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Learners' performance in arithmetic equivalences and linear equations

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A research project submitted to the Faculty of Science, School of Education, University of the Witwatersrand, in partial fulfilment of the requirements for the degree of Master of Science by combination of coursework and research.

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(Signature of candidate)

24th day of March, 2017

Abstract

This study investigates learners' performance in solving arithmetic equivalences and arithmetic and algebraic equations and was influenced by the notion of the didactic cut (Filloy & Rojano, 1989). Data was collected from two township schools in Johannesburg using a written test. With a Vygotskian perspective on learning, learners' performance was investigated in two ways: through a response pattern analysis of 106 test scripts as well as through an error analysis on 46 scripts. The response pattern analysis identified seven clusters of responses, each of which suggested a different performance pattern. Two clusters of responses suggest evidence of the didactic cut and that learners struggled with the concept of negativity. A purposive sample of 46 test scripts was analysed further to investigate the actual errors that learners made. Common errors within the two most relevant response pattern analyses were also investigated. Using a combination of typological and inductive methods to categorise learners' errors, equality and negativity errors were most prominent. Findings revealed that there were very few learners who used arithmetic strategies to solve arithmetic equations and that instead, they used algebraic procedures. The most unexpected finding was that learners appear to memorise the structure of solutions and hence manipulate their procedures in order to obtain familiar structured solutions.

Key words: Equality, equal sign, solving linear equations, negativity, learner error, response patterns

Dedication

I dedicate this study to my husband Ian, the love of my life, and my two beautiful children, Aidan Craig and Rachael Kate. I will make up for all the time I spent in front of my computer doing homework.

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Learners' performance in arithmetic equivalences and linear equations

Chapter 1 : Background and Rationale

1.1 Introduction

Mathematics is portrayed as a difficult and intimidating school subject (Arcavi, 2008) that is also poorly understood both on an international and national scale (Department of Basic Education (DBE), 2015; Kieran, 2006). It has been suggested that one way to improve learners' understanding of algebra is to have a smoother transition from arithmetic to algebra (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). For this transition, content areas in mathematics such as equality and the solving of linear equations have been identified as key understandings for abstract algebraic thought (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). This study therefore aims to gain a deeper appreciation of how learners perform when answering questions that relate to equality and solving linear equations. Learner performance in this study is defined as how a learner responds to test items. In order to investigate this, Grade 10 learners' test scripts were coded and analysed in two ways. Firstly, responses to six questions from a convenience sample of 106 learners' scripts were coded as correct, incorrect or missing. Patterns in these responses were then identified. Secondly, 46 learners' scripts were purposively chosen for a deeper analysis of the errors made in response to the six items.

In this chapter, I outline the focus of my study and situate it in the transition of arithmetic to algebra. I begin by explaining four words/ phrases that are used in this report and then discuss the background and rationale for my study. I present algebra as a difficult but important topic in the teaching and learning of mathematics and discuss it within the South African curriculum. My research questions are then presented.

1.2 Defining some phrases used in this report

Although I defined learner performance as how a learner responds to test items, *performance,* as used in this study, has two components: a) performance in terms of a series of responses to test items, and b) performance in terms of errors made.

The phrases arithmetic equivalence, arithmetic equation and algebraic equation are the focus of my study and hence I want to define them upfront. I then define the word *understanding* and *structure* as used in this report because I use them in a different way to how they are used in mathematics education literature.

Linear equations can be presented in many forms. In this report I refer to the following three forms of linear equations:

1) *Arithmetic equivalences:* These are equations that do not contain a letter, for example $6 + 9 = _ + 7$
2) *Arithmetic equations:* These are equations that contain one letter on only one side of the equal sign, for example $3x + 4 = 10$
3) *Algebraic equations*: These are equations that contain one letter but on both sides of the equal sign, for example: $3x + 4 = x + 10$

Understanding is a loaded word. It is widely accepted that learners should understand mathematics (Godino, 1996) but what is meant by *understanding* varies (Godino, 1996; Sierpinska, 1990; Skemp, 1978). I have considered definitions by Skemp (1978) on *relational* and *instrumental* understanding; Sierpinska's (1990) description of *acts of understandings* and Perkins' (1998) ideas on understanding in terms of performance and relationships. Some researchers (for example Kilpatrick, Swafford, and Findell (2001) and Sfard (2008)) have bypassed the word *understanding*, and have instead described how it could be recognised. It was initially an intended outcome of my study to define what it means to understand equations within school mathematics. This, however, proved to be a very complex task and thus one which I chose not to pursue. For the purposes of this study I consider evidence of understanding to be reflected in learners' ability to produce mathematically coherent responses to test items.

Structure is a word that emerged later in this study as I grappled with the analysis of learners' errors. I use the word *structure* to refer to the idea of learners paying attention to (or not paying attention to) the visual form of an equation. I do not use it with reference to other theoretical underpinnings, for example as in Mason, Stephens, and Watson (2009).

1.3 Background, context and rationale

Many learners see Mathematics as a subject that involves multiple rules and procedures which are often seen as meaningless (Hart et al., 1981; Kieran, 1992; Watson, 2009). Despite being a challenging school subject for most learners, if they do not have Mathematics at the Further Education and Training[1] (FET) level they are denied access into many tertiary programmes (Stephens et al., 2013). Mathematics is a gateway subject (Usiskin, 2004) into the Science, Technology, Engineering and Mathematical (STEM) career paths (Stephens et al., 2013) and poor performance in mathematics is preventing many learners from entering these programmes (Howie & Pietersen, 2001). Howie and Pietersen (2001) state that in 2001, only 12% of higher education students in South Africa were able to enter degree and diploma programmes in the engineering, life, physical and mathematical sciences. This problem is not unique to South Africa. Only 14% of high school students in the United States, in 2004, enrolled in a STEM field, of which only 1% was in Mathematics (Chen, 2009). Wang (2013) states that there has been and will continue to be a great increase in the demand for graduates in STEM fields. It is predicted that by the year 2018, 90% of occupations will require a degree that is dependent on mathematics (Wang, 2013). In today's world we need to improve learners' performance in mathematics, particularly in algebra, in order to increase the number of learners that enter the STEM disciplines.

Algebra is a generalised form of arithmetic and lays the foundation for more complex mathematical content (Arcavi, 2008). Although this branch of mathematics is widely researched and is a major part of the South African school curriculum, it is viewed as

[1] In South Africa the Further Education and Training level comprises of Grade 10, 11 and 12.

a difficult topic for learners (Watson, 2009). Researchers have suggested that one way to improve the poor performance in secondary mathematics is to bridge the gap between arithmetic and algebra by focusing on specific and difficult algebraic concepts such as equality, the use of brackets, negativity and the meaning of letters (Essien & Setati, 2006; Kieran, 1992; Knuth, Stephens, McNeil, & Alibali, 2006). This transition from arithmetic to algebra is more commonly researched amongst younger (typically aged 8-12) learners (Herscovics & Linchevski, 1994; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; Knuth et al., 2006).

The diagnostic reports for the Grade 9 Annual National Assessment (ANA) (DBE, 2014) as well as the Grade 12 Mathematics examination suggest that many high school learners (aged 13-17) have not yet made the shift into abstract algebraic thought:

> The algebraic skills of the learners were poor. They struggled with Mathematics in Grades 11 and 12 because they could not do the basic mathematics of Grades 8, 9 and 10. If this problem can be rectified, candidates will perform much better in the Grade 12 examination.
>
> (DBE, 2016, p. 151)

In 2013 the national average for the ANA was 14% with only 2% of learners obtaining more than 50% for the assessment (DBE, 2013). It is obvious that these results are a national crisis and it does shed some light on the high failure and drop-out rates in Grades 10 and 11 (Equal Education National Council, 2015). Even the 2015 National Senior Certificate (NSC) diagnostic report for mathematics states that in Grade 12, learners struggle with basic algebraic concepts such as the ability to do substitution; apply the distributive law; add like terms; factorise and solve linear equations (DBE, 2016). This struggle is not limited to South Africa (Araya et al., 2010; Knuth et al., 2008). It is important to note that struggling with basic algebraic concepts is not the only factor that is attributed to poor performance. There are many interrelated problems that influence the learning (and teaching) of mathematics (Setati & Adler, 2000).

Although mathematics, and in particular, algebra teaching and learning is a complex integration of social, cultural, political and linguistic aspects (Gorgorió & Planas, 2001), the focus of this study is on learner performance in algebra. This study forms part of the Wits Maths Connect Secondary (WMCS) project which is briefly discussed below as it has influenced many aspects of my research which is further discussed in Chapter 3.

1.4 Wits Maths Connect Secondary Project

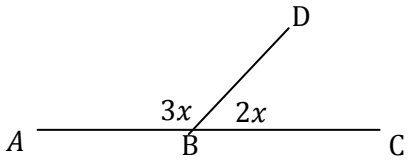
The Wits Maths Connect Secondary (WMCS) project began in 2010 with the goal of improving both the quality of mathematics teaching as well as learner performance. A Learning Gains (LG) project was conducted between 2013 and 2015 and the findings were published in 2015 (see Pournara, Hodgen, Adler, and Pillay (2015)). They report on the learning gains made by Grade 10 learners who were taught by teachers who participated in the Wits Professional Development Program. They found that although the learning gains were small, they were statistically significant and hence provide empirical evidence that improving teachers' mathematical knowledge can impact learning. In 2016 the WMCS project started conducting a new study focusing on Grade 9 learning gains. The new LG project was in its beginning and pilot phase when data was collected for my study. The pilot phase involved designing a test instrument and testing for its reliability and validity. I played a part in the development of the test items, administering the test and the initial coding of the responses. I was also involved in the interviews that took place. These interviews investigated whether the test items achieved what was intended. For my study the aspects of developing test items as well as investigating the gains made by learners are not a focus. Since my focus is on learner performance on test items, it would be useful for me to elaborate on the algebraic content and its position within the South African Mathematics curriculum.

1.5 Algebra and equations in the curriculum

Algebra is not a topic that exists in isolation of other mathematical topics, for example, there are elements of algebra in finance, calculus, geometry and

trigonometry. All these topics require some form of the manipulation of variables, solving equations, or simply understanding the relationship between two variables. These topics are all dependent on knowledge of, and the use of, basic algebraic skills. Algebra, on its own, covers a large percentage of both the General Education and Training[2] (GET) and FET curricula and this is evident in the suggested distribution of marks in the Grade 12 final Paper 1 examination (DBE, 2011a, 2011b). Although algebra is explicitly allocated 30% in Paper 1 alone, algebraic skills are tested in other topics and hence algebra covers more than a third of the examination. It is important to note that it is not only Paper 1 that requires algebraic manipulation. In Paper 2 foundations of algebra are needed to simplify and solve trigonometric expressions and geometric problems. An example of a Grade 9 geometry problem that requires algebraic knowledge would be:

If ABC is a straight line, determine the value of $C\hat{B}D$.



A learner who does not grasp mathematical facts, procedures, definitions, or concepts will find further mathematical topics more challenging (Schoenfeld, 2007) which in turn would result in making additional errors. Based on the large amount of time allocated to solving equations in Grade 9 (DBE, 2011a), the Grade 9 year is clearly a year where the teaching of equations is a focus. One would therefore expect that by the end of Grade 9, or at the beginning of Grade 10, learners would have a relatively sound grasp of equations. Below is a list of the algebraic content from Learning Outcome (LO) 2.3: Algebraic expressions and LO 2.4: Algebraic equations that are covered in Grade 8 and Grade 9. This content is thus expected to have been mastered by a Grade 10 learner:

- Identifying variables, constants, like and unlike terms, coefficients and exponents.

[2] In South Africa, the General Education and Training level comprises of Grade 7, Grade 8 and Grade 9.

- Recognise, interpret and use rules and laws such as the communicative, associative and distributive laws.
- Operating on, simplifying, factorising and substituting numerical values into polynomials and algebraic fractions.
- Set up, analyse, interpret and solve equations by inspection as well as by using additive and multiplicative inverses, the laws of exponents and factorisation.

(DBE, 2011a, 2011b)

1.6 Research problem and focus

The focus of this study is on equality and solving equations. Through my involvement in the WMCS project I have become aware of learners' difficulties in seeing the relationship between number patterns, expressions, solving equations and dealing with functions. It appears that learners see the above content as isolated areas of study, each with their own set of rules and procedures. I have become sensitive to learners' battles with the use and simplification of brackets, negatives, simplifying expressions and applying the additive and multiplicative inverses when solving equations. It appears that learners have difficulty in applying the 'rules' they learn in the correct places and that they confuse them with other previously learnt 'rules'. For example, consider the question to simplify $2a + 5a$. I have seen learners respond to this with $7a^2$ rather than $7a$. A possible reason for the incorrect response is that the 'new rule' for the learner is to add exponents when the bases are the same and they appear to use this rule whether asked to multiply or not. This over generalised rule appears to eliminate the 'old rule' of adding like terms. A possible reason that the previously learnt rules and concepts get confused, forgotten or eliminated is that they have only been partially learnt. Consider the question: Solve $2x + 1 = 23$. I have seen learners respond to this as follows:

Step 1: $2x = 23 - 1$

Step 2: $2x = 22$

Step 3: $x = 22 - 2$

Step 4: $x = 20$

A possible reason for the error made in step 3 is that the learner has over generalised the additive inverse rule or confused it with the multiplicative inverse rule. Literature confirms that learners encounter many obstacles as they progress through algebraic studies: obstacles such as knowing how to deal with variables (Kieran, 2006; Knuth et al., 2008); conjoining (Booth, 1988; Hart et al., 1981); application of the distributive law (Vermeulen, Olivier, & Human, 1996); viewing the equal sign as a relational symbol (Essien & Setati, 2006; Kieran, 1981; Knuth et al., 2008) and solving equations (Kieran, 2006). Kieran (1981) states that the concept of equivalence is difficult to grasp and not only important for primary school but that the view of the equal sign and solving linear equations are two important components within the topic of algebra in high school. This is especially true in Grades 8 and 9 when one's view of the equal sign needs to shift from being operational to include a relational view (Knuth et al., 2008; Schliemann, 2013). This means that learners need to shift their view from seeing the equal sign as a *to do something* symbol to seeing it as representing sameness (Kieran, 1992). These appear to be problematic concepts for learners to grasp (Cai & Moyer, 2008) and hence make the transition from arithmetic to algebra difficult.

From my experience as a teacher, many high school learners struggle with algebraic equations. I have noticed that when solving linear equations many learners battle to keep the balance of an equation, they do not add or subtract the same number on both sides of the equation; they use the additive inverse when they should be using the multiplicative inverse and they transform an equation into an expression. Where one would expect that perhaps the procedural fluency (Kilpatrick et al., 2001) (rather than the conceptual understanding) has been mastered by Grade 12, unfortunately it is not the case (DBE, 2015). This observation is not unique to South Africa (Araya et al., 2010) which suggests that the errors learners are making are "normal and necessary" (Brodie, 2014, p. 221) for the development of more complex concepts. This implies that learners *need* to make errors and encounter obstacles in order to learn. Literature has suggested that a reason for learners' struggles in solving equations is that they do not have a relational view of the equal sign but instead only

have an operational view (Essien & Setati, 2006; Kieran, 1981; Knuth et al., 2008; Schliemann, 2013).

I will be limiting the focus of this study to equation questions that include only addition, subtraction and multiplication (without brackets), for example $3x + 5 = x - 1$. I also limit my focus to equations that have a letter on one or both sides of the equal sign. This decision was influenced and inspired by the study conducted by Filloy and Rojano (1989) who identified a *didactic cut* (discussed in more detail in Section 2.6.3) between equations with letters on both sides of the equal sign and equations with only numerical values on one side.

1.7 Research questions

Two research questions are posed, each relating to a component of performance.

1) What patterns arise in learners' responses to items on solving arithmetic equivalences and arithmetic and algebraic equations?
2) What errors do learners make when solving arithmetic equivalences and arithmetic and algebraic equations?

The first question relates to learner performance in terms of learners' final responses to test items and does not take into account the detail of their responses. In Chapter 4 this research question will be further refined using the language of the study. The second question relates to learner performance in terms of the actual errors made within their responses. These questions were motivated by the discussion in Section 1.3 which alludes to a gap in research with older learners (aged 13-18) regarding the understanding of important concepts which are needed for a less bumpy transition from arithmetic to algebra. In high school[3] issues around equality are not made explicit to learners and at FET level issues around negativity and the meaning and interpretation of letters are assumed and not re-taught. This makes learning algebra and solving linear equations more difficult. This study aims to address the gap mentioned above and investigate how learners' perform in terms of response patterns and errors made. This is done in the hope that teachers and

[3] High school in South Africa comprises of Grades 8-12

researchers can have a deeper understanding of learners' responses to questions regarding solving equations. Knowing the common errors that learners make informs pedagogy (Ryan & Williams, 2007) and designing interventions.

1.8 Conclusion

In this chapter I have outlined the background, rationale and context of my study. Discussing how my study relates to the South African Mathematics curriculum has shown how both algebra and equations are relevant and important to mathematics education. My definition of performance has two components: a) performance in terms of a series of responses and b) performance in terms of errors made was the basis for my two research questions.

This research report has six chapters, two of which are devoted to analysing test scripts. The outline of this study is given below.

Chapter 2 provides a review of the literature pertaining to the theoretical frameworks that underpin my study. I provide a discussion on the nature of and the learning of mathematics, algebra and equations. Literature on error analysis in mathematics as well as specific errors relating to equality, negativity and letters are also discussed in Chapter 2. Chapter 3 discusses the research design, methodology and methods that were used for the data collection. Data was collected from test scripts and was analysed in two ways. I provide definitions and examples for the error categories used when conducting the detailed error analysis and elaborate on the measures taken to ensure rigour in my research. Issues relating to ethics as well as some of the limitations of my study are also discussed in Chapter 3. Chapter 4 focuses on answering my first research question which deals with identifying and discussing patterns found in learners' final responses when the responses are coded as correct, incorrect or missing. Chapter 5 focuses on answering my second research question which constitutes a detailed error analysis. Chapter 5 reveals that errors relating to equality and negatives are the most common errors made. Chapter 6 is a concluding chapter where I relate the findings from Chapter 4 to the findings in Chapter 5.

Chapter 2 : Literature Review

2.1 Introduction

This chapter gives a comprehensive review of literature that relates to the theoretical and conceptual frameworks that guided my study. I have separated this chapter into three parts: Part 1) My overarching perspective of learning; Part 2) Mathematics and algebra and Part 3) Solving linear equations.

I begin by presenting the socio-constructivist theory as my perspective on learning. A Vygotskian approach was chosen due to the semiotic focus of my study. Much of my study is about the equal sign, the minus sign and letters, and the errors learners make when dealing with these signs and symbols. These symbols are important in mathematics and algebra since the appropriate use of them creates a smoother transition from arithmetic into algebra as well as aids in the solving of linear equations. Matz's (1980) algebraic processes of reduction and deduction provided me with a structure to classify the key elements of solving equations as well as provided me with the language to help describe some of the learners' errors.

Part 1: Overarching perspective of learning

This section is a discussion on my perspective on learning. I take a Vygotskian perspective on learning mathematics and draw on some of his ideas on semiotic mediation.

2.1 Vygotsky and socio-constructivism

A fundamental assumption made when adopting a Vygotskian perspective is that learning is mediated and cannot be separated from the environment (Vygotsky, 1978). According to Vygotsky (1978), two central elements of teaching and learning are social interaction and semiotic mediation. From this perspective, learning is a social process and knowledge is constructed by building on prior knowledge. Learning is mediated through people as well as with the use of *signs*, *tools* and

artifacts. This study does not focus on social interaction and so I do not draw on several key Vygotskian ideas such as the role of language, the role of the more knowledgeable other or the Zone of Proximal Development. Rather, I draw on some of Vygotsky's ideas about semiotic mediation through the use of signs.

2.2 Semiotic mediation: Signs

Semiotic mediation is the use of signs, tools and artifacts to mediate learning. This means that semiotic mediators such as language; common/everyday signs (for example a stop sign, the facebook icon and emoticons); esoteric signs (for example the equal sign); or mnemonic aids (for example BODMAS) can be used to aid learning. The aspect of semiotic mediation on which I focus is that of signs. For clarity, from here on, I will refer to signs from a Vygotskian perspective as *signs* (italicised) and when referring to the addition sign "+", minus sign "−"and the equal sign "=" the word 'sign' will not be italicised. The semiotic mediation through the use of *signs* is therefore where learners attach meaning to a sign or symbol that is not instantly relevant to their life. I will be focusing on the sense learners make when working with the equal sign and minus sign. Vygotsky (1978) defined the role of *signs* as oriented inward, that it is "a means of internal activity aimed at mastering oneself" or mastering one's behaviour (Vygotsky, 1978, p. 55). *Signs* are used to mediate the psychological or mental world of an individual. Where a tool or artifact would be something external, a *sign* is something that can be internalised and used mentally. *Signs* are symbols that cause an intended reaction but do not interfere directly with the process. For example, the equal sign would be a *sign* that causes the intended reaction of solving an equation but does not interfere with the procedure that the learner executes. For a learner to solve for an unknown and execute the procedure correctly s/he would need to have internalised the correct meaning of the equal sign.

Part 2: Mathematics and algebra

This section provides a discussion on mathematics and algebra. I present a discussion on what mathematics is, doing and learning mathematics, as well as the importance

of error analysis in mathematics education research and practice. I then present an in-depth discussion of algebra, the transition from arithmetic to algebra and identify key elements needed to make this transition less difficult. These key elements include equality, negativity and the meaning of letters.

2.3 What is mathematics and how is it learnt?

2.3.1 Perspective on the nature of mathematics

Mathematics is a body of knowledge that includes theorems, proofs, definitions, conjecturing, procedures and finding patterns. Mathematics is also about the activity and solving both theoretical and real world problems. My view of mathematics is that it is *fallible* (Lerman, 1990). This means that I regard mathematics as being a combination of experiences that a learner can access, draw upon and use. It is the process that is important and not just the content (Lerman, 1990). Mathematics is characterised by the activities involved in doing mathematics and not solely on how to do the mathematics. It is a human activity, a social construction invented by humans, where they can create, accept or reject ideas. Mathematics does not depend on procedures alone.

In contrast to viewing mathematics as fallible, my experience has been that mathematics on a school level is viewed as *absolute* (Lerman, 1990). An absolutist view of mathematics is concerned with the procedures and reinforces the belief that mathematics can be, and is possibly best learnt, by rote. It appears that many learners value content and answers over the process and that the different mathematical topics are seen as disjoint sections, each with a set of rules. Learners do not appear to, for example, view theorems and definitions as man-made or fallible. Although in my experience learners get an absolutist view of mathematics, the GET curriculum states that mathematics is viewed as "…a human activity that involves observing… relationships... between mathematical objects…" (DBE, 2011a, p. 8). This implies that the processes used in school mathematics are intended to be emphasised and embraced and that school mathematics is in fact seen as fallible. It

appears that there is a contradiction or gap between the aims of the intended curriculum and how it is experienced. While curriculum is not a focus of my study, I acknowledge that a gap exists between the intended, implemented and the attained curriculum (Jansen and Taylor (2003); Taylor (2000)).

Although I hold the view that mathematics is fallible and more than just content and procedure, this study focuses on a key aspect of school mathematics which *is* very procedural but also very important. The discussion below is of researchers who also see mathematics as a human activity and offer frameworks that involve the 'doing' and learning mathematics.

2.3.2 Doing and learning mathematics

To be proficient at doing and learning mathematics, Kilpatrick et al. (2001) offer a framework to describe mathematical proficiency. They suggest five interwoven strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (Kilpatrick et al., 2001). This study focuses mainly on procedural fluency. Procedural fluency is the skill in carrying out procedures flexibly, accurately, efficiently and appropriately (Kilpatrick et al., 2001). A different framework is offered by Ball (2002), who encourages her learners to develop three core mathematical practices: justification, generalisation and representation. She asserts that learners who develop these three practices will be mathematically competent. What Kilpatrick et al (2001) and Ball (2002) have in common is that they view mathematics as a process and not solely consisting of rules and "strings of symbols" (Davis & Hersh, 1998, p. 319). Viewing mathematics as fallible, where the process and activity are important, the errors learners make when doing mathematics should therefore be welcomed as opportunities for learning.

2.3.3 Error analysis in mathematics education research and practice

From a socio-constructivist perspective errors are embraced and seen as opportunities for learning (Nesher, 1987). There is much literature on analysing learners' mathematical errors, especially on the nature of learners' errors and

possible reasons for their underlying misconceptions (see for example: Borasi (1987); Erlwanger (1973); Godden, Mbekwa, and Julie (2013); Nesher (1987); Olivier (1989); Ryan and Williams (2007)). The literature mentioned above is evidence that analysing learners' errors has been of great interest to the mathematical community for many years and therefore this inquiry adds to an important area of mathematics education research. Another important reason to study learners' errors is for the pedagogical implications (Corder, 1982). Mathematical errors can provide valuable insights for a teacher into their learners' thinking (Brodie, 2014). Research can inform teachers on how instruction could be altered and how tasks could be designed to address the problems that the error analysis helped diagnose (Hiebert & Carpenter, 1992).

2.3.4 The nature of learner error

I draw on Nesher's (1987) theory of errors and hence view errors as rational and rooted in some (mis)understanding of prior knowledge. I also adopt the view that errors are "normal and necessary" (Brodie, 2014, p. 221) and that "each error has the potential to become a significant milestone in learning" (Nesher, 1987, p. 39). It is important to note that although the errors made may at times seem illogical to the teacher or researcher, there is often an underlying cause which makes the error not a random but logical, rational attempt (Ben-Zeev, 1998) that "originated in a consistent conceptual framework based on earlier acquired knowledge"(Olivier, 1989, p. 8). All the authors mentioned above agree that learner errors should be seen as an opportunity for teaching and learning and not as a problem that should be avoided. In contrast to errors being systematic with some underlying conceptual cause, slips are defined as a lapse of concentration while solving a task (Nesher, 1987) and are easily rectified when pointed out (Olivier, 1989). For the purposes of the response pattern analysis presented in this study (see Chapter 4), slips are considered as incorrect responses (since the final reponse was incorrect). However, for the purpose of the error analysis (see Chapter 5), slips have been coded as correct responses as, from my interpretation, the error made was not due to a misunderstanding but rather to an oversight. Where slips can be dealt with easily,

dealing with systematic errors is more complex (Luneta & Makonye, 2013). There are three reasons for this: 1) errors are persistent and resistant to instruction (Godden et al. (2013); Smith, Disessa, and Roschelle (1994)); 2) errors arise even though they are not explicitly taught (Hatano, 1996); 3) similar errors occur amongst learners in different classes, schools and even countries (Hodgen, Küchemann, Brown, & Coe, 2009). Errors pertaining to solving linear equations, the minus sign and letters are discussed in Sections 2.5 and 2.6 .

2.4 What is algebra and how is it learnt?

In this section I explain what it means to 'do' algebra and discuss my view of algebra as generalised arithmetic. I highlight some of the differences between arithmetic and algebra and identify three key elements (equality, negativity and the meaning of letters) that are needed for a smoother transition from arithmetic to algebra.

2.4.1 Algebra as generalised arithmetic

There are different schools of thought about what constitutes school algebra but there is a general consensus that it can be approached from the following perspectives: functional; generalised arithmetic; problem solving; procedural or structural (Bednarz, Kieran, & Lee, 1996; Lins & Kaput, 2004; Sfard, 1995; Usiskin, 1988). In both the GET and FET curriculum, rational expressions and equations (a traditional view of school algebra) and a functional orientation to algebra (a reformist view) are employed. This is evident when learners are expected, for example, to use substitution to generate tables of ordered pairs.

Due to my study focusing on learner performance in solving linear equations, which is influenced by the transition from arithmetic to algebra, I have adopted the perspective of algebra as generalised arithmetic. I acknowledge that from a teaching perspective taking this view alone is inadequate becasue it does not promote connenctions between representations in the way that, for example, a functional approach might. For example, from a generalised arithmetic perspective, in the

equation $2x + 4 = 3x - 2$ the letter x is seen as a value that is not yet known rather than as a variable belonging to two functions ($f(x) = 2x + 4$ and $f(x) = 3x - 2$). For the purpose of this study a generalised arithmetic approach is appropriate. Generalised arithmetic is explained by Kieran (2004, p. 24) as "the unknown takes priority over the variable". Expressions and equations are viewed as generalised forms of numeric processes (meaning that it expresses general arithmetic rules using letters (Watson, 2009)) rather than, for example, as functional relationships.

