_x)}{2(X_x T_t - X_t T_x)}, \\
u_t \ &: \ 0 = \frac{2U_t U_u (X_x^2 - T_x^2) + 2U_u U_x (T_t T_x - X_t X_x)}{2(X_x T_t - X_t T_x)}, \\
1 \ &: \ 0 = \frac{U_t^2 (X_x^2 - T_x^2) + U_x^2 (X_t^2 - T_t^2) + 2U_t U_x (T_t T_x - X_t X_x)}{2(X_x T_t - X_t T_x)}.
\end{aligned}$$

Using the mathematical software *Mathematica* (see [17]) to solve this over-determined system of equations, we recover the transformation

$$X = f(x) + g(t), \qquad T = f(x) - g(t), \qquad U = u,$$

for f and g arbitrary functions of x and t respectively.

A particular case of the transformation is

$$X = t + x, \qquad T = t - x, \qquad U = u.$$

This is the standard transformation to canonical form which we know, from Section 3.2, maps $\overline{L}$, given by (3.2), to L, given by (3.14).

Example 2. In order to further test this method, we use the Lagrangian

$$L = \frac{-u^2}{2x^3}(4u_x^2x^2 - 4uu_xx + u^2 - 4u_t^2), \tag{3.15}$$

which was constructed to be equivalent to (3.2) in Example 2 of the previous section. We wish to use the *Form of the Equivalent Lagrangian*, given by equation (3.13), in order to recover the transformation used in the construction of (3.15). This transformation maps (3.2) to (3.15) and hence maps the associated differential equations from one to the other. Thus we can use this transformation to find solutions of the differential equation associated with (3.15), if the solutions of (3.1) are known.

Using the *Form of the Equivalent Lagrangian*, we substitute (3.15) for L in equation (3.13), which gives us the equation

$$\begin{aligned}
\frac{-u^2}{2x^3}(4u_x^2x^2 - 4uu_xx + u^2 - 4u_t^2) &= \left[\frac{1}{2(X_xT_t - X_tT_x)}\right]\{u_xu_t[2U_u^2(T_tT_x - X_tX_x)] \\
&+ u_x^2[U_u^2(X_t^2 - T_t^2)] + u_t^2[U_u^2(X_x^2 - T_x^2)] \\
&+ u_x[2U_uU_x(X_t^2 - T_t^2) + 2U_tU_u(T_tT_x - X_tX_x)] \\
&+ u_t[2U_tU_u(X_x^2 - T_x^2) + 2U_uU_x(T_tT_x - X_tX_x)] \\
&+ [U_t^2(X_x^2 - T_x^2) + U_x^2(X_t^2 - T_t^2) + 2U_tU_x(T_tT_x - X_tX_x)]\}.
\end{aligned}$$

As before, we can separate the above equation into six different equations by separating by the derivative terms of u. We can do this because X, T and U are dependent on x, t and u only, and not on their derivatives (this will henceforth be referred to as *point dependency*). We arrive at the six equations

$$\begin{aligned}
u_t u_x &: \quad 0 = \frac{2U_u^2(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\
u_x^2 &: \quad \frac{-2u^2}{x} = \frac{U_u^2(X_t^2 - T_t^2)}{2(X_xT_t - X_tT_x)}, \\
u_t^2 &: \quad \frac{2u^2}{x^3} = \frac{U_u^2(X_x^2 - T_x^2)}{2(X_xT_t - X_tT_x)}, \\
u_x &: \quad \frac{2u^3}{x^2} = \frac{2U_uU_x(X_t^2 - T_t^2) + 2U_tU_u(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\
u_t &: \quad 0 = \frac{2U_tU_u(X_x^2 - T_x^2) + 2U_uU_x(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\
1 &: \quad \frac{-u^4}{2x^3} = \frac{U_t^2(X_x^2 - T_x^2) + U_x^2(X_t^2 - T_t^2) + 2U_tU_x(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}.
\end{aligned}$$

Using software (see [17]) to solve these equations simultaneously, we discover that one solution is the transformation

$$\begin{aligned}
X &= f(t + \ln\ x) - g(t - \ln\ x), \\
T &= f(t + \ln\ x) + g(t - \ln\ x), \\
U &= \pm\frac{u^2}{x},
\end{aligned}$$

where f and g are arbitrary functions of x and t. If we choose f such that $f(t + \ln\ x) = \frac{1}{2}(t + \ln\ x)$ and g such that $g(t - \ln\ x) = -\frac{1}{2}(t - \ln\ x)$, we have the same transformation as the one used in Example 2 of the previous section that resulted in the construction of the Lagrangian (3.15). Thus the *Method of Equivalent Lagrangians* is validated for this example. In the next section, we apply the method to a particular Klein-Gordon equation, to which we recover a transformation map from the one-dimensional linear wave equation with dissipation.

