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The Dynamics of Open String - Membrane Systems

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg in fulfilment of the requirements for the degree of Master of Science

Supervisor: Professor Robert de Mello Koch

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

_____ (Signature of candidate)

_____day of _____20_____

Abstract

In this dissertation, the interacting Cuntz chain Hamiltonian for an open string - giant graviton system with an arbitrary number of strings attached is derived, thus generalizing the single string results of hep-th/0701067. The open strings considered carry angular momentum on an S^3 embedded in the S^5 of the $AdS_5 \times S^5$ background. In the process, we construct operators in the $\mathcal{N} = 4$ super Yang-Mills theory dual to states with open strings ending on boundstates of sphere giant gravitons. The techniques we develop facilitate the computation of one-loop anomalous dimensions of these operators. The problem of computing the one loop anomalous dimensions is replaced with the problem of diagonalizing an interacting Cuntz oscillator Hamiltonian. Our Cuntz oscillator dynamics illustrates how the Chan-Paton factors for open strings propagating on multiple branes can arise dynamically.

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1 Introduction

The AdS/CFT correspondence, originally conjectured by Maldacena, suggests a duality between a string theory defined on a certain space (the product of a negatively curved space and a closed manifold) and a conformal field theory on the boundary of that space. A concrete proposal for operators in the gauge theory dual to giant gravitons (membranes which are extended in the AdS_5 or S^5 space of the $AdS_5 \times S^5$ background) was made by [1], [2] who motivated identifying these giants with Schur polynomials labeled by Young diagrams. This was then extended by [3] who identified systems in which open strings are attached to giant gravitons with operators known as restricted Schur polynomials. Here, attaching open strings corresponds to exciting the giant gravitons. The technology necessary to calculate correlators of restricted Schur polynomials dual to giant gravitons with strings attached was developed in [4]. Before the introduction of this and other related technology in [4], [5], the dynamics of membranes (in particular, giant gravitons with open strings attached) was not within our means to explore. Indeed, the membrane is traditionally treated as no more than a static, unchanging boundary condition for open strings. With the new technology built upon restricted Schur polynomials, probing the dynamics of open string -membrane systems becomes tractable, at least in principle. Up to this point however, only the dynamics of giants with a single string attached have been explored, [5]. Our primary aim is to further develop and bring to fruition the technology required to treat a general open - string giant graviton system with not just one string attached but any number of strings and thus allow the exploration of the resultant dynamics in a universal manner. To this end, we develop techniques to allow the construction of restricted Schur Polynomials dual to states with open strings stretching between two giants. In practice, this requires obtaining projection operators (termed intertwiners) that act on representations of the symmetric group and extract off-diagonal blocks of the matrix. We then extend this to allow the calculation of restricted characters used in constructing operators dual to a system of an arbitrary number of giants with any number of strings attached, including some or all of the

strings being stretched between the giants. Next, we treat the dynamics of this quite general system. In order to do this, boundary interaction terms of the Cuntz chain Hamiltonian are derived for the case of two or more strings attached to a giant (in a manner analogous to [5]). In particular, the boundary interactions describing the transfer of a unit of momentum from an open string to a giant (termed hop off), from a giant to an open string (termed hop on) and what is termed the "kissing interaction" are explored. This is done by elucidating all possible transitions (and their interaction strengths) between states of the open string - giant system, a process which entails deriving the necessary identities, inverting them and normalizing the states. The full Cuntz chain Hamiltonian thus obtained provides some very valuable physical insights which we explore, such as how the Chan-Paton factors of the open strings arise dynamically. The techniques developed in the course of this endeavor also allow the further investigation of the emergence of a new Yang-Mills theory from the matrix degrees of freedom of the original gauge theory. In addition, the means to more convincingly demonstrate the duality between membranes and Schur polynomials is provided. We begin with a brief review of the AdS/CFT correspondence in section 2, followed by a review of some aspects of giant gravitons in section 3, excited giants in section 4 and the Cuntz oscillator chain (and Cuntz chain Hamiltonian) in section 5. The main results of this dissertation are presented in sections 6 to 9. In section 6 we derive the intertwiners discussed above. We then present a straightforward and general algorithm to compute restricted characters in section 7. In section 8 we derive the general Cuntz chain Hamiltonian for multiple strings attached to an arbitrary number of branes. The interpretation of our results follows in section 9 and we conclude with a discussion of the results in section 10. The new results derived in this dissertation have been presented in the arXiv e-print [6] which has been submitted to JHEP for publication.

2 The AdS/CFT Conjecture

2.1 Overview

The AdS/CFT correspondence, also known as the gauge theory - gravity correspondence is an intriguing duality between a theory with gravity and one without gravity [7], [8], [9]. In its original form as conjectured by Maldacena, the correspondence relates Type IIB string theory defined on $AdS_5 \times S^5$ and $\mathcal{N} = 4$, 3+1 dimensional Super - Yang Mills theory. Type IIB string theory (along with all other string theories) includes a massless spin 2 particle which is naturally interpreted as the graviton. Thus Type IIB string theory, as with the other string theories, can be considered as a theory containing gravity. $\mathcal{N} = 4$ SYM theory is a quantum field theory with a U(N) gauge symmetry. One of most useful aspects of the AdS/CFT correspondence (and one that simultaneously frustrates efforts to explore the duality more exhaustively) is the fact that it is a strong-weak coupling duality. It allows us to study string theory in the strong coupling domain by studying the weakly coupled dual gauge theory and vice versa. The fact that the string theory is defined on a space with a different dimensionality to that of the gauge theory can be related to one of the profound physical ideas that accompany the conjecture - the holographic principle [10], [11]. The other deep physical idea related to the AdS/CFT conjecture is the proposal that gauge theories at large N are in some way equivalent to string theories [12].

2.2 Holographic Principle

In a loose sense, the holographic principle states that the entropy (number of degrees of freedom) in a theory of quantum gravity scales like the surface area of the system and not the volume. This idea was originally explicated in the context of black holes. Black holes, in addition to being viewed as classical gravitational systems, can also be viewed as thermodynamic systems with temperature and entropy as illustrated by Bekenstein and Hawking. The identification of parameters essentially follows from the linking of the laws of black hole mechanics with the laws of thermodynamics. The temperature of

the black hole, T is associated with the surface gravity, κ and the entropy, S with the area of the horizon, A in the following way:

$$S = \frac{A}{4G},$$

$$T = \frac{\kappa}{2\pi}.$$

The identification of entropy with the area of the horizon does however present a difficulty. From the perspective of statistical mechanics, we would expect the number of degrees of freedom (and hence entropy) of the system to scale like the volume of the system and not the area. A possible resolution comes in identifying a d-dimensional theory of gravity with a $d-1$ dimensional local field theory. This provides consistency since an area in d dimensions is like a volume in $d-1$ dimensions. This motivates the identification of a theory containing gravity in the volume of some d - dimensional space with a local field theory on the boundary of the space.

The interpretation of the dimensionality mismatch between the space in which the string theory lives and the space in which the gauge theory lives in the AdS/CFT conjecture now becomes clear. The boundary of $AdS_5 \times S^5$ space is 3+1 dimensional Minkowski space and thus the correspondence links a theory with gravity defined on the volume of a space to a theory without gravity defined on the boundary of the space and we see that the AdS/CFT correspondence is a concrete realization of the holographic principle.

2.3 Large N Gauge Theories and String Theories

The notion that gauge theories and string theories may be related and in certain limits equivalent underlies the AdS/CFT correspondence. One of the most powerful motivations for this idea comes from comparing the perturbative expansion of a large N gauge theory in $1/N$ (keeping $\lambda = g^2 N$ constant) and the perturbative (loop) expansion in string theory. The perturbative gauge theory expansion has the form:

$$\sum_{g\geq 0} N^{2-2g} f_g(\lambda). \tag{1}$$

The form of this expansion arises in the following way. The Feynman diagrams of a Yang Mills theory can be written in a ribbon (double line) notation. In this notation each matrix index is replaced by a labeled dot. To obtain the values of correlators, pairs of dots are joined with ribbons. Each line linking the dots corresponds to a Kronecker delta with the indices of the joined dots. A closed loop in these diagrams corresponds to a single power of N as a result of the contraction of the indices of the Kronecker delta's. In these diagrams the closed loops formed by the ribbons could be considered to constitute a triangulation (or a generalization thereof) of a surface. It turns out that the power of N associated with a particular ribbon diagram can be linked to a surface with a particular topology, with the diagrams at leading order in N being associated with surfaces with the topology of a sphere (or plane). This link comes in the form of the Euler characteristic of the diagram (a topological invariant) - which allows the power of N associated with a particular diagram to be written in terms of the genus, g (number of handles) of the corresponding surface. A perturbative expansion of the gauge theory can now be organized in terms of powers of N (via the genus) and powers of λ and this leads directly to the expansion above. The link to string theory now becomes apparent - it seems natural to identify these surfaces with the worldsheets of strings. The similarity of the loop expansion in string theory to the above serves to illustrate this:

$$\sum_{g\geq 0} g_s^{2g-2} Z_g. \tag{2}$$

Here g_s=$1/N$ and g corresponds to the genus of the worldsheet of the string. Thus we can see that (2) is also a genus expansion. Note also that Z_g is a function of the string tension and equivalently of the string length, l_s. We therefore expect a relation between λ and l_s. The form of this relation is discussed in section 2.6.

2.4 Anti-de Sitter Space

Anti-de Sitter space is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. AdS_d space can be thought of as a hypersurface with the following equation embedded in $d+1$ dimensional flat space:

$$(x^0)^2 + (x^d)^2 - \sum_{i=1}^{d-1}(x^i)^2 = R^2.$$

Here R is a constant which is identified as the radius of curvature of the AdS_d space. This has metric

$$ds^2 = -(dx^0)^2 - (dx^d)^2 + \sum_{i=1}^{d-1}(dx^i)^2.$$

For example, if we consider $d = 3$ and utilize global co-ordinates which have form[35]:

$$x^0 = R\cosh\mu\cos t, \qquad x^3 = R\cosh\mu\sin t,$$

$$x^1 = R\sinh\mu\cos\theta,$$

$$x^2 = R\sinh\mu\sin\theta.$$

we obtain the following form of the metric:

$$ds^2 = R^2\left(-\cosh^2\mu dt^2 + d\mu^2 + \sinh^2\mu d\theta^2\right). \tag{3}$$

where $\mu \geq 0,\ 0 \leq t, \theta \leq 2\pi$.

This co-ordinate chart covers the entire hyperboloid. The boundary of the AdS space in this co-ordinate system is obtained by taking $\mu \to \infty$. For $\mu \to \infty$ the metric becomes:

$$ds^2 = R^2 e^{2\mu}\left(-dt^2 + d\theta^2\right).$$

Thus when $d = 3$, the boundary of the AdS space in global co-ordinates

corresponds to $R\text{x}S^1$ (up to a scaling). Similarly, for $d = 5$, we have that the boundary of AdS_5 is conformally equivalent to $R\text{x}S^3$. Since the AdS/CFT correspondence relates a conformal field theory on the boundary of the AdS space to a gravitational theory defined on the volume of the space, we expect a conformal field theory defined on $R\text{x}S^3$ to be dual to a gravitational theory on AdS space. In section 3 we will see that considering $\mathcal{N} = 4$ super Yang Mills defined on $R\text{x}S^3$ is particularly useful.

We could also utilize Poincare co-ordinates (which are local) which have the form[35]:

$$x^0 = \frac{1}{2r}(R^2 + x^2 + r^2 - t^2), \qquad x^3 = R\frac{t}{r},$$

$$x^1 = R\frac{x}{r},$$

$$x^2 = \frac{1}{2r}(-R^2 + x^2 + r^2 - t^2).$$

and thus obtain the following form of the metric:

$$ds^2 = \frac{R^2}{r^2}(-dt^2 + d\vec{x}^2 + dr^2). \tag{4}$$

The boundary of the AdS space in this case corresponds to $r = 0$. This can be seen by introducing the co-ordinate r' and setting

$$r' = \log\frac{1}{r},$$

i.e.

$$r = e^{-r'}.$$

The metric now becomes:

$$ds^2 = R^2\left(e^{2r'}\left(-dt^2 + d\vec{x}^2\right) + dr^2\right).$$

So we see that as $r' \to \infty$ (i.e. $r \to 0$), we have:

$$ds^2 \sim e^{2r'}\left(-dt^2 + d\vec{x}^2\right).$$

which is conformally equivalent to 2 dimensional Minkowski space for $d = 3$. In general, the boundary of AdS_d space in Poincare co-ordinates corresponds to $d-1$ dimensional Minkowski space (up to a scaling). Thus, in terms of the AdS/CFT correspondence, we see that $\mathcal{N} = 4$ super Yang Mills (defined on Minkowski space) makes contact with the dual gravitational theory in Poincare co-ordinates. Finally note that as $r' \to -\infty$ ($r \to \infty$), $g_{00} \to 0$ i.e. the geometry has a horizon.

2.5 Motivation

The primary motivation for the AdS/CFT conjecture is obtained by considering a system of N parallel, coincident (or near coincident) D3 branes in Type IIB string theory and then taking a low energy limit. This system of D3 branes admits two possible descriptions. The first description entails considering the D3 branes as massive charged objects that curve space and thus act as a source for supergravity fields (we thus consider a supergravity solution carrying D3 brane charge). Naturally, this description can only be trusted at large N and large 't Hooft coupling where the background becomes approximately flat. The second description, views the D3 branes as hypersurfaces on which open strings are allowed to end. It is noteworthy that in the first description, only closed strings appear whereas in the second description both open and closed strings appear.

Consider the first description, the metric for the D3 brane then has the form:

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{R^4}{r^4}\right)^{-1/2}(dr^2 + r^2 d\Omega_5^2).$$

Now, the low energy limit yields very long wavelength supergravity modes propagating in the bulk region where space is approximately flat (space is flat as a consequence of the fact that N and the 't Hooft coupling are large).

In addition, all the modes of Type IIB string theory close to the stack of branes will be red-shifted due to the very large gravitational potential and thus appear to be shifted to low energy (from the perspective of an external observer). The supergravity modes do not interact with the D3 branes since their wavelengths are much larger than the gravitational size of the branes in the limit considered. Also, the modes very near to the horizon find it increasingly difficult to escape the gravitational potential and return to the bulk. The near horizon geometry of the D3 branes is given by taking the limit $r \ll R$:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2.$$

This is nothing but the metric of $AdS_5 \times S^5$ space. Thus, in this description, we have two decoupled sectors in the theory - long wavelength supergravity modes freely propagating in flat 10 dimensional Minkowski space and all the modes of Type IIB string theory in the $AdS_5 \times S^5$ geometry.

The second description, involves both open and closed strings in the Type IIB string theory in which the open strings end on D-branes. The closed string states in the bulk are described in the low energy limit by Type IIB Supergravity whereas the open string states are described in the low energy limit by $\mathcal{N} = 4$ SYM theory. In this low energy limit, the interaction Lagrangian mixing the bulk and brane sectors falls away and the two sectors again decouple, with the supergravity theory in the bulk becoming free. Thus, the two decoupled sectors are now 3+1 dimensional $\mathcal{N} = 4$ SYM theory and long wavelength supergravity modes propagating in the bulk.

In both descriptions, one of the decoupled sectors of the theory is that of long wavelength supergravity modes propagating in flat 10 dimensional Minkowski space. Since the two descriptions and their low energy physics should be equivalent, this leads us to conclude that the other sector of the descriptions should also match. Hence, the conjecture: Type IIB superstring theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ 3+1 dimensional SYM theory.

2.6 Identification of Parameters in AdS/CFT

The mapping of parameters of the string theory into those of the gauge theory (and vice versa) is as follows:

$$g_s = g_{YM}^2, \tag{5}$$

$$\left(\frac{R}{l_s}\right)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda. \tag{6}$$

The parameters in the Type IIB string theory have the following description: g_s is the string coupling constant and l_s is the string length. The parameter R is the radius of curvature of the AdS_5 and S^5 spaces constituting the $AdS_5 \times S^5$ geometry. The parameters relating to the $\mathcal{N} = 4$, 3+1 dimensional SYM theory with gauge group U(N) are N and g_{YM}^2, the coupling constant. The above equations explicitly demonstrate the strong-weak coupling nature of the duality. The string theory is only tractable when the string length is much smaller than the radius of curvature of the space (corresponding to large 't Hooft coupling, λ) and the string coupling constant is small (which corresponds to small g_{YM}^2). This is due to the fact that only when the string coupling, g_s is small can higher order terms in the loop expansion be neglected. Further, when evaluating the leading term in the loop expansion, $l_s \ll R$ (i.e. λ large) implies that curvature corrections can be dropped. In contrast the gauge theory is only tractable when λ is small.

3 Giant Gravitons

Since the AdS/CFT correspondence is a strong - weak coupling duality in the 't Hooft coupling of the field theory, the quantities used to begin to probe (and test) the duality must be such that calculations performed at weak coupling can be faithfully extrapolated to strong coupling (they should only receive small or no corrections). Quantities protected by supersymmetry such as those associated with BPS states serve this purpose. Giant gravitons are half-BPS states (half of the supersymmetries are preserved) which have facilitated the calculation of many useful quantities on both sides of the correspondence. Giant gravitons are D3 branes which are extended in the AdS_5 or S^5 space of the $AdS_5 \times S^5$ background. These giant gravitons are gravitons propagating in the bulk that have been spatially extended (blown up) as a result of the five form flux present [13]. The magnitude of the enlargement is determined by the angular momentum of the giant. This can be understood as follows: the worldvolume of the D3 brane has 3 spatial dimensions and one time dimension and thus, in the membrane action, the volume element couples to a potential with 4 indices. This volume element changes sign at antipodal points of the brane (along with the term in the membrane action which couples to the potential) and thus each pair of antipodal points on the brane constitutes a dipole. The presence of the five form flux then leads to the spherical enlargement of the giant graviton. This process can be considered to be the analog (as in [13]) of a dipole moving on a plane or the surface of a sphere in the presence of a constant magnetic field. In such a system, the dipole is stretched in the direction perpendicular to its direction of motion by an amount proportional to the magnitude of its momentum (in the case of a spherical surface - angular momentum). It turns out that for gravitons moving on an $AdS_5 \times S^5$ background of radius R, where the number of units of five form flux is N, and the angular momentum of the giant, L, is fixed, the gravitons expand as [13]:

$$r = \sqrt{\frac{L}{N}} R. \tag{7}$$

In the case of the S^5 space, the gravitons can only expand until they reach the

maximum radius possible - that of the S^5 itself, R. This then implies an upper bound on angular momentum of the giant, corresponding to N. The giant gravitons are classically stable as a result of the fact that the force due to the five form flux precisely balances the tension of the brane. A concrete proposal for operators in the gauge theory dual to giant gravitons was made by [1], [2] who motivated identifying these giants with Schur polynomials labeled by Young diagrams. Although various aspects of this proposal are still being explored and extended, by now there is a wealth of evidence supporting it. To see why the Schur polynomials were originally singled out, we begin by considering $\mathcal{N} = 4$ SYM theory on $R\text{x}S^3$ (which is conformally equivalent to R^4). The conformal equivalence of $R\text{x}S^3$ to R^4 can be seen by considering the metric on R^4:

$$ds^2 = dr^2 + r^2 d\Omega_3^2.$$

After setting $\tau = \log r$, we obtain

$$ds^2 = e^{2\tau}(d\tau^2 + d\Omega_3^2).$$

After Weyl rescaling:

$$ds^2 = d\tau^2 + d\Omega_3^2.$$

This is nothing but the metric on $R\text{x}S^3$. Recall that in section 2.4 we illustrated how the boundary of AdS_5 in global co-ordinates is $R\text{x}S^3$. Thus, the $\mathcal{N} = 4$ SYM theory defined on $R\text{x}S^3$ will make contact with the dual gravitational theory in global co-ordinates.

