Resurgence in the $\frac{1}{2}$ BPS Sector

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Declaration

I declare that this dissertation is my own, unaided work. It is submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

We study matrix models as a toy model for $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory which is a quantum field theory. In particular we are interested in the gauge/gravity duality which conjectures an equivalence between $\mathcal{N} = 4$ SYM and IIB string theory on $AdS_5 \times S^5$. We discuss the planar 't Hooft limit where we fix $\lambda = g_{YM}^2 N$ while taking $N \to \infty$. In this limit we find $1/N^2$ in the matrix model is equivalent to $\hbar$ of the string theory. When we study the N dependence of ribbon graphs, we find that the $\frac{1}{N}$ expansion in the gauge theory can be interpreted as a sum over surfaces suggestive of the perturbation expansion of a closed string theory. We then consider a non-planar but large N limit, allowing us to discuss the giant graviton. We find that the group representation theory of the symmetric group and unitary group organizes the physics of giant gravitons. We compute two, three and multi point functions of giant graviton operators. The large N expansion of giant graviton correlators is considered. Giant gravitons are described using operators with a bare dimension of order N. In this case the usual $1/N$ expansion is not applicable and there are contributions to the correlator that are non-perturbative in character. The machinery needed to determine the non-pertubative physics form the pertubative contributions is the origin of the term resurgence. By writing the (square of the) correlators in terms of the hypergeometric function ${}_2F_1(a, b; c; 1)$, we clarify the structure of the $1/N$ expansion.

Acknowledgements

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1. Introduction

Pertubation theory is a systematic technique to explore the weak coupling limit of a quantum field theory. When the coupling is small enough, it gives accurate results. It makes the assumption that the observable being studied admits a power series expansion in the coupling constant of the theory. Although this assumption leads to a useful practical calculational scheme, it is usually wrong. Most obervables recieve non-pertubative contributions, which can't be expanded as a power series in the coupling. A remarkable recent discovery, called resurgence, provides a deep and unexpected link between pertubative and non-pertubative contributions [1]. Indeed, the non-pertubative contributions are completely determined by the pertubation series.

In many quantum field theories, the sources of the non-pertubative contributions can be identified. An important example is provided by instanton contributions [2]. These are non-trivial saddle points of the Euclidean action that play a crucial role in determining the vacuum structure of the theory. For theories of quantum gravity, where we do not have a complete description in terms of an action, determing these non-pertubative contributions may seem like a hopeless task. The goal of this MSc is to show that, by using the ideas and framework of resurgence, these non-pertubative contributions can be identified.

The setting for our study is the $\frac{1}{2}$ BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory. This sector of the theory preserves one half of the supersymmetry of the theory and, as a result, the correlation functions of the theory do not recieve any $\hbar$ corrections. This implies that it is possible to compute these correlators exactly [3, 4], expand them and then study the features of the resulting expansion. The dual description of this sector of the theory is also well understood making it the ideal setting in which to explore holography.

$\mathcal{N} = 4$ super Yang-Mills theory is holographically dual to string theory on asymptotically AdS$_5\times$S^5 spacetime [5, 6, 7]. This implies that all excitations appearing in the spectrum of string theory must appear in the CFT Hilbert space. The usual perturbative spectrum (which consists of supergravity excitations, as well as closed strings) is captured by the planar limit of the dual CFT [8]. There are also many non-perturbative objects, including branes[9, 10, 11, 12] and LLM spacetime geometries [13], an interesting example being spacetimes containing black holes, that must be found in the CFT Hilbert space [14, 15]. These non-perturbative configurations are dual to operators with a bare dimension that grows parametrically with N ($\sim N$ for branes or $\sim N^2$ for new spacetime geometries). To explain how this works, we consider the half-BPS sector where a useful basis for the operators of the theory is given by the Schur polynomials. Consider a Schur polynomial labeled by a Young diagram consisting of a single column, $\chi_{(1^J)}(Z)$ of J boxes. For $J \sim O(1)$ the operator is dual to a collection of (point like) KK-gravitons. As J is increased to $J \sim O(\sqrt{N})$ long single trace operators dual to stringy states start to participate. Increasing J further to $O(N)$ we obtain a giant graviton brane. Thus, the dual to the single CFT operator $\chi_{(1^J)}(Z)$ transitions through different physical descriptions (particles, strings and branes) as the parameter J is varied. It is natural to ask how these different partial representations are combined into a single coherent description.

The character of the large N expansion changes in transition between these different partial representations. For $J \ll \sqrt{N}$ we can take the usual 't Hooft limit and the large N theory is just the planar limit. As N goes to infinity and $J \gg \sqrt{N}$ one must sum much more than just the planar diagrams (see [9, 16] for clear and relevant discussions). For this reason we will refer to these limits as large N but non-planar limits. In large N but non-planar limits one does not have the usual $1/N$ expansion. The ribbon graph expansion is not of much help because enormous combinatorial factors imply that the usual higher genus suppression is overwhelmed by the sheer number of diagrams of a given topology[9]. Different trace structures do mix and it is not at all clear how the large N expansion can usefully be organized. This is a key question that we wish to address, albeit in the limited setting of a specific example. A nice class of correlators that we will use to explore this issue are three point functions of $\frac{1}{2}$-BPS operators as well as extremal n-point functions of $\frac{1}{2}$-BPS operators. There are rigorous non-renormalization theorems[18] that prove that these correlators do not receive 't Hooft coupling corrections. Thus, they can be computed exactly, in the free field theory limit. Even this problem is one of considerable complexity, due to the very large dimensions of the operators. Fortunately, using group representation theory, this problem has been solved exactly, as we briefly review in Section 3 and Section 5. Our goal here is to explain the structure of the $1/N$ expansion for some correlation functions of giant graviton branes. In this way we will take a small first step towards defining the structure of the $1/N$ expansion in large N but non-planar limits.

Since $\hbar$ of the dual gravitational system is $1/N$, the different large N limits that can be taken lead to different classical configurations of the gravitational theory [6]. This is inline with conventional wisdom: when performing a path integral quantization there are many possible saddle points so that typically a quantum system has many perturbative series, each associated to a different classical configuration. These series are the basic building block in many computations. Although summing a few terms gives a good approximation, these series are almost always divergent. One needs a theory that can organize these different series into a coherent description of the quantum system. This is precisely what the theory of resurgence does. The first step entails converting the divergent series into meaningful objects by Borel resummation in the perturbation parameter. Typically one considers a loop expansion and the small parameter is $\hbar$. The second step entails exhibiting a relation between the different series, which is manifested through the Stokes phenomenon[1]. This relation implies that, given a specific series, the discontinuities of its Borel transform encode the information about other series in the problem. In this way, one can (for example) synthesize the usual perturbative expansion, together with the expansions in the (typically many) different instanton sectors, to recover exact results. From this point of view, the Stokes lines of the perturbative expansion simply demarcate where contributions from other saddle points become dominant.

Given this discussion, it seems that resurgence has a crucial role to play in understanding the large N limit of Yang-Mills theories. Specifically, resurgence should be relevant to understand how the different representations (i.e. the different possible large N limits) fit together to provide a complete and coherent description. If the ideas of resurgence are relevant, there should be a Stokes phenomenon present as the parameter J (and not $\hbar$) is varied. As a first step in exploring this possibility, we will look for and exhibit this Stokes phenomenon in this dissertation. To approach this problem we use the exact WKB method[19]. The different perturbative series

[1] The Stokes phenomenon is the basic fact that, in general, Borel resummations are discontinuous along rays in the complex plane. These rays are the Stokes and anti-Stokes lines.

that appear are the WKB series around different classical trajectories. The basic objects are the (Borel resummed) perturbative series in $\hbar$. The series can be characterized by two types of data: their classical limit and their discontinuity structure [19, 20, 21], which is encoded in the action of the so-called Stokes automorphisms. A simple characterization of the Stokes discontinuities is in terms of Voros symbols, which are simply the exponent of the WKB series. Our analysis starts with the observation that the (square of the) correlators we compute can be expressed in terms of the hypergeometric ${}_2F_1(a,b,c,x)$ function. This is a useful observation because the differential equation obeyed by the hypergeometric function is easily mapped into a Schrödinger equation, which can be approached using an exact WKB analysis. The relevant Schrödinger equation has $1/N$ playing the role of $\hbar$ so that the WKB expansion of the wave function of this Schrödinger equation gives the $1/N$ expansion of our correlator. Fortunately, the relevant Schrödinger equation has been studied in detail in [22, 23, 24, 25, 26]. In particular, the Voros symbols have been studied and their singularity structure in the WKB plane is well understood. The relevant WKB solutions have been proved to be Borel summable[27]. The solutions do exhibit Stokes phenomena in the parameter J and this has been studied in detail: the Stokes lines and Stokes regions for this equation can be described quite explicitly and connection formulas relating solutions in different Stokes domains are known. This implies that the singularities of the Borel transforms of the WKB solutions are well understood and that the Alien calculus for this problem is completely worked out[22, 23, 24, 25, 26]. These are the only ingredients needed to give the trans-series expansion for the hypergeometric function and hence of our giant graviton correlators. In Section 7 the exact WKB method is applied to unravel the structure of the large N expansion of the giant graviton correlation functions.

A key result of this project is the expansion of extremal n-point correlation functions of normalized Schur polynomial operators O_J, labeled either by a single column or a single row containing J boxes with J of order N. It is useful to introduce the parameters $j \equiv \frac{J}{N}$ which are held fixed as we take $N \to \infty$. The expansion is of the form

$$\langle O_{J_1} \cdots O_{J_k} O^\dagger_{J_1+\cdots+J_k} \rangle = \frac{e^{\alpha N}}{\sqrt{N}} \sum_{n=k-1}^{\infty} c_n N^{-n} \tag{1.1}$$

The coefficients α and c_n are functions of the fixed parameters $j_1, j_2, \cdots j_k$. We find that α can be both positive and negative. The series (1.1) is an asymptotic series. For the special case of three point functions we discuss the Borel summation of the series and gives the Stokes region in which the resummation converges. The details of the Stokes regions depend on the parameters (functions of the j_is) appearing in the Schrödinger equation, so that we can dissect the (complex) parameter space into regions with the topology of the Stokes graph constant in each region. The boundary of the region relevant for the three giant graviton correlator has a transparent physical interpretation and corresponds to points at which a giant shrinks to zero size or expands to maximal size. These are exactly the limits of the giant graviton description, so that this parametric Stokes phenomenon does indeed seem to be connected to the transition from one physical representation to another. Identifying the coupling $g_s = N^{-1}$ we see that (1.1) is a particularly simply transeries with a single nonperturbative term parameter $e^{\alpha N} = e^{\frac{\alpha}{g_s}}$. These non-perturbative contributions have been identified[28] with instantons in the tiny graviton matrix model[29] description of giant gravitons. Finally in Section 8 we discuss our results and suggest some possible directions for further study.

2. The Planar Limit of Matrix Models as a Free String Theory

In this chapter, the techniques needed to study the planar limit of matrix models are reviewed. Our goal is to demonstrate the equivalence of the planar limit of a matrix model and a free string theory. Firstly, we will study the free theory by defining the model and the generating function of interest. After a thorough study of the free theory, the more complicated case of an interacting theory is examined, in the next section. This amounts to adding an interaction term to the free theory and studying its effect perturbatively. We will introduce the Feyman diagram expansion. The Feynman diagrams for the matrix model are called ribbon graphs. Using ribbon graphs to study correlation functions of the trace operators, we will show that the large N limit simplifies dramatically. The key to this simplification is factorization, which implies that the theory is in a classical limit. We also define the 't Hooft limit and use it to define the genus expansion.

2.1 Free field theory

The elementary field of the theory is a Hermitian matrix M. Recall that Hermitian matrices have the following defining properties:

- There are N diagonal elements which are all real.
- The elements below the diagonal are the complex conjugates of the elements above the diagonal. There are a total of N^2 matrix elements in M. There are N elements on the diagonal and $N^2 - N$ elements off the diagonal. The off diagonal elements are all complex numbers. Thus, there are N^2-N real numbers needed to specify the off diagonal elements. In conclusion, M is specified by a total of N^2 real numbers.

We define the model that we will study as follows

$$\left\langle M_{ij}....M_{kl}\right\rangle_0 = \mathcal{N}\int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)} M_{ij}..M_{kl}. \tag{2.1}$$

The subscript 0 reminds us that this is a free theory. M is an N $\times$ N complex Hermitian matrix, $\mathcal{N}$ is a normalization factor and ω is a number. $[dM]$ is the measure of the field and $\int [dM]$ is defined as:

$$\int [dM]... = \mathcal{N}\prod_{i=1}^{N}\int_{-\infty}^{\infty} dM_{ii} \prod_{k,l=1 k>l}^{N}\int_{-\infty}^{\infty} dM_{kl}^{r}\int_{-\infty}^{\infty} dM_{kl}^{i}... \tag{2.2}$$

The normalization of the measure is fixed by requiring

$$\int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)} =< 1 >_0 = 1. \tag{2.3}$$

To study the correlators defined in (2.1) it is useful to introduce the following generating function

$$Z_0[J] = \mathcal{N} \int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)+\mathrm{Tr}(JM)} \tag{2.4}$$

where $\mathrm{Tr}(JM) = J_{ij}M_{ji}$ and repeated indices are summed over. We will now summarize a few identities that will prove to be very useful when we compute correlators.

$$\frac{\partial M_{ij}}{\partial M_{kl}} = \delta_{ik}\delta_{jl}. \tag{2.5}$$

$$\begin{aligned}
\frac{d}{dM_{ij}}\mathrm{Tr}(JM) &= \frac{d}{dM_{ij}}(J_{kl}M_{lk}) \\
&= J_{kl}\frac{dM_{lk}}{dM_{ij}} \\
&= J_{kl}\delta_{li}\delta_{kj} \\
&= J_{ji}.
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\frac{d}{dJ_{ij}}e^{\mathrm{Tr}(JM)} &= e^{\mathrm{Tr}(JM)}\frac{d}{dJ_{ij}}\mathrm{Tr}(JM) \\
&= e^{\mathrm{Tr}(JM)}\frac{d}{dJ_{ij}}(J_{kl}M_{lk}) \\
&= e^{\mathrm{Tr}(JM)}M_{lk}\delta_{ki}\delta_{lj} \\
&= e^{\mathrm{Tr}(JM)}M_{ji}.
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\frac{d}{dJ_{ij}}\frac{d}{dJ_{kl}}e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)+\mathrm{Tr}(JM)}\bigg|_{J=0} &= e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)}\frac{d}{dJ_{ij}}\frac{d}{dJ_{kl}}e^{\mathrm{Tr}(JM)}\bigg|_{J=0} \\
&= e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)}\frac{d}{dJ_{ij}}\left(e^{\mathrm{Tr}(JM)}M_{lk}\right)\bigg|_{J=0} \\
&= e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)}e^{\mathrm{Tr}(JM)}M_{ji}M_{lk}\bigg|_{J=0} \\
&= e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)}M_{ji}M_{lk}.
\end{aligned} \tag{2.8}$$

In the second and third lines of the last identity derived above, we used (2.7). To evaluate $Z_0[J]$ we need to evaluate a Gaussian integral. This is most easily done by completing the square in the exponent of (2.4)

$$\begin{aligned}
-\frac{\omega}{2}\mathrm{Tr}(M^2) + \mathrm{Tr}(JM) &= -\frac{\omega}{2}\left[\mathrm{Tr}(M^2) - \frac{2}{\omega}\mathrm{Tr}(JM)\right] \\
&= -\frac{\omega}{2}\left[\mathrm{Tr}(M^2 - \frac{2}{\omega}JM)\right] \\
&= -\frac{\omega}{2}\left[\mathrm{Tr}\left(M - \frac{J^2}{\omega}\right)^2 - \mathrm{Tr}\left(\frac{J^2}{\omega^2}\right)\right] \\
&= -\frac{\omega}{2}\mathrm{Tr}\left(M - \frac{J}{\omega}\right)^2 + \frac{1}{2\omega}\mathrm{Tr}(J^2).
\end{aligned} \tag{2.9}$$

(2.4) now reduces to

$$Z_0[J] = \mathcal{N}\int [dM] e^{-\frac{\omega}{2}\mathsf{Tr}((M-\frac{J}{\omega})^2)+\frac{1}{2\omega}\mathsf{Tr}(J^2)}. \tag{2.10}$$

To perform the Gaussian integral it is useful carry out the following change of variables

$$M' = M - \frac{J}{\omega} \Rightarrow dM' = dM \quad \text{and} \quad \text{then} \quad [dM] = [dM']. \tag{2.11}$$

When performing this change of variables, it is important to keep in mind that J is a fixed constant matrix which does not depend on M. After this change of variables, we find

$$\begin{aligned} Z_0[J] &= \mathcal{N}\int [dM'] e^{-\frac{\omega}{2}\mathsf{Tr}(M'^2)+\frac{1}{2\omega}\mathsf{Tr}(J^2)} \\ &= e^{\frac{1}{2\omega}\mathsf{Tr}(J^2)} \end{aligned} \tag{2.12}$$

where we have used

$$\int [dM'] e^{-\frac{\omega}{2}\mathsf{Tr}(M'^2)} = 1. \tag{2.13}$$

This equation defines our measure. The two point function can be obtained by taking derivatives of the generating function, as usual

$$\langle M_{ij} M_{kl} \rangle_0 = \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}} Z_0[J]\Big|_{J=0}. \tag{2.14}$$

We now use eqn.(2.12) to rewrite this as

$$\begin{aligned} \langle M_{ij} M_{kl} \rangle_0 &= \frac{d}{dJ_{ij}}\frac{d}{dJ_{kl}} e^{\frac{1}{2\omega}tr(J^2)}\Big|_{J=0} \\ &= \frac{d}{dJ_{ij}}\left[\frac{1}{2\omega}e^{\frac{1}{2\omega}\mathsf{Tr}(J^2)}\frac{d}{dJ_{kl}}\mathsf{Tr}(J^2)\right]_{J=0}. \end{aligned} \tag{2.15}$$

To complete this computation, note that

$$\begin{aligned} \frac{d}{dJ_{kl}}\mathsf{Tr}(J^2) &= \frac{d}{dJ_{kl}}(J_{ij}J_{ji}) \\ &= J_{ij}\frac{d}{dJ_{kl}}J_{ji} + J_{ji}\frac{d}{dJ_{kl}}J_{ij} \\ &= J_{ij}\delta_{jk}\delta_{il} + J_{ji}\delta_{ik}\delta_{jl} \\ \frac{d}{dJ_{kl}}\mathsf{Tr}(J^2) &= 2J_{lk}. \end{aligned} \tag{2.16}$$

Using eqn.(2.16) in (2.15) we find

$$\begin{aligned} \langle M_{ij} M_{kl} \rangle_0 &= \frac{d}{dJ_{ji}}\left[\frac{J_{kl}}{\omega}e^{\frac{1}{2\omega}\mathsf{Tr}(J^2)}\right]_{J=0} \\ &= \left[\frac{1}{\omega}\frac{d}{dJ_{ji}}J_{kl} + \frac{J_{kl}}{2\omega}\frac{d}{dJ_{ji}}\mathsf{Tr}(J^2)\right]e^{\frac{1}{2\omega}tr(J^2)}\Big|_{J=0} \\ &= \left[\frac{\delta_{il}\delta_{jk}}{\omega} + \frac{J_{kl}J_{ji}}{\omega}\right]_{J=0} \\ \langle M_{ij} M_{kl} \rangle_0 &= \frac{1}{\omega}\delta_{il}\delta_{jk}. \end{aligned} \tag{2.17}$$

This result is the propagator of the matrix model. This propagator is one of the building blocks of the Feynman rules for the matrix model. We represent the propagator using the following ribbon diagram

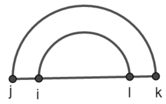

Figure 2.1: The propagator for $\langle M_{ji}M_{lk}\rangle_0$.

where the indices of each matrix are placed in pairs in order on a line. Pairs of indices are connected by a ribbon which is then translated into Kronecker delta symbols. In the ribbon graph shown, the index "i" is connected to index "l", which is translated into the Kronecker delta δ_{ij} and the index "j" is connected to "k" which translates into the Kronecker delta δ_{jk}. For each ribbon we include a factor of $\frac{1}{\omega}$.

The general correlator is computed as follows

$$\langle M_{ij}...M_{kl}\rangle_0 = \frac{d}{dJ_{ji}} \cdots \frac{d}{dJ_{lk}} Z_0[J]\bigg|_{J=0}. \tag{2.18}$$

The gauge theory has a local gauge invariance. The physical observables in the theory are gauge invariant operators. The matrix model represents local gauge symmetry as a $U(N)$ symmetry under which we have $M \to U^\dagger M U$. Since we have declared that this is a gauge symmetry, the physical observables of the theory should be invariant under this symmetry. The trace of any power of M is invariant under the global symmetry and traces of powers of M generate the complete set of physical observables of the model. With this motivation in mind, we will now study correlators of traces of powers of M. One of the simplest correlators we might consider is given by

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle_0 &= \langle M_{ij}M_{ji}\rangle \\ &= \frac{1}{\omega}\delta_{ii}\delta_{jj} \\ &= \frac{1}{\omega}N^2. \end{aligned} \tag{2.19}$$

All indices are repeated and hence summed. Thus, the result for this correlator is not a tensor, but rather it is a polynomial in N. This is in fact generally true: any correlator of traces of powers of M is given by a polynomial in N. This strongly motivates us to improve our ribbon graph rules. The indices of any gauge invariant quantities are all summed. Indicate these sums by solid lines

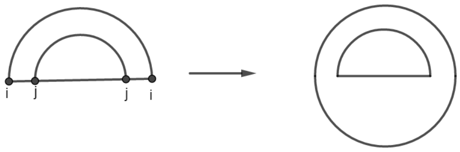

Figure 2.2: The ribbon graph notation for $\left\langle \mathsf{Tr}(M^2) \right\rangle_0$.

In this figure each closed loop is a sum over a Kronecker delta and hence contributes a factor of N, while each ribbon contributes $\frac{1}{\omega}$ as usual. Since we have two closed loops and one ribbon, we obtain the result

$$\left\langle \mathsf{Tr}(M^2) \right\rangle_0 = \frac{N^2}{\omega}. \tag{2.20}$$

The above discussion suggests that when computing correlators of gauge invariant operators, it is useful to specialize the Feyman rules for these correlators as follows

- Draw two dots for each pair of indices on the operators.
- In the gauge invariant operator, all indices are contracted. Connect the contracted indices with a line.
- Wick's theorem tells us that any correlator is computed by replacing pairs of fields by the propagator of the matrix model. We carry this step out by connecting pairs of indices with ribbons. Every possible way of performing these connections gives a different graph that must be summed.
- Each ribbon contributes a factor of $\frac{1}{\omega}$.
- Each closed loop contributes a factor of N.