Even though algebra can be viewed as a generalised form of arithmetic, there are fundamental differences between them. Both involve the use and understanding of abstract symbols and operations such as the equal sign (Sfard, 1991) but they operate on different levels of abstraction: arithmetic is limited to number and numerical computations (Norton & Cooper, 2001; Sfard & Linchevski, 1994) whereas algebra incorporates generalisations of numeric structures as well as the manipulation of these structures (Kieran, 1992). These differences require a shift in thinking.

2.4.2 Doing and learning algebra

2.4.2.1 Manipulating symbols

Doing algebra involves manipulating symbols, simplifying and solving equations, as well as modelling, graphing and generalising situations. I, however, only focus on manipulating symbols and solving equations. Derbyshire (2006) argues that the shift to symbols and symbolism in algebra is motivated by the need and desire to solve equations and determine unknown quantities. An additional difference is that arithmetic procedures, as taught in primary schools, typically use the equal sign as an operational or *unidirectional symbol* (Essien, 2009). In high school learners' view of the equal sign is expected to expand to include a relational and multidirectional view (Norton & Cooper, 2001). Learners are expected to differentiate between using the equal sign as a relational symbol and an operational symbol. This shift in understanding appears not only to be problematic for learners (Kieran, 1981), but also makes transforming equations difficult (Kieran, 1992).

2.4.2.2 Aiming for algebraic thinking: the transition from arithmetic to algebra

Love (1986, p.49) states that thinking algebraically is

> "not merely 'giving meaning to the symbols', but another level beyond that: concerning itself with those modes of thought that are essentially algebraic- for example, handling the as yet unknown, inverting and reversing operations…Being aware of these processes and in control of them".
>
> (cited in Kieran (2004, p. 25))

Algebraic thinking is the close examination of numerical relationships and processes in order to generalise or, for example, to solve an equation (Caspi & Sfard, 2012). The transition between arithmetic and algebra requires multiple cognitive adjustments that all learners are expected to make (Kilpatrick et al., 2001). In order to prepare learners for algebraic thinking, research has suggested the transition between arithmetic and algebra (Filloy & Rojano, 1989) should focus on the meaning of letters and operations. In arithmetic $5+\frac{1}{3}=5\frac{1}{3}$ but in algebra $5+a\neq 5a$. When interpreting $5\frac{1}{3}$ and $5a$, the numeric example represents addition and the algebraic example represents the repeated addition of a five times, $a+a+a+a+a$.

One of the biggest differences between arithmetic and algebra relates to the procedures involved in any given calculation. When doing arithmetic calculations, for example $2+5$, the procedure of adding 5 to 2 is different from the object, 7, that is obtained (Herscovics & Linchevski, 1994). When presented with unlike terms, for example $a+b$, the procedure of adding b to a is the same as the object, $a+b$, that is obtained. When adding like terms, for example, adding $3a$ to $5a$, the result $8a$ is a different object from $3a+5a$. Although $8a$ and $3a+5a$ are equivalent, they look different. Hence, when doing arithmetic one does not need to visualise, understand or use a group of numbers as an object, whereas in algebra one is constantly acting on unknowns as objects (Sfard, 1991). Arithmetic is procedural but in algebra the

procedure is part of the object (Kieran, 1992). This shift in thinking explains why learners battle, for example, with the concept of adding like terms.

Herscovics and Linchevski (1994, p. 59) refer to the difficulties in the transition from working with numbers in arithmetic to dealing with unknowns in algebra as a "cognitive gap" (p.63). Looking more specifically within this gap Filloy and Rojano (1989) argue that there exists a didactic cut between solving equations of the form $Ax + B = C$ and $Ax + B = Cx + D$. This is discussed in more detail in Section 2.6 when I elaborate on the content involved and errors made when solving equations. The manipulation process involved in solving equations is as much of a "conceptual object in algebra learning as are the typical algebra objects" (Kieran, 2004, p. 25) such as letters, the minus sign and the equal sign. The letters, minus sign and equal sign are all symbols that mediate learning. These signs are intended to mediate the mental world of the learner, be internalised and used mentally. Meaning should be attached to these symbols so that they can be used in mathematically appropriate ways.

2.5 Algebraic elements needed for a smooth transition from arithmetic to algebra

Literature identifies three elements as being fundamental to the transition from arithmetic to algebra are: equality, negativity and the meaning of letters.

2.5.1 Working with the equal sign

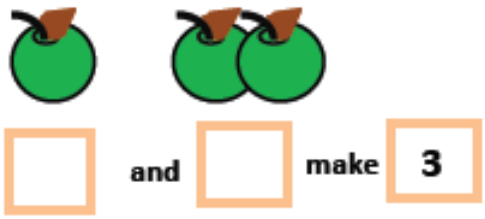
2.5.1.1 Views of the equal sign

One of the core algebraic ideas needed for the successful transition from arithmetic to algebra is related to the view of the equal sign (Essien and Setati (2006); Knuth et al. (2008)). It is vitally important that the equal sign is viewed as a symbol of equivalence when solving equations. Stephens et al. (2013) offer three different views of the equal sign: 1) an operational view, 2) a relational-computational view and 3) a relational-structural view. A learner with an operational view will respond to $7 + 5 = _ + 2$, with 12 in the blank space because $7 + 5$ yields 12. The equal sign

here is a signal to get an answer. Using the same example, a learner with a relational-computational view would reason that the left hand side computes to 12 so the right hand side should also give you 12. This learner would write 10 in the blank space. A relational-structural view sees a relationship between the left and right hand side. The value 2 is three less than 5 and so the blank space should be three more than 7 if the balance of the equation is to be maintained. This view requires a deeper and more flexible understanding of the equal sign as it draws on a learners "ability to 'see' abstract ideas hidden behind symbols" (Stephens et al., 2013, p. 174). Having a relational-computational or a relational-structural view of the equal sign requires relational thinking. The term *relational thinking* is being used in this report as: "attending to relations and fundamental properties of arithmetic operations... rather than focusing exclusively on procedures for calculating answers" (Carpenter, Levi, Franke, & Zeringue, 2005, p. 53). This means that to think relationally is to look at the expression on either side of the equal sign as well as looking at equation as a whole rather than viewing the equation as a procedure that needs to be carried out in a series of steps.

2.5.1.2 Equality

Stacey and MacGregor (1999) report that learners view the equal sign as a cue to perform an action. This is evident in the way the equal sign is assigned the words *makes* or *gives*. Learners are introduced to these words in Grade 1 mathematics:



(DBE, 2011c, p. 111)

The sign/ symbol '=' is also introduced in Grade 1 and it is seen when dealing with money, for example, $\mathrm{R5} + \mathrm{R1} = \mathrm{R6}$ (DBE, 2011c), or when answering number sentences, for example:

1 + 1 = □	2 – 1 = □
2 + 2 = □	4 – 2 = □
3 + 3 = □	6 – 3 = □
4 + 4 = □	8 – 4 = □

(DBE, 2011c, p. 169)

Throughout primary school learners are confronted with the equal sign operating as a *to do something* symbol. Learners begin to understand the symbol '=' as a signal to find the answer, hence often having an operation on the left hand side of the symbol and a single value on the right. Even when introduced to integers or when beginning to solve linear equations by inspection (at the end of Grade 7), learners deal with the equal sign in an operational way. For example: $7 - 4 = 7 + (-4) = 3$ (DBE, 2011a, p. 67) and

a) Solve x if $x + 4 = 7$, where x is a natural number. (What must be added to 4 to give 7?)
b) Solve x if $x + 4 = -7$, where x is an integer. (What must be added to 4 to give -7?)
c) Solve x if $2x = 30$, where x is a natural number. (What must be multiplied by 2 to give 30?)

(DBE, 2011a, p. 64)

In high school, the equal sign needs to be seen as a relational symbol rather than an operational one. Knuth et al. (2006) argue that this shift, from an operational to a relational view, is an essential aspect when learning and understanding algebra. This means that learners need to view the equal sign in terms of 'sameness' where the left hand side is equivalent to the right hand side.

Much research has shown that many primary school learners view the equal sign as a unidirectional or as a *to do something* symbol (Carpenter, Franke, and Levi (2003); Essien and Setati (2006); Kieran (1981); Knuth et al. (2006)). Essien and Setati (2006) discuss a unidirectional view of the equal sign which incorporates both right-to-left

and left-to-right reasoning. The issue is not in which direction the learners are operating but rather that they only operate in one direction. In order to have a relational view of the equal sign, learners need to discern the detail (Mason, 1998) in both the left and right sides and work in both directions at the same time. Moreover, research suggests that without a relational view of the equal sign, maintaining equality becomes meaningless and learners are reduced to memorising many rules to transform equations (Falkner, Levi, & Carpenter, 1999). Stephens et al. (2013) argue that when a calculation involves small positive numbers, the calculations are straight forward and distinguishing between a relational-computational and relational-structural view is difficult. When large numbers are used, it is easier to observe whether learners need to calculate (having a relational-computational view) or whether they are able to determine the missing number by observing the structure (having a relational-structural view). Hence, using large numbers may encourage learners to think relationally about how the two sides of the equal sign relate rather than where learners just do a procedural calculation.

2.5.2 Working with the minus sign

Gallardo and Rojano (1994) and Vlassis (2004) have done extensive research in learners' understandings of negative numbers and the minus sign. Gallardo and Rojano (1994) identify three main functions of the minus sign and Vlassis (2004) uses the term *negativity* to describe having a deep and flexible understanding of its different functions. Dealing with the minus sign in high school involves conceptual changes from what learners experienced in primary school (Vlassis, 2004) and in algebra, learners are expected to be flexible with the multiple functions of it. In this report I use the term *minus sign* in the same way that Vlassis (2004) uses it, where it describes the actual symbol " – " without any reference to its function.

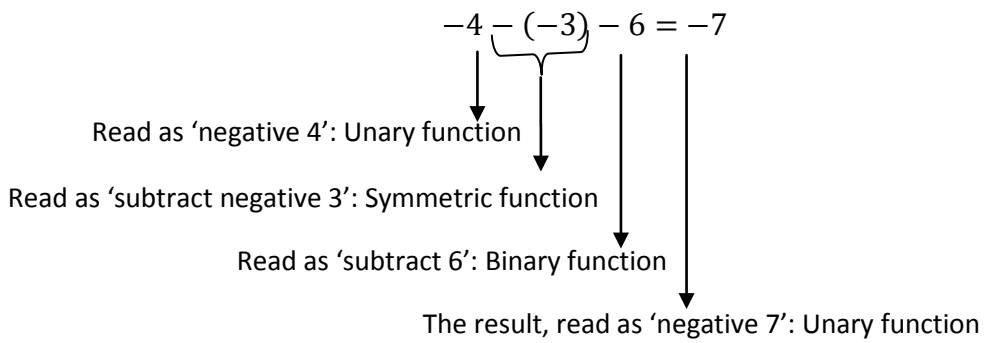
2.5.2.1 Triple nature of the minus sign

Gallardo and Rojano (1994) identify three main functions of the minus sign: The unary, binary and symmetric function. In primary school learners would probably

have only encountered the minus sign as an operation, for example as in *subtract* or *take away.*

The unary function of the minus sign is largely about the formal definition and understanding of negative numbers. It is where the minus sign is referred to as a "structural signifier" (Vlassis, 2004, p. 472). For example, considering the value -5, the '–' in front of the 5 is a sign that identifies 5 as a negative number rather than having subtracted 5 from another value. This is evident in, for example, solutions such as $x = -5$ or in an expression, for example $-5 + 4$ which is read as 'negative five add four rather than subtract five add four'.

The binary function is where the symbol '–' signifies an operation, for example in $13 - 8$ where 8 is subtracted or taken away from 13. The minus sign here is indicating a difference between two numbers which requires an action and is referred to as an "operational symbol" (Vlassis, 2004, p. 472). In primary school learners' only experience with the minus sign is as an operation. Subtraction in number sentences such as $4 - 1 = __$ are introduced in Grade 1 (DBE, 2011c) and according to the curriculum document, learners do not appear to encounter the minus sign performing a different function until Grade 8. In Grade 8, learners' previous understandings are challenged when the binary function of the minus sign can produce a negative number, hence including the unary function. In addition, high school learners are also confronted with number sentences that include the third function of the minus sign: the symmetric function (Gallardo & Rojano, 1994). This is where the minus sign includes "taking the opposite of a number" Vlassis (2008, p. 561). The example below shows all three functions in a single number sentence:



Moreover, in algebra the unary function of the minus sign in $'-x'$ causes much confusion. When solving linear equations it is vital that learners recognise that $(-x) + x = 0$ and that to solve an algebraic equation of the type $Ax + B = Cx + D$, the additive inverse of Cx, which is $-Cx$, needs to be applied. The additive inverse is a key concept in solving equations and is discussed further in Section 2.6.

In light of the above discussion on the multiple functions of the minus sign, it is not surprising that when learners reach high school and are faced with the symbol $'-'$, they produce many different errors.

2.5.2.2 Reasoning that creates errors in negativity

Learner errors in negativity are not only common in literature but were also common in my data. Although these errors are not the focus of my study the prevalence of negativity errors found in my data required me to further categorise them. The errors learners make relate to the types of reasoning they used and so I discuss three forms of reasoning around negativity found in literature. These forms of reasoning informed the error categories I used when analysing my data. The different types of reasoning discussed are: right-to-left reasoning; bracket reasoning and signs-rule reasoning.

2.5.2.2.1 Right-to-left reasoning

It appears that right-to-left reasoning (Vlassis, 2004) is used when an expression is simplified by operating from right to left. For example, when $3 - 8$ is read from right to left is $8 - 3$ which yields 5. Vlassis (2004, p. 477) argues that learners reverse the order to make operating with the minus sign more 'comfortable'. Although I agree with this interpretation, if the leading number is a negative number, for example $-8 + 3 \rightarrow 5$, then the same error cannot receive the same interpretation. What is possibly done with $-8 + 3 = 5$ is that the operations are reversed in order to obtain a positive answer. Ryan and Williams (2007) offer an explanation for the error $3 - 8 \rightarrow 5$ in that learners seem to be overgeneralising the commutative property of addition and applying it to subtraction. So where $a + b = b + a$,

learners overgeneralise this to $a - b = b - a$. This interpretation could also then be applied to $-8 + 3 \rightarrow 5$, which is seen to be the same as $-3 + 8 \rightarrow 5$. Hall (2002) categorised these types of errors as *number line errors* which, in my experience, does not describe the underlying problem. When learners reason from right to left or overgeneralise the commutative property of addition, it is because, from my experience, they require a positive solution and not because they went in the wrong direction on a number line. It is important to note that in primary school arithmetic, learners work with positive numbers and so subtracting a larger value from a smaller one does not make sense to them. Reasoning from right to left and overgeneralising the commutative property of addition therefore produces errors that are understandable. These types of errors are discussed in literature (Ryan & Williams, 2007) and are prevalent in my data. In Chapter 3, I use the category *minus sign used incorrectly* to incorporate both $3 - 8 \rightarrow 5$ and $-8 + 3 \rightarrow 5$ errors.

2.5.2.2.2 Bracket reasoning

Bracket reasoning (Vlassis, 2004) is characterised by inserting brackets (explicitly or mentally) for example $-5 + 3 \rightarrow -(5 + 3) \rightarrow -8$. The minus sign is detached from the 5 and the numerals 5 and 3 are operated on. Ryan and Williams (2007) argue that learners view the signs and numbers as separate objects and Herscovics and Linchevski (1994) term the error *detachment of the minus sign.*

When there are multiple terms in an expression, for example, $2x - 4 + x - 2$, Vlassis (2004) argues that learners re-group the like terms but do not focus on the operations and leave the signs in their original place. For example, the expression $2x - 4 + x - 2$ is reduced to $(2x - x) + (4 - 2)$ where the x and the 4 swap places so that the like terms are together. This error could be explained by learners overgeneralising the commutative property for addition but also by over generalising the associative property for addition and ignoring the operation. In the associative property, different groupings of addition do not affect the result, for example: $a + (b + c) = (a + b) + c$. In the numeric example: $-5 + 3 \rightarrow -(5 + 3) \rightarrow -8$ as

well as the algebraic example: $2x - 4 + x - 2 \rightarrow (2x - x) + (4 - 2)$, where bracket reasoning is applied, learners do not appear to have the signs in focus.

2.5.2.2.3 Signs-rule reasoning

Another form of reasoning offered in the literature is where there is an explicit focus on the operations. The error made with this form of reasoning is characterised by overgeneralising the *signs-rule* (Gallardo & Rojano, 1994; Vlassis, 2004). The signs-rule is the multiplication rule for integers: *a negative and a negative make a positive* and *a negative and a positive make a negative*. Using the same example as above, $-5 + 3$ could be simplified to -8 because a *negative and a positive make a negative* and 5 and 3 make 8. Another example is $-10 - 4 \rightarrow 14$ because a *negative and a negative make a positive*, and 10 and 4 make 14.

2.5.2.2.4 Too many signs reasoning

Gallardo and Rojano (1994) highlight a form of reasoning learners provide when learners deliberately leave out (or ignore) certain minus signs. They provide the example of a learner reducing $-a - (-b)$ to $-a - b$ because the minus sign in the bracket, attached to b, "is not required" (p. 162).

2.5.3 Working with letters

Letters are a source of difficulty for learners as they have different meanings depending on the context in which they are used (Watson, 2009). For example, the letter x in $2x + 5$ could vary and be any value whereas in $2x + 5 = 11$ there is only one value of x, $x = 3$ that will maintain equivalence. Küchemann (1981) discusses six interpretations of letters: 1) letter evaluated; 2) letter as a specific unknown; 3) letter as a generalised number; 4) letter as a variable number; 5) letter as an object and 6) letter not used. My study does not focus on the different types of errors made with letters but rather in a general sense that *letter errors* are made.

When learners make errors with letters they usually combine unlike terms. The source of this error has roots in arithmetic and once again shows how the transition from arithmetic to algebra is problematic. In arithmetic, simplification results in a single numerical value and so it appears that learners who respond to $5a - 2$ with $3a$ are actually 'deleting' a and treating the expression as $5 - 2$, and then attach the letter back onto the result (Kieran, 1992). Lewis (1980) talks about how viewing algebra as purely manipulating symbols has the potential to give the false idea that the aim in mathematical activity is "to get rid of things, rearrange and replace, until you have '$x =$ something' with no x in the something" (Lewis, 1980, p. 6). Watson (2009, p. 21) cites findings from Johnson (1989) and states that they "found that those who used the language 'getting rid of' were more likely to engage in superficial manipulation of symbols".

Küchemann (1981) would describe this error as a *letter ignored* where learners ignore, for example, the letter a in $5a - 2$ and then operating on the numerals 5 and -2. This error is also called *conjoining* (Kieran, 1992) . Literature relates this error to a *lack of closure* where learners need to have a single term as an answer (see for example Booth (1988); MacGregor and Stacey (1997)). Kieran (1992, p. 398) explains that learners who make this error are "over generalising certain mathematically valid operations, arriving at a single generic deletion operation that often produces incorrect results".

From my experience in the classroom and in working in the WMCS project, there appear to be different forms of conjoining. For example: $a + 3 = 3a$; $a + 3 = 4a$ and $a + 3a = 3a^2$. There does not appear to be a distinction between different types of conjoining in the literature and hence no language to distinguish them. For this reason, and because these types of errors are not a main focus of my analysis, I chose to group them together and call them *letter errors*.

Section 2.5 discusses three elements (equality, negativity and letters) that are needed to make a smoother transition from arithmetic to algebra. These signs and symbols are required to mediate learning, their meanings and uses are to be

internalised and used mentally. Of these signs and symbols, the equal sign is of most interest to this study as Knuth et al. (2006) suggests that learners' purely operational understanding of the equal sign prevents them from being successful in solving linear equations.

Part 3: Solving linear equations

This part of Chapter 2 begins with how I define arithmetic and algebraic equations. I then discuss key concepts (in addition to what was discussed in Section 2.5) that are needed to solve linear equations. These include balance and inverse operations as well as the reduction and the deduction process in algebra. I then discuss the didactic cut and common strategies used for solving linear equations.

2.6 Linear Equations

2.6.1 Definitions

An equation is created when two expressions are equated to each other. An equation that is linear is one where the highest power of the variable is 1. I adopt Filloy and Rojano (1989) definitions of arithmetic and algebraic linear equations where arithmetic equations are equations with a letter on only one side of the equal sign, for example, $Ax + B = C$, and algebraic equations with letters on both sides, for example, $Ax + B = Cx + D$. An equation in which there is no presence of a letter, for example $3 + __ = 11$, will be referred to as an arithmetic equivalence. In an arithmetic equivalence the structure can vary in the position of the missing number, just as in an arithmetic and algebraic equation, the structure can vary depending on the position of x.

There are many different ways in which an equation can be structured and made more complex. Examples of how to increase the cognitive complexity would be to a) have a letter on either side of the equal sign, b) inserting brackets that would need to be simplified, c) include fractions or d) introduce a negative (Hart et al., 1981; Watson, 2009). The structure could also vary in the number of terms on each side of

the equal sign and by which operations are used (Watson, 2009). Due to the nature of this study and the scope of the types of equations, I limit my focus to:

a) Arithmetic equivalence (for example $8 + 6 = _ + 3$);

b) Arithmetic equations (for example $2x + 5 = 19$) and

c) Algebraic equations (for example $2x + 5 = x + 19$).

The scope has been further limited to exclude division and the use of brackets.

2.6.2 Additional concepts needed when solving equations

Linsell (2009) offers four basic concepts that learners need to be familiar with in order to be proficient in solving linear equations. These are: 1) dealing with number and arithmetic structure; 2) dealing with letters (i.e. lack of closure); 3) notation, conventions and signs in general (such as the minus sign) and 4) dealing with equality (i.e. the equal sign). These are the same elements that were discussed in the transition from arithmetic to algebra. I now discuss two concepts that are related to equality but specific to solving equations: inverse operations and balance.

2.6.2.1 Inverse and balance

In mathematics it is imperative to know both the additive and multiplicative inverses. When solving equations, inverse operations need to be applied for the sole purpose of maintaining balance. Learners need to be able to determine an inverse and understand operations in order to solve equations when the guess and check strategy or number bonds no longer work (Watson, 2009). An additive inverse is a number that when added to the negative of that number yields the value zero. For example, the additive inverse of 3 is -3 because $3 + (-3) = 0$. In general, the additive inverse of a is $-a$ because $a + (-a) = 0$. A multiplicative inverse is the number one needs to multiply by in order to obtain the value 1. For example, the multiplicative inverse of 3 is $\frac{1}{3}$ because $3 \times \frac{1}{3} = 1$. In general, the multiplicative inverse of a is $\frac{1}{a}$ because $a \times \frac{1}{a} = 1$. All transformations involved in solving equations must maintain equivalence and balance. This means that the left and right hand sides of the equal sign need to stay the same. To keep something the same, one

could add 0 or one could multiply by 1. Hence the additive inverse yields 0 and the multiplicative inverse yields 1.

Errors related to inverses occur when learners are unable to identify what the inverse should be. For example, when $4x = 9$ is simplified to $x = 9 - 4$, we can see that the additive inverse was used rather than the multiplicative inverse. Hall (2002) calls this error *the other inverse* as when learners need to use the multiplicative inverse (for example), they do not and instead they use *the other inverse*, the additive inverse. Errors related to inverses are focused around identifying the value and operation that, when used, will maintain balance. Balance errors on the other hand are about how the inverse is applied. For example, if a learner responded to $3x + 7 = 15$ with $3x + 7 - 7 = 15 + 7$, we can see that -7 was chosen because when added to $+7$ the result is 0. The learner then adds 7 on the right hand side instead of subtracting. This error is termed *redistribution* (Kieran, 1992) and describes the situation when a learner explicitly adds different values to the left and right and hence does not maintain balance. This error suggests that learners are unsure of the structural relationship between addition and subtraction (Kieran, 1992).

Another variation of confusing the rule: *do to the left as you do to the right* is called *switching addends* (Kieran, 1992) and occurs when a learner tries to create symmetry with the value that they are trying to manipulate. For example, knowing that the $+7$ in $3x + 7 = 15$ *i*s what needs to be operated on, an additional $+7$ (the same value as was on the left hand side) is added to the right hand side of the equal sign: $3x = 7 + 15$. What was on the left hand side is now on the right hand side. Both the redistribution error and the switching addends error would result in the same incorrect answer. The difference is that the redistribution error explicitly adds different values to the left and right, whereas it appears that a value is 'moved' in the switching addends error. It looks as though the learner is moving a value to the right hand side of the equal sign in order for it to look like the left hand side and so does not change the sign.

These two concepts (inverse and balance) are didactically different to dealing with letters, addition of real numbers and the minus sign as they require deductive thinking. Adding like terms and operating with a minus sign are both actions that are involved in simplifying expressions. This is why they are concepts which, if understood, would smooth the transition from arithmetic to algebra as well as aid solving linear equations.

2.6.2.2 Deduction and reduction process in algebra

Matz (1980) states that when shifting from arithmetic to algebraic thought, learners are required to extend their operational view of the equal sign to incorporate two algebraic uses of the equal sign. The first use is in a *tautology* and the second in a *constraint equation.* Tautologies are statements of equivalent expressions, for example, $5(x+2) = 5x + 10$ or $5x + 4 + x - 1 = 6x + 3$. Matz (1980) states that simplifying expressions "produces a single chained sequence of tautologies where each link in the chain is some simplification or change in form of its predecessor" (p. 137). She refers to the transformations that produced each link as a *reduction*. From a different point of explanation, a tautology results in a solution that includes any real value of x (i.e., $x \in R$). An example of such an equation would be: $4x + 6 = 2(x+3) + 2x$ which is also known as an identity. Transformations such as adding like terms, dealing with numbers or the minus sign, or even applying the distributive rule would count as a reduction. A constraint equation on the other hand is an equation that is to be solved, for example, $5\,(x+2) = x + 18$. The solution of a (linear) constraint equation is limited to only one value. For example $x = 2$ is the only value that will satisfy the statement $5\,(x+2) = x + 18$. The processes involved in obtaining that solution go beyond mere reductions. In order to isolate x one needs to perform certain operations to both the left and right hand side. The term *deduction* is used to describe the transformation that produces a constraint equation in a different form. It is the understanding of inverses and balance that gives rise to a deduction. These two types of processes involved in solving linear equations (deductions and reductions) are demonstrated in Figure 2.1. Each reduction and deduction has been numbered individually for easy reference.

