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# Solutions of fractional differential models by using Sumudu transform method and its hybrid

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ABSTRACT

This paper presents the Sumudu transform method and its hybrid for the construction of solutions of differential equations, both with integer-order and fractional derivatives. The paper discusses the construction of solutions of fractional differential equations with varying delay proportional to the independent variable by using two methods. The paper considers the mathematical model for the decay of Iodine 135 as an application of fractional differential equations in nuclear physics. The application indicates that fractional differential equations with variable delay proportional to the independent variable are a useful tool for the modeling of many anomalous phenomena in nature and in the theory of complex systems.

## 1. Introduction

In recent decades, there are exceptional advancement efforts on the construction of approximate analytical and numerical solutions of differential equations. Solutions of differential equations have been considered by using several analytical and numerical methods including Legendre spectral method,[1] artificial neural networks,[2] operational matrix technique[3] Legendre collocation technique,[4] Haar wavelet collocation method,[5] iterative Laplace transform method,[6] hyperbolic-NILT method,[7] reproducing kernel algorithm,[8] variational iteration algorithm,[9] Adomian decomposition method,[10] finite difference method,[11] Homotopy Analysis,[12] Homotopy-Pade,[13] Homotopy Perturbation,[14] variational principle,[15] Jacobi elliptic function expansion,[16] implicit–explicit,[17] tanh–coth,[18] Fourier pseudo-spectral,[19] Hirotas bilinear,[20] Finite volume,[21] lattice Boltzmann,[22] generalized $(G'/G)$- expansion method,[23] and hybrids of Sumudu Transform (ST).[24] Most of these approaches are not hitch free due to their association with certain impediments that include divergent results, un-realistic assumptions, calculation of Adomians polynomials, very lengthy calculations, limited convergence and non-compatibility with the nonlinearity of physical problem.[25] The research efforts to overcome those identified setbacks in some existing mathematical approaches breed Lie group method[26,27] and generalized exponential rational function method.[28–30] Fractional calculus extends differentiation to non-integer-orders and provides an amazing platform for the description of various kinds of systems.[31–33] Time delay in differential equations play amazing roles by making the differential equations to be more precise as models for real-life systems. Delay differential equations are the essential platform for the consideration of the hereditary properties of the physical systems. In delay differential equations, the system at a given time and its previous evolution determine the rate of change.[34] Delay differential equations have numerous applications in the modeling of chemical processes such as mechanical systems,[35] transmission lines and industrial processes,[36] population growth,[37] economic growth,[38] neural networks[39] and several other systems. Integral transforms are a class of powerful methods in solving differential equations of integer and non-integer orders.[40] The integral transforms have broad applications in various research fields including solid physics,[41] heat transfer,[42] convective diffusion,[43] nuclear physics,[44,45] fluid flow,[46] signal process,[47] Tribology,[48] elasticity[49] and population dynamics.[50] Consider a given radioactive isotope whose atoms decay with a fixed percentage in each fixed period. Ordinary differential equations (ODEs) are the most widespread formalism to model dynamical systems in science and engineering. The number of decays that occur in a fixed time in relation to the number of atoms and the length of time can be

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**Abbreviations**

| | |
|---|---|
| ABC | Atangana–Baleanu fractional derivative in the Caputo sense |
| CFC | Caputo–Fabrizio fractional derivative in the Caputo sense |
| HSV | Hybrid Sumudu Variational |
| ODEs | Ordinary Differential Equations |
| ST | Sumudu Transform |

described by using the ODEs. The mathematical model for the decay of Iodine *135* ($^{135}I$) is given by

$$\frac{dI}{dt} = \lambda - \alpha I, \tag{1.1}$$

which is an ODE with $I(t)$ denoting the density of $^{135}I$ $\left(\text{atoms/cm}^3\right)$, $\alpha$ denoting the decay constant and $\lambda$ is a constant that denotes the product of thermal neutron flux, macroscopic fission cross-section and effective fission yield of the isotope.[51,52] The Eq. (1.1) is solved by

$$I(t) = I(0)e^{-\alpha t} + \lambda/\alpha\left(1 - e^{-\alpha t}\right), \tag{1.2}$$

where $I(0)$ refers to the amount of $^{135}I$ at the time $t = 0$. At equilibrium, the rate of production of $^{135}I$ equals its removal and $^{135}I$ undergoes complete decay to xenon, which is being termed $^{135}$Xe reservoir. The decay process of $^{135}I$ is the wellspring of $^{135}$Xe, which is an artificial element that has great influences on the nuclear reactor operation.[53] The equilibrium concentration of $^{135}I$ can be determined by setting $\frac{dI}{dt} = 0$ in (1.1), which gives

$$I(t) = \lambda/\alpha. \tag{1.3}$$

This indicates that the concentration of $^{135}I$ remains constant at equilibrium. It can be deduced from (1.3) that the equilibrium concentration of $^{135}I$ is proportional to the fission reaction rate and consequently to the reactor power level.

This paper presents an integral transform method and its hybrid for the construction of solutions of differential equations. The paper presents two suitable methods for solving differential equations of integer and non-integer orders. They are suitable methods with brief computation steps that produce exact solutions. The paper considers an application in nuclear physics by introducing the time delay and fractional order derivatives into the mathematical model for the description of decay of $^{135}I$. The presence of these two types of memories in the model enriches the dynamics of the radioactive model. Exact solutions are obtained for the radioactive models with fractional order derivatives and approximate analytical solutions for the radioactive model with time delay and fractional order derivative.

## 2. Preliminaries

This section presents the definitions of the terms and some existing results that are pertinent to this paper. Throughout this paper, $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers respectively.