Using these rules, the computation of $\langle \mathsf{Tr}(M^3) \rangle_0$ is not possible since in the new streamlined notation we link dots that are labeled by the same index with a solid line (as shown in the figure 2.2) and repeated indices are summed, they do not take a value. Indeed, $\langle \mathsf{Tr}(M^3) \rangle_0 = \left\langle M_{ij} M_{kl} M_{mn} \right\rangle$. The computation of $\langle \mathsf{Tr}(M^4) \rangle_0$ is as follows:

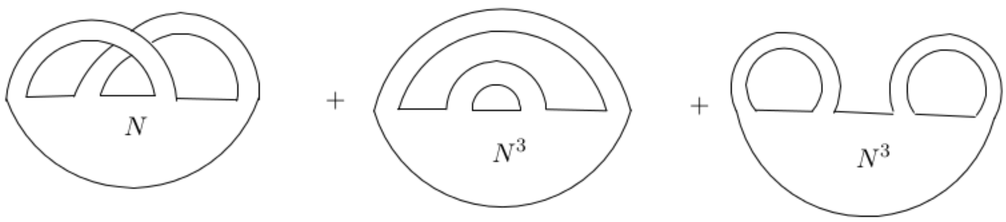

Figure 2.3: The propagator for $\left\langle \mathsf{Tr}(M^4) \right\rangle_0$.

In figure 2.3, the second diagram and last diagram are planar diagrams while the first one is a non-planar diagram. The planar diagrams have more closed loops and hence they dominate the large N limit. Our discussion up to this point has been concerned with the free theory. In the next section, we will add terms to the free theory which will allow the fields to interact with each other. It is in this more general setting that we will develop the connection to string theory.

2.2 Interacting theory

We now consider an interacting matrix model. To obtain an interacting theory, add a term quartic in M. The generating function for the interacting theory is then given by

$$Z[J] = \mathcal{N} \int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4) + \mathrm{Tr}(JM)}. \tag{2.21}$$

In order to derive the correlators in the interacting theory, we will first compute the generating function. It is useful to use the above normalization condition, so $Z[J=0]=1$ when $g=0$. We will start this calculation by expanding $e^{-g\mathrm{Tr}(M^4)}$ which appears in the above expression. Then we change the M's into derivatives with respect to J.

$$\begin{aligned} e^{-g\mathrm{Tr}(M^4)} &= \sum_{k=0}^{\infty} \frac{(-g\mathrm{Tr}(M^4))^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} (\mathrm{Tr}(M^4))^k \end{aligned} \tag{2.22}$$

$$\int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4) + \mathrm{Tr}(JM)} \longrightarrow \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} \left(\frac{d}{dJ_{ji}} \frac{d}{dJ_{kj}} \frac{d}{dJ_{lk}} \frac{d}{dJ_{il}} \right)^k \int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) + \mathrm{Tr}(JM)} \tag{2.23}$$

Using this in (2.21) yields

$$\begin{aligned} Z[J] &= \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} \left(\frac{d}{dJ_{ji}} \frac{d}{dJ_{kj}} \frac{d}{dJ_{lk}} \frac{d}{dJ_{il}} \right)^k \mathcal{N} \int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) + \mathrm{Tr}(JM)} \\ &= \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} \left(\frac{d}{dJ_{ji}} \frac{d}{dJ_{kj}} \frac{d}{dJ_{lk}} \frac{d}{dJ_{il}} \right)^k e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}. \end{aligned} \tag{2.24}$$

Note that we can also write

$$Z[J=0] = \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} \langle \mathrm{Tr}(M^4)^k \rangle_0. \tag{2.25}$$

The correlators in the interacting theory are again given by

$$\langle M_{ij} \cdots M_{kl} \rangle_0 = \frac{d}{dJ_{ij}} \cdots \frac{d}{dJ_{kl}} Z[J] \bigg|_{J=0}. \tag{2.26}$$

The interaction is represented diagrammatically with a 4-point vertex as follows

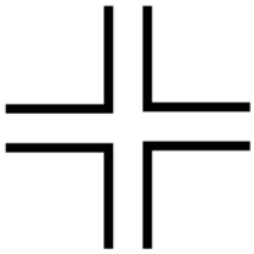

Figure 2.4: Diagram for the interaction vertex.

The rules for the diagrams of the interacting theory associate a factor of $-g$ with each vertex. The four legs of the vertex represents each of the fields in the interaction term, which is a trace of M to the power of 4. We can now compute the correlators in a perturbative expansion, treating g as a small parameter. For example Tr(M^2) is computed as follows:

$$\begin{aligned}
\left\langle \mathrm{Tr}(M^2)\right\rangle &= \frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}Z[J=0] \\
&= \frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}\sum_{k=0}^{\infty}\left[\frac{-g}{k!}\frac{d}{dJ_{ij}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}\right]^k e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}\Bigg|_{J=0} \\
&= \frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}} - g\frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}\frac{d}{dJ_{ij}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}\Bigg|_{J=0} + O(g^2) \\
&= \left\langle \mathrm{Tr}(M^2)\right\rangle_0 - g\left\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\right\rangle_0 + O(g^2) \\
\left\langle \mathrm{Tr}(M^2)\right\rangle &= \frac{N^2}{\omega} - \frac{g}{\omega^3}\left(2N^5 + 9N^3 + 4N\right) + O(g^2).
\end{aligned} \tag{2.27}$$

In the second line we have used (2.25) and in the third line we have written out the sum over k. In the fourth line we have again replaced the $\frac{d}{dJ}$'s by M's. We can easily check that this correlator is reproduced by summing the following diagrams

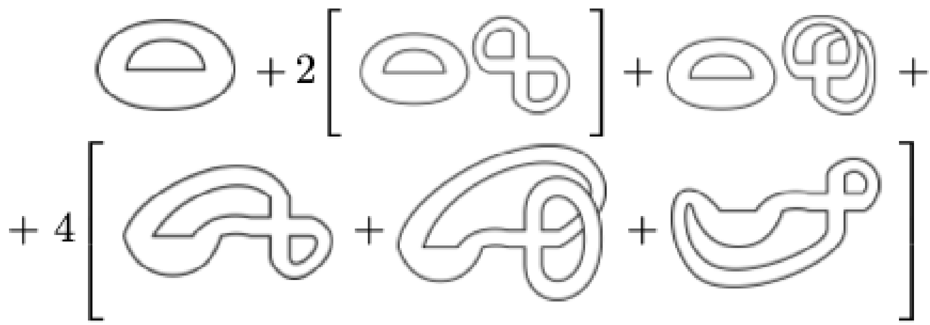

Figure 2.5: Diagram for $\left\langle \mathrm{Tr}(M^2)\right\rangle$ to order g^2 in the interacting theory.

From this diagram we have

$$\left\langle \text{Tr}(M^2)\right\rangle = \frac{N^2}{\omega} - \frac{g}{\omega^3}\left(2N^5 + N^3 + 4(N^3 + N + N^3)\right) + O(g^2)$$
$$= \frac{N^2}{\omega} - \frac{g}{\omega^3}\left(2N^5 + 9N^3 + 4N\right) + O(g^2). \tag{2.28}$$

This reproduce (2.27). If we normalize the generating function of correlation functions to 1 when $J = 0$, for any value of g, we find that correlators are given by dropping vacuum graph contributions. As an example, consider the computation of the two point function in the new normalization

$$\left\langle \text{Tr}(M^2)\right\rangle_{nn} = \frac{\left\langle \text{Tr}(M^2)\right\rangle}{\sum_{k=0}^{\infty} \frac{(-g)^k}{k!}\left\langle \text{Tr}(M^4)^k\right\rangle_0}$$
$$= \frac{\frac{N^2}{\omega} - \frac{g}{\omega^3}\left(2N^5 + 9N^3 + 4N\right) + \mathcal{O}(g^2)}{1 - \frac{g}{\omega^2}\left(2N^3 + 2N\right) + \frac{g^2}{2\omega^4}\left(4N^6 + 40N^4 + 61N^2\right) + \mathcal{O}(g^3)} \tag{2.29}$$

where the subscript nn stands for new normalisation. Assuming that g is a small parameter, expand this expression to first order in g to obtain

$$\left\langle \text{Tr}(M^2)\right\rangle_{nn} \simeq \left(\frac{N^2}{\omega} - \frac{g}{\omega^3}(2N^5 + 9N^3 + 4N) + \mathcal{O}(g^2)\right)\left(1 + \frac{g}{\omega^2}(2N^3 + N) + \mathcal{O}(g^3)\right)$$
$$\left\langle \text{Tr}(M^2)\right\rangle_{nn} \simeq \frac{N^2}{\omega} - \frac{g}{\omega^3}\left(8N^3 + 4N\right) + \mathcal{O}(g^2). \tag{2.30}$$

It is not difficult to check that this result is reproduced by removing all vacuum contributions. We have now developed enough background that we could consider the large N limit of the interacting theory, so we need not pursue this further. In the next section we will describe a significant simplification present in the large N limit, known as factorization. Factorization is one of the main motivations for the gauge / gravity duality.

2.3 Factorization

Factorization is a property of the large N theory. It represents a profound simplification of the theory. To explain what factorization is, in the simplest possible setting, we consider a statistical description of our system. The system can be in any one of a set of states, labelled by an integer i. The probability for the system to be in state i is given by μ_i, with

$$\sum_i \mu_i = 1 \quad \text{where } \mu_i \geq 0 \quad \forall\ i. \tag{2.31}$$

The system has a set of observables O_I. The value of observable O_I in state i is denoted by $O_I(i)$. As usual in statistical mechanics, the expectation value of O_I is

$$< O_I >= \sum_i \mu_i O_I(i). \tag{2.32}$$

Using the ribbon graph formalism we have developed above, it is straight forward to check that, in the large N limit, the expectation value of a product of the observables is equal to the product of the expectation values of the individual observables. In equations, we have

$$< O_{I_1} O_{I_2} ... O_{I_n} >=< O_{I_1} >< O_{I_2} > ... < O_{I_n} > . \tag{2.33}$$

Using (2.32) and (2.33) we find the following

$$\sum_i \mu_i O_{I_1}(i) O_{I_2}(i) ... O_{I_n}(i) = \sum_{i_1} \mu_{i_1} O_{I_1}(i_1) \sum_{i_2} \mu_{i_2} O_{I_2}(i_2) ... \sum_{i_n} \mu_{i_n} O_{I_n}(i_n). \tag{2.34}$$

The above expression forces us to conclude that the system is in a definite state i^* so that

$$\mu_i = \begin{cases} 1 & i = i^* \\ 0 & i \neq i^*. \end{cases} \tag{2.35}$$

The interpretation of this conclusion is not difficult. When computing the path integral, we sum over all possible states the system can occupy. In the limit that $\hbar \to 0$ we find that the contribution to the integral is peaked more and more sharply around a single state. This makes sense because as we send $\hbar \to 0$ we obtain classical physics and classically the system is in a definite state. Thus, factorization is telling us that the large N limit is a classical limit. In this limit we can simplify the computation of correlators.

To conclude this discussion we will give two examples which illustrate factorization in the large N limit of the free theory. Note that

$$\langle \mathrm{Tr}(M^2) \rangle = \frac{N^2}{\omega} \tag{2.36}$$

$$\langle \mathrm{Tr}(M^4) \rangle = \frac{1}{\omega^2}(2N^3 + N). \tag{2.37}$$

At the leading order for large N we have

$$\begin{aligned} \langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^2) \rangle &= \frac{N^4}{\omega^2} + O(N^2) \\ &=< \mathrm{Tr}(M^2) >< \mathrm{Tr}(M^2) > + O(N^2) \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} \langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4) \rangle &= \frac{2}{\omega^3} N^5 + O(N^3) \\ &=< \mathrm{Tr}(M^2) >< \mathrm{Tr}(M^4) > + O(N^3). \end{aligned} \tag{2.39}$$

2.4 1/N expansion

A naive approach to the study of the large N limit quickly runs into difficulties. Indeed, consider the correlator (2.30) up to second order in g

$$\langle \mathrm{Tr}(M^2) \rangle_{nn} = \frac{N^2}{\omega} - \frac{g}{\omega^3}\Big(8N^3 + 4N\Big) + \frac{g^2}{\omega^5}\Big(144N^4 + 224N^2\Big) + \mathcal{O}(g^3). \tag{2.40}$$

Notice that at highers order in perturbation theory in the small parameter g, we find ever increasing powers of N. In this case the large N limit simply does not make sense. To obtain a sensible limit, we must use a double scaling limit, as suggested by 't Hooft [30], in which the coupling is taken to zero as we take N to infinity. Concretely, in this double scaling limit we take $N \to \infty$, while holding $\lambda = gN$ fixed and small. The new parameter λ is called the 't Hooft coupling. Expressing the above expansion of the correlator in terms of λ, we find

$$\begin{aligned}\langle Tr(M^2)\rangle_{nn} &= \frac{N^2}{\omega} - \frac{gN^3}{\omega^3}\left(8 + \frac{4}{N^2}\right) + \frac{g^2N^4}{\omega^5}\left(144 + \frac{224}{N^2}\right) + \mathcal{O}(g^3) \\ &= \frac{N^2}{\omega} - \frac{N^2\lambda}{\omega^3}\left(8 + \frac{4}{N^2}\right) + \frac{N^2\lambda^2}{\omega^5}\left(144 + \frac{224}{N^2}\right) + \mathcal{O}(\lambda^3). \end{aligned} \tag{2.41}$$

In the last line we see an expansion in two small quantities λ and $\frac{1}{N^2}$. In QFT, λ is equivalent to $\hbar$ (Planck's constant) and $\frac{1}{N^2}$ is $\hbar$ in the dual string theory. Thus $\frac{1}{N^2} \equiv \hbar \to 0$ at large N. Therefore the large N theory emerges as a classical limit of a string theory.

From the expansion in (2.41), it is now clear that there is a well defined leading term at large N given by N^2. Further, there are small corrections to this leading result, controlled by two small parameters: $\frac{1}{N^2}$ and λ. It is now sensible to ask what the physical interpretation of this expansion is. Towards this end, we use the rescaled variable $M = \sqrt{N}M'$. The "action" of the theory now becomes

$$-\frac{\omega}{2}\mathsf{Tr}(M^2) - g\mathsf{Tr}(M^4) = -\frac{\omega N}{2}\mathsf{Tr}(M'^2) - N\lambda\mathsf{Tr}(M'^4). \tag{2.42}$$

Using (2.21) and the above expression we find

$$Z[0] = \mathcal{N}\int [dM] e^{-\frac{\omega N}{2}\mathsf{Tr}(M'^2) - N\lambda\mathsf{Tr}(M'^4)}. \tag{2.43}$$

The rescaling has produced the following change in our Feyman rules: each ribbon (i.e. each propagator) now comes with a factor of $\frac{1}{\omega N}$ and each vertex now comes with $-\lambda N$. Each closed loop in the ribbon graph continues to contribute a factor of N. In order to interpret the N dependencies of a ribbon graph, we denote by E, F and V the number of edges, faces and vertices respectively, in the diagram. The N dependence of any graph is then given by N^{F-E+V}. Thinking of the ribbon graph as a triangulation of a surface, the number $F-E+V$ is a topological invariant of the underlying surface, called the Euler characteristic. To show this, we will consider a triangulation and argue that $F-E+V$ is invariant under smooth deformations of the underlying surface. First, consider stretching an edge so that it creates a new face

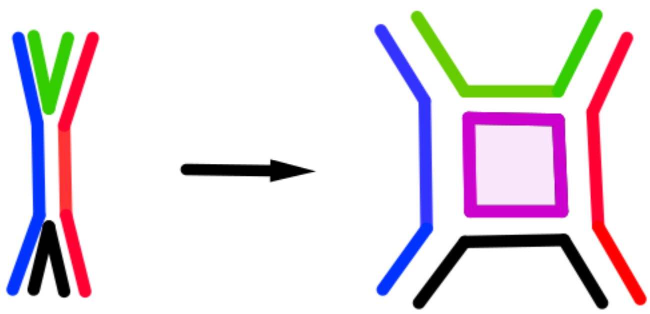

Figure 2.6: A side of the triangulation is stretched to produce a new face.

Before we perform the stretch we have F faces, E edges and V vertices. After stretching the edge into a face we have a new triangulation with $F' = F + 1$, $V' = V + 2$ and $E' = E + 3$, so that $F' - E' + V' = F - E + V$. This shows that the Euler characteristic is invariant under the deformation considered. Another deformation we could consider is to shrink an edge to nothing.

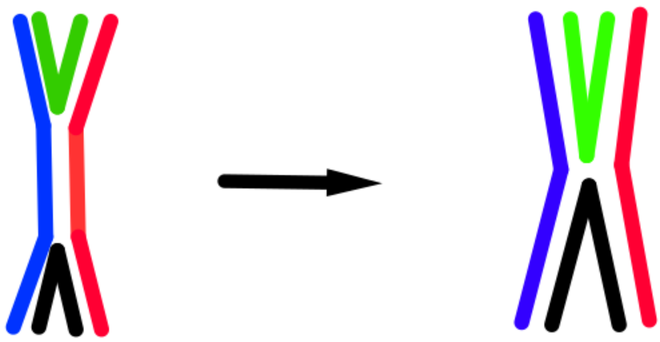

Figure 2.7: An edge of the triangulation is shrunk to nothing.

Before we shrink the edge to nothing we have F faces, E edges and V vertices. After the deformation we have $F' = F$ faces, $E' = E - 1$ edges and $V' = V - 1$ vertices. Thus, $F' - E' + V' = F - E + V$. This again shows that $F - E + V$ is a topological invariant.

Now we will consider consider a deformation that changes the topology of the surface. We wish to cut two holes in the surface and glue the edges of the holes together. In this way we add a handle to the surface.

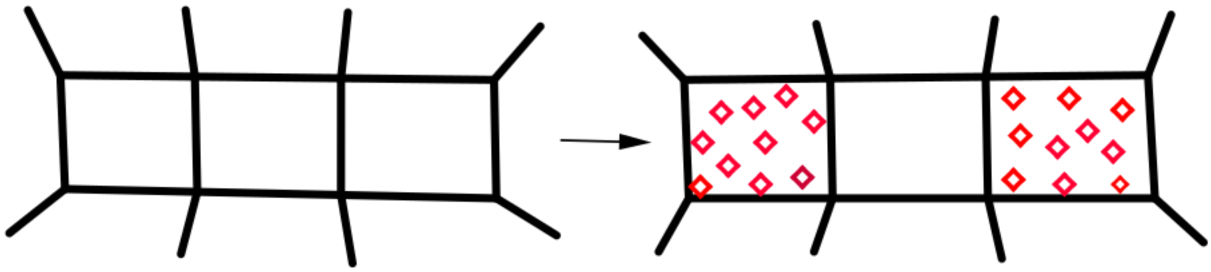

Figure 2.8: The first step in adding a handle entails cutting two holes as shown.

In the process of cutting two holes and then gluing, we lose 4 edges, 2 faces and 4 vertices so that $E' = E - 4$, $F' = F - 2$ and $V' = V - 4$. Thus, we find that $F' - E' + V' = (F-2) - (E-4) + V - 4 = F - E + V - 2$. This means that adding a handle to the surface reduces the Euler characteristic by 2. For example, adding a handle to a sphere produces a torus and adding a handle to a torus creates a pretzel.

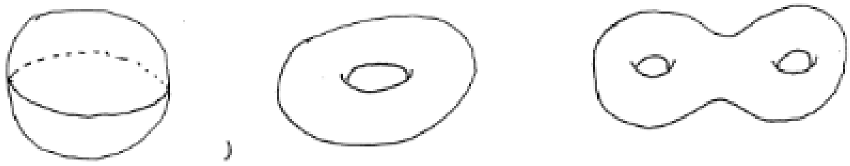

Figure 2.9: The figure shows a sphere (of genus 0), a torus (of genus 1) and a pretzel (of genus 2). The Euler characteristic is given by $2-2g$ where g is the genus.

Every closed orientable two dimensional surface is topologically equivalent to a sphere with some number of handles attached. The number of handles is called the genus of the surface. Our Feynman diagram expansion can be understood as an expansion in the genus of the surface. This expansion in terms of the genus of the surface strongly suggests a link to string theory, upon identifying the surface triangulated by the ribbon graph with the worldsheet of a string. In the string picture, quantum corrections are encoded by summing worldsheets of higher genus. The one loop contribution corresponds to the genus 1 worldsheet, the two loop contribution to a genus 2 worldsheet and so on.

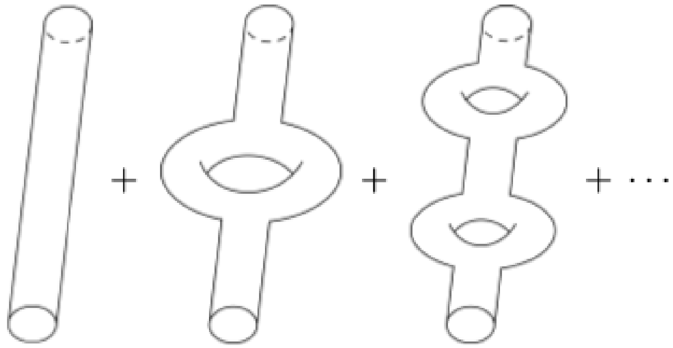

Figure 2.10: Loop expansion of the two point function in string theory.

This completes our discussion of the usual motivation for the connection between the planar limit of a matrix model and string theory. We would now like to consider large N but non-planar limits of the theory. Towards this end, in the next chapter we will review some background maths useful to accomplish this goal.

3. Background Mathematics

This chapter develops the tools and techniques we need to explore the physics of giant gravitons. This requires the use of group representation theory of the symmetric and unitary group. We will provide a dictionary between half-BPS representations in the $\mathcal{N}=4$ super Yang Mills theory with gauge group $U(N)$ and giant graviton states in the dual gravity description. It turns out that the space of Schur polynomials of $U(N)$, which are labeled by Young diagrams with no more than N rows, can be mapped to the space of half-BPS representations [3].

3.1 Half-BPS Operators

The $\mathcal{N}=4$ super Yang-Mills theory can be written in terms of an $\mathcal{N}=1$ vector multiplet and three chiral multiplets. The scalars of the chiral multiplets are three complex scalar fields, given as linear combinations of the six hermittian scalars as follows

$$Z = X^1 + iX^4, \quad X = X^2 + iX^5, \quad Y = X^3 + iX^6 \tag{3.1}$$

where all the fields takes values in the adjoint of $U(N)$.

The operators in $\mathcal{N}=4$ SYM theory built from a single complex matrix Z are half BPS operators. This means these operators are invariant under half of the super symmetry of the theory. In $\mathcal{N}=4$ SYM, the half-BPS operators are constructed from traceless symmetric $S0(6)$ tensor combinations of the six hermittian scalars. We will focus on the half BPS operators that are invariant under an $S0(4)$ subgroup of the $S0(6)$ symmetry. They include both single trace operators and multi-trace operators. For a fixed $\mathcal{R}$ charge n, there is a distinct operator for each partition of n of the form

$$\prod_{l_i,k_i}[\mathrm{Tr}(Z^{l_i})]^{k_i} \tag{3.2}$$

where the integers l_i, k_i define a partition of n as described below

$$n = \sum_{i=1}^{m} l_i k_i. \tag{3.3}$$

We have then a one to one correspondence between half-BPS representations of charge n and partitions of n. The Schur polynomials of degree n, for the unitary group $U(N)$, gives a useful basis in the space of local operators constructed using n fields. A Schur polynomial in the complex matrix Z will be associated to each short representation. We can consider the calculation of correlators of holomorphic Schur polynomials and their conjugates. These correlators enjoy a non-renormalization theorem. Consequently weak coupling computations can be trusted in the strongly coupled limit of the theory.