3.5 Main Application

The equation

$$U_{TT} + U_T - U_{XX} = 0 \tag{3.16}$$

is the one-dimensional case of the linear wave equation with dissipation, [1]. It has the well-known Lagrangian, after multiplication by a variational factor,

$$\overline{L} = \frac{1}{2}e^T(U_T^2 - U_X^2). \tag{3.17}$$

We map $\overline{L}$ to the Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}uu_t + \frac{1}{8}u^2, \tag{3.18}$$

giving rise to the Euler equation

$$u_{tt} - u_{xx} - \frac{1}{4}u = 0, \tag{3.19}$$

which we note to be a Klein-Gordon equation, see [2]. The Noether symmetries of L are of the form $X = \xi\partial_x + \tau\partial_t + \phi\partial_u$, with gauge (f, g), and satisfy the equation

$$\begin{aligned} 0 \;&= \tfrac{u\phi}{4} - \tfrac{\phi u_t}{2} + \tfrac{1}{2}uu_x\xi_t - u_tu_x\xi_t + \tfrac{1}{2}uu_tu_x\xi_u - u_t^2u_x\xi_u + u_x^3\xi_u \\ &+u_x^2\xi_x + \tfrac{1}{2}uu_t\tau_t - u_t^2\tau_t + \tfrac{1}{2}uu_t^2\tau_u - u_t^3\tau_u \\ &+u_tu_x^2\tau_u + u_tu_x\tau_x - \tfrac{u\phi_t}{2} + u_t\phi_t - \tfrac{1}{2}uu_t\phi_u + u_t^2\phi_u - u_x^2\phi_u - u_x\phi_x \\ &+ \left(\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - \tfrac{1}{2}uu_t + \tfrac{1}{8}u^2\right)(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u) - (f_t + u_tf_u + g_x + u_xg_u)\,, \end{aligned}$$

[17]. This separates into an over-determined system of partial differential equations whose solution is

$$\begin{array}{lll} X_1 = \partial_t, & f_1 = 0, & g_1 = 0, \\ X_2 = \partial_x, & f_2 = 0, & g_2 = 0, \\ X_3 = t\partial_x + x\partial_t, & f_3 = 0, & g_3 = \tfrac{1}{4}u^2, \\ X_\infty = F(x,t)\partial_u, & f_\infty = -\tfrac{1}{2}uF + uF_t, & g_\infty = -uF_x, \end{array}$$

where F satisfies $\frac{1}{4}F + F_{xx} - F_{tt} = 0$, [17]. This Lie algebra is isomorphic to the Noether algebra corresponding to the Lagrangian $\overline{L}$, [4]. Hence L and $\overline{L}$ are equivalent Lagrangians.

We can therefore use the equation (1.9) given in Definition 1 (again assuming zero guage) in order to find the transformation $X = X(x,t,u)$, $T = T(x,t,u)$ and $U = U(x,t,u)$ that maps $\overline{L}$ to L.

Assuming that $X_u = 0$ and $T_u = 0$ as before, we have that $J = X_xT_t - X_tT_x$. Again, we use the equations (3.5) and (3.6) and solve simultaneously for U_X and U_T. Using the same procedure as before, we eventually deduce that

$$\begin{aligned} L &= \left[\frac{e^T}{2(X_xT_t - X_tT_x)}\right]\{u_xu_t[2U_u^2(T_tT_x - X_tX_x)] \\ &+ u_x^2[U_u^2(X_t^2 - T_t^2)] + u_t^2[U_u^2(X_x^2 - T_x^2)] \\ &+ u_x[2U_uU_x(X_t^2 - T_t^2) + 2U_tU_u(T_tT_x - X_tX_x)] \\ &+ u_t[2U_tU_u(X_x^2 - T_x^2) + 2U_uU_x(T_tT_x - X_tX_x)] \\ &+ [U_t^2(X_x^2 - T_x^2) + U_x^2(X_t^2 - T_t^2) + 2U_tU_x(T_tT_x - X_tX_x)]\}. \end{aligned}$$