**Definition 2.1.** Consider

$$\Lambda = \left\{ w(t) : \exists\, Q, \xi_1, \xi_2 > 0, |w(t)| < Qe^{|t|/\xi_j}, \text{if } t \in (-1)^j \times [0,\infty) \right\},$$

which is a set of functions with $w(t) \in \Lambda$ for all real $t \geq 0$.[54] The ST for a given function $w(t)$ is defined as

$$\mathcal{S}[w(t)] = \int_0^\infty w(tu)e^{-t}dt, \ \ u \in (-\xi_1, \xi_2).$$

$\mathcal{S}[w(t)]$ will be denoted by $W(u)$. The inverse ST of $W(u)$ is the function $w(t)$. ST is the theoretical dual to the Laplace transform,

$$\mathcal{L}[w(t)] = \int_0^\infty w(t)e^{-st}dt, \ \ s > 0.$$

for a given function $w(t)$. ST is little well known and not widely used compared to the Laplace transform. However, the two integral transforms rival in problem solving. ST has scale and unit preserving properties, which makes it possible to be used to solve problems without resorting to a new frequency domain and a good candidate for the integral production-depreciation problems.[54] Denoting $\mathcal{L}[w(t)]$ by $L(u)$, the duality relations between the two integral transforms are

$$W(1/s) = sL(s), \quad L(1/u) = uW(u).$$

ST is defined for a given function that fulfills the Dirichlet conditions and using ST, the Lagrange multiplier can be easily obtained. ST is a linear function. Recall that a function $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ is said to be linear provided that for every $u, v \in \mathbb{R}$,

$$\mathcal{T}(\theta u + \sigma v) = \theta\mathcal{T}(u) + \sigma\mathcal{T}(v),$$

where $\theta$ and $\sigma$ are arbitrary real constants. For an integer order derivative, its ST is expressed as

$$S\left[\frac{dw(t)}{dt}\right] = \frac{1}{u}\left[W(u) - w(0)\right],$$

and for the $n$th-order derivative, the ST is given as

$$S\left[\frac{d^k w(t)}{dt^k}\right] = \frac{1}{u^k}\left[W(u) - \sum_{n=0}^{k-1} u^n \frac{d^n w(t)}{dt^n}|_{t=0}\right].$$

**Definition 2.2.** Let $a > 0, b > 0$ be positive real numbers and $0 < \mu < 1$. The left and right sided Caputo fractional derivatives of order $\mu$ are defined respectively as Ref. 55

$${}^{C}_{a}D^{\mu}w(t) = \frac{1}{\Gamma(1-\mu)}\int_a^t (t-\xi)^{-\mu}w'(\xi)d\xi$$

and

$${}^{C}D^{\mu}_{b}w(t) = \frac{-1}{\Gamma(1-\mu)}\int_t^b (\xi - t)^{\mu}w'(\xi)d\xi.$$

The ST of Caputo-fractional derivatives of order $\mu$ is given by Ref. 56

$$S\left[{}^{C}_{0}D^{\mu}w(t)\right](u) = u^{-\mu}\left(W(u) - w(0)\right). \tag{2.1}$$

**Definition 2.3.** The left and right sided Caputo–Fabrizio fractional derivatives in the Caputo sense (CFC) of order $\mu$ are defined respectively as Ref. 57

$${}^{CFC}_{a}D^{\mu}w(t) = \frac{M(\mu)}{(1-\mu)}\int_a^t w'(\xi)e^{\varphi(t-\xi)}d\xi$$

and

$${}^{CFC}D^{\mu}_{b}w(t) = -\frac{M(\mu)}{(1-\mu)}\int_t^b w'(\xi)e^{\varphi(\xi - t)}d\xi,$$

where $M(\mu)$ is the normalization function and $\varphi = -\frac{\mu}{1-\mu}$. The ST of CFC of order $\mu$ is given by Ref. 56

$$S\left[{}^{CFC}_{0}D^{\mu}w(t)\right](u) = \frac{M(\mu)}{1-\mu+\mu u}\left(W(u) - w(0)\right). \tag{2.2}$$

**Definition 2.4.** The left and right sided Atangana–Baleanu fractional derivatives in the Caputo sense (ABC) of order $\mu$ are defined respectively as Ref. 58

$${}^{ABC}_{a}D^{\mu}w(t) = \frac{B(\mu)}{(1-\mu)}\int_a^t w'(\xi)E_\mu\left[\varphi(t-\xi)\right]d\xi$$

and

$${}^{ABC}D^{\mu}_{b}w(t) = -\frac{B(\mu)}{(1-\mu)}\int_t^b w'(\xi)E_\mu\left[\varphi(\xi - t)\right]d\xi,$$

where $B(\mu)$ is the normalization function and $\varphi = -\frac{\mu}{1-\mu}$. The ST of ABC of order $\mu$ is given by Ref. 56

$$S\left[{}^{ABC}_{0}D^{\mu}w(t)\right](u) = \frac{B(\mu)}{1-\mu+\mu u^{\mu}}\left(W(u) - w(0)\right). \tag{2.3}$$

**Proposition 2.1.** *Let $f, g : [0, \infty) \to \mathbb{R}$, then the classical convolution product is given by*

$$(f * g)(t) = \int_0^t f(t-x)g(x)dx.$$

*The ST for the convolution product is given by*

$$\begin{aligned} S[(f * g)(t)] &= uS[f(t)]S[g(t)] \\ &= uF(u)G(u). \end{aligned}$$

**Definition 2.5.** One parameter Mittag-Leffler function $E_\mu(t)$ is defined as

$$E_\mu(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\mu+1)}, \mu > 0.$$

The following results about Mittag-Leffler functions and ST are well known[59]:

(i) $S\left[E_\mu\left(-at^\mu\right)\right] = \frac{1}{1+au^\mu}$;
(ii) $S\left[1 - E_\mu\left(-at^\mu\right)\right] = \frac{au^\mu}{1+au^\mu}$.

**Definition 2.6.** Lagrange multipliers can literally be described as auxiliary variables that are introduced into a problem of constrained minimization to find their solutions. The concept was introduced by Joseph Louis Lagrange in the early 1770s. They are auxiliary variables that aid in transforming constrained optimization problems into unconstrained optimization problems.[60]

## 3. Main results

### 3.1. Sumudu transform method

Consider the fractional form of (1.1), which is given by Ref. 61

$$\frac{d^\mu I}{dt^\mu} = \lambda - \alpha I. \tag{3.1}$$

In this section, this paper considers when (3.1) is given by the

(i) Caputo fractional derivative;
(ii) Caputo–Fabrizio fractional derivative in the Caputo sense (CFC) and
(iii) Atangana–Baleanu fractional derivative in the Caputo sense (ABC).