3.2 Free Field Combinatorics

The basic two point function is obtained using the free field correlator

$$< Z_{ij}(x)Z^*_{kl}(y) >= \frac{\delta_{ik}\delta_{jl}}{(x-y)^2}. \tag{3.4}$$

We will focus on the dependence of the correlators on the rank N of the $U(N)$ gauge group. The space time dependence of the correlator is determined by conformal symmetry and is trivial, so we will often suppress this dependence. Evaluating the free field contractions in $U(N)$ frequently involves evaluating sums of the form

$$\sum_{i_1,i_2...i_n} \delta^{i_1}_{i_{\sigma(1)}}\delta^{i_2}_{i_{\sigma(2)}}...\delta^{i_n}_{i_{\sigma(n)}} \tag{3.5}$$

where $i_1, ..., i_n$ run from 1 to N and $\sigma \in S_n$ is a permutation. If σ is the identity, $i_{\sigma(j)} = i_j$ and the above sum produces N^n. Notice that the identity is a product of n one-cycles. Next consider a permutation σ with one cycle of length 2 and the remaining cycles of length 1. For example, consider $\sigma = (12)(3)(4)(5).....(n)$. It is clear that we have $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(k) = k$ for $k = 3, ..., n$. The above sum now gives

$$\sum_{i_1,i_2...i_n} \delta^{i_1}_{i_2}\delta^{i_2}_{i_1}\delta^{i_3}_{i_3}\delta^{i_4}_{i_4}...\delta^{i_n}_{i_n} = N^{n-1} \tag{3.6}$$

since the first two Kronecker deltas are only non zero when $i_1 = i_2$. Notice that σ has $n-1$ cycles, which is again equal to the power of N we obtain upon performing the sum. In general, the sum in (3.5) is

$$\sum_{i_1,i_2...i_n} \delta^{i_1}_{i_{\sigma(1)}}\delta^{i_2}_{i_{\sigma(2)}}...\delta^{i_n}_{i_{\sigma(n)}} = N^{C(\sigma)}. \tag{3.7}$$

In the last formula, $C(\sigma)$ denotes the number of cycles in σ. We can improve our notation by denoting a collection of n indices $i_1, i_2, ..., i_n$ with a multi-index notation $I(n)$. Further, we use $I(\sigma(n))$ to denote the same collection of n indices after their labels are shuffled by a permutation σ. In this notation (3.7) becomes

$$\sum_I (\delta)^{I(n)}_{I(\sigma(n))} = N^{C(\sigma)}. \tag{3.8}$$

3.3 Symmetric groups

The symmetric group S_n is isomorphic to the group of permutations of the integers $\{1, ..., n\}$. To develop the representation theory of the symmetric group it is useful to introduce the group algebra. To obtain the group algebra, we consider formal sums over symmetric group elements. The product on this algebra is the usual composition law for permutations. An interesting function on the group algebra is the delta function, which is 1 when the argument is the identity and 0 otherwise. The expansion of this function in group characters of S_n is

$$\delta(\rho) = \frac{1}{n!}\sum_R d_R \chi_R(\rho) \tag{3.9}$$

where the sum is over the irreducible representations R of S_n. The label R can be associated with a Young diagram constructed using n boxes. $\chi_R(\rho)$ is the character of group element ρ in the representation R. The character of the group element is given by the trace of the matrix representing the group element. Finally, d_R is the dimension of representation R.

In any irreducible representation R, the product of an element C of the group algebra, which commutes with every element of the group, can be factorised into a product of characters as follows

$$\chi_R(C\sigma) = \frac{\chi_R(C)\chi_R(\sigma)}{d_R}. \tag{3.10}$$

The element C is constructed out of averages over the symmetric group of the form $\sum_\rho g(\rho)\rho$ where $g(\rho)$ is a class function. A class function is a function that takes the same value for all elements in a conjugacy class. Now, we consider the operator

$$\sum_\sigma \chi_R(\sigma^{-1}) D_S(\sigma) \tag{3.11}$$

where $D_S(\sigma)$ is the matrix representing σ in the irreducible representation S. We will see that this operator defines a projection operator. The matrix written down in (3.11) commutes with any permutation τ acting in the representation S. So, by Schur's Lemma, it must be a constant multiple of the unit matrix, i.e

$$\sum_\sigma \chi_R(\sigma^{-1}) D_S(\sigma) = cI \tag{3.12}$$

where c is a constant. Thus,

$$\sum_\sigma \chi_R(\sigma^{-1}) D_S(\sigma\alpha) = \sum_\sigma \chi_R(\sigma^{-1})\frac{\chi_R(\sigma)}{d_R} D_S(\alpha) \tag{3.13}$$

where we have used the result explained in eqn.(3.10). The orthogonality relation for characters is

$$\sum_\sigma \chi_R(\sigma^{-1})\chi_S(\sigma) = \delta_{RS}\; n!. \tag{3.14}$$

From eqns.(3.13) and (3.14) one can easily establish that

$$\sum_\sigma \chi_R(\sigma^{-1}) D_S(\sigma\alpha) = \frac{\delta_{RS}\; n!}{d_S} D_S(\alpha). \tag{3.15}$$

where d_S is the dimension of representation S of S_n. Finally, we take a trace to find

$$\sum_\sigma \chi_R(\sigma^{-1})\chi_S(\sigma\alpha) = \frac{\delta_{RS}\; n!}{d_S}\chi_S(\alpha). \tag{3.16}$$

3.4 The Schur polynomial

The symmetric group S_n is a group of finite order n! while the unitary group $U(N)$ has an infinite number of elements. The connection between the symmetric group and the unitary group is known as Schur-Weyl duality. Let V be the fundamental representation of $U(N)$. The space $Sym(V^{\otimes n})$ is also a representation of $U(N)$. It admits a commuting action of the symmetric group S_n. We can then organize the actions of the symmetric group and the unitary group into collections of states which are irreducible representations of both. This is the reason why Young diagrams can be used to label both $U(N)$ and S_n representations. We will develop some results following from this connection.

The Schur polynomials can be defined as the characters of the unitary group in their irreducible representations

$$\chi_R(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma U) \tag{3.17}$$

where $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in representation R. The trace which appears in this equation is given by

$$\mathrm{Tr}(\sigma U) = \sum_{i_1,i_2,..,i_n} U^{i_1}_{i_{\sigma(1)}} U^{i_2}_{i_{\sigma(2)}} ... U^{i_n}_{i_{\sigma(n)}}. \tag{3.18}$$

By taking $U = 1$ we find $\mathrm{Tr}(\sigma) = N^{C(\sigma)}$. Therefore

$$\chi_R(1) = Dim_N(R) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)N^{C(\sigma)} \tag{3.19}$$

where $Dim_N(R)$ is the dimension of representation R of the unitary group $U(N)$. In the context of this dissertation it is convenient to consider the extension of the Schur polynomials from unitary to complex matrices

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma Z) \tag{3.20}$$

where the representation R of S_n can be labelled by Young Diagrams with n boxes and $\mathrm{Tr}(\sigma Z)$ is given by

$$\mathrm{Tr}(\sigma Z) = \sum_{i_1,i_2,..,i_n} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} ... Z^{i_n}_{i_{\sigma(n)}} \tag{3.21}$$

$$= \sum_I Z\begin{pmatrix} I(n) \\ I(\sigma(n)) \end{pmatrix} \tag{3.22}$$

where we used the compact notation introduced in section 3.1.2. We can also consider the fusion coefficients of $U(N)$. We denote these coefficients by $g(R_1, R_2; S)$. They give the multiplicity of the representation S in the tensor product of the representation $R_1 \otimes R_2$. Consequently, we have the character decomposition

$$\chi_S(\sigma_1 \; o \; \sigma_2) = \sum_{R_1\in Rep}\sum_{R_2\in Rep} g(R_1, R_2; S)\chi_{R_1}(\sigma_1)\chi_{R_2}(\sigma_2). \tag{3.23}$$

This coefficient can also be computed using the Littlewood-Richardson rule which counts the numbers of times the representation S appears in the tensor product of R_1 and R_2. The coefficient $g(R_1, R_2; S)$ can be written in terms of a $U(N)$ group integral as follows

$$g(R_1, R_2; S) = \int dU \chi_{R_1}(U)\chi_{R_2}(U)\chi_S(U^\dagger). \tag{3.24}$$

In general we have

$$\begin{aligned} g(R_1, R_2, ...R_n; S) &= \int dU \chi_{R_1}(U)\chi_{R_2}(U)....\chi_{R_n}(U)\chi_S(U^\dagger) \\ &= \int dU (\prod_{i=1}^{n} \chi_{R_i}(U))\chi_S(U^\dagger). \end{aligned} \tag{3.25}$$

The product of Schur polynomials of irreducible representations R_1 and R_2 can be written as

$$\chi_{R_1}(U)\chi_{R_2}(U) = \sum_S g(R_1, R_2; S)\chi_S(U). \tag{3.26}$$

Repeated use of this expansion in the integral (3.25) gives

$$g(R_1, R_2, ...R_n; S) = \sum_{S_1, S_2, ..., S_{n-2}} g(R_1, R_2; S)g(S_1, R_3; S_2), ...g(S_{n-2}, R_n; S). \tag{3.27}$$

This concludes our quick review of the representation theory of the symmetric and unitary groups. In the next section we turn to consider relevant mathematical tools from the theory of resurgence.

4. Resurgence

In this chapter we will develop mathematical tools that allow us to recover non-perturbative contributions given the pertubative contribution to the quantity of interest. The tools we will review are collectively refered to as methods of resurgence. Resurgence is a collection of ideas in mathematics which show that the coefficients of an asymptotic series contain information about nonperturbative contributions. Our goal is to develop these ideas for the $\frac{1}{2}$-BPS sector of $N = 4$ super Yang-Mills theory. We begin by introducing the Borel transform and resummation methods of asymptotic series to see how to associate values to (factorially) divergent sums [1].

4.1 Borel Resummation

To simplify the discussion, we will use a concrete toy model to develop the ideas and methods motivating resurgence. For the toy model, we consider evaluating the integral

$$I(g) = \int_{-\infty}^{\infty} d\phi e^{\frac{-\phi^2}{2} - g\phi^4}. \tag{4.1}$$

We can study $I(g)$ using pertubation theory. This amounts to expanding the integrand as a power series in g

$$\begin{aligned} I(g) &= \int_{-\infty}^{\infty} d\phi e^{\frac{-\phi^2}{2}} \sum_{n=0}^{\infty} \frac{(-g)^n \phi^{4n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int_{\infty}^{\infty} d\phi \phi^{4n} e^{\frac{-\phi^2}{2}} \\ &= \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (4n-1)!!}{n!} g^n \\ &\equiv \sqrt{2\pi} \sum_{n=0}^{\infty} a_n g^n. \end{aligned} \tag{4.2}$$

Truncating the sum, we find the following approximation to $I(g)$ at n^{th} order in pertubation theory

$$I_n = \sqrt{2\pi} \sum_{i=0}^{n} a_n g^n. \tag{4.3}$$

It is easy to verify that, for example, when $g = 0.1$ and $n = 6$ we find a very good agreement with the exact answer. Proceeding to higher order, we find that the accuracy becomes worse and worse and I_n starts to diverge. See figures 4.1 and 4.2 which illustrate numerically obtained results. The plots show $I_n(g)$ versus n. Clearly, the pertubation expansion defines an asymptotic expansion. To really prove this is the case, we shall study the behaviour of the coefficients a_n for large values of n.

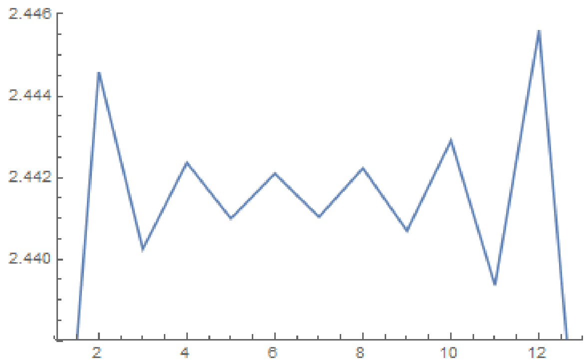

Figure 4.1: $I_n(g)$ versus n for $g = 0.01$.

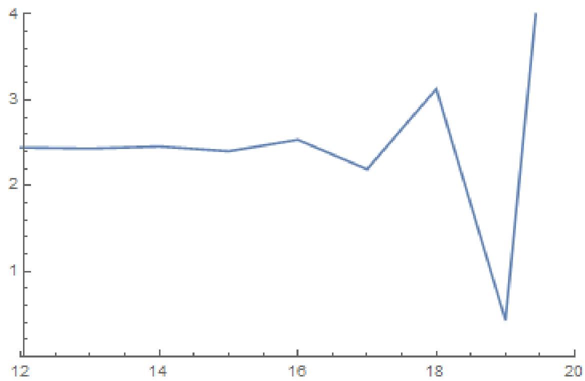

Figure 4.2: $I_n(g)$ versus n for $g = 0.01$.

It is easy to see that the accuracy becomes worse as we increase the order in pertubation theory.

To prove that the pertubation series is an asymptotic expansion, we need to consider the behaviour of the coefficients a_n as $n \to \infty$. Towards this end we will employ Stirling's approximation. Rewrite a_n as follows

$$a_n = \frac{(-1)^n (4n)!}{2^{2n} n! (2n)!}. \tag{4.4}$$

Using Stirling's approximation we have

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{4.5}$$

$$(2n)! \sim \sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n} \tag{4.6}$$

$$(4n)! \sim \sqrt{8\pi n}\left(\frac{4n}{e}\right)^{4n}. \tag{4.7}$$

Thus,

$$\begin{aligned} a_n &\sim \frac{(-1)^n(\frac{4n}{e})^{4n}}{\sqrt{\pi n}2^{2n}(\frac{n}{e})^n(\frac{2n}{e})^{2n}} \\ &\sim (-4n)^n. \end{aligned} \tag{4.8}$$

Equation (4.8) is obtained after a little computation using Stirling approximation. For the pertubation series to converge, we need $4ng < 1$. But $4ng$ becomes arbitrarly large for n large enough, no matter how small g is. Thus, the series expansion for $I(g)$ does not converge. The pertubation theory fails to give a good approximation for $I(g)$ since the sum on the RHS of (4.2) diverges for every value of g. This proves that this pertubation series is indeed an asymptotic expansion. The Borel transform of the formal power series is given by

$$I_B(s) = \sum_{m=0}^{\infty} \frac{I_m}{m!} s^m. \tag{4.9}$$

From the Borel transform we can, under favorable circumstances, get $I(g)$ by Borel resummation. Borel resummation is a way to construct a function that has a given asymptotic expansion.

Denote $\mathbb{C}[[z^{-1}]]$ the set of all the formal power series in z^{-1}. Consider a formal power series $F(z)$ for $z \sim \infty$ defined by

$$F(z) = \sum_{n=0}^{\infty} a_n z^{n-1}. \tag{4.10}$$

The Borel transform of $F(z)$ is defined by

$$B[F](\zeta) = \sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!}. \tag{4.11}$$

We will now spend some time developing the mathematics of the Borel transform. Given two formal power series $F_1(z)$ and $F_2(z)$, the product $F_1(z)F_2(z)$ is a formal power series of the form

$$F_1(z)F_2(z) = \left(\sum_{n=0}^{\infty} b_n z^{n-1}\right)\left(\sum_{n=0}^{\infty} c_n z^{n-1}\right) \tag{4.12}$$

$$= \sum_{n=0}^{\infty} d_n z^{n-1} \tag{4.13}$$

where

$$d_n = \sum_{p+q=n-1} b_p c_q \quad n \geq 1. \tag{4.14}$$

When we pass to the Borel transform, the natural multiplication of formal power series becomes a convolution $(*)$ in $\mathbb{C}[\zeta]$. Indeed, the Borel transform of this product is given by

$$(B[F_1] * B[F_2])(\zeta) = \sum_{n,m\geq 0} \frac{b_n c_m}{n!m!} \int_0^\zeta d\zeta_1 \zeta_1^n (\zeta - \zeta_1)^m \tag{4.15}$$

To prove this we need to make use of the formula for the beta function

$$B(n+1, m+1) = \int_0^1 d\zeta_1 \zeta_1^n (1-\zeta_1)^m = \frac{n!m!}{(n+m+1)!}. \tag{4.16}$$

which implies

$$\int_0^\zeta d\zeta_1 \zeta_1^n (\zeta - \zeta_1)^m = B(n+1, m+1)\zeta^{n+m+1}. \tag{4.17}$$

Thus, (4.15) becomes

$$\begin{aligned}
(B[F_1] * B[F_2])(\zeta) &= \sum_{n,m\geq 0} \frac{b_n c_m}{n!m!} B(n+1, m+1)\zeta^{n+m+1} \qquad (4.18) \\
&= \sum_{n,m\geq 0} \frac{b_n c_m}{n!m!} \frac{n!m!}{(n+m+1)} \zeta^{n+m+1} \\
&= \sum_{n,m\geq 0} \frac{b_n c_m}{(n+m+1)!} \zeta^{n+m+1} \\
&= \sum_{n\geq 1} \frac{d_n}{n!} \zeta^n. \qquad (4.19)
\end{aligned}$$

Note that the Borel transform (4.11) has a finite radius of convergence if and only if $F(z)$ is of Gevrey-1 type. A formal power series $F(z) = \sum_{n=0}^\infty a_n z^{-n-1}$ is of Gevrey-1 type if there exists somes constants A, α such that $|a_n| \leq \alpha A^n n!$. When the formal power series $F(z) = \sum_{n=0}^\infty a_n z^{-n-1}$ is of Gevrey order-$\frac{1}{m}$ there exist somes constants A, α such that $|a_n| \leq \alpha A^n (n!)^m$. In what follows, we will assume that F is of Gevrey-1 type and its Borel transform $B[F]$ defines a convergent expansion about the origin. After having improved the convergence of the original formal series $F \longrightarrow B[F]$, we need an operator that can be used to define the analytic extension of the original formal power series, that is the Laplace transform along θ,

$$\mathcal{S}_\theta[F](z) = \int_0^{e^{i\theta}\infty} d\zeta e^{-z\zeta} F(z) \tag{4.20}$$

We can compute the Laplace transform S on the real positive line in the direction $\theta = 0$ using the usual Laplace transform as follows:

$$\mathcal{S}_0[\zeta^\alpha] = \int_0^\infty d\zeta e^{-z\zeta} \zeta^\alpha \tag{4.21}$$

Changing variable to $v = z\zeta$ in (4.21) gives

$$\begin{aligned}
\mathcal{S}_0[\zeta^\alpha] &= \frac{1}{z^{\alpha+1}} \int_0^\infty dv e^{-v} v^\alpha \\
&= \frac{\Gamma(\alpha+1)}{z^{\alpha+1}}. \qquad (4.22)
\end{aligned}$$

The inverse Laplace transform of this is

$$(\mathcal{S}_0)^{-1}[z^{-\alpha-1}] = \frac{\zeta^{\alpha}}{\Gamma(\alpha+1)}. \tag{4.23}$$

We want to compute the Laplace tranform along direction θ of the formal power series

$$F(z) = \sum_{n=0}^{\infty}(-n)^n n! z^{-n-1}. \tag{4.24}$$

This will give the analytic continuation to the half-plane $Re(z) > 0$. The Borel transform of (4.24) is

$$B[F](\zeta) = \sum_{n=0}^{\infty}(-1)^n \zeta^n = \frac{1}{1+\zeta}, \quad |\zeta| < 1. \tag{4.25}$$

The Laplace transform along $\theta = 0$ is

$$\mathcal{S}_0[F](z) = \int_0^{\infty} d\zeta e^{-z\zeta}\frac{1}{1+\zeta}. \tag{4.26}$$

Change variables from ζ to $u = 1+\zeta$ to get

$$\mathcal{S}_0[F](z) = e^z \int_0^{\infty} due^{-zu}\frac{1}{u}. \tag{4.27}$$

Again change variables to $zu = t$ to find

$$\mathcal{S}_0[F](z) = e^z \int_z^{\infty} dte^{-t}t^{-1} = e^z\Gamma(0,z) \tag{4.28}$$

where $\Gamma(0,z)$ is the incomplete gamma function. The last expression above gives the analytic contiuation to the half-plane $Re(z) > 0$.

When computing the Laplace transform in direction θ, we might run into singular points. Since we cannot integrate through the singular point we must deform the contour to dodge the singularity points. A suitable method to achieve this uses the Stokes automorphism.

4.2 Stokes automorphism and Alien Derivatives

To avoid singular points along a direction θ we will relate two Borel transforms in the directions θ^+ and θ^-. This allows us to understand how the resummed series jumps across a Stokes lines (see the figure below).

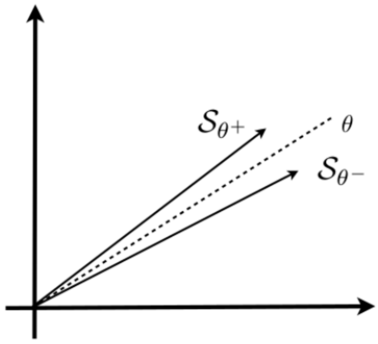

Figure 4.3: Lateral Borel summation along θ.

Considering this lateral Borel summation, the Stokes automorphism $\sum_\theta$ can be defined as follows

$$S_{\theta^-}F = S_{\theta^+} \circ \sum_{\theta} F \tag{4.29}$$

The Stokes automorphism can also be written as

$$S_{\theta^+} = S_{\theta^-} \circ \sum_{\theta} = S_{\theta^-} \circ (Id - Disc_\theta) \tag{4.30}$$

$$S_{\theta^+} - S_{\theta^-} = -S_{\theta^-} \circ Disc_\theta \tag{4.31}$$

where $Disc_\theta$ encodes the full discontinuity across θ. If there is no singularity we have $S_{\theta^+} = S_{\theta^-}$ and this leads to $\sum_\theta B[F] = B[F]$ where $Disc_\theta = 0$. F is called a resurgence constant and the Borel transform of F has no singularity along θ. This gives a convergent power series. The difference between the θ^+ and θ^- transforms is nothing but a sum over contours around each singular point and the discontinuity of S across θ is given as an infinite sum of contributions coming from each one of the singular points.