Substituting (3.18) for L and then separating by derivative terms of u (which is possible by point dependency), we arrive at the six equations

$$\begin{aligned} u_tu_x &: \; 0 = \frac{2U_u^2e^T(T_tT_x - X_tX_x)}{2(X_xT_t - X_tT_x)}, \\ u_x^2 &: \; -\frac{1}{2} = \frac{U_u^2e^T(X_t^2 - T_t^2)}{2(X_xT_t - X_tT_x)}, \\ u_t^2 &: \; \frac{1}{2} = \frac{U_u^2e^T(X_x^2 - T_x^2)}{2(X_xT_t - X_tT_x)}, \\ u_x &: \; 0 = \frac{e^T[2U_uU_x(X_t^2 - T_t^2) + 2U_tU_u(T_tT_x - X_tX_x)]}{2(X_xT_t - X_tT_x)}, \\ u_t &: \; -\frac{1}{2}u = \frac{e^T[2U_tU_u(X_x^2 - T_x^2) + 2U_uU_x(T_tT_x - X_tX_x)]}{2(X_xT_t - X_tT_x)}, \end{aligned}$$

$$1 \quad : \quad \frac{1}{8}u^2 = \frac{e^T[U_t^2(X_x^2 - T_x^2) + U_x^2(X_t^2 - T_t^2) + 2U_tU_x(T_tT_x - X_tX_x)]}{2(X_xT_t - X_tT_x)}.$$

For this overdetermined system of equations, the software (see [17]) yields the result

$$\begin{aligned} X &= f_1(t+x) - f_2(t-x), \\ T &= f_1(t+x) + f_2(t-x), \\ U &= -e^{-\frac{1}{2}t}u. \end{aligned}$$

A particular choice of the functions f_1 and f_2 is

$$f_1(t+x) = \frac{1}{2}(t+x), \qquad f_2(t-x) = \frac{1}{2}(t-x),$$

from which we recover the transformation

$$X = x, \qquad T = t, \qquad U = -e^{-\frac{1}{2}t}u.$$

This is a known transformation mapping (3.16) to (3.19), as is noted in [1]. We note that this transformation also transforms the Noether symmetries of $\overline{L}$ to those of L.

3.6 Discussion and Conclusion

In this chapter, we set about extending the *Method of Equivalent Lagrangians* to some classes of PDEs in two independent variables. We began by elaborating on the work done in [5], in which a Lagrangian equivalent to a given Lagrangian was constructed using a known transformation and Definition 1. This aspect of the work took the form of two Illustrative Examples. Secondly, we found the *Form of the Equivalent Lagrangian*, which is the general form for a Lagrangian equivalent to the usual Lagrangian of the one-dimensional linear wave equation. In the next two Illustrative Examples, we made use of this form in order to apply the *Method of Equivalent Lagrangians*. Using the equivalent Lagrangians constructed in Section 3.2, we sought to recover the transformations, X, T and U, that map the usual Lagrangian of the $(1 + 1)$ linear wave equation to those equivalent Lagrangians, namely the transformations used to construct the equivalent Lagrangians in Section 3.2. The successful recovery of these two transformations through the application of our method serves to confirm the validity of the *Method of Equivalent Lagrangians* for these two examples. In the final part of this chapter, we applied the *Method of Equivalent Lagrangians* to a Klein-Gordon equation, whose Lagrangian is equivalent to that of the one-dimensional linear wave equation with dissipation. The method yielded a transformation which is noted in [1] as mapping these two equations from one to the other. Thus the *Method of Equivalent Lagrangians* has been successful for the main application of this chapter. The transformation found can be used to find the solutions and conserved quantities of our particular Klein-Gordon equation, since those of the linear wave equation with dissipation are well known. This section has extended the *Method of Equivalent Lagrangians* to some classes of PDEs in two independent variables, and validated the method for specific examples. In the same way, the method can be easily extended to other classes of PDEs in two independent variables, as long as Lagrangians for these equations exist. In the next chapter, we extend the *Method of Equivalent Lagrangians* to some classes of PDEs in three and four independent variables.

Chapter 4

Generalisation to PDEs in Three and Four Independent Variables

4.1 Introduction

We now generalise the *Method of Equivalent Lagrangians* to some classes of PDEs in three and four independent variables. As the number of independent variables increases, the work becomes, computationally, increasingly complex. Therefore we shall restrict our investigation to the standard linear wave equations with the usual Lagrangians in three and four independent variables respectively. We begin by looking at the case of three independent variables. It is well known that the two-dimensional linear wave equation can be written in polar co-ordinates using the regular transformation from Cartesian to polar co-ordinates. Applying this transformation to Definition 1, we are able to construct a Lagrangian which is equivalent to the usual Lagrangian of the two-dimensional linear wave equation.