Dilations on R^4 can be seen to correspond to time translations (after Wick rotation $\tau = it$) on $R\text{x}S^3$. To see this consider the dilation $r \to e^a r$:

$$\tau = \log r \to \log e^a r = \log r + a = \tau + a.$$

As a result, conformal dimensions of $\mathcal{N} = 4$ SYM on R^4 map into energies of

the theory on RxS^3 and by the state operator correspondence of conformal field theory we have that the generator of dilations on R^4 maps into the Hamiltonian for the $\mathcal{N} = 4$ SYM theory on RxS^3. The utility of considering the theory on RxS^3 is that in the limit (8) (in which R charge is denoted by J) the half-BPS states (and small deformations of these states) decouple from the full $\mathcal{N} = 4$ SYM theory, and can be described in terms of the quantum mechanics of a single complex matrix [14]. The Schur polynomials we will later describe will be built out of complex matrices governed by this matrix quantum mechanics. Note that the Hamiltonian in (8) is obtained via a suitable time slicing of $AdS_5 \times S^5$ [14].

$$H = \lim_{\epsilon \to 0} \frac{(\Delta - J) + 2\Delta\epsilon}{2\epsilon}. \tag{8}$$

The action for the theory on RxS^3 (with radius of the S^3 set to 1) is as follows:

$$S = \frac{N}{4\pi\lambda} \int dt \int_{S^3} \frac{d\Omega_3}{2\pi^2} \left(\frac{1}{2}(D\phi^i)(D\phi^i) + \frac{1}{4}([\phi^i, \phi^j])^2 - \frac{1}{2}\phi^i\phi^i + \ldots \right).$$

Note that terms involving gauge fields and fermions have been dropped and only scalar fields retained since it is these that we are presently interested in. The following complex scalar fields are formed from the original six real scalar fields present in the theory:

$$Z = \phi^1 + i\phi^2, \quad Y = \phi^3 + i\phi^4, \quad X = \phi^5 + i\phi^6. \tag{9}$$

We now return to the question of what exemplifies Schur polynomials as promising candidates for the gauge theory operators dual to half-BPS giant gravitons. To answer this we consider a specific kind of giant graviton dual to a half-BPS operator built out of one of the complex matrices, Z. We denote the number of Zs in the operator by n. Depending on the number of Zs present in the operator, it has a different dual in the gravitational theory. For $n = O(1)$ the operator is dual to a point like graviton, for $n = O(\sqrt{N})$ the operator is dual to a string and for $n = O(N)$, the operator is dual to a giant graviton. This can be seen in the following way. We know that a graviton propagating in the bulk of the $AdS_5 \times S^5$ background expands to a radius

given by (7). In addition, from relation (6) in section 2.6 we have that:

$$R = (4\pi)^{\frac{1}{4}} N^{\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s. \tag{10}$$

Combining (7) and (10) we have:

$$r = \sqrt{\frac{L}{N}} (4\pi)^{\frac{1}{4}} N^{\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s. \tag{11}$$

We identify the number of Zs in the gauge theory operator with the momentum of the object in the gravitational theory to which it is dual. Thus when $n = O(1)$ we have that L is $O(1)$, when $n = O(\sqrt{N})$, L is $O(\sqrt{N})$ etc. Now, when L is $O(1)$ we see from (11) that

$$\begin{aligned} r &\sim \sqrt{\frac{1}{N}} N^{\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s, \\ &= N^{-\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s. \end{aligned}$$

Thus in the limit $N \to \infty$, $r \to 0$. In other words the radius of the object in the dual gravitational theory goes to zero and we have a point like particle. When L is $O(\sqrt{N})$, we have:

$$\begin{aligned} r &\sim \frac{1}{N^{\frac{1}{4}}} N^{\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s, \\ &= (g_{YM})^{\frac{1}{2}} l_s. \end{aligned}$$

Thus for $L \sim O(\sqrt{N})$, the size of the object in the dual gravitational theory is on the order of the string length l_s and we therefore identify these objects as strings.

Finally, when L is $O(N)$,we have:

$$\begin{aligned} r &\sim N^{\frac{1}{4}} (g_{YM})^{\frac{1}{2}} l_s, \\ &= R. \end{aligned}$$

Thus for $L \sim O(N)$, the size of the object in the dual gravitational theory is on the order of the radius of curvature of the $AdS_5 \times S^5$ background and we therefore identify these objects as giant gravitons.

Now, suitable operators in the gauge theory dual to the aforementioned objects in the gravitational theory should be orthogonal. For $O(1)$ Zs a suitable set of operators is simply a product of traces of the Zs, one for each partition of n. Orthogonality follows in this case since the two point function of the Zs is diagonal, $\langle Z^\dagger_{ij}(t) Z_{kl}(t) \rangle \propto \delta_{il}\delta_{jk}$, and the non-planar diagrams are suppressed for $O(1)$ Zs.

However, for $O(N)$ Zs comprising the half-BPS operator, large combinatoric factors overcome the suppression of the non-planar diagrams and the products of traces are no longer orthogonal (their two point functions are no longer diagonal). However, Schur polynomials which have the form shown below seem to satisfy all requirements:

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \dots Z^{i_{n-1}}_{i\sigma(n-1)} Z^{i_n}_{i\sigma(n)}.$$

The Schur polynomial is labeled by R which is a Young diagram of n boxes. Young diagrams of n boxes are in one-to-one correspondence with the irreducible representations of the symmetric group S_n and thus a Schur polynomial labeled by R is associated with a particular irreducible representation of the symmetric group. The factor $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in the irreducible representation R. The Schur polynomials do have diagonal two point functions as demonstrated by Corley, Jevicki and Ramgoolam [1] and are thus suitable operators for $n = O(N)$ Zs. It therefore seems natural to identify this Schur polynomial operator as being dual to a giant graviton in the dual gravitational theory. For operators that are built out of Z, X, and Y see the recent paper [15]. There is also physically motivated evidence based on the characteristics of giant gravitons and their dynamics. One such piece of evidence is provided by considering the cutoff in angular momentum for a giant graviton expanding in the S^5 space of the $AdS_5 \times S^5$ background as discussed previously. If we identify Young diagrams of a single column

(totally anti-symmetric rep) with spherical giants then this cutoff in angular momentum is manifest. This is due to the fact that the number of boxes in a column is identified as angular momentum of the giant and there are at most $n = N$ rows in such a column (matching the cutoff on angular momentum described above). For a giant graviton expanded in the AdS_5 space of the $AdS_5 \times S^5$ background, we see that identification with a single row of a Young diagram (totally symmetric rep) yields similarly encouraging results. This identification does not place a limit on the size of the giant in AdS_5 space and rightly so (the giant's size should be unbounded in this case) but it does place a limit on the number of AdS giants in that the Young diagram can have at most N rows and thus N AdS giants. This cut off is also necessary for consistency with the dual gravitational theory. This is due to the fact that one unit of five form flux is lost when passing through each AdS giant. Thus, if there are N or more AdS giants the flux at the center of the AdS space will become zero or negative. A positive five form flux is required to support an AdS giant however. Given the identification of a spherical giant graviton with a column of $O(N)$ boxes, it seems natural to identify a Young diagram with $O(1)$ columns in which each column has $O(N)$ boxes as a bound state of spherical giants. Similarly, we can identify a Young diagram consisting of $O(1)$ rows each containing $O(N)$ boxes with a bound state of AdS giants. Treating the dynamics of such systems requires that the giants can be excited; we review the technology and notation relating to excited giants next.

4 Excited Giant Gravitons

4.1 Restricted Schur Polynomials

On the string theory side, exciting a giant graviton corresponds to attaching open strings to the giant. In the dual field theory, the Schur polynomial operators described previously are extended to include open strings in their description. This is done in the following way. Firstly, for each open string attached to the giant we replace a matrix Z in the Schur polynomial by an open string word denoted by $((W^a)^i_j)$. This open string word simply corresponds to the product of $O(\sqrt{N})$ matrices each of which could in principle be fermions, Higgs fields or covariant derivatives of these fields. Secondly, the overall coefficient of the polynomial is modified from $\frac{1}{n!}$ to $\frac{1}{(n-k)!}$, where k is the number of open strings attached. Finally, the conventional trace operation used in obtaining the character is replaced with an operation known as a restricted trace, a concept which will be elaborated upon shortly. The resultant operator shown below is known as a restricted Schur polynomial.

$$\chi^{(k)}_{R,R_1}(Z, W^{(1)}, \ldots, W^{(k)}) = \frac{1}{(n-k)!} \sum_{\sigma \in S_n} Tr_{R_1}(\Gamma_R(\sigma)) Tr(\sigma Z^{\otimes n-k} W^{(1)} \ldots W^{(k)}),$$

$$Tr(\sigma Z^{\otimes n-k} W^{(1)} \ldots W^{(k)}) = Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \ldots Z^{i_{n-k}}_{i\sigma(n-k)} (W^{(1)})^{i_{n-k+1}}_{i\sigma(n-k+1)} \ldots (W^{(k)})^{i_n}_{i\sigma(n)}.$$

In this definition, R_1 is an irreducible representation of S_{n-k} and we therefore associate it with a Young diagram with $n-k$ boxes. Here n is $O(N)$ and k is $O(1)$. Consider the case where all the open strings attached to the giants are distinguishable (this corresponds to all the open string words in the restricted Schur polynomial operator being distinct). The representation R of S_n will subduce a representation of the $S_{n-k} \otimes (S_1)^k$ subgroup which will be reducible for the case $\sigma \in S_{n-k}$. Now, the restricted trace operation corresponds to tracing over only those indices belonging to a particular irreducible component of R (i.e. tracing over a particular block of the matrix $\Gamma_R(\sigma)$ under restricting S_n to $S_{n-k} \otimes (S_1)^k$). In order to see the subtleties involved we need

to consider the following cases:

First consider the situation where the irreducible representation R_1 only appears once under restricting from S_n to $S_{n-k} \otimes (S_1)^k$. Consider for example restricting S_n to $S_{n-2} \otimes (S_1)^2$. Further, suppose that under restricting to S_n to $S_{n-2} \otimes (S_1)^2$ we have:

$$R \to R_1 \oplus R_2 \oplus R_3.$$

For $\sigma \in S_{n-2}$, a suitable choice of basis yields:

$$\begin{pmatrix} \Gamma_{R_1}(\sigma)_{i_1 j_1} & 0 & 0 \\ 0 & \Gamma_{R_2}(\sigma)_{i_2 j_2} & 0 \\ 0 & 0 & \Gamma_{R_3}(\sigma)_{i_3 j_3} \end{pmatrix}. \tag{12}$$

Thus, in this case, $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ is uniquely defined i.e. simply trace over $\Gamma_{R_1}(\sigma)$. If $\sigma \notin S_{n-2}$, we utilize the same basis used previously and still trace over the same block of $\Gamma_R(\sigma)$ even though $\Gamma_R(\sigma)$ need not be block diagonal. $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ does not have an obvious group theoretic interpretation (for one thing it is basis dependent, unlike the character), nonetheless we interpret operators defined using $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$, where the trace is taken over an on - diagonal block, as being dual to a system with one open string attached to each giant.

Now consider the situation where the irreducible representation R_1 appears more than once. Considering the same restriction as above, S_n to $S_{n-2} \otimes (S_1)^2$, suppose we now have:

$$R \to R_1 \oplus R_1 \oplus R_2.$$

For $\sigma \in S_{n-2}$ we have in a suitable basis:

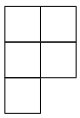
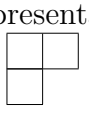
$$\Gamma_R(\sigma) = \begin{pmatrix} \Gamma_{R_1}(\sigma)_{i_1 j_1} & 0 & 0 \\ 0 & \Gamma_{R_1}(\sigma)_{i_1 j_1} & 0 \\ 0 & 0 & \Gamma_{R_2}(\sigma)_{i_2 j_2} \end{pmatrix}. \tag{13}$$

$\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ is no longer uniquely defined as a result of the fact that the representation R_1 appears with multiplicity greater than one. The operators defined using $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ are still interpreted as being dual to a system with one open string attached to each giant however. The primary difference is that the particular diagonal block that is traced over is determined by what subgroups are used in the restriction. These subgroups are the set of elements of the permutation group that leave an index invariant, $\sigma(i) = i$. Choosing the index to be the index of an open string, we can associate the subgroups participating with specific open strings. The subgroups are specified by dropping boxes from R, so that we can now associate boxes in R with specific open strings. This leads to a convenient graphical notation which has been developed in [4, 5]. There is an obvious generalization to the case that a representation R_1 appears n times after restricting to the subgroup.

Now, if we assume that the irreducible representation R_1 appears more than once and the chain of subductions of R (i.e. the order in which we drop boxes from R) indicates that the trace must be taken over an off - diagonal block of $\Gamma_R(\sigma)$, we can provide another valid definition of $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$. Consider $\Gamma_R(\sigma)$ in the basis utilized above for a general element $\sigma \in S_n$.

$$\Gamma_R(\sigma) = \begin{pmatrix} A^{(1,1)}_{i_1 j_1} & A^{(1,2)}_{i_1 j_2} & A^{(1,3)}_{i_1 j_3} \\ A^{(2,1)}_{i_2 j_1} & A^{(2,2)}_{i_2 j_2} & A^{(2,3)}_{i_2 j_3} \\ A^{(3,1)}_{i_3 j_1} & A^{(3,2)}_{i_3 j_2} & A^{(3,3)}_{i_3 j_3} \end{pmatrix}. \tag{14}$$

We take the trace over the off diagonal blocks $A^{(1,2)}_{i_1 j_2}$ and $A^{(2,1)}_{i_2 j_1}$ as a valid definition for $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ for example. Again $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ does not have an obvious group theoretic interpretation. This is most easily illustrated by an example:

Consider an irreducible representation of S_5, $R =$  and and irreducible representation of S_3, $R_1 =$  which is subduced from R. The full set of

irreducible representations that R subduces via the removal of two boxes are as follows:

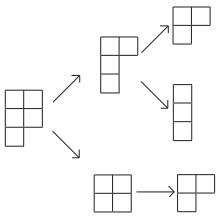


Figure 1: Irreducible representations subduced from R by removing two boxes

$\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ can now be seen to correspond to either tracing over block A or block B of the $\Gamma_R(\sigma)$ matrix below:

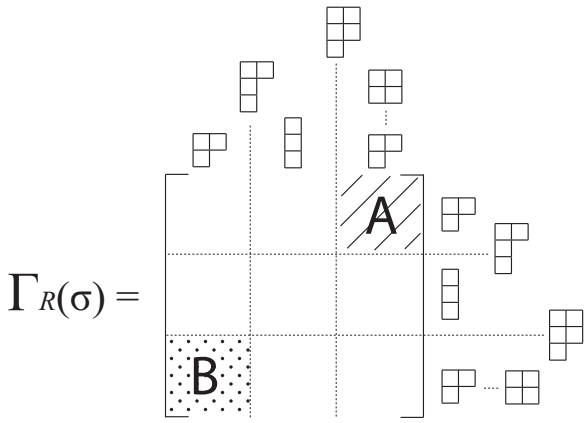
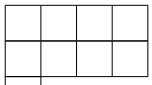


Figure 2: Off-Diagonal blocks associated to R_1

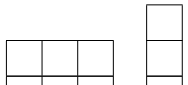
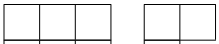
Here we see how the irreducible representations and in particular, the specific chain of subductions involved, identify which block of the matrix $\Gamma_R(\sigma))$ to trace over (as discussed above). We interpret operators defined using restricted characters which correspond to traces over off diagonal blocks as being dual to systems where open strings are stretched between the giants.

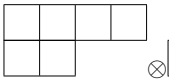
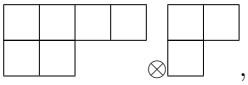
If any of the strings are identical, one needs to decompose with respect to a larger subgroup and to pick a representation for the strings which are indistinguishable. Thus, for example, if we consider a bound state of a giant system with three identical strings attached, we would consider an $S_{n-3} \otimes S_3$ subgroup of S_n. The restricted Schur polynomial would be given by $\chi^{(3)}_{R,R_1}$ with

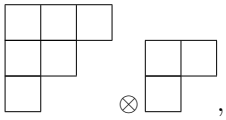
R an irreducible representation of S_n and R_1 an irreducible representation of $S_{n-3} \otimes S_3$. The S_3 subgroup would act by permuting the indices of the three identical strings; the S_{n-3} subgroup would act by permuting the indices of the Zs out of which the giant is composed. Write $R_1 = r_1 \times r_2$ with r_1 are irreducible representation of S_{n-3} and r_2 an irreducible representation of S_3. As an example, if we take R to be an irreducible representation of S_9

$R =$  , $\dim(R) = 84$

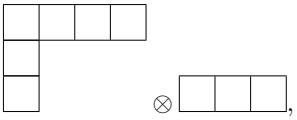
then we can have

$R_1 =$  , $\dim(R_1) = 5$, $R_1 =$  , $\dim(R_1) = 10$,

$R_1 =$  , $\dim(R_1) = 9$, $R_1 =$  , $\dim(R_1) = 18$,

$R_1 =$  , $\dim(R_1) = 32$,

or

$R_1 =$  , $\dim(R_1) = 10$.

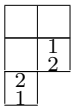
By summing the dimensions of these representations, it is easy to see that we have indeed listed all of the representations that are subduced by R.

In practice the restricted trace is obtained by utilizing projection operators in both cases. The operators defined in [4] are utilized for tracing over on-diagonal blocks (one string attached to each membrane). The operators used to calculate the restricted trace for the off diagonal blocks, which we term intertwiners, will be explained in detail in section 6. Thus we have:

$$\mathrm{Tr}_{R_1} = \mathrm{Tr}\left(\Pi\Gamma_R(\sigma)\right).$$

where Π is a product of projection operators and/or intertwiners, used to implement the restricted trace. Π is defined by the sequence of irreducible representations used to subduce R_1 from R, as well as the chain of subgroups to which these representations belong. Since the row and column indices of the block that we trace over (denoted by R_1 in the above formula) need not coincide, we need to specify this data separately for both indices. The graphical notation - which we review briefly below - summarizes this information.