The logarithm of the Stokes automorphism defines the Alien derivative Δ_ω as follows

$$\sum_{\theta} = \exp\left(\sum_{\omega\in\Omega} e^{-\omega z}\Delta_\omega\right) \tag{4.32}$$

where Ω is the set of singularities in the complex plane along the θ direction. Using (4.32) we can rewrite (4.30) as follows

$$\begin{aligned} S_{\theta^+}B[F](z) &= S_{\theta^-}B[F](z) + S_{\theta^-}\exp\Big(\sum_{\omega\in\Omega} e^{-\omega z}\Delta_\omega\Big)B[F](z) \\ &= S_{\theta^-}B[F](z) + \sum_{l=1}^{\infty}\sum_{m_1\cdots m_l\geq 1} e^{(\omega_{m_1}+\cdots+\omega_{m_l})} S_{\theta^-}(\Delta_{\omega_{m_1}}\cdots\Delta_{\omega_{m_l}}B[F](z)) \end{aligned} \tag{4.33}$$

To understand how the Alien derivative works, we will construct some examples. Consider a formal power series

$$F(z) = \frac{\alpha}{2\pi i(z-\omega)} + \frac{1}{2\pi i}G(z-\omega)\log(z-\omega) \tag{4.34}$$

The Borel transform of this is

$$BF(\zeta) = \frac{\alpha}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i} BG(\zeta - \omega)\log(\zeta - \omega) \tag{4.35}$$

where at $\zeta = \omega$ there is a simple singularity. Perform a contour integral along a closed path C_ω around ω. This contour integral is equal to the sum of the integrals going from 0 to $\infty e^{i\theta^-}$ and from 0 to $\infty e^{i\theta^+}$ plus the integral over a path that closes these paths at infinity. We have

$$\begin{aligned}\oint_{C_\omega} d\zeta e^{-z\zeta} BF(\zeta) &= \int_0^{\infty e^{i\theta^-}} d\zeta e^{-z\zeta} BF(\zeta) + \int_{\infty e^{i\theta^+}}^{0} d\zeta e^{-z\zeta} BF(\zeta) + \int_{\infty e^{i\theta^-}}^{\infty e^{i\theta^+}} d\zeta e^{-z\zeta} BF(\zeta) \\ &= S_{\theta^-}F(z) - S_{\theta^+}F(z)\end{aligned} \tag{4.36}$$

Choose C_ω to be small and use (4.35) to obtain

$$\oint_{C_\omega} d\zeta e^{-z\zeta}\left[\frac{\alpha}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i} BG(\zeta - \omega)\log(\zeta - \omega)\right] = S_{\theta^-}F(z) - S_{\theta^+}F(z) \tag{4.37}$$

Let's change the variable to $\zeta = u + \omega$ such that C_ω will change into a path C_0 that encloses the origin in the u-plane. Thus,

$$\begin{aligned}S_{\theta^-}F(z) - S_{\theta^+}F(z) &= \oint_{C_0} du e^{-z(u+\omega)}\frac{\alpha}{2\pi i u} + \oint_{C_0} du \frac{1}{2\pi i} BG(u)\log(u) \\ &= \frac{\alpha e^{-z\omega}}{2\pi i}\oint_{C_0} du \frac{e^{-zu}}{u} + e^{-z\omega}\int_0^\infty du e^{-zu} BG(u) \\ &= \alpha e^{-z\omega} + e^{-z\omega} G(z)\end{aligned} \tag{4.38}$$

Now use (4.37) to obtain

$$(S_{\theta^+} - S_{\theta^-})F(z) = e^{-z\omega} S_{\theta^-}(\Delta_\omega F(z)) \tag{4.39}$$

From this we see that when the Alien derivative act on a simple resurgent function of the form (4.33) we have

$$S\Delta_\omega F(z) = \alpha + SG(z) \tag{4.40}$$

where the Borel transform has a simple singularity and the difference between the two Borel sums from integrating across the singular point gives the Alien derivative Δ_ω. Note that Δ_ω is a linear operator such that $\Delta_\omega = 0$ if the Borel transform of F has no singularity and $\Delta_\omega \neq 0$ if the Borel transform of F has a singularity or a pole.

Consider now the series expansion, as an example,

$$F(z) = \sum_{g=0}^{\infty} \frac{g!}{g+1}\frac{1}{A^g z^{g+1}} \tag{4.41}$$

The Borel transform of this series is

$$BF(\zeta) = \sum_{g=0}^{\infty} \frac{1}{g+1} \left(\frac{\zeta}{A}\right)^g = -\frac{A}{\zeta} \log(1 - \frac{\zeta}{A}) \tag{4.42}$$

We have singularities at $\zeta = \{0, A\}$ where $\zeta = 0$ is a pole and $\zeta = A$ a branch point. Motivated by this example we will now introduce a few more ideas.

A holomorphism function $BF(\zeta)$ in an open disc $D \subset \mathbb{C}$ has a simple singularity at ω if there exists $\alpha \in \mathbb{C}$ and two germs of analytic functions at the origin such that

$$BF(\zeta) = \frac{\alpha}{2\pi(\zeta - \omega)} + \frac{1}{2\pi i} BG(\zeta - \omega) \log(\zeta - \omega) + reg(\zeta - \omega) \tag{4.43}$$

For $\alpha = 0$ we have

$$\frac{BG(s - A)}{2\pi i} = -\frac{A}{s} \Rightarrow BG(s - A) = -\frac{2\pi i A}{s} \tag{4.44}$$

From this,

$$BG(s) = -\frac{2\pi i}{1 + \frac{s}{A}} - 2\pi i \sum_{g=0}^{\infty} \left((-1)^g (\frac{s}{A}\right)^g, \quad \left|\frac{s}{A}\right| < 1 \tag{4.45}$$

and the formal series is

$$G(z) = -2\pi i \sum_{g=0}^{\infty} g! \frac{1}{(-A)^g} \frac{1}{z^{g+1}} \tag{4.46}$$

Using (4.40) and the fact that the singularity at $\omega = A$ is on the real axis, together with $\alpha = 0$, we find that the Borel sums give us back the asymptotic expansion

$$\Delta_\omega F = G(z) = -2\pi i \sum_{g=0}^{\infty} g! \frac{1}{(-A)^g} \frac{1}{z^{g+1}}. \tag{4.47}$$

4.3 Transseries

In this section we will introduce some basic concepts regarding the transseries expansion. We start by defining a log-free transseries as a formal power series

$$T = \sum_j C_j g_j \tag{4.48}$$

where the coefficients C_j are real and g_j are log-free trans-monomials which are a symbol of the form

$$g = z^a e^T \tag{4.49}$$

with $a \in \mathbb{R}$. The height of a trans-monomial is defined as the number of times we compose the formal exponential symbol, i.e $ze^{e^z} + z$ has height 2.

Transseries inherits almost all the standard properties of usual power series when treated as formal sums. For example, differentiation of a trans-series is defined by the standard differentiation of the trans-monomials

$$g' = (z^a e^T)' = az^{a-1}e^T + z^a T' e^T, \tag{4.50}$$

$$T' = (\sum_j C_j g_j)' = \sum_j C_j g_j' \tag{4.51}$$

A general transseries is obtained by using the symbol $\log_m z$ such that

$$\log_m z = \log \circ \cdots \circ \log z \tag{4.52}$$

where we composed the logarithm $m \in \mathbb{N}$ times and the integer m is called depth of the transseries. Note that finite depth transseries arise naturally when we are dealing with instanton contributions to physical observables. The question we should ask now is how we can obtain a consistent formal solution to our physical problem since the perturbative power series expansion is usually insuficient to recover the solution. We know from the study of the analytic properties of the Borel transform of the pertubative series that the non-perturbative terms have to be included to obtain a consistent solution. Therefore we consider a transseries of the form

$$F(z, \sigma) = \sum_{n=0}^{\infty} \sigma^n F^{(n)}(z) \tag{4.53}$$

where $F^{(n)}(z)$ are n-instanton contributions defined as

$$F^{(n)}(z) = e^{-nAz}\Phi_n(z) \tag{4.54}$$

A is the instanton action, σ is a complex parameter to keep track of the resurgence symbols e^{-Az} and $\Phi_n(z)$ is a formal asymptotic power series. The problem we want to solve in the next section is how to compute the Alien derivative of the transseries (4.53). We will relate the Alien derivative to the standard derivation through a bridge equation.

4.4 Bridge equation

The Alien derivatives can be constructed using the bridge equation, which is obtained from the commutation relation of the dotted Alien derivatives and the partial derivative of the transeries $F(z, \sigma)$. The dotted Alien derivative is defined as $\dot{\Delta}_\omega = e^{-\omega x}\Delta_\omega$. Note that the dotted Alien derivatives $\dot{\Delta}_{lA} F(z, \sigma)$ and the partial derivative of the transseries $\partial_\sigma F(z, \sigma)$ are proportional since they satisfy the same differential equation

$$\dot{\Delta}_{lA} F(z, \sigma) = S_l(\sigma)\partial_\sigma F(z, \sigma) \tag{4.55}$$

where $S_l(\sigma)$ is the proportionality factor which depends on σ. The name "bridge" comes from the fact that this equation relates the alien calculus with the ordinary calculus. To determine the proportionality factor $S_l(\sigma)$ apply the dotted alien derivative to the transseries $F(z,\sigma)$

$$\dot{\Delta}_{lA}F(z,\sigma)=\sum_{n=0}^{\infty}\sigma^n e^{-(l+n)Az}\Delta_{lA}\Phi_n(z) \tag{4.56}$$

The ordinary derivative of $F(z,\sigma)$ is

$$\partial_\sigma F(z,\sigma)=\sum_{n=0}^{\infty}n\sigma^{n-1}e^{-nAz}\Phi_n(z) \tag{4.57}$$

To match these two equations term by term we expect that $S_l(\sigma)$ has the form

$$S_l(\sigma)=\sum_{k=0}^{\infty}S_l^{(k)}\sigma^k, \quad S_l^{(k)}\in\mathbb{C} \tag{4.58}$$

Now we can determine $S_l^{(k)}$ by computing the RHS for a simple resurgent function of the form (4.55), using (4.57) and (4.58)

$$S_l(\sigma)\partial_\sigma F(z,\sigma)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}nS_l^{(k)}\sigma^{n+k-1}e^{-nAz}\Phi_n(z) \tag{4.59}$$

Comparing this equation with (4.56) we find that $k=1-l$ and this leads to

$$S_l^{(k)}=\begin{cases}0 & \text{if} \quad l>1\\ S_l(\sigma)=S_l\sigma^{1-l} & \text{if} \quad l<1\end{cases}$$

Using the last equation above in (4.59) we obtain

$$S_l(\sigma)\partial_\sigma F(z,\sigma)=\sum_{n=1}^{\infty}nS_l\sigma^{n-l}e^{-nAz}\Phi_n(z) \tag{4.60}$$

Let's change the variable to $p=n-l$

$$S_l(\sigma)\partial_\sigma F(z,\sigma)=\sum_{p=1-l}^{\infty}(p+l)S_l\sigma^p e^{-(l+p)Az}\Phi_{l+p}(z), \quad l\leq 1 \tag{4.61}$$

Comparing this equation with (4.56) we find

$$\Delta_{lA}\Phi_n(z)=\begin{cases}0 & \text{if} \quad l>1\\ S_l(n+l)\Phi_{n+l}(z) & \text{if} \quad l\leq 1\end{cases} \tag{4.62}$$

For a simple resurgent function $\Phi_n(z)$ with a singularity at $\omega=lA$, by making use of (4.40) and (4.43), the Borel transform of this function Φ_n is

$$B\Phi_n(\zeta)=\frac{1}{2\pi i}B\Phi_{n+l}(\zeta-lA)S_l(l+n)\log(\zeta-\omega) \tag{4.63}$$

Note from this that the singular behaviour of the Borel transform $B\Phi_n(\zeta)$ close to $\omega = lA$ is entirely governed by $B\Phi_{n+k}(\zeta)$. It is important to note that the bridge equation not only allow us to reconstruct the entire behaviour of our transseries close to a singular point but also makes manifest the appearance of the Stokes phenomenon along the singular lines $\theta = 0$ and $\theta = \pi$. To understand this, let us go back to the expression for the Stokes automorphism in term of the alien derivative (4.32)

$$\sum_\theta = \exp\left(\sum_{\omega\in\Omega} e^{-\omega z}\Delta_\omega\right) \tag{4.64}$$

where Ω is the set of singularities in the complex plane along the θ direction. Let us specialise it to the singular direction $\theta = 0$. Since the singular behaviour of our Borel transform $B\Phi_n(\zeta)$ is close to $\omega = lA$ we can rewrite the last expression above as

$$\sum_0 = \exp\left(\sum_{l=1}^{\infty} e^{-lAz}\Delta_{lA}\right) \tag{4.65}$$

We know from (4.62) that $\Delta_{lA}\Phi_n(z) = 0$ for all n if $l > 1$. Using this, the above equation simplifies to

$$\begin{aligned}\sum_0 \Phi_n(z) &= (1 + e^{-Az}\Delta_A + \frac{1}{2!}e^{-2Az} + \cdots)\Phi_n(z)\\ &= \sum_{l=0}^{\infty}\frac{1}{l!}\Delta_A^l\Phi_n(z)\end{aligned} \tag{4.66}$$

We can also use the second part of equation (4.62) to conclude

$$\Delta_A\Phi_n(z) = S_1(n+1)\Phi_{n+1}(z) \tag{4.67}$$

and this leads to

$$\Delta_{S_0}^l\Phi_n(z) = S_1^l(n)_l\Phi_{n+l}(z) \tag{4.68}$$

where we use the Pochhammer symbol $(n)_l = \prod_{i=1}^{l}(n+i)$. Therefore

$$\sum_0 \Phi_n(z) = \sum_{l=0}^{\infty} C_{n+l}^n S_1^l(n)e^{-lAz}\Phi_{n+l}(z) \tag{4.69}$$

where $C_{n+l}^n = \frac{n!}{l!(n-l)!}$. We can now use (4.66) and (4.69) in (4.30) to obtain

$$\begin{aligned}S_{0^+}F(z,\sigma) &= S_{0^-}\circ\sum_0 F(z,\sigma)\\ &= (1 + e^{-Az}\Delta_A + \cdots)\left(\sum_{n=0}^{\infty} C_{n+l}^n e^{-nAz}\Phi_{n+l}(z)\right)\\ &= S_{0^-}\circ\left(\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} C_{n+l}^n S_1^l\sigma^{ne^{-(n+l)Az}\Phi_{n+l}(z)}\right)\end{aligned} \tag{4.70}$$

Changing the variable $(n+l)$ to m we have

$$S_{0^+}F(z,\sigma) = S_{0^-}\left(\sum_{m=0}^{\infty} e^{-mAz}\Phi_n(\sigma+S_1)^m\right) \tag{4.71}$$

Now use (4.53) to conclude

$$F(z,\sigma+S_1) = \sum_{m=0}^{\infty} e^{-mAz}\Phi_m(\sigma+S_1)^m \tag{4.72}$$

Plug this back into (4.71) to obtain

$$S_{0^+}F(z,\sigma) = S_{0^-}F(z,\sigma+S_1) \tag{4.73}$$

We obtain the same results using the following Bridge equation

$$\dot{\Delta}_{lA}F(z,\sigma) = S_l\sigma^{1-l}\partial_\sigma F(z,\sigma), \quad l \leq 1 \tag{4.74}$$

Therefore the Stokes automorphism along $\theta = 0$ is

$$\sum_0 F(z,\sigma) = \exp(e^{-Az}\Delta_A) = F(z,\sigma+S_1) \tag{4.75}$$

where the singularity is at $l = 1$. Now consider the Stokes automorphism along the direction $\theta = \pi$. Let us again go back to the definition for Stokes automorphism in term of alien derivative and apply this to $\theta = \pi$. We have,

$$\sum_\pi = \exp\left(\sum_{l=1}^{\infty} e^{lAz}\Delta_{-lA}\right) \tag{4.76}$$

Here the singular points are $\{-A,-2A,-3A,\cdots,-lA,\cdots\}$ so that the Stokes automorphism is

$$\sum_\pi = 1 + e^{Az}\Delta_{-A} + e^{2Az}\Delta_{-2A} + \cdots \tag{4.77}$$

Note that the contributions to each resurgent symbol e^{lAz} in $\sum_\pi$ come from

$$\Delta_{-l_1A}\cdots\Delta_{-l_NA}\Phi_n(z) \tag{4.78}$$

Using (4.62) and the Pochhammer symbol we have

$$\begin{aligned}\Delta_{-l_1A}(\Delta_{-l_2A}\Phi_n(z)) &= \Delta_{-l_1A}(S_{-l_2}(n-l_2)\Phi_{n-l_2}) \\ &= S_{-l_2}(n-l_2)S_{-l_1}(n-l_2-l_1\Phi_{n-l_2}-l_1)\end{aligned} \tag{4.79}$$

and

$$\begin{aligned}\Delta_{-l_2A}(\Delta_{-l_1A}\Phi_n(z)) &= \Delta_{-l_2A}(S_{-l_1}(n-l_1)\Phi_{n-l_1}) \\ &= S_{-l_1}(n-l_1)S_{-l_2}(n-l_1-l_2\Phi_{n-l_1}-l_2)\end{aligned} \tag{4.80}$$

where we use the fact that the alien derivatives at different points do not commute, i.e,

$$[\Delta_{-l_1 A}, \Delta_{-l_2 A}]\Phi_n(z) = S_{-l_1} S_{-l_2}(l_1 - l_2)(n - l_1 - l_2)\Phi_{n-l_1-l_2} \tag{4.81}$$

Therefore, the multiple alien derivatives can be written as

$$\Delta_{-l_1 A} \cdots \Delta_{-l_N A}\Phi_n = \prod_{i=1}^{N} S_{-li} \prod_{i=1}^{N} (n - \sum_{j=1}^{i} k_j)\Phi_{n-\sum_i k_i} \tag{4.82}$$

Note that this expression vanishes as soon as $\sum_{i=1}^{N} l_i \geq n$. The Stokes automorphism along the direction $\theta = \pi$ on the transseries $F(z, \sigma)$ is then given by

$$\sum_{\pi} F(z, \sigma) = \exp\left(\sum_{l=1}^{\infty} e^{lAz}\Delta_{-lA}\right) F(z, \sigma) \tag{4.83}$$

where we use (4.76). Using (4.74) we can rewrite this equation as

$$\sum_{\pi} F(z, \sigma) = \exp\left(\sum_{l=1}^{\infty} S_{-l}\sigma^{1+l}\partial_\sigma F(z, \sigma)\right) \tag{4.84}$$

Assuming that all the Stokes constants vanish except $S_{-l} \neq 0$, the Stokes automorphism along the direction $\theta = \pi$ is simply given by

$$\begin{aligned} \sum_{\pi} F(z, \sigma) &= \exp\left(S_{-l}\sigma^{1+l}\partial_\sigma F(z, \sigma)\right) \\ &= \sum_{n=0}^{\infty} S_{-l}^{n} \frac{\sigma^{(1+l)^n}}{n!} \partial_\sigma^n F(z, \sigma) \end{aligned} \tag{4.85}$$

with

$$\partial_\sigma^n F(z, \sigma) = \sum_{i=0}^{\infty} i(i-1)\cdots(i-k+1)\sigma^{i-k}e^{-iAz}\Phi_i(z) \tag{4.86}$$

Finally the Stokes automorphism along the direction $\theta = \pi$ is

$$\begin{aligned} \sum_{\pi} F(z, \sigma) &= \sum_{i=0}^{\infty}\sum_{k=0}^{i} C_i^k (S_{-l}\sigma^{1+l})^k \sigma^{i-k} e^{-iAz}\Phi_i(z) \\ &= \sum_{i=0}^{\infty} (\sigma + S_{-l}\sigma^{1+l})^i e^{-iAz}\Phi_i(z) \\ &= F(z, \sigma + S_{-l}\sigma^{1+l}) \end{aligned} \tag{4.87}$$

where $C_i^k = \frac{i!}{k!(i-k)!}$. This concludes our discussion of resurgence. In the next section we will introduce the correlators of most importance in this dissertation.

5. CFT Correlators

The contents of this section, as well as the sections 6, 7 and 8 have been reported in [17].

The space of half-BPS representations can be mapped to the space of Schur polynomials of $U(N)$, that is, to the space of Young diagrams characterizing representations of $U(N)$[3].There are rigorous theorems[18] that imply that extremal correlation functions of Schur polynomials do not receive any 't Hooft coupling corrections and hence they are given exactly by their values in free field theory. The Schur polynomials that correspond to giant graviton branes have a single column with order N boxes, while those corresponding to dual giant graviton branes have a single row with order N boxes[3]. Computing correlators of these operators is still a highly non-trivial task, even in the free field theory limit, because the number of fields in each operator is going to infinity as we take $N \to \infty$. Fortunately, using techniques based on group representation theory, this problem has been completely solved in[3, 31] for operators constructed using a single field (say Z) and in[32] for operators constructed using more than one matrix (see also [33, 34, 35]). We give a quick review of these results in this section and then use them to explore different possible behaviors of these correlators at large N.

Let V denote the N dimensional vector space carrying the fundamental representation of $U(N)$. The space $\mathrm{Sym}(V^{\otimes n})$ is also a representation of $U(N)$ but it carries in addition a commuting action of S_n. These actions can be simultaneously diagonalized leading naturally to the operators of interest to us, the Schur polynomials. After diagonalizing, the representations of both groups can be labeled by a Young diagram that has n boxes. A further consequence is that the two point function is also diagonalized. The simplest way to achieve the diagonalization is by using a projection operator. The Schur polynomials are given by

$$\chi_R(Z) = \mathrm{Tr}(P_R Z^{\otimes n}) \tag{5.1}$$

where P_R is a projection operator

$$P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma \tag{5.2}$$

It is rather natural to study operators with a two point function normalized to 1. The Schur polynomial has two point function

$$\chi_R(X)\chi_S^\dagger(Y) = \frac{\delta_{RS} f_R}{|X-Y|^{2n}} \tag{5.3}$$

where R is a Young diagram with n boxes and f_R is the product of the factors of the boxes in R. Recall that a box in row i and column j has factor $N - i + j$. The normalized version of the Schur polynomial is given by

$$O_R(x) = \frac{\chi_R(x)}{\sqrt{f_R}} \tag{5.4}$$

The normalized three point correlator is given by

$$\langle O_R(Z)(x_1) O_S(Z)(x_2) O_T(Z^\dagger)(x_3)\rangle = \sqrt{\frac{f_T}{f_R f_S}} \frac{g_{RST}}{|x_1 - x_3|^{2n_R}|x_2 - x_3|^{2n_S}} \tag{5.5}$$

Recall that $g(R,S;T)$ is called a Littlewood-Richardson coefficient. It was introduced in equation (3.24). Young diagrams R and S have n_R and n_S boxes respectively. These are the correlators we will study in this article. In what follows the spatial dependence plays no role and consequently from now on we omit it. This dependence is easily reinstated using simple dimensional analysis. Use A_J to denote the antisymmetric representation with J boxes (i.e. the Young diagram A_J has a single column) and S_J to denote the symmetric representation with J boxes (i.e. S_J has a single row). It is straight forward to see that

$$\langle O_{A_{J_1}} O_{A_{J_2}} O_{A_J} \rangle = \sqrt{\frac{(N-J_1)!(N-J_2)!}{(N-J)!N!}} \tag{5.6}$$

where $J = J_1 + J_2$. We stress that this expression is the exact answer, valid for any values of J_1 and J_2. For operators in the planar limit we would hold J_1, J_2 fixed as we take $N \to \infty$ in which case expanding the correlator leads to a well behaved power series in N^{-1}. To make this point we can consider $J_1 = J_2 = 2$ in which case

$$\begin{aligned}
\langle O_{A_2} O_{A_2} O_4 \rangle &= \sqrt{\frac{(N-2)(N-3)}{(N-1)N}} \\
&= \sqrt{\frac{\left(1-2\frac{1}{N}\right)\left(1-3\frac{1}{N}\right)}{1-\frac{1}{N}}} \\
&= 1 - 2\frac{1}{N} - \frac{1}{N^2} - \frac{1}{N^3} - \frac{3}{2}\frac{1}{N^4} - 3\frac{1}{N^5} - 7\frac{1}{N^6} + O\left(N^{-7}\right)
\end{aligned} \tag{5.7}$$

This expansion in $1/N$ converges absolutely in the range $0 \le \frac{1}{N} < \frac{1}{3}$, that is, for $N > 3$. This planar limit is the regime in which we study perturbative string theory, so it is perhaps not too surprising that we can perform a $1/N$ expansion. This result is however, better than we may have expected: most perturbative expansions are only asymptotic expansions. If we were to increase J_1 and J_2 the radius of converges would shrink further.