This is the Lagrangian associated with the two-dimensional wave equation in polar co-ordinates. Following this, we reverse the procedure, applying the *Method of Equivalent Lagrangians* in an attempt to recover the transformation that maps one Lagrangian to the other, namely the transformation from Cartesian to polar co-ordinates. For the three-dimensional linear wave equation, we show that the well-known transformation from Cartesian to *spherical* co-ordinates satisfies the equations derived from Definition 1 when two known equivalent Lagrangians for this wave equation are used.

4.2 The Two-Dimensional Linear Wave Equation

The two-dimensional linear wave equation is well known and is given by

$$U_{TT} - U_{XX} - U_{YY} = 0, \tag{4.1}$$

where T, X and Y are the independent variables, and $U = U(T, X, Y)$ is the dependent variable, [1]. The usual Lagrangian for this equation is

$$\overline{L} = \frac{1}{2}U_T^2 - \frac{1}{2}U_X^2 - \frac{1}{2}U_Y^2, \tag{4.2}$$

[1]. We begin by constructing a Lagrangian $L = L(t, x, y)$ equivalent to $\overline{L}$, using the well-known transformation from Cartesian to polar co-ordinates, which is given by

$$T = t, \qquad X = x\cos y, \qquad Y = x\sin y.$$

After this we work backwards, starting with equation (1.9) that relates L and $\overline{L}$ as equivalent Lagrangians, and using it to derive the transformation from Cartesian to polar co-ordinates given above.

4.2.1 Constructing an Equivalent Lagrangian

In order to find a Lagrangian L which is equivalent to $\overline{L}$, we use Definition 1. To this we apply the transformation from Cartesian to polar co-ordinates, in order to find J, and substitute (4.2) for $\overline{L}$.

Hence we have that

$$\begin{aligned} J &= \begin{vmatrix} T_t & X_t & Y_t \\ T_x & X_x & Y_x \\ T_y & X_y & Y_y \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos y & \sin y \\ 0 & -x\sin y & x\cos y \end{vmatrix} \\ &= x\cos^2 y + x\sin^2 y \\ &= x. \end{aligned}$$

By the multivariate case of the chain rule, we have the matrix equation

$$\begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix} = \begin{pmatrix} T_t & X_t & Y_t \\ T_x & X_x & Y_x \\ T_y & X_y & Y_y \end{pmatrix} \begin{pmatrix} U_T \\ U_X \\ U_Y \end{pmatrix}. \tag{4.3}$$

Hence,

$$\begin{aligned} \begin{pmatrix} U_T \\ U_X \\ U_Y \end{pmatrix} &= \begin{pmatrix} T_t & X_t & Y_t \\ T_x & X_x & Y_x \\ T_y & X_y & Y_y \end{pmatrix}^{-1} \begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos y & \sin y \\ 0 & -x\sin y & x\cos y \end{pmatrix}^{-1} \begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos y & -\frac{1}{x}\sin y \\ 0 & \sin y & \frac{1}{x}\cos y \end{pmatrix} \begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix},$$

which gives us the following expressions for U_T, U_X and U_Y.

$$U_T = u_t, \qquad U_X = u_x \cos y - u_y \frac{1}{x} \sin y, \qquad U_Y = u_x \sin y + u_y \frac{1}{x} \cos y.$$

Substituting these into equation (1.9), along with our expression for J, we have that a Lagrangian equivalent to (4.2) is

$$L = \frac{1}{2} x u_t^2 - \frac{1}{2} x u_x^2 - \frac{1}{2x} u_y^2. \tag{4.4}$$

It is well known that this Lagrangian is isomorphic to the usual Lagrangian of the two-dimensional linear wave equation, namely (4.2).

Now that we have used our definition of equivalent Lagrangians in order to find L, we reverse the process, applying the *Method of Equivalent Lagrangians* in order to recover the required transformation.

4.2.2 Finding Transformations - The Reverse Procedure

In the previous section, we found that the Lagrangian L, given by (4.4), is equivalent to $\overline{L}$, the usual Lagrangian of the two-dimensional linear wave equation, given by (4.2). Now, using the *Method of Equivalent Lagrangians*, we wish to find the transformation, $T = T(t, x, y)$, $X = X(t, x, y)$ and $Y = Y(t, x, y)$, that maps $\overline{L}$ to L. The transformation is known to us by construction. Thus in order to verify the method for this particular example, we wish to recover the usual transformation of Cartesian to polar co-ordinates.