Caputo fractional derivative is the most used operator probably because it admits the traditional initial and boundary conditions in the formulation of the problems. CFC contains a nonsingular kernel through which one can express the non-locality of real-world situations properly. Also, the ABC is a nonlocal fractional derivative with a nonsingular kernel that relates to a variety of applications. Certain models of dissipative phenomena can adequately be described by using fractional derivatives with nonsingular kernels.[31] These three operators will be applied in this paper to examine the non-local complex dynamical behavior of $^{135}I$. Srivastava et al.[62] studied the existence of solution of fractional problems. The qualitative behaviors such as stability, exponential stability, asymptotic stability, Mittag-Leffler stability and boundedness of solutions of the delay differential equations with fractional derivative have been investigated. The ST is one of the best integral transforms. This paper examines the decay models for $^{135}I$ that involve Caputo, CFC, ABC type fractional derivatives. This paper presents how to obtain the solutions of these models by using the ST method and its hybrid and the paper shows the efficiency of these methods in producing the results that are reliable.

**Case 1**: Model for the decay of $^{135}I$ with Caputo fractional derivative

Consider

$${}_0^C D^\mu I(t) = \lambda - \alpha I(t), \tag{3.2}$$

taking its ST and applying (2.1) lead to

$$S\left[{}_0^C D^\mu I(t)\right] = \lambda - \alpha S\left[I(t)\right]$$

$$u^{-\mu}\left(I(u) - I(0)\right) = \lambda - \alpha I(u)$$

$$\left[u^{-\mu} + \alpha\right] I(u) = u^{-\mu} I(0) + \lambda$$

$$I(u) = \frac{I(0)}{1+\alpha u^\mu} + \frac{\lambda u^\mu}{1+\alpha u^\mu}.$$

Taking the inverse ST leads to

$$I(t) = I(0)S^{-1}\left[\frac{1}{1+\alpha u^\mu}\right] + \frac{\lambda}{\alpha} S^{-1}\left[\frac{\alpha u^\mu}{1+\alpha u^\mu}\right].$$

According to Definition 2.5(i),

$$I(t) = I(0)E_\mu\left(-\alpha t^\mu\right) + \frac{\lambda}{\alpha}\left(1 - E_\mu\left(-\alpha t^\mu\right)\right). \tag{3.3}$$

**Case 2**: Model for the decay of $^{135}I$ with CFC

Consider

$${}_0^{CFC} D^\mu I(t) = \lambda - \alpha I(t), \tag{3.4}$$

taking its ST and applying (2.2) lead to

$$S\left[{}_0^{CFC} D^\mu I(t)\right] = \lambda - \alpha S\left[I(t)\right]$$

$$\frac{M(\mu)}{1-\mu+\mu u}\left(I(u) - I(0)\right) = \lambda - \alpha I(u)$$

$$\left[\frac{M(\mu)}{1-\mu+\mu u} + \alpha\right] I(u) = \frac{M(\mu)I(0)}{1-\mu+\mu u} + \lambda$$

$$I(u) = \frac{\frac{M(\mu)I(0)}{1-\mu+\mu u}}{\left[\frac{M(\mu)}{1-\mu+\mu u} + \alpha\right]} + \frac{\lambda}{\left[\frac{M(\mu)}{1-\mu+\mu u} + \alpha\right]}.$$

Taking the inverse ST leads to

$$\begin{aligned} I(t) &= S^{-1}\left[\frac{\frac{M(\mu)I(0)}{1-\mu+\mu u}}{\left[\frac{M(\mu)}{1-\mu+\mu u} + \alpha\right]}\right] + S^{-1}\left[\frac{\lambda}{\left[\frac{M(\mu)}{1-\mu+\mu u} + \alpha\right]}\right] \\ &= I(0)S^{-1}\left[\frac{M(\mu)}{M(\mu) + \alpha(1-\mu+\mu u)}\right] \\ &\quad + S^{-1}\left[\frac{\lambda(1-\mu+\mu u)}{M(\mu) + \alpha(1-\mu+\mu u)}\right]. \end{aligned}$$

According to Definition 2.5(i),

$$\begin{aligned} I(t) = &\frac{I(0)M(\mu)}{M(\mu) + \alpha(1-\mu)} exp\left(\frac{-\mu\alpha t}{M(\mu) + \alpha(1-\mu)}\right) \\ &+ \frac{\lambda\left[M(\mu) + \alpha(1-\mu) + M(\mu)exp\left(\frac{-\mu\alpha t}{M(\mu)+\lambda(1-\mu)}\right)\right]}{\alpha\left(M(\mu) + \alpha(1-\mu)\right)}. \end{aligned}$$

**Case 3**: Model for the decay of $^{135}I$ with ABC

Consider

$${}_0^{ABC} D^\mu I(t) = \lambda - \alpha I(t), \tag{3.5}$$

taking its ST and applying (2.3) lead to

$$S\left[{}_0^{ABC} D^\mu I(t)\right] = \lambda - \alpha S\left[I(t)\right]$$

$$\frac{B(\mu)}{1-\mu+\mu u^\mu}\left(I(u) - I(0)\right) = \lambda - \alpha I(u)$$

$$\left[\frac{B(\mu)}{1-\mu+\mu u^\mu} + \alpha\right] I(u) = \frac{B(\mu)I(0)}{1-\mu+\mu u^\mu} + \lambda$$

$$I(u) = \frac{\frac{B(\mu)I(0)}{1-\mu+\mu u^\mu}}{\left[\frac{B(\mu)}{1-\mu+\mu u^\mu} + \alpha\right]} + \frac{\lambda}{\left[\frac{M(\mu)}{1-\mu+\mu u^\mu} + \alpha\right]}$$

$$
\begin{aligned}
&= \frac{B(\mu)I(0)}{B(\mu)+\alpha(1-\mu+\mu u^{\mu})} + \frac{\lambda(1-\mu+\mu u^{\mu})}{B(\mu)+\alpha(1-\mu+\mu u^{\mu})} \\
&= \frac{B(\mu)I(0)}{B(\mu)+\alpha(1-\mu)} \frac{1}{1+\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}} \\
&+ \frac{\lambda(1-\mu)}{B(\mu)+\alpha(1-\mu)} \frac{1}{1+\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}} \\
&+ \frac{\lambda}{B(\mu)+\alpha(1-\mu)} \frac{\mu u^{\mu}}{1+\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}}.
\end{aligned}
$$