For the case that we have k strings, we label the words describing the open strings $1, 2, ..., k$. Denote the chain of subgroups involved in the reduction by $\mathcal{G}_k \subset \mathcal{G}_{k-1} \subset \cdots \subset \mathcal{G}_2 \subset \mathcal{G}_1 \subset S_n$. $\mathcal{G}_m$ is obtained by taking all elements S_n that leave the indices of the strings $W^{(i)}$ with $i \leq m$ inert. To specify the sequence of irreducible representations employed in subducing R_1, place a pair of labels into each box, a lower label and an upper label. The representations needed to subduce the row label of R_1 are obtained by starting with R. The second representation is obtained by dropping the box with upper label equal to 1; the third representation is obtained from the second by dropping the box with upper label equal to 2 and so on until the box with label k is dropped. The representations needed to subduce the column label are obtained in exactly the same way except that instead of using the upper label, we now use the lower label. For example, block A of $\Gamma_R(\sigma))$ in Figure 2 is labeled by:



For further details and explicit examples, see [4].

4.2 Open String Words

The open string words built out of the Z and Y matrices are of the form:

$$W_j^i = (YZ...ZYZ...ZYZ...ZY)_j^i.$$

We can label these open string words as:

$$(W(\{n_1, n_2, \cdots, n_{L-1}\}))^i_j = (YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_{L-1}}Y)^i_j. \tag{15}$$

where $\{n_1, n_2, \cdots, n_{L-1}\}$ are Cuntz lattice occupation numbers (the Cuntz oscillator chain will be discussed in the next section). The giant is built out of Zs; the first and last letters of the open string word W are not Zs. We will always use L to denote the number of Y fields in the open string word and $J = n_1 + n_2 + \cdots + n_{L-1}$ to denote the number of Z fields in the open string word. The number of fields in each word is $J + L \approx L$ in the case that $J \ll L$ which we will assume in this dissertation. For the words $W^{(1)}, W^{(2)}$ to be dual to open strings, we need to take $L \sim O(\sqrt{N})$. We do not know how to contract the open strings words exactly; when contracting the open string words, only the planar diagrams are summed. To suppress the non-planar contributions we take $\frac{L^2}{N} \ll 1$. To do this we consider a double scaling limit in which the first limit takes $N \to \infty$ holding $\frac{L^2}{N}$ fixed and the second limit takes the effective genus counting parameter $\frac{L^2}{N}$ to zero. Taking the limits in this way corresponds, in the dual string theory, to taking the string coupling to zero, in the string theory constructed in a fixed giant graviton background. Since the two strings are distinguishable they are represented by distinct words and hence, in the large N limit, we have

$$\langle W^{(i)}(W^{(j)})^\dagger \rangle \propto \delta^{ij}.$$

When computing a correlator of two restricted Schur polynomials, the fields belonging to the giants in the two systems of excited giant gravitons are contracted amongst each other, the fields in the first open string of each are contracted amongst each other and the fields in the second open string are contracted amongst each other. We drop the contributions coming from contractions between Zs in the open strings and Zs associated to the brane system, as well as contractions between Zs in different open string words. When computing two point functions in free field theory, if the number of

boxes in the representation R is less than[1] $O(N^2)$ and the numbers of Z's in the open string is $O(1)$, the contractions between any Zs in the open string and the rest of the operator are suppressed in the large N limit[17]. Contractions between Zs in different open string words are non planar and are hence subleading (clearly there are no large combinatoric factors that modify this).

An important parameter of our excited giant graviton system is $N - b_0$. This parameter can scale as $O(N)$, $O(\sqrt{N})$ or $O(1)$. In section 8.3, we will see that when $N-b_0$ is $O(1)$ the sphere giant boundary interaction is $O(\frac{1}{N})$, when $N-b_0$ is $O(\sqrt{N})$ the boundary interaction is $O(\frac{1}{\sqrt{N}})$ and when $N - b_0$ is $O(N)$, the boundary interaction is $O(1)$. Since we want to explore the dynamics arising from the boundary interaction, we will assume that $N - b_0$ is $O(N)$.

As will be discussed in section 5, the giant boundstate and the open string can exchange momentum. Thus the value of J is not a parameter that we can choose, but rather, it is determined by the dynamics of the problem. Cases in which J becomes large correspond to the situation in which a lot of momentum is transferred from the giant to the open string, presumably signaling an instability. See [18] for a good physical discussion of this instability. In cases where J is large, back reaction is important and the approximations we are employing are no longer valid. This will happen when J becomes $O(\sqrt{N})$ since the assumption that we can drop non-planar contributions when contracting the open string words breaks down. Essentially this is because as more and more Zs hop onto the open string, it is starting to grow into a state which is eventually best described as a giant graviton itself. One can also no longer neglect the contractions between any Zs in the open string and the rest of the operator, presumably because the composite system no longer looks like a string plus giant (which can be separated nicely) but rather, it starts to look like one large deformed threebrane. Thus, the fact that our approximation breaks down has a straight forward interpretation: We have set up our description by assuming that the operator we study is dual to a threebrane with an open string attached. This implies that our operator can

[1]When the number of operators in the Young diagram is $O(N^2)$, the operator is dual to an LLM geometry[16].

be decomposed into a "threebrane piece" and a "string piece". These two pieces are treated very differently: when contracting the threebrane piece, all contractions are summed; when contracting the string piece, only planar contractions are summed. Contractions between the two pieces are dropped. When a large number of Zs hop onto the open string our operator is simply not dual to a state that looks like a threebrane with an open string attached and our approximations are not valid. We are not claiming that this operator can not be studied using large N techniques - it may still be possible to set up a systematic $1/N$ expansion. We are claiming that the diagrams we have summed do not give this approximation.

5 Cuntz Chain Hamiltonian

Our goal is to compute the one loop anomalous dimensions of operators in the $\mathcal{N} = 4$ super Yang Mills theory which are dual to open strings ending on bound states of giant gravitons. It has been demonstrated that the one loop anomalous dimensions of operators dual to closed string states give rise to an integrable spin chain [19], [20]. Bethe Ansatz techniques can be utilized to solve the resulting integrable spin chain model describing the full planar one loop spectrum of anomalous dimensions, [20]. Attempts to utilize a similar approach for operators dual to open strings are hindered by the fact that the open string and the giant graviton to which it is attached can exchange momentum and thus the number of sites in the spin chain lattice is no longer fixed (i.e. it becomes a dynamical variable). This difficulty was overcome with the introduction of the Cuntz oscillator chain which has a fixed number of lattice sites [21]. The Cuntz chain uses one of the matrices defined above, Y say, to define a lattice which is populated by another of the previously described matrices, Z for instance. This is in contrast to how the spin chain is obtained - under restricting to the SU(2) sector, one of the matrices, Y say, is mapped into spin down and the other, Z, is mapped into spin up. Many encouraging results utilizing the Cuntz oscillator approach have been obtained - the coherent state expectation value of the Cuntz chain Hamiltonian reproduces the open string action for an open string attached to a sphere giant in $AdS_5 \times S^5$ [21], [18] for an open string attached to an AdS giant in $AdS_5 \times S^5$ [22] and for an open string attached to a sphere giant in a deformed $AdS_5 \times S^5$ background[23]. This parallels the spin chain results for the closed string where the low energy description of the spin chain relevant for closed string states appearing on the field theory side matches perfectly with the low energy limit of the string action in $AdS_5 \times S^5$ [24]. This is an important result because it shows how a string action can emerge from large N gauge theory.

The Hamiltonian for this Cuntz oscillator chain consists of two parts: the bulk term which describes the transposition of adjacent Y's and Zs in the Cuntz oscillator chain (i.e. the open string word (17)) and boundary inter-

action terms. It is useful to decompose the potential for the scalars into D terms and F terms. The advantage of this decomposition is that it is known that at one loop, the D term contributions cancel with the gauge boson exchange and the scalar self energies[25]. Consequently we will only consider the planar interactions arising from the F term. The F term interaction preserves the number of Y's (the lattice is not dynamical) and allows impurities (the Zs) to hop between neighboring sites. The bulk term has the form:

$$H_{bulk} = 2\lambda \sum_{l=1}^{L} \hat{a}_l^\dagger \hat{a}_l - \lambda \sum_{l=1}^{L-1} (\hat{a}_l^\dagger \hat{a}_{l+1} + \hat{a}_l \hat{a}_{l+1}^\dagger), \tag{16}$$

where

$$\hat{a}_i \hat{a}_i^\dagger = I, \qquad \hat{a}_i^\dagger \hat{a}_i = I - |0\rangle\langle 0|.$$

See [18] for the derivation of this result. The boundary interaction terms arise from the interaction of the open string with the giant to which it is attached. This interaction introduces sources and sinks for the impurities at the boundaries of the lattice. The boundary interaction allows Zs to hop from the string onto the giant, or from the giant onto the string. Since the number of Zs gives the angular momentum of the system in the plane that the giant is orbiting in, the boundary interaction allows the string and the brane to exchange angular momentum. We can classify the different types of boundary interaction depending on whether momentum flows from the string to the brane or from the brane to the string. Consider the interaction that allows a Z to hop from the first or last site of either string onto the giant. In this process the string loses momentum to the giant graviton. We call this a "hop off" process because a Z has hopped off the string. The opposite process in which a Z hops off the brane and onto the string is called a "hop on" process. In the "hop on" process the giant loses momentum to the string. In addition to these momentum exchanging processes, there is also a boundary interaction in which a Z belonging to the giant "kisses" the first (or last) Y in the open string word so that no momentum is exchanged. We call this the kissing interaction. To derive the boundary interactions

and hence the full Cuntz chain Hamiltonian for multiple strings attached to an arbitrary number of giant gravitons, we need to be able to compute the two point functions of restricted Schur polynomials dual to such systems. In addition to the technology developed in [4] we need to be able to calculate restricted characters for stretched string states. We treat this in the next section.

6 Intertwiners

We now describe how to obtain the intertwiner projection operators used to construct the gauge theory operators dual to states with open strings stretching between giant gravitons.

6.1 Strings stretching between two branes

The Gauss Law is a strict constraint on the allowed excited brane configurations [3] : since the branes we consider have a compact world volume, the total charge on any given brane must vanish. This implies that to construct a state with strings stretching between two branes, we need at least two strings in the brane plus string system. Thus, in constructing the restricted Schur polynomial, we will need to remove at least two boxes. For concreteness, consider the case of two sphere giants, so that our restricted Schur polynomial is built with the Young diagram R that has two columns and each column has $O(N)$ boxes. R has a total of $n = O(N)$ boxes. Denote the two boxes to be removed in constructing the restricted Schur polynomial by box 1 and box 2. To attach strings stretching between these two giants, the two boxes must belong to different columns. Assume that box 1 belongs to column 1 and box 2 to column 2. After restricting S_n to an S_{n-1} subgroup, representation R subduces irreducible representation R' (whose Young diagram is obtained by removing box 1 from R) and irreducible representation S' (whose Young diagram is obtained by removing box 2 from R). If we now further restrict to an S_{n-2} subgroup, one of the irreducible representations subduced by R' is R'' (whose Young diagram is obtained by removing box 2 from R') and one of the irreducible representations subduced by S' is S'' (whose Young diagram is obtained by removing box 1 from S'). Note that R'' and S'' have the same Young diagram (and hence the same dimension) but act on distinct states in the carrier space of R. The two possible intertwiners we can define map between the states belonging to R'' and the states belonging to S''.

The precise form of the intertwiners depends on the basis used for the S_{n-2} irreducible representations $\Gamma_{R''}(\sigma)$ and $\Gamma_{S''}(\sigma)$. In writing down the intertwiner, we assume that $\Gamma_{R''}(\sigma)$ and $\Gamma_{S''}(\sigma)$ represent σ with the same matrix.

With this assumption, it is possible to put the elements of the basis of the carrier space of R'' into one to one correspondence with the elements of the basis of the carrier space of S'': $|i, R''\rangle \leftrightarrow |i, S''\rangle$. We will use this correspondence below. In a suitable basis, we have

$$\Gamma_R(\sigma) = \begin{bmatrix} \Gamma_{R''}(\sigma) & 0 & \cdots \\ 0 & \Gamma_{S''}(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix},$$

for $\sigma \in S_{n-2}$. In constructing the restricted Schur polynomial, we also consider more general $\sigma \in S_n$. In this case, if $\sigma \notin S_{n-2}$, $\Gamma_R(\sigma)$ will not be block diagonal. Even in this more general case, we will use the labels of the S_{n-2} subduced subspaces to label the carrier space of irreducible representation R. Denote the projection operator that projects from the carrier space of R to the R'' subspace by $P_{R\to R'\to R''}$, and the projection operator that projects from the carrier space of R to the S'' subspace by $P_{R\to S'\to S''}$. Clearly, the intertwiner which maps from S'' to R'' must take the form

$$I_{R'',S''} = P_{R\to R'\to R''} O P_{R\to S'\to S''} = \begin{bmatrix} 0 & M & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}. \tag{17}$$

The second possible intertwiner that we can construct is given by

$$I_{S'',R''} = P_{R\to S'\to S''} O P_{R\to R'\to R''} = \begin{bmatrix} 0 & 0 & \cdots \\ M & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

We want to find a unique specification for O so that M is simply the identity matrix. For $\sigma \in S_{n-2}$ we have

$$\Gamma_R(\sigma) I_{R'',S''} = \begin{bmatrix} 0 & \Gamma_{R''}(\sigma) M & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$I_{R'',S''}\Gamma_R(\sigma) = \begin{bmatrix} 0 & M\Gamma_{S''}(\sigma) & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

Now, by assumption, $\Gamma_{R''}(\sigma) = \Gamma_{S''}(\sigma)$ since we have $\sigma \in S_{n-2}$. Thus,

$$\big[\Gamma_R(\sigma), I_{R'',S''}\big] = \begin{bmatrix} 0 & \big[\Gamma_{R''}(\sigma), M\big] & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}. \tag{18}$$

Applying Schur's Lemma (for irreducible representation R'') to the right hand side implies that M is the identity matrix if and only if $\big[\Gamma_R(\sigma), I_{R'',S''}\big] = 0$ for all $\sigma \in S_{n-2}$. Clearly, for $\sigma \in S_{n-2}$ we have $\big[\Gamma_R(\sigma), P_{R\to R'\to R''}\big] = \big[\Gamma_R(\sigma), P_{R\to S'\to S''}\big] = 0$ so that

$$0 = \big[\Gamma_R(\sigma), I_{R'',S''}\big] = P_{R\to R'\to R''}\big[\Gamma_R(\sigma), O\big]P_{R\to S'\to S''}.$$

Thus, we will require

$$\big[\Gamma_R(\sigma), O\big] = 0, \qquad \forall \sigma \in S_{n-2}. \tag{19}$$

If we specify a condition that determines the normalization of the intertwiner, then this normalization condition and (19) provide the specification for O that we were looking for. The normalization of the intertwiner is fixed by demanding that

$$\mathrm{Tr}\,(M) = \dim(R''),$$

with $\dim(R'')$ the dimension of irreducible representation R''. This provides a unique definition of the intertwiner.

For the example we are considering here, imagine that the S_{n-1} subgroup is obtained as

$$\mathcal{G} = \{\sigma \in S_n | \sigma(n) = n\},$$

and further that the S_{n-2} subgroup is obtained as

$$\mathcal{H} = \{\sigma \in \mathcal{G} | \sigma(n-1) = n-1\}.$$

Then the intertwiner is given by

$$I_{R'',S''} = \mathcal{N} P_{R\to R'\to R''} \Gamma_R(n, n-1) P_{R\to S'\to S''},$$

with

$$\mathcal{N}^{-1} = \frac{\mathrm{Tr}_{R'',S''}(\Gamma_R(n,n-1))}{\dim(R'')} \equiv \sum_{i=1}^{\dim(R'')} \frac{\langle R'', i|\Gamma_R(n,n-1)|S'', i\rangle}{\dim(R'')}.$$

This last equation makes use of the correspondence between the bases of the carrier spaces R'' and S''. Using the technology developed in section 7, we find

$$\frac{\mathrm{Tr}_{R'',S''}(\Gamma_R(n,n-1))}{\dim(R'')} = \sqrt{1 - \frac{1}{(c_1 - c_2)^2}},$$

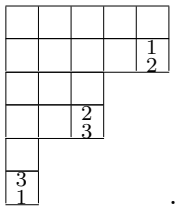
where c_1 and c_2 are the weights associated with box 1 and box 2 respectively. Note that the above trace is invariant under simultaneous similarity transformations of R'' and S''. It will however, change under general similarity transformations so that this last result is dependent on our choice of basis.

6.2 The General Construction

In the previous section we have developed our discussion of the intertwiner using a system of two branes with strings stretching between them. Our conclusion however, is completely general. For any system of branes with strings stretching between the branes, the intertwiner is always given, up to normalization, by the product (projection operator)×(group element)×(projection operator). The Gauss Law forces the net charge on any given brane's worldvolume to vanish. This implies that for every string leaving a brane's worldvolume, there will be a string ending on the worldvolume. Thus, starting with any particular brane with a stretched string attached, we can follow the string to the next brane, switch to the stretched string leaving that brane, follow it and so on, until we again reach the first brane. If we move along k stretched strings before returning to the starting point, the group element is $\Gamma_R(n, n-k+1)$. The normalization factor easily follows using the results of section 7.

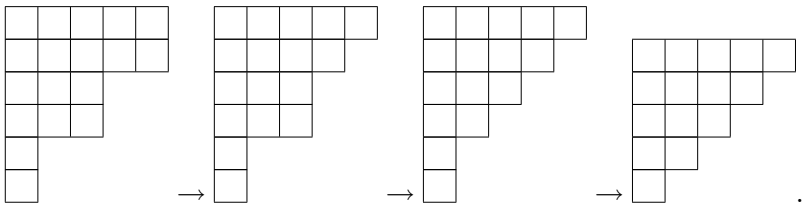
6.3 Example

Consider the excited brane system described by the diagram (see section 4 for a summary of our graphical notation)



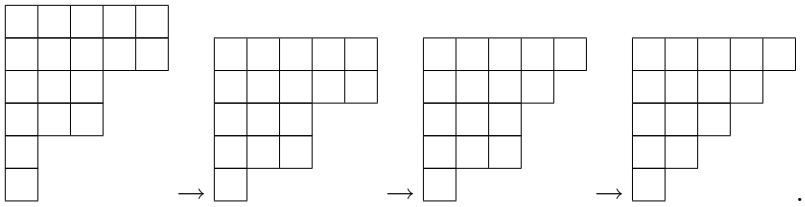
.

The boxes are labeled by the upper index in each box and the weight of box i is denoted c_i. The projector $P_{R\to R_1'''}$ projects through the following sequence of irreducible representations



.

The projector $P_{R\to R_2'''}$ projects through the following sequence of irreducible representations



.