Note: We are using two Lagrangians which we know to be equivalent by construction. If this were not the case, however, we could check that L is equivalent to $\overline{L}$ by verifying that they have isomorphic 10-dimensional Noether symmetry algebras.

We begin by computing J. For the sake of simplification, we assume that $T_x = T_y = X_t = Y_t = 0$. In other words, $T = T(t)$, $X = X(x,y)$ and $Y = Y(x,y)$. From this assumption it follows that

$$\begin{aligned} J &= \begin{vmatrix} T_t & X_t & Y_t \\ T_x & X_x & Y_x \\ T_y & X_y & Y_y \end{vmatrix} \\ &= \begin{vmatrix} T_t & 0 & 0 \\ 0 & X_x & Y_x \\ 0 & X_y & Y_y \end{vmatrix} \\ &= T_t(X_xY_y - X_yY_x). \end{aligned}$$

The multivariate case of the chain rule, given by (4.3), relates u_t, u_x and u_y to U_T, U_X and U_Y. Using this equation we have that

$$\begin{aligned} \begin{pmatrix} U_T \\ U_X \\ U_Y \end{pmatrix} &= \begin{pmatrix} T_t & 0 & 0 \\ 0 & X_x & Y_x \\ 0 & X_y & Y_y \end{pmatrix}^{-1} \begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{T_t} & 0 & 0 \\ 0 & \frac{Y_y}{(X_xY_y - X_yY_x)} & \frac{-Y_x}{(X_xY_y - X_yY_x)} \\ 0 & \frac{-X_y}{(X_xY_y - X_yY_x)} & \frac{X_x}{(X_xY_y - X_yY_x)} \end{pmatrix} \begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix} \\ &= \frac{1}{(X_xY_y - X_yY_x)} \begin{pmatrix} \frac{u_t(X_xY_y - X_yY_x)}{T_t} \\ -(Y_xu_y - Y_yu_x) \\ (X_xu_y - X_yu_x) \end{pmatrix}, \end{aligned}$$

which gives us expressions for U_T, U_X and U_Y. By Definition 1, making the substitutions for L, $\overline{L}$ and J, we have the equation

$$\frac{1}{2}xu_t^2 - \frac{1}{2}xu_x^2 - \frac{1}{2x}u_y^2 = \frac{1}{2}(U_T^2 - U_X^2 - U_Y^2)T_t(X_xY_y - X_yY_x). \tag{4.5}$$

Writing U_T, U_X and U_Y in terms of u_t, u_x and u_y, using the expressions in the matrix above, gives us

$$\begin{aligned}\frac{1}{2}xu_t^2 - \frac{1}{2}xu_x^2 - \frac{1}{2x}u_y^2 &= \frac{1}{2T_t(X_xY_y - X_yY_x)}(u_t^2(X_xY_y - X_yY_x)^2 \\ &- u_x^2(Y_y^2 + X_y^2) - u_y^2(Y_x^2 + X_x^2) + 2u_xu_y(Y_xY_y + X_xX_y)).\end{aligned}$$

Since T, X and Y are point dependent, we can separate by derivatives of u to obtain the following over-determined system of four equations.

$$\begin{aligned} u_t^2 &: \ T_tx = (X_xY_y - X_yY_x), \\ u_x^2 &: \ T_tx(X_xY_y - X_yY_x) = Y_y^2 + X_y^2, \\ u_y^2 &: \ T_t(X_xY_y - X_yY_x) = x(Y_x^2 + X_x^2), \\ u_xu_y &: \ 0 = 2(Y_xY_y + X_xX_y). \end{aligned}$$

Solving this system using mathematical software (see [17]) yields the result

$$T = t + a, \qquad X = F(t) + x\cos y, \qquad Y = G(t) + x\sin y,$$

where a is a constant and F and G are arbitrary functions of t. If we let $F = G = 0$, we recover the transformation

$$T = t, \qquad X = x\cos y, \qquad Y = x\sin y,$$

which is the standard transformation from Cartesian to polar co-ordinates, as anticipated. This transformation can in turn be used to find solutions and conserved quantities of the PDE associated with (4.4), since those associated with the two-dimensional linear wave equation are known. In the next section we consider the three-dimensional linear wave equation.