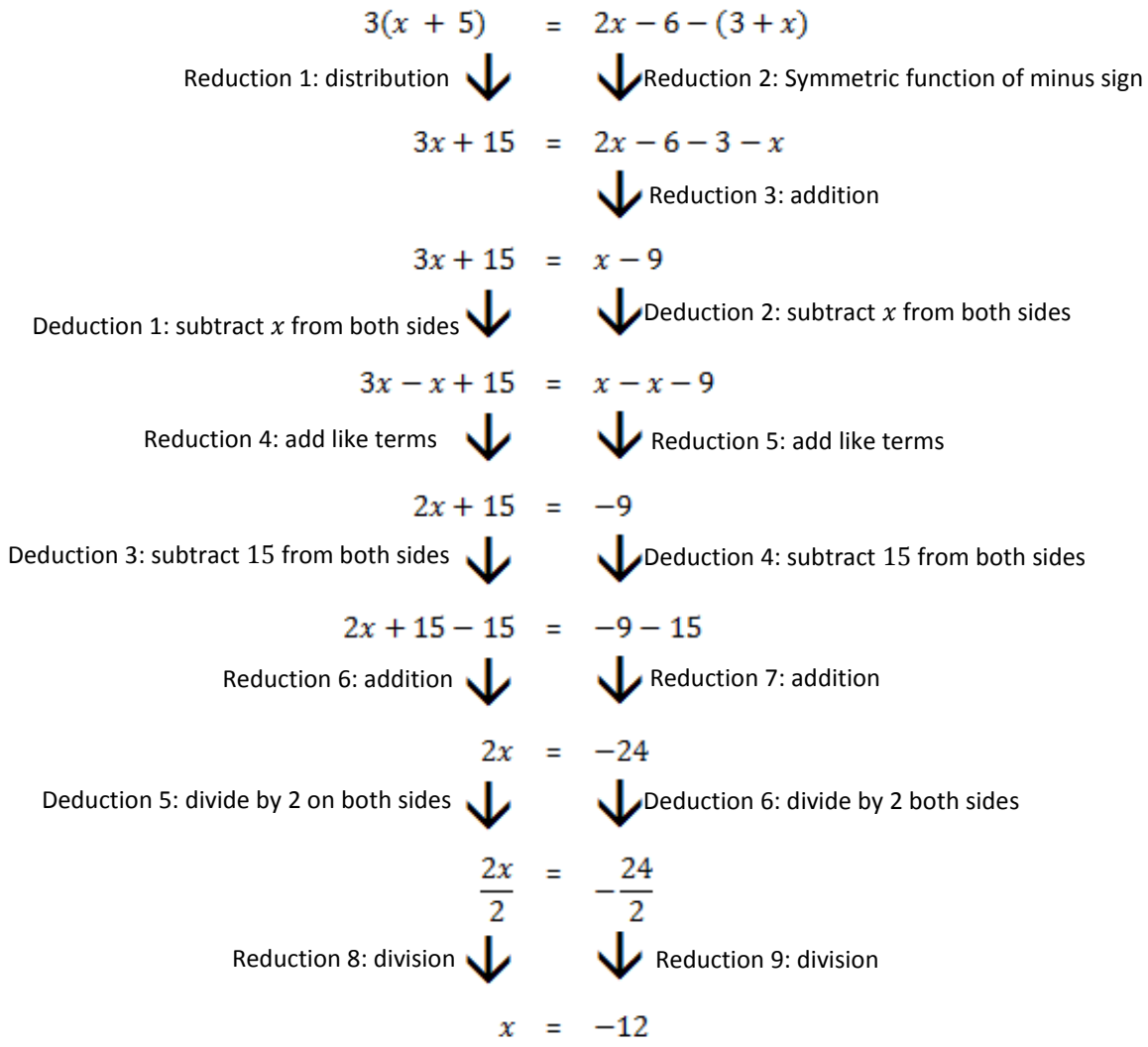


Figure 2.1: Reduction and deduction process

In terms of learning to solve linear equations, the deduction-reduction process involved can be confusing to learners who may not know which step or process they are working on, or which step to do first. For a low performing learner, deciding whether s/he should first add like terms (apply a reduction) or *do to the left as you do to the right* (apply a deduction) could be confusing. For example, in Figure 2.1 deductions 1 and 2 must be performed in the same step but reductions 4 and 5 could be done at different times in different steps. Also, for learners who are operating purely on memorised rules, it may be confusing to know when to operate on one side of the equal sign and when to operate on both sides. An error that was evident in my data related to learners' syntax around the deductions and the way they wrote them out. It is important that learners learn to write mathematically

(Powell, 2001). In these errors the learners' thinking appears to be correct but their writing was not mathematically correct. Kieran (1981) refers to *short cut* errors where, for example, the additive inverse is added on both sides of the equal sign but in separate steps or lines. Learners take short cuts, and abbreviate their steps. It is possible that learners make this error because they are attending to each process systematically, being explicit in all they do but not noticing that they have not maintained balance. The equal sign is not viewed as a relational symbol but rather as a symbol that separates each of their steps, and in a way, each piece of their thinking (Kieran, 1992). Learners continue to get the correct result despite not having a relational view of the equal sign. For example:

$$2x + 1 = 5$$

Deduction 1: Subtract 1 from left hand $2x + 1 - 1 = 5$

$$2x = 5 - 1$$ Deduction 2: Subtract 1 from right hand side

$$2x = 4$$

$$x = 2$$

Deduction 1 and 2 should take place at the same time in order to maintain equivalence. When learners respond as above, they *have* in fact *done to the left as they have done to the right*, what is missing is the understanding of maintaining balance.

2.6.3 Didactic cut

Filloy and Rojano (1989) conducted a study on 12-13 year olds where they investigated learners shifting from arithmetic to algebraic thought and focused on linear equations. They argue that to answer an arithmetic equation correctly, one does not need to act on the letter or treat it as an object but that it can be solved using arithmetic. When an equation is in the form $Ax + B = C$ "the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown) and the right side represents the consequence of having performed such operations" (Filloy & Rojano, 1989, p. 19). This type of equation can be solved using arithmetic strategies or by undoing each operation on the left. In contrast, an

algebraic equation of the form $Ax + B = Cx + D$, requires learners to engage with letters, operating on the unknown which in turn requires learners to view the equal sign as a relational symbol rather than just an operational one. The jump from arithmetic equations to algebraic equations has been theorised by what Filloy and Rojano (1989) call the "didactic cut". Herscovics and Linchevski (1994) similarly identified a cognitive gap in the transition from arithmetic to algebra when investigating Grade 7 learners' performance prior to any formal introduction to algebra. This gap can be characterised as learners' "inability to operate spontaneously with or on the unknown" (p.59). In a later study, Linchevski and Herscovics (1996) extended their study to include equations where a letter was on both sides of the equal sign. They found that whether the equation was of the form $Ax + B = C$ or $Ax + B = Cx + D$, there was a fundamental shift in the learners' procedures. Learners could only solve these equations by relying on numerical substitution. Although this is perhaps expected from Grade 7 learners, it does reinforce what Filloy and Rojano (1989) state about learners approaching the arithmetic equations ($Ax + B = C$) numerically.

A series of studies by Tall, de Lima and Healy (see for example de Lima and Tall (2008) and Tall, de Lima, and Healy (2014)) investigated learners' shifts from linear equations to quadratic equations. Their study involved 15-16 year old learners from six secondary schools in Brazil. These studies found that in solving linear equations learners did not solve arithmetic equations ($Ax + B = C$) by 'undoing' the operations nor by applying the inverse operations to both sides of the equal sign. Instead, they found that learners were mentally moving the algebraic objects and numbers around and applying rules such as *change sides, change signs* and hence experienced both arithmetic and algebraic equations ($Ax + B = Cx + D$) as equally difficult. The presence of the didactic cut was not evident in these studies and a possible reason for this is the difference in age of the learners and the likely possibility that they have been exposed to more algebra and algebraic procedure. Although Lima and Tall (2008) argue that their findings do not support the notion of the didactic cut, they did find that learners struggled to maintain balance when isolating the variable. This concurs with the findings of both Filloy and Rojano (1989)

and Linchevski and Herscovics (1996) that learners battle to operate on the unknown. I was therefore motivated to investigate how learners in a South African context would perform on arithmetic and algebraic equations. The learners in my sample are of similar age to those in Lima and Tall's (2008) study and have been learning algebra for at least 2 years.

2.6.4 Strategies for solving equations

There are various methods one can use to solve equations (Kieran, 1992). I only elaborate on three types of strategies as they are most relevant to my study: 1) arithmetic strategies, 2) inverse strategies and 3) the transposing strategy.

2.6.4.1 Arithmetic strategies

In South African high schools, the first method learners are exposed to when solving linear equations is the trial and improvement method (DBE, 2011a). This method is alternatively known as a *guess and check* method (Linsell, 2009). This strategy may work well for solving equations of the form $Ax + B = C$ and where the value of the unknown is a positive whole number. This strategy may also encourage learners' accuracy when substituting but Herscovics and Linchevski (1994) show that learners use this strategy as a crutch to solve more complex equations. This method does not demonstrate an understanding of equality and presumably, if the same learner were given an equation where the solution was not a whole number, s/he would struggle to determine the solution. This strategy is arithmetic and is what Filloy and Rojano (1989) argue as being the strategy used to solve arithmetic equations and is a strategy where learners do not need any formal knowledge of algebra.

2.6.4.2 Inverse operation strategies

In solving algebraic equations, Filloy and Rojano (1989) argue that it is with this type of equation that the arithmetic strategies fall short. To solve an equation with a letter on both sides of the equal sign, learners are required to operate with inverse operations. A strategy suited to working with inverse operations is called the balance

approach method (Kieran, 2006). The balance approach method is arguably the most conceptually benefiting (Kieran & Drijvers, 2006). For example, solving $2x + 3 = 21$ would typically be approached by subtracting three from both sides of the equation, and then dividing both sides by 2. The written procedure may resemble the following:

Solve for x: $2x + 3 = 21$

Step 1: $2x + 3 - 3 = 21 - 3$

Step 2: $2x = 18$

Step 3: $\frac{2x}{2} = \frac{18}{2}$

Step 4: $x = 9$

2.6.4.3 Transposition method

The method that, in my experience, is most common in South African classrooms is the *transposition* method which involves 'moving' letters and numbers from one side of the equal sign to the other. This method often emerges as the short cut for the balance method. Learners may decide that, using the example above, part of Step 1 is a waste of time or a longer route and this could cause them to skip the '$3 - 3$' and instead have $2x = 21 - 3$. The generalisation and rule that then emerges is that if you have a '$+3$', it 'moves' to the other side and becomes '-3'. Although short cuts and tricks can be useful in mathematics, and even though they can enhance learners' procedural proficiency, it is important that a learner understands that if one has an addition of three on one side of the equal sign it is equivalent to subtracting three on both sides of the equal sign.

Researchers have collectively identified multiple strategies that learners use to solve equations of the form $Ax + B = C$ and $Ax + B = Cx + D$ (Araya et al. (2010); Kieran (1992); Kieran and Drijvers (2006); Linsell (2009); Vlassis, (2002)). There is evidence that a hierarchical nature exists in terms of the strategies used and that the development of these strategies are an indication of a learners' understanding of algebra (Linsell, 2009).

2.7 Conclusion

From a socio-constructivist perspective signs and symbols are important in mediating learning. This chapter has reviewed literature to show how signs and symbols such as letters, the minus sign and the equal sign are important. There needs to be a shift in thinking when dealing with both a minus sign and an equal sign as they are viewed differently when used in algebra. The appearance of a letter on both sides of an equation also creates the need for a conceptual shift in understanding equality and variables. This chapter has discussed the shift from arithmetic to algebra and highlighted key concepts needed for both learning algebra and more specifically, to solve linear equations.

Equality and working with the equal sign is a crucial component when solving linear equations and I have discussed how in primary school an operational view dominates. Also related to equation solving is the understanding of inverses and balance, both of which are described, exemplified and then discussed in terms of what errors could be made. The reduction and deduction processes are additional elements that are presented as crucial but difficult for learners when solving equations. Different strategies to solve equations were also presented: arithmetic strategies, inverse operation strategies and the common strategy used in schools, the transposition strategy.

Working with the minus sign is another key component needed for solving linear equations. The three functions of the minus sign (Unary, binary and symmetric) as well as three forms of reasoning about the minus sign are discussed. Errors that result from these forms of reasoning are presented and exemplified.

In discussing these key concepts I have used literature to foreground some of the common errors made when doing mathematics. The error categories used to analyse my data are discussed In Chapter 3 and are elaborated on and analysed in Chapters 4 and 5. Where Chapter 2 discussed the literature pertaining to my study, Chapter 3 now discusses the methods used to answer my research questions.

Chapter 3 : Methodology

3.1 Introduction

This chapter focuses on the design of the study and gives an in-depth explanation of the methods chosen that enabled me to answer my research questions. As a reminder, my research questions are:

1) What patterns arise in learners' responses to items on solving arithmetic equivalences and arithmetic and algebraic equations?
2) What errors do learners make when solving arithmetic equivalences and arithmetic and algebraic equations?

This chapter is important because it knits together all the methodological elements of my study that describe how I gained information on learners' performance when solving arithmetic equivalences and arithmetic and algebraic linear equations. I begin this chapter with a discussion of my research design which is followed by the research methodology. I discuss my sample selection, the instruments used, my coding scheme and its limitations as well as the limitations of the design of my study in general. Methodologically I make a contribution in terms of a response pattern analysis. I show that there is value is doing such an analysis but also that certain types of data organisation can become idiosyncratic. I show how different organisations of data can highlight different information and can tell a different story.

3.2 Research design

A research paradigm is an established model of thinking that is widely accepted by a research community (Bertram & Christiansen, 2013). It is a combination of one's ontological, epistemological and methodological beliefs (Hatch, 2002). In other words it is the overlap of one's view of reality (ontological belief), how that reality is investigated (epistemological belief) and what instruments are used for the investigation (methodological beliefs). Because my ontological belief is that knowledge is created by social and contextual understanding and interactions, this paradigm best fits my research. It is also suitable because the ways in which one

comes to understand an insider's view is by studying the participants through qualitative methods such as document analysis (Hatch, 2002).

3.3 Research methodology

The purpose of this study is to gain insight into how learners perform when solving arithmetic equivalences and arithmetic and algebraic equations. In order to increase my understanding of learner performance, I focus on six questions from the LG test by identifying response patterns and analysing the errors made. Data collection involved a document analysis where 106 learners from two townships schools in Johannesburg were given a written test. The data was analysed in two phases. The first phase involved coding learners' final answers to the six items from the test. This phase was about identifying response patterns (as opposed to identifying and classifying errors) and relates to my first research question. Phase 2 involved selecting learners based on the response patterns found in phase 1 and conducting a detailed error analysis of the errors made. As mentioned in Section 1.4, my study is part of the LG project. This has meant that the two schools used, the Grade tested (Grade 10) and the test itself, were pre-determined. My study was adapted to accommodate these fixed components.

3.4 Sample

There were 106 Grade 10 learners from two secondary schools involved in my study. For ethical reasons the learners and the schools can't be named. Learners have been assigned numbers and are referred to as, for example, Learner 1 and the schools are referred to as School A and School B. Both schools are situated in a township in Johannesburg and cater for learners from poor socio-economic backgrounds. The schools are no-fee schools and the home languages of the learners are not English even though the language of learning and teaching is English. These schools were selected because of the existing relationship they have with the WMCS project. This study does not compare the two schools in the analysis.

3.5 Data collection instrument: test

The test used was developed by the WMCS project for the LG study and I was involved in all aspects of its design. My study fits into the piloting phase of this new test instrument for the LG project but I do not focus on test and item development. My study focuses on learner performance on the items that were being piloted. Using the Solo Taxonomy (Biggs & Collis, 2014), 71 test items were developed that covered Grade 8 and Grade 9 curricula content but were limited to integer work, substitution, simplifying expressions, factorising, equality, solving equations, function, and pattern. My study is only concerned with six of the nine items that relate to equality and solving linear equations. Although the content is aimed at Grade 8 and 9, the test was given to Grade 10 learners because Grade 9 learners at the beginning of the year would not have been taught the content. The test was administered after school in February under test conditions and calculators were not allowed to be used. At this time of year the Grade 10 learners had not yet done any work on equations and so would be relying on what they had learnt in Grade 9. I have not taken into account or differentiated between learners who are in Grade 10 for the first time and those who are repeating. The test items that were used in my study are listed below and discussed in more detail in Chapter 4 as part of my analysis.

Question 1: Write down the missing number in the space provided.

Item 1) $7 + 5 = _ + 2$

Item 2) $4747 + 3945 = _ + 3943$

Item 3) $4747 + n = _ + (n - 2)$

Question 2: Solve for the unknown.

Item 4) $3x - 1 = 5$

Item 5) $3x - 1 = 5 - x$

Item 6) $1 - 3x = 5 - x$

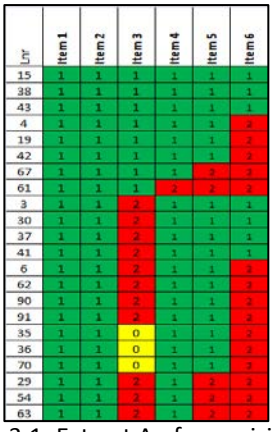
3.6 How the test data was coded and analysed

Although I used the same test scripts as the WMCS project, my analysis of the scripts had a different focus. I analysed the test data in two phases. Firstly, I performed a response pattern analysis on 106 scripts. This involved coding the final answers as *missing*, *correct* or *incorrect*. When this data is organised in a particular way, clusters of responses are foregrounded. I sorted the data in three different ways to show how different information becomes evident depending on the sort and how some information is visible irrespective of the type of sort. Secondly I analysed the errors made in 46 of the 106 scripts. Since items 1-3 and items 4-6 are different in nature, two coding schemes were used. The two phases of analysis and the two coding schemes for phase 2 are discussed below in separate sections.

3.7 Phase 1: Error pattern coding and how it is analysed

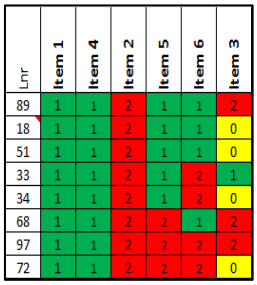
In phase 1, for the response pattern analysis, only final responses were coded. They were coded as missing (0: yellow), correct (1: green) or incorrect (2: red). Colour coding the categories in this way provided me with a visual, colour-coded summary of the learners' responses. The data was then organised in three different ways: 1) according to item number, 2) according to idem difficulty and 3) by posing a question. These resulted in three patterns that foregrounded different *clusters of responses.*

For example, Figure 3.1 is an extract of data that was sorted according to item number, meaning that column 1 is item 1, column 2 is item 2, etc. In this extract we see that there is much green (correct responses) but also that there is a red vertical strip (incorrect responses) in column 3. This pattern suggests that these learners can answer arithmetic equivalences correctly (Item 1 and 2) as well as most of the arithmetic and algebraic equations (items 4, 5 and 6) but could not correctly answer the item that did not point them towards a known procedure (item 3).



Lnr	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
15	1	1	1	1	1	1
38	1	1	1	1	1	1
43	1	1	1	1	1	1
4	1	1	1	1	1	2
19	1	1	1	1	1	2
42	1	1	1	1	1	2
67	1	1	1	1	2	2
61	1	1	1	2	2	2
3	1	1	2	1	1	1
30	1	1	2	1	1	1
37	1	1	2	1	1	1
41	1	1	2	1	1	1
6	1	1	2	1	1	2
62	1	1	2	1	1	2
90	1	1	2	1	1	2
91	1	1	2	1	1	2
35	1	1	0	1	1	2
36	1	1	0	1	1	2
70	1	1	0	1	1	2
29	1	1	2	1	2	2
54	1	1	2	1	2	2
63	1	1	2	1	2	2

Figure 3.1: Extract A of organising data



Lnr	Item 1	Item 4	Item 2	Item 5	Item 6	Item 3
89	1	1	2	1	1	2
18	1	1	2	1	1	0
51	1	1	2	1	1	0
33	1	1	2	1	2	1
34	1	1	2	1	2	0
68	1	1	2	2	1	2
97	1	1	2	2	2	2
72	1	1	2	2	2	0

Figure 3.2: Extract B of organising data

In contrast to this pattern, the extract in Figure 3.2, shows data being sorted according to item difficulty, meaning that the first column is of the item the learners found least difficult and the last column is the item the learners as a whole found most difficult. We can see that there are groups of learners who were able to answer the more difficult questions correctly but not an easier one (the item in the third column).

The third way that I sorted data was vastly different to the previous two as I posed a question and then filtered out the learners who responded in the way I was investigating. For example, one of the questions asked was: How did learners who could correctly answer the most difficult item perform on the other five items?

By taking a high level look at my data I was able to identify different clusters of responses that were suggestive of these learners' performance and understanding. This method of analysis informed my sample selection for the error analysis.

3.8 Phase 2: Detailed error analysis

A mix between a typological and an inductive data analysis was used to analyse the data for phase 2 (Hatch, 2002). Chapter 2 provided a detailed discussion of my perspective on the nature of errors and offered definitions and examples of each of the error classifications that are used in this study. I used categories of errors derived from literature as a preliminary way to classify the errors. Additional subcategories were added in order to classify errors found that were not found in the literature.

Since items 1-3 are different in nature to items 4-6, two coding schemes were employed.

3.8.1 Scripts selection process for detailed error analysis

From the 106 scripts that were analysed in terms of the response patterns, there were 46 scripts purposively selected for further coding and analysing in terms of the nature of errors made. In selecting these scripts a decision was made to exclude the 60 learners (57%) who got all the equations items incorrect. This decision was made with guidance from my supervisor and another senior academic. The response pattern analyses suggested that these learners had very little understanding when solving linear equations and as mentioned earlier in Chapter 3, there were also many responses that were incoherent. The following figures are examples of such responses:

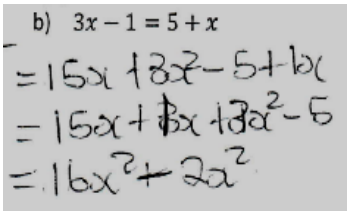


Figure 3.3: Learner 102 Illogical response A

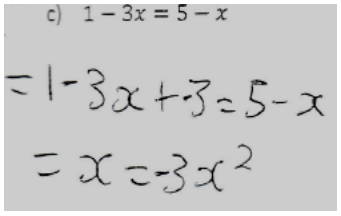


Figure 3.4: Learner 106 Illogical response B

Learner 102, as seen in Figure 3.3, changed the equation to be an expression and got $15x + 3x^2 - 5x + 1x$. Presumably the learner multiplied $3x$ by 5 (the 5 coming from the right hand side of the equation) but does not multiply the -1 in the same way. I am uncertain as to how the learner obtained $3x^2$. It appears that $16x^2$ (and the $-2x^2$) is the combination of $15x$ and $1x$ (and the $3x^2$ and -5), where the learner added coefficients and multiplied the letters. Learner 106, as seen in Figure 3.4, appears to simplify $1 - 3x - 3$ to just x and $5 - x$ to $-3x^2$. I am unsure as to how the learner got either term. There were still illogical responses in the data I selected and these were coded as *unknown strategy*. This is discussed further in Chapter 5.

Following from the response pattern analysis, I concentrated on 46 of the 106 scripts. I focused on all the scripts where at least one equation item had been answered correctly. The 46 scripts were analysed using a coding framework that I built based on literature on errors. Responses for items 1-3 were coded with only one error code as learners were required to fill in the blank space. However, items 4-6 required more than just an answer and would typically be completed in two or three lines of working. Hence, responses for Items 4-6 were coded with more than one code, where appropriate, because a learner could make multiple errors in a response. A single error, however, was assigned only one code for the purpose of avoiding multiple interpretations of the error as well as avoiding a false indication of the number of errors made. When alternative interpretations were possible, this was discussed but not coded or counted as part of the number of errors made. Repeats of the same error within an item were not coded twice. This decision was made because a learner may make the same error repeatedly in the question and this would make the error look prominent when in fact it is not. There are 7 categories for coding items 1-3 and six categories for coding items 4-6. These are described below.

3.8.2 Categories for items 1-3

Items 1-3 were coded using six main categories: *No response; correct; pre-relational; operational view; repeat value* and *unknown strategy*. The operational view category was disaggregated into two further categories: 1) *Operational view: left-to-right* and *2) Operational view: focus on one value*.

3.8.2.1 No response

This code was given when learners did not attempt the question. There are several reasons learners may not attempt a question. One such reason is that not enough time was given to complete the test. A second reason may be that learners have not responded to an item because of a lack of content knowledge, meaning they are aware that they do not know certain mathematic truths nor know how to use them.

Learners may be so overwhelmed with what they do not know that they just leave out the question.

3.8.2.2 Correct response

The correct response for item 1 was 10 because $7 + 5 = \mathbf{10} + 2$; for item 2 the correct response was 4749 because $4747 + 3945 = \mathbf{4749} + 3943$ and the correct response for item 3 was also 4749 because $4747 + n = \mathbf{4749} + (n - 2)$.

3.8.2.3 Pre-relational

As a reminder, a relational view of the equal sign is defined in terms of the left and right hand side of the equal sign having a relationship of equality. A pre-relational response is therefore one where it appears that the learner does attempt to maintain balance. For example, if one focuses on the second term of both the left and right sides of the equal sign in the arithmetic equivalence $7 + 5 = _ + 2$, one subtracts 3 from 5 to get 2. One could then (incorrectly) say that in order to maintain balance one must subtract 3 from 7 to get the missing number to be 4. A response such as this is considered pre-relational because a relationship is being inferred. This is, however, the wrong relationship and 3 should have been added to 7 in order to maintain balance.

3.8.2.4 Operational view: left-to-right reasoning

In contrast to a relational view of the equal sign, an operational view is viewing the equal sign as a symbol that requires an action. This *left-to-right reasoning* category is where a learner does not consider the whole right hand side when calculating the missing number, so, for example, the learner may say that the blank space is for 'the answer' after having operated on the left hand side. For example, in $7 + 5 = _ + 2$, the value 12 would be placed in the blank space because $7 + 5 = 12$, so the $+2$ is not considered to be a part of the calculation that must take place. The learner focuses on operating on two numbers on the left and places the result of operating on these numbers in the blank space on the right.

3.8.2.5 Operational view: focus on one value

As with the error above, in this error the learner will use the blank space as the answer for operating on two numbers. This error is different from the one above in that the two numbers focused on are the ones on the right hand side and the sum of these numbers is the number on the left hand side closest to the equal sign. For example, if the blank space in $7 + 5 = _ + 2$ was given the value 3, my interpretation was that this value came from $5 = 3 + 2$ where the learner appears to have focused on the 5 as 'the answer'. My interpretation is that learners who made this error were working with an operational view of the equal sign, but from right to left rather than left to right as above. Responses such as these concur with the findings made by Essien and Setati (2006), where Grade 8 and 9 learners viewed the equal sign as a *to do something* signal (Kieran, 1981) or a unidirectional symbol (Essien & Setati, 2006) where one only works in one direction. Learners need to work in both directions simultaneously in order to have a relational view of the equal sign. A variation to this error is: $7 + 5 = \mathbf{5} + 2$ where $7 = 5 + 2$ and so focusing on the 7 as the answer.

3.8.2.6 Repeat value

If a learner filled in the missing number by repeating the first number on the left hand side, I called the error a repeat value. For example $7 + 5 = \mathbf{7} + 2$ or $4747 + 3945 = \mathbf{4747} + 3943$.

3.8.2.7 Unknown strategy

There were many responses that I could not interpret and these were categorised as unknown strategy. This was a large category and I was not able to subdivide it as there were many different responses and they appeared illogical to me. Here is an example, $4747 + 3945 = \mathbf{816} + 3943$.

3.8.3 Categories for items 4-6

There are six categories for coding items 4-6: *no response, correct response, errors relating to equality, negativity, letter* and *number*. The errors relating to equality were divided into six subcategories and errors relating to negativity into three subcategories. The no response and correct response categories are discussed first. The error categories are discussed in order of prevalence to the findings (which are discussed in Chapter 5).

In my data, there were errors found that could have been assigned to both the *letter error* and *negativity error* category. For example, if $1-3x$ was simplified to $2x$ two errors were made: the learner added unlike terms (letter error) and also subtracted from right to left (minus sign used incorrectly). I made the decision, however, to only assign one code per error so that the number of errors learners made was not exaggerated. This decision requires a justification as to which error got reported on. I decided that a negativity error would take precedence over a letter error because literature tells us that learners who struggle with negativity would get the same answer (2) for $1-3$ and $3-1$ (Gallardo, 2002; Gallardo & Rojano, 1990; Vlassis, 2004). This means that the learners are focusing on the fact that they need to subtract but not on what needs to be subtracted. Therefore, whether the question was $1-3$ or $1-3x$, I assert that if 2 or $2x$ is given as the answer, the learner is focusing on the binary function of the minus sign rather than on what should be subtracted and whether they are like terms.

Moreover, in a response to a question there could have been more than one occurrence of an error made. For example, if a learner conjoins on both the left and right hand side of the equal sign, I did not count the letter error twice. So when a specific error is made more than once, I did not count the multiple appearances.

3.8.3.1 No response

This response is the same as the *no response* category for items 1-3.

3.8.3.2 Correct

This category of responses is one where a response contained no content errors, meaning that there were no errors relating to number, negativity, addition, simplifying expressions, adding unlike terms or understanding equality. It is a response where the procedure, strategy and method used was mathematically correct and would be endorsed by the mathematical community. There are two exceptions to the above criteria. Firstly, the notational error of having the equal sign on the left hand side of an equation was not physically counted as an error and hence it was ignored when considering whether an item was correct or not. This is discussed in more detail in Section 5.4.3. Secondly, if a learner made a single slip in their procedure it was overlooked and their response was considered correct. This is in contrast to how the same response was dealt with in the response pattern analysis. As a reminder, a slip is a mistake made that is not reflective of an underlying conceptual cause (as discussed in Section 2.3.4). It is a lapse in the learner's concentration and is considered as unintentional. For example, in Figure 3.5 the learner's only error was in line 2 when s/he wrote $5 + 6$ instead of $5 + 1$. We can see from the crossed out version that adding one was what was implied. Since this is not based on a misconception, it would be easily rectified and if given to the learner a second time, s/he would probably obtain a correct solution.