We could also consider the case that $J_1 = O(N)$ with $\frac{J_1}{N}$ fixed and much less than 1. The result (5.6) is exact, so it continues to hold in this limit. Since J_1 is order N, $O_{A_{J_1}}$ is a giant graviton. We can then take $J_2 = 2n = O(1)$, so that $O_{A_{J_2}}$ is some collection of point gravitons. The correlator (5.6) then describes the emission or absorption of gravitons by a giant graviton. In this case we find

$$\langle O_{A_{J_1}} O_{A_{J_2}} O_{A_J} \rangle \simeq \left(1 - \frac{J_1}{N}\right)^n + O(N^{-1}) \tag{5.8}$$

and there is again no obstacle to carrying out a $1/N$ expansion. A simple case study is provided by taking $J_2 = 2$ in which case we have

$$\begin{aligned}
\langle O_{A_{J_1}} O_{A_{J_2}} O_{A_J} \rangle &= \sqrt{\frac{(N-J_1)!(N-2)!}{(N-J_1-2)!N!}} \\
&= \sqrt{\frac{(N-J_1)(N-J_1-1)}{N(N-1)}} \\
&= (1-j_1) - \frac{j_1}{2N} + \frac{j_1(3j_1-4)}{(8-8j_1)N^2} + O\left(N^{-3}\right)
\end{aligned} \tag{5.9}$$

where $j_1 = \frac{J_1}{N}$. This is again an absolutely convergent expansion for $\frac{1}{N} < 1 - j_1$.

To obtain giant graviton correlators we should set $J_i = N j_i$ and hold j_i fixed as we take $N \to \infty$. Giant gravitons are spherical D3-brane states which are not part of the perturbative string spectrum[10, 11, 12], so we might expect that this correlator does not have a $1/N$ expansion. In this limit the normalized correlator behaves as[36]

$$\langle O_{A_{J_1}} O_{A_{J_2}} O_{A_J} \rangle = \sqrt{\frac{(N-J_1)!(N-J_2)!}{(N-J)!N!}} \simeq e^{-N j_1 j_2} \tag{5.10}$$

The exponential on the right hand side of the above correlator does not admit a $\frac{1}{N}$ expansion and it therefore constitutes a genuine non-perturbative contribution. The corresponding result for the dual giant graviton correlator is

$$\langle O_{S_{J_1}} O_{S_{J_2}} O_{S_J} \rangle = \sqrt{\frac{(N+J_1-1)!(N+J_2-1)!}{(N+J-1)!(N-1)!}} \simeq e^{N j_1 j_2} \tag{5.11}$$

which is again a non-perturbative contribution. One of the main results of this article is the trans-series expansion of (the square of) these giant graviton correlators. The starting point for this analysis uses Gauss' Hypergeometric Theorem, which says

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{5.12}$$

Clearly then, we can write

$$\left(\langle O_{A_{J_1}} O_{A_{J_2}} O_{A_J} \rangle\right)^2 = {}_2F_1(-J_1, J_2; N - J_1 + 1; 1) \tag{5.13}$$

$$\left(\langle O_{S_{J_1}} O_{S_{J_2}} O_{S_J} \rangle\right)^2 = {}_2F_1(J_1, -J_2; N + J_1; 1) \tag{5.14}$$

It is possible to transform the hypergeometric differential equation into the Schrödinger equation and then use any of the techniques developed for quantum mechanics. To explore the structure of the $1/N$ expansion of the giant graviton correlators, we will use known results for the exact WKB expansion for the hypergeometric function.

The language of Schur polynomials generalizes to the case of multi matrix models. The Schur polynomials are replaced by restricted Schur polynomials. For concreteness focus on restricted

Schur polynomials constructed from two complex matrices Z and Y. These restricted Schur polynomials are labeled by three Young diagrams[1] $\chi_{R,(r,s)}(Z,Y)$. For an operator constructed using n Z fields and m Y fields, the Young diagram r has n boxes, s has m boxes and R has $n+m$ boxes[32]. A giant graviton operator would be given by the restricted Schur polynomial $\chi_{A_{n+m},(A_n,A_m)}(Z,Y)$, while a dual giant graviton operator is given by $\chi_{S_{n+m},(S_n,S_m)}(Z,Y)$. The normalized correlator of three giant gravitons is given by[37]

$$\langle \chi_{A_{n_1+m_1},(A_{n_1},A_{m_1})}\chi_{A_{n_2+m_2},(A_{n_2},A_{m_2})}\chi^\dagger_{A_{n_{12}+m_{12}},(A_{n_{12}},A_{m_{12}})}\rangle$$

$$= \sqrt{\frac{(N-n_1-m_1)!(N-n_2-m_2)!}{N!(N-n_{12}-m_{12})!}}\sqrt{\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}} \tag{5.15}$$

where $n_{12}=n_1+n_2$ and $m_{12}=m_1+m_2$. This result is a product of two square root factors. The first factor has the same form as the one matrix result. The second factor is always ≤ 1. To see this, consider the binomial expansion of

$$(1+x)^m = \sum_{k=0}^{m}\binom{m}{k}x^k, \qquad \text{where} \qquad \binom{m}{k} = \frac{m!}{k!(m-k)!}$$

By comparing the coefficient of x^{r+s} coming from the expansion of $(1+x)^m$ times the expansion of $(1+x)^n$ to the coefficient of x^{r+s} coming from the expansion of $(1+x)^{m+n}$, we learn that

$$\binom{m}{r}\binom{n}{s} \quad + \quad \text{non negative integers} \quad = \binom{m+n}{r+s}.$$

Thus

$$\frac{\binom{m}{r}\binom{n}{s}}{\binom{m+n}{r+s}} \leq 1,$$

which proves that the second factor is ≤ 1. Notice that when $m_1=m_2=0$, the second factor is identically equal to 1 so that our result correctly reduces to the one matrix result we discussed above. It is equally easy to compute the correlation function for three dual giant gravitons. The result is[37]

$$\langle \chi_{S_{n_1+m_1},(S_{n_1},S_{m_1})}\chi_{(S_{n_2+m_2},(S_{n_2},S_{m_2})}\chi^\dagger_{S_{n_{12}+m_{12}},(S_{n_{12}},S_{m_{12}})}\rangle$$

$$= \sqrt{\frac{(N+n_{12}+m_{12}-1)!(N-1)!}{(N+m_1+n_1-1)!(N+n_2+m_2-1)!}}\sqrt{\frac{n_{12}!m_{12}!(n_1+m_1)!(n_2+m_2)!}{n_1!n_2!m_1!m_2!(n_{12}+m_{12})!}} \tag{5.16}$$

It is again easy to verify that if we set $m_1=0=m_2$, we recover the correct one matrix result.

There are again a number of physical processes described by our correlators (5.15) and (5.16): a three point correlators of point like graviton or of string states, a three point function involving two giant gravitons and one string or point like graviton or a three point correlator involving only

[1] In general we also need some extra multiplicity labels. These labels however will not be needed for the giant graviton correlators we study so they will be omitted from our discussion for simplicity.

giant gravitons. We will only quote the result for a correlator involving three giant gravitons. Setting $n_1 = n_2 = n = N\mathfrak{n}$ and $m_1 = m_2 = N\mathfrak{m}$ we find

$$\langle \chi_{A_{n+m},(A_n,A_m)}\chi_{A_{n+m},(A_n,A_m)}\chi^\dagger_{A_{2n+2m},(A_{2n},A_{2m})}\rangle \sim e^{-2N(\mathfrak{n}+\mathfrak{m})}\sqrt{\frac{2(\mathfrak{n}+\mathfrak{m})}{\pi N\mathfrak{n}\mathfrak{m}}} \tag{5.17}$$

$$\langle \chi_{S_{n+m},(S_n,S_m)}\chi(S_{n+m},(S_n,S_m))\chi^\dagger_{S_{2n+2m},(S_{2n},S_{2m})}\rangle \sim e^{2N(\mathfrak{n}+\mathfrak{m})}\sqrt{\frac{2(\mathfrak{n}+\mathfrak{m})}{\pi N\mathfrak{n}\mathfrak{m}}} \tag{5.18}$$

Both of these correlators again are quantities that can not be expanded in a power series in N^{-1}.

It is also interesting to consider extremal n-point functions since these are also protected. The result is

$$\langle O_{R_1}O_{R_2}\cdots O_{R_k}O_T^\dagger\rangle = g_{R_1R_2\cdots R_kT}\sqrt{\frac{f_T}{f_{R_1}f_{R_2}\cdots f_{R_k}}} \tag{5.19}$$

where $g_{R_1R_2\cdots R_kT}$ counts how many times T appears in $R_1\otimes R_2\otimes\cdots\otimes R_k$. We will see, in explicit examples considered later, that these correlators also exhibit interesting behavior that is non-perturbative with respect to the $1/N$ expansion.

6. Review of Exact WKB

In this section we will review the exact WKB solution to the Schrödinger equation[19]. This is useful because as we review in the next section, the hypergeometric differential equation can be mapped into the Schrödinger equation with a specific potential. The exact WKB method starts from the usual WKB expansion to write the wave function as an infinite series. The Borel sums of these WKB solutions exhibit parametric Stokes phenomena[24] (see summary of the first two chapters in appendix B), which is a Stokes phenomena in the asymptotic behavior of WKB solutions with a change in the parameters of the problem[1]. The space of parameters can be partitioned into regions by the Stokes graph. The vertices of the Stokes graphs are singular points as well as turning points associated to the Schrödinger equation. The Borel sum of the WKB solutions converge within each face of the Stokes graph, but are discontinuous across the Stokes lines. This parametric Stokes phenomena is nicely captured in Voros coefficients, which describe the relative normalization of wave functions normalized[2] at well chosen distinct points. The whole analysis can be phrased in terms of Stokes' automorphims and Alien derivatives, introduced by Ecalle[38] in his theory of resurgence. This analysis explicates the singularities of the Borel sum and these are the seeds of the non-perturbative contribution to the wave function.

6.1 Orientation

Broadly speaking the collection of ideas that we are drawing on go under the name of resurgence. Since much of the background maybe a little unfamiliar, in this section we will give a very brief overview of the relevant ideas. For very helpful background reading, aimed at theoretical physicists, we suggest the reader consults [1, 39, 40, 41].

Use g to denote the coupling constant. The perturbative expansion of an interesting observable $\mathcal{O}$ will take the form

$$\mathcal{O} = \sum_{n=0}^{\infty} c_n g^n \tag{6.1}$$

Typically the coefficients c_n grow as $n!$ so that this series does not converge, but rather it defines an asymtotic expansion. In this situation, we would typically carry out a Borel resummation. This is a two step process, in which we first perform a Borel transform of the sum and then we perform a Laplace transform. The Borel transform of a given term $\mathcal{B}[g^{n+1}](s) = \frac{s^n}{\Gamma(n+1)}$ so that

$$\mathcal{B}[\mathcal{O}](s) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n)} s^{n-1} \tag{6.2}$$

[1]One could also consider Stokes phenomenon arising as a consequence of changing $\hbar$. This is typically considered when analyzing the Borel resummation of WKB solutions and it is not what we are considering here.

[2]By a wave function normalized at a point x_0, we mean the overall amplitude of the wave function $\psi(x)$ is chosen so that $\psi(x_0) = 1$.

This sum is much better behaved that the original sum and, if it converges in some region it can be used to define a function analytic in s, except possibly at a few singular points in the complex s plane. We can then perform an inverse map of the Borel transform (which is the Laplace transform) to complete the resummation

$$\mathcal{S}_\theta\mathcal{O} = \int_0^{e^{i\theta}\infty} \mathcal{B}[\mathcal{O}](s)e^{-\frac{s}{g}}\, ds \tag{6.3}$$

This Laplace transform is not well defined if singularities of the Borel sum lie on the contour of the s integration. Indeed, the result of the transform becomes ambiguous since it will depend on whether we go above or below the singularity. By slowly increasing θ so that the contour of integration moves past a singularity, we find a jump in the value of the Borel resummed observable. This is nothing but the familiar Stokes phenomenon, signaling a change in the behaviour of the asymptotics of the observable $\mathcal{O}$. A pole in $\mathcal{B}[\mathcal{O}](s)$ would produce a jump proportional to $e^{-\frac{A}{g}}$ where A sets the location of the pole. The form of the jump is reminiscent of an instanton contribution and indeed, it can be reproduced in this way. One finds that A is the classical action of an instanton. This is a rather remarkable claim: by Borel resumming the perturbative series we can learn about non-perturbative phenomena in the problem[3]. Further, it explains how to make sense of the full non-perturbative structure of the problem: the usual perturbative series should be replaced by a trans-series, which takes the form

$$\mathcal{O} = \sum_{n=0}^{\infty} \sigma^n \mathcal{O}^{(n)}(g) \tag{6.4}$$

where $\mathcal{O}^{(n)}(g)$ is the contribution of the n-instanton sector. It takes the form

$$\mathcal{O}^{(n)} = e^{-\frac{nA}{g}} \sum_{m=0}^{\infty} c_m^{(n)} g^m \tag{6.5}$$

These sums are themselves asymptotic and need to be resummed. However the trans-series restores uniqueness to the Laplace transform: although each of the individual sums $\mathcal{O}^{(n)}$ jump as we pass a singularity, the complete sum $\mathcal{O}$ does not. The parameter σ is called a trans-series parameter and its role is to track instanton number.

Our goal is to determine the trans-series expansion for the giant graviton correlators we wrote down in the previous section. This will explain the structure of the large N expansion for these correlators and it will make it clear what the non-perturbative contributions to the correlator are.

[3]This is a result with a lot of history[2, 42, 43, 44, 45, 46] that has recently been understood in a remarkable way [53].

6.2 WKB Solutions

We study the Schrödinger problem

$$\left(-\frac{d^2}{dx^2} + N^2 Q\right)\psi = 0 \tag{6.6}$$

where

$$Q = \sum_{j=0} N^{-j} Q_j(x) \qquad Q_0(x) = \frac{F(x)}{G(x)} \tag{6.7}$$

We assume that $G(x)Q_j(x)$ are polynomials in x. The small parameter N^{-1} plays the role of $\hbar$. As usual, a turning point of the classical motion is given by a zero of $Q_0(x)$. A simple turning point is a simple zero of $Q_0(x)$. The poles of $Q_0(x)$ are singular points of the differential equation (6.6). In the exact WKB analysis the poles and zeros of $Q_0(x)$ will play an important role. The usual WKB ansatz

$$\psi(x) = e^{\int^x dx' S(x',N)} \tag{6.8}$$

leads to a solution of the Schrödinger equation as long as S solves the Riccati equation

$$\frac{dS}{dx} + S^2 = N^2 Q \tag{6.9}$$

To solve (6.9), plug the ansatz

$$S(x,N) = \sum_{j=-1}^{\infty} N^{-j} S_j(x) \tag{6.10}$$

into (6.9) and equate each power of N to zero (see appendix (B.2) for the computation). This process yields

$$S_{-1}^2 = Q_0 \tag{6.11}$$

as well as the following recursion relation

$$S_{j+1} = -\frac{1}{2S_{-1}}\left(\frac{dS_j}{dx} + \sum_{k=0}^{j} S_{j-k}S_k - Q_{j+2}\right) \qquad j = -1, 0, 1, 2, \ldots \tag{6.12}$$

There are two possible solutions for S_{-1}

$$S_{-1}^{(\pm)} = \pm\sqrt{Q_0(x)} \tag{6.13}$$

and hence there are two possible formal series solutions to the Riccati equation

$$S^{(\pm)}(x,N) = \sum_{j=-1}^{\infty} N^{-j} S_j^{(\pm)}(x) \tag{6.14}$$

Here "formal series" means formal Laurent series in N^{-1}. These functions are multivalued and holomorphic on the complex plane once the zeros and poles of $Q_0(x)$ are removed. It is known that the series (6.14) is divergent in general. In the framework of the exact WKB analysis, the Borel resummation of the WKB solution is used to arrive at exact results. It is useful to introduce

$$
\begin{aligned}
S_{\text{odd}}(x,N) &= \frac{1}{2}\left(S^{(+)} - S^{(-)}\right) = \sum_{j=-1}^{\infty} N^{-j} S_{\text{odd},j}(x) \\
S_{\text{even}}(x,N) &= \frac{1}{2}\left(S^{(+)} + S^{(-)}\right) = \sum_{j=0}^{\infty} N^{-j} S_{\text{even},j}(x)
\end{aligned}
\tag{6.15}
$$

An identity that we will use below is

$$
-\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} = S_{\text{even}} \tag{6.16}
$$

All that is needed to prove this identity is a simple application of (6.9). Our two possible solutions are $S = S^{(\pm)} = S_{\text{even}} \pm S_{\text{odd}}$. If we use (6.16) we can simplify our solution as follows

$$
\begin{aligned}
\psi_{\pm}(x) &= e^{\int_{x_0}^{x} dx' (S_{\text{even}}(x',N) \pm S_{\text{odd}}(x',N))} \\
&= e^{\int_{x_0}^{x} dx' \left(-\frac{1}{2}\frac{d}{dx'}\log S_{\text{odd}}(x',N) \pm S_{\text{odd}}(x',N)\right)} \\
&= \sqrt{\frac{S_{\text{odd}}(x_0,N)}{S_{\text{odd}}(x,N)}}\, e^{\pm\int_{x_0}^{x} dx' S_{\text{odd}}(x',N)}
\end{aligned}
\tag{6.17}
$$

Since we can always multiply or divide ψ by a constant[4] we can equally well take

$$
\psi_{\pm}(x) = \frac{1}{\sqrt{S_{\text{odd}}(x,N)}} e^{\pm\int_{x_0}^{x} dx' S_{\text{odd}}(x',N)} \tag{6.18}
$$

Recall that since

$$
S_{\text{odd}} = N\sqrt{Q_0(x)} + O(1) \tag{6.19}
$$

the WKB wave function $\psi(x)$ blows up at the simple zeroes of $Q_0(x)$. This is nothing but the familiar breakdown of the WKB approximation at the turning points of the classical motion.

The Borel sum of the WKB solutions $\psi(x)$ have been studied in [27]. Consider the complex plane of parameters $\mathcal{C}_p$ of the Schrödinger equation. The asymptotic behavior of the solutions to the Schrödinger equation (in x-space) depends on where the parameters take their values on the complex $\mathcal{C}_p$ plane, hence the name "parametric" Stokes phenomenon. These regions are bounded in x-space by Stokes lines. The Stokes line is the integral curve of $\text{Im}(\sqrt{Q_0})dx = 0$, emanating from a turning point. Each Stokes line can either end on a singular point or on a turning point. The complex x-plane is dissected into Stokes regions, each of which is bounded by Stokes curves. The graph formed by taking the singular points and turning points as vertices and the Stokes

[4] This is because (6.6) does not fix the normalization of ψ.

lines as edges is called a Stokes graph. If every edge of the Stokes graph starts on a turning point and ends on a singular point, we say that the graph is non-degenerate. The singularities of the Borel sum (which is our main interest) will lie on these Stokes curves, so it is useful to consider them in detail. Three Stoke's lines meet at each simple turning point. To see this, note that close to a simple turning point at $x = a_0$ we have

$$\sqrt{Q_0} = \sqrt{x - a_0} R_0(x) \tag{6.20}$$

with $R_0(x)$ a polynomial. At the turning point imagine that $R_0(x = a_0) = A_0 e^{i\phi_0}$. Change variables as follows: $(x - a_0) = re^{i\phi}$, hold ϕ fixed and let r vary. In this case

$$dx = dre^{i\phi} \tag{6.21}$$

and thus

$$\sqrt{Q_0} dx = A_0^{\frac{3}{2}} e^{i\phi_0 + i\frac{3}{2}\phi} dr \tag{6.22}$$

The condition for the Stokes line $\mathrm{Im}(\sqrt{Q_0})dx = 0$ becomes

$$\phi = \frac{2}{3}(\pi n - \phi_0) \tag{6.23}$$

There are 3 distinct directions (for $n = 0, 1, 2$) and hence there are 3 Stokes lines meeting at each turning point. We will assume that all of the singular points are poles of order 2 (this is indeed the case of interest to us as we will see in the next section). In this case, an analysis which is very similar to what we just did above, leads to the conclusion that two Stokes lines end on each singular point. To proceed further we should specify the potential which would determine the number of singular points and turning points and hence the details of the Stokes graph. Following this logic, we will characterize the Stokes graph for our problem in Section 7. To complete our review of the exact WKB method, in the next subsection we will describe the jumps in the WKB solution as we pass through a Stokes line.

6.3 Borel Resummation and Voros Coefficients

Under some conditions (that we will spell out below) a suitably normalized WKB solution is Borel summable in each Stoke's region. There are singularities on the edge of each Stoke's region that we would like to identify. This can be accomplished by studying the Stokes phenomena of the WKB resummed solutions across the Stokes curves, since the singularities are the origin of the Stokes phenomenon. If we take a solution to (6.6) and continue it along non-trivial paths in the space of parameters, we find that the solutions transform under a non-trivial monodromy group, which is another way to describe the Stokes phenomenon. One can introduce Voros coefficients, which relate WKB solutions normalized at a turning point to WKB solutions normalized at a singular point. The importance of the Voros coefficients follows because they capture this non-trivial monodromy group, and consequently they provide a complete characterization of the singularities of the Borel sum. Denote the Borel sum of ψ in region D by Ψ^D. Consider $\psi_\pm$ with

x_0 chosen to be a simple turning point $x_0 = a_0$. Consider the solutions $\Psi^I_\pm$ and $\Psi^{II}_\pm$ which are the Borel sums of $\psi_\pm$ in two distinct regions I and II. If $\mathrm{Re}\left(\int_{a_0}^x \sqrt{Q_0}dx\right) > 0$ on the boundary Stokes curve between regions I and II, then we have[19]

$$\begin{aligned} \Psi^I_+ &= \Psi^{II}_+ + i\Psi^{II}_- \\ \Psi^I_- &= \Psi^{II}_- \end{aligned} \tag{6.24}$$

We say that ψ_+ is dominant and ψ_- is recessive on the Stokes curve. The above formulas are called connection formulas and they clearly exhibit the Stokes phenomenon for the dominant solution. The recessive WKB solution does not have Stokes phenomena across the Stokes curves.