4.3 The Three-Dimensional Linear Wave Equation

We now turn our attention to some classes of PDEs in four independent variables. The three-dimensional linear wave equation is the well-known second-order PDE

$$U_{TT} - U_{XX} - U_{YY} - U_{ZZ} = 0, \tag{4.6}$$

see [1]. Here T, X, Y and Z are the independent variables, and $U = U(T, X, Y, Z)$ is the dependent variable. The usual Lagrangian for this equation is

$$\overline{L} = \frac{1}{2}U_T^2 - \frac{1}{2}U_X^2 - \frac{1}{2}U_Y^2 - \frac{1}{2}U_Z^2, \tag{4.7}$$

[1]. It is well known that the Lagrangian for the three-dimensional wave equation in spherical co-ordinates is given by

$$L = \frac{1}{2}x^2 \sin y \; u_t^2 - \frac{1}{2}\sin y \; x^2 \; u_x^2 - \frac{1}{2}\sin y \; u_y^2 - \frac{1}{2\sin y}u_z^2. \tag{4.8}$$

L and $\overline{L}$ are naturally equivalent because they are Lagrangians of the same equation. However, it can also be verified that they generate isomorphic Noether algebras of point symmetries.

In this section, we shall apply the *Method of Equivalent Lagrangians* to L and $\overline{L}$, in an attempt to recover the transformation that maps $\overline{L}$ to L. We shall demonstrate that equation (1.9) of Definition 1 is satisfied by the transformation from Cartesian to polar co-ordinates, which is to be expected from our choice of Lagrangians.

As before, we make use of Definition 1, which relates two equivalent Lagrangians to each other by means of the transformation that maps one to the other. Substituting

L and $\overline{L}$ into equation (1.9) gives us

$$\begin{aligned}&\tfrac{1}{2}x^2 \sin y\ u_t^2 - \tfrac{1}{2}\sin y\ x^2\ u_x^2 - \tfrac{1}{2}\sin y\ u_y^2 - \tfrac{1}{2\sin y}\ u_z^2 \\ &= (\tfrac{1}{2}U_T^2 - \tfrac{1}{2}U_X^2 - \tfrac{1}{2}U_Y^2 - \tfrac{1}{2}U_Z^2)J. \end{aligned} \tag{4.9}$$

We first calculate the Jacobian, J. We assume that $T_x = T_y = T_z = X_t = Y_t = Z_t = 0$. In other words, $T = T(t)$, $X = X(x,y,z)$, $Y = Y(x,y,z)$ and $Z = Z(x,y,z)$. Furthermore, we assume that $T = t$. Hence $T_t = 1$. From these assumptions it follows that

$$\begin{aligned} J &= \begin{vmatrix} T_t & X_t & Y_t & Z_t \\ T_x & X_x & Y_x & Z_x \\ T_y & X_y & Y_y & Z_y \\ T_z & X_z & Y_z & Z_z \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & X_x & Y_x & Z_x \\ 0 & X_y & Y_y & Z_y \\ 0 & X_z & Y_z & Z_z \end{vmatrix} \\ &= X_xY_yZ_z - X_xY_zZ_y - X_yY_xZ_z + X_yY_zZ_x + X_zY_xZ_y - X_zY_yZ_x. \end{aligned}$$

Now, by the multivariate case of the chain rule (in four dimensions this time), we have the matrix equation

$$\begin{pmatrix} u_t \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} T_t & X_t & Y_t & Z_t \\ T_x & X_x & Y_x & Z_x \\ T_y & X_y & Y_y & Z_y \\ T_z & X_z & Y_z & Z_z \end{pmatrix} \begin{pmatrix} U_T \\ U_X \\ U_Y \\ U_Z \end{pmatrix}. \tag{4.10}$$

Hence,

$$\begin{aligned}
\begin{pmatrix} U_T \\ U_X \\ U_Y \\ U_Z \end{pmatrix} &= \begin{pmatrix} T_t & X_t & Y_t & Z_t \\ T_x & X_x & Y_x & Z_x \\ T_y & X_y & Y_y & Z_y \\ T_z & X_z & Y_z & Z_z \end{pmatrix}^{-1} \begin{pmatrix} u_t \\ u_x \\ u_y \\ u_z \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_x & Y_x & Z_x \\ 0 & X_y & Y_y & Z_y \\ 0 & X_z & Y_z & Z_z \end{pmatrix}^{-1} \begin{pmatrix} u_t \\ u_x \\ u_y \\ u_z \end{pmatrix} \\
&= \frac{1}{J} \begin{pmatrix} u_t(X_xY_yZ_z - X_xY_zZ_y - X_yY_xZ_z + X_yY_zZ_x + X_zY_xZ_y - X_zY_yZ_x) \\ u_z(Y_xZ_y - Y_yZ_x) - u_y(Y_xZ_z - Y_zZ_x) + u_x(Y_yZ_z - Y_zZ_y) \\ u_y(X_xZ_z - X_zZ_x) - u_z(X_xZ_y - X_yZ_x) - u_x(X_yZ_z - X_zZ_y) \\ u_z(X_xY_y - X_yY_x) - u_y(X_xY_z - X_zY_x) + u_x(X_yY_z - X_zY_y) \end{pmatrix},
\end{aligned}$$