Taking the inverse ST leads to

$$
\begin{aligned}
I(t) &= \frac{B(\mu)I(0)}{B(\mu)+\alpha(1-\mu)} \mathcal{S}^{-1}\left[\frac{1}{1+\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}}\right] \\
&+ \frac{\lambda(1-\mu)}{B(\mu)+\alpha(1-\mu)} \mathcal{S}^{-1}\left[\frac{1}{1+\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}}\right] \\
&+ \frac{\lambda}{\alpha}\mathcal{S}^{-1}\left[\frac{\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}}{1+\frac{\alpha\mu u^{\mu}}{B(\mu)+\alpha(1-\mu)}}\right].
\end{aligned}
$$

According to Definition 2.5(i) and (ii),

$$
\begin{aligned}
I(t) &= \frac{B(\mu)I(0)}{B(\mu)+\alpha(1-\mu)} E_{\mu}\left(-\frac{\alpha\mu t^{\mu}}{B(\mu)+\alpha(1-\mu)}\right) \\
&+ \frac{\lambda(1-\mu)}{B(\mu)+\alpha(1-\mu)} E_{\mu}\left(-\frac{\alpha\mu t^{\mu}}{B(\mu)+\alpha(1-\mu)}\right) \\
&+ \frac{\lambda}{\alpha}\left[1-E_{\mu}\left(-\frac{\alpha\mu t^{\mu}}{B(\mu)+\alpha(1-\mu)}\right)\right].
\end{aligned}
$$

### 3.2. Hybrid sumudu variational method

A hybrid of ST with variational iterative method is referred to as Hybrid Sumudu Variational (HSV) method.[44] Delays are inherently bonded with several dynamical systems. However, in general, dynamical systems with delays often defy the well known analytical methods for obtaining their exact solutions. HSV method is an efficient approximate analytical method for obtaining the solutions of the delay differential equations. HSV method often leads to the exact solutions of the dynamical systems. Aibinu and Momoniat[44] presented HSV method for Pantograph-type equations, which is a special kind of delay differential equations. This paper considers a most general form of delay differential equations that are given by Caputo fractional derivative, CFC and ABC. This paper presents HSV method for the delay differential equations that are given by the Caputo fractional derivative. The procedures of the presentation of HSV method for the differential equations that are given by CFC and ABC are similar to what is presented in this paper.

#### 3.2.1. Fractional derivative differential equations with linear coefficients

Consider

$$
{}_{a}^{C}D^{\mu}w(t) + \Phi[w(t)] + \Psi[w(t-\tau)] = g(t), \tag{3.6}
$$

where $\tau > 0$ is the delay in time, $\Phi$ is a linear operator, $\Psi$ is a nonlinear operator and $g(t)$ is a given continuous function. Taking the ST of (3.6) as

$$
\mathcal{S}\left[{}_{a}^{C}D^{\mu}w(t)\right] = \mathcal{S}\left[g(t) - \Phi[w(t)] - \Psi[w(t-\tau)]\right],
$$

and implementing (2.1) with $a = 0$, leads to

$$
u^{-\mu}(W(u) - w(0)) = \mathcal{S}\left[g(t) - \Phi[w(t)] - \Psi[w(t-\tau)]\right].
$$

Consequently, the iteration of HSV formula is given as

$$
\begin{aligned}
W_{n+1}(u) = W_n(u) + \phi(u)\Bigg(&\frac{W_n(u)-w(0)}{u^{\mu}} \\
&- \mathcal{S}\left[g(t) - \Phi[w(t)] - \Psi[w(t-\tau)]\right]\Bigg), n \in \mathbb{N}.
\end{aligned} \tag{3.7}
$$

Considering $\mathcal{S}[\Phi[w(t)] + \Psi[w(t-\tau)]]$ as the restricted term and taking the classical variation operator on both sides of (3.7) gives

$$
\delta W_{n+1}(u) = \delta W_n(u) + \phi(u)\frac{1}{u^{\mu}}\delta W_n(u),
$$

and consequently the Lagrange multiplier is obtained as

$$
\phi(u) = -u^{\mu}. \tag{3.8}
$$

Substitute (3.8) into (3.7) and then take its inverse ST to obtain the explicit iteration formula

$$
\begin{aligned}
w_{n+1}(t) &= w_n(t) + \mathcal{S}^{-1}\Bigg[-u^{\mu}\Bigg(\frac{W_n(u)-w(0)}{u^{\mu}} \\
&\quad - \mathcal{S}\left[g(t) - \Phi\left[w_n(t)\right] - \Psi\left[w_n(t-\tau)\right]\right]\Bigg)\Bigg] \\
&= w_1(t) + \mathcal{S}^{-1}\Bigg[u^{\mu}\mathcal{S}\Bigg[g(t) \\
&\quad - \Phi\left[w_n(t)\right] - \Psi\left[w_n(t-\tau)\right]\Bigg]\Bigg],
\end{aligned}
$$

with the initial approximation that is given as
$w_1(t) = \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{-w(0)}{u^{\mu}}\right)\right] = w(0)\mathcal{S}^{-1}[1] = w(0).$

#### 3.2.2. Fractional derivative differential equations with variable coefficients

Consider

$$
{}_{a}^{C}D^{\mu}w(t) + \beta\Phi_1[w(t)] + \psi(t)\Phi_2[w(t)] + \Psi[w(t-\tau)] = g(t), \tag{3.9}
$$

where $\beta$ is a constant, $\psi(t)$ is a variable coefficient, $\Phi_1$ and $\Phi_2$ denote linear operators and other terms remain as defined in (3.6). Taking the ST of (3.9) and by similar computations as in Section 3.2.1, it produces the iteration

$$
\begin{aligned}
W_{n+1}(u) = W_n(u) + \phi(u)\Bigg(&\frac{W_n(u)-w(0)}{u^{\mu}} - \mathcal{S}\Bigg[g(t) - \beta\Phi_1[w(t)] \\
&- \psi(t)\Phi_2[w(t)] - \Psi[w(t-\tau)]\Bigg]\Bigg), n \in \mathbb{N}.
\end{aligned} \tag{3.10}
$$