The intertwiner is now given by

$$I_{12} = \mathcal{N} P_{R\to R_2'''} \Gamma_R\left((n, n-2)\right) P_{R\to R_1'''},$$

where

$$\mathcal{N}^{-1} = \frac{\mathrm{Tr}_{R_2''',R_1'''}\left(\Gamma_R\left((n,n-2)\right)\right)}{\dim(R_1''')} = \frac{1}{c_2-c_3}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}},$$

is easily computed using the methods of section 7. To understand the order of the projection operators, note that

$$\begin{aligned} \mathrm{Tr}_{R_1''',R_2'''}\Big(\Gamma_R(\sigma)\Big) &= \sum_i \langle i, R_1''' | \Gamma_R(\sigma) | i, R_2''' \rangle \\ &= \mathrm{Tr}\,(\mathcal{N}^{-1} P_{R\to R_2'''} \Gamma_R(n, n-2) P_{R\to R_1'''} \Gamma_R(\sigma)), \end{aligned}$$

so that the row (column) index of the trace is column (row) index of the intertwiner respectively.

7 Restricted Characters

Starting from S_n, define a chain of subgroups $\mathcal{G}_i$ $i = 1, ..., d$ as follows

$$\mathcal{G}_1 = \{\sigma \in S_n | \sigma(n) = n\} \tag{20}$$

$$\mathcal{G}_i = \{\sigma \in \mathcal{G}_{i-1} | \sigma(n-i+1) = n-i+1\}, \qquad i = 2, 3, ..., d. \tag{21}$$

In this section we will give a simple algorithm for the computation of

$$\chi_{R_1,R_2}\Big((p_1, p_2, ..., p_m)\Big) \equiv \mathrm{Tr}_{R_1,R_2}\Big(\Gamma_R\big((p_1, p_2, ..., p_m)\big)\Big)$$

with R_1 and R_2 irreducible representations of $\mathcal{G}_d$ subduced from R, $(p_1, p_2, ..., p_m)$ is an element of S_n specified using the cycle notation and $n - d < p_i \leq n$ $\forall i$. We call χ_{R_1,R_2} a *restricted character*. If $R_1 = R_2$, we will simply write χ_{R_1}. We have already seen that restricted characters determine the normalization of the intertwiners. Further, they are also needed in the derivation of the hopping identities that determine the interactions between strings and the branes to which they are attached.

In section 7.1 we will derive the algorithm for the computation of the restricted character. Subsequently, we describe in section 7.2 a graphical notation which considerably simplifies the computation. The remainder of section 7 then develops this diagrammatic notation further.

7.1 Computing Restricted Characters

Consider an irreducible representation R of S_n labeled by a Young diagram which has at least two boxes, either of which can be dropped to leave a valid Young diagram. Label these two boxes by 1 and 2. Denote the weights of these boxes by c_1 and c_2. Denote the irreducible representation of S_{n-2} obtained by dropping box 1 and then box 2 by R_1''. Denote the irreducible representation of S_{n-2} obtained by dropping box 2 and then box 1 by R_2''. Our first task is to compute

$$\mathrm{Tr}_{R_1'',R_2''}\left(\Gamma_R\left((n, n-1)\right)\right).$$

Using the subgroup swap rule obtained in [4], we can write

$$\begin{aligned}\chi_{R_1''}\left((n,n-1)\right) &= \left[1-\frac{1}{(c_1-c_2)^2}\right]\chi_{R_2''}\left((n,n-1)\right)+\frac{1}{(c_1-c_2)^2}\chi_{R_1''}\left((n,n-1)\right) \quad (22)\\ &+ \sqrt{1-\frac{1}{(c_1-c_2)^2}}\frac{1}{c_1-c_2}\left[\chi_{R_1'',R_2''}\left((n,n-1)\right)+\chi_{R_2'',R_1''}\left((n,n-1)\right)\right].\end{aligned}$$

A second application of the subgroup swap rule gives

$$\begin{aligned}\chi_{R_2'',R_1''}\left((n,n-1)\right) &= \left[1-\frac{1}{(c_1-c_2)^2}\right]\chi_{R_1'',R_2''}\left((n,n-1)\right)+\frac{1}{(c_1-c_2)^2}\chi_{R_2'',R_1''}\left((n,n-1)\right)\\ &+ \sqrt{1-\frac{1}{(c_1-c_2)^2}}\frac{1}{c_1-c_2}\left[\chi_{R_2''}\left((n,n-1)\right)-\chi_{R_1''}\left((n,n-1)\right)\right]. \quad (23)\end{aligned}$$

Now, substituting the results[4]

$$\chi_{R_1''}\left((n,n-1)\right)=\frac{1}{c_1-c_2}\dim(R_1''), \qquad \chi_{R_2''}\left((n,n-1)\right)=\frac{1}{c_2-c_1}\dim(R_2''),$$

into (22) and (23) and solving, we obtain

$$\chi_{R_1'',R_2''}\left((n,n-1)\right)=\sqrt{1-\frac{1}{(c_1-c_2)^2}}\dim(R_1'')=\chi_{R_2'',R_1''}\left((n,n-1)\right).$$

Next, consider an irreducible representation of S_n labeled by Young diagram R . Choose three boxes in this Young diagram, and label them 1, 2 and 3 respectively. Choose the boxes so that dropping box 1 gives a legal Young diagram R' labeling an irreducible representation of S_{n-1}, dropping box 1 and then box 2 gives a legal Young diagram R'' labeling an irreducible representation of S_{n-2}, and dropping box 1, then box 2 and then box 3 again gives a legal Young diagram R''' labeling an irreducible representation of S_{n-3}. We will compute

$$\chi_{R'''}\left((n,n-2)\right)=\mathrm{Tr}_{R'''}\left(\Gamma_R\left((n,n-2)\right)\right).$$

In what follows, we will frequently need to refer to vectors belonging to the carrier spaces of specific representations subduced by R when boxes are dropped from R. A convenient notation is to list the labels of the boxes that must be dropped from R in the order in which they must be dropped.

Thus, the ket $|i,123\rangle$ is the ith ket belonging to the carrier space of the S_{n-3} irreducible representation obtained by dropping box 1, then box 2 and then box 3 from R; the ket $|j,231\rangle$ is the jth ket belonging to the carrier space of the S_{n-3} irreducible representation obtained by dropping box 2, then box 3 and then box 1 from R (assuming of course that the boxes can be dropped from R in this order, giving a legal Young diagram at each step). Start by writing

$$\chi_{R'''}((n,n-2)) = \sum_{i=1}^{\dim(R''')} \langle i,123|\Gamma_R\left((n,n-2)\right)|i,123\rangle$$
$$= \sum_{i=1}^{\dim(R''')} \langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)\Gamma_R\left((n,n-1)\right)\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle.$$

Noting that $\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle$ must belong to the carrier space of R', and using the completeness relation ($1_{R'}$ is the identity on the R' carrier space)

$$1_{R'} = \sum_{k=1}^{\dim(R')} |k,1\rangle\langle k,1|,$$

we have

$$\chi_{R'''}\left((n,n-2)\right) = \sum_{i=1}^{\dim(R''')}\sum_{j,k=1}^{\dim(R')} \langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)|k,1\rangle\langle k,1|\Gamma_R\left((n,n-1)\right)|j,1\rangle$$
$$\times\langle j,1|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle.$$

Now, decompose R' into a direct sum of S_{n-2} irreducible representations $R' = \oplus R''_\beta$. Use the label β to denote the box that must be dropped from R' to obtain R''_β. Thus, we can write

$$1_{R'} = \sum_{k=1}^{\dim(R')} |k,1\rangle\langle k,1| = \sum_\beta \sum_{k=1}^{\dim(R''_\beta)} |k,1\beta\rangle\langle k,1\beta|,$$

and hence

$$\chi_{R'''}\left((n,n-2)\right) = \sum_{i=1}^{\dim(R''')} \sum_{\beta_1,\beta_2} \sum_{k=1}^{\dim(R''_{\beta_1})} \sum_{j=1}^{\dim(R''_{\beta_2})} \langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)|k,1\beta_1\rangle$$

$$\times\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle\langle j,1\beta_2|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle.$$

Now, introduce the operator $O(2)$ obtained by summing all two cycles of the S_{n-2} subgroup of which the R''_β are irreducible representations. This operator is a Casimir of S_{n-2}. If the Young diagram R''_β has r_i boxes in the ith row and c_i boxes in the ith column, then when acting on the carrier space of R''_β we have[26]

$$O(2)|i,1\beta\rangle = \left[\sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}\right]|i,1\beta\rangle \equiv \lambda_\beta|i,1\beta\rangle.$$

Clearly, for the problem we study here, $\lambda_{\beta_1} = \lambda_{\beta_2}$ if and only if R_{β_1} and R_{β_2} have the same shape as Young diagrams. From the definition of the $\mathcal{G}_2$ subgroup given above, it is clear that

$$\left[O(2),\Gamma_R\left((n,n-1)\right)\right] = 0.$$

It is now a simple matter to see that

$$\begin{aligned}
\lambda_{\beta_1}\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle &= \langle k,1\beta_1|O(2)\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle \\
&= \langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)O(2)|j,1\beta_2\rangle \\
&= \lambda_{\beta_2}\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle
\end{aligned}$$

so that $\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle$ vanishes if R_{β_1} and R_{β_2} do not have the same shape. A completely parallel argument, using a Casimir of S_{n-3}, can be used to show that $\langle j,1\alpha_1\alpha_2|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle$ is only non-zero if $\alpha_1=2$, $\alpha_2=3$ or $\alpha_1=3$, $\alpha_2=2$. Thus,

$$\chi_{R'''}\left((n,n-2)\right) = \sum_{i=1,j,k}^{\dim(R''')} \Big[\langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)|k,123\rangle\langle k,123|\Gamma_R\left((n,n-1)\right)|j,123\rangle$$

$$
\begin{aligned}
&\times \quad \langle j,123|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle + \langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)|k,132\rangle \\
&\times \quad \langle k,132|\Gamma_{R}\left((n,n-1)\right)|j,132\rangle\langle j,132|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle\Big] \\
&= \quad \left[\frac{1}{(c_2-c_3)^2}\frac{1}{c_1-c_2} + \left(1-\frac{1}{(c_2-c_3)^2}\right)\frac{1}{c_1-c_3}\right]\dim(R'''').
\end{aligned}
$$

This example illustrates the general algorithm to be used to compute restricted characters:

- The group element whose trace is to be computed, can be decomposed into a product of two cycles of the form $\Gamma_R\left((i,i+1)\right)$. A complete set of states is inserted between each factor.
- Using appropriately chosen Casimirs, one can argue that the only non-zero matrix elements of each factor, are obtained when the order of boxes dropped to obtain the carrier space of the bra matches the order of boxes dropped to obtain the carrier space of the ket, except for the $(n-i+1)^{\text{th}}$ and $(n-i+2)^{\text{th}}$ boxes, whose order can be swapped.
- We can plug in the known value of the restricted character, which we have computed for precisely the two cases arising in the previous point.

7.2 Strand Diagrams

Strand diagrams are a graphical notation designed to compute restricted characters. Strand diagrams keep track of two things:

- The order in which boxes are to be dropped and the identity (= position within the Young diagram) of the boxes.
- The group element whose trace we are computing.

If we are to drop n boxes, we draw a picture with n columns. The columns are populated by labeled strands - each strand represents one of the boxes that are to be dropped. We label the strands by the upper index in the box. Here we make use of the graphical notation summarized in section 4. Whatever appears in the first column is to be dropped first; whatever appears in the

second column is to be dropped second and so on. The strands are ordered at the top of the diagram, according to the order in which they must be dropped to get the row index. The strands are ordered at the bottom of the diagram according to the column index. The strands move from the top of the diagram to the bottom of the diagram, without breaking, so that strands ends at the top connect to the corresponding strand ends at the bottom. To connect the strands (which in general are in a different order at the top and bottom of the diagram) we need to weave the strands, thereby allowing them to swap columns. The allowed swaps depends on the specific group element whose trace we are computing. To determine the allowed swaps, write the group element as a product of cycles of the form $(i, i+1)$. For example, we would write

$$(n, n-2) = (n, n-1)(n-1, n-2)(n, n-1).$$

Each time we drop a box, we are considering a new subgroup. The action of the permutation group can be visualized as a permutation of n indices. The subgroups are obtained by considering elements that hold certain indices fixed (see (20) and (21)). Choose the subgroups involved so that when box i is dropped, $n-i+1$ is held fixed. Clearly then, each column j is associated with the index $n-j+1$. Each cycle $(i, i+1)$ is drawn as a box which straddles the columns associated with indices i and $i+1$. When the strands pass through a box, they may do so without swapping or by swapping columns. Each box is associated with a factor. Imagine that the strands passing through the box, reading from left to right, are labeled n and m. The weights associated with these boxes are c_n and c_m respectively. If the strands do not swap inside the box the factor for the box is

$$f_{\text{no swap}} = \frac{1}{c_n - c_m}.$$

If the strands do swap inside the box, the factor is

$$f_{\text{swap}} = \sqrt{1 - \frac{1}{(c_n - c_m)^2}}.$$

Denote the product of the factors, one from each box, by F. We have

$$\mathrm{Tr}_{R_1,R_2}\Big(\Gamma_R(\sigma)\Big) = \sum_i F_i \dim(R_1),$$

where the index i runs over all possible paths consistent with the boundary conditions. With a little thought, the astute reader should be able to convince herself that this graphical rule is nothing but a convenient representation of the computation of the last subsection.

7.3 Strand Diagram Examples

In this section we will illustrate the use of strand diagrams in the computation of restricted characters. For our first example, we consider the computation of

$$\chi_1 = \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline & & 1\\\hline & 2 & 3\\\hline 3 & 1\\\cline{1-2} 2\\\cline{1-1}\end{array}}\Big(\Gamma_{\begin{array}{|c|c|c|}\hline & & \\\hline & & \\\cline{2-3}\end{array}}((6,4))\Big).$$

Writing $(6,4) = (6,5)(4,5)(6,5)$ we obtain the strand diagram shown in Figure

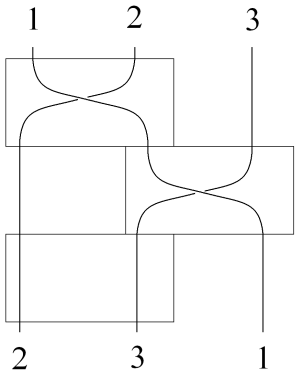


Figure 3: The strand diagram used in the computation of χ_1.

3. The factors for the upper most, middle and lower most boxes are

$$\sqrt{1-\frac{1}{(c_1-c_2)^2}}, \qquad \sqrt{1-\frac{1}{(c_1-c_3)^2}}, \qquad \frac{1}{c_2-c_3}$$

respectively. Thus,

$$\begin{aligned}\chi_1 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}}\frac{1}{c_2-c_3}\dim(\ydiagram{2,1}) \\ &= 2\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}}\frac{1}{c_2-c_3}.\end{aligned}$$

The alert reader may worry that our recipe is not unique. Indeed we could also have written $(6,4) = (4,5)(6,5)(4,5)$. In this case, we obtain the strand diagram given in Figure 4. In this case, the factors for the upper most, middle and lower most boxes are

$$\frac{1}{c_2-c_3}, \qquad \sqrt{1-\frac{1}{(c_1-c_2)^2}}, \qquad \sqrt{1-\frac{1}{(c_1-c_3)^2}}$$

respectively. This gives exactly the same value for χ_1. Next, we consider the

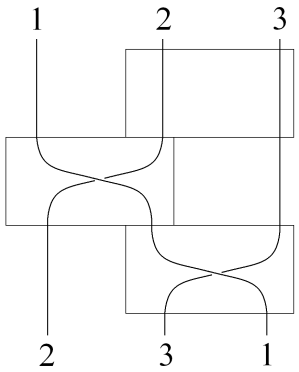


Figure 4: A second strand diagram that can be used in the computation of χ_1.

computation of

$$\chi_2 = \mathrm{Tr}\ydiagram{3,2,1}\Big(\Gamma\ydiagram{3,2,1}((6,4))\Big).$$

This example is interesting as more than one path contributes. Writing $(6,4) = (4,5)(6,5)(4,5)$ we obtain the strand diagrams shown in Figure 5. The

product of factors for the diagram on the left is

$$\frac{1}{c_1 - c_3}\left[1 - \frac{1}{(c_2 - c_3)^2}\right].$$

The product of factors for the diagram on the right is

$$\frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_3)^2}.$$

Thus,

$$\begin{aligned}\chi_2 &= \left(\frac{1}{c_1 - c_3}\left[1 - \frac{1}{(c_2 - c_3)^2}\right] + \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_3)^2}\right)\dim(\ydiagram{2,1}) \\ &= 2\left(\frac{1}{c_1 - c_3}\left[1 - \frac{1}{(c_2 - c_3)^2}\right] + \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_3)^2}\right).\end{aligned}$$

The reader can check that the same value for χ_2 is obtained by decomposing $(6,4) = (6,5)(4,5)(6,5)$.

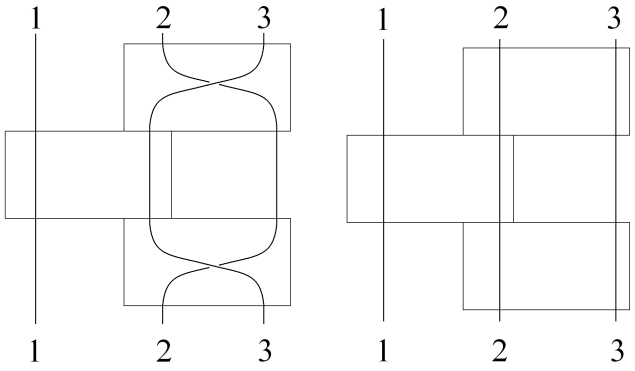


Figure 5: The strand diagrams used in the computation of χ_2.

Finally, consider

$$\chi_3 = \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}}\Big(\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}}(1)\Big).$$

Since we consider the identity element, the strand diagram has no boxes and

hence $\chi_3 = \dim(\ydiagram{2,2}) = 2$. Since $(4,5)(4,5) = 1$ we could also have written

$$\chi_3 = \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline & & 1\\\hline & 2\\\cline{1-2} 3\\\cline{1-1}\end{array}}\Big(\Gamma_{\begin{array}{|c|c|c|}\hline & & \\\hline & \\\cline{1-2} \\\cline{1-1}\end{array}}((4,5)(4,5))\Big).$$

In this case there are two strand diagrams given in Figure 6. Summing the contributions from these two strand diagrams we obtain

$$\chi_3 = \frac{1}{(c_2-c_3)^2}\dim(\ydiagram{2,2}) + \left(1 - \frac{1}{(c_2-c_3)^2}\right)\dim(\ydiagram{2,2}) = \dim(\ydiagram{2,2}) = 2.$$

Once again, the two ways of writing the restricted character give the same result. Note that the trace

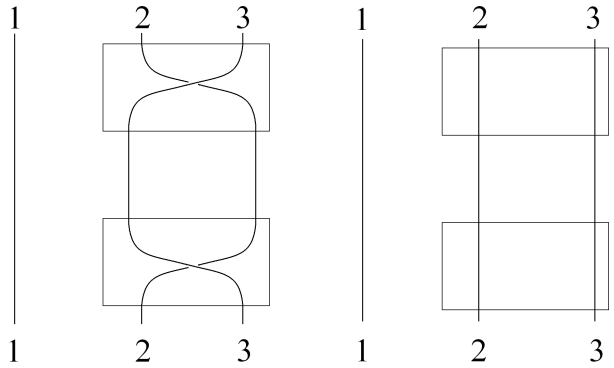


Figure 6: The strand diagrams used in the computation of χ_3.