Consider WKB solutions normalized at a regular singular point, located at $x = r$. In this case, Q_0 has a double pole at $x = r$ and, to simplify the analysis that follows, we assume that $(x-r)^2 Q_j$ for $j > 0$ are holomorphic at $x = r$. With this assumption it follows that S_{odd} has a simple pole at $x = r$. To define the WKB solution at the regular singular point, we subtract this pole from S_{odd} and handle it analytically on its own. To do this it is useful to introduce the expansion

$$\rho = \rho_0 + N^{-1}\rho_1 + N^{-2}\rho_2 + \dots \tag{6.25}$$

of the function

$$\rho = \mathrm{Res}_{x=r}\ \sqrt{Q} \qquad \text{where} \qquad Q = \sum_{j=0} N^{-j} Q_j \tag{6.26}$$

Proposition 3.6 of [25] computes the residue (this formula assumes that ρ is an even function of N as explained in Appendix A)

$$\mathrm{Res}_{x=r}\ S_{\mathrm{odd}} = \sigma N \tag{6.27}$$

where

$$\sigma = \rho\sqrt{1 + \frac{1}{4\rho^2 N^2}} \tag{6.28}$$

The WKB solutions normalized at the regular singular point $x = r$ are given by

$$\psi^{(r)}_\pm = \frac{(x-r)^{\pm\sigma N}}{\sqrt{S_{\mathrm{odd}}}} e^{\pm\int_r^x \left(S_{\mathrm{odd}} - \frac{\sigma N}{x-r}\right)dx} \tag{6.29}$$

The integrand in the above exponential is free of singularities throughout the integration domain because we have subtracted the pole at $x = r$ from S_{odd}. The factor $(x-r)^{\pm\sigma N}$ upfront comes from an analytic treatment of the pole contribution. This manipulation is performed so that the integrand $S_{\mathrm{odd}} - \frac{\sigma N}{x-r}$ is regular at the singular point, ensuring that the formula (6.29) is well defined.

We will now consider the recessive WKB solution at the regular singular point $x = r$. Assume that $\mathrm{Re}(\rho_0) > 0$. Then $\psi^{(r)}_+$ is recessive on any Stokes curve flowing into $x = r$. By the connection formula, (6.24), the recessive WKB solution does not have Stoke's phenomena on the Stokes curves. We now quote a Theorem from [22]

Theorem 1: Set $\tilde{\psi}_+^{(r)} = (x-r)^{-\frac{1}{2}-\sigma N}\psi_+^{(r)}$. There is a neighborhood U of $x=r$ such that $\tilde{\psi}_+^{(r)}$ is Borel summable in $U-\{r\}$ and $x=r$ is a removable singularity of the Borel sum $\tilde{\Psi}_+^{(r)}$. Hence it is holomorphic in $U \times \{N; \mathrm{Re}(N) \gg 0\}$. Moreover

$$\tilde{\Psi}_+^{(r)}(r,N) = \tilde{\psi}_+^{(r)}(r,N) = (\sigma N)^{-\frac{1}{2}} \tag{6.30}$$

holds.

The significance of this theorem is easy to appreciate: the factor $(x-r)^{\frac{1}{2}+\sigma N}$ does not admit an expansion in $1/N$. The above theorem implies that this factor appears in the WKB solution $\psi_+^{(r)}$, but this is the only non-perturbative contribution and it appears as a multiplicative factor. Indeed, as soon as it is removed (to obtain $\tilde{\psi}_+^{(r)}$) the result is Borel summable. If $\mathrm{Re}(\rho_0) < 0$ we have to exchange $+$ and $-$.

We are now ready to introduce the Voros coefficient[19] which will play an important role in the next section. The Voros coefficient V_j describes the discrepancy between WKB solutions normalized at a turning point a (denoted $\psi_\pm$) and those normalized at a singular point b_j (denoted $\psi_\pm^{(b_j)}$) where j specifies which singular point we consider. The definition is

$$\psi_\pm^{(b_j)} = e^{\pm V_j}\psi_\pm \tag{6.31}$$

This completes our review of the exact WKB solutions. In the next section we apply the method to the hypergeometric differential equation.

7. Application of Exact WKB to Giant Graviton Correlators

We have seen in Section 5 that the normalized three point function of giant graviton correlators can be expressed in terms of the hypergeometric function ${}_2F_1(a,b;c;1)$. In this section we will map the hypergeometric differential equation into a Schrödinger equation. We can then apply the results of the previous section to perform an exact WKB analysis. Under the mapping to the Schrödinger equation, $1/N$ maps to $\hbar$ so that the semi-classical expansion for the Schrödinger equation is the $1/N$ expansion of our correlators. This implies that through this map we are able to understand the structure of the $\frac{1}{N}$ expansion in this large N but non-planar limit.

7.1 Mapping to the Schrödinger Equation

The hypergeometric differential equation is

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0 \tag{7.1}$$

Notice that it has regular singular points at $b_0=0$, $b_1=1$ and $b_2=\infty$. The parameters of the hypergeometric function are

$$a=\frac{1}{2}+\alpha N \qquad b=\frac{1}{2}+\beta N \qquad c=1+\gamma N \tag{7.2}$$

where N is taken to be large. This particular parametrization of a,b,c follows [22] and will simplify many of the formulas that follow. Introduce the wave function ψ as follows

$$\psi=x^{\frac{1}{2}(1+\gamma N)}(1-x)^{\frac{1}{2}(1+(\alpha+\beta-\gamma)N)}w \tag{7.3}$$

Plugging this into (7.1) we find that ψ obeys the following Schrödinger equation

$$\left(-\frac{1}{N^2}\frac{d^2}{dx^2}+Q(x)\right)\psi=0 \tag{7.4}$$

where

$$Q(x)=Q_0(x)+N^{-2}Q_2(x) \tag{7.5}$$

$$Q_0(x)=\frac{(\alpha-\beta)^2x^2+2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2} \tag{7.6}$$

$$Q_2(x)=-\frac{x^2-x+1}{4x^2(x-1)^2} \tag{7.7}$$

An important and non-trivial feature of this mapping is that we see that $\frac{1}{N}$ plays the role of $\hbar$. This is in complete agreement with the usual holographic dictionary between CFT parameters and the parameters of the dual gravity, so one may wonder if this Schrödinger equation has a natural gravitational origin. We will not explore this possibility in this article. Since $Q_0(x)$ is quadratic, there are two turning points $\{a_0, a_1\}$, given by the zeros of the numerator of (7.7).

To properly define the coefficient $S_{-1} = \sqrt{Q_0}$ of the WKB solution, we need to explain what branch of $\sqrt{Q_0}$ we use. The branch cut runs between the two turning points avoiding the singular points b_k. The branch we use is specified by choosing

$$\sqrt{Q_0} \sim \frac{\gamma}{2x} \quad \text{at} \quad x = 0 \tag{7.8}$$

$$\sqrt{Q_0} \sim \frac{\alpha+\beta-\gamma}{2(x-1)} \quad \text{at} \quad x = 1 \tag{7.9}$$

$$\sqrt{Q_0} \sim \frac{\beta-\alpha}{2x} \quad \text{at} \quad x = \infty \tag{7.10}$$

7.2 Stokes Graph

The (complexified) position space with coordinate x on which the wave function is defined is divided up into regions by the Stokes graph. The Stokes graph of (7.1) is the graph drawn on the sphere with vertices given by the turning points $\{a_0, a_1\}$ and the regular singular points $\{b_0, b_1, b_2\}$ and edges given by Stokes lines. The WKB solutions jump discontinuously across the Stokes lines, which is the usual Stokes phenomenon. The Stokes graph of (7.1) is well understood[26]. Since this will be needed in what follows, we review the relevant results of [26] in this section.

The topology of the Stokes graph can change depending on the values of the parameters appearing in the potential. We imagine that the parameters α, β and γ are arbitrary complex numbers taking values on the Riemann sphere $\mathcal{C}_p$. We can divide this space up into regions, such that the topology of the Stokes graph is fixed in each region. Towards this end, introduce the following three sets

$$E_0 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 | \alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma) = 0\}$$

$$E_1 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 | \mathrm{Re}(\alpha)\mathrm{Re}(\beta)\mathrm{Re}(\gamma-\alpha)\mathrm{Re}(\gamma-\beta) = 0\}$$

$$E_2 = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 | \mathrm{Re}(\alpha-\beta)\mathrm{Re}(\alpha+\beta-\gamma)\mathrm{Re}(\gamma) = 0\} \tag{7.11}$$

To get some insight into the definition of the above open sets, note that

$$\alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma) \neq 0 \tag{7.12}$$

is the condition that there are two distinct turning points and further that neither turning points coincides with a singular point. The conditions

$$\mathrm{Re}(\alpha)\mathrm{Re}(\beta)\mathrm{Re}(\gamma-\alpha)\mathrm{Re}(\gamma-\beta) \neq 0 \neq \mathrm{Re}(\alpha-\beta)\mathrm{Re}(\alpha+\beta-\gamma)\mathrm{Re}(\gamma) \tag{7.13}$$

ensure that there is no Stokes curve connecting distinct turning points (the first condition) or the same turning point (the second condition). If turning points are connected by a Stokes curve, the Stokes geometry is said to be degenerate. The conditions under which the Stokes graph is degenerate is summarized in the following theorem

Theorem 2: Assume that (α, β, γ) is not contained in E_0. (i) If two distinct turning points a_0 and a_1 are connected by a Stokes curve, then (α, β, γ) belong to E_1. Conversely, if (α, β, γ) is contained in $E_1 - E_2$, the Stokes geometry of (7.1) has a Stokes curve which connects two distinct turning points a_0 and a_1. (ii) If a Stokes curve forms a closed curve with a single turning point as the base point, then (α, β, γ) belongs to E_2. Conversely if (α, β, γ) is contained in $E_2 - E_1$, the Stokes geometry of (7.1) has a Stokes curve which forms a closed path with a turning point as the base point.

To proceed further we need to define the following sets of parameters

$$
\begin{aligned}
\omega_1 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \mathrm{Re}(\alpha) < \mathrm{Re}(\gamma) < \mathrm{Re}(\beta)\} \\
\omega_2 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \mathrm{Re}(\alpha) < \mathrm{Re}(\beta) < \mathrm{Re}(\gamma) < \mathrm{Re}(\alpha) + \mathrm{Re}(\beta)\} \\
\omega_3 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \mathrm{Re}(\gamma) < \mathrm{Re}(\alpha) < \mathrm{Re}(\beta)\} \\
\omega_4 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \mathrm{Re}(\gamma) < \mathrm{Re}(\alpha) + \mathrm{Re}(\beta) < \mathrm{Re}(\beta)\}
\end{aligned} \tag{7.14}
$$

as well as the involutions

$$
\begin{aligned}
\iota_0 &: (\alpha, \beta, \gamma) \to (-\alpha, -\beta, -\gamma) \\
\iota_1 &: (\alpha, \beta, \gamma) \to (\gamma - \beta, \gamma - \alpha, \gamma) \\
\iota_2 &: (\alpha, \beta, \gamma) \to (\beta, \alpha, \gamma)
\end{aligned} \tag{7.15}
$$

The relevance of these involutions follows because they are symmetries of the potential $Q(x)$, so parameters related by the involution give the same solution. Let G be the group generated by ι_j $j = 0, 1, 2$. G is then a discrete group of symmetries of Q. Define the open subsets

$$
\Pi_h = \bigcup_{r \in G} r(\omega_h) \qquad h = 1, 2, 3, 4 \tag{7.16}
$$

The union of the Π_h covers most of $\mathbb{C}^3$:

$$
\bigcup_{h=1}^{4} \Pi_h = \mathbb{C}^3 - \mathcal{U} \tag{7.17}
$$

where

$$\begin{aligned}\mathcal{U} = \{(\alpha, \beta, \gamma) | &\mathrm{Re}(\alpha)\mathrm{Re}(\beta)\mathrm{Re}(\gamma) \\ &\times \mathrm{Re}(\gamma - \alpha)\mathrm{Re}(\gamma - \beta)\mathrm{Re}(\alpha - \beta)\mathrm{Re}(\alpha + \beta - \gamma) = 0\}\end{aligned} \tag{7.18}$$

The topological structure of the Stokes graph can be summarized by the triple of integers (n_0, n_1, n_2) where n_j counts how many Stokes curves flow into the regular singular point b_j. The topological structure of the Stokes graph is summarized in the following theorem

Theorem 3: Let $\hat{n} = (n_0, n_1, n_2)$ denote the order sequences of the Stokes graph with parameters (α, β, γ). If $(\alpha, \beta, \gamma) \in \Pi_1$ then $\hat{n} = (2, 2, 2)$. If $(\alpha, \beta, \gamma) \in \Pi_2$ then $\hat{n} = (4, 1, 1)$. If $(\alpha, \beta, \gamma) \in \Pi_3$ then $\hat{n} = (1, 4, 1)$. If $(\alpha, \beta, \gamma) \in \Pi_4$ then $\hat{n} = (1, 1, 4)$.

7.3 Voros Coefficients

From the discussion in Section 6.3, it is straightforwards to see that the Voros coefficient accounting for the discrepancy between the WKB solutions normalized at turning point a and those normalized at singular point b_k are given by

$$V_k(\alpha, \beta, \gamma) = \int_{b_k}^{a} (S_{\mathrm{odd}} - NS_{-1})dx \tag{7.19}$$

The residues of S_{odd} and NS_{-1} at the singular points coincide which implies that the $V_k(\alpha, \beta, \gamma)$ are well defined and that we can develop a formal power series in N^{-1}. The explicit power series are[23]

$$\begin{aligned}
V_0 &= -\frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n N^{1-n}}{n(n-1)} \left[(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} + \frac{1}{(\gamma-\alpha)^{n-1}} + \frac{1}{(\gamma-\beta)^{n-1}}\right) + \frac{2}{\gamma^{n-1}}\right] \\
V_1 &= \frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n N^{1-n}}{n(n-1)} \left[(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma-\alpha)^{n-1}} - \frac{1}{(\gamma-\beta)^{n-1}}\right) + \frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right] \\
V_2 &= \frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n N^{1-n}}{n(n-1)} \left[(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}} - \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma-\alpha)^{n-1}} + \frac{1}{(\gamma-\beta)^{n-1}}\right) - \frac{2}{(\beta-\alpha)^{n-1}}\right]
\end{aligned} \tag{7.20}$$

where B_n are the Bernoulli numbers defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \tag{7.21}$$

Noting the asymptotic growth of the Bernoulli numbers

$$B_{2k} \sim 4\left(\frac{k}{\pi e}\right)^{2k} \sqrt{\pi k} \tag{7.22}$$

it is clear that the series expansions given above are asymptotic series. The Borel transforms of the above series are well defined and are given by[26]

$$\mathcal{B}[V_0](y) = -\frac{1}{4}\left[g_1(\alpha;y) + g_1(\beta;y) + g_1(\gamma-\alpha;y) + g_1(\gamma-\beta;y)\right] + g_0(\gamma;y)$$
$$\mathcal{B}[V_1](y) = \frac{1}{4}\left[-g_1(\alpha;y) - g_1(\beta;y) + g_1(\gamma-\alpha;y) + g_1(\gamma-\beta;y)\right] + g_0(\alpha+\beta-\gamma;y)$$
$$\mathcal{B}[V_2](y) = \frac{1}{4}\left[-g_1(\alpha;y) + g_1(\beta;y) + g_1(\gamma-\alpha;y) - g_1(\gamma-\beta;y)\right] - g_0(\beta-\alpha;y) \qquad (7.23)$$

where

$$g_0(t;y) = \frac{1}{y}\left(\frac{1}{e^{\frac{y}{t}}-1} + \frac{1}{2} - \frac{t}{y}\right) \qquad (7.24)$$
$$g_1(t;y) = \frac{1}{e^{\frac{y}{2t}}-1} + \frac{1}{e^{\frac{y}{2t}}+1} - \frac{2t}{y} \qquad (7.25)$$

These functions have singularities which signals both the Stokes phenomenon of the asymptotic series and non-perturbative behaviour in the field theory. Both functions have simple poles at $y = 2tm\pi i$ with m any nonzero integer. The residues of these poles are

$$\underset{y=2tm\pi i}{\text{Res}}\, g_0(t;y) = \frac{1}{2\pi mi} \qquad \underset{y=2tm\pi i}{\text{Res}}\, g_1(t;y) = \frac{(-1)^m}{\pi mi} \qquad (7.26)$$

The results can be used to compute alien derivatives and the Stokes automorphims for the WKB solutions. The interested reader can find a clear readable account in [26].

7.4 Trans-series Expansion of Giant Graviton Three Point Function

Our primary goal in this section is to relate the Borel sum of the WKB solution to ${}_2F_1(a,b;c;x)$ near $x=1$. Our approach is based on the study [22] which established the relationship between the Borel sum of the WKB solution to ${}_2F_1(a,b;c;x)$ near $x=0$. Specifically we will study the leading contribution to the WKB solution and show that it reproduces the leading behavior of the correlator, which is non-perturbative in $1/N$. The relation between the hypergeometric function and the WKB solution normalized at $x=0$ is[22]

$$F(\frac{1}{2}+\alpha N, \frac{1}{2}+\beta N, 1+\gamma N; x) = \sqrt{\frac{\gamma}{2}}N^{1/2}e^{-Nh_0}x^{-\frac{1}{2}(1+\gamma N)}(1-x)^{-\frac{1}{2}-\frac{\alpha+\beta-\gamma}{2}N}$$
$$\times \frac{1}{\sqrt{S_{odd}}}\exp\left[N\int_0^x S_{-1}dx + \int_0^x (S_{odd} - NS_{-1})dx\right] \qquad (7.27)$$

Note that we have normalized the wave function using the value of the hypergeometric function at $x=0$, i.e. $F(a,b;c;x=0)=1$. For the leading order at large N, we only need the first

other contexts too. In the recent article [47] non–perturbative interpolating functions to probe the physics of the cusp and twist-two anomalous dimensions were constructed. Finite N $\mathcal{N}=4$ SYM is expected to be S-duality invariant. To probe this physics, [47] accounts for both non–planar and instanton contributions by constructing modular invariant interpolating functions. At the two ends the anomalous dimensions scale as $\sim\sqrt{\lambda}$ or $\sim\lambda^{\frac{1}{4}}$. The cusp anomalous dimension emerges in the large spin limit while the twist two operators are considered in the small spin limit. The two descriptions are long spinning "spiky" strings versus small circular strings. It again seems natural to guess there is a Stokes line separating these distinct saddles that must be crossed as the spin varies.

In the usual planar limit, the $1/N$ expansion has a compelling physical interpretation [30]. The Feynman diagram expansion is in terms of ribbon graphs. The small parameter of the expansion is $\frac{1}{N^2}$ and the power of N multiplying a given term has a nice interpretation as the genus of the worldsheet corresponding to the ribbon graph. Is there an interpretation for the series multiplying the non-perturbative term in (8.1)? An approach towards this problem is suggested by recalling that the giant gravitons are spherical D3 branes and their excitations can be described in terms of open strings. The non-perturbative factor in (8.1) is naturally associated to the spherical D3 brane[1], while the series multiplying this factor is naturally associated to the open string theory living on the giant graviton. From this point of view, powers of N would be associated to the genus of world sheets for the open strings. This provides a natural explanation of why the perturbative factor is a series in $\frac{1}{N}$ and not $\frac{1}{N^2}$. We hope to further develop this point of view.

The mapping to the Schrödinger equation has allowed us to find the form of the $\frac{1}{N}$ expansion and to argue that the asymptotic series coming from the $\frac{1}{N}$ expansion of giant graviton correlators is Borel summable. If all that we are interested in is the form of the $\frac{1}{N}$ expansion, then because our correlators are ratios of $\Gamma(\cdot)$ functions, we can simply use the known asymptotic expansion

$$\Gamma(z)=e^{-z}z^{z-\frac{1}{2}}\sqrt{2\pi}\left(1+\frac{1}{12z}+\frac{1}{288z^2}+\cdots\right) \tag{8.2}$$

Plugging this into the giant graviton correlators we easily recover the form we obtained from the WKB analysis. Repeating this logic, we find the following form for the expansion of the general giant and dual giant extremal correlators

$$\langle O_{A_{J_1}}\cdots O_{A_{J_k}}O^\dagger_{A_{J_1+J_2+\cdots+A_k}}\rangle=e^{-N\sum_{i>j=1}^{k-1}j_ij_i}\sum_{n=0}^{\infty}c_nN^{-n+k-1} \tag{8.3}$$

$$\langle O_{S_{J_1}}\cdots O_{S_{J_k}}O^\dagger_{S_{J_1+J_2+\cdots+A_k}}\rangle=e^{N\sum_{i>j=1}^{k-1}j_ij_i}\sum_{n=0}^{\infty}c_nN^{-n+k-1} \tag{8.4}$$

Given our experience with the three point functions, we conjecture that these series will be Borel summable in the physically allowed range of parameters, which is $j_i>0$ for $i=1,2,\cdots,k$ and $j_1+\cdots j_k\le 1$ for the giant gravitons. It would be interesting to explicitly prove this.

[1]It is naturally interpreted as the exponential of the D3 brane action. The D3 action is multiplied by the tension of a D-brane which behaves like $\sim\frac{1}{g_s}\sim N$ since $g_s\sim\frac{1}{N}$.

We have considered general extremal correlators between giant gravitons and between dual giant gravitons. The complete class of extremal correlation functions of Schur polynomials is much more general. It would also be interesting to study correlators involving operators with a dimension of order N^2. These would have a gravitational interpretation in terms of physics in an LLM geometry[13], so that one is probing a back reacted version of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. It is interesting to ask what the structure of the large N expansion in this case is? Once again the extremal correlation functions can be evaluated exactly. In simple examples [48] for well chosen backgrounds, the only effect on the extremal correlators is a renormalization $N \to N_{\rm eff}$. The perturbative expansion in this LLM background becomes an expansion in $\frac{1}{N_{\rm eff}^2}$, which suggests that the closed string coupling constant g_s has been renormalized. This same effect has also been observed beyond the half-BPS sector [49, 50, 51, 52]. Does this renormalization of N persist when non-perturbative corrections are considered? This could be probed by studying giant graviton correlators in the LLM background. If the closed string coupling is renormalized, then the tension of the D3-brane $\sim \frac{1}{g_s}$ should be renormalized and we do expect the renormalization of N to persist.

The true power of resurgence only comes into play when we have many non-perturbative sectors as well as a perturbative sector. Resurgence then relates the series in these different sectors (see [53] for recent results and references). Extremal giant graviton and dual giant graviton correlators have only a single sector. Further, the form of our extremal correlators makes it likely that we need to go beyond the half-BPS sector for correlators that have more than a single sector.

Finally, it maybe worth rexamining the "analytic bootstrap" for the exact WKB method, formulated in [19]. The method considers WKB periods, which are (Borel resummed) perturbative series in $\hbar$. These periods are determined by their classical limit and their discontinuity structure, which is encoded in the Stokes automorphisms. This data defines a Riemann-Hilbert problem, which can be solved in terms of a TBA-like system, uncovering a remarkable correspondence between ordinary differential equations and integrable models[54, 55]. It would be interesting to revisit the analytic bootstrap, considering the role of discontinuities associated with the parametric Stokes phenomenon.