Considering $\mathcal{S}\left[\psi(t)\Phi_2[w(t)] + \Psi[w(t-\tau)]\right]$ as the restricted term in taking the classical variation operator on both sides of (3.10) yields

$$
\delta W_{n+1}(u) = \delta W_n(u) + \phi(u)\frac{1}{u^{\mu}}\delta W_n(u),
$$

from which the Lagrange multiplier is obtained as

$$
\phi(u) = -u^{\mu}.
$$

Substituting for $\phi(u)$ in (3.10) and taking its inverse ST yields the explicit iteration formula

$$
\begin{aligned}
w_{n+1}(t) &= w_n(t) + \mathcal{S}^{-1}\Bigg[-u^{\mu}\Bigg(\frac{W_n(u)-w_0}{u^{\mu}} - \mathcal{S}\Bigg[g(t) \\
&\quad - \beta\Phi_1[w(t)] - \psi(t)\Phi_2[w(t)] - \Psi\left[w_n(t-\tau)\right]\Bigg]\Bigg)\Bigg] \\
&= w_1(t) + \mathcal{S}^{-1}\Bigg[u^{\mu}\mathcal{S}\Bigg[g(t) - \beta\Phi_1[w(t)] \\
&\quad - \psi(t)\Phi_2[w(t)] - \Psi\left[w_n(t-\tau)\right]\Bigg]\Bigg],
\end{aligned}
$$

where $w_1(t) = \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{-w(0)}{u^{\mu}}\right)\right] = w(0)\mathcal{S}^{-1}[1] = w(0).$

### 3.3. Application of hybrid sumudu variational method

The efficiency of the HSV method will be demonstrated by applying it to find the solution of a mathematical model of a radioactive material.

#### 3.3.1. Radioactive model with Caputo fractional derivative without delays

The exact solution of (3.2) has been obtained by using the ST method. Here, HSV method will be applied to solve (3.2) to justify the potency of the HSV method in producing the exact solutions. Taking the ST of (3.2) as

$$\mathcal{S}\left[{}_0^C D^{\mu} I(t)\right] = \mathcal{S}\left[\lambda - \alpha I(t)\right],$$

and applying (2.1), lead to

$$u^{-\mu}\left(I(u) - I(0)\right) = \mathcal{S}\left[\lambda - \alpha I(t)\right].$$

Consequently, the iteration of HSV formula is

$$I_{n+1}(u) = I_n(u) + \phi(u)\left(\frac{I_n(u) - I(0)}{u^{\mu}} - \mathcal{S}\left[\lambda - \alpha I_n(t)\right]\right), n \in \mathbb{N}. \tag{3.11}$$

Considering $\mathcal{S}[\alpha I(t)]$ as the restricted term and taking the classical variation operator on both sides of (3.7) gives

$$\delta I_{n+1}(u) = \delta I_n(u) + \phi(u)\frac{1}{u^{\mu}}\delta I_n(u),$$

and consequently gives the Lagrange multiplier as

$$\phi(u) = -u^{\mu}. \tag{3.12}$$

Substitute (3.12) into (3.11) and then take its inverse ST to obtain the explicit iteration formula

$$\begin{aligned} I_{n+1}(t) &= I_n(t) + \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{I_n(u) - I(0)}{u^{\mu}} - \mathcal{S}\left[\lambda - \alpha I_n(t)\right]\right)\right] \\ &= I_1(t) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\lambda - \alpha I_n(t)\right]\right], \end{aligned}$$

with the initial approximation that is given as $I_1(t) = \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{-I(0)}{u^{\mu}}\right)\right] = I(0)\mathcal{S}^{-1}[1] = I(0)$. Therefore, the iteration is given by

$$\begin{cases} I_{n+1}(t) = I(0) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\lambda - \alpha I_n(t)\right]\right], \ n \in \mathbb{N}, \\ I_1(t) = I(0). \end{cases}$$

Accordingly,

$$\begin{aligned} I_2(t) &= I_1(t) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\lambda - \alpha I_1(t)\right]\right] \\ &= I(0) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\lambda - \alpha I(0)\right]\right] \\ &= I(0) + (\lambda - \alpha I(0))\,\mathcal{S}^{-1}\left[u^{\mu}\right] \\ &= I(0) + (\lambda - \alpha I(0))\frac{t^{\mu}}{\Gamma(\mu+1)} \\ &= I(0)\left(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)}\right) + \frac{\lambda t^{\mu}}{\Gamma(\mu+1)}. \end{aligned}$$

$$\begin{aligned} I_3(t) &= I(0) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\lambda - \alpha I_2(t)\right]\right] \\ &= I(0) + \mathcal{S}^{-1}\Bigg[u^{\mu}\mathcal{S}\Bigg[\lambda \\ &\qquad - \alpha\left\{I(0)\left(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)}\right) + \frac{\lambda t^{\mu}}{\Gamma(\mu+1)}\right\}\Bigg]\Bigg] \\ &= I(0) + \mathcal{S}^{-1}\left[u^{\mu}\left(\lambda - \alpha\left\{I(0)\left(1 - \alpha u^{\mu}\right) + \lambda u^{\mu}\right\}\right)\right] \\ &= I(0) + \mathcal{S}^{-1}\left[\lambda\left(u^{\mu} - \alpha u^{2\mu}\right) - I(0)\left(\alpha u^{\mu} - \alpha^2 u^{2\mu}\right)\right] \\ &= I(0) + \lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}\right) \\ &\qquad - I(0)\left(\frac{\alpha t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)}\right) \\ &= I(0)\left(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)} + \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)}\right) \\ &\qquad + \lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}\right) \end{aligned}$$

$$\begin{aligned} I_4(t) &= I(0) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\lambda - \alpha I_3(t)\right]\right] \\ &= I(0) + \mathcal{S}^{-1}\Bigg[u^{\mu}\mathcal{S}\Bigg[\lambda \\ &\qquad - \alpha\Bigg\{I(0)\left(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)} + \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)}\right) \\ &\qquad + \lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}\right)\Bigg\}\Bigg]\Bigg] \\ &= I(0) + \mathcal{S}^{-1}\Bigg[u^{\mu}\Bigg(\lambda - \alpha\Bigg\{I(0)\left(1 - \alpha u^{\mu} + \alpha^2 u^{2\mu}\right) \\ &\qquad + \lambda\left(u^{\mu} - \alpha u^{2\mu}\right)\Bigg\}\Bigg)\Bigg] \\ &= I(0) + \mathcal{S}^{-1}\Big[\lambda\left(u^{\mu} - \alpha u^{2\mu} + \alpha^2 u^{3\mu}\right) \\ &\qquad - I(0)\left(\alpha u^{\mu} - \alpha^2 u^{2\mu} + \alpha^3 u^{3\mu}\right)\Big] \\ &= I(0) + \lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)} + \frac{\alpha^2 t^{3\mu}}{\Gamma(3\mu+1)}\right) \\ &\qquad - I(0)\left(\frac{\alpha t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)} + \frac{\alpha^3 t^{3\mu}}{\Gamma(3\mu+1)}\right) \\ &= I(0)\left(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)} + \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)} - \frac{\alpha^3 t^{3\mu}}{\Gamma(3\mu+1)}\right) \\ &\qquad + \lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)} - \frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)} + \frac{\alpha^2 t^{3\mu}}{\Gamma(3\mu+1)}\right). \end{aligned}$$