from which we can read the expressions for U_T, U_X, U_Y and U_Z in terms of u_t, u_x, u_y and u_z. Substituting these into equation (4.9), along with our expression for J, we have the equation

$$\begin{aligned}
& \frac{1}{2}x^2 \ \sin y \ u_t^2 - \frac{1}{2}\sin y \ x^2 \ u_x^2 - \frac{1}{2}\sin y \ u_y^2 - \frac{1}{2\sin y} \ u_z^2 \\
= \ & \frac{1}{2}(-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z)[u_t^2 \\
- \ & \frac{(u_zX_yY_x - u_yX_zY_x - u_zX_xY_y + u_xX_zY_y + u_yX_xY_z - u_xX_yY_z)^2}{(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z)^2} \\
- \ & \frac{(u_zY_yZ_x - u_yY_zZ_x - u_zY_xZ_y + u_xY_zZ_y + u_yY_xZ_z - u_xY_yZ_z)^2}{(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z)^2} \\
- \ & \frac{(u_zX_yZ_x - u_yX_zZ_x - u_zX_xZ_y + u_xX_zZ_y + u_yX_xZ_z - u_xX_yZ_z)^2}{(-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z)^2}].
\end{aligned}$$

Since X, Y and Z are point dependent, we can separate by derivatives of u to obtain the following over-determined system of equations.

u_t^2:

$$\frac{1}{2}x^2\sin y = \frac{1}{2}(-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z),$$

u_x^2 :

$$\begin{aligned} -\frac{1}{2}x^2\sin y & \\ = \;& \frac{X_z^2\left(Y_y^2+Z_y^2\right)+(Y_zZ_y-Y_yZ_z)^2-2X_yX_z(Y_yY_z+Z_yZ_z)+X_y^2\left(Y_z^2+Z_z^2\right)}{2(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z)}, \end{aligned}$$

u_y^2 :

$$\begin{aligned} -\frac{1}{2}\sin y & \\ = \;& \frac{X_z^2\left(Y_x^2+Z_x^2\right)+(Y_zZ_x-Y_xZ_z)^2-2X_xX_z(Y_xY_z+Z_xZ_z)+X_x^2\left(Y_z^2+Z_z^2\right)}{2(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z)}, \end{aligned}$$

u_z^2 :

$$\begin{aligned} -\frac{1}{2\sin y} & \\ = \;& \frac{X_y^2\left(Y_x^2+Z_x^2\right)+(Y_yZ_x-Y_xZ_y)^2-2X_xX_y(Y_xY_y+Z_xZ_y)+X_x^2\left(Y_y^2+Z_y^2\right)}{2(X_zY_yZ_x - X_yY_zZ_x - X_zY_xZ_y + X_xY_zZ_y + X_yY_xZ_z - X_xY_yZ_z)}, \end{aligned}$$

u_xu_y :

$$\begin{aligned} 0 \;=\; & \left[\frac{1}{-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z}\right] \\ & \times \Big[X_z^2(Y_xY_y + Z_xZ_y) + (Y_zZ_x - Y_xZ_z)(Y_zZ_y - Y_yZ_z) \\ & \quad - X_z(X_yY_xY_z + X_xY_yY_z + X_yZ_xZ_z + X_xZ_yZ_z) + X_xX_y\left(Y_z^2+Z_z^2\right)\Big], \end{aligned}$$

u_xu_z :

$$\begin{aligned} 0 \;=\; & \left[\frac{1}{-X_zY_yZ_x + X_yY_zZ_x + X_zY_xZ_y - X_xY_zZ_y - X_yY_xZ_z + X_xY_yZ_z}\right] \\ & \times \Big[X_xX_z\left(Y_y^2+Z_y^2\right) + (Y_yZ_x - Y_xZ_y)(-Y_zZ_y + Y_yZ_z) \\ & \quad + X_y^2(Y_xY_z + Z_xZ_z) - X_y(X_zY_xY_y + X_xY_yY_z + X_zZ_xZ_y + X_xZ_yZ_z)\Big], \end{aligned}$$