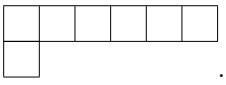
$$\chi_3 = \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline & & \frac{1}{2}\\\hline & \frac{2}{1}\\\cline{1-2} 3\\\cline{1-1}\end{array}}\Big(\Gamma_{\begin{array}{|c|c|c|}\hline & & \\\hline & \\\cline{1-2} \\\cline{1-1}\end{array}}((1))\Big),$$

clearly vanishes because we are tracing the identity over an off the diagonal block. This is reflected graphically by the fact that there is no strand diagram that can be drawn - the order of strands at the top of the diagram does not match the order of strands at the bottom of the diagram and since we consider the identity element, the strand diagram has no boxes.

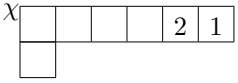
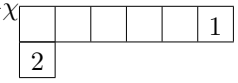
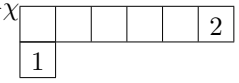
See appendix A for an application of strand diagrams to obtaining irreducible matrix representations of S_n.

7.4 Tests of the Restricted Character Results

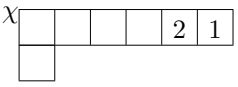
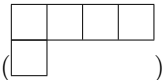
By summing well chosen restricted characters, one can recover the characters of S_n which are known. This provides a number of tests that our restricted character formulas pass. As an example, consider the computation of $\chi_R((6,7))$ for

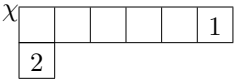
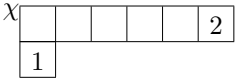
$R =$ .

From the character tables for S_7 we find $\chi_R((6,7)) = 4$. In terms of restricted characters

$\chi_R((6,7)) = \chi$  $((6,7)) + \chi$  $((6,7)) + \chi$  $((6,7))$.

Using the algorithm given above, it is straight forward to verify that

χ  $((6,7)) = \dim($  $) = 4,$

χ  $((6,7)) = \frac{1}{6}, \quad \chi$  $((6,7)) = -\frac{1}{6},$

which do indeed sum to give 4. The reader is invited to check some more examples herself.

As a further check of our methods, we have computed the restricted characters $\mathrm{Tr}_{R_1,R_2}(\Gamma_R[\sigma])$ numerically. This was done by explicitly constructing the matrices $\Gamma_R[\sigma]$. Each representation used was obtained by induction. One induces a reducible representation; the irreducible representation that participates was isolated using projection operators built from the Casimir obtained by summing over all two cycles. See appendix B.2 of [4] for more details. The resulting irreducible representations were tested by verifying

the multiplication table of S_n. The intertwiners were computed using the projection operators of [4] and the results of section 6; the normalization of the intertwiner was computed numerically.

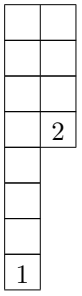
8 General Cuntz Chain Hamiltonian

8.1 Overview and Notation

We now derive the full Cuntz chain Hamiltonian for a general open string - giant system with any number of open strings attached and some or all of the strings stretching between the giants. As discussed in section 5, the full Cuntz chain Hamiltonian consists of a bulk term (18) describing the hopping of Z's into and out of lattice sites set up by the Y's in the open string word (i.e. the bulk of the open string) and boundary interaction terms describing the transfer of angular momentum to the giant from the open string and vice versa as well as the kissing interaction in which no momentum is transferred. Since we already have the bulk term (18), we now need to derive the boundary interaction terms. To make our discussion concrete, we mostly consider the specific example of two strings attached to a bound state of two sphere giants[2]. Note however, that most of the formulas we derive (and certainly all the techniques we develop) are applicable to the general problem. Both the strings and the branes that we consider are distinguishable. In this case there are a total of six possible states. For a bound state of two sphere giant gravitons, we need to consider restricted Schur polynomials labeled by Young diagrams with two columns each with $O(N)$ boxes. Denote the number of boxes in the first column by b_0+b_1 and the number of boxes in the second column by b_0. Thus we choose b_0 to be $O(N)$ (b_1 is chosen to be $O(1)$). It is natural to interpret the number of boxes in each column as the momentum of each giant. We can use the state operator correspondence (see section 8.2.5 and appendix B for further discussion) to associate a Cuntz chain state with each restricted Schur polynomial. The Cuntz chain states have six labels in total: the first two labels are b_0 and b_1 which determine the momenta of the two giants; the next two labels are the branes on which the endpoints of string one are attached and the final two labels are the branes on which the endpoints of string two are attached. We label the strings by '1' and '2'. The brane corresponding to column 1 of the

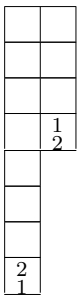
[2]In Appendix C we consider a boundstate of three sphere giants with two open strings attached.

Young diagram is labeled 'b' (for big brane) and the brane corresponding to column 2 of the Young diagram is labeled 'l' (for little brane). Since the second column of a Young diagram can never contain more boxes that the first column, and since the radius of the giant graviton is determined by the square root of its angular momentum, these are accurate labels. Consider a state with string 1 on big brane and string 2 on little brane. The restricted Schur polynomial (written using the graphical notation of [4],[5]) together with the corresponding Cuntz chain state are (in this case, $b_0 = 3$ and $b_1 = 4$)



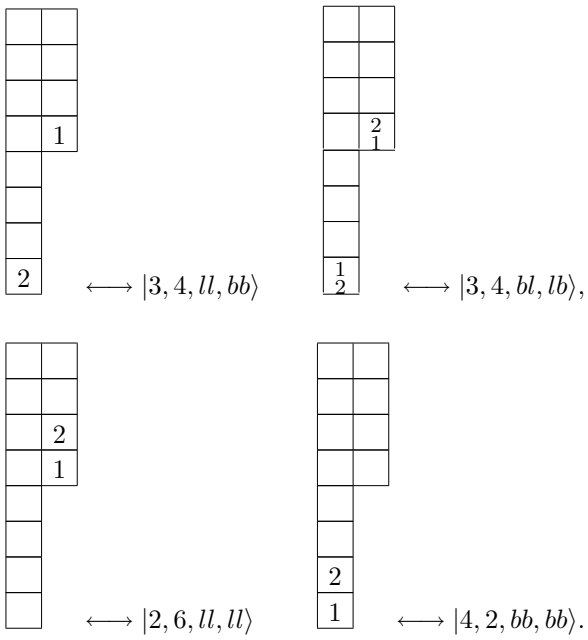
$\longleftrightarrow |3, 4, bb, ll\rangle.$

We will call states with strings stretching between branes "stretched string states". When labeling the Cuntz chain state corresponding to a stretched string state, we will write the end point label corresponding to the upper index first. Thus,



$\longleftrightarrow |3, 4, lb, bl\rangle.$

The remaing four states are



The construction of the operators dual to excitations described by strings stretching between the branes is facilitated by the results of sections 6 and 7. In the notation of [4], we assume that when the restricted Schur polynomial is to be reduced, string 1 is removed first and string 2 second. This implies that, when using the graphical notation, removing the box occupied by string 1 first will always leave a valid Young diagram. This choice is arbitrary, but useful for explicit computation. Once we have the form of the Hamiltonian, we can always change to a "physical basis".

We now derive a set of identities that allow us to compute the term in the Hamiltonian describing the "hop off" process. These identities make extensive use of the technology for computing restricted characters which we have developed in sections 6 and 7. Concretely, these identities allow us to express objects like $\chi^{(2)}_{R,R''}(Z, ZW^{(1)}, W^{(2)})$ in terms of $\chi^{(2)}_{S,S''}(Z, W^{(1)}, W^{(2)})$ where S is a Young diagram with one more box than R[3]. This then allows us to describe the boundary interaction that results when the hopping interaction

[3]The number of primes on the label of the restricted Schur polynomial indicates how many boxes are dropped, i.e. R'' is obtained by dropping two boxes from R.

(described by the bulk hamiltonian) causes a Z to hop past a Y marking an endpoint of the open string (17) and subsequently be transferred to the giant (i.e. hop off). Since the Hamiltonian must be Hermitian, we can obtain the "hop on" term by daggering the "hop off" term. Finally, we obtain the momentum conserving boundary interaction by expressing the kiss as a hop on followed by a hop off. This determines the complete Cuntz oscillator chain Hamiltonian needed for a one loop computation of the anomalous dimensions of operators dual to excited giant graviton bound states.

8.2 Hopping Identities

In this section, we derive identities that can be used to obtain the Cuntz chain Hamiltonian that accounts for the $O(g^2_{YM})$ correction to the anomalous dimension of our operators. To construct the "hop off" process, we use the fact that whenever a Z field hops past the borders of the open string word W, the resulting restricted Schur polynomial decomposes into a sum of two types of systems, one is a giant with a closed string and another is a string-giant system where the giant is now bigger. In the large N limit only the second type needs to be considered. The identities we derive in this section express this decomposition. The irreducible representations which play a role in the derivation of the identities are illustrated in Figure 7. The basic structure of the derivation of these identities is very similar. For this reason, we explicitly derive an identity in the next subsection and simply state the remaining identities. In contrast to the case of a single string attached[5], here it does make a difference if the first or last sites of the string participate in the hopping. The identities needed in these two cases are listed separately. We have performed extensive numerical checks of the identities, which we describe next. Finally, we explain how to express the leading large N form of the identities, in terms of states of the Cuntz chain.

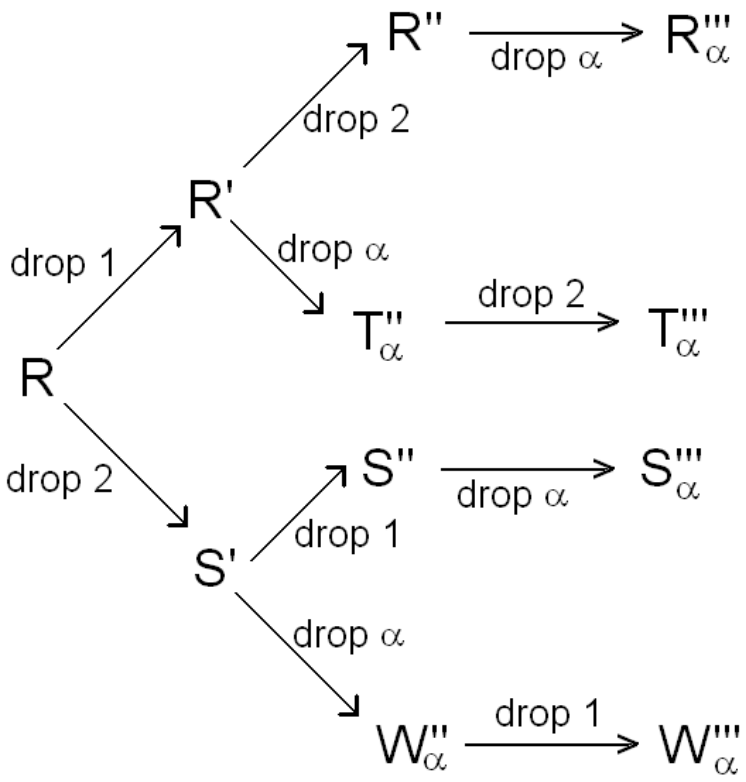


Figure 7: This figure shows the irreducible representations that are used in the hopping identities. Starting from R, the figure shows which irreducible representation is obtained when boxes in R are dropped.

8.2.1 Derivation of a Hopping Identity

Our starting point is the restricted Schur polynomial

$$\chi^{(2)}_{R,R''}\Big|_1\Big|_2 = \frac{1}{(n-2)!}\sum_{\sigma\in S_n} \mathrm{Tr}_{R''}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}(W^{(1)})^{i_n}_{i_{\sigma(n)}}.$$

There are two labeled boxes in R; dropping box 1 gives irreducible representation R'; dropping box 2 gives irreducible representation R''. Since R' is an irreducible representation of the S_{n-1} subgroup $\mathcal{G}_1 = \{\sigma \in S_n | \sigma(n) = n\}$, we say that the open string described by the word $W^{(1)}$ is associated to box 1. Since R'' is an irreducible representation of the S_{n-2} subgroup $\mathcal{G}_2 = \{\sigma \in \mathcal{G}_1 | \sigma(n-1) = n-1\}$, we say that the open string described by the word $W^{(2)}$ is associated with

box 2. Notice that, in the chain of subductions used to define the restricted Schur polynomial, the box associated with $W^{(1)}$ is dropped before the box associated to $W^{(2)}$. We have indicated this with the notation $\Big|_1\Big|_2$. Rewrite the sum over S_n as a sum over $\mathcal{G}_1$ and its cosets

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \frac{1}{(n-2)!}\sum_{\sigma\in\mathcal{G}_1}\Big[\mathrm{Tr}_{R''}\left(\Gamma_{R'}(\sigma)\right)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}\mathrm{Tr}\,(W^{(1)})\\ &+ \mathrm{Tr}_{R''}\left(\Gamma_R((1,n)\sigma)\right)(W^{(1)}Z)^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}+\cdots+\\ &+ \mathrm{Tr}_{R''}\left(\Gamma_R((n-2,n)\sigma)\right)Z^{i_1}_{i_{\sigma(1)}}\cdots(W^{(1)}Z)^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}+\\ &+ \mathrm{Tr}_{R''}\left(\Gamma_R((n-1,n)\sigma)\right)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}((W^{(1)}W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}\Big].\end{aligned}$$

The first term on the right hand side is

$$\frac{1}{(n-2)!}\sum_{\sigma\in\mathcal{G}_1}\mathrm{Tr}_{R''}\left(\Gamma_{R'}(\sigma)\right)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}\mathrm{Tr}\,(W^{(1)})=\chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)}).$$

Using the methods of section 7, we know that

$$\mathrm{Tr}_{R''}\left(\Gamma_R((n-1,n)\sigma)\right)=\frac{1}{c_1-c_2}\mathrm{Tr}_{R''}\left(\Gamma_{R'}(\sigma)\right),$$

so that the last term on the right hand side is

$$\frac{1}{(n-2)!}\sum_{\sigma\in\mathcal{G}_1}\mathrm{Tr}_{R''}\left(\Gamma_R((n,n-1)\sigma)\right)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(1)}W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}$$

$$=\frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)}).$$

Focus on the remaining terms on the right hand side. Each of these terms makes the same contribution. We need to evaluate

$$\mathrm{Tr}_{R''}\left(\Gamma_R((j,n)\sigma)=\sum_{i=1}^{\dim(R'')}\langle i,12|\Gamma_R((j,n))\Gamma_{R'}(\sigma)|i,12\rangle.\right.$$

Using the techniques of section 7, it is straight forward to show that (the sum on α in the next equation is a sum over all boxes that can be removed from R'' to leave a valid Young diagram; the relevant S_{n-3} subgroup is given

by $\{\sigma \in \mathcal{G}_2 | \sigma(j) = j\}$)

$$\begin{aligned}
\mathrm{Tr}_{R''}(\Gamma_R((j,n)\sigma) &= \sum_\alpha \sum_{i,k=1}^{\dim(R'''_\alpha)} \langle i, 12\alpha|\Gamma_R((j,n))|k,12\alpha\rangle\langle k,12\alpha|\Gamma_{R'}(\sigma)|i,12\alpha\rangle \\
&+ \sum_\alpha \sum_{i,k=1}^{\dim(R'''_\alpha)} \langle i, 12\alpha|\Gamma_R((j,n))|k,1\alpha 2\rangle\langle k,1\alpha 2|\Gamma_{R'}(\sigma)|i,12\alpha\rangle \\
&= \sum_\alpha \frac{1}{c_1-c_\alpha}\left[1+\frac{1}{(c_1-c_2)(c_2-c_\alpha)}\right]\mathrm{Tr}_{R'''_\alpha}(\Gamma_{R'}(\sigma)) \\
&+ \sum_\alpha \frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\mathrm{Tr}_{T'''_\alpha,R'''_\alpha}(\Gamma_{R'}(\sigma)).
\end{aligned}$$

Thus, summing the remaining $n-2$ terms we obtain

$$\begin{aligned}
&\sum_\alpha \frac{1}{c_1-c_\alpha}\left[1+\frac{1}{(c_1-c_2)(c_2-c_\alpha)}\right]\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1 \\
+ &\sum_\alpha \frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha,R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1.
\end{aligned}$$

A straight forward application of the subgroup swap rule gives

$$\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1 = \left[\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right.$$

$$+\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})+\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\frac{1}{c_2-c_\alpha}\left(\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right.$$

$$\left.\left.+\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right)\right]\Big|_1\Big|_2,$$

$$\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1 = \left[\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right.$$

$$-\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})+\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\frac{1}{c_2-c_\alpha}\left(\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right.$$

$$\left.\left.-\chi^{(2)}_{R',T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right)\right]\Big|_1\Big|_2.$$

Thus, we finally obtain

$$\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 = \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)}) + \frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)})$$

$$
\begin{aligned}
&+ \quad \sum_\alpha \left[\frac{1}{c_1 - c_\alpha} \left(1 - \frac{1}{(c_2 - c_\alpha)^2} \right) \chi^{(2)}_{R', T'''_\alpha}(Z, W^{(1)} Z, W^{(2)}) \right. \\
&+ \quad \frac{1}{c_1 - c_2} \frac{1}{(c_2 - c_\alpha)^2} \chi^{(2)}_{R', R'''_\alpha}(Z, W^{(1)} Z, W^{(2)}) \\
&+ \quad \frac{1}{c_1 - c_2} \frac{1}{c_2 - c_\alpha} \sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}} \chi^{(2)}_{R' \to R'''_\alpha T'''_\alpha}(Z, W^{(1)} Z, W^{(2)}) \\
&+ \quad \left. \frac{1}{c_1 - c_\alpha} \frac{1}{c_2 - c_\alpha} \sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}} \chi^{(2)}_{R' \to T'''_\alpha R'''_\alpha}(Z, W^{(1)} Z, W^{(2)}) \right] \Big|_1 \Big|_2 .
\end{aligned}
$$

The above identity is relevant for interactions in which the impurity hops out of the last site of the string. For the hopping interaction in which the impurity hops out of the first site of the string, the right hand side of our identity should be written in terms of $ZW^{(1)}$. This identity is easily derived by rewriting the sum over S_n in terms of right cosets of $\mathcal{G}_1$ instead of left cosets as we have done above.

The identity derived above is relevant for the description of interactions in which string 1 exchanges momentum with the branes in the boundstate. To derive identities that allow string 2 to exchange momentum with the branes in the boundstate, we first use the subgroup swap rule to swap strings 1 and 2. We then rewrite the sum over S_n in terms of a sum over S_{n-1} and its cosets and then employ character identities as above. We give a complete set of identities in the next two subsections.