AppendixA. Residue of S_{odd}

To determine the WKB solution normalized at the regular simgular point $x = 1$, we need to evaluate the resdiue of S_{odd} at $x = 1$. This has been carried out in detail in [25] - see Proposition 3.6. Here we will give a quick summary of the argument, both to make the paper self contained and to stress the differences between our case and the case of [25]. We will make use of the Ricatti equation (6.9) which we repeat for convenience

$$\frac{dS}{dx} + S^2 = N^2 Q \tag{A.1}$$

First, note that

$$\sqrt{Q(x)} = \frac{\sqrt{F(x)}}{2x(x-1)} \tag{A.2}$$

with

$$F(x) = (\alpha - \beta)^2 x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2 - N^{-2}\left(x^2 - x + 1\right) \tag{A.3}$$

It is trivial to see that

$$\mathrm{Res}_{x=1}\ \sqrt{Q(x)} = \frac{\sqrt{(\alpha-\beta)^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma) + \gamma^2 - N^{-2}}}{2} \equiv \rho(N) \tag{A.4}$$

The formal solution to the Riccati equation is given by a power sum

$$S(x, N) = \sum_{j=-1}^{\infty} S_j(x) N^{-j} \tag{A.5}$$

Plugging this sum into the Riccati equation leads to (6.11) and (6.12). From (6.11) we see that $S_{-1}(x)$ has a pole of order 1 at $x = 1$, while from (6.12) we see that $S_j(x)$ with $j \geq 0$ has a pole at $x = 1$. Consequently, for all j we have the Laurent expansions

$$S_j(x) = \frac{f_{j,-1}}{x-1} + \sum_{n \geq 0} f_{j,n}(x-1)^n \tag{A.6}$$

Thus, the residue of $S(x, N)$ at $x = 1$ is given by

$$\mathrm{Res}_{x=1}\, S(x, N) = \sum_{j=-1}^{\infty} f_{j,-1} N^{-j} \tag{A.7}$$

To perform the sum on the right hand side, plug (A.6) into (A.5), and then plug (A.5) into the Riccati equation. Equating the coefficient of $(x-1)^{-2}$ to zero, we find

$$-\sum_{j=-1}^{\infty} f_{j,-1} N^{-j} + \left(\sum_{j=-1}^{\infty} f_{j,-1} N^{-j}\right)^2 = N^2 \rho(N)^2 \tag{A.8}$$

This quadratic equation is easily solved to obtain

$$\sum_{j=-1}^{\infty} f_{j,-1} N^{-j} = \frac{1}{2} + N\rho(N)\sqrt{1 + \frac{1}{4N^2\rho(N)^2}} = \mathrm{Res}_{x=1}\, S(x,N) \tag{A.9}$$

For the residue of S_{odd} we need to extract the odd powers of N which gives

$$\mathrm{Res}_{x=1}\, S_{\mathrm{odd}}(x,N) = \frac{1}{2}\left(N\rho(N)\sqrt{1 + \frac{1}{4N^2\rho(N)^2}} + N\rho(-N)\sqrt{1 + \frac{1}{4N^2\rho(-N)^2}}\right) \tag{A.10}$$

For even $\rho(N)$ we recover the result of [25] which says

$$\mathrm{Res}_{x=1}\, S_{\mathrm{odd}}(x,N) = N\rho(N)\sqrt{1 + \frac{1}{4N^2\rho(N)^2}} \tag{A.11}$$

AppendixB. Parametric Stokes phenomena of the Gauss hypergeometric differential equation with a large parameter

B.1 Introduction

In this Appendix we will review some of the relevant ideas and results of the paper [24]. The main idea of this paper is to exhibit parametric Stokes phenomena of the Gauss hypergeometric differential equation with a large parameter, in terms of a WKB analysis. The hypergeometric differential equation can be mapped into the Schrödinger equation with a specific potential. The Borel sums of these WKB solutions exhibit parametric Stokes phenomenon, which is a Stokes phenomena in the asymptotic behaviour of WKB solutions with a change in the parameters. The space of parameters are bounded by Stokes lines. To analyse the Stokes phenomena of WKB solutions with respect to parameters in differential equations, the notion of Voros coefficients have been introduced. Differents forms of Voros coefficients for the Weber equations have been computed by Shen and Silverstone[56] and Takei[57].

The paper show that Voros coefficients can be defined for the Gauss hypergeometric equation [57, 58] with somes modifications. The system of difference equations that characterise these coefficients was computed. This system can be solved using the method of formal differential operators of infinite order (Candelpergher, Coppo and Delabaere, [59]). One finds that Voros coefficients undertake such Stokes phenomena as exhibited through the Borel resummation method. To analyze Stoke phenomena of Voros coeffients, it is also important to know how the Stokes graphs depends on the parameters. For the equation we study it is possible to give a characterisation of the types of Stokes graph in terms of the parameters.

B.2 The Gauss hypergeometric differential equation with a large parameter

Consider the following Schrodinger-type equation

$$\left(-\frac{d^2}{dx^2} + N^2 Q\right)\Psi = 0 \tag{B.1}$$

Here N is a large parameter and α, β and γ are complex parameters,

$$Q(x) = Q_0(x) + N^{-2} Q_1(x) \tag{B.2}$$

with

$$Q_0(x) = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2} \tag{B.3}$$

and

$$Q_1(x) = \frac{-x^2 - x + 1}{4x^2(x-1)^2} \tag{B.4}$$

Consider the hypergeometric differential equation

$$x(1-x)\frac{d^2\omega}{dx^2} + [c - (a+b+1)x]\frac{d\omega}{dx} - ab\omega = 0 \tag{B.5}$$

where a, b and c are complex parameters. Introduce a large parameter N by setting $a = \frac{1}{2} + N\alpha$, $b = \frac{1}{2} + N\beta$ and $c = 1 + N\gamma$ where α, β and γ are complex parameters. We obtain

$$x(1-x)\frac{d^2\omega}{dx^2} + [1 + \gamma n - ((\alpha+\beta)N + 2)x]\frac{d\omega}{dx} - \left(\frac{1}{2} + N\alpha\right)\left(\frac{1}{2} + N\beta\right)\omega = 0 \tag{B.6}$$

Next we eliminate the first-order term of (B.6) by taking the unknown function

$$\Psi = x^{(1+\gamma\eta)/2}(1-x)^{(1+(\alpha+\beta-\gamma)\eta)/2}\omega \tag{B.7}$$

Then we have (B.1). The WKB solutions of (B.1) is

$$\Psi = \exp\left(\int_{x_0}^{x} S(x,N)dx\right) \tag{B.8}$$

where S solves the Riccati equation

$$S^2 + \frac{dS}{dx} = N^2Q(x) \tag{B.9}$$

associated with (B.1). Let

$$S = S_{\text{odd}} + S_{\text{even}} \tag{B.10}$$

where

$$S_{\text{odd}} = \sum_{k=0}^{\infty} N^{-2k+1}S_{2k-1}(x) \tag{B.11}$$

$$= S_{-1}N^1 + S_1N^{-1} + S_3N^{-3} + \cdots \tag{B.12}$$

and

$$S_{\text{even}} = \sum_{k=0}^{\infty} N^{-2k}S_{2k}(x) \tag{B.13}$$

$$= S_0 + S_2N^{-2} + S_4N^{-4} + \cdots \tag{B.14}$$

Then (B.9) implies

$$(S_{\text{odd}} + S_{\text{even}})^2 + \frac{d}{dx}(S_{\text{odd}} + S_{\text{even}}) = N^2 Q(x) \tag{B.15}$$

$$S_{\text{odd}}^2 + S_{\text{even}}^2 + 2S_{\text{odd}}S_{\text{even}}\frac{d}{dx}(S_{\text{odd}} + S_{\text{even}}) = N^2 Q(x) \tag{B.16}$$

Using the terms of odd powers of N^{-1} from both sides, we find

$$2S_{\text{odd}}S_{\text{even}} + \frac{d}{dx}S_{\text{odd}} = 0 \tag{B.17}$$

Thus,

$$S_{\text{even}} = \frac{-\frac{d}{dx}S_{\text{odd}}}{2S_{\text{odd}}} \tag{B.18}$$

$$= -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} \tag{B.19}$$

Using (B.10) the two possible solutions are

$$S = S^{\pm} = S_{\text{even}} \pm S_{\text{odd}} \tag{B.20}$$

Substituting (B.18) in (B.20) yields

$$S = S^{\pm} = -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} \pm S_{\text{odd}} \tag{B.21}$$

Using (B.21) the WKB solutions ((B.8)) can be rewritten as

$$\begin{aligned}
\Psi_{\pm} &= \exp\left\{\int_{x_0}^{x}\left(-\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} \pm S_{\text{odd}}\right)dx\right\} \\
&= \exp\left\{-\frac{1}{2}\log S_{\text{odd}}(x,\eta)\Big|_{x_0}^{x} \pm \int_{x_0}^{x} S_{\text{odd}}dx\right\} \\
&= \exp\left\{\log S_{\text{odd}}(x_0,\eta)^{1/2} - \log S_{\text{odd}}(x,\eta)^{1/2} \pm \int_{x_0}^{x} S_{\text{odd}}dx\right\} \\
&= \exp\log\sqrt{\frac{S_{\text{odd}}(x_0,\eta)}{S_{\text{odd}}(x,\eta)}} \times \exp\left(\pm\int_{x_0}^{x} S_{\text{odd}}dx\right)
\end{aligned} \tag{B.22}$$

Then the WKB solution of (B.1) can be expressed as

$$\Psi_{\pm} = \sqrt{\frac{S_{\text{odd}}(x_0,N)}{S_{\text{odd}}(x,N)}} \times \exp\left(\pm\int_{x_0}^{x} S_{\text{odd}}dx\right) \tag{B.23}$$

By dividing $\Psi_{\pm}$ by $\sqrt{S_{\text{odd}}(x_0,N)}$, we end up with

$$\Psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}(x,N)}}\exp\left(\pm\int_{x_0}^{x} S_{\text{odd}}dx\right) \tag{B.24}$$

where x_0 is a fixed point. Note that each S is holomorphic except at a zero point (which is called a turning point of (B.1)) and a singular point of Q_0. A Stokes curve emanating from the turning point $a_h(h = 0, 1)$ is a curve defined by

$$Im \int_{a_h}^{x} \sqrt{Q_0} dx = 0 \tag{B.25}$$

A Stokes curve flows into a singular point or a turning point. We say that the Stokes geometry of (B.1) is non-degenerate if any Stokes curve does not flow into a turning point. A Stokes graph of (B.1) can be defined as a two-colored sphere graph consisting of all Stokes curves (emanating from a_0 and a_1) as edges, $\{a_0, a_1\}$ as vertices of the first color and $\{b_0, b_1, b_2\}$ as vertices of the second color [65] where $b_0 = 0$, $b_1 = 1$ and $b_2 = \infty$ are the singular points of (B.1).

Let us define the sets $E_j(j = 0, 1, 2)$ of the parameters α, β, γ as follows:

$$E_0 = \{(\alpha, \beta, \gamma) \in \mathcal{C}^3 \backslash \alpha \cdot \beta \cdot \gamma \cdot (\alpha - \beta) \cdot (\alpha - \gamma) \cdot (\beta - \gamma) \cdot (\alpha + \beta - \gamma) = 0\} \tag{B.26}$$
$$E_1 = \{(\alpha, \beta, \gamma) \in \mathcal{C}^3 \backslash Re\alpha \cdot Re\beta \cdot Re(\gamma - \alpha) \cdot (\gamma - \beta) = 0\}, \tag{B.27}$$
$$E_2 = \{(\alpha, \beta, \gamma) \in \mathcal{C}^3 \backslash Re(\alpha - \beta) \cdot Re(\alpha + \beta - \gamma) \cdot Re\gamma = 0\} \tag{B.28}$$

If (α, β, γ) is not contained in E_0, the two distinct turning points a_0, a_1 and the singular points are mutualy distinct. If (α, β, γ) is contained in the set $E_1 \cup E_2$, the Stokes geometry is degenerate. Assume that (α, β, γ) is not contained in the sets $E_0 \cup E_1 \cup E_2$. The topological type of Stokes graph is characterized by its order sequence $\hat{n} = (n_0, n_1, n_2)$, where n_j are the number of Stokes curves that flow into 0, 1 and ∞ respectively. For the equation we consider $n_0 + n_1 + n_2 = 6$.

B.3 Voros Coefficients

Here we assume that (α, β, γ) is not contained in the set $E_0 \cup E_1 \cup E_2$ and choose a closed path $C_j(j = 0, 1, 2)$ going around a turning point a with the base point $b_j(j = 0, 1, \infty)$ in a counterclockwise direction . The following integrals, which are called Voros coefficients can be defined:

$$V_0 = V_0(\alpha, \beta, \gamma; N) := \frac{1}{2} \int_0^a (S_{\text{odd}} - NS_{-1}) dx \tag{B.29}$$

$$V_1 = V_1(\alpha, \beta, \gamma; N) := \frac{1}{2} \int_1^a (S_{\text{odd}} - NS_{-1}) dx \tag{B.30}$$

$$V_2 = V_2(\alpha, \beta, \gamma; N) := \frac{1}{2} \int_\infty^a (S_{\text{odd}} - NS_{-1}) dx \tag{B.31}$$

These integrals are well-defined for every homotopy class of the path of integration since the residues of S_{odd} and NS_{-1} coincide [25]. Note that V_0, V_1 and V_2 are independent of the choice of the turning point a. We can then define a Voros coefficient as a formal power series of N^{-1} defined by the following integral

$$V_l = V_l(\alpha, \beta, \gamma; \eta) := \frac{1}{2} \int_l^a (S_{\text{odd}} - \eta S_{-1}) dx \tag{B.32}$$

where l is a singular point of (B.1) and a its turning point. Let

$$\Psi_{\pm}^{(j)} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{b_j}^{x} (S_{\text{odd}} - \eta S_{-1}) dx \pm \eta \int_{a}^{x} S_{-1} dx \right) \tag{B.33}$$

be the WKB solutions normalized at the singular point b_j. For $j = 0, 1$ and 2, $V_j(\alpha, \beta, \gamma; N)$ describe the discrepency between WKB solutions normalized at a and those normalized at singular points $b_0 = 0$, $b_1 = 1$ and $b_2 = \infty$ respectively, that is, when we set

$$\Psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{a}^{x} S_{\text{odd}} dx \right) \tag{B.34}$$

and

$$\Psi_{\pm}^{(j)} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{b_j}^{x} (S_{\text{odd}} - \eta S_{-1}) dx \pm \eta \int_{a}^{x} S_{-1} dx \right) \tag{B.35}$$

we have

$$\Psi_{\pm}^{(j)} = \exp(\pm V_j) \Psi_{\pm} \Rightarrow \Psi_{\pm} = \exp(\mp V_j) \Psi_{\pm}^{(j)} \tag{B.36}$$

Next, we need the explicit forms of V_j. For this purpose, we need to specify the branch of $S_{-1}(x) = \sqrt{Q_0}$ precisely. Consider the case where (α, β, γ) is contained in the set ω where ω is as follows:

$$\omega = \{(\alpha, \beta, \gamma) \in \mathcal{C}^3 \backslash 0 < Re\alpha < Re\beta < Re\gamma < Re\alpha + Re\beta\} \tag{B.37}$$

We consider

$$Q_0(0.5, 1, 1; x) = \frac{(x-2)^2}{(4x(x-1))^2} \tag{B.38}$$

where we take

$$(\alpha, \beta, \gamma) = (0.5 + \delta' i, 1 - \epsilon - \delta i, 1) \tag{B.39}$$

in ω_2 (δ', ϵ and δ are small positive numbers). The branch of $\sqrt{Q_0}$ is chosen by taking a segment connecting the turning points as branch cut for $\sqrt{Q_0}$ so that

$$\sqrt{Q_0} \sim \frac{\beta - \alpha}{2x} \tag{B.40}$$

as $x \to \infty$. In this case, near 0 and 1, $\sqrt{Q_0}$ takes the form

$$\sqrt{Q_0} \sim \frac{\gamma}{2x} \tag{B.41}$$

and

$$\sqrt{Q_0} \sim \frac{\alpha + \beta - \gamma}{2(x-1)} \tag{B.42}$$

respectively. Using the above choice of the branch of $\sqrt{Q_0}$, we can write the explicit forms of the Voros coefficients as follows:

Theorem The Voros coefficients V_j have the following forms:

$$V_0(\alpha,\beta,\gamma;N) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n N^{1-n}}{n(n-1)} \times \left[(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right] \tag{B.43}$$

$$V_1(\alpha,\beta,\gamma;N) = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n N^{1-n}}{n(n-1)} \times \left[(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right] \tag{B.44}$$

$$V_2(\alpha,\beta,\gamma;N) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n N^{1-n}}{n(n-1)} \times \left[(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)-\frac{2}{(\beta-\alpha)^{n-1}}\right] \tag{B.45}$$

where B_n are the Bernoulli numbers defined by

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n \tag{B.46}$$

In what follows we will derive (B.43). Apply the ladder operator for the hypergeometric differential equation in (B.5):

$$H_1(a,b,c) = x\frac{d}{dx}+a : S(a,b,c) \to S(a+1,b,c) \tag{B.47}$$

$$H_2(a,b,c) = x\frac{d}{dx}+b : S(a,b,c) \to S(a,b+1,c) \tag{B.48}$$

$$B_3(a,b,c) = x\frac{d}{dx}+c : S(a,b,c+1) \to S(a,b,c) \tag{B.49}$$

where $S(a,b,c)$ denotes the solution space of (B.5). These operators yields the following difference equations:

$$S(\alpha+\eta^{-1},\beta,\gamma;x,\eta) - S(\alpha,\beta,\gamma;x,\eta) = -\frac{1}{2(1-x)}+\frac{d}{dx}\log\left\{-\frac{1}{2}\gamma\eta+\frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)\eta)+xS(\alpha,\beta,\gamma;x,\eta)+\alpha\eta\right\} \tag{B.50}$$

$$S(\alpha,\beta+N^{-1},\gamma;x,N)-S(\alpha,\beta,\gamma;x,N)$$
$$=-\frac{1}{2(1-x)}+\frac{d}{dx}\log\left\{-\frac{1}{2}\gamma N+\frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+xS(\alpha,\beta,\gamma;x,N)+\beta N\right\} \tag{B.51}$$

$$S(\alpha,\beta,\gamma+N^{-1};x,N)-S(\alpha,\beta,\gamma;x,N)$$
$$=\frac{1}{2(1-x)}+\frac{1}{2x}-\frac{d}{dx}\log\left\{\frac{1}{2}\gamma N+\frac{x}{2(1-x)}(\alpha+\beta-\gamma)N+xS(\alpha,\beta,\gamma;x,N)\right\} \tag{B.52}$$

Now, we will prove (B.50). Using the change of variable $a=\frac{1}{2}+N\alpha$, $b=\frac{1}{2}+N\beta$ and $c=1+N\gamma$ in (B.6) we obtain

$$H_1\left(\frac{1}{2}+N\alpha,\frac{1}{2}+N\beta,1+N\gamma\right)=x\frac{d}{dx}+\frac{1}{2}+N\alpha:S(\alpha,\beta,\gamma;N)\to S(\alpha+N^{-1},\beta,\gamma;N) \tag{B.53}$$

Let $\tau(\alpha,\beta,\gamma;x,N)$ be a solution of the Riccati equation (B.9). Then we have

$$x(1-x)\left(\frac{d\tau}{dx}+\tau^2\right)+[1+\gamma N-((\alpha+\beta)N+2)x]\tau-\left(\frac{1}{2}+N\alpha\right)\left(\frac{1}{2}+N\beta\right)\omega=0 \tag{B.54}$$

Let $\hat{\tau}$ be the logarithmic derivative of

$$\left(x\frac{d}{dx}+\alpha N+\frac{1}{2}\right)e^{\int\tau dx}=\left(x\tau+\alpha N+\frac{1}{2}\right)e^{\int\tau dx} \tag{B.55}$$

where

$$\hat{\tau}=\tau+\frac{d}{dx}\log\left(x\tau+\alpha N+\frac{1}{2}\right) \tag{B.56}$$

Replace α by $\alpha+N^{-1}$ in (B.54):

$$x(1-x)\left(\frac{d\tau}{dx}+\tau^2\right)+[1+\gamma N-((\alpha+N^{-1}+\beta)N+2)x]\tau-\left(\frac{1}{2}+N(\alpha+N^{-1})\right)\left(\frac{1}{2}+N\beta\right)\omega=0 \tag{B.57}$$

Since τ is a solution of (B.54), $\hat{\tau}=\tau+\frac{d}{dx}\log\left(x\tau+\alpha N+\frac{1}{2}\right)$ is also a solution of (B.57). If S is a formal solution of (B.9),

$$\tau=S-\frac{1+\gamma N}{2x}+\frac{1+(\alpha+\beta-\gamma)N}{2(1-x)} \tag{B.58}$$

is a formal solution of (B.54) and

$$\hat{S}=\hat{\tau}+\frac{1+\gamma N}{2x}-\frac{1+(\alpha+\beta-\gamma)N}{2(1-x)} \tag{B.59}$$

is a formal solution of the equation obtained from (B.9) after replacing α by $\alpha + N^{-1}$. Then,

$$\hat{S} = S(\alpha + N^{-1}, \beta, \gamma; x, N) \tag{B.60}$$

Combining (B.59) and (B.60) we have

$$S(\alpha + N^{-1}, \beta, \gamma; x, N) = \hat{\tau} + \frac{1 + \gamma N}{2x} - \frac{1 + (\alpha + \beta - \gamma)N}{2(1 - x)} \tag{B.61}$$

Thus,

$$S(\alpha + N^{-1}, \beta, \gamma; x, N) - S(\alpha, \beta, \gamma; x, N) = \hat{\tau} + \frac{1 + \gamma N}{2x} - \frac{1 + (\alpha + \beta - \gamma)N}{2(1 - x)} - S(\alpha, \beta, \gamma; x, N) \tag{B.62}$$

Inserting (B.56) into (B.62) yields

$$\begin{aligned} &S(\alpha + \eta^{-1}, \beta, \gamma; x, \eta) - S(\alpha, \beta, \gamma; x, \eta) = \\ &\tau + \frac{d}{dx} \log\left(x\tau + \alpha\eta + \frac{1}{2}\right) + \frac{1 + \gamma\eta}{2x} - \frac{1 + (\alpha + \beta - \gamma)\eta}{2(1 - x)} - S(\alpha, \beta, \gamma; x, \eta) \end{aligned} \tag{B.63}$$

Substituting (B.58) in the last equation above, we have

$$\begin{aligned} &S(\alpha + \eta^{-1}, \beta, \gamma; x, \eta) - S(\alpha, \beta, \gamma; x, N) \\ &= S(\alpha, \beta, \gamma; x, N) - \frac{1 + \gamma\eta}{2x} + \frac{1 + (\alpha + \beta - \gamma)N}{2(1 - x)} \\ &+ \frac{d}{dx} \log\left(x(S - \frac{1 + \gamma N}{2x} + \frac{1 + (\alpha + \beta - \gamma)N}{2(1 - x)}) + \alpha N + \frac{1}{2}\right) \\ &+ \frac{1 + \gamma N}{2x} - \frac{1 + (\alpha + \beta - \gamma)N}{2(1 - x)} - S(\alpha, \beta, \gamma; x, N) \end{aligned} \tag{B.64}$$

$$= -\frac{1}{2(1 - x)} + \frac{d}{dx} \log\left(x(S - \frac{1 + \gamma N}{2x} + \frac{1 + (\alpha + \beta - \gamma)N}{2(1 - x)}) + \alpha N + \frac{1}{2}\right) \tag{B.65}$$