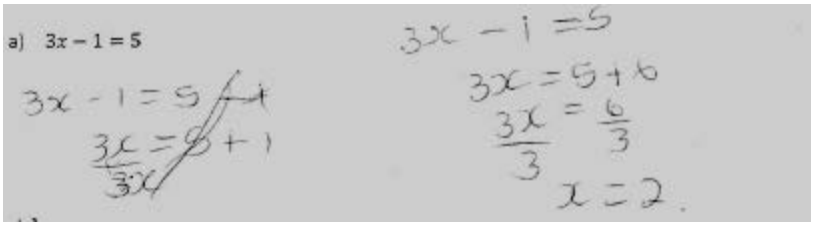


Figure 3.5: Learner 22 Response to item 4

Although these two exceptions (notational error and a slip) are not considered or coded as errors nor were they counted in the total error count, they were noted and are reported in Chapter 5.

In contrast to the method of coding in phase 1, a final answer being correct does not guarantee a *correct* code and so in phase 2 the term *false positives* is introduced. This term describes a response that was coded as correct because the final answer was correct but is in fact incorrect because of the incorrect mathematics that was presented before the final answer. False positives are treated as incorrect responses in phase 2 of data analysis. Although the final answer is correct there are errors in the procedure used and these errors are analysed. The number of false positives that occurred was counted and is reported in Section 3.10.3 since it is a consequence of coding rather than an error category. For example, Figure 3.6 shows Learner 104's response to item 4 which in the response pattern analysis was coded as correct because the learner has the correct final answer of $x = 2$. On closer inspection, we see that the learner adds unlike terms and simplifies $\frac{1}{2}$ to 2 and hence the correct code was a false positive.

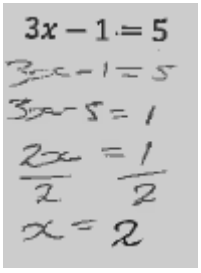


Figure 3.6: Learner 104 False positive

3.8.3.3 Equality

Since this is the essence of my study, these codes are more refined. When solving linear equations a learner needs to solve for an unknown, with the solution being in the form $x = c$. Learners need first to recognise the difference between an expression and an equation and then need to isolate x by applying additive and multiplicative inverses in conjunction with maintain balance. It is then also important that the way in which the learners write out their procedure is mathematically correct. These key features needed to solve a linear equation have been used as categories to analyse the errors that relate to equality.

3.8.3.3.1 Balance errors

These errors are rooted in learners not maintaining balance. Balance errors are those where the learner does something different on the left and right hand sides of the equal sign. For example: $2x = 9x$

$$\frac{2x}{\mathbf{2}} = \frac{9x}{\boldsymbol{x}}$$

$$x = 9$$

Dividing by 2 on the left and x on the right does not maintain the balance of the equation. It is possible that the terms are not divided by the same value because it would produce an unfamiliar solution with a letter x on both sides. Another example would be: $2x + 4 = 3$

$$2x + 4 - \mathbf{4} = 3 + \mathbf{4}$$

In this example 4 is subtracted on the left but added on the right. Not all learners are as explicit in their procedure, meaning that many times learners do not show the $+4 - 4$. By being explicit, I am able to infer that the learner applied two different operations on the left and right hand side and hence did not maintain balance. If a learner's procedure is not explicit, their response may look as follows:

$$2x + 4 = 3$$

$$2x = 3 + \mathbf{4}$$

Although this reduction is identical to the reduction of $2x + 4 - 4 = 3 + 4$, the learner may be operating on the right in such a way that it balances out how the left hand side looks. There is a term with an addition of 4 on the left hand side of $2x + 4 = 3$ and so by adding 4 to the right side the equation appears balanced with $+4$ on each side. Although this error does not maintain balance it is conceivable that the learners need the symmetry precisely because they do *know* about the symmetric property of equality (Kieran, 2006). My interpretation therefore is that a learner would make this error because s/he wants the left and right sides to *look* the same. This then implies that s/he does understand that when an inverse is applied both sides should be the same, or in their understanding should look the same, but

the learner is unable to execute it correctly. This would therefore be an inverse error.

3.8.3.3.2 Inverse errors

Learners battle with inverse in multiple ways (Kieran, 2006). One being that they use the multiplicative inverse instead of the additive inverse, for example:

$$3x = 12$$
$$x = 12 - 3$$

Another way learners use inverses incorrectly is when a learners' deduction process is based on wanting symmetry, for example:

$$2x - 1 = x$$
$$2x - 1 = x - 1$$

The learner needed the -1 on the right of the second step to have symmetry (Kieran, 1992). I am aware that this could be argued to be a balance error because the learner did not do the same to the left and right hand side. I am, however, taking the view that the learner wants the left and right sides to look the same. This suggests that their error is based in not knowing what an inverse is; why it's needed or how it's applied. I do assert that the learner does have an idea of the left and right hand side needing to be (or look) the same. The example below is a different version of a learner requiring symmetry:

$$2x - 1 = x$$
$$2x = x - 1$$

In the above example the symmetry is on a separate line. Hall (2002) refers to this error as *change addendum.* I have chosen to group the different versions of symmetry together to show that these errors are due to misunderstanding what an inverse is and how it can be used. If one takes the view that learners have incorrectly applied the *change sides, change signs* rule then this would still be coded as an inverse error. I argue that changing sides but not changing the sign is an inverse error rather than a balance error because when one 'moves' a number to 'the other

side' it is in order to isolate x and isolating x is about knowing that a certain value (the inverse) has the power to isolate x.

3.8.3.3.3 Familiar structure

A familiar structure error is a reduction error. This occurred when a learner manipulated his/her reduction by ignoring the letter on one side of the equation. I argue that this was done in order to obtain a solution that was more familiar to him/her. Although balance may be maintained, certain details are ignored in the carrying out of the reduction process. Learners would arrive at a statement in the form $Ax = Bx$ but treat it as $Ax = B$, ignoring the second appearance of the letter x. Their solution would therefore be in the form $x = C$ rather than $x = Cx$. I argue that this is done because $x = C$ looks familiar. The errors made in this category are only found in items 5 and 6 since they are the only questions that involve having a letter on both sides of the equal sign.

3.8.3.3.4 Syntax errors

The errors in this category are about how learners write out their procedure. Learners may be able to identify the correct inverse and may apply it to both sides of the equation and obtain the correct answer. The error is that the *doing to the left* is done on a separate line to *doing to the right* and so balance is only obtained over a series of steps instead of at each step.

3.8.3.3.5 Incomplete Procedure

The category is considered an error because learners could not correctly continue in their procedure. Not only were learners' responses incomplete but more significantly all these responses were left in the form $Ax = Bx$. Learners did not continue the deduction and reduction process. A possible reason for responses to be left in this form is that learners do not know how to operate on the unknown (Herscovics & Linchevski, 1994). It is this structure of an equation that Filloy and Rojano (1989) argue is what makes transitioning from arithmetic to algebra difficult.

3.8.3.3.6 Changing an equation into expression

This error is one where learners converted an equation into an expression. With an equation in the form $Ax + B = C$, as in item 4, part of the procedure to isolate x is to add like terms (B and C). If a learner subtracts C from both sides of the equal sign, s/he should be left with $Ax + B - C = 0$. Leaving off the '$= 0$' would result in changing the equation into an expression. Learners do this because zero is not perceived to be a number, and so there is no number. Learners view the '$= 0$' as nullity rather than totality, which Gallardo and Hernández (2005) argue is the duality of the number zero that adds to learners' difficulties in the transition from arithmetic to algebra.

In summary, this section on equality errors has disaggregated the strategies used and errors made by learners when solving linear equations. Errors are made in maintaining balance and identifying and applying an inverse operation. Other errors are made in the way learners write out their procedure and in how they manipulate their procedure to have their solution look like something they have seen before. Moreover, errors are made in terms of not completing a procedure because the structure of the equation is not familiar or because they interpret the structure $Ax + B - C = 0$ to be the same as $Ax + B - C$.

3.8.3.4 Negativity

An important aspect of solving equations involves dealing with subtraction and negative numbers. Because this is a very prominent error in the research literature as well as in my data, I subdivided this error category into three subcategories. I acknowledge that the categories could be further disaggregated but my study does not focus on negativity and so I have kept the number of subcategories to a minimum. The different forms of reasoning about the minus sign (as discussed in Section 2.5.2) formed a basis on which my codes relating to minus sign errors were developed. I created three error categories: using the minus sign, but incorrectly; acknowledging the minus sign, but not operating with it; and ignoring the minus sign

completely. These three categories incorporate a broad range of errors and as future research could be disaggregated further.

3.8.3.4.1 Minus sign used incorrectly

The first category describes responses where the minus sign was acknowledged and used but not correctly. For example $7 - 9 \rightarrow 2$, the process involved to get an answer of 2 involves subtracting 7 from 9. The subtraction, or binary function of the minus sign, was acknowledged and used but from right to left and not left to right. The reasoning behind this error is what Vlassis (2004) calls *right-to-left*[4] reasoning. This form of reasoning does not include similar variations of errors, for example $-9 + 7 \rightarrow 2$. I therefore extend her *right-to-left reasoning* category to be about learners acknowledging and using the minus sign but in such a way that their answer is positive rather than negative. Examples of the variations that are included in the *minus sign used incorrectly* category, with their anticipated incorrect reduction, are shown below:

- $-8 + 2 \rightarrow 6$ (treated as $8 - 2$)
- $2 - 8x \rightarrow 6x$ (treated as $8 - 2$ and attach the x)
- $2x - 8x \rightarrow 6x$ (treated as $8x - 2x$)

In the above examples, subtraction did take place but it was done incorrectly and always resulted in a positive answer when it should have been negative. This error category is indicative of viewing the minus sign in its binary function (by applying subtraction) but not viewing it in its unary function (for example where the solution of-6 is not embraced). In the example $2 - 8x \rightarrow 6x$, there are actually two different errors being made: adding unlike terms and using the minus sign incorrectly. As a reminder, negativity errors take precedence over letter errors and hence this error was coded as a negativity error rather than a letter error.

[4] Right-to-left reasoning in understanding the minus sign should not be confused with left-to-right reasoning of the equal sign as mentioned in Section 3.8.2.4.

3.8.3.4.2 Minus sign acknowledged but not used (or only partially used)

This category characterises errors where numbers are added as though they are positive but the result is given as a negative. Examples of the variations that are included in the *minus sign acknowledged but not used* category, with their anticipated incorrect reduction, are shown below:

- $-7+2 \rightarrow -9$
- $2-7 \rightarrow -9$
- $-7-2 \rightarrow -5$
- $7-x \rightarrow -7x$

This category is in contrast to the category above (minus sign used incorrectly) in that the unary function of the minus sign is acknowledged. Acknowledging a minus sign suggests that dealing with a negative number or answer is not foreign to the learner, meaning that the unary function of a minus sign is accepted. Literature talks about this form of error being a result of bracket reasoning (Gallardo and Rojano (1990); Vlassis (2004)). So for example $-7+2$ is treated as $-(7+2) \rightarrow -9$ where the minus sign is acknowledged (as it appears in the answer) but has not been operated with. The minus sign is being detached (Herscovics & Linchevski, 1994) or isolated. It is being viewed as an object on its own in isolation and is detached from the number 7 (Ryan & Williams, 2007) and then 7 and 2 are added together. The detached minus sign is then reattached. The example $-7-2$ is treated as $-(7-2)$ which then yields -5. In this example the minus sign between the 7 and 2 is used as subtraction and the other is detached from the 7.

Bracket reasoning would not account for the same error made when $-7+2 \rightarrow -9$ is rephrased as: $2-7 \rightarrow -9$. The reasoning behind the error now could be due to overgeneralising the signs-rule, where a *negative and a positive make a negative* and 7 and 2 make 9, hence -9 is the result. What the errors, with both forms of reasoning, have in common is that the minus sign was not used in its binary function. No subtraction took place, only addition and in the solution a minus sign appeared.

In light of the discussion above, the category *minus sign acknowledged but not used (or partially used)* was created in order to accommodate multiple forms of reasoning.

3.8.3.4.3 Minus sign ignored

This error is characterised by ignoring the minus sign in both its unary and binary functions. Examples of the variations that are included in the *minus sign ignored* code are shown below with their anticipated incorrect reductions:

- $-10 + 4 \rightarrow 14$
- $-5x - 2x \rightarrow 7x$
- $5 - 2x \rightarrow 7x$
- $10 - 4x + 3 \rightarrow 4x + 13$

To further exemplify errors that were called minus sign ignored, when learners added like terms: $5x - 2x$; $2x - 5x$ and $-2x - 5x$ were reduced to $7x$ and both $-1 - 5$ and $1 - 5$ were reduced to 6. In all these examples one can apply the interpretation of ignoring the signs and operating on the numbers. To exemplify when unlike terms were added the following expressions: $1 - x$; $x - 5$ and $5 - x + 3x$ were reduced to $1x$; $5x$ and $8x$ respectively. As a reminder, if unlike terms were added as well as there being a negativity error, the negativity error would take precedence over the conjoining error. In the above examples, the interpretation that I used is that the minus sign was not used nor acknowledged and that it was only the numbers that were operated on. Despite the reasoning behind the error, the minus sign ignored error is where, from one line of working to the next, the minus sign disappears and never returns.

There are multiple ways to interpret the responses that were assigned to my three categories. Different interpretations may have yielded different error categories and my categories could have been disaggregated further. In order to have as few negativity error categories as possible (because my focus is not on negativity), yet still disaggregate the negativity errors (because of their prevalence in my data); I

developed broad error categories where I clustered different errors into one. These three categories, *minus sign used incorrectly*, *minus sign acknowledged but not used* and *minus sign ignored*, were informed by my data as well as the view that learners struggle to operate with the minus sign (Vlassis, 2004).

In summary, this section on negativity errors has disaggregated the different ways I have interpreted learners' errors in dealing with the minus sign. Errors are made because learners are not flexible in negativity (Vlassis, 2004) and blur the different functions of the minus sign.

3.8.3.5 Letter errors

Another critical element to solving linear equations is that of like terms. Because an equation is defined as being two expressions equal to each other, errors that learners bring when simplifying expressions become errors when solving equations. One such error is that of conjoining (Kieran, 2006; Watson, 2009). When needing to solve for the unknown in $3x - 1 = 2x + 5$, learners add unlike terms: 3x and -1 on the left hand side and obtain 2x and then add $2x$ and 5 to get $7x$ on the right hand side, reducing $3x - 1 = 2x + 5$ to $2x = 7x$. The category *letter errors* was assigned to responses where the learner added unlike terms, but also accommodates responses where learners multiply like terms instead of adding them, for example when $3a + 2a$ is reduced to $5a^2$ or $6a^2$. Given that the use of letter is not a focus of this study, I do not subdivide this category nor differentiate between the different types of conjoining.

I have taken the different forms of reasoning that produce different errors and combined them in such a way that all the negativity errors found in my data are accounted for.

3.8.3.6 Number errors

Number errors are errors in basic arithmetic and counting (for example when simplifying fractions). Because there is a code that deals with negatives, I need to be explicit about what is considered a number error and what is considered a minus

sign error. Errors that deal with addition, subtraction, multiplication and division where the solutions are positive real numbers, are considered number errors. A learner who adds 5 and -8 and gets 3 could deal with the numbers 5 and 8 but made an error with their sign. This is not a number error but rather a negativity error. Fortunately, no learner made both a number error and a negativity error (for example $-4 - 9 = 15$) and so no decisions needed to be made regarding having two categories being assigned to a single error.

Section 3.8 has provided a detailed description of the error categories used when analying my data. The difference between the types of questions asked in items 1-3 and 4-6 required two coding schemes. To code items 1-3 the following categories were used: *no response; correct response; pre-relational; operational view; repeat value* and *unknown strategy*. The operational view category was disaggregated into two further categories: *operational view: left-to-right and operational view: focus on one value*. To code items 4-6 these categories were used: *no response, correct response, equality errors; negativity errors, letter errors* and *number errors were used*. The errors relating to equality were further divided into six subcategories and errors relating to negativity were further disaggregated into three subcategories. In the sections that follow I move from discussing how my data was coded and analysed to discuss the rigour in my research. I discuss the difficulties experienced and note the limitations of my study as well as how I attended to the ethical considerations.

3.9 Rigour in qualitative research

Validity and reliability are two central issues within all research that determine the strength of a study (Cohen, Manion, & Morrison, 2007). They are, however, used in different ways depending on the research paradigm employed (Cohen et al., 2007). Data is not valid or invalid (Henning, Van Rensburg, & Smit, 2004) and it is therefore the inferences that are drawn from the data that would determine its validity. Maxwell (1992) argues that in qualitative research, validity and reliability are closely related and suggests 'understanding' as a replacement for validity and reliability.

Similarly, Guba and Lincoln (1982) incorporate validity and reliability in the idea of *trustworthiness*. Trustworthiness involves credibility, transferability, dependability and confirmability. These are viewed as modifications of the typical categories (reliability, objectivity and internal and external validity) used in quantitative research (Opie & Sikes, 2004). For this study, I shall use these four criteria offered by Guba and Lincoln (1982) to discuss the trustworthiness of this study.

Credibility refers to the accuracy of describing the settings, data and the manner in which the study was conducted (Guba & Lincoln, 1982). Describing the settings of a study relates to being able to justify the sample chosen, and choosing the appropriate subjects. In Section 3.4 my sample for both phases of the analysis are described and justified. Guba and Lincoln (1982) suggest different ways to encourage the credibility of one's study including peer debriefing, member checks, and referential adequacy materials and triangulation. In terms of peer debriefing and member checks, in Section 3.8.1 I mentioned that a senior academic, other than my supervisor was consulted regarding my sample and the coding of the response pattern analysis. I also had a second coder for the coding of errors in my error analysis (this is discussed in more depth as the confirmability of my study). In my analysis and discussion of errors made, I provide many references to learner work which is considered referential adequacy material. Triangulation typically refers to using multiple methods of collecting data, for example, through a test as well as an interview. Although this form of triangulation is not present in this report, I did conduct two different types of an analysis on the learner's responses to the test items: a response pattern analysis and an error analysis. The above discussion shows how this study has provided an accurate account and description of the sample and the data analysis hence providing evidence of the credibility of my study.

In a non-naturalistic paradigm or rationalist paradigm, transferability would be about the generalisability or external validity of a study (Cohen et al., 2007; Guba & Lincoln, 1982). Since a naturalistic paradigm involves people who have different perceptions and understandings, generalisability is not possible (Guba & Lincoln, 1982). Guba and Lincoln (1982) argue that some transferability of one's findings is possible through

purposive sampling and providing a *thick description* of the study in order for it to be transferred to a similar context. Providing a thick description means to "provide enough information about a context" (p. 248). The 'context' not only includes *how* the data was collected, sorted, coded and analysed but also *why*. It relates to describing the sample, the context and the tools used. Throughout this study, I provide thick descriptions of how my study can be transferable. In Chapter 1 I provided a thick description of the background context; rationale and research problem of my study. In Chapter 2 I provided a thick description of the error categories used in my coding. In this chapter I provided a thick description of how data was collected and gave examples and explanations of how the data was coded and analysed. In Chapter 4 I will describe in detail how the data was sorted in three different ways and how I used the response pattern analysis to select my sample purposively for phase 2 of the data analysis. In both Chapters 4 and 5 I provide thick interpretations of the cluster of errors found as well as thick explanations of the actual errors found in the detailed analysis.

Phase 1 of my study used convenience sampling (Opie & Sikes, 2004) as the two schools are part of the LG project. Phase 2, however, made use of purposive sampling (Opie & Sikes, 2004). I provide accurate and in-depth accounts for each phase of my data collection so that the study can be transferred to another similar setting. My findings should therefore be transferable to other schools within South Africa, using Grade 10 learners with similar mathematical errors and error patterns.

Dependability is synonymous with the notion of reliability in a rationalist paradigm (Cohen et al., 2007; Guba & Lincoln, 1982), which refers to the extent to which a study can be replicated. In qualitative research, this is difficult to achieve since one is dealing with human participants. Repeating this study with the same or similar participants and under the same or similar conditions and with the same or similar researcher would yield different results. Using Guba and Lincoln's (1982) terms, to increase the dependability of a study it is necessary to have a paper trail or evidence that points to each stage or phase of one's study and that clearly outlines each methodological step and decision made. Throughout this study I provide evidence of

the aspects of dependability. The paper trail is about more than just having the test scripts. I have codes, have defined them and given examples of how I coded errors. I have spreadsheets from my coded scripts I have spreadsheets and in Chapter 4 I explain how I sorted the data to obtain the different patterns.

Confirmability is the last tenet of Guba and Lincoln's (1982) notion of trustworthiness. This is concerned with being objective rather than subjective. This means that all my findings and interpretations need to be reasonable and evident to others. For my qualitative data to be confirmable, I must ensure that others would code the errors in the same way that I did. I therefore explain each code in detail and give an example of each so that others can recode. Since items 4-6 required more steps than items 1-3, there were more opportunities for learners to make errors. This means that coding the errors made when responding to items 4-6 would be more complex. It was therefore possible that I would not be as consistent with coding responses to items 4-6 as I was with responses to items 1-3. I therefore had a second coder code items 4-6 to test the reliability and consistency of my coding as well the quality of my explanations of each code. A post-doctoral fellow who is involved in the LG project coded 15% of my data[5] (sevem scripts). In terms of the percentage of errors, he coded 27% of them. Of the 21 items ($7 \text{ scripts} \times 3 \text{ items}$) that he coded, we agreed completely on 13 of them (62% agreement on the whole item). This is a conservative percentage as within an item we may have agreed on, for example, two out of the three codes (see Appendix A). Looking at the reliability in a different way, there was a maximum of 49 codes given and we agreed on 38 of them (78% agreement per code). There were 11 discrepancies which led me to refine the description of my codes, elaborating on some and being more explicit on others. One example of the discrepancies is highlighted in Figure 3.7.

[5] These scripts were purposively chosen to ensure a wide range of errors were being coded. A random selection may have resulted in a 100% reliability due to all the items being correct or missing or having the same error. This would not have been beneficial to improve my coding scheme and explanations of codes.

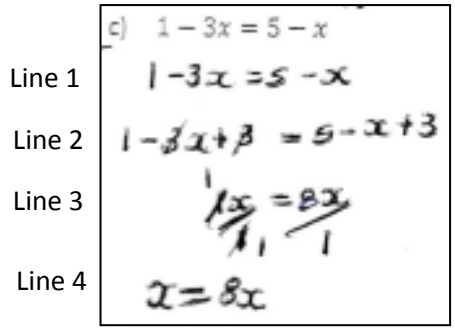


Figure 3.7: Learner 12 Response to item

The discrepancy in the coding of this response was between line 2 and line 3. In line 2, Learner 12 has added 3 to both the left and right hand side which is not an error since the learner has maintained balance. My interpretation was that this was done in an attempt to isolate x, that the $+3$ is acting as the inverse of the coefficient of x. We can see that the coefficient of x, which is -3, and the $+3$ are crossed out. I coded this error in line 2 as an equality error, and more specifically, an inverse error, meaning that the learner sees $+3$ as the inverse of the -3 in front of x. From line 2 to line 3 the learner then appears to have obtained $1x$ by conjoining the 1 and the 'isolated' x. By contrast, the post-doctoral fellow did not code the addition of 3 on both sides of the equal sign because balance was maintained and it was not an error. He then assumed that the $1x$ was obtained by conjoining $-3x$ and 3. Since the $+3$ was not given as part of the question but instead deliberately inserted by the learner, we both agreed that the learner's intention was to isolate x and so the inverse error code remained. See Appendix I for more details on how other discrepancies were dealt with. Opie and Sikes (2004) term this intercoder-reliability and according to Guba and Lincoln (1982), having a second person code one's data is important in research. It was particularly important in my study because the agreement percentage (Cohen et al. (2007); Creswell (2013)) informs the extent to which my coding scheme can be applied by someone else on the same set of data with the same results. This increases the confirmability of my interpretation of the errors made. Confirmability is also about being clear and upfront about the limitations of my study. These are discussed below.

3.10 Difficulties and consequences of coding

3.10.1 Difficulties and limitations in coding items 1-3

Coding items 1-3 was relatively simple because learners had to fill in the blank spaces and hence I needed to code a single value. The only difficulty encountered was making sense of errors that sometimes appeared to be illogical. When one knows that the correct answer to, for example, $7 + 5 = __ + 2$ and $7 + 5 = 10 + 2$ it becomes difficult to make sense of why a learner would say, for example, $7 + 5 = 2 + 2$. Errors such as these were coded as unknown errors. Literature suggests that learners with a relational view of the equal sign would fill in 10 in the blank space and a learner with an operational view would fill in 12 (Knuth et al. (2005); Stephens et al. (2013); (Watson, 2009)). There was one learner who gave the answer 11. This answer could be due to an operational view of the equal sign as well as an arithmetic slip ($7 + 5 = 11$ rather than 12). This error could also have been due to only an arithmetic slip where the learner does in fact have a relational view of the equal sign. The learner could have made a mistake in thinking that $7 + 5 = 13$ but having noticed the $+2$ on the right wrote down the answer 11 because $11 + 2 = 13$. In item 2, this same learner gave the following response: $4747 + 3945 = 4693 + 3943$. Learners were not allowed to use calculators, so if the learner had done column addition and used $7 + 5 = 13$ in the units column, but did all other additions and subtractions correctly, s/he would obtain the 4693 as the value that makes the left hand side equal to the right hand side. Because this item was testing the view of the equal sign and not a learner's arithmetic ability, this response was coded as correct which could then be seen as a relational view. Although this was only one learner it demonstrates how I was systematic and thorough in the coding process.

3.10.2 Difficulties and limitations in coding items 4-6

Despite being able to justify why this form of coding is reasonable and valid for my study, there are limitations to coding. As seen by the example of the difference in coding between the post-doctoral fellow and myself, there are many difficulties

when coding errors. Two difficulties I encountered and how they were resolved are discussed briefly below.

3.10.2.1 Multiple interpretations

When analysing a learner's response to an item, the error could be interpreted in multiple ways depending on who and how one views the errors. For example, Figures 3.8; 3.9 and 3.10 are Learner 72's responses to items 4, 5 and 6 respectively:

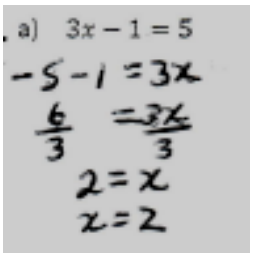


Figure 3.8: Learner 72 Response to item 4

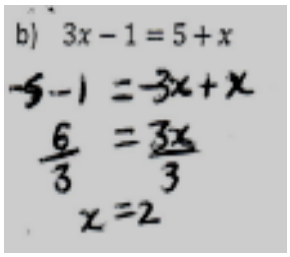


Figure 3.9: Learner 72 Response to item 5

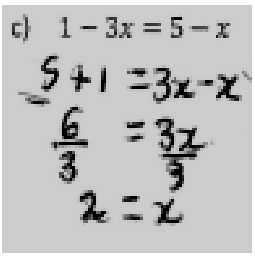


Figure 3.10: Learner 72 Response to item 6

The statement $-5-1$ should simplify to -6 but the learner's response is 6 in both items 4 and 6. There are multiple interpretations one can use to explain how the learner obtained 6 and not -6. One interpretation is that the learner focuses explicitly on the signs and incorrectly uses the rule *a negative and a negative make a positive;* hence $-1-5 \rightarrow 6$. This interpretation assumes that the learner is paying strict attention to the negative signs and acting on them. An alternate interpretation is that the learner avoids dealing with negatives, and attends to the numbers and ignores the signs, i.e.: $-1-5 \rightarrow 1+5 \rightarrow 6$. The responses to $-5-1$ in items 4 and 5 could be interpreted in either way which causes a problem when coding. To decide on which interpretation to use, I looked at how the learner responded to $-3x+x$ in item 5 and $3x-x$ in item 6. In item 5, $-3x+x$ should reduce to $-2x$ and in item 6, $3x-x$ should reduce to $2x$ but the learner reduced both to $3x$. Viewing these errors in light of the second interpretation, it appears that the learner ignored the minus sign in both items 5 and 6 and attended to the visible numbers (3 and 'nothing' in front of x) and then attached the letter onto the sum, obtaining $3x$ in both cases. We will never know what the learner thought when responding to

these items. We can only ensure that our interpretations of the errors made are done so consistently. For the coding of my data the second interpretation was employed because it appeared to fit consistently with errors that involved both numerals and letters.