$u_z u_y$:

$$
\begin{aligned}
0 \;=\; & \left[\frac{1}{-X_z Y_y Z_x + X_y Y_z Z_x + X_z Y_x Z_y - X_x Y_z Z_y - X_y Y_x Z_z + X_x Y_y Z_z}\right] \\
& \times \Big[- X_x X_z (Y_x Y_y + Z_x Z_y) + (Y_y Z_x - Y_x Z_y)(Y_z Z_x - Y_x Z_z) \\
& \quad + X_x^2 (Y_y Y_z + Z_y Z_z) + X_y \left(X_z \left(Y_x^2 + Z_x^2 \right) - X_x (Y_x Y_z + Z_x Z_z) \right) \Big].
\end{aligned}
$$

It is easy to check that these equations are satisfied by the transformation map from Cartesian to spherical co-ordinates given by

$$
\begin{aligned}
T &= t, \\
X &= x \sin y \cos z, \\
Y &= x \sin y \sin z, \\
Z &= x \cos y.
\end{aligned}
$$

4.4 Discussion and Conclusion

In this chapter we have generalised the *Method of Equivalent Lagrangians* to PDEs in three and four independent variables, using the standard two- and three-dimensional linear wave equations respectively. We began by constructing a Lagrangian equivalent to the usual Lagrangian of the two-dimensional linear wave equation, using the known transformation from Cartesian to polar co-ordinates. This gave us the Lagrangian associated with the two-dimensional wave equation in polar co-ordinates. Following this, we applied our method as before, and successfully recovered the transformation from Cartesian to polar co-ordinates. For the case of four independent variables, we started with two Lagrangians which we know to be equivalent, and attempted to use Definition 1 to find the known transformation from Cartesian to spherical co-ordinates. Although we did not go as far as to solve the system of equations resulting from the application of our method, we demonstrated that they would be satisfied by the transformation from Cartesian to spherical co-ordinates, as anticipated. Hence we have verified the *Method of Equivalent Lagrangians* for two PDEs in three and four independent variables. In the same way, the method can be applied to other PDEs in three and four independent variables, provided that Lagrangians exist which are equivalent to other Lagrangians associated with PDEs possessing known solutions and conserved quantities. If these conditions are satisfied, then the transformations mapping one Lagrangian to the other can be found, and these can be used to transform the known solutions and conserved quantities to those of the PDE in question.

Conclusion

In this study, we have applied the notion of Equivalent Lagrangians to determine transformations that map differential equations one to another in order generate solutions, conservation laws, inter alia. In the first chapter, we presented the fundamental results and definitions required for the application of the method, along with an illustrative example in which we used the method to find solutions and conserved quantities of the harmonic oscillator ODE. In the second chapter, we built on the work done in [6] by applying the method to two second-order linear ODEs: the *Kummer Equation* and the *Combined Gravity-Inertial-Rossby Wave Equation*. For these two ODEs, the *Method of Equivalent Lagrangians* successfully yielded transformation maps which can be used to find solutions and conserved quantities. The procedure also holds for PDEs of any number of independent variables. In the following chapters, we demonstrated this for PDEs in two, three and four independent variables. In the third chapter, we began by following the procedure detailed in [5], in which an equivalent Lagrangian is constructed, given an original Lagrangian and a transformation map. We then found the form of a Lagrangian equivalent to the usual Lagrangian of the linear wave equation in $(1+1)$ dimensions. This form was then used in two illustrative examples demonstrating the *Method of Equivalent Lagrangians*, as well as in the main application of this section, in which a transformation was recovered that can be used to find the solutions and conserved quantities of a particular Klein-Gordon equation. In the fourth chapter, we generalised the method to certain classes of PDEs in three and four independent variables, using the two- and three- dimensional linear wave equations respectively. As a consequence of the procedure, we recovered two well-known transformations, namely the mapping from Cartesian to polar co-ordinates, and the mapping from Cartesian to spherical co-ordinates. The advantages of the *Method of Equivalent Lagrangians* are that it enables one to avoid the integration techniques normally required to solve differential equations and that once a transformation map is recovered, it can be used to find all solutions and conserved quantities for a given differential equation. This powerful application makes up for the cumbersome nature of the procudure. A possible drawback, however, is that a necessary requirement of the method is the existence of a Lagrangian for the equation under investigation.

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