$$\begin{aligned} I_n(t) &= I(0)\Bigg(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)} + \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)} - \frac{\alpha^3 t^{3\mu}}{\Gamma(3\mu+1)} \\ &\qquad + \cdots + \frac{\alpha^n t^{n\mu}}{\Gamma(n\mu+1)}\Bigg) + \lambda\Bigg(\frac{t^{\mu}}{\Gamma(\mu+1)} \\ &\qquad - \frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)} + \frac{\alpha^2 t^{3\mu}}{\Gamma(3\mu+1)} - \cdots + \frac{\alpha^{n-1} t^{n\mu}}{\Gamma(n\mu+1)}\Bigg) \\ &= I(0)\Bigg(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)} + \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)} - \frac{\alpha^3 t^{3\mu}}{\Gamma(3\mu+1)} \\ &\qquad + \cdots + \frac{(-\alpha t^{\mu})^n}{\Gamma(n\mu+1)}\Bigg) + \frac{\lambda}{\alpha}\Bigg\{1 - \Bigg(1 - \frac{\alpha t^{\mu}}{\Gamma(\mu+1)} \\ &\qquad + \frac{\alpha^2 t^{2\mu}}{\Gamma(2\mu+1)} - \frac{\alpha^3 t^{3\mu}}{\Gamma(3\mu+1)} + \cdots + \frac{(-\alpha t^{\mu})^n}{\Gamma(n\mu+1)}\Bigg)\Bigg\} \\ &= I(0)\sum_{k=0}^{n}\frac{(-\alpha t^{\mu})^k}{\Gamma(k\mu+1)} + \frac{\lambda}{\alpha}\left(1 - \sum_{k=0}^{n}\frac{(-\alpha t^{\mu})^k}{\Gamma(k\mu+1)}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} I(t) &= \lim_{n\to\infty} I_n(t) = I(0)E_{\mu}\left(-\alpha t^{\mu}\right) \\ &\quad + \frac{\lambda}{\alpha}\left(1 - E_{\mu}\left(-\alpha t^{\mu}\right)\right). \end{aligned} \tag{3.13}$$

Observe that (3.13) is the same as (3.3). This proves the potency of the HSV method as an efficient approximate analytical method that produces exact solutions of the differential equations, both with integer-order and fractional derivatives.

#### 3.3.2. Radioactive model with Caputo fractional derivative and delays

In this section, delays are introduced into the mathematical model with Caputo fractional derivative that describes the decay of $^{135}I$. The presence of delays in mathematical models can improve their adequacy in the description of the systems under consideration. The delays in mathematical models can render the ST method to be impotent for solving them. The effectiveness of the HSV method in solving the delay differential equations of various forms will be demonstrated. An illustration will be given to show the efficiency of the HSV method over the ST method by applying the HSV method to solve the delay equation with Caputo fractional derivative that models the decay of $^{135}I$. Consider an analogue of (3.2) for the decay of $^{135}I$ with a time delay that is given by

$${}_0^C D^{\mu} I(t) = \lambda - \alpha I(\zeta t), \tag{3.14}$$

where $\zeta > 0$ ($\zeta \neq 1$). Eq. (3.14) is a typical pantograph-type with fractional order derivative.[63] For $0 < \zeta < 1$,(3.14) is a special case of delay differential equations with varying delay $\tau(t) = (1 - \zeta)t > 0$.

**Remark 3.1.** Efforts to solve (3.14) by using the ST method prove to be abortive due to the presence of the time delay in the equation. HSV method is presented in Section 3.2 and it will be applied to solve (3.14).

Taking the ST of (3.14) and applying (2.1) lead to

$$u^{-\mu}\left(I(u)-I(0)\right)=\lambda-\alpha\mathcal{S}\left[I(\zeta t)\right].$$

Consequently, the iteration of HSV method is

$$I_{n+1}(u)=I_n(u)+\phi(u)\left(\frac{I_n(u)-I(0)}{u^{\mu}}+\alpha\mathcal{S}\left[I(\zeta t)\right]-\lambda\right),n\in\mathbb{N}. \tag{3.15}$$

Considering $I(\zeta t)$ as the restricted term and taking the classical variation operator on both sides of (3.15) gives

$$\phi(u)=-u^{\mu}.$$

Substitute $\phi(u)$ into (3.15) and take its inverse ST to obtain

$$\begin{aligned}I_{n+1}(t) &= I_n(t)+\mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{I_n(u)-I(0)}{u^{\mu}}+\alpha\mathcal{S}\left[I_n(\zeta t)\right]-\lambda\right)\right]\\ &= I_1(t)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\mathcal{S}\left[I_n(\zeta t)\right]-\lambda\right)\right],\end{aligned}$$

where $I_1(t)=\mathcal{S}^{-1}[I(0)]=I(0)\mathcal{S}^{-1}[1]=I(0)$, is the initial approximation. Consequently, the successive iteration formula is obtained as

$$\begin{cases}I_1(t)=I(0),\\ I_{n+1}(t)=I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\mathcal{S}\left[I_n(\zeta t)\right]-\lambda\right)\right].\end{cases} \tag{3.16}$$

The order of convergence of (3.16) has been discussed where the error and order of convergence are mathematically derived for a general class of problems.[64] Notice that $I_1(\zeta t)=I(0)$, therefore

$$\begin{aligned}I_2(t) &= I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\mathcal{S}\left[I_1(\zeta t)\right]-\lambda\right)\right]\\ &= I(0)-\mathcal{S}^{-1}\left[\left(\alpha I(0)-\lambda\right)u^{\mu}\right]\\ &= I(0)-\left(\alpha I(0)-\lambda\right)\mathcal{S}^{-1}\left[u^{\mu}\right]\\ &= I(0)-\left(\alpha I(0)-\lambda\right)\frac{t^{\mu}}{\Gamma(\mu+1)}\\ &= I(0)\left(1-\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}\right)+\lambda\frac{t^{\mu}}{\Gamma(\mu+1)}.\end{aligned}$$