3.10.2.2 Ambiguities in coding

Another problem I faced was being explicit in my definitions, especially for the categories involving negated letters, for example, $-3x$. This was challenging because sometimes an error suggested a problem with both letters and negatives. For example if $1 - 3x$ is reduced to $2x$, the learner may have a) added unlike terms suggesting a problem with letters and b) may have added from right to left to avoid getting a negative answer. This was dealt with by being explicit that the negative error overrides the letter error when unlike terms and negatives are involved because letters are not a focus of my study.

3.10.3 A consequence of coding: False positives

There were some final responses to items that were correct but on closer inspection what was written in leading up to the final response was not mathematically correct and did not follow from step to step. This type of response is referred to as a false positive. This is an important issue for my study because in the response pattern analysis I only coded final responses and did not take into account the procedure used. In the error analysis, however, the procedure was analysed. Figure 3.11 is an example of a false positive.

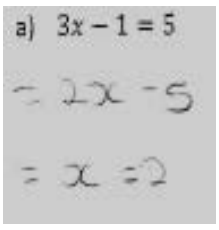


Figure 3.11: Learner 62 False positive

In phase 1, responses such as these were coded as correct because of their final answer, but when analysing the actual errors made in phase 2, these responses were

coded as incorrect. An interesting finding with the 12 false positives (9% of the data) is that there were 6 instances where it appears that the learners became 'stuck' in the algebraic procedure yet their final statement was correct. Perhaps these learners realised that their algebraic procedure was not helping them get to the solution and abandoned the algebra and instead used arithmetic strategies to obtain the correct answer. Figure 3.11 shows Learner 62 making three errors. The learner first conjoins the left hand side and then converts the equation into an expression. The final statement does not follow from the previous line and hence that error was categorised as unknown because I do not know how the learner could go from $2x - 5$ to $x = 2$. This is the correct answer and so I assume that learners who did the same as Learner 62, changed strategies to solve for x.

Filloy and Rojano (1989) state that learners use arithmetic to solve equations such as Item 4: $3x - 1 = 5$ and continue to say that learners then struggle when needing to solve an equation with a letter on both sides of the equal sign. This is because their arithmetic methods no longer help them. Responses such as the one above were categorised as *unknown strategy* because I do not know exactly how the learners went from one line to the next.

3.11 Ethical considerations

Bertram and Christiansen (2013) identify two key ethical principles for conducting research. The first is "autonomy" (p. 66) which means that all participation needs be voluntary and that consent needs to be granted by both parents and learners to participate in the study. The second principle is that of "non-maleficence" (p. 66) which means that the researcher needs to make every effort to protect the participants and not bring any harm to them. To achieve this, participants need to be ensured of confidentiality and anonymity (see Appendix E). This is a vitally important part of my research as I am dealing with minors. To ensure that my intentions and behaviour is ethical, I requested and received ethical clearance from the GDE (see Appendix B) as well as from the Wits Ethics Committee (Protocol number 2016ECE005M) (see Appendix C) for me to analyse the learners' errors. Permission to

conduct the written assessment was already granted (Protocol number 2016ECE003S) (see Appendix A). After receiving permission from the principals of the two schools (see Appendix D), I informed the Grade 10 learners and parents of the nature of my research and what would be expected of them if they agreed to participate (see Appendices E and G). Learners and parents who agreed returned a consent letter acknowledging that their involvement was voluntary and confidential.

3.12 Limitations

In this section I identify four issues that limit my study. The first is about the transferability of my findings: since my study is very local and on small scale, my findings cannot be generalised or transferred to the larger population. The second relates to the scope of the mathematical content as I am only focusing on certain types of equations and so am not investigating the effect of brackets, negatives or fractions. These are areas that learners find troublesome and hence would affect their understanding of solving equations, but it goes beyond the scope of my study. Although I refer to this issue as a limitation, I believe the narrow scope is beneficial as it reduces the focus of my study and increases the chance of developing new understanding of how learners attempt questions involving equations. The third limitation relates to the implemented curriculum: not focusing on how the content was taught is a limitation because it is potentially a very large influencing factor on learners' understanding. Lastly, in terms of my methodology and decision to code final responses in the response pattern analysis, coding only final answers in learners' responses was a problem. This is because there are often many places that a learner could make a mistake. The items 1-3 analysed in this study were only 'one number' answers and so did not pose any problems in this regard but in items 4-6, there was more than one line of working out required.

3.13 Conclusion

This chapter provided an outline of the research design, methodology and methods that were used for the data collection. Data was collected using test scripts and was analysed in two phases.

This chapter has provided detailed discussions on the sample, context and data collection methods used in this study and explained how the data was coded and analysed. A thick description regarding the categories used was provided and I discussed how certain errors, such as equality errors, took precedence over other errors, for example negativity errors. This chapter also provided a thick description of how rigour was established, of the difficulties and limitations of the study as well as how ethical considerations were dealt with.

Where Chapter 3 detailed how the data was collected and analysed, Chapters 4 and 5 provide an analysis of the data and give an in-depth discussion of the findings. Chapter 4 focuses on phase 1 of the analysis which deals with identifying response patterns. Chapter 5 focuses on phase 2 of the analysis which constitutes a detailed error analysis.

Chapter 4 : Response Pattern Analysis

4.1 Introduction

The previous chapter discussed the methods used for data collection as well as the measures taken to ensure rigour in my research. In light of the discussions in Chapter 3, my first research question: *What patterns in learners' responses emerge when solving arithmetic equivalences and arithmetic and algebraic equations?* is now being reworded to: *What response patterns emerge when solving arithmetic equivalences and arithmetic and algebraic equations?*

This chapter presents a thick description of the results and analysis of learners' final responses to the six test items and answers the research question above. I provide three different response pattern analyses of the data. The data was coded using three codes (correct, incorrect and missing) and then organised in three different ways resulting in three patterns that foregrounded different clusters of responses. In Section 4.5 I discuss the limitations to this method of coding and then in Chapter 5 some of the clusters are analysed further in terms of the errors made. Literature on the didactic cut informed the framework that I used on the data and hence guided my choices on which clusters to focus on in Chapter 5. The response pattern analyses served a dual purpose for my study. Firstly, by taking a high level look at a strategically selected set of test items, I was able to identify different clusters of responses that were suggestive of learners' difficulties with arithmetic, negatives, letters, having a relational view of the equal sign and executing algebraic procedures when solving linear equations. Secondly, and as a consequence of identifying these clusters, this method of analysis informed my sample selection for the error analysis.

As a reminder, the six items are:

Item 1: $7 + 5 = _ + 2$

Item 2: $4747 + 3945 = _ + 3943$

Item 3: $4747 + n = _ + (n - 2)$

Item 4: $3x - 1 = 5$

Item 5: $3x - 1 = 5 - x$

Item 6: $1 - 3x = 5 - x$

4.2 Coding final responses

Learners' responses were coded as missing (0: yellow), correct (1: green) or incorrect (2: red). Only final responses were coded which means that only the last statement that the learner wrote was considered. I did not check whether the final answer followed on from the previous step. Coding the responses in this way and assigning a colour to each code provided me with a visual, colour-coded summary of the learners' responses. Depending on how the data was organised or sorted, I saw different clusters of responses. Figure 4.1 is a bar chart showing the percentage of items that were answered correctly, incorrectly and not attempted. In the next section I analyse and discuss each item separately and use Figure 4.1 as my point of reference and then look at the different patterns and clusters.

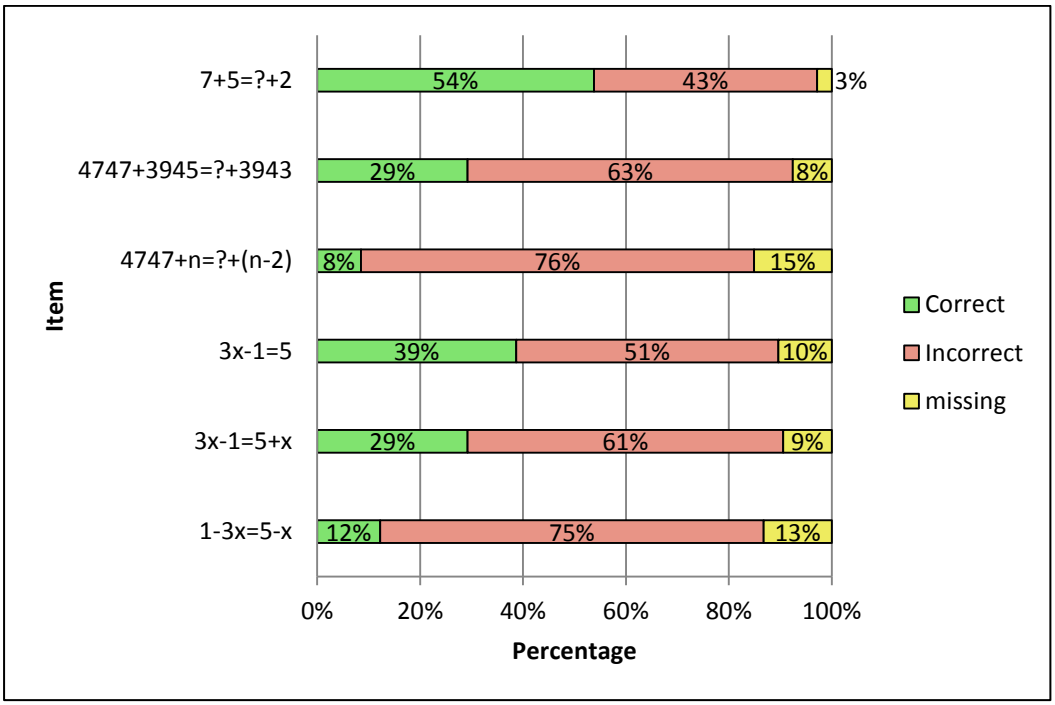


Figure 4.1: Percentages of responses per item in phase 1

What stands out in the above figure is that the colour red is dominant. This means that there were more incorrect responses than correct ones. Looking at the statistics, the ratio of correct to incorrect is approximately $2:5$. Also, only three percent of the learners got all six items correct and only 7% got more than four of the six items correct. All six items were answered incorrectly by 21% of the learners and 11% had a combination of incorrect and missing responses. So, to put

these statistics into perspective, 32% of my sample did not get any item correct, and a further 25% could only answer one or two of the arithmetic equivalences but none of the equations. This mathematical content should be taught in Grade 8 (DBE, 2011a) and hence suggests that these learners do not have a grasp on how to solve a linear equation. This finding supports the literature about learners in secondary schools in South Africa who are not coping with Mathematics (Spaull, 2013), algebraic manipulation (Pournara, Hodgen, Sanders, & Adler, 2016), solving equations (Makgakga, 2012) and understanding the meaning of the equal sign (Molefe, Brodie, Sapire, & Shalem, 2010).

4.3 Analysis of the error patterns per item

4.3.1 Item 1: $7 + 5 = _ + 2$

Item 1 tests whether learners have an operational or relational view of the equal sign. It is the only item where there are more correct responses (54%) than incorrect (43%) and missing (3%) responses. This was expected as, in the curriculum, this type of question is first encountered in term 1 of Grade 4 (DBE, 2011b) and hence would be considered to be of low cognitive demand (Stein, Smith, & Henningsen, 2000) when given at Grade 10 level. Although the blank space (missing number) could have been filled in with knowledge of bonds of 12, recognising that the whole left and whole right hand side must be equal is indicative of a relational view of the equal sign (Knuth et al., 2008). It is therefore surprising that as many as 49 learners (46%) did not get the correct response to a question that could have been answered by adding the left hand side and comparing it to the right. The large percentage of incorrect responses suggests that learners did not adopt a relational view of the equal sign in answering this item.

4.3.2 Item 2: $4747 + 3945 = _ + 3943$

Item 2 has the same structure as item 1 but the numbers are larger. Carpenter et al. (2003) argue that a relational view is more easily observed when large numbers are used. The intention was for learners to attempt this question by noticing the

structure and not by adding the numbers as they may have done in item 1. They were expected to notice that 3943 is two less than 3945 and so to maintain balance, the missing number needed to be two more than 4747. This item had only 29% of correct responses and 63% of incorrect responses. These results suggest that learners are either not able to deal with the arithmetic of large numbers or that they have an operational view of the equal sign.

4.3.3 Item 3: $\mathbf{4747 + n = _ + (n - 2)}$

Item 3 is a generalised version of item 2 involving not only a large number but a letter (n) and a blank space. Carpenter et al's (2003) assertion that a relational view of the equal sign is more easily observed when large numbers are present could be extended to include the presence of letters because letters (or algebra) can be seen as generalised arithmetic (Kieran, 1992). Only 8% of the items were correctly answered and this item had the highest percentage of incorrect (76%) and missing responses (15%) suggesting that the item was too difficult. The fact that the item did not point learners to a known procedure was forcing them to display relational thinking, which learners where not able to do. These results suggest that the majority of these Grade 10 learners do not yet have a relational view of the equal sign.

4.3.4 Item 4: $\mathbf{3x - 1 = 5}$

Item 4 is an algebraic equation with a letter on one side of the equal sign. This type of question is first encountered in term 2 of Grade 8 (DBE, 2011a) and so it is not surprising that, next to item 1, this item has the second highest percentage of correct responses (39%). What was surprising was that there were more missing responses (10%) but fewer incorrect responses (51%) than what was seen from the number of responses in item 2. This suggests that perhaps learners felt more confident in responding to the arithmetic equivalences because they were dealing with whole numbers and not letters. Having fewer incorrect responses possibly suggests that these learners are more proficient in the procedure involved when solving equations than dealing with arithmetic and the equal sign in a relational way.

A deeper look at the actual errors made when responding to this item is done in Chapter 5.

4.3.5 Item 5: $3x - 1 = 5 - x$

Item 5 is an algebraic equation with a letter on both sides of the equal sign. The high percentage of incorrect responses, (61%), suggests that these learners were not able to use arithmetic strategies to determine the value of x (Filloy & Rojano, 1989). This type of equation can be solved using arithmetic strategies or by undoing each operation on the left, however, being an equation of the form $Ax + B = Cx + D$ requires operating on the unknown which in turn requires a transition in the learners' concept of equality. It is therefore not surprising that item 5 had fewer correct (29%) and more incorrect responses (61%) than item 4, the equation with a letter only on the left side.

4.3.6 Item 6: $1 - 3x = 5 - x$

This item also has a letter on both sides of the equal sign but in addition the structure of the left hand side has been changed. The letter is now the second term and is negated. There was a big drop in correct responses from item 5 to item 6 (29% to 12%) which was not expected because equations with letters on both sides of the equal sign was first encountered in term 4 of Grade 8 (DBE, 2011a). However, it was not surprising in light of what researchers have found regarding a) letters on both sides (Filloy & Rojano, 1989), b) negatives (Pournara et al., 2016) and c) varying structure of the equation (Dreyfus & Hoch, 2004).

The above discussion was an analysis where I focused on each of the six test items and gave the percentages of responses per item. The three sections that follow focus on rows which exhibit similar patterns of responses without focusing on individual learners. I sorted the data in three different ways resulting in three patterns and discuss the clusters of responses that stand out from the rest of the responses.

4.4 Analysis of responses through different organisations of data

I organised my data in three ways by using the sort function on Excel. I therefore obtained three patterns: Patterns A, B and C which are discussed below.

4.4.1 Pattern A: Sorted by item number

The data in Pattern A was first sorted in item order (item 1-6) and then the rows were sorted with correct responses followed by incorrect responses and missing (see Figure 4.2). This means that item 1 was sorted first with all the correct responses at the top of the figure and all the incorrect and missing responses at the bottom of the figure. After item 1 was sorted items 2, 3, 4, 5 and lastly item 6 was sorted. Each item's sort was dependent on the previous item's sort which is why, for example, in Figure 4.2 the green cluster in the middle of the figure is not closer to the top with the other green cluster, it also explains why, for example, item 6's correct responses are not all grouped together.

Looking at Figure 4.2 we see three clusters: Cluster 1, 2 and 3. Cluster 1 is a cluster of correct responses with a strip of incorrect responses to item 3. These learners can answer the arithmetic equivalences correctly as well as most of the arithmetic and algebraic equations. The item that none of these learners could answer correctly is the item that did not point towards a known procedure. This may suggest that these learners can memorise rules and apply them to familiar questions and are able to add large numbers. This may suggest that they do not have a relational view of the equal sign as they could not answer the item using relational thinking, nor by applying algebraic procedures.

Cluster 2 is a smaller cluster and is a different cluster to Cluster 1 because item 2 is incorrect. As in Cluster 1, this may suggest that these learners can memorise rules and apply them to familiar questions but their incorrect responses to item 2 suggest that they may have difficulty in adding large numbers. I assume that these learners do not have a relational view of the equal sign as they did not respond correctly to item 2 or item 3. They also struggle with numbers because they could

not solve item 2 arithmetically. As in Cluster 1, despite my assumption of not having a relational view of the equal sign, learners were able to solve some of the arithmetic and algebraic equations.

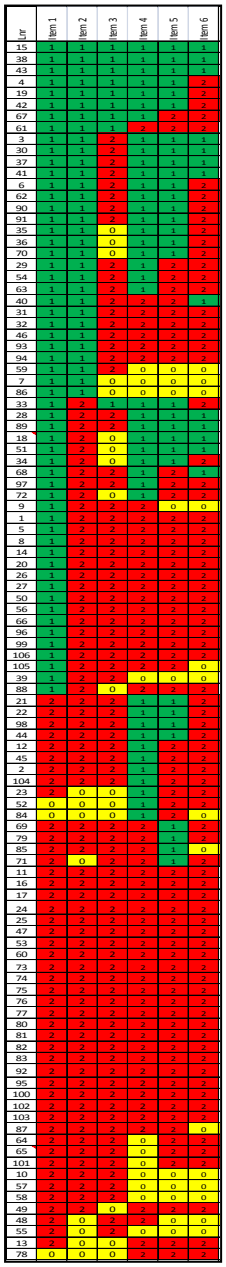


Figure 4.2: Pattern 1:

Sorted by item number

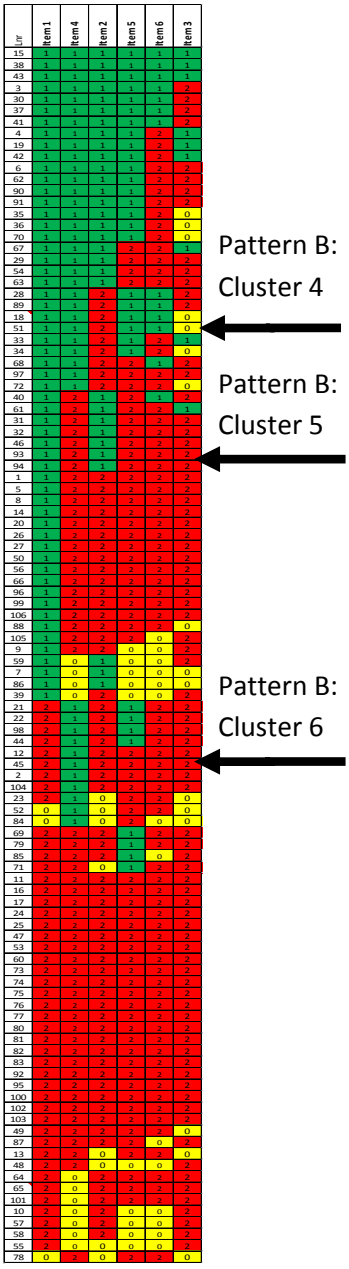


Figure 4.3: Pattern 2:

Sorted by item difficulty

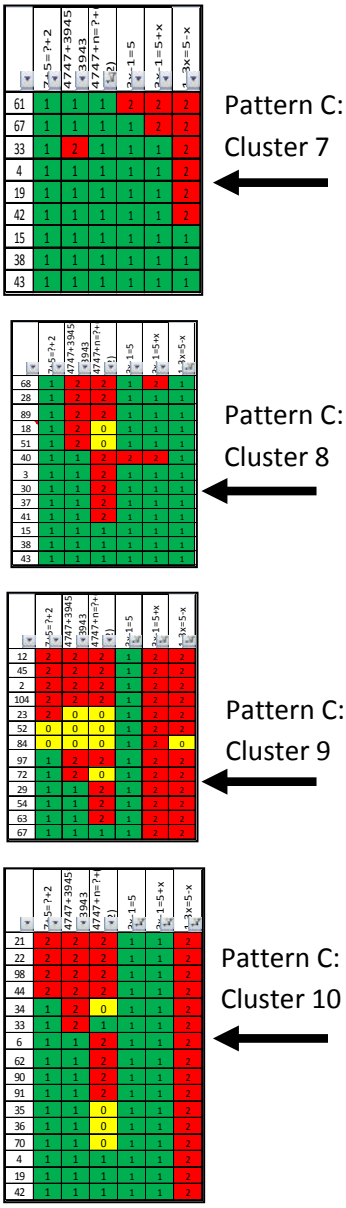


Figure 4.4: Pattern 3:

Sort by posing a question

Cluster 3 is an even smaller cluster than the previous two and represents incorrect responses with a strip of correct ones. These learners got some equations correct but everything else incorrect. This means that they can do some of the algebraic

procedure for item 4 and some for item 5 but none of them could solve item 6 and none of them could correctly answer items 1-3. I assert that these learners do not have a relational view of the equal sign as they were unable to answer questions arithmetically or algebraically. The fact that they could solve only one equation (except the four learners who could solve two) supports the literature that says that not having a relational view hinders learners' performance in the solving of equations (McNeil et al., 2004). The learners in all three clusters did not get all the equation items correct and a possible reason for this is that the learners are only remembering partial rules or procedures (Kieran (2006); Ryan and Williams (2007); Watson (2009)). The actual errors made when solving these items are analysed in Chapter 5.

4.4.2 Pattern B: Sorted by item difficulty

For Pattern B (see Figure 4.3 above), the columns were organised in order of item difficulty (items 1, 4, 2, 5, 6, 3) with the easiest item first. The easiest item was determined by the number of correct responses and not by what the difficulty level was assumed to be when the items were designed. When sorting from easiest to most difficult, item 2 and 5 had the same number of correct responses and so I decided to have item 2 sorted before item 5. This was decided because item 5 was designed to be more difficult than item 2. If item 5 is sorted before item 2 a new pattern is formed but there are few learners in each cluster and many clusters. After sorting the columns, the rows were sorted with correct responses followed by incorrect and missing responses, as was done in Pattern A.

From Figure 4.3 we see that there are three main clusters: Clusters 4, 5 and 6. Because the items are sorted in increasing difficulty levels all the clusters represent learners who got a more difficult question correct (in relation to the rest of the learners) but an easier question incorrect.

Cluster 4 shows learners who got both the equations with a letter on both sides correct but got the arithmetic equivalence with large numbers incorrect. This

suggests that these learners found executing a procedure easier than calculating (relationally or otherwise) $4747 + 3945 = _ + 3943$. This cluster corresponds with Cluster 2 from Pattern A.

Cluster 5 highlights a group of learners who got the arithmetic equivalence with large numbers correct but not the arithmetic equation (the equation with a letter on only one side). Since there was a letter present in all the items these learners got incorrect, it suggests that these learners may have difficulty in dealing with letters. This also suggests that they may have been working arithmetically rather than algebraically as they are getting the arithmetic equivalences correct but not able to execute the algebraic procedures. This cluster is important as it contrasts strongly with those who could execute a procedure which is discussed next.

Cluster 6 contrasts with Cluster 5 in that there are some learners who on two occasions get a perceived difficult question correct. This means that these learners got correct what most other learners got wrong. The perceived difficult questions in both these cases are solving algebraic equations. These learners appear to be able to execute a procedure but not able to cope with an item ($7 + 5 = _ + 2$) that is considered the easiest by the rest of the group. Where Cluster 5 emphasises the difficulty some learners have when letters are present, Cluster 6 emphasises the difficulty learners have in dealing relationally with numbers. This Cluster corresponds with Cluster 3 from Pattern A.

In both patterns we can see a group of incorrect responses (between Cluster 2 and 3 in Pattern A and between Cluster 5 and 6 in Pattern B), but it is not foregrounded as the other clusters were. This group of responses are from learners who only answered item 1 correctly and nothing else. These learners were not able to solve any of the equations correctly suggesting that they not only struggle with algebraic procedures but that they could not operationally follow through on the rules or steps for executing the procedure. Moreover, their incorrect responses to item 2 suggest that they are unable to deal with large numbers. This group of learners does not display a relational view of the equal sign as they could not relationally solve item 2

or item 3 and could not solve the equations either through algebraic procedure or with arithmetic strategies such as trial and improvement or substitution (Kieran, 1992).

Also in both Patterns A and B, the missing responses are scattered between other responses or, due to the nature of the sort, fell at the bottom of the pattern. In both Figures 4.2 and 4.3 the missing responses are occurring in horizontal clusters as opposed to vertical ones. This suggests a problem with learners' content knowledge rather than with the nature of the question. There are many assumptions one can make about why questions were left blank, for example, not having enough time, being bored or a lack of content knowledge. Another possible reason is that for a learner who is not confident with algebraic manipulation, the reduction and deduction processes involved in solving equations (Matz, 1980) may prove to be too daunting (Hall, 2002). This would explain why there were more learners who did not attempt the equation questions.

4.4.3 Pattern C: Sorted by posing a question

The clusters in this pattern are different to the ones above because they are a result of using the Excel filter function rather than the sort function. For the clusters found in Pattern C (see Figure 4.4), I first posed a question and then identified the responses that fit the criteria. I did this by ordering the items in number order (rather than difficulty order) and then filtered certain responses. The four clusters are discussed below.

4.4.3.1 Cluster 7: Getting the non-familiar question correct

The question posed to generate Cluster 7 was: *How did learners who could correctly answer the item that didn't point to a known procedure, perform in the other five items?* This question was chosen because, based on the low number of correct responses, item 3 was more difficult than the other five items but was also the question that more explicitly tested a relational view of the equal sign. Nine learners got item 3 correct but they also got many of the other items correct too. Item 6 was

the most poorly answered item in this cluster, which is surprising. I say this because whichever method learners used to correctly answer item 3, it is possible that the same method could have been used to solve item 6. This gives the impression that item 6 (the equation with a negated letter on both sides) was actually more difficult for this group of learners (except for the learners who got all the items correct). A possible reason for this is that perhaps dealing with more complex algebraic terms is more of a hindrance than understanding equality. This would imply that even with a relational view of the equal sign, to be successful at solving equations, proficiency in simplifying algebraic expressions is required simultaneously with a relational view. This is supported by Kilpatrick et al. (2001) and evident in procedural fluency and conceptual understanding (and the other strands) being intertwined and neither preceding nor being of more importance than the other strands. This cluster of learners contrasts with those who respond incorrectly to item 3 but correctly to item 6. This is discussed next in Cluster 8.

4.4.3.2 Cluster 8: Procedurally fluent in solving some algebraic equations

In contrast to Cluster 7, this cluster shows learners who could solve item 6, the algebraic equation with a negated letter on either side of the equal sign. Thirteen learners got item 6 correct and there were 11 learners who answered all three equation questions correctly. There were only three learners who could answer item 3 correctly. These three learners were also the only three to answer all six items correctly. The small number of learners getting item 3 correct was surprising because they could have used the same strategy (whether it was trial and improvement or algebraic procedures) as used for item 6 to answer item 3. This suggests that even though there are strategies available and known by these learners, they did not use them. This further suggests that learners have memorised which strategies are used for a certain type of question, meaning that these learners lack in strategic competence (Kilpatrick et al., 2001) as they do not show a flexible approach to solving non-routine problems such as , for example, item 3.

4.4.3.3 Cluster 9: Possible evidence of the didactic cut

The question asked that produced this cluster was: *How did learners perform in the arithmetic equivalences items if they could correctly solve the arithmetic equation but not the algebraic equations?* To obtain this cluster, responses were chosen where item 4, the equation with a letter on only one side was correct but item 5 and 6 (the two equations with a letter on both sides) were answered incorrectly. This criteria was informed by the notion of the didactic cut (Filloy & Rojano, 1989) and I was interested to see whether learners who met this criteria did in fact get all the arithmetic equivalences correct. There were 13 learners who were able to solve the equation with a letter on one side of the equal sign, but unable to solve the equations with a letter on both sides. Although this might be evidence that supports the presence of the didactic cut, there were six learners who did not answer the arithmetic questions correctly.