8.2.2 Identities Relevant to Hopping off the first site of the string

$$
\begin{aligned}
\chi^{(2)}_{R,R''}(Z, W^{(1)}, W^{(2)}) \Big|_1 \Big|_2 \quad &= \quad \chi^{(1)}_{R', R''}(Z, W^{(2)}) \mathrm{Tr}\,(W^{(1)}) + \frac{1}{c_1 - c_2} \chi^{(1)}_{R', R''}(Z, W^{(2)} W^{(1)}) \\
&+ \quad \sum_\alpha \left[\frac{1}{c_1 - c_\alpha} \left(1 - \frac{1}{(c_2 - c_\alpha)^2} \right) \chi^{(2)}_{R', T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \right. \\
&+ \quad \frac{1}{c_1 - c_2} \frac{1}{(c_2 - c_\alpha)^2} \chi^{(2)}_{R', R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \qquad (24) \\
&+ \quad \frac{1}{c_1 - c_2} \frac{1}{c_2 - c_\alpha} \sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}} \chi^{(2)}_{R' \to T'''_\alpha R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \\
&+ \quad \left. \frac{1}{c_1 - c_\alpha} \frac{1}{c_2 - c_\alpha} \sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}} \chi^{(2)}_{R' \to R'''_\alpha T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \right] \Big|_1 \Big|_2
\end{aligned}
$$

The form of this identity is rather intuitive. The first term on the right hand side contributes to the process in which the bound state emits string 1; the second term describes the process in which the two open strings join to form one long open string. In both of these processes, the box which string 1 occupied on the left hand side does not appear on the right hand side. These two processes will not contribute to our Cuntz chain Hamiltonian; they are relevant for the description of interactions which change the number of open strings attached to the boundstate and do not contribute at the leading order of the large N expansion.

It is instructive to consider the form of this identity for well separated branes. For well separated branes, we have $|c_1 - c_2| \gg 1$. For $|c_1 - c_\alpha| \sim 1$, $|c_2 - c_\alpha| \gg 1$ so that of the last four terms only the first one contributes, giving $\approx \frac{1}{c_1 - c_\alpha} \chi^{(2)}_{R', T''_\alpha}(Z, ZW^{(1)}, W^{(2)})$. Thus, string 2 stays in box 2 and string 1 is close to where it started. Note that dropping terms of order $(c_1 - c_2)^{-1}$ or $(c_\alpha - c_2)^{-1}$ we obtain

$$\chi^{(2)}_{R,R''}(Z, W^{(1)}, W^{(2)})\Big|_1\Big|_2 \approx \chi^{(1)}_{R',R''}(Z, W^{(2)}) \mathrm{Tr}\,(W^{(1)}) + \sum_\alpha \frac{1}{c_1 - c_\alpha} \chi^{(2)}_{R', T''_\alpha}(Z, ZW^{(1)}, W^{(2)}),$$

which is the identity of [5].

Next, consider the stretched string identities

$$\begin{aligned}\chi^{(2)}_{R\to R''S''}(Z, W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\, \chi^{(1)}_{R',R''}(Z, W^{(2)} W^{(1)}) \\ &+ \sum_\alpha \left[\frac{1}{c_1 - c_\alpha} \frac{1}{c_2 - c_\alpha} \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\, \chi^{(2)}_{R', R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \right. \\ &\left. + \frac{1}{c_1 - c_\alpha} \sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}} \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\, \chi^{(2)}_{R' \to T''_\alpha R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \right] \Big|_1\Big|_2 \end{aligned} \tag{25}$$

$$\begin{aligned}\chi^{(2)}_{R\to S''R''}(Z, W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\, \chi^{(1)}_{S',S''}(Z, W^{(2)} W^{(1)}) \\ &+ \sum_\alpha \left[\frac{1}{c_1 - c_\alpha} \frac{1}{c_2 - c_\alpha} \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\, \chi^{(2)}_{S', S'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \right. \end{aligned} \tag{26}$$

$$+\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_\alpha)^2}}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S'\to W'''_\alpha S'''_\alpha}(Z,ZW^{(1)},W^{(2)})\Bigg]\Big|_1\Big|_2.$$

Notice that in contrast to (24), (25) and (26) do not have a term on the right hand side corresponding to emission of string 1. This is what we would expect for an operator dual to a state with two strings stretching between branes, since if string 1 is emitted, it leaves a state with string 2 stretched between branes; this state is not allowed as it violates the Gauss Law. The process in which the two open strings join at their endpoints is allowed. In this process, it is the box with the upper 1 label that is removed. Thus, we can identify the Chan-Paton label for the side of the string defining the first lattice site of the Cuntz chain with the upper label for the string, in our diagrammatic notation. This corresponds to the first label of the restricted Schur polynomial. We will see further evidence for this interpretation when we interpret the final form of the Hamiltonian.

If we again consider the limit of two well separated branes, we find that (25) becomes

$$\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2\approx\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})+\sum_\alpha\frac{1}{c_1-c_\alpha}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,ZW^{(1)},W^{(2)})\Big|_1\Big|_2.$$

In this case, the box with upper 1 label and lower 2 label moves from box 1 to box α (which are close to each other in the Young diagram) and box with upper 2 label and lower 1 label stays where it is.

The first three identities that we have discussed corresponded to an interaction in which an impurity from the first site of string 1 interacts with the brane. The next three identities that we discuss correspond to an interaction in which an impurity from the first site of string 2 interacts with the brane. The first three terms of the identity

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2=&\left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(1)}_{S',S''}(Z,W^{(1)})\mathrm{Tr}\,(W^{(2)})\\&+\frac{1}{(c_1-c_2)^2}\chi^{(1)}_{R',R''}(Z,W^{(1)})\mathrm{Tr}\,(W^{(2)})+\frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)})\\&+\sum_\alpha\left[\frac{1}{c_2-c_\alpha}\left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right.\end{aligned}$$

$$+\frac{1}{c_2-c_\alpha}\frac{1}{(c_1-c_2)^2}\chi^{(2)}_{R',R''_\alpha}(Z,W^{(1)},ZW^{(2)}) \qquad (27)$$
$$+\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)},ZW^{(2)})\Big]\Big|_1\Big|_2$$

change the number of open strings attached to the boundstate. The first two terms correspond to gravitational radiation; for both of these terms, string 2 is emitted as a closed string. The third term corresponds to a process in which the two open strings join to give a single open string. The order of the open string words in this term is not the same as the order in the corresponding term of (24). The term above is natural because it is the first site of string 2 that is interacting; the order in (24) also looks natural because in that case it is the first site of string 1 that is interacting. Notice that the above identity is rather different to (24). Physically this is surprising - since in both cases it is the first site of the string interacting, these identities should presumably look identical. This mismatch between the two identities is a consequence of the fact that we have treated string 1 and string 2 differently when constructing the operator. See section 9 for further discussion of this point.

If we again consider the limit of two well separated branes, we find that (27) becomes (take $|c_1-c_2|\gg 1$, $|c_1-c_\alpha|\gg 1$ and $|c_2-c_\alpha|\sim 1$)

$$\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 \approx \chi^{(1)}_{S',S''}(Z,W^{(1)})\mathrm{Tr}(W^{(2)})+\sum_\alpha \frac{1}{c_2-c_\alpha}\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)}).$$

This again reproduces the identity of [5]. Thus, the content of the formula for well separated branes matches the corresponding limit of (24). This is satisfying, because in this limit the order in which the strings are attached does not matter. This follows because the swap factor of [5] behaves as $|c_1-c_2|^{-1}$.

The remaining two identities are stretched string identities. In contrast to what we found for the stretched string identities (27), (28), there are terms corresponding to gravitational radiation in these identities. We interpret this as a signal that there is some mixing between the operators we have defined (which as explained above, made some arbitrary choices) to get to a "physical

basis". See section 9 for more details. The first term in both identities

$$\begin{aligned}\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\,\chi^{(1)}_{S',S''}(Z,W^{(1)}W^{(2)})\\ &+\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{R',R''}(Z,W^{(1)})-\chi^{(1)}_{S',S''}(Z,W^{(1)})\right)\mathrm{Tr}\,(W^{(2)})\\ &+\sum_\alpha\left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_1}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\,\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right.\\ &+\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_1)^2}}\,\chi^{(2)}_{S'\to S'''_\alpha W'''_\alpha}(Z,W^{(1)},ZW^{(2)})\\ &\left.+\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\,\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right]\Big|_1\Big|_2,\end{aligned}\tag{28}$$

$$\begin{aligned}\chi^{(2)}_{R\to S''R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\,\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)})\\ &+\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{R',R''}(Z,W^{(1)})-\chi^{(1)}_{S',S''}(Z,W^{(1)})\right)\mathrm{Tr}\,(W^{(2)})\\ &+\sum_\alpha\left[\frac{1}{c_2-c_\alpha}\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\,\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right.\\ &+\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_2)^2}}\,\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)},ZW^{(2)})\\ &\left.+\frac{1}{c_2-c_1}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\,\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right]\Big|_1\Big|_2,\end{aligned}\tag{29}$$

corresponds to two open strings joining to form one long open string. The order of the open string words in these terms again looks natural given that it is the first site of string 2 that is interacting. They will again not contribute in the leading order of the large N expansion. It is satisfying that the content of the large distance limit of (28)

$$\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 \approx \chi^{(1)}_{S',S''}(Z,W^{(1)}W^{(2)})+\sum_\alpha\frac{1}{c_2-c_\alpha}\chi^{(2)}_{S'\to S'''_\alpha W'''_\alpha}(Z,W^{(1)},ZW^{(2)}),$$

is in complete agreement with the large distance limit of (25).

8.2.3 Identities Relevant to Hopping off the last site of the string

In this subsection, impurities hop between the last site of the strings and the threebrane. There are again six possible identities that we could consider. The first three identities describe an interaction between the last site of string 1 and the threebrane. The first identity

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)}) + \frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)}) \\ &+\sum_\alpha\left[\frac{1}{c_1-c_\alpha}\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right. \\ &+\frac{1}{c_1-c_2}\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)}) \\ &+\frac{1}{c_1-c_2}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)}) \\ &\left.+\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right]\Big|_1\Big|_2\end{aligned} \tag{30}$$

can be obtained from (24) by (i) swapping the labels on the twisted string states on the right hand side and (ii) swapping the order of the open string words in the second term on the right hand side. This is exactly what we would expect - it is now the last site of the string that is interacting; to swap the first and last sites, we must swap Chan-Paton indices i.e. we must swap the labels on the twisted string states. The discussion of this identity now parallels the discussion of (24) and is not repeated.

Consider next the stretched string identities

$$\begin{aligned}\chi^{(2)}_{R\to S''R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)}) \\ &+\sum_\alpha\left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right. \\ &\left.+\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right]\Big|_1\Big|_2,\end{aligned} \tag{31}$$

$$\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2=\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{S',S''}(Z,W^{(1)}W^{(2)})$$
$$+\sum_\alpha\left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right.$$
$$\left.+\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_\alpha)^2}}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S'\to S''_\alpha W''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right]\Big|_1\Big|_2. \tag{32}$$

It is satisfying that identity (31) can be obtained from (25) and (32) from (26) by swapping the labels for stretched string states on both sides, and reversing the order of the open string words in the first term on the right hand side. The discussion of these identities now parallel the discussion of (25) and (26) and is not repeated.

The remaining three identities describe an interaction between the last site of string 2 and the threebrane. The identity

$$\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2=\left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(1)}_{S',S''}(Z,W^{(1)})\mathrm{Tr}\,(W^{(2)})$$
$$+\frac{1}{(c_1-c_2)^2}\chi^{(1)}_{R',R''}(Z,W^{(1)})\mathrm{Tr}\,(W^{(2)})+\frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})$$
$$+\sum_\alpha\left[\frac{1}{c_2-c_\alpha}\left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(2)}_{S',S''_\alpha}(Z,W^{(1)},W^{(2)}Z)\right.$$
$$+\frac{1}{c_2-c_\alpha}\frac{1}{(c_1-c_2)^2}\chi^{(2)}_{R',R''_\alpha}(Z,W^{(1)},W^{(2)}Z) \tag{33}$$
$$\left.+\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T''_\alpha R''_\alpha}(Z,W^{(1)},W^{(2)}Z)\right]\Big|_1\Big|_2$$

can be obtained from (27) by (i) swapping the labels on the twisted string states on the right hand side and (ii) swapping the order of the open string words in the second term on the right hand side. Finally, the stretched string identities

$$\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2=\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})$$
$$+\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{R',R''}(Z,W^{(1)})-\chi^{(1)}_{S',S''}(Z,W^{(1)})\right)\mathrm{Tr}\,(W^{(2)})$$

$$
\begin{aligned}
&+\sum_\alpha \left[\frac{1}{c_2-c_\alpha}\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R''_\alpha}(Z,W^{(1)},W^{(2)}Z) \right. \\
&+\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_2)^2}}\chi^{(2)}_{R'\to T''_\alpha R''_\alpha}(Z,W^{(1)},W^{(2)}Z) \\
&\left. -\frac{1}{c_1-c_2}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S''_\alpha}(Z,W^{(1)},W^{(2)}Z) \right] \Big|_1\Big|_2
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\chi^{(2)}_{R\to S''R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{S',S''}(Z,W^{(2)}W^{(1)}) \\
&-\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{S',S''}(Z,W^{(1)})-\chi^{(1)}_{R',R''}(Z,W^{(1)})\right)\mathrm{Tr}\,(W^{(2)}) \\
&+\sum_\alpha \left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_1}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S''_\alpha}(Z,W^{(1)},W^{(2)}Z) \right. \\
&+\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_1)^2}}\chi^{(2)}_{S'\to W''_\alpha S''_\alpha}(Z,W^{(1)},W^{(2)}Z) \\
&\left. +\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R''_\alpha}(Z,W^{(1)},W^{(2)}Z) \right] \Big|_1\Big|_2
\end{aligned}
\tag{35}
$$

can be obtained from (27) and (28) by swapping the labels for stretched string states on both sides, and reversing the order of the open string words in the first term on the right hand side.

8.2.4 Numerical Test

An important result of this dissertation are the identities presented in the previous two subsections, since they determine the hop-off interaction. The hop-on interaction follows from the hop-off interaction by Hermitian conjugation and the kissing interaction by composing the hop-on and the hop-off interactions. Thus, the complete boundary interaction and the corresponding back reaction on the brane are determined by these identities. For this reason, we have tested the identities numerically. In this subsection we will explain the check we have performed.

Our formulas are identities between restricted Schur polynomials. They

must hold if we evaluate them for *any*[4] numerical value of the matrices Z and W. Our check entails evaluating our identities for randomly generated matrices $W^{(1)}$, $W^{(2)}$ and Z, to check their validity. Evaluating a restricted Schur polynomial entails evaluating a restricted character as well as a product of traces of a product of the matrices $W^{(1)}$, $W^{(2)}$ and Z.

The restricted character $\mathrm{Tr}_{R'',S''}(\Gamma_R[\sigma])$ or $\mathrm{Tr}_{R''}(\Gamma_R[\sigma])$ was computed by explicitly constructing the matrices $\Gamma_R[\sigma]$. Each representation used was obtained by induction. One induces a reducible representation; the irreducible representation that participates was isolated using projection operators built from the Casimir obtained by summing over all two cycles. See appendix B.2 of [4] for more details. The resulting irreducible representations were tested by verifying the multiplication table of S_n. The restricted trace is then evaluated with the help of a projection operator or an intertwiner. The intertwiner was computed using the results of section 5.

The trace $\mathrm{Tr}(\sigma Z^{\otimes n-1}W^{(1)}W^{(2)}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}} (W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}} (W^{(1)})^{i_n}_{i_{\sigma(n)}}$ for any given $\sigma \in S_n$ is easily expressed as a product of traces of powers of Z, $W^{(1)}$ and $W^{(2)}$.

In total we verified over 50 specific instances of our identities, which provides a significant check of each identity.

8.2.5 Identities in terms of Cuntz Chain States

The state-operator correspondence is available for any conformal field theory. Using this correspondence, we can trade our (local) operators for a set of states. Concretely, this involves quantizing with respect to radial time. Considering a fixed "radial time" slice we obtain a round sphere. The states dual to the restricted Schur polynomial operators are the states of our Cuntz chain. Thus, we need to rewrite the identities obtained in this section as statements in terms of the states of the Cuntz oscillator chain. The states of the Cuntz oscillator chain are normalized. Normalized states correspond to operators whose two point function is normalized. Using the technology of [4] it is a simple task to compute the free equal time correlators of the restricted

[4] In particular, not necessarily Hermitian.

Schur polynomials. After making use of the free field correlators to write our identities in terms operators with unit two point functions, we find that not all terms are of the same order in N. We drop all terms which are subleading in N. These terms are naturally interpreted in terms of string splitting and joining processes, so that they will be important when interactions that change the number of open strings are considered.

The discussion for all of the identities above is rather similar, so we will be content to discuss a specific example which illustrates the general features. Consider the right hand side of (24). From the equal time correlator (there are a total of h_i fields in open string word $W^{(i)}$; f_R is the product of the weights of the Young diagram R; d_R is the dimension of R as an irreducible representation of the symmetric group; n_R is the number of boxes in Young diagram R)

$$\begin{aligned}&\langle\chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\mathrm{Tr}\,(W^{(1)})^\dagger\rangle\\ &=\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}}h_1N^{h_1+h_2-1}n_{R''}f_{R'}\frac{d_{R''}}{d_{R'}}\end{aligned}\tag{36}$$

we know that the operator $\chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)})$ corresponds to the state (all Cuntz chain states are normalized to 1)

$$\sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}}h_1N^{h_1+h_2-1}n_{R''}f_{R'}\frac{d_{R''}}{d_{R'}}}|R',R'',W^{(2)};W^{(1)}\rangle.$$

The result (36) is not exact. When computing $\langle\mathrm{Tr}\,(W^{(1)})\mathrm{Tr}\,(W^{(1)})^\dagger\rangle$ we have only summed the leading planar contribution. When computing $\langle\chi^{(1)}_{R',R''}(Z,W^{(2)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\rangle$ we have only kept the F_0 contribution in the language of [4].