Notice that $I_2(\zeta t)=I(0)\left(1-\frac{\alpha(\zeta t)^{\mu}}{\Gamma(\mu+1)}\right)+\lambda\frac{(\zeta t)^{\mu}}{\Gamma(\mu+1)}$, consequently

$$\begin{aligned}I_3(t) &= I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\mathcal{S}\left[I_2(\zeta t)\right]-\lambda\right)\right]\\ &= I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\mathcal{S}\left[I(0)\left(1-\frac{\alpha(\zeta t)^{\mu}}{\Gamma(\mu+1)}\right)+\lambda\frac{(\zeta t)^{\mu}}{\Gamma(\mu+1)}\right]-\lambda\right)\right]\\ &= I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\left(I(0)\left(1-\alpha(\zeta u)^{\mu}\right)+\lambda(\zeta u)^{\mu}\right)-\lambda\right)\right]\\ &= I(0)-\mathcal{S}^{-1}\left[I(0)\left(\alpha u^{\mu}-\alpha^2\zeta^{\mu}u^{2\mu}\right)+\alpha\lambda\zeta^{\mu}u^{2\mu}-\lambda u^{\mu}\right]\\ &= I(0)-I(0)\left(\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}-\zeta^{\mu}\frac{\alpha^2t^{2\mu}}{\Gamma(2\mu+1)}\right)\\ &\quad -\lambda\zeta^{\mu}\frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}+\lambda\frac{t^{\mu}}{\Gamma(\mu+1)}\\ &= I(0)\left(1-\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}+\zeta^{\mu}\frac{\alpha^2t^{2\mu}}{\Gamma(2\mu+1)}\right)\\ &\quad +\lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)}-\zeta^{\mu}\frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}\right).\end{aligned}$$

Notice that

$$\begin{aligned}I_3(\zeta t) &= I(0)\left(1-\zeta^{\mu}\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}+\zeta^{3\mu}\frac{\alpha^2t^{2\mu}}{\Gamma(2\mu+1)}\right)\\ &\quad +\lambda\left(\zeta^{\mu}\frac{t^{\mu}}{\Gamma(\mu+1)}-\zeta^{3\mu}\frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}\right),\end{aligned}$$

therefore

$$\begin{aligned}I_4(t) &= I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha\mathcal{S}\left[I_3(\zeta t)\right]-\lambda\right)\right]\\ &= I(0)-\mathcal{S}^{-1}\Bigg[u^{\mu}\Bigg(\alpha\mathcal{S}\Bigg[I(0)\Bigg(1-\zeta^{\mu}\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}+\zeta^{3\mu}\frac{\alpha^2t^{2\mu}}{\Gamma(2\mu+1)}\Bigg)+\lambda\Bigg(\zeta^{\mu}\frac{t^{\mu}}{\Gamma(\mu+1)}\\ &\quad -\zeta^{3\mu}\frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}\Bigg)\Bigg]-\lambda\Bigg)\Bigg]\\ &= I(0)-\mathcal{S}^{-1}\left[u^{\mu}\left(\alpha I(0)\left(1-\alpha\zeta^{\mu}u^{\mu}+\alpha^2\zeta^{3\mu}u^{2\mu}\right)+\alpha\lambda\left(\zeta^{\mu}u^{\mu}-\alpha\zeta^{3\mu}u^{2\mu}\right)-\lambda\right)\right]\\ &= I(0)-\mathcal{S}^{-1}\left[\alpha I(0)\left(u^{\mu}-\alpha\zeta^{\mu}u^{2\mu}+\alpha^2\zeta^{3\mu}u^{3\mu}\right)+\alpha\lambda\left(\zeta^{\mu}u^{2\mu}-\alpha\zeta^{3\mu}u^{3\mu}\right)-\lambda u^{\mu}\right]\\ &= I(0)-\alpha I(0)\left(\frac{t^{\mu}}{\Gamma(\mu+1)}-\zeta^{\mu}\frac{\alpha t^{2\mu}}{\Gamma(2\mu+1)}+\zeta^{3\mu}\frac{\alpha^2t^{3\mu}}{\Gamma(3\mu+1)}\right)-\alpha\lambda\left(\zeta^{\mu}\frac{t^{2\mu}}{\Gamma(2\mu+1)}\right.\\ &\quad \left.-\zeta^{3\mu}\frac{\alpha t^{3\mu}}{\Gamma(3\mu+1)}\right)+\lambda\frac{t^{\mu}}{\Gamma(\mu+1)}\\ &= I(0)\left(1-\alpha\frac{t^{\mu}}{\Gamma(\mu+1)}+\alpha^2\zeta^{\mu}\frac{t^{2\mu}}{\Gamma(2\mu+1)}-\alpha^3\zeta^{3\mu}\frac{t^{3\mu}}{\Gamma(3\mu+1)}\right)+\lambda\left(\frac{t^{\mu}}{\Gamma(\mu+1)}\right.\\ &\quad \left.-\alpha\zeta^{\mu}\frac{t^{2\mu}}{\Gamma(2\mu+1)}+\alpha^2\zeta^{3\mu}\frac{t^{3\mu}}{\Gamma(3\mu+1)}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}I_n(t) &= I(0)\left(1-\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}+\zeta^{\mu}\frac{(\alpha t^{\mu})^2}{\Gamma(2\mu+1)}-\zeta^{3\mu}\frac{(\alpha t^{\mu})^3}{\Gamma(3\mu+1)}+\cdots+\zeta^{\frac{n}{2}(n-1)\mu}\frac{(-\alpha t^{\mu})^n}{\Gamma(n\mu+1)}\right)\\ &\quad +\frac{\lambda}{\alpha}\left\{1-\left(1-\frac{\alpha t^{\mu}}{\Gamma(\mu+1)}+\zeta^{\mu}\frac{(\alpha t^{\mu})^2}{\Gamma(2\mu+1)}-\zeta^{3\mu}\frac{(\alpha t^{\mu})^3}{\Gamma(3\mu+1)}+\cdots+\zeta^{\frac{n}{2}(n-1)\mu}\frac{(-\alpha t^{\mu})^n}{\Gamma(n\mu+1)}\right)\right\}\\ &= I(0)\sum_{k=0}^{n}\zeta^{\frac{k}{2}(k-1)\mu}\frac{(-\alpha t^{\mu})^k}{\Gamma(k\mu+1)}+\frac{\lambda}{\alpha}\left(1-\sum_{k=0}^{n}\zeta^{\frac{k}{2}(k-1)\mu}\frac{(-\alpha t^{\mu})^k}{\Gamma(k\mu+1)}\right)\end{aligned}$$

and

$$I(t)=\lim_{n\to\infty}I_n(t). \tag{3.17}$$

For the graphical display of the obtained solutions, real values are assigned to the parameters as follows: $I_0=1,\mu=0.85,\alpha=0.3,\lambda=0.1$ and $\zeta=0.5$.