4.4.3.4 Cluster 10: Issues with negativity

Cluster 10 was obtained by asking the question: *How did learners who could correctly answer an algebraic equation with negatives and a letter on both sides, perform in the other items?* This question was of interest to me because learners could correctly solve one of the algebraic equations with a letter on both sides of the equal sign. This suggests evidence that the second appearance of a letter was not what hindered learners from getting the correct answer. There were 16 learners who responded correctly to item 4 and item 5 but incorrectly to item 6. Since both item 5 and item 6 have letters on both sides of the equal sign, it is possible that the presence of letter on both sides was not the problem. As mentioned in Chapter 2, in the item analysis, the difference between item 5 and item 6 is that the letters that needed to be dealt with were both attached to a minus sign. This cluster suggests that 15% of my sample may struggle more with negatives than with the presence of a letter on both sides. It is worth noting the high number of correct responses to item 1 and 2. This means that the learners were able to view the equal sign in an appropriate way to obtain the correct answer and were also able to execute an algebraic procedure to solve the first two equation items. The findings for this

cluster suggest that the presence of a negated unknown is more of a stumbling block than large numbers or letters on both sides. Learner's errors with negatives are discussed in more detail in Chapter 5.

If the response patterns in Pattern C were to be placed in relation to all the other responses (as in Pattern A and B) then I would need to manually move certain learners to certain positions in the sort. By *cutting* and *pasting* I could potentially create any pattern I wanted and that would jeopardise the rigour and validity of my findings. To avoid idiosyncratic behaviour I constrained the actions I performed on the data to only Excels filters and sorts.

This chapter has largely been about the different information that is highlighted depending on how one organises the data and depending on what questions are being posed. In both Patterns A and B the same information was used but it is interesting that the different patterns brought about different clusters and hence highlighted different performance patterns. For example, Cluster 6 was highlighted in Pattern B but not in Pattern A. At a deeper look, a cluster equivalent to Cluster 6 in Pattern B lies between Cluster 2 and 3 in Pattern A but was not foregrounded in Pattern A. In contrast to the organisation of data in Patterns A and B, Pattern C shows clusters that were purposefully foregrounded. Having coded and organised my data in three different ways, I now discuss the limitations to the method of coding used in phase 1 of my data analysis as well as the way in which one sorts data.

4.5 Limitations

Coding learners' final responses as correct, incorrect and missing and focusing only on the last line (or something similar) was very useful for me in terms of taking a bird's eye view of the learners' responses as well as for selecting my sample for the detailed analysis. No coding scheme is perfect and all have limitations. Below I briefly discuss two limitations of coding final responses. These are a) responses

that were incorrect due to a single error that was considered a slip and b) final responses that were correct but the method or procedure was incorrect.

4.5.1 Responses that were incorrect due to a slip

A learner may have made a slip and hence got their final answer wrong warranting being placed in the incorrect response category for phase 1 analysis. When selecting the 46 scripts to be analysed in terms of the errors made, if all three equations items (item 4-6) were incorrect, the script was not chosen and the learners responses were not further analysed. I decided (on the advice of my supervisor) to not select learners who got all the equation items wrong because it suggested that the learner had very little proficiency in solving linear equations. It is possible that one of the incorrect responses was in fact due to a slip and so perhaps the script should have been chosen. This is a limitation because these scripts could have given me insight into the learners' performance. This scenario represents a small sample of learners and their errors probably emerged in the reduced sample anyway.

4.5.2 Correct final responses with an incorrect method or procedure

In contrast to a slip, some final responses were coded as correct in phase 1 because the final answer was the correct answer. As discussed in Chapter 3, a limitation to this coding was obtaining false positives where although learners' final answers were correct the answer did not follow from previous steps in their procedure. There were 12 instances (amongst 10 learners) where the correct answer did not follow mathematically from the previous steps. These 12 responses were therefore incorrect. This means that if the false positive was the only equation item the learner got correct their script should not have been chosen. In retrospect none of these 10 scripts should have been chosen because they would have had all the equation items incorrect.

The sections that follow discuss how the above analysis informed my sample selection for phase 2, the detailed error analysis.

4.6 Conclusion

The analysis done in this chapter enabled three high level looks on a strategically selected set of test items. I identified different clusters of learners' final responses and used these to inform my sample selection for phase 2. In terms of learner performance and based on a pattern of responses, I identified certain difficulties that learners may have: difficulties with arithmetic, negatives, letters, having a relational view of the equal sign and executing algebraic procedures when solving linear equations. My data suggests that the transition from arithmetic to algebra does not happen in a linear fashion and that some learners can solve equations even if they do not have a relational view of the equal sign.

A secondary outcome of this analysis is that I was provided with a systematic way of making strategic selections of data to analyse in terms of the nature of errors made. In the chapter that follows findings from a detailed error analysis are presented. I discuss all the error categories but focus on the two most prominent categories: equality and negativity.

Chapter 5 : Error Analysis

5.1 Introduction

This study investigated learners' errors when solving linear equations and arithmetic equivalences and the purpose of this phase of the study is to answer the research question: *What errors do learners make when solving arithmetic equivalences and arithmetic and algebraic equations?*

In Chapter 2 I provided a detailed discussion of my perspective on the nature of errors and definitions and examples of each of the error classifications that were used in this study. In Chapter 3 I discussed the design of the test items and provided a justification for which items were analysed. In the previous chapter, Chapter 4, I provided a high level look at the final responses from 106 learner test scripts and discussed certain error patterns that were found.

In this chapter I provide an in-depth analysis and a thick description of the actual errors made, in the responses to six items, from 46 Grade 10 learners' test scripts. I also analyse the errors made in two of the clusters identified in Chapter 4. Since items 1-3 are different in nature to items 4-6, two coding schemes were employed and the analysis of both is discussed separately in Sections 5.3 and 5.4. Both sections begin with a reminder of the coding scheme, followed by an overview of the findings of the error analysis. The errors are then analysed by selecting the most significant and prominent errors that occurred. Although I discuss all the categories, I focus on the equality and negativity error categories.

As a reminder, the term *structure* emerges in this chapter. I refer to the structure of equations and solutions as being the visual form of an equation and do not use it with reference to any other theoretical underpinnings.

5.2 Overview of responses

Figure 5.1 shows the percentages of each code per item of the selected 46 scripts. We can see a very different picture to that in Figure 4.1 and Figure 4.2 in Chapter 4. The ratio of correct to incorrect responses is now 5:4, compared to the 2:5 (as mentioned in Section 4.3). Phase 1 focused on the combination of responses across the whole sample. Phase 2 focuses on each of the incorrect responses and discusses the nature of the error made.

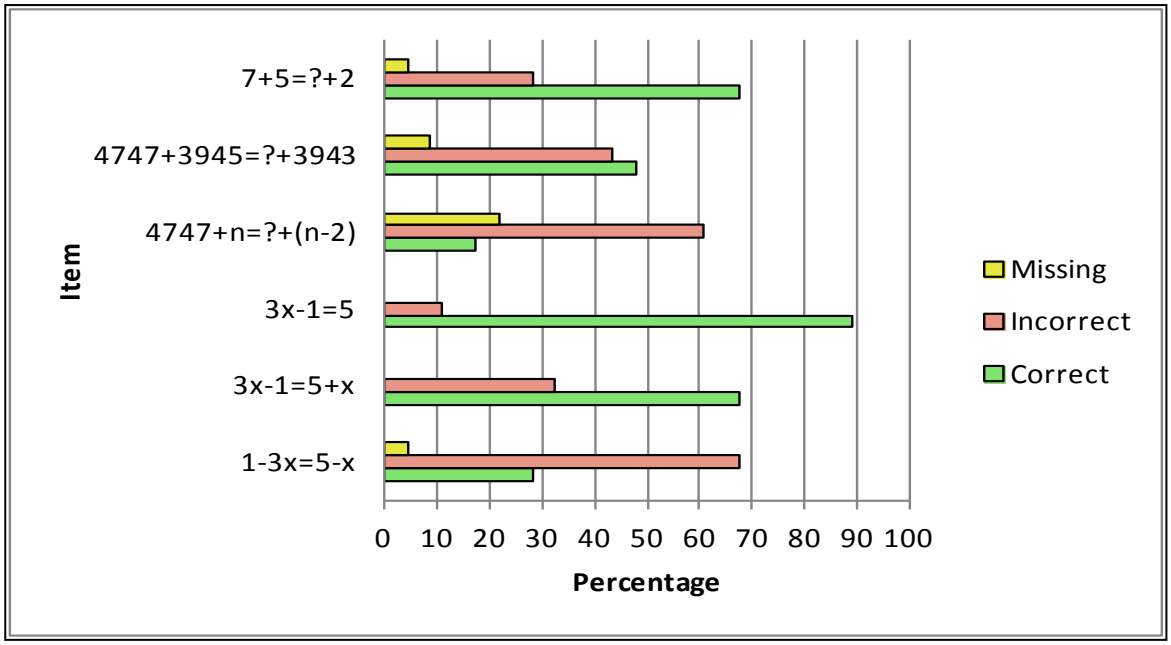


Figure 5.1: Percentages of responses per item in phase 2

5.3 Error analysis for items 1-3

5.3.1 Coding scheme for items 1-3

As a reminder, the 138 responses from items 1-3 were coded according to six categories, four of which are considered error categories. The category *operational view* was sub-divided and the justification for this division was given in Section 3.8. Table 5.1 is a reminder of the categories used:

	Category
1	Missing
2	Relational view: correct response
3	Pre-relational view
4	Operational view a) Left-to-right reasoning b) Focusing on one value
5	Repeat value
6	Unknown

Table 5.1: Error categories for items 1-3

5.3.2 Analysis of responses to items 1-3

Responses to items 1-3 were single responses and did not require learners to show their working out or their procedure. Learners had to fill in the blank space and therefore there were fewer errors and less room for my own interpretation in coding the responses.

5.3.2.1 Overview of findings

In this section I present my findings from analysing items 1-3 and provide a discussion on the errors made. I begin with an overview of the findings and then discuss each error in the order of prevalence found. There were 138 responses ($46 \text{ scripts} \times 3 \text{ items}$) that were analysed. Responses were correct, missing or incorrect. Table 5.2 shows the number of these responses.

	Item 1	Item 2	Item 3	Total number	Total percentage
Missing	2	4	10	16	12%
Correct	30	24	8	62	45%
Incorrect	14	18	28	60	43%
Total	46	46	46	138	100%

Table 5.2: Number of responses to items 1-3

The number of correct responses decreases from item 1 to item 3 and the number of incorrect and missing responses increases from item 1 to item 3. This was expected since item 3 was designed to be a more cognitively demanding item (Stein et al.,

2000), but also, in hindsight, appears to be more unfamiliar. There were 62 correct responses (45%) and almost half of these came from item 1.

Since these items were designed to test learners' view of the equal sign and see how many had a relational or operational view of the equal sign, the 45% of correct responses suggests that some learners do have a relational view. I argue that one could correctly answer these questions using arithmetic and not necessarily hold a relational view. In my opinion, it is more likely that the 17% of learners who got item 3 correct, have a relational view of the equal sign. Figure 5.2 disaggregates the 60 incorrect responses into five categories. The sections that follow are discussions about the errors.

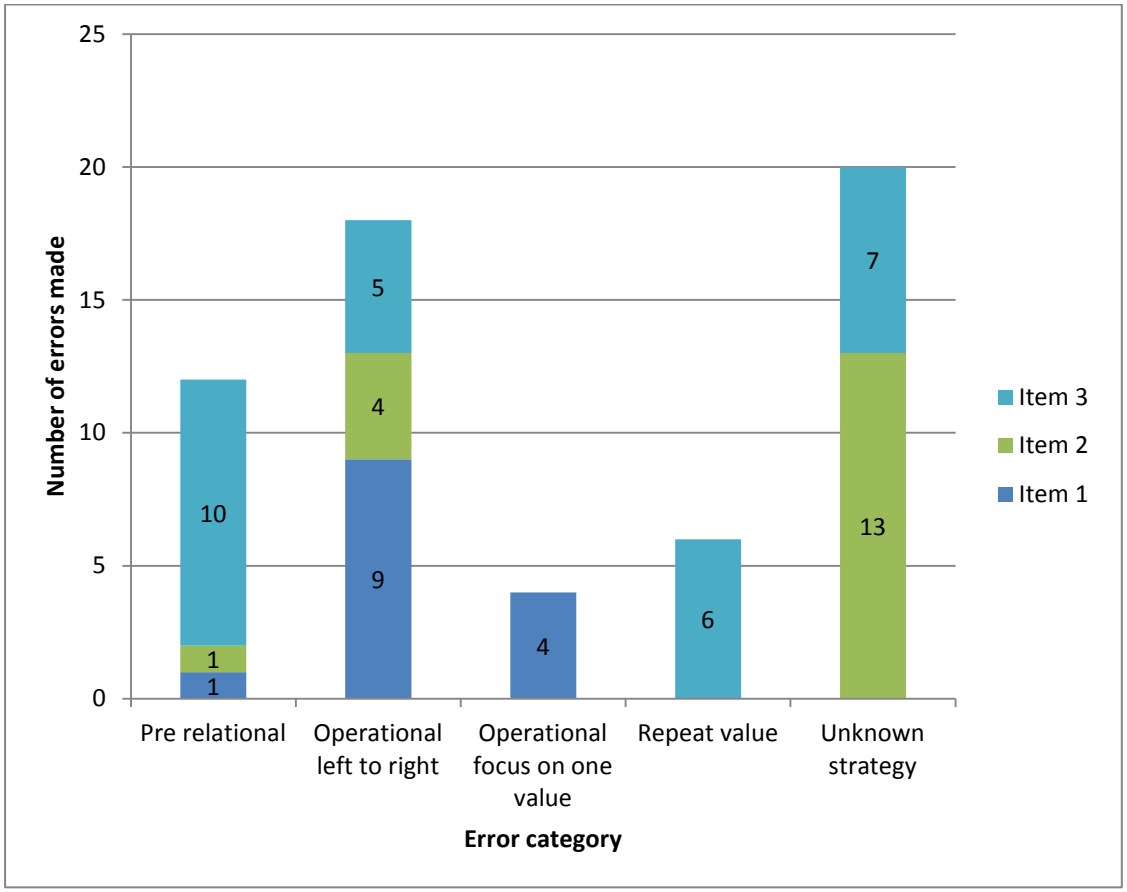


Figure 5.2: Item 1-3 responses per category

5.3.2.2 Pre-relational

A pre-relational response is one where the learner appears to attempt having, as Stephens et al. (2013) call it, a relational-structural view of the equal sign. I say 'attempt' because the learners made an error while trying to deal with the items in this way. Stephens et al. (2013) state that a relational-structural view is more easily observed when large numbers are used and item 3 not only had large numbers but also the letter n and a blank space. It is then not surprising that item 3 had 10 times more of this error than the other items. I assert that some learners are moving towards having a relational view of the equal sign and approached item 3 from a structural point of view. They did, however, make the mistake of subtracting 2 instead of adding 2 to 4747. This is possibly due to the fact that on the right 2 is subtracted from n and so the learners (in their attempt to maintain balance) also subtracted 2 from 4747.

5.3.2.3 Operational: Left to right

This error was a more prominent error than the others, accounting for 18% of the errors. This finding concurs with the findings made by Essien and Setati (2006), where Grade 8 and 9 learners viewed the equal sign as a *to do something* signal (Kieran, 1981) or unidirectional symbol (Essien & Setati, 2006). This error was more prominent in item 1. One possible reason is that the small numbers resemble the types of questions learners saw in primary school (Behr (1980); Carpenter et al. (2003); Essien (2009)).

5.3.2.4 Operational: Focusing on one value

This category is different from the left-to-right reasoning in that, from my interpretation of the learners' errors, they focused on one number to be 'the answer' and filled in the blank space accordingly. In the operational: left to right reasoning category my interpretation is that the learners are focusing on the operation on one side of the equal sign and treating the blank space as the final answer, so not focusing on or taking into account the added value after the blank space. It was therefore important for this category to be separated from the one

above. This error of focusing on only one value was only made in response to item 1 suggesting that the smaller values encouraged such an error. Learners responded to item 1 with a) $7 + 5 = \mathbf{3} + 2$ or b) $7 + 5 = \mathbf{5} + 2$. It is possible that that those who responded with $7 + 5 = \mathbf{3} + 2$ were looking at what should go in the blank space that would add up to the 5. This suggests that the learners' attention was on the 5 and not $7 + 5$ (Mason, 1998). Regarding the learners who responded with $7 + 5 = \mathbf{5} + 2$, it is possible that that they tried to fill in the blank space to add up to 7 so they ignore the +5. Again, the learners who made this error were not taking into account the operations on both sides and instead only operated on the right and ignored the operation on the left. This is an operational view of the equal sign where the learner could only operate in one direction, hence suggesting having a unidirectional view of the equal sign (Essien & Setati, 2006).

5.3.2.5 Repeat value

This error was only made in item 3. This may suggest that learners do not know how to deal with the item and so repeated the value 4747. Perhaps this error was not made in items 1 and 2 because items 1 and 2 pointed to a strategy using arithmetic whereas item 3 had a letter as well as a blank space, not pointing to a known strategy.

5.3.2.6 Unknown strategy

There were many responses that I could not interpret. For example, all 13 unknown responses to item 2 were different. A sample of some of the responses are: 1214; 262 and 3942. There were seven unknown responses to item 3 that were also all different. Four examples are: 2350; 4730; 2323 and 869. There were only 2 learners who had unknown responses for both items 2 and 3. The fact that there were only *unknown strategy* responses to items 2 and 3 suggests that learners were not able to deal with large numbers nor able to make sense of the question. These learners appear to not be able to work arithmetically or algebraically. There were 20% of errors that were illogical to me despite my efforts of attempting various ways to make sense of them.

This section has presented my findings from analysing items 1-3 and provided a discussion on the errors made. In terms of learner performance, the two most common errors made were due to an operational view of the equal sign and an unknown strategy. The section that follows focuses on the error analysis for items 4-6 which required more rigour.

5.4 Error analysis for items 4-6

5.4.1 Coding scheme for items 4-6

I coded responses to items 4-6 according to seven categories, five of which are considered error categories. The negativity errors and equality errors category were sub-divided and justifications for the divisions were elaborated on in Section 3.8. Table 5.3 is a reminder of the categories used to classify the different errors made:

	Category
1	Blank/ no response
2	Correct
3	Equality errors a) Notation: misuse of the equal sign b) Inverse errors c) Balance errors d) Familiar structure (ignoring certain detail to obtain a familiar solution) e) Incomplete(structure unfamiliar) f) Syntax g) Change to expression
4	Negativity errors a) Minus sign ignored b) Minus sign used incorrectly c) Minus sign acknowledged but not used
5	Letter errors
6	Number errors
7	Unknown error (stuck and use arithmetic or illogical)

Table 5.3: Error categories for items 4-6

5.4.2 Analysis of responses to items 4-6

In this section I focus on the two most prominent error categories found in my study: *equality errors* and *negativity errors*. Although I discuss the other errors and categories, I do not focus as much on them. I begin this section with an overview of the findings and then discuss the errors within the equality and negativity category. This is followed by a discussion of the remaining categories.

5.4.2.1 Overview of error analysis

As a reminder, the *correct* category includes responses where learners made a slip. This decision was made because, from my interpretation, the error was not due to a misunderstanding but was rather an oversight. Of the 84 correct responses, 10 were due to a slip. In contrast to the method of coding in phase 1, a final answer being correct did not guarantee a *correct* code. In Chapter 4 obtaining false positives was discussed and now in Chapter 5 the false positives are treated as incorrect responses and the errors made are analysed. False positives were not assigned a code but the number of occurrences was noted in Chapter 4. There were 138 responses (46 scripts $\times$ 3 items) that were analysed. Responses were first coded as correct, missing or incorrect. Table 5.4 shows the number of these responses.

	Item 4	Item 5	Item 6	Total number	Total percentage
Correct	36	27	21	84	61%
Missing	0	0	2	2	1%
Incorrect	10	19	23	52	38%
Total	46	46	46	138	100%

Table 5.4: Number of responses per code for items 4-6

The number of correct responses decrease from item 4 to item 6 and the number of incorrect items increases from item 4 to item 6. This was expected since item 6 was designed to be a more cognitively demanding item (Stein et al., 2000). There were 2 instances (1%) where an item was not attempted. Because there was nothing to analyse in these blank responses the number of missing responses was not included

in the total number of errors made. From the 52 incorrect responses the errors were analysed and a total of 113 errors were counted (see Appendix J).

Table 5.5 shows the distribution or errors per item. For example, 10 learners answered item 4 incorrectly and within these 10 incorrect responses, 21 errors were made.

	Item 4	Item 5	Item 6	Total
Number of incorrect responses	10	19	23	52
Number of errors	21	39	53	113

Table 5.5: Number of errors and incorrect responses

There was an increase in the number of incorrect responses as well as errors from item 4 to item 6. Since the items were designed to increase in cognitive demand and because the structure changed from item 4 to 5, the increase was not surprising.

As a reminder, a response could have more than one error, but each error was only coded once. Each of these errors fits into only one of the following error categories: equality; negativity; letter number; or unknown. Figure 5.3 shows a stacked bar graph of the percentage of errors per error category and within each item. The bulk of this chapter is devoted to analysing these 52 responses with 113 errors.

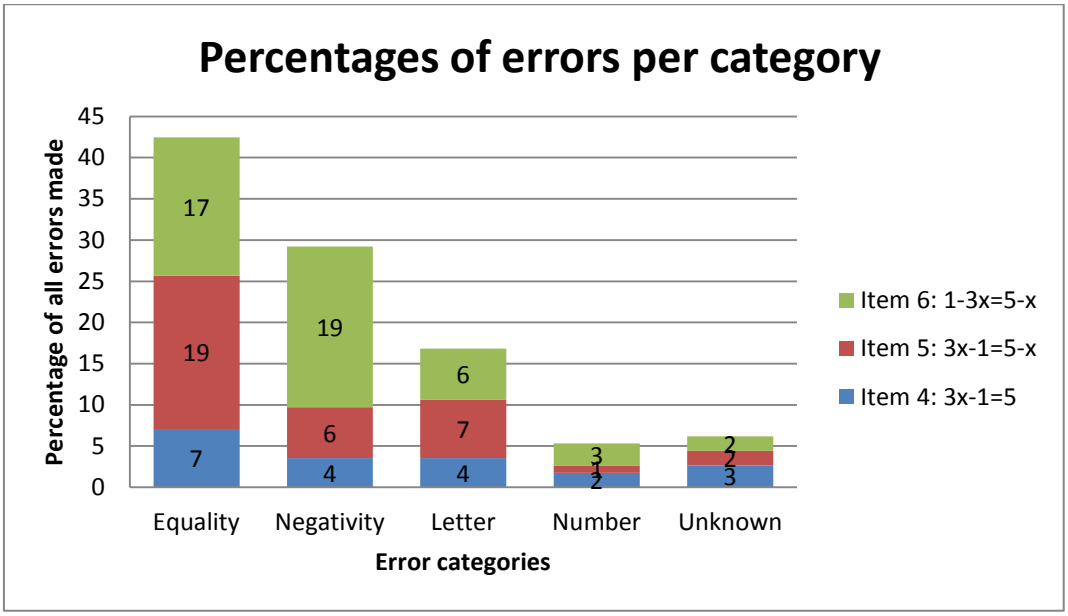


Figure 5.3: Percentage of errors per error category for items 4-6

We see from Figure 5.3 that the equality errors are the most prominent with 48 of the 113 errors (42%). This is followed by errors relating to negativity (29%). It is not surprising that item 6 has more negativity errors than the other items since the letter on the left hand side of item 6 was subtracted, whereas in both items 4 and 5 the letters were added. It is also not surprising that items 5 and 6 had the most equality errors as these items had a letter on both sides of the equal sign and so required more use of inverses and maintaining balance. The next section provides a discussion on the different types of equality errors found.

5.4.2.2 Equality errors

This category had 6 subcategories namely: inverse errors; balance errors; familiar structure; incomplete procedure; syntax; and changing an equation into an expression. These have been listed in the order of highest to lowest number of errors found and are discussed in the above order. Figure 5.4 shows the percentage of equality errors per subcategory.

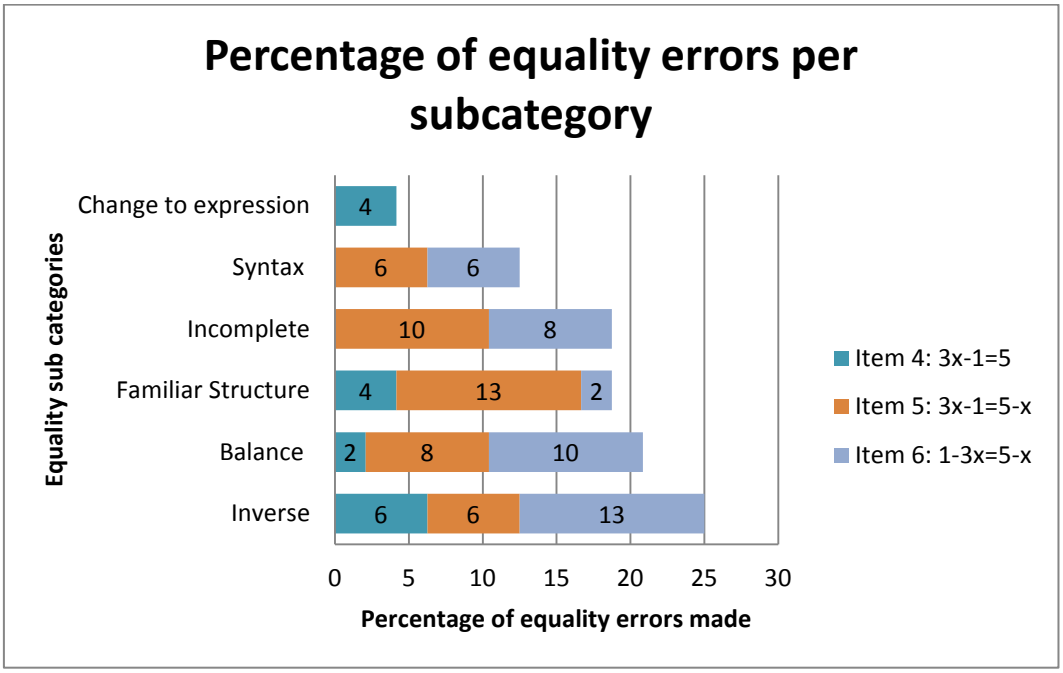


Figure 5.4: Percentage of equality errors

5.4.2.2.1 Inverse

The inverse error was a much larger category containing 11% of all errors made and accounted for 25% of all the equality errors made. The learners who made this error were not able to identify what the additive inverse should be and either used the multiplicative inverse (instead of the additive inverse) or did not change the sign when 'moving the number to the other side'. For example, Figure 5.5 shows learner 44 moving the coefficient of x to the other side by using an additive inverse rather than the multiplicative inverse. In Figure 5.6, Learner 23 has a $+1$ on the right hand side instead of -1. In the example of Learner 44, Hall (2002, p. 11) calls this type of inverse error "the other inverse" and in the example of Learner 23, Kieran (1992, p. 402) names this type of inverse error "switching addends".

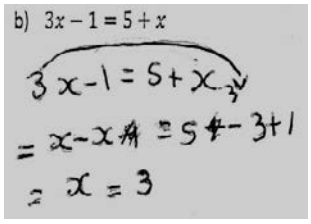


Figure 5.5: Learner 44 response to item 5

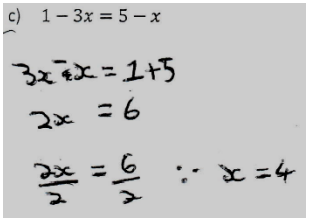


Figure 5.6: Learner 23 response to item 6

5.4.2.2.2 Balance

I identified 2 different types of balance errors: the first relating to using two inverses to isolate x on one side and a constant on the other side, and the second relating to substituting values for x that do not make the left hand side equal to the right hand side. There were 7 learners who did not maintain balance when executing the procedures. They operated on objects on the left and right hand side of the equal sign in different ways. This accounted for 20% of the equality errors. In trying to apply an additive inverse, these learners would, for example, subtract a number from the left and then add the number on the right. Kieran (1984) called this error a *redistribution error*.

My own interpretation of why learners may do differently to the left and right is to obtain a familiar solution. By 'familiar' I mean that the solution learners wrote down includes statements that they would have seen before in class. Solutions to

equations are in the form $x = C$ and very seldom $C = C$ or $x = x$. Statements such as these are confusing for learners (Behr, 1980). A statement such as $Ax = Bx$ is not considered a solution by the learner and hence not familiar. For example, in Figure 5.7 Learner 104 divided the left hand side by 2 and the right hand side by x, and after ignoring the minus sign, obtains a solution of $x = 5$.