In what follows, we set

$$I(\alpha, \beta, \gamma; x_0, N) = \frac{1}{2} \int_{\gamma_{x_0}} (S(\alpha + N^{-1}, \beta, \gamma; x, N) - (S(\alpha, \beta, \gamma; x, N))dx \tag{B.66}$$

$$J(\alpha, \beta, \gamma; x_0, N) = \frac{1}{2} \int_{\gamma_{x_0}} (S(\alpha, \beta + N^{-1}, \gamma; x, N) - (S(\alpha, \beta, \gamma; x, N))dx \tag{B.67}$$

$$K(\alpha, \beta, \gamma; x_0, N) = \frac{1}{2} \int_{\gamma_{x_0}} (S(\alpha, \beta, \gamma + N^{-1}; x, N) - (S(\alpha, \beta, \gamma; x, N))dx \tag{B.68}$$

$$I_{-1}(\alpha, \beta, \gamma; x_0, N) = \frac{1}{2} \int_{\gamma_{x_0}} (S_{-1}(\alpha + N^{-1}, \beta, \gamma; x, N) - (S_{-1}(\alpha, \beta, \gamma; x, N))dx \tag{B.69}$$

$$J_{-1}(\alpha, \beta, \gamma; x_0, N) = \frac{1}{2} \int_{\gamma_{x_0}} (S_{-1}(\alpha, \beta + N^{-1}, \gamma; x, N) - (S_{-1}(\alpha, \beta, \gamma; x, N))dx \tag{B.70}$$

$$K_{-1}(\alpha,\beta,\gamma;x_0,N)=\frac{1}{2}\int_{\gamma_{x_0}}(S_{-1}(\alpha,\beta,\gamma+N^{-1};x,N)-(S_{-1}(\alpha,\beta,\gamma;x,N))dx \tag{B.71}$$

Here γ_{x_0} is a path that runs from x_0 for a sufficiently small number (very close to the singular point $b_0=0$), encircles a in a counterclockwise manner and returns to x_0. The notation $\hat{x_0}$ is used to distinguish the two branch points (the branch of S_{-1} at the starting point of γ_{x_0} is different from the branch at its end point). So $\hat{x_0}$ corresponds to the branch of S_{-1} at the starting point of γ_{x_0}. Futhermore, since each coefficient of $S_{\text{even}}=S-S_{\text{odd}}$ is single valued at $x=a$ and

$$\text{Res}_{x=a}S_{\text{even}}=\text{Res}_{x=a}S_0=-\frac{1}{4} \tag{B.72}$$

holds for (B.10), we have

$$\frac{1}{2}\int_{\gamma_{x_0}}(S_{\text{odd}}-NS_{-1})dx=\frac{1}{2}\int_{\gamma_{x_0}}(S-NS_{-1}-S_0) \tag{B.73}$$

Using (B.43), (B.44) and (B.45) and $S_0=\frac{1}{2(x-1)}$ we can rewrite (B.66), (B.67) and (B.68) as follows

$$
\begin{aligned}
I(\alpha,\beta,\gamma;x_0,N) &= \frac{1}{2}\int_{\gamma_{x_0}} \frac{d}{dx}\Bigg(\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad xS(\alpha,\beta,\gamma;x,N) + \alpha N\Bigg\}\Bigg)dx \qquad \text{(B.74)} \\
&= \frac{1}{2}\int_{\gamma_{x_0}} d\Bigg(\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad xS(\alpha,\beta,\gamma;x,N) + \alpha N\Bigg\}\Bigg) \qquad \text{(B.75)} \\
&= \frac{1}{2}\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x_0}{2(1-x_0)}(1+(\alpha+\beta-\gamma)\eta)+ \\
&\quad x_0 S(\alpha,\beta,\gamma;\hat{x_0},N) + \alpha N\Bigg\} \\
&\quad -\frac{1}{2}\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad x_0 S(\alpha,\beta,\gamma;x_0,N) + \alpha N\Bigg\} \qquad \text{(B.76)} \\
J(\alpha,\beta,\gamma;x_0,N) &= \frac{1}{2}\int_{\gamma_{x_0}} \frac{d}{dx}\Bigg(\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad xS(\alpha,\beta,\gamma;x,\eta) + \beta\eta\Bigg\}\Bigg)dx \qquad \text{(B.77)} \\
&= \frac{1}{2}\int_{\gamma_{x_0}} d\Bigg(\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad xS(\alpha,\beta,\gamma;x,N) + \beta N\Bigg\}\Bigg) \qquad \text{(B.78)} \\
&= \frac{1}{2}\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x_0}{2(1-x_0)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad x_0 S(\alpha,\beta,\gamma;\hat{x_0},N) + \beta N\Bigg\} \\
&\quad -\frac{1}{2}\log\Bigg\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(1+(\alpha+\beta-\gamma)N)+ \\
&\quad x_0 S(\alpha,\beta,\gamma;x_0,N) + \beta N\Bigg\} \qquad \text{(B.79)}
\end{aligned}
$$

$$
\begin{aligned}
K(\alpha,\beta,\gamma;x_0,N) &= \frac{1}{2}\int_{\gamma_{x_0}} \frac{d}{dx}\Bigg(\log\left\{\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(\alpha+\beta-\gamma)\eta + \right.\\
&\left. xS(\alpha,\beta,\gamma+N^{-1};x,N) + \alpha\eta\right\}\Bigg) dx \qquad \text{(B.80)}\\
&= \frac{1}{2}\int_{\gamma_{x_0}} d\Bigg(\log\left\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(\alpha+\beta-\gamma)N + \right.\\
&\left. xS(\alpha,\beta,\gamma+N^{-1};x,N)\right\}\Bigg) \qquad \text{(B.81)}\\
&= \frac{1}{2}\log\left\{ -\frac{1}{2}\gamma N + \frac{x_0}{2(1-x_0)}(\alpha+\beta-\gamma)N + \right.\\
&\left. x_0 S(\alpha,\beta,\gamma+N^{-1};\hat{x_0},N)\right\}\\
&-\frac{1}{2}\log\left\{ -\frac{1}{2}\gamma N + \frac{x}{2(1-x)}(\alpha+\beta-\gamma)N + \right.\\
&\left. x_0 S(\alpha,\beta,\gamma+N^{-1};x_0,N)\right\} \qquad \text{(B.82)}
\end{aligned}
$$

Now, fix the semiaxis $Re(x-1) < 0$ as a branch cut of the logarithmic function and use the following conventions:

$$
\begin{aligned}
-\alpha &= e^{-\pi i}\alpha,\\
-\beta &= e^{\pi i}\beta,\\
-\gamma &= e^{\pi i}\gamma'\\
\alpha-\gamma &= e^{\pi i}(\gamma-\alpha),\\
\beta-\gamma &= e^{-\pi i}(\gamma-\beta),\\
\gamma-\alpha-\gamma &= e^{\pi i}(\alpha+\beta-\gamma),\\
\alpha-\beta &= e^{\pi i}(\beta-\alpha)
\end{aligned} \qquad \text{(B.83)}
$$

The above conventions correspond to:

$$
\begin{aligned}
0 &< \arg\alpha \le \pi,\\
-\pi &< \arg\beta \le 0,\\
-\pi &< \arg\gamma \le 0,\\
-\pi &< \arg(\gamma-\alpha) \le 0,\\
0 &< \arg(\gamma-\beta) \le \pi,\\
-\pi &< \arg(\alpha+\beta-\gamma) \le 0
\end{aligned} \qquad \text{(B.84)}
$$

We will now compute the explicits forms of $V_j (j=0,1,2)$. For this purpose, let us compute the leading terms and subleading terms of the Laurent series of S at $x=b_0$ on the Riemann surface of Q_0. Insert Eqn (B.11) into (B.10) to obtain the following recursion relations:

$$S_{-1}^2 = Q_0(x) \tag{B.85}$$

$$2S_{-1}S_0 + \frac{dS_{-1}}{dx} = 0 \tag{B.86}$$

$$2S_{-1}S_1 + S_0^2 + \frac{dS_0}{dx} = Q_1(x) \tag{B.87}$$

$$2S_{-1}S_n + \sum_{k=0}^{n-1} S_k S_{n-k-1} + \frac{dS_{n-1}}{dx} = 0 \tag{B.88}$$

It follows from (B.85) that

$$S_{-1} = \sqrt{Q_0(x)} = \frac{\gamma}{2x} \tag{B.89}$$

From (B.86) we have

$$2S_{-1}S_0 = -\frac{dS_{-1}}{dx} \tag{B.90}$$

$$= \frac{\gamma}{2x^2} \tag{B.91}$$

This gives

$$S_0 = \frac{1}{2x} \tag{B.92}$$

Substituting (B.89) and (B.92) back in (B.54) we find the leading term of S, that is,

$$\frac{1+\gamma N}{2x} \tag{B.93}$$

Following the same procedure, the subleading terms of S is

$$-\frac{2\gamma^2 N^2 - 1}{4(1+\gamma N)} \tag{B.94}$$

Hence, near $x = 0$ on the first sheet we have:

$$x_0 S(\alpha, \beta, \gamma, x_0; N) = x_0 \left\{ \frac{1+\gamma N}{2x_0} - \frac{2\gamma^2 N^2 - 1}{4(1+\gamma N)} + O(x_0) \right\} \tag{B.95}$$

On the second sheet we have:

$$x_0 S(\alpha, \beta, \gamma, \hat{x_0}; N) = x_0 \left\{ \frac{1-\gamma N}{2x_0} - \frac{2\gamma^2 N^2 - 1}{4(1-\gamma N)} + O(x_0) \right\} \tag{B.96}$$

Using the last two equations above in (B.83) we obtain the asymptotic behaviour of I, J, K as $x_0 \to 0$ as follows:

$$I(\alpha,\beta,\gamma;N) = \frac{1}{2}\log\frac{\gamma-\alpha-(N^{-1}/2)}{\alpha+(N^{-1}/2)} + \frac{1}{2}\log x_0 + O(x_0) \tag{B.97}$$

$$J(\alpha,\beta,\gamma;N) = \frac{1}{2}\log\frac{\gamma-\beta-(N^{-1}/2)}{\beta+(N^{-1}/2)} + \frac{1}{2}\log x_0 + O(x_0) \tag{B.98}$$

$$\begin{aligned} K(\alpha,\beta,\gamma;N) &= \frac{1}{2}\log\frac{\gamma(\gamma+N^{-1})}{(\gamma-\alpha+(N^{-1}/2))(\gamma-\beta+(N^{-1}/2))} \\ &\quad - \frac{1}{2}\log x_0 + O(x_0). \end{aligned} \tag{B.99}$$

Proposition: The Voros coefficient V_0 satisfy the following system of difference equations as a formal power series in η^{-1}:

$$\begin{aligned} V_0(\alpha+\eta^{-1},\beta,\gamma;\eta) &= \frac{1}{2}\log\frac{\gamma-\alpha-(\eta^{-1}/2)}{\alpha+(\eta^{-1}/2)} \\ &- \frac{\eta}{2}\Big\{\alpha\log\alpha-(\alpha+N^{-1})\log(\alpha+N^{-1}) \\ &+ (\gamma-\alpha)\log(\gamma-\alpha)-(\gamma-\alpha-N^{-1})\log(\gamma-\alpha-N^{-1})\Big\} \end{aligned} \tag{B.100}$$

$$\begin{aligned} V_0(\alpha,\beta+N^{-1},\gamma;N) &= \frac{1}{2}\log\frac{\gamma-\beta-(N^{-1}/2)}{\beta+(N^{-1}/2)} \\ &- \frac{N}{2}\Big\{\beta\log\beta-(\beta+N^{-1})\log(\beta+N^{-1}) \\ &+ (\gamma-\beta)\log(\gamma-\beta)-(\gamma-\beta-N^{-1})\log(\gamma-\alpha-N^{-1})\Big\} \end{aligned} \tag{B.101}$$

$$\begin{aligned} V_0(\alpha,\beta,\gamma+N^{-1};N) &= \frac{1}{2}\log\frac{\gamma(\gamma+N^{-1})}{(\gamma-\alpha+(N^{-1}/2))(\gamma-\beta+(N^{-1}/2))} \\ &- \frac{N}{2}\Big\{(\gamma-\alpha)\log(\gamma-\alpha)-(\gamma-\alpha+N^{-1})\log(\gamma-\alpha+N^{-1}) \\ &+ (\gamma-\beta)\log(\gamma-\beta)-(\gamma-\beta+N^{-1})\log(\gamma-\beta+N^{-1}) \\ &- 2\gamma\log\gamma+2(\gamma+N^{-1})\log(\gamma+N^{-1})\Big\} \end{aligned} \tag{B.102}$$

Let

$$w = u(\alpha,\beta,\gamma;N) + v(\alpha,\beta,\gamma;N) \tag{B.103}$$

be the left hand side of (B.100). Then

$$u(\alpha,\beta,\gamma;N) = \frac{1}{2}\log\frac{\gamma-\alpha-(N^{-1}/2)}{\alpha+(N^{-1}/2)} \tag{B.104}$$

and

$$v(\alpha,\beta,\gamma;N) = -\frac{N}{2}\Big\{\alpha\log\alpha - (\alpha+N^{-1})\log(\alpha+N^{-1}) + (\gamma-\alpha)\log(\gamma-\alpha) \\ - (\gamma-\alpha-N^{-1})\log(\gamma-\alpha-N^{-1})\Big\} \tag{B.105}$$

We can decompose (B.100) as follows:

$$V_{01}(\alpha+N^{-1},\beta,\gamma;N) - V_{01}(\alpha,\beta,\gamma;N) = u(\alpha,\beta,\gamma;N) \tag{B.106}$$
$$V_{11}(\alpha+N^{-1},\beta,\gamma;N) - V_{11}(\alpha,\beta,\gamma;N) = v(\alpha,\beta,\gamma;N) \tag{B.107}$$

If V_{01} and V_{02} exist and are the solutions of (B.100), then $V_0 = V_{01} + V_{02}$ is also a solution (B.100). Hence,

$$V_{02}(\alpha,\beta,\gamma;N) = \frac{N}{2}\Big\{\alpha\log\alpha + (\gamma-\alpha)\log(\gamma-\alpha)\Big\} \tag{B.108}$$

To find the solution $V_{01}(\alpha,\beta,\gamma;N)$ of (B.100) we use the method developed by Candelpergher-Coppo-Delabaere in[59]:

$$V_{01}(\alpha+N^{-1},\beta,\gamma;N) - V_{01}(\alpha,\beta,\gamma;N) = (e^{N^{-1}\partial_\alpha}-1)V_{01}(\alpha,\beta,\gamma;N) \tag{B.109}$$

where the expansion of the inverse of $(e^{N^{-1}\partial_\alpha}-1)$ is given by:

$$(e^{N^{-1}\partial_\alpha}-1)^{-1} = N\left(\frac{\partial}{\partial\alpha}\right)^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}N^{-n}\left(\frac{\partial}{\partial\alpha}\right)^n \tag{B.110}$$

with B_n the n-th Bernoulli number and ∂_α^{-1} is the indefinite integral operator. Using (B.109), the solution $V_{01}(\alpha,\beta,\gamma;\eta)$ of (B.100) can be written as:

$$V_{01}(\alpha,\beta,\gamma;N) = (e^{N^{-1}\partial_\alpha}-1)^{-1}u(\alpha,\beta,\gamma;N) + g_0(\beta,\gamma;N^{-1}) \tag{B.111}$$

where $g_0(\beta,\gamma;\eta^{-1})$ is an arbitrary formal power series in N^{-1} which is independent of α. The following formulas are also useful:

$$\partial_\alpha(e^{N^{-1}\partial_\alpha}-1)^{-1}\log\left(1+\frac{1}{\alpha N}\right) = \frac{1}{\alpha} \tag{B.112}$$
$$\partial_\alpha(e^{N^{-1}\partial_\alpha/2}-1)^{-1}\log\left(1+\frac{1}{2\alpha N}\right) = \frac{1}{\alpha}. \tag{B.113}$$

Equations (B.112) and (B.113) can be proved as follows:

$$\begin{aligned}\frac{\partial}{\partial\alpha}\log\left(1+\frac{1}{\alpha N}\right) &= \frac{1}{\alpha(N\alpha+1)} \\ &= -\frac{1}{\alpha}+\frac{N}{N\alpha+1} \\ &= \frac{1}{\alpha}+\frac{1}{\alpha+N^{-1}}\end{aligned} \tag{B.114}$$

$$\frac{\partial}{\partial\alpha}\log\left(1+\frac{1}{\alpha N}\right) = (e^{N^{-1}\partial_\alpha}-1)\frac{1}{\alpha} \tag{B.115}$$

Thus,

$$\frac{1}{\alpha} = \partial_\alpha (e^{N^{-1}\partial_\alpha} - 1)^{-1} \log\left(1 + \frac{1}{\alpha N}\right) \tag{B.116}$$

Similarly we have (B.113). Let us rewrite equation (B.111) using (B.104). We have

$$\begin{aligned} V_{01}(\alpha,\beta,\gamma;N) &= (e^{N^{-1}\partial_\alpha} - 1)^{-1}\frac{1}{2}\log\frac{(\gamma-\alpha-(N^{-1}/2))}{\alpha+(N^{-1}/2)} \\ &= (e^{N^{-1}\partial_\alpha} - 1)^{-1}\frac{1}{2}\left\{\log(\alpha+N^{-1}/2) - \log(\gamma-\alpha-N^{-1}/2)\right\} \end{aligned} \tag{B.117}$$

Taking the α-derivative we obtain

$$\begin{aligned} \partial_\alpha &= -\partial_\alpha (e^{N^{-1}\partial_\alpha} - 1)^{-1}\frac{1}{2}\left\{\log(\alpha+N^{-1}/2) - \log(\gamma-\alpha-N^{-1}/2)\right\} \\ &= -\frac{\partial_\alpha}{2}(e^{N^{-1}\partial_\alpha} - 1)^{-1}\log\left\{\alpha\left(1+\frac{1}{2\alpha N}\right)\right\} \\ &\frac{\partial_\alpha}{2}(e^{N^{-1}\partial_\alpha} - 1)^{-1}\log\left\{(\gamma-\alpha)\left(1-\frac{1}{2(\gamma-\alpha)N}\right)\right\} \\ &= \frac{\partial_\alpha}{2}(e^{N^{-1}\partial_\alpha} - 1)^{-1}\{\log(\gamma-\alpha)-\log\alpha\} \\ &+ \frac{\partial_\alpha}{2}(e^{N^{-1}\partial_\alpha} - 1)^{-1}\log\left\{\left(1-\frac{1}{2(\gamma-\alpha)N}\right) - \log\left\{\left(1-\frac{1}{2\gamma N}\right)\right\}\right. \\ &= \frac{\partial_\alpha}{2}(e^{N^{-1}\partial_\alpha} - 1)^{-1}\{\log(\gamma-\alpha)-\log\alpha\} + \\ &\frac{\partial_\alpha}{2}[(e^{(1/2)N^{-1}\partial_\alpha} - 1)^{-1} + (e^{(1/2)N^{-1}\partial_\alpha} + 1)^{-1}]\log\left\{\left(1-\frac{1}{2(\gamma-\alpha)N}\right) - \log\left\{\left(1-\frac{1}{2\gamma N}\right)\right\}\right. \end{aligned} \tag{B.118}$$

Rewrite the first term of this last equation above using the inverse operator defined in (B.110):

$$\begin{aligned} &\frac{\partial_\alpha}{2}(e^{N^{-1}\partial_\alpha} - 1)^{-1}\{\log(\gamma-\alpha)-\log\alpha\} = \\ &\frac{N}{2}\left(\frac{\partial}{\partial\alpha}\right)^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}N^{-n}\left(\frac{\partial}{\partial\alpha}\right)^n\{\log(\gamma-\alpha)-\log\alpha\} \\ &= \frac{N}{2}\{\log(\gamma-\alpha)-\log\alpha\} + \frac{B_1}{2}\left(\frac{1}{\gamma-\alpha}+\frac{1}{\alpha}\right) + \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n}{n}n^{-n+1}\left(\frac{1}{(\gamma-\alpha)^n}+\frac{1}{\alpha^n}\right) \end{aligned} \tag{B.119}$$

The following identity

$$(e^{(1/2)N^{-1}\partial_\alpha} + 1)^{-1} = (e^{(1/2)N^{-1}\partial_\alpha} - 1)^{-1} - 2(e^{N^{-1}\partial_\alpha} - 1)^{-1} \tag{B.120}$$

and the equations (B.112) and (B.113) are are useful to rewrite the second term of (B.118):

$$
\begin{aligned}
&\frac{\partial_\alpha}{2}[(e^{(1/2)N^{-1}\partial_\alpha}-1)^{-1}+(e^{(1/2)N^{-1}\partial_\alpha}+1)^{-1}]\log\left\{\left(1-\frac{1}{2(\gamma-\alpha)N}\right)-\log\left\{\left(1-\frac{1}{2\gamma N}\right)\right\}\right. \\
&=\frac{1}{2}(e^{(1/2)N^{-1}\partial_\alpha}-1)^{-1}-2(e^{N^{-1}\partial_\alpha}-1)^{-1}\left(-\frac{1}{\gamma-\alpha}-\frac{1}{\alpha}\right) \\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n B_n n^{1-n}}{n!}(2^{1-n}-2)\partial_\alpha^{n-1}\left(-\frac{1}{\gamma-\alpha}-\frac{1}{\alpha}\right) \\
&=\frac{B_1}{2}\left(-\frac{1}{\gamma-\alpha}-\frac{1}{\alpha}\right)+\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n n^{1-n}}{n!}(2^{1-n}-2)\left(\frac{1}{(\gamma-\alpha)^n}+\frac{1}{\alpha^n}\right)
\end{aligned} \tag{B.121}
$$

Combine (B.119) and (B.121) to obtain

$$
\partial_\alpha V_{01}(\alpha,\beta,\gamma;N)=\frac{1}{2}\{N\log(\gamma-\alpha)-\log\alpha\}+\sum_{n=2}^{\infty}\frac{B_n n^{1-n}}{n!}(2^{1-n}-2)\left(\frac{1}{(\gamma-\alpha)^n}+\frac{1}{\alpha^n}\right) \tag{B.122}
$$

Using (B.108) and (B.116) together with the fact that $V_0=V_{01}+V_{02}$ we have

$$
V_0(\alpha,\beta,\gamma;\eta)=\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n n^{1-n}}{n!}(2^{1-n}-2)\left(\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{\alpha^{n-1}}+\frac{2}{\gamma^{n-1}}\right)+g_0(\beta,\gamma) \tag{B.123}
$$

Solving (B.101) and (B.102) by the same method as for (B.100) yields respectively:

$$
V_0(\alpha,\beta,\gamma;N)=\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n n^{1-n}}{n!}(2^{1-n}-2)\left(\frac{1}{(\gamma-\beta)^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{2}{\gamma^{n-1}}\right)+g_1(\alpha,\gamma) \tag{B.124}
$$

$$
V_0(\alpha,\beta,\gamma;N)=\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n n^{1-n}}{n!}(2^{1-n}-2)\left(\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}+\frac{2}{\gamma^{n-1}}\right)+g_2(\alpha,\beta) \tag{B.125}
$$

Combining (B.123), (B.124) and (B.125) we obtain (B.43).

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