Fig. 1 displays the graph of (1.2), which is the exact solution for (1.1). The model is given by the integer-order derivative and both ST and HSV methods produced its exact solution (1.2). The figure shows that the rate of change of the decay process is rapid at the beginning but it gradually reduces. Fig. 2 displays (3.3), which is the exact solution for (3.2). The application of ST and HSV methods to the model with fractional derivative produce the exact solution (3.3). A gradual change in the rate of decay process is observed in the figure. Fig. 3 displays (3.17), which is the approximate solution for (3.14). The model involves time delay and it is given by the fractional derivative. The HSV method is applied to obtain its approximate solution (3.17). A gradual change is observed in the decay process. Recall that half-life is the interval of time required for one-half of the atomic nuclei

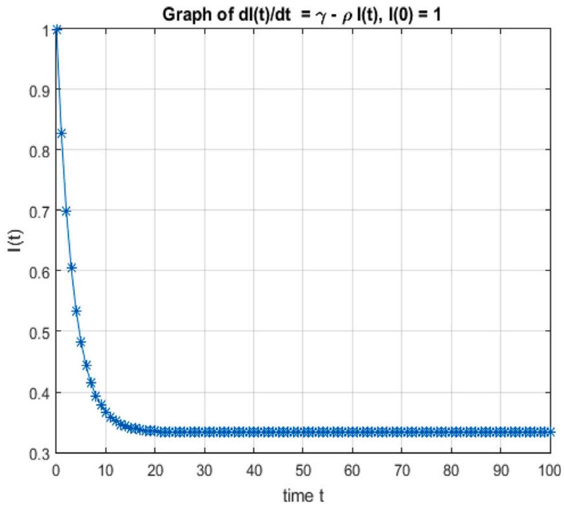


**Fig. 1.** Model given by the integer-order derivative.

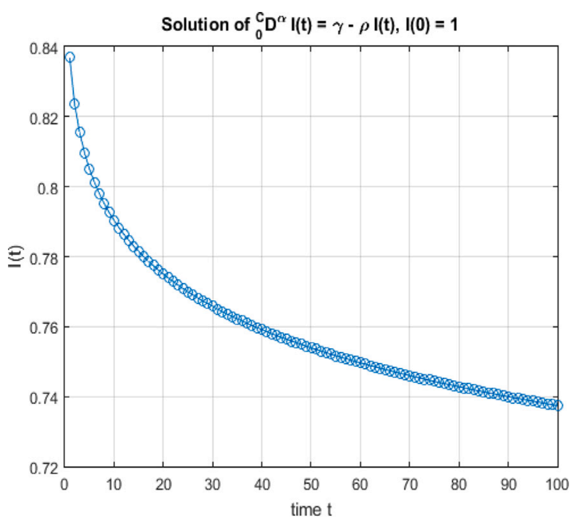


**Fig. 2.** Model given by the Caputo fractional derivative.

of a radioactive sample to decay. The graphs display the half-life of $^{135}I$ over the specified period. Figs. 1 and 2 show the comparison between the model given that is by the fractional derivative and the model that is given by the ODE, respectively while Figs. 1 and 3 show the comparison between the model that is given by the fractional derivative, which associated with time delay and the model given that is by the ordinary differential equation, respectively. The presence of the fractional derivatives and time delay in the models provide more degree of freedom in the experimental simulations.

## 4. Conclusion

The paper displayed the efficiency of the ST and HSV methods. ST method is an integral transform method and HSV is its hybrid, which is an approximate analytical method. ST is the theoretical dual to the Laplace transform, and for each property of Laplace transform, corresponding property can be obtained for the ST. The obtained results show that ST and HSV methods often yield accurate solutions with few computation steps, and they do not impose any restriction on the solutions. Moreover, the paper considered a radioactive model that is given by the integer-order derivative. The paper proposed and studied the radioactive models with fractional derivatives and time delay, which

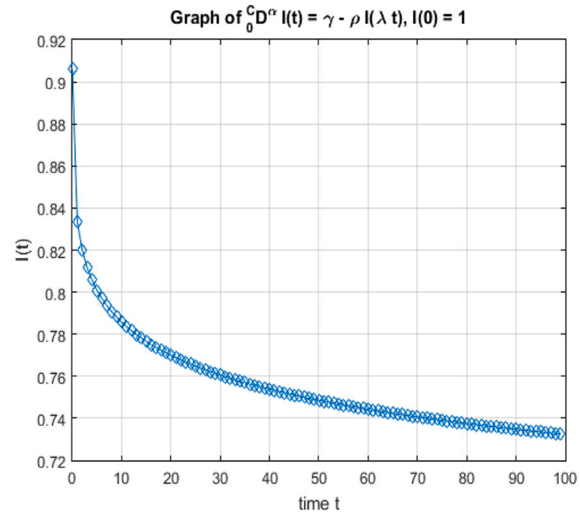


**Fig. 3.** Model with time delay and given by the Caputo fractional derivative.

are analogue of the model with integer-order derivative. The paper introduced two types of memories, which are the fractional derivatives and time delay into the radioactive models to enrich its dynamics. The two types of memories improve the vitality and suitability of the convectional radioactive model. Gradual changes are observed in the rate of the decay process of the models with memories. The study has various applications such as in the quantitative and qualitative analysis of the operation of nuclear and conventional power plants, degradation processes and characterization of nuclear reactor operation.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

Data will be made available on request.

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