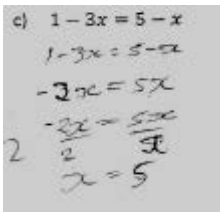


Figure 5.7: Learner 104 response to item 6

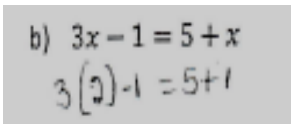


Figure 5.8: Learner 63 response to item 5

Another error that was coded as a balance error was when learners used arithmetic to determine x and substituted values for x but the left and right hand sides were not equal. For example, in Figure 5.8, Learner 63 substituted $x = 2$ on the left and $x = 1$ on the right and these did not make the left and right sides equal. This learner does not appear to connect the letter on the left and right as having the same value but also, does not appreciate that the two sides need to be equal. This error was coded as a balance error. Although there were only two learners who made this version of a balance error, the error is important to note because the learners did not use algebra in their responses to the three items and instead used arithmetic strategies to determine the value of x. They both got item 4 correct but then substituted different values for x that did not make both sides of the equal sign equivalent. This supports the findings that Filloy and Rojano (1989) had regarding the didactic cut and learners not being able to apply their arithmetic strategies to equations with a letter on both sides of the equal sign.

Both the inverse and balance errors together account for 20% of all the errors made and 56% of the equality errors. Both these errors are concerned with the deduction process involved when solving equations and suggests that when solving equations these learners overgeneralise rules that can be applied to a procedure, for example applying *a negative and a negative make a positive* when adding. As stated by Watson (2009, p. 4), learners "do not think about the meaning of the situation in

which they [the rules] might be successfully applied" so although the learners may have some idea of what the procedure entails, they don't know what the rule means and hence can't execute the procedure properly. It appears that these learners know *what* to 'get rid of' but do not *how* to do so. What they perhaps do know is a series of steps that need to be followed that do the 'getting rid of'. A learner who correctly applies the *change sides, change sign*s rule, although is applying a rule, one cannot tell whether they have any knowledge of inverses or balance.

5.4.2.2.3 Familiar structure

There were six learners who got to the statement $Ax = Bx$ and then manipulated their reduction process or their final solution to obtain a familiar-looking solution. It appears that these learners manipulated their procedure by just ignoring the x on the right hand side. Although balance was maintained and what was done on the left was done on the right, in the final step/solution they ignored the second appearance of the letter and ended up with a solution that again looked familiar. For example (see Figure 5.9), after making a conjoining error and adding unlike terms, Learner 69 divided both left and right hand sides of the equation in order to isolate x. The original final solution was $x = 3x$ which suggests that Learner 69 realised that this did not 'look right' and so ignored the second x and crossed it out leaving a 'nice' solution of $x = 3$ that has a familiar structure.

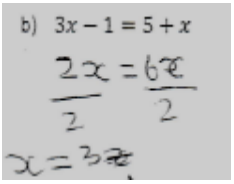


Figure 5.9: Learner 69 response to item 5

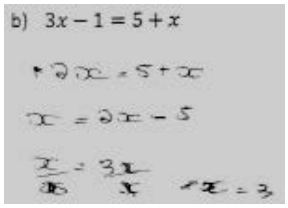


Figure 5.10: Learner 85 response to item 5

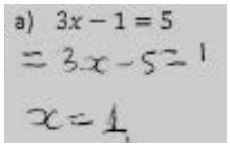


Figure 5.11: Learner 69 response to item 4

Figure 5.10 shows Learner 85's similar procedure but after adding unlike terms and using the minus sign in $2x - 5$ incorrectly, by applying right to left reasoning, the learner was left with $x = 3x$ and divided both sides by x. I did not code a division by x as an error because mathematically it is allowed with a restriction of $x \neq 0$. At the time of the testing these learners had not encountered this and so I did not code it

as an error. The division by x should have reduced the equation to $1 \neq 3$ but the learner wrote $x = 3$. I assume that the learner already knew what the structure of the final solution should look like and so without even discerning the detail (Mason, 2008) of what the equation should reduce to, s/he wrote down what was familiar, $x = 3$.

Another example of an error that was coded as familiar structure is given in Figure 5.11 which is Learner 69's response to item 1. S/he correctly manipulated the equation by subtracting 5 from both sides and was left with an equation of the form $Ax + B = C$ and then wrote $x = C$. This suggests that a familiar structure of the equation is a cue that learners look for when they solve equations. It appears that some learners can execute procedures correctly until there is a single number on the right hand side, that is, until the manipulated version looks like a *canonical equation* (McNeil et al. (2006); Sherman and Bisanz (2009)) and then the number on the right becomes 'the answer'. This echoes the operational view of the equal sign where you must do something and then the number is 'the answer'. Lewis (1980) talks about how much of the mathematics is about 'getting rid of things' and that for many learners the aim of the mathematics game is to get a "x = something with no x in the something" (p. 6).

In summary, these errors were coded as an error in equality because the learner could not correctly isolate x. It was further coded as *familiar structure* because I interpreted the error as being made for the purpose of obtaining a familiar structured solution. This error accounted for 8% of the errors and was made by six different learners in nine instances.

5.4.2.2.4 Incomplete procedure

The category *incomplete procedure* does not reflect the rich insight that can be gained from these 11 responses. These responses were incomplete but 9 of the 11 incomplete responses were left in the form $Ax = Bx$, suggesting that learners were unable to identify what actions needed to be put in place to end up with a letter on

only one side of the equal sign. Having a solution in the form $Ax = Bx$ is possibly an unfamiliar structured statement. The learners should have the skills and a strategy that can be used to resolve the problem (DBE, 2011a) but not completing their procedure suggests that the learners do not recognise that they could use these skills. A learner's ability to manipulate algebraic symbols successfully requires an understanding of the structural properties of mathematical operations and relationships (Booth, 1989). Not understanding the structural properties therefore means that manipulating symbols will not be done successfully. In the previous discussion on the familiar structure error, it appeared that learners were forcing some (incorrect) manipulation in order to obtain a structure that was familiar but that they, presumably, do not understand. A possible reason for learners not knowing what procedure they could use to get them 'unstuck' is related to not being exposed to many, if any, equations of the same structure.

5.4.2.2.5 Syntax

The *syntax error* was made only in items 5 ($3x - 1 = 5 - x$) and 6 ($1 - 3x = 5 - x$) which have a letter on both sides of the equal sign. Kieran (1981) talks about this error as a *shortcut error* and states that it is a different version of an inverse error. A possible reason that this error is present in items 5 and 6 and not in item 4 is because in items 5 and 6 there are more objects that need to be operated on and so learners appear to try to be more explicit and careful in their working, yet violate the symmetric property of equality. The more explicit workings suggest that some learners understand what must happen but, by not having a relational view of the equal sign, they are not able to articulate each step on paper in a mathematically acceptable way. These errors make up 5% of all the errors made. Some learners made an error in their syntax but their solution was correct, meaning that their incorrect syntax did not affect their answer. For example (see Figure 5.12), Learner 90 added 1 to both sides and subtracted x from both sides but subtracted the x on different lines. An x was subtracted on line 1 on the right and then on line 2 on the left. In these two lines of working out, the learner has not maintained balance but rather obtained it over a series of steps. Some learners are not as explicit as Learner

90, meaning that they leave out lines 1 and 2 and write down only lines 3 and 4, giving the impression that they are procedurally and syntactically fluent. Having Learner 90 (and others) be explicit in their deduction has provided me with insight into how they were thinking.

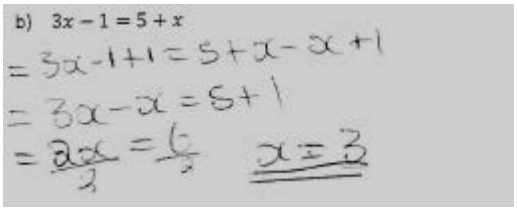


Figure 5.12: Learner 90 response to item 5

At first I coded these errors as slips but a slip is unintentional and can be self-corrected. These syntax errors, in my opinion, would not be corrected as I assume that they were written purposefully demonstrating what detail the learner was attending to first (Mason, 2008).

Booth (1989) states that the struggles learners face when learning syntax are the result of a limited understanding of mathematical structures. This section has shown how learners make many errors for the sake of obtaining a recognisable structure. The syntax errors reported here could therefore be a consequence of a limited understanding of mathematical structures.

5.4.2.2.6 Change equation to expression

Ryan and Williams (2007) found that learners are often unable to deal with an expression as an answer and so try to solve it as though it were an equation. This highlights two issues: the first is about equality and the second about the *duality of zero* (Gallardo & Hernández, 2005).

Literature has reported learners adding '$= 0$' to an expression hence changing an expression into an equation (see Kieran (1992); Papaieronymou (2007); Wagner, Rachlin, and Jensen (1984); and Watson (2009)). I was therefore surprised to find that some learners changed the equation into an expression. This is an error that I did not find in the literature. The error *change equation to an expression* was only

made in responses to item 4. This is a small subcategory but accounted for 5% of the equality errors. The fact that this error was only made in item 4 does suggest that the structure of the item 'encourages' the error. The learners who made this error did so by first correctly applying the inverse operation or 'moving the number to the other side" but then were left with an $expression = nothing$. Since there is no number on the right hand side learners continue their procedure by simplifying an expression. Gallardo and Hernández (2005) discuss how zero can be viewed as *nullity* or *totality*. It appears that if learners are faced with an $expression = nothing$ and then change the equation into an expression, the learners are viewing the zero as nullity. Changing an equation into an expression is fundamentally about equality and knowing what the relationship is between the left and right hand side of an equation. This error suggests that learners do not have a relational view of the equal sign. In items 5 and 6, if the learner 'moves' the number to the other side, the term with x would still be on the right hand side and hence the learner would not be 'encouraged' to create an expression. An example of this error can be found in Figure 5.13 where Learner 85 responds to item 4 by conjoining $3x$ and -1 and reducing this to $2x$. The learner then correctly subtracted 5 from both sides of the equation to maintain balance and should then have been left with $0 = 2x - 5$, but instead the learner wrote '$= 2x - 5$' changing the equation into an expression. The learner then simplified the expression by adding the two unlike terms from right to left in order to, I assume, avoid a negative answer.

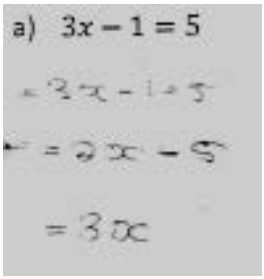


Figure 5.13: Learner 85 response to item 4

In summary, the errors mentioned above suggest that some learners may battle with executing the procedures but also that when they try to be explicit and *do to the left as they do to the right*, they then make additional syntax errors. It also appears that many errors are made in an attempt to obtain a familiar solution and avoid

structures that are atypical. A big difference between attending to equations and expressions is that in equations learners need to use the deduction process and not just the reduction process (Kieran, 1992; Matz, 1980). One of the biggest struggles learners encounter when reducing an expression relates to dealing with negatives and subtraction, this error category is discussed next.

5.4.2.3 Negativity errors

Negativity errors are prominent in my study and therefore they have been subdivided into three subcategories, namely: minus sign ignored; minus sign used inappropriately; and minus sign acknowledged but not used.

When dealing with equations it is impossible to avoid dealing with negatives and subtraction because, the inverse of addition is subtraction and inverses are a key element needed in solving equations. Even applying the rule *change sides, change signs* means that a positive sign would change to a minus sign. The problems encountered regarding negatives are not related to equations though, but rather to the reduction process when simplifying expressions. I classified three ways that learners incorrectly reduced the left and/or right sides of an equation when a minus sign was present. These errors are listed and discussed in order of highest to lowest number of errors found and are discussed in this order too.

Figure 5.14 shows the percentages of the negativity errors per subcategory and within each item. The number of negativity errors increases from item 4 to item 6 but also, there is a larger jump in the number of errors from items 5 to 6 than from items 4 to 5. This is surprising because it is in item 5 where the additional appearance of x (that is subtracted) is first introduced. This suggests that the change in order, from $3x - 1$ on the left in item 5 to $1 - 3x$ on the left in item 6 caused more problems for the learners that the additional appearance of x.

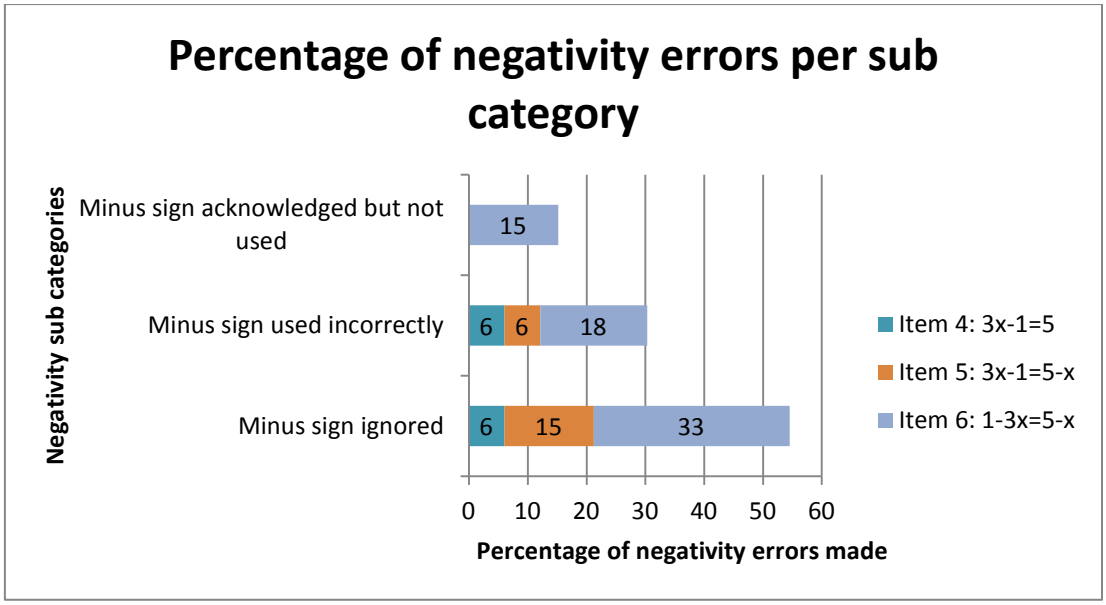


Figure 5.14: Percentage of negativity errors per subcategory

5.4.2.3.1 Minus sign ignored

In this category, learners ignore the minus sign and only operate (by adding) on the numbers that they could see, meaning that they separate the sign or operation from the number or letter. This category of error was the most prominent negativity error accounting for 54% of the negativity errors and 16% of all errors made. These learners appeared to not pay attention to the signs and operation which supports the findings from Pournara et al. (2016). If a minus sign is ignored, it suggests that the learner does not know how to deal with it appropriately. Perhaps this is because of the different functions of a minus sign (Gallardo & Rojano, 1990). The multiple meanings of the minus sign could be causing learners to be confused and hence they ignore it. For example, in Figure 5.15, Learner 72 reduced $-5-1$ to 6. I interpreted this as the learner ignoring the sign and operating on the 5 and 1. As mentioned in Chapter 3 an alternative interpretation could be that the learner paid explicit attention to the signs and overgeneralised the signs-rule. This was not my interpretation because as we can see in Learner 72's reduction of $-3x+x$, the reduction was not $-3x$ (using *a negative and a positive makes a negative*), but rather it appears that the learner ignored the negative and operated on the 3 in $-3x$ and the 'no number' in front of x.

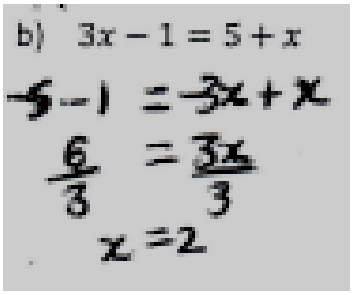


Figure 5.15: Learner 72 response to item 5

In contrast to learners not paying attention to the sign, some learners did pay attention to it, but either used it incorrectly or didn't appear to know how to use it and so just attached it on to their solution. These two situations are discussed below.

5.4.2.3.2 Minus sign used inappropriately

An error that was coded as a *minus sign used inappropriately* could be referred to as right-to-left reasoning (Vlassis, 2004). Vlassis (2004) states that learners operate from right to left in order to perform a more "comfortable" procedure (p.477). For example in Figure 5.16, Learner 33 reduced $1 - 3x$ to $2x$ by operating on the objects from right to left for the purpose, I assume, of obtaining a positive result.

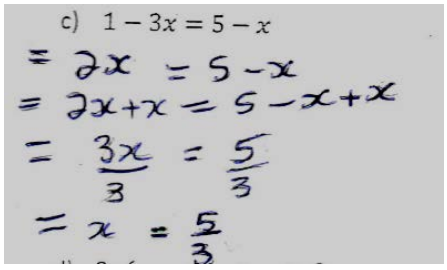


Figure 5.16: Learner 33 response to item 6

I have extended the right-to-left reasoning category by including all errors where the learner avoids a negative answer or rather, has used the negative or subtraction sign in an inappropriate way to get a positive answer. For example in Figure 5.17, Learner 36 reduced $-3x + x$ to $2x$. The learner subtracted 1 from 3 instead of 3 from 1 but the two terms were not written in such a way that the learner's actions were from right to left, yet the response was still positive.

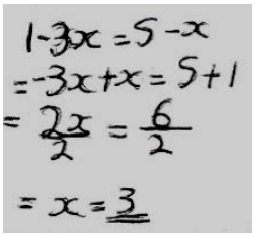


Figure 5.17: Learner 36 response to item 6

In both examples above, the learners are using the minus sign inappropriately by subtracting, what is thought to be, a smaller number from a larger number in order to obtain a positive, more comfortable, solution. So although the learners are using the symbol as an operational sign, they do so by subtracting the apparently smaller from the larger. It is not surprising that there were three times more *minus sign used inappropriately* errors in item 6 because it was the only item where the numeral in the second term was larger than the numeral in the first term, hence 'encouraging' right-to-left reasoning. So, because the left hand side of item 6 was $1 - 3x$ and not $3x - 1$ (as it was in item 4 and 5) there were more negativity errors. In both items many learners reduced the above to $2x$. In items 4 and 5 this would be considered a conjoining error but in item 6 when the left hand side is $1 - 3x$ instead of $3x - 1$ and is also reduced to 2x, it is not only a conjoining error but a right-to-left reasoning error where the learner uses the minus sign inappropriately. As mentioned in Chapter 3 a single error was not given more than one error code and so negativity errors would override conjoining errors.

5.4.2.3.3 Minus sign acknowledged but not used

This category is an extension of the *minus sign ignored* category above. The *minus sign acknowledged but not used* code was assigned to responses where after ignoring the minus sign and operating on the numbers, learners attached a minus sign to their reduction statement. Although it appears that these learners did not use the minus sign as it was intended to be used, they clearly did notice the symbol as they attached it. This use of the minus sign echoes Gallardo's (2002) unary function of the minus sign where the sign is attached to a number to form a negative number. For example, in Figure 5.18, Learner 4 reduced $-3x + x$ to $-4x$. It appears that the

learner ignored the minus sign and added $3x$ and x to get the $4x$ but was aware of the minus sign and so attached it to the $4x$. Again, an alternative interpretation is that the learner paid explicit attention to the signs and perhaps misapplied the signs-rule.

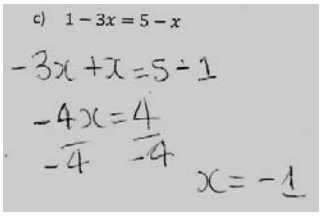


Figure 5.18: Learner 4 response to item 6

This error was only found in responses to item 6 which was surprising because there were negatives in the other two items. The difference between items 5 and 6 is that item 5 has $3x - 1$ and item 6 has $1 - 3x$. Having this particular error occur only in responses to item 6 suggests that the change in structure where the $3x$ is the second term and not the leading term may play a part in learners being made more aware of the negative in front of $3x$ but not knowing how to operate on it and so ignore it and then place it back in the solution as an afterthought (Herscovics and Linchevski (1994); Vlassis (2004)).

By looking more deeply at the negativity errors, it appears that the change in structure from item 5 to item 6 has meant that more learners are applying right-to-left reasoning and hence using the negative inappropriately. There are more learners ignoring the fact that the $3x$ has a minus sign in front of it and instead treating it as $+3x$. There are also learners who are noticing that the letter is negated but not sure of how to operate on it and so ignore it and then attach it back to the numeric calculation.

In terms of learner performance, this section has shown that many errors occur when negative numbers or negated letters are included in an equation. According to literature, this finding should have been expected (Gallardo and Rojano (1990); Gallardo (2002); Thompson and Dreyfus (1988); Vlassis (2004)). However, the number and variety of negativity errors was surprising because the learners in my

should have been learning and dealing with the minus sign and letters for at least 2 years. It is important to note that these errors were made by 50% of the learners, meaning that 23 of the 46 learners made at least one error relating to the minus sign.

5.4.2.4 Remaining categories and other findings

This section briefly discusses the three remaining error categories (letter, number and unknown) as well as an interesting finding regarding learners' non mathematical use of the equal sign.

An error that is prevalent in learners' responses when solving linear equations involves adding unlike terms. Although errors of this nature are not a focus of my study they do hinder learners in successfully executing the procedure to solve equations. This category was not subdivided, but could have been according to, for example, Küchemann's six categories of letter interpretation (Hart et al., 1981) had I had a different research focus. This single classification contained 17% of all the errors made and means that 24% of the learners made at least one letter error.

Number errors are errors made in arithmetic and they accounted for the fewest number of errors (5%). Many of the number errors were fraction based and hence not surprising (Post, Behr, & Lesh, 1982). There were a small percentage of errors that were illogical to me (6%). Even though the scripts selected for the error analysis had at least one equation question correct, it was inevitable that I would encounter responses that did not make sense to me.

Using the equal sign in a non-mathematical way is a notational error where learners put the symbol down on the left hand side of the page suggesting that the symbol means 'and then I do this'. These learners are not using the equal sign consistently as a symbol for equivalence (Kieran, 1981). The decrease in the misuse of the equal sign from item 4 (21%) to item 5 (16%) and then to item 6 (11%) is surprising. If writing the symbol on the left hand side was a habit then I would expect the percentages to be more similar. In terms of the percentage of learners, 35% of the learners at some

point did not use the equal sign appropriately. If this non-mathematical use of the equal symbol is not just a bad habit then using it as a 'and now I do this' symbol suggests an operational view of the equal sign.

5.5 Synthesis: Error analysis of selected clusters of responses

I do not look at errors made in relation to all the clusters identified in Chapter 4, instead I focus on errors made in Cluster 9 and Cluster 10 only. These two clusters were chosen because they relate to the transition from arithmetic to algebra: Cluster 9 is a cluster of responses where learners could correctly solve algebraic equations and I wanted to investigate their performance in the arithmetic items. Cluster 10 is a cluster of responses where only one of the equations with a letter on both sides was answered correctly. I was interested to know what learners struggled with if they could only sometimes solve an equation with a letter on both sides. Moreover, these two clusters were of interest to me because of what the patterns suggested about the presence of a didactic cut.

5.5.1 Cluster 9: Possible evidence of the didactic cut

The question posed for Cluster 9 was: *How did learners perform in the arithmetic equivalences items if they could correctly solve the arithmetic equation but not the algebraic equations?* As a reminder, in Chapter 4, this cluster was formed in Pattern C when responses to items 5 and 6 (the two equations with a letter on both sides) were answered incorrectly but item 4 (the equation with a letter on only one side) was answered correctly . In Chapter 4 I stated that my findings might support the presence of the didactic cut (Filloy & Rojano, 1989). This was because it was assumed that a) item 4 was answered correctly using arithmetic strategies and that b) items 5 and 6 were incorrect due to not being able to use arithmetic strategies with letters on both sides.

The analysis of errors in this cluster showed that there were very few learners who used arithmetic strategies to solve item 4. Only two learners used arithmetic strategies and then appear to have got 'stuck' in items 5 and 6 when employing the

same strategy, yet half of the learners got item 1 correct. In terms of learner performance, this finding suggests that learners are using procedures for the arithmetic equations which, based on the notion of the didactic cut, would typically be solved using arithmetic strategies. Figure 5.19 shows the percentage of errors made in responses to items 1-3 in Cluster 9. The most common error when responding to items 1-3 was due to an operational view of the equal sign, which is made up of 16% for *focus on one value* and 21% for *left-to-right reasoning*.

There were a large percentage of responses that were coded as *unknown* meaning that I could not logically follow how the learners got their answers. Figure 5.20 shows the percentage of the errors made in response to items 4-6 in Cluster 9. The two most common errors were due to equality and negativity and in particular, the balance error and minus sign ignored were the most common. This suggests that the concept of equality and the procedures involved in solving linear equations as well as the presence of a minus sign are all problematic for the learners in this cluster. These findings suggest that not having a relational view of the equal sign may hinder learners' performance when solving linear equations with a letter on both sides. Although there were few *number errors* in Cluster 9 we do not see evidence of arithmetic proficiencies and strategies hindering or helping learners with solving equations.

5.5.2 Cluster 10: Issues with negativity

As a reminder, the question that generated this cluster was: *How did learners who could correctly answer an algebraic equation with negatives and a letter on both sides, perform in the other items?* In contrast to Cluster 9, Cluster 10 was obtained by selecting learners who could correctly answer items 4 and 5 but not item 6.

In Chapter 4 I suggested that the presence of a letter on both sides was not the problem for these learners since both item 5 and item 6 had a letter on both sides. It was also suggested that these learners struggle more with negatives than with the

presence of a letter on both sides. Figure 5.20 shows that the large percentage of negativity errors (34%) supports this assumption. The *minus sign ignored* and *minus sign used incorrectly* category were the most prominent negative errors. There was, however, a much larger percentage of errors made in Cluster 10 that was related to equality (47%), with the inverse error being most prominent (19%). The negative error was a code used when reducing either the left or right side of the equation and the equality error was for the deductive process.

These findings suggest that learners struggle with the deduction process when many negatives are involved. Figure 5.19 shows the percentage of errors made in responses to items 1-3 in Cluster 10. The most common error when responding to items 1-3 was due to a unidirectional view of the equal sign. This category was only made up of the left-to-right reasoning This means that 38% of the learners in Cluster 9 filled in the value 12 for $7 + 5 = _ + 2$. These learners did not consider the whole right hand side of the arithmetic equivalence and placed the sum of 7 and 5 in the blank space.