We have also factorized $\langle\chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\mathrm{Tr}\,(W^{(1)})^\dagger\rangle$ as $\langle\chi^{(1)}_{R',R''}(Z,W^{(2)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\rangle\times\langle\mathrm{Tr}\,(W^{(1)})\mathrm{Tr}\,(W^{(1)})^\dagger\rangle$ which is valid at large N. Similarly, (again we sum only the leading order at large N)

$$\langle\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})^\dagger\rangle=\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}}N^{h_1+h_2-1}n_{R''}f_{R'}\frac{d_{R''}}{d_{R'}}$$

implies that $\chi^{(1)}_{R',R''}(Z, W^{(2)}W^{(1)})$ corresponds to the state

$$\sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} N^{h_1+h_2-1} n_{R''} f_{R'} \frac{d_{R''}}{d_{R'}}} |R', R'', W^{(2)}W^{(1)})\rangle.$$

Finally, the correlators (again we sum only the leading order at large N)

$$\langle \chi^{(2)}_{R',T''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \chi^{(2)}_{R',T''''_\alpha}(Z, ZW^{(1)}, W^{(2)})^\dagger \rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{T''''_\alpha}}{d_{R'}} f_{R'},$$

$$\langle \chi^{(2)}_{R',R''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \chi^{(2)}_{R',R''''_\alpha}(Z, ZW^{(1)}, W^{(2)})^\dagger \rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{R''''_\alpha}}{d_{R'}} f_{R'},$$

$$\langle \chi^{(2)}_{R'\to T''''_\alpha R''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \chi^{(2)}_{R'\to T''''_\alpha R''''_\alpha}(Z, ZW^{(1)}, W^{(2)})^\dagger \rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{T''''_\alpha}}{d_{R'}} f_{R'},$$

$$\langle \chi^{(2)}_{R'\to R''''_\alpha T''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \chi^{(2)}_{R'\to R''''_\alpha T''''_\alpha}(Z, ZW^{(1)}, W^{(2)})^\dagger \rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{T''''_\alpha}}{d_{R'}} f_{R'}$$

imply the correspondences

$$\chi^{(2)}_{R',T''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{T''''_\alpha}}{d_{R'}} f_{R'}} |R', T_{\alpha''''}, ZW^{(1)}, W^{(2)}\rangle,$$

$$\chi^{(2)}_{R',R''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{R''''_\alpha}}{d_{R'}} f_{R'}} |R', R''''_\alpha, ZW^{(1)}, W^{(2)}\rangle,$$

$$\chi^{(2)}_{R'\to T''''_\alpha R''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{T''''_\alpha}}{d_{R'}} f_{R'}} |R', T'''''_\alpha R''''_\alpha, ZW^{(1)}, W^{(2)}\rangle$$

$$\chi^{(2)}_{R'\to R''''_\alpha T''''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T''''_\alpha}} N^{h_1+h_2-1} n^2_{R'} \frac{d_{T''''_\alpha}}{d_{R'}} f_{R'}} |R', R''''_\alpha T''''_\alpha, ZW^{(1)}, W^{(2)}\rangle$$

Consider the factor

$$n^2_{R'} \frac{d_{R''''_\alpha}}{d_{R'}} = \frac{(\text{hooks})_{R'}}{(\text{hooks})_{R''''_\alpha}},$$

where $(\text{hooks})_R$ is the product of the hook lengths of Young diagram R. It is straight forward to compute this ratio of hook lengths, which is generically of order N^2 implying that $\frac{d_{R''''_\alpha}}{d_{R'}}$ is of order 1. Using this observation, it is equally easy to verify that $\frac{d_{T''''_\alpha}}{d_{R'}}$ and $\frac{d_{R''}}{d_{R'}}$ are also both $O(1)$. Given these results, it is

simple to see that the sum of operators

$$\begin{aligned}
& \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}\,(W^{(1)}) + \frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)}) \\
+ \quad & \sum_\alpha \left[\frac{1}{c_1-c_\alpha}\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z,ZW^{(1)},W^{(2)}) \right. \\
+ \quad & \frac{1}{c_1-c_2}\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,ZW^{(1)},W^{(2)}) \\
+ \quad & \frac{1}{c_1-c_2}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,ZW^{(1)},W^{(2)}) \\
+ \quad & \left. \frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,ZW^{(1)},W^{(2)}) \right] \Big|_1\Big|_2
\end{aligned}$$

corresponds to the following sum of normalized states

$$\begin{aligned}
& \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} N^{h_1+h_2-1} n^2_{R'} f_{R'}} \left[\sqrt{\frac{h_1 d_{R''}}{n_{R'} d_{R'}}}|R',R'',W^{(2)};W^{(1)}\rangle \right. \\
+ \quad & \frac{1}{c_1-c_2}\sqrt{\frac{d_{R''}}{n_{R'}d_{R'}}}|R',R'',W^{(2)}W^{(1)}\rangle \\
+ \quad & \sum_\alpha \left[\frac{1}{c_1-c_\alpha}\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\sqrt{\frac{d_{T'''_\alpha}}{d_{R'}}}|R',T'''_\alpha,ZW^{(1)},W^{(2)}\rangle \right. \\
+ \quad & \frac{1}{c_1-c_2}\frac{1}{(c_2-c_\alpha)^2}\sqrt{\frac{d_{R'''_\alpha}}{d_{R'}}}|R',R'''_\alpha,ZW^{(1)},W^{(2)}\rangle \\
+ \quad & \frac{1}{c_1-c_2}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\sqrt{\frac{d_{T'''_\alpha}}{d_{R'}}}|R',T'''_\alpha R'''_\alpha,ZW^{(1)},W^{(2)}\rangle \\
+ \quad & \left. \left. \frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\sqrt{\frac{d_{T'''_\alpha}}{d_{R'}}}|R',R'''_\alpha T'''_\alpha,ZW^{(1)},W^{(2)}\rangle \right] \right].
\end{aligned}$$

Recalling that $h_1 = O(\sqrt{N})$ and $n_{R'} = O(N)$, it is clear that the first two terms are subleading. These two terms correspond to gravitational radiation (first term) and string joining (second term); they are the only terms that correspond to an interaction that changes the number of open strings attached to the excited giant system. Although we have illustrated things with an example, this conclusion is general - for all of the identities obtained in this section, terms that do not correspond to two strings attached to the giant system can be dropped in the leading large N limit. This is consistent with

the results of [4].

8.3 Hop-Off Interaction

We now have the ability to express objects like $\chi^{(2)}_{R,R''}(Z, ZW^{(1)}, W^{(2)})$ in terms of $\chi^{(2)}_{S,S''}(Z, W^{(1)}, W^{(2)})$ where S is a Young diagram with one more box than R. This is easily achieved by inverting the identities derived above. To get the hop-off interaction in the Hamiltonian, we rewrite the identities in terms of normalized Cuntz chain states.

$+1 \to 1$ *Hop-off interaction:* This term in the Hamiltonian describes the hop off process in which a Z hops out of the first site of string 1. We write $+1 \to 1$ to indicate that the string before the hop has one extra Z in its first site.

$$H_{+1\to 1}\begin{bmatrix} |b_0-1, b_1, bb, ll\rangle \\ |b_0-1, b_1, ll, bb\rangle \\ |b_0-1, b_1, bl, lb\rangle \\ |b_0-1, b_1, lb, bl\rangle \\ |b_0-2, b_1+2, ll, ll\rangle \\ |b_0, b_1-2, bb, bb\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M_1 \begin{bmatrix} |b_0-1, b_1+1, bb, ll\rangle \\ |b_0, b_1-1, ll, bb\rangle \\ |b_0-1, b_1+1, bl, lb\rangle \\ |b_0, b_1-1, lb, bl\rangle \\ |b_0-1, b_1+1, ll, ll\rangle \\ |b_0, b_1-1, bb, bb\rangle \end{bmatrix},$$

where

$$M_1 = \begin{bmatrix} -(b_1)_1^2 & \frac{1}{b_1(b_1+1)^2} & 0 & \frac{(b_1)_0}{b_1+1} & \frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{b_1(b_1+1)} \\ -\frac{1}{(b_1+2)(b_1+1)^2} & -(b_1)_1^2 & -\frac{(b_1)_2}{b_1+1} & 0 & -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & -\frac{(b_1)_1}{b_1+1} \\ -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & \frac{(b_1)_1}{b_1+1} & -(b_1)_1(b_1)_2 & 0 & -\frac{b_1}{(b_1+1)^2} & \frac{1}{(b_1+1)^2} \\ -\frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{b_1(b_1+1)} & 0 & -(b_1)_0(b_1)_1 & \frac{1}{(b_1+1)^2} & \frac{b_1+2}{(b_1+1)^2} \\ -\frac{(b_1)_2}{b_1+1} & 0 & \frac{1}{b_1+2} & 0 & -(b_1)_1(b_1)_2 & 0 \\ 0 & \frac{(b_1)_0}{b_1+1} & 0 & -\frac{1}{b_1} & 0 & -(b_1)_0(b_1)_1 \end{bmatrix},$$

and

$$(b_1)_n = \frac{\sqrt{b_1+n-1}\sqrt{b_1+n+1}}{b_1+n}.$$

The term in the Hamiltonian describing the process in which the Z hops out of the last site of string 1 is described by swapping the labels of the endpoints

of the open strings. Concretely, it is given by

$$H_{1+\to 1}\begin{bmatrix} |b_0-1,b_1,bb,ll\rangle \\ |b_0-1,b_1,ll,bb\rangle \\ |b_0-1,b_1,lb,bl\rangle \\ |b_0-1,b_1,bl,lb\rangle \\ |b_0-2,b_1+2,ll,ll\rangle \\ |b_0,b_1-2,bb,bb\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M_1 \begin{bmatrix} |b_0-1,b_1+1,bb,ll\rangle \\ |b_0,b_1-1,ll,bb\rangle \\ |b_0-1,b_1+1,lb,bl\rangle \\ |b_0,b_1-1,bl,lb\rangle \\ |b_0-1,b_1+1,ll,ll\rangle \\ |b_0,b_1-1,bb,bb\rangle \end{bmatrix},$$

where M_1 is the matrix given above. We write $1+\to 1$ to indicate that the string before the hop has one extra Z in its last site.

$+2\to 2$ *Hop-off interaction:* This term in the Hamiltonian describes the hop off process in which a Z hops out of the first site of string 2.

$$H_{+2\to 2}\begin{bmatrix} |b_0-2,b_1+1,bb,ll\rangle \\ |b_0-1,b_1-1,ll,bb\rangle \\ |b_0-2,b_1+1,bl,lb\rangle \\ |b_0-1,b_1-1,lb,bl\rangle \\ |b_0-2,b_1+1,ll,ll\rangle \\ |b_0-1,b_1-1,bb,bb\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M_2 \begin{bmatrix} |b_0-1,b_1,bb,ll\rangle \\ |b_0-1,b_1,ll,bb\rangle \\ |b_0-1,b_1,bl,lb\rangle \\ |b_0-1,b_1,lb,bl\rangle \\ |b_0-1,b_1,ll,ll\rangle \\ |b_0-1,b_1,bb,bb\rangle \end{bmatrix},$$

where

$$M_2 = \begin{bmatrix} -(b_1)_1^2 & -\frac{1}{(b_1+2)(b_1+1)^2} & -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & -\frac{(b_1)_1}{b_1+1} & 0 & -\frac{(b_1)_2}{b_1+1} \\ \frac{1}{b_1(b_1+1)^2} & -(b_1)_1^2 & \frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{(b_1+1)b_1} & \frac{(b_1)_0}{b_1+1} & 0 \\ 0 & -\frac{(b_1)_2}{b_1+1} & -(b_1)_1(b_1)_2 & 0 & 0 & \frac{1}{b_1+2} \\ \frac{(b_1)_0}{b_1+1} & 0 & 0 & -(b_1)_0(b_1)_1 & -\frac{1}{b_1} & 0 \\ -\frac{(b_1)_1}{b_1(b_1+1)} & -\frac{(b_1)_1}{b_1+1} & \frac{1}{(b_1+1)^2} & \frac{b_1+2}{(b_1+1)^2} & -(b_1)_1(b_1)_0 & 0 \\ \frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & -\frac{b_1}{(b_1+1)^2} & \frac{1}{(b_1+1)^2} & 0 & -(b_1)_2(b_1)_1 \end{bmatrix}.$$

Notice that these interactions (as is the case for all of the boundary interactions) are highly suppressed for a maximal giant[27]. The term in the Hamiltonian describing the process in which the Z hops out of the last site of string 2 is described by swapping the labels of the endpoints of the open

strings.

The function $(b_1)_n$ also appears in the Hamiltonian relevant for a single string attached to a giant[5]. Notice that $(b_1)_n$ vanishes when $b_1 = 1 - n$, but tends to 1 very rapidly as b_1 is increased from this value. The diagonal terms in the Hamiltonian with a $(b_1)_1$ factor will thus vanish when $b_1 = 0$. The radius of each giant is determined by their momentum. Since b_1 is the difference in momentum of the two giants, $b_1 = 0$ corresponds to coincident giants. Thus, $(b_1)_n$ is switching off short distance interactions. The hop-off Hamiltonian does not generate illegal Young diagrams from legal ones precisely because these interactions are switched off.

Finally, note that the structure of the hop-on and hop-off interactions, clearly reflect the fact that the open strings attached to the giants are orientable.

8.4 Hop-On Interaction

Since $\mathcal{N} = 4$ super Yang-Mills theory is a unitary conformal field theory, we know that the spectrum of anomalous dimensions of the theory is real. This implies that the energy spectrum of our Cuntz chain Hamiltonian must be real and hence the Hamiltonian must be Hermitian. Thus, the hop on term in the Hamiltonian is given by the Hermitian conjugate of the hop off term.

To give an example, we will now derive the term in the Hamiltonian describing the process in which a Z from the brane hops into the first site of string 1. Let $|\psi\rangle$ denote the state with a brane of momentum $P_{\text{brane}} = P$ and a string of momentum $P_{\text{string}} = p$ and $|\phi\rangle$ denote the state with $P_{\text{brane}} = P + 1$ and $P_{\text{string}} = p - 1$. Then,

$$H_{+1\to 1}|\psi\rangle = -\lambda\sqrt{1 - \frac{b_0}{N}}M_1|\phi\rangle,$$

and

$$\langle\phi'|H_{+1\to 1}|\psi\rangle = -\lambda\sqrt{1 - \frac{b_0}{N}}\langle\phi'|M_1|\phi\rangle = -\lambda\sqrt{1 - \frac{b_0}{N}}(M_1)_{\phi'\phi}.$$

Daggering we find (keep in mind that M_1 is real)

$$\begin{aligned}\langle\psi|H_{1\to+1}|\phi'\rangle &= (\langle\phi'|H_{+1\to1}|\psi\rangle)^\dagger \\ &= -\lambda\sqrt{1-\frac{b_0}{N}}\langle\phi|(M_1)^T|\phi'\rangle \\ &= -\lambda\sqrt{1-\frac{b_0}{N}}\left((M_1)^T\right)_{\phi\phi'}.\end{aligned}$$

Thus we obtain

$$H_{1\to+1}|\phi\rangle = -\lambda\sqrt{1-\frac{b_0}{N}}N_1|\psi\rangle,$$

with $N_1 = (M_1)^T$.

8.5 Kissing Interaction

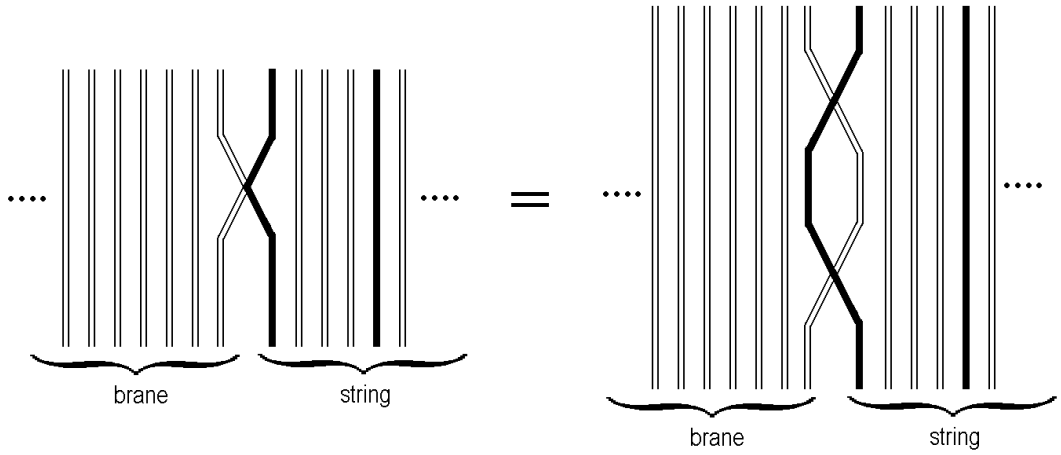


Figure 8: The Feynman diagram on the left of this figure shows the kissing interaction. The white ribbons are Z fields, the black ribbons are Y fields. The interacting black ribbon shown marks the beginning of the string; there are 3 Zs in the first site of the string. The Feynman diagram on the right of this figure shows a hop-on interaction followed by a hop-off interaction. If you shrink the composite hop-on/hop-off interaction to a point, you recover the kissing interaction.

The kissing interaction corresponds to the Feynman diagram shown on the left in Figure 8. Notice that the number of Z fields in the giant is

unchanged by this process so that the string and brane do not exchange momentum by this process. As far as the combinatorics goes, we can model the kissing interaction as a hop on (the string) followed by a hop off. We know both the hop on and hop off terms so the kissing interaction follows. This is illustrated by the Feynman diagram shown on the right in Figure 8. The kissing interaction must be included for both endpoints of both strings.

A straight forward computation easily gives

$$H_{\text{kissing}} = \lambda \left(1 - \frac{b_0}{N}\right) \mathbf{1},$$

for each endpoint of either string. In this formula $\mathbf{1}$ is the identity operator.

The fact that the kissing interaction comes out proportional to the identity operator is a non-trivial check of our hop-on and hop-off interactions. Indeed, the contraction of the F term vertex which leads to the kissing interaction removes an adjacent Z and Y and then replaces them in the same order. Thus, the kissing interaction had to come out proportional to the identity. The careful reader may worry that this is not in fact true - indeed, the restricted Schur polynomial includes terms in which the open string word is traced and terms in which the two open string words are multiplied. For these terms there is no Z next to the word to "do the kissing". Precisely these terms were considered in section 8.2.5. They do not contribute at large N.

9 Interpretation

The operators we are studying are dual to giant gravitons with open strings attached. Since the giant gravitons have finite volume, the Gauss Law implies that the total charge on each giant must vanish - there must be the same number of strings leaving each brane as there are arriving on each brane. These operators do indeed satisfy these non-trivial constraints[3], providing convincing evidence for the proposed duality. The low energy dynamics of the open strings attached to the giant gravitons is a Yang-Mills theory. This new *emergent* $3+1$ dimensional Yang-Mills theory is not described as a local field theory on the S^3 on which the original Yang-Mills theory is defined - it is local on a new space, the world volume of the giant gravitons[3],[28]. This new space emerges from the matrix degrees of freedom participating in the Yang-Mills theory. Reconstructing this emergent gauge theory may provide a simpler toy model that will give us important clues into reconstructing the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ quantum gravity. In this section, our goal is to make contact with this emergent Yang-Mills dynamics.