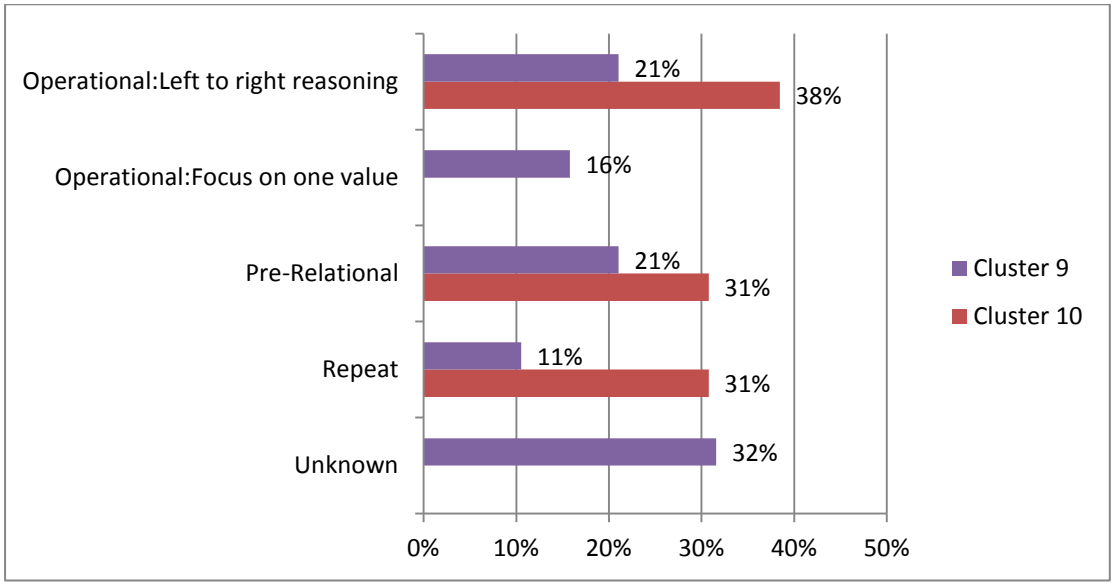


Figure 5.19: Percentage of Cluster 9 and 10 errors in items 1-3

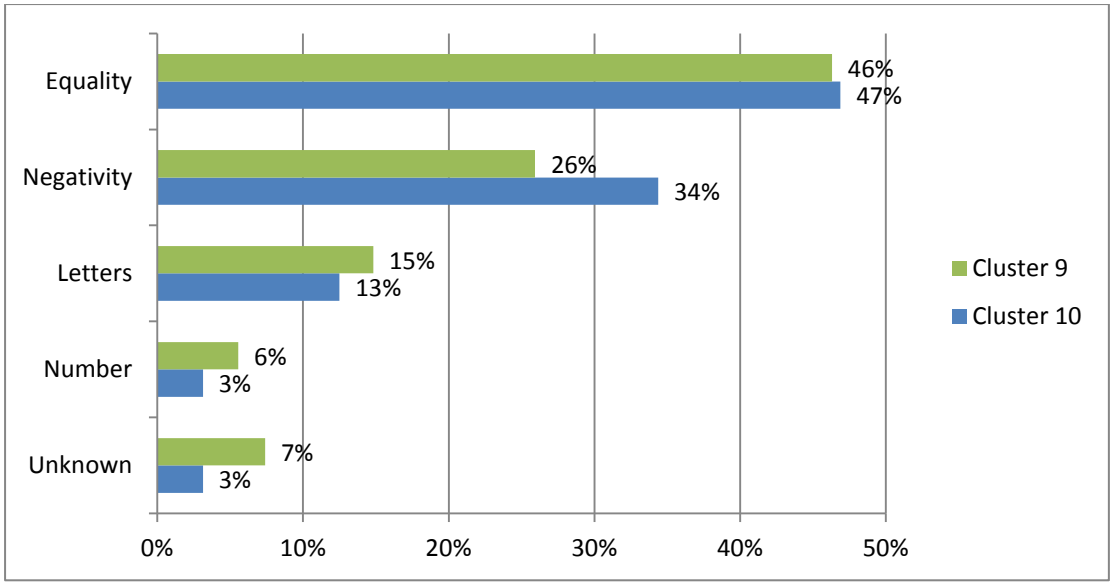


Figure 5.20: Percentage of Cluster 9 and 10 errors in items 4-6

In this section I focused on the errors made in Clusters 9 and 10 from Chapter 4. The response pattern analysis in Cluster 9 was suggestive of the presence of the didactic cut. The error analysis, however, showed that there were very few learners who used arithmetic strategies to solve arithmetic equations and that they use algebraic procedures instead. My findings from analysing the errors in Cluster 9 showed that balance and minus sign ignored were the most common errors made. The data also suggests that not having a relational view of the equal sign may hinder learners' performance when solving linear equations with a letter on both sides as well as on only one side of the equal sign. Also, I did not find evidence of arithmetic proficiencies or strategies hindering or helping learners with solving equations. The response pattern analysis in Cluster 10 suggested that learners struggle with negativity more than equality and more than the presence of a letter on both sides of the equal sign.

5.6 Conclusion

This chapter focused mainly on identifying the different types of equality errors and to some extent the different types of negativity errors. I used responses from 46 test scripts to classify and describe the types of errors learners make when solving linear equations. Grade 10 learners make many different errors when solving linear

equations. Even with a focused set of items many different errors were produced, the majority of which were based on equality. Some of these errors were related to learners not executing the procedures correctly. Other errors were based on notation and syntax and in the way learners write mathematical statements. It also appears that some learners memorised what solutions to linear equations should look like and either recognise that the response they have does not fit that familiar structure or they ignore the elements that conflict with that structure. Learners also appear to struggle with both the deduction and reduction process of algebra.

The response pattern analysis done in Cluster 9 was suggestive of the presence of the didactic cut but the error analysis showed that learners were using algebraic procedures rather than arithmetic strategies to solve arithmetic equations. The data did suggest that not having a relational view of the equal sign may hinder learners' performance when solving linear equations but did not find evidence of arithmetic proficiencies or strategies hindering or helping learners with solving equations. The error analysis done on the responses in Cluster 10 concurred with the response pattern analysis which suggested learners struggle with negativity more than with equality and the presence of a letter on both sides of the equal sign.

In the chapter that follows, I conclude this research report by summarising my findings and answering my two research questions. In Chapter 6 I also discuss the limitations of this study and present the reader with reflections on key issues that stood out for me.

Chapter 6 : Conclusion

6.1 Introduction

The main focus of this research report was to investigate learners' performance in solving arithmetic equivalences and linear equations. Performance consisted of two parts: 1) a response pattern analysis where learners' performance in a series of final responses were investigated and 2) a detailed error analysis. To investigate this I posed two research questions:

1) What response patterns emerge when solving arithmetic equivalences and arithmetic and algebraic equations?
2) What errors do learners make when solving arithmetic equivalences and arithmetic and algebraic equations?

In order to answer these questions, a test developed by WMCS was administered to 106 learners, chosen out of convenience, and responses to six test items were analysed. The analysis was done in two phases. The first was a response pattern analysis on all 106 learners' responses to six items. This analysis involved organising data in three ways and obtaining different patterns of responses. The patterns not only provided me with different insights into learners' performance but also provided a systemic way of selecting scripts to analyse further. The second phase of data analysis involved conducting a detailed error analysis on 46 purposively selected scripts.

In this chapter I summarise my findings and answer my research questions. I then discuss some of the findings that were unexpected. I also highlight some of the limitations of my study and end this report by offering suggestions for future research.

6.2 What response patterns emerge when solving arithmetic equivalences and arithmetic and algebraic equations?

I sorted my data in three different ways and obtained three different response patterns. As a reminder, only final answers were coded and these were coded as correct, incorrect or missing. What was evident in all three patterns was the anomaly between learners being able to execute algebraic procedures proficiently and the same learners not coping with arithmetic equivalences (or vice versa). I have used the word *anomaly* because in Chapter 2 I discussed in detail the transition from arithmetic to algebra and how understanding equality is seen to be a key concept for success in algebra and in particular, solving equations. My data suggests that some learners are able to solve equations proficiently without having a relational view of the equal sign, while other learners, who appear to have a relational view of the equal sign, did not solve linear equations correctly.

From my three response pattern analyses, there appears to be an additional tension in the type of question that learners can answer correctly. The response pattern analysis of Clusters 2 and 3 suggested that 20% of learners could execute algebraic procedures but not solve arithmetic equivalences. Other learners appear to be stuck in arithmetic and have not been able to progress into proficiently executing algebraic manipulations, dealing with letters and solving equations. There are also learners who can do aspects of each: the easy arithmetic equivalences and some equations. Some learners' performance supports the notion of the didactic cut in that they could not solve equations with a letter on both sides but could solve the equation with a letter on one side. Other learners' responses suggested that it was the presence of a negative that blocked their success and not two occurrences of the letter. Literature suggests that performance in solving equations needs or relies on proficiency in arithmetic (Caspi & Sfard, 2012).

Based on my findings, learners' performance in arithmetic, algebra and equivalence do not necessarily follow sequentially nor are they dependent on each other. My findings suggest that the transition from arithmetic into algebra in terms

of solving arithmetic equivalences and arithmetic and algebraic equations is not 'all or nothing' or even a linear transition. Rather it is a complex process that involves the intertwining of different concepts such as equality, negativity and the meaning of letters.

6.3 What errors do learners make when solving arithmetic equivalences and arithmetic and algebraic equations?

Learners' errors on items 1-3 suggest that many learners still operate with an operational view of the equal sign. The two most common errors made in responding to items 1-3 were: an operational view of the equal sign and a strategy not known to me. Not being able to identify the learners' strategies means that I was not able to use literature to interpret their response to items 1-3 and hence the responses appeared illogical.

Learners' errors on items 4-6 suggest that they struggle with equality and negativity. These two dominant error categories were therefore subdivided and the errors were further categorised. The equality errors suggest that some learners were making deliberate manipulation errors in order to obtain familiar solutions and to avoid structures that are atypical. It was also found that when learners try to be explicit and *do to the left as they do to the right* they then make additional syntax errors, for example by adding an inverse on the left hand side of the equal sign in one step and then on the right hand side in a different step. Although the rule *do to the left as you do to the right* is obeyed, balance is not maintained between steps. All these findings support the assertion that learners battle with executing the procedures which involve both reduction and deduction processes in algebra. One of the biggest struggles learners encountered was reducing an expression was related to dealing with the minus sign. For example, the minus sign was either used inappropriately, completely ignored or was acknowledged but not used. Although it was expected that there would be errors related to negativity, the number and variety of negativity errors was surprising since the learners had been dealing with the minus sign for at

least two years. Hence it was an important and unexpected finding that negativity errors were made by half the learners in my study.

6.4 Is there evidence to support the didactic cut?

The notion of the didactic cut was an important concept from the outset of this study. It not only motivated the focus of my study but informed my item selection, the focus of the response pattern analysis as well as the sample selection for phase 2 of my data analysis. It also influenced which clusters of responses from Chapter 4 were further analysed in terms of errors made.

In Cluster 9 only the arithmetic equations were solved correctly and not the algebraic equations. This suggested the presence of the didactic cut. After analysing the errors made, the most common were due to balance and ignoring the minus sign. To support the notion of didactic cut, I would have expected to see more evidence of learners using arithmetic strategies to solve the algebraic equations. Although my findings do not support those of Filloy and Rojano (1989), my findings do concur with those of Tall et al. (2014) where arithmetic and algebraic equations appear to be equally difficult for learners and that they struggled to maintain balance when isolating the unknown. Since both Tall et al. (2014) and my study focus on older learners (15-16 year olds) it is possible that the extra practice of solving linear equations and using algebraic methods (as opposed to arithmetic ones) makes the two equations equally difficult.

6.5 Limitations

In this section I elaborate on five limitations of this study. These relate to: the population chosen; the sample and generalisability of the findings; the design of my study; the choice of mathematical scope; and coding issues. The limitations are discussed in the order that they occurred and so although the limitation related to coding is possibly the most important, it is not discussed first.

Firstly, it was a limitation of my study that I did not focus on the teacher or teaching, especially since I took a Vygotskian perspective on learning. The teacher is a fundamental part of the learning process and a fundamental aspect that needs to be researched if improving learning is a goal. For the scope of this report, however, focusing on learners only was appropriate.

Secondly, my data was collected from only two schools which is a small sample and so my findings can't be generalised to all Grade 10 learners in South Africa. Despite this being a limitation, conducting a response pattern analysis on six items of 106 test scripts and then analysing the errors made in the six items of 46 of these scripts is not a small scale project for a Masters Research Report. Although my findings may not be generalisable, subsequent work done in the LG project during the course of 2016 suggests that my findings apply in different schools within the LG project. Having selected 46 of the 106 scripts was a limitation as I 'lost' the errors that were in the excluded scripts. That being said, with the range of errors I did find, I am confident that many of these errors were still represented in the 46 selected scripts.

Thirdly, with a focused set of test items, certain content was excluded. I am therefore unable to comment on errors made when solving linear equations that, for example, contain brackets or fractions.

The fourth limitation is related to coding. I chose to allow only one code per error and I decided on which errors took precedence over others. I also limited the number of negativity error categories. Although these were strategic decisions, all decisions were made in order to maintain focus and ensure rigorous and systematic research. I acknowledge that different decisions could have led to slightly different outcomes.

Lastly, at the outset of my study the intention was to analyse test scripts and conduct interviews with learners. I conducted interviews with eight learners but given the depth of analysis which I chose to do on the test scripts, I was not able to include the analysis of the interviews within this report. This means that I cannot

make claims about learner thinking. I do look forward to analysing the interview data and linking it to the error analysis in the near future.

6.6 Reflections

I have learnt much from conducting this research but will focus on issues that stood out for me that relate to the methodology and findings of my study.

6.6.1 Reflections on methodology and analysis

In reflecting on the research methodology of my study, the most significant learning experience was in terms of coding errors. I found it very difficult and frustrating to code and recode learner responses. As with the nature of research, the number and naming of error categories changed multiple times during the process of the analysis. This happened because choosing the categories was influenced by factors such as my focus, what was present in the data and the literature. For example, with the negativity error categories, I limited the number of categories because the minus sign was not a focus of my study but with the equality category I expanded the number of categories because it was a focus. In addition, having a second person code a portion of my data was a new experience for me, and although it required more work, it was satisfying knowing that I was rigorous and consistent in my coding.

The response pattern analysis was a powerful tool used in my study as it was suggestive of, for example. the presence of the didactic cut and learners' struggles with negativity. It was only through the deeper error analyses that these suggestions were verified (with reference to learners' struggles with negativity) or not (with reference to the presence of the didactic cut).

6.6.2 Reflections on findings

Before I had started this research report I had an idea of what errors to expect. However, I was surprised at the finding that learners appear not only memorise rules but also to memorise the structure of a solution. On the one hand it is encouraging

that learners are paying attention to the structure but on the other hand I am left with the feeling that the structure of an equation's solution is being noticed without understanding. Perhaps, just as learners memorise rules without fully understanding them or applying them appropriately, they too memorise a solution structure without understanding what it means to have the solution.

One of the less prominent errors made was learners changing an equation into an expression. This highlighted the duality of zero where ' 0' can be viewed in terms of nullity or totality (Gallardo & Hernández, 2005). This stood out for me because I had never expected learners to convert an equation to an expression but rather expected the error to be converting an expression to an equation.

In Section 6.5 I mentioned that not being able to generalise my findings was a limitation. However, the errors I found resonate with international research. In isolation, the equality and negativity errors can connect with international research, but together, looking at negativity within equality, is a contribution that I make to the mathematics education research community. Perhaps international research does not appear to put these two concepts together because the research done on equality is done on younger learners (as mentioned in Chapter 1) where they have not yet encountered the different functions of the minus sign.

6.7 Future research

Having conducted this research and being excited by many of the findings I have many suggestions for future research. These are presented in point form:

- The unexpected finding that learners manipulate mathematics for the purpose of obtaining a familiar structured solution points towards researching learners' understanding of the structure of equations and solutions in more detail.
- Having found that negativity is still a problem area for Grade 10 learners there is scope to investigate at a much deeper level how learners view the minus sign.

- I suggest investigating and disaggregating letter errors, especially drawing a distinction between the different types of conjoining as well as how and when learners see x as $1x$ or $0x$.
- Investigating why and when learners change an equation into an expression and vice versa and whether zero is a visual cue to solve or factorise the given expressions.
- Following from the idea that zero might be a visual cue for learners *to do something* I would suggest researching the importance of, and the difficulty that learners have with the appearance of the number 0.

6.8 Conclusion

In conclusion, learners find mathematics difficult and the literature has suggested focusing on equality, negativity and the meaning of letters to improve learner performance in algebra. The literature also suggests conducting error analyses in mathematics as it can inform pedagogy. Taking these suggestions, I conducted a qualitative study on learner performance that involved thick descriptions of: the context of my study; the research methodology; and the analysis of my data.

Performance as used in this report consisted of two parts: 1) a response pattern analysis where learners' performance in a series of final responses were investigated and 2) a detailed error analysis. Analysing learner responses to six test items revealed that when learners have difficulty with the mathematics that is in front of them, they reduce/manipulate to an expression or solution with which they are familiar. Sometimes this includes adding unlike terms or conjoining and sometimes it includes ignoring the minus sign. I started this report knowing that learners see algebra as a subject that involves multiple rules and procedures that are often memorised, but I did not expect to find that they also seem to memorise the structure of a solution and use this when they are faced with unfamiliar equations. Although I was aware that negativity is a difficult concept for learners I was surprised at the number of errors made that related to the minus sign. Both these findings therefore warrant future research.

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Appendices

Appendix A: GDE group clearance

For administrative use:
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GDE GROUP RESEARCH APPROVAL LETTER

Date:	19 February 2016
Validity of Research Approval:	19 February 2016 to 30 September 2016
Names of Researchers:	Dr C. Pournara; Askew M; Sanders Y. Ntow F; Alshwaik J. and Owusu J.
Name of Supervisor/s:	Prof. K. Brodie
Address of 1st Researcher:	P.O. Box 4531; Pinegowrie; 2123
Telephone/ Fax Number/s:	011 717 3253; 082 696 8381; 076 158 3758; 072 325 5304; 063 332 9080; 060 883 8678; 078 033 8863; 082 696 8381
Email addresses:	craig.pournara@wits.ac.za; Michael.askew@wits.ac.za; Yvonne.gubler@gmail.com; Forster.ntow@wits.ac.za; Jehad.alshwaik@wits.ac.za; James.owusu@wits.ac.za.
Research Topic:	Development of test instrument to measure learning gains of Grade 9 learners in Mathematics
Number and type of schools:	TWO Secondary Schools
District/s/HO	Johannesburg East

Re: Approval in Respect of Request to Conduct Research

Hlwds
2016/02/22

Appendix B: GDE ethics clearance

For administrative use only:
Reference no: D2016 / 432
enquiries: Diane Buntting 011 843 6503

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GDE RESEARCH APPROVAL LETTER

Date:	**11 March 2016**
Validity of Research Approval:	**11 March 2016 to 30 September 2016**
Name of Researcher:	**Sanders Y.L.**
Address of Researcher:	**P.O. Box 66355; Broadway; 2020**
Telephone / Fax Number/s:	**011 622 1966; 072 325 5304; 086 508 8134**
Email address:	**yvonne.gubler@gmail.com**
Research Topic:	**Grade 10 learners understanding of equality and solving equations**
Number and type of schools:	**TWO Secondary Schools**
District/s/HO	**Johannesburg East**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved. A separate copy of this letter must be presented to the Principal, SGB and the relevant District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted. However participation is VOLUNTARY.

The following conditions apply to GDE research. The researcher has agreed to and may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

CONDITIONS FOR CONDUCTING RESEARCH IN GDE

1. *The District/Head Office Senior Manager/s concerned, the Principal/s and the chairperson/s of the School Governing Body (SGB.) must be presented with a copy of this letter.*
2. *The Researcher will make every effort to obtain the goodwill and co-operation of the GDE District officials, principals, SGBs, teachers, parents and learners involved. Participation is voluntary and additional remuneration will not be paid;*
3. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal and/or Director must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*

2016/03/11

Appendix C: Wits ethics clearance

Wits School of Education



27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

Date: 29 March 2016

Student Number: 0402970F

Protocol Number: 2016ECE005M

Dear Yvonne Sanders

Application for Ethics Clearance: Master of Science

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Grade 10 Learners' understandings of equality and solving linear equations

The committee recently met and I am pleased to inform you that **clearance was granted**.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

MMaseko

Wits School of Education
011 717-3416
cc Supervisor – Dr Craig Pournara

Appendix D: School consent form

Wits Maths Connect
University of the Witwatersrand, Johannesburg. Wits School of Education. Faculty of Humanities.
• Marang West Block, 27 St. Andrews Road, Parktown, Johannesburg, 2193
• Private Bag 3, Wits 2050 • Tel: +27 11 717 3412 • Fax:+27 11 717 3109 • www.wits.ac.za/WitsMathsConnect

Date

Dear Mr XXX

I write to you to invite XXX Secondary School to participate in research that is being conducted through the School of Education at the University of the Witwatersrand.

The research focuses on learning gains and I'd like to give you some background to the project. In 2013 we investigated the impact of the Transition Maths 1 and 2 courses on learners' gains in Mathematics over one year. We invited 5 project schools to participate and we tested approximately 800 Grade 10 learners in February and October of that year. There were 21 teachers who participated in the study, some of whom participated in a TM course and some who didn't. The results showed that the learners taught by teachers who participated in a TM course made larger gains than those taught by teachers who did not participate in the course. However, we treat these results as indicative evidence rather than conclusive evidence that the TM courses have an impact on learning gains.

In 2016 to 2019 we wish to conduct a more rigorous study of the impact of the TM1 course on learning gains. The first step of this research is to develop a new test for learners. We are requesting the participation of your school in 2016 to assist us. It will involve the following:

1. We will invite approximately 60 Grade 10 Mathematics learners to write a test in XXX 2016. This will take approximately 75 min at a time convenient to the school.
2. We will conduct one-on-one interviews with approximately 20 learners who wrote the test. These interviews will take place at a mutually convenient time in XXX 2016. Each interview will last 30 min.
3. We may request to conduct interviews with approximately 10 additional learners in XXX 2016.
4. We will invite approximately 120 Grade 9 learners to write a test in XXX 2016. This will take approximately 75 min at a time convenient to the school.
5. We will invite approximately 120 Grade 9 Mathematics learners to write a test in XXX 2016. This will take approximately 75 min at a time convenient to the school. The learners selected for this test will not be from the group who wrote the test in XXX.

The research participants will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in participating in this study. The participants will not be paid for this study.

The names of the research participants and identity of the school will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study. All research data will be destroyed 5 years after completion of the project.

Please let me know if you require any further information. I look forward to your response as soon as is convenient.

Yours sincerely,



Dr Craig Pournara
Project Manager
Wits Maths Connect Secondary Project
Marang Block West, Wits Education Campus
craig.pournara@wits.ac.za, 082 696 8381

Appendix E: Information sheet for learners

INFORMATION SHEET LEARNERS

Protocol number: 2016ECE005M Date _____ 2016

Dear Learner

My name is Yvonne Sanders and I am a Master Student in the School of Education at the University of the Witwatersrand. I am doing research on Learners' understandings of the equal sign and solving equations.

In my research I want to find out how learners solve linear equations. I am interested in the words learners use to describe the procedures and would like understand why learners do solve equations in the way that they do. To do this research I first need to give learners a test and analyse their errors.

I was wondering whether you would be part of the group who will write the test. Your test responses will help me to see the number of different types of errors made.

I will also be inviting some learners to be interviewed about their answers to the test questions. I need to conduct interviews in order to understand what they were thinking when answering the questions. These interviews will be audio-recorded.

The test and the interviews will take place at a time that is agreed with your school. They will take place on the school property.

Although we talk about a *test*, this test is not for marks and you are not expected to study for it. Your participation is voluntary, which means that you don't have to do it. Also, if you decide halfway through that you would prefer to stop, this is completely your choice and will not affect you negatively in any way.

We will not be using your own name but I will make one up so no-one can identify you. All information about you will be kept confidential in all our writing about the study. Also, all collected information will be stored safely and destroyed 5 years after we have completed the project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

Please feel free to contact me if you have any questions.

Thank you

Ms Yvonne Sanders

yvonne.sanders@wits.ac.za

072 325 5304

Note:
Interview data was not included in this study.

Appendix F: Learner consent form

LEARNER'S CONSENT FORM

Please fill in the reply slip below if you agree to participate in my study.

My name is: _____ Grade and class: _____

Permission for questionnaire/test

Circle one

I agree to write a test for this study. YES/NO

Permission to be interviewed

I would like to be interviewed for this study. YES/NO

I know that I can stop the interview at any time and don't have to answer all the questions asked. YES/NO

Permission to be audiotaped

I agree to be audiotaped during the interview YES/NO

I know that the audiotapes will be used for this project only YES/NO

Informed Consent

I understand that:

1) My name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.

2) I do not have to answer every question and can withdraw from the study at any time.

3) I can ask not to be audiotaped.

4) All the data collected during this study will be destroyed 5 years after completion of the project.

Sign:_____ Date:_____

Note:
Interview data was not included in this study.

Appendix G: Information sheet for parents

INFORMATION SHEET PARENTS

Protocol number: 2016ECE005M Date xxxx 2016

Dear Parent

My name is Yvonne Sanders and I am a Master Student in the School of Education at the University of the Witwatersrand. I am doing research on learners' understandings of the equal sign and solving equations.

In my research I want to find out how learners solve linear equations. I am interested in the words learners use to describe the procedures and would like understand why learners do solve equations in the way that they do. To do this research I first need to give learners a test and analyse their errors.

I was wondering whether you would give consent for your child to be part of the group who will write the test. Your child's test responses will help me to see the number of different types of errors made.

I will also be inviting some learners to be interviewed about their answers to the test questions. I need to conduct interviews in order to understand what they were thinking when answering the questions. These interviews will be audio-recorded.

The test and the interviews will take place at a time that is agreed with the school. They will take place on the school property.

Although we talk about a *test*, this test is not for marks and learners are not expected to study for it. Your child's participation is voluntary, which means that s/he doesn't have to do it. Also, if your child decides halfway through that s/he would prefer to stop, this is completely his/her choice and will not affect him/her negatively in any way.

We will not be using your child's own name but I will make one up so no-one can identify your child. All information about your child will be kept confidential in all our writing about the study. Also, all collected information will be stored safely and destroyed 5 years after we have completed the project.

Your child has also been given an information sheet and consent form. At the end of the day it is your child's decision to join us in the study.

Please feel free to contact me if you have any questions.

Thank you

Ms Yvonne Sanders

yvonne.sanders@wits.ac.za

072 325 5304

Note:
Interview data was not included in this study.

Appendix H: Parent consent form

PARENT'S CONSENT FORM

Please fill in and return the reply slip below by **xxx 2016** indicating your willingness to allow your child to participate in my research project.

I, _____ the parent of _____

Permission for questionnaire/test

Circle one

I agree for my child to write a test for this study. YES/NO

Permission to be interviewed

I agree for my child to be interviewed for this study. YES/NO

I know that s/he can stop the interview at any time and doesn't have to answer all the questions asked. YES/NO

Permission to be audiotaped

I agree for my child to be audiotaped during the interview YES/NO

I know that the audiotapes will be used for this project only YES/NO

Informed Consent

I understand that:

My child's name and information will be kept confidential and safe and that my child's name and the name of the school will not be revealed.

My child does not have to answer every question and can withdraw from the study at any time.

My child can ask not to be audiotaped.

All the data collected during this study will be destroyed 5 years after completion of the project.

Sign_____ Date_____

Note:
Interview data was not included in this study.

Appendix I: Inter-coder summary sheet

Learner And Item	Researchers codes	Second coder's codes	Researcher's comments	Max no. codes	No. agreed codes
2a	2; 3a	2;3a	Agreed	2	2
2b	3a;3c;3g	3a;3c;3g	Agreed	3	3
2c	7	7	Agreed	1	1
12a	2	2	Agreed	1	1
12b	5; 3d	5; 3d	Agreed	2	2
12c	5;3c;3g,4a	5;4a,3g	This error could be viewed in two ways. Based on the focus of my study, my coding remained and the second coder agreed.	4	3
54a	2;3a	2;3a	Agreed	2	2
54b	4a;3c;5;3d	4a;3c;5;3d	Agreed.	4	4
54c	4a;4b;6d	4c; 3c;7	I needed to be more explicit in the difference between code 4a and 4c. The discrepancy between code 4b and 7 meant that I needed to elaborate on code 4b.	3	1
63a	2;3a	2;3a	Agreed	2	2
63b	3c	3c; 3g	I needed to be more explicit in what counted as incomplete. These two items were answered using substitution and did not have a final statement such as "x=…".	2	1
63c	3c	3c;3g		2	1
67a	2;3a	2;3a	Agreed	2	2
67b	5;3g	5;3g	Agreed	2	2
67c	4a	5	I needed to be more deliberate in the difference between codes 4a and 5	1	0
68a	2;3a	2;3a	Agreed	2	2
68b	3a;2	3a;3f; 7	I needed to elaborate on code 2 (a slip that is considered correct) in order to avoid ambiguity in the coding	3	1
68c	3a;2	3a; 3f;7	Same as above	3	1
84a	3a3b;5	3a;3b	Second coder slip, he forgot code 5	3	2
84b	3a;5;3b,3g	5;3a;3b,3g	Agreed	4	4
84c	0	0	Agreed	1	1
			Total:	49	38
Number of 'not counted' errors : 20 Number of counted errors: 30					

Appendix J: Number of errors per item per category

		No. errors Item 4	No. errors Item 5	No. errors Item 6	Total no. errors	Percentage %
Codes and categories not counted as errors						% of the sample
1	Missing	0	0	2	2	1%
2	Correct	36	27	21	84	61%
3a	Equality: misuse of equal sign	21	16	11	48	35%
Error codes and categories						% of errors
3b	Inverse	3	3	6	12	11%
3c	Balance	1	4	5	10	9%
3d	Familiar Structure	2	6	1	9	8%
3e	Incomplete	0	5	4	9	8%
3f	Syntax	0	3	3	6	5%
3g	Change to expression	2	0	0	2	2%
4a	Minus sign ignored	2	5	11	18	16%
4b	Minus sign used incorrectly	2	2	6	10	9%
4c	Minus sign acknowledged	0	0	5	5	4%
5	Letter error	4	8	7	19	17%
6	Number error	2	1	3	6	5%
7	Unknown error	3	2	2	7	6%
	Total no. of errors per item	21	39	53	113	
	% of errors per item	19%	35%	47%	100%	100%