9.1 Dynamical Emergence of Chan-Paton Factors

Return to the $H_{+1\to 1}$ hop-off interaction obtained in section 8.3. Recall that this corresponds to the interaction in which a Z hops out of the first site of string 1. If we expand the matrix M_1 for large b_1, we find

$$M_1 = \sum_{n=0}^{\infty} M_1(n) b_1^{-n}.$$

The leading order $M_1(0)$ is simply $-\mathbf{1}$ with $\mathbf{1}$ the 6×6 identity matrix. The Z simply hops off the string and onto the brane without much rearranging of the system. This is the dominant process. Next, consider the term of order

b_1^{-1}. It is simple to compute

$$M_1(1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

The radius of the giant graviton R_g is related to its momentum P by $R_g = \sqrt{\frac{P}{N}}$. The giant orbits with a radius $R = \sqrt{1 - R_g^2}$. For the two giants in the bound state we are considering we have $P_1 = b_0$ and $P_2 = b_0 + b_1$. Using the fact that $b_0 = O(N)$ and $b_1 = O(1)$ it is simple to verify that both the difference in the radii of the two giants and the difference in the radii of their orbits is proportional to b_1. Thus, a b_1^{-1} dependence indicates a potential with an inverse distance dependence which is the correct dependence for massless particles moving in 3+1 dimensions. In Figure 9 we have represented the transitions implied by $M_1(1)$ graphically. Transitions between any two adjacent Young diagrams are allowed.

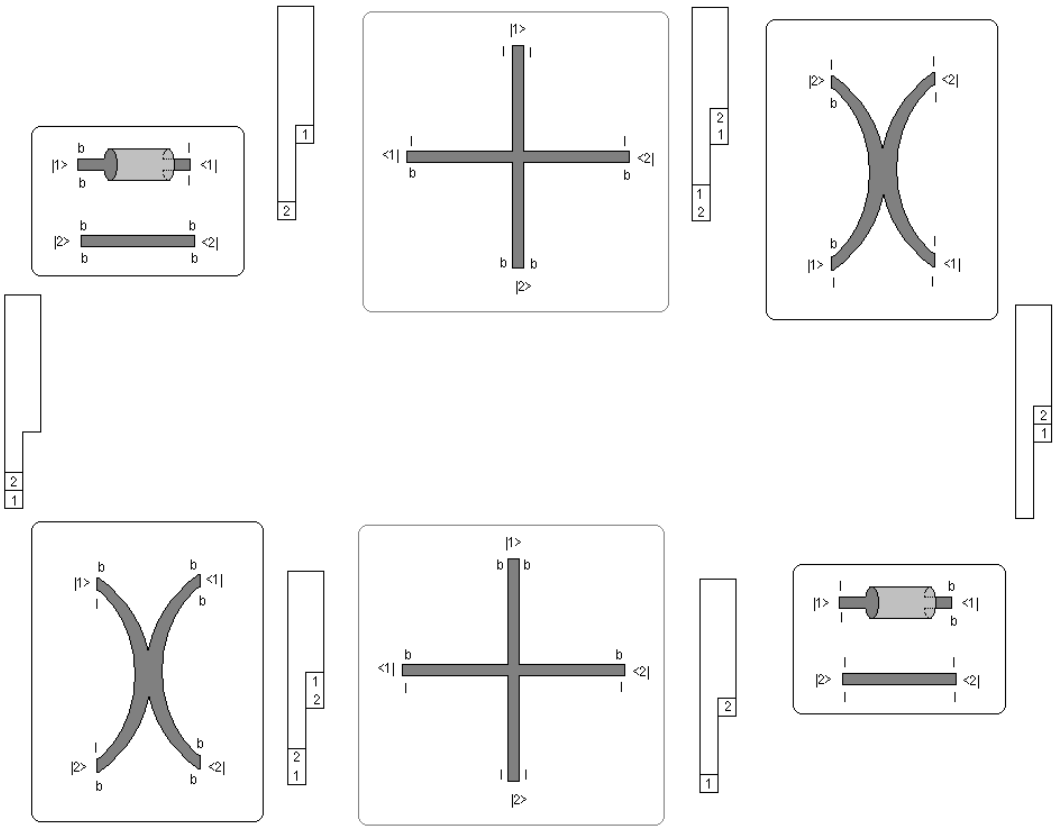
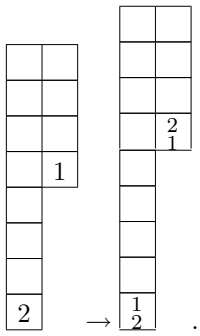


Figure 9: The order b_1^{-1} terms in the hop-off interaction. This interaction allows a transition between the operators described by any two adjacent Young diagrams. The figures between the Young diagram show the open string diagram relevant for the clockwise transition. The kets are associated to the open string states before the transition; the bras to the states after the transition. The end point labels 'b' and 'l' are for big brane and little brane.

As an example, consider the transition



The upper label of string 1 has moved. In all of the transitions shown, the upper index of string 1 always moves, so that it is natural to associate the upper index of string 1 with the first site of string one, and to look for an interpretation of this interaction in terms of open string processes that involve the upper index of string 1. The figures between the Young diagram show that there is indeed a natural interpretation for these transitions. *It is clear that our Cuntz oscillator dynamics illustrates how the Chan-Paton factors for open strings propagating on multiple branes arise dynamically.* Drawing all possible ribbon diagrams correctly accounts for both $M_1(0)$ and $M_1(1)$.

9.2 Physical Basis

Although the interpretation of the b_1^{-1} terms is encouraging, there are extra higher order corrections ($M_1(2)b_1^{-2}$, $M_1(3)b_1^{-3}$ and higher orders) that do not seem to have a natural open string interpretation. In addition to this, the interaction we have obtained depends on the open string words describing each open string, the Young diagram describing the brane bound state system as well as the order in which the strings were attached. This dependence on the order in which the strings are attached is not physically sensible.

It is natural to expect that the resolution to these two puzzles is connected. Recall that when constructing the restricted Schur polynomial we have assumed that when computing reductions, string 1 is removed first and string 2 second. This arbitrary choice defines a basis for the Cuntz oscillator chain. We interpret the unphysical features of our interactions, described in

the previous paragraph, as reflecting a property of the basis it is written in and not as an inherent problem with the interaction. In this section we will define a new physical basis, singled out by the requirement that the boundary interaction does not depend on the order in which the open strings are attached.

A few comments are in order. A basis for the $\frac{1}{2}$ BPS states (giants with no open strings attached) is provided by the taking traces of Z or by taking subdeterminants or by the Schur polynomials. These are three perfectly acceptable bases, since using any single one of these bases we can generate, by taking linear combinations of the elements of the basis considered, a member from every $\frac{1}{2}$ BPS multiplet[1]. From a physical point of view, these different bases are not on an equal footing: the Schur polynomial is the most useful. Indeed, the Schur polynomials diagonalize the matrix of two point correlators (Zamolodchikov metric) so that they can be put into correspondence with the (orthogonal) states of a Fock space. In the same way, the basis for excited giants gravitons we have been considering is a perfectly acceptable basis. However, it is the operators in the physical basis (defined below) that have a good physical interpretation.

Denote our two strings by string A and string B. The state obtained by attaching string A first will be denoted by $|b_0, b_1, x_A y_A, x_B y_B\rangle$, where $x_A y_A$ are the endpoints of string A and $x_B y_B$ are the endpoints of string B. The state obtained by attaching string B first will be denoted by $|b_0, b_1, x_B y_B, x_A y_A\rangle\rangle$. In each subspace of sharp giant graviton momentum (definite b_0 and b_1), we can write the following relation between these two sets of states

$$\begin{bmatrix} |b_0, b_1, bb, ll\rangle \\ |b_0, b_1, ll, bb\rangle \\ |b_0, b_1, bl, lb\rangle \\ |b_0, b_1, lb, bl\rangle \\ |b_0, b_1, ll, ll\rangle \\ |b_0, b_1, bb, bb\rangle \end{bmatrix} = PT \begin{bmatrix} |b_0, b_1, bb, ll\rangle\rangle \\ |b_0, b_1, ll, bb\rangle\rangle \\ |b_0, b_1, bl, lb\rangle\rangle \\ |b_0, b_1, lb, bl\rangle\rangle \\ |b_0, b_1, ll, ll\rangle\rangle \\ |b_0, b_1, bb, bb\rangle\rangle \end{bmatrix},$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and}$$

$$T = \begin{bmatrix} \left(1 - \frac{1}{(b_1+1)^2}\right) & \frac{1}{(b_1+1)^2} & -\frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & 0 & 0 \\ \frac{1}{(b_1+1)^2} & \left(1 - \frac{1}{(b_1+1)^2}\right) & \frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & \frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & 0 & 0 \\ \frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & \left(1 - \frac{1}{(b_1+1)^2}\right) & -\frac{1}{(b_1+1)^2} & 0 & 0 \\ \frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)}\sqrt{1 - \frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)^2} & \left(1 - \frac{1}{(b_1+1)^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix T is determined by the subgroup swap rule of [4]. It is satisfying that $PT \times PT = 1$. It is straight forward to check that

$$H_{+1\to 1} = A_{2\to 1}\, PT\, H_{+2\to 2}\, A_{1\to 2}\, PT,$$

where

$$\begin{bmatrix} |b_0-2, b_1+2, bb, ll\rangle \\ |b_0-1, b_1, ll, bb\rangle \\ |b_0-2, b_1+2, bl, lb\rangle \\ |b_0-1, b_1, lb, bl\rangle \\ |b_0-2, b_1+2, ll, ll\rangle \\ |b_0-1, b_1, bb, bb\rangle \end{bmatrix} = A_{2\to 1} \begin{bmatrix} |b_0-1, b_1, bb, ll\rangle \\ |b_0-1, b_1, ll, bb\rangle \\ |b_0-1, b_1, bl, lb\rangle \\ |b_0-1, b_1, lb, bl\rangle \\ |b_0-1, b_1, ll, ll\rangle \\ |b_0-1, b_1, bb, bb\rangle \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} |b_0-2, b_1+1, bb, ll\rangle \\ |b_0-1, b_1-1, ll, bb\rangle \\ |b_0-2, b_1+1, bl, lb\rangle \\ |b_0-1, b_1-1, lb, bl\rangle \\ |b_0-2, b_1+1, ll, ll\rangle \\ |b_0-1, b_1-1, bb, bb\rangle \end{bmatrix} = A_{1\to 2} \begin{bmatrix} |b_0-2, b_1+1, bb, ll\rangle \\ |b_0-2, b_1+1, ll, bb\rangle \\ |b_0-2, b_1+1, bl, lb\rangle \\ |b_0-2, b_1+1, lb, bl\rangle \\ |b_0-3, b_1+3, ll, ll\rangle \\ |b_0-1, b_1-1, bb, bb\rangle \end{bmatrix}.$$

Denote the similarity transformation which takes us to the physical basis by

S. In this basis, we denote $H_{+1\to 1}$ by $\hat{H}_{+1\to 1}$ and $H_{+2\to 2}$ by $\hat{H}_{+2\to 2}$. Clearly

$$\hat{H}_{+1\to 1} = SH_{+1\to 1}S^{-1}, \qquad \hat{H}_{+2\to 2} = SH_{+2\to 2}S^{-1}.$$

The transformation S is now determined by the requirement

$$\hat{H}_{+1\to 1} = P\hat{H}_{+2\to 2}P.$$

We have not yet been able to solve this equation for S. Due to the presence of $A_{1\to 2}$ and $A_{2\to 1}$ in the relation between $H_{+1\to 1}$ and $H_{+2\to 2}$, it seems that S must mix subspaces of different giant momenta (b_0, b_1). In this case the physical basis will not have sharp giant momentum and hence the resulting states will not have a definite radius. This is not too surprising: the open strings will pull "dimples" out of the giant graviton's world volume so that the giant with an open string attached does not have a definite radius. We leave the interesting question of determining the transformation S for the future.

10 Discussion

A bound state of giant gravitons can be excited by attaching open strings. The problem of computing the anomalous dimensions of these operators can be replaced with the problem of diagonalizing a Cuntz oscillator Hamiltonian. In this dissertation we have developed the technology needed to construct this Cuntz oscillator Hamiltonian to one loop. Firstly, we have given an algorithmic construction of the operators dual to excitations described by open strings which stretch between the branes. This involved giving an explicit construction of the intertwiner which is used to construct the relevant restricted Schur polynomial. Secondly, we have developed methods that allow an efficient evaluation of any restricted character. Our method expresses the restricted character graphically as a sum of strand diagrams. Finally, we have explained how to derive the boundary interaction terms from identities satisfied by the restricted Schur polynomials. Since the excited giant graviton operators are small excitations of BPS states, we expect that our results can be extrapolated to strong coupling and hence can be compared with results from the dual string theory. The form of our Cuntz oscillator Hamiltonian provides evidence that the excitations of the giant gravitons have the detailed interactions of an emergent gauge theory. In particular, we have demonstrated the dynamical emergence of the Chan-Paton factors of the open strings. We have also started to clarify the dictionary relating the states of the Cuntz oscillator chain to the states of string field theory on D-branes in $AdS_5 \times S^5$. Although we have mainly considered a bound state of two sphere giants with two open strings attached, our methods are applicable to an arbitrary bound state of giant gravitons with any number of open strings attached.

Our result is a generalization of the spin chains considered so far in the literature: usually the spin chain gives a description of closed strings. Our Cuntz oscillator describes the dynamics of an open string interacting with a giant graviton. Both the state of the string (described by the Cuntz chain occupation numbers) and the state of the giant graviton (the shape of the Young diagram) are dynamical in our approach.

It is worth emphasizing that the new emergent gauge symmetry is distinct from the original gauge symmetry of the theory[3]. The excited giant graviton operators[3] are obtained by taking a trace over the indices of the symmetric group matrix $\Gamma_R(\sigma)$ appearing in the sum

$$\frac{1}{(n-k)!}\sum_{\sigma\in S_n}\Gamma_R(\sigma)\mathrm{Tr}\,(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}),\qquad \text{where}$$

$$\mathrm{Tr}\,(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}) = Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_{n-k}}_{i_{\sigma(n-k)}}(W^{(1)})^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}\cdots(W^{(k)})^{i_n}_{i_{\sigma(n)}}.$$

The color indices of the original super Yang-Mills theory are all traced: every term in the above sum is a color singlet with respect to the gauge symmetry of the original Yang-Mills theory. The color indices of the new gauge theory arise from the labeling of the partial trace over $\Gamma_R(\sigma)$. In some sense we are "substituting" symmetric group indices for the original gauge theory indices. We call this mechanism "*color substitution*".

There are a number of directions in which this work can be extended. For Young diagrams with m columns we expect an emergent Yang-Mills theory with gauge group $U(m)$. It would be nice to repeat the calculations we performed here in that setting[5]. Another interesting calculation would involve studying the dynamics of two giant gravitons with strings stretched between them. In general, the boundary terms will certainly have different values at each boundary (as anticipated in [18]) in which case there will be a net flow of Zs from one brane to the other. This flow of Z's will produce a force between the two giants, conjectured to be an attractive force in[18].

A very concrete application of our methods is the construction of the gauge theory operator dual to the fat magnon[29][6]. The fat magnon is a bound state of a giant graviton and giant magnons (fundamental strings). Essentially, due to the background five form flux, the giant magnon becomes fat by the Myers effect[31]. The fat magnon has the same anomalous dimension as the giant magnon. It would be nice to explicitly recover this anomalous dimension using our technology.

[5]For the $m = 3$ case, see Appendix C.

[6]The fat magnon in the plane wave background is the hedgehog of [49]

Finally, there is now a proposal for gauge theory operators dual to brane-anti-brane states[32]. This proposal was made, at the level of the free field theory, by identifying the operators that diagonalize the two point functions of operators built from Z and $Z^\dagger$. Since these states are non-supersymmetric, corrections when the coupling is turned on are expected to be important for the physics. It would be interesting to extend the technology developed in this dissertation to this non-supersymmetric setting.

A Representations of S_n from Strand Diagrams

Using Strand diagrams, it is possible to write down the irreducible matrix representations of S_n. We will treat the simplest nontrivial example of S_3. First consider the  irreducible representation. Start by numbering the boxes in the Young diagram labeling the irreducible representation, with an ordering in which the boxes are to be removed, so that one is left with a legal Young diagram after each box is removed. These labeled Young diagrams are in one-to-one correspondence with the matrix indices of the matrices in the irreducible representation. For our example,

$i = 1, \leftrightarrow$  $\qquad\qquad$ $i = 2, \leftrightarrow$  .

i.e.

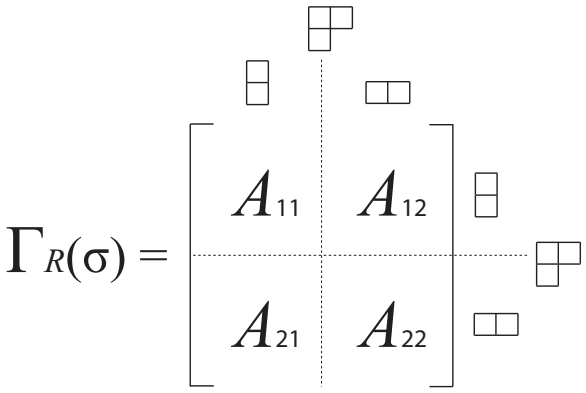


Figure 10: Labeling of matrix elements by Young diagrams

Each matrix element of $\Gamma_{\square\square\atop\square}((12))$ is given by a single strand diagram

$$\left[\Gamma_{\square\square\atop\square}((12))\right]_{11} = \mathrm{Tr}_{\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}}((12)) = \frac{1}{c_1 - c_2} = \frac{1}{2},$$

$$\left[\Gamma_{\square\square\atop\square}((12))\right]_{12} = \mathrm{Tr}_{\begin{array}{|c|c|}\hline 3&\frac{1}{2}\\\hline \frac{2}{1}\\\cline{1-1}\end{array}}((12)) = \sqrt{1 - \frac{1}{(c_1 - c_2)^2}} = \frac{\sqrt{3}}{2},$$

$$\left[\Gamma_{\begin{array}{|c|c|}\hline&\\\hline\\\cline{1-1}\end{array}}((12))\right]_{21} = \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}}((12)) = \sqrt{1-\frac{1}{(c_1-c_2)^2}} = \frac{\sqrt{3}}{2},$$

and

$$\left[\Gamma_{\begin{array}{|c|c|}\hline&\\\hline\\\cline{1-1}\end{array}}((12))\right]_{22} = \mathrm{Tr}_{\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}}((12)) = \frac{1}{c_1-c_2} = -\frac{1}{2},$$

so that

$$\Gamma_{\begin{array}{|c|c|}\hline&\\\hline\\\cline{1-1}\end{array}}((12)) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

In exactly the same way we obtain

$$\Gamma_{\begin{array}{|c|c|}\hline&\\\hline\\\cline{1-1}\end{array}}((23)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These two elements can now be used to generate the complete irreducible representation.

Next consider $\begin{array}{|c|c|c|}\hline&&\\\hline\end{array}$. There is only one valid labeling $\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}$, so that the representation is one dimensional. It is straight forward to obtain

$$\mathrm{Tr}_{\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}}((12)) = \frac{1}{c_1-c_2} = 1, \qquad \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}}((23)) = \frac{1}{c_2-c_3} = 1,$$

which are the correct results. Finally, consider $\begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}$. Again, there is only one valid labeling so that the representation is again one dimensional. We find

$$\mathrm{Tr}_{\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}}((12)) = \frac{1}{c_1-c_2} = -1, \qquad \mathrm{Tr}_{\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}}((23)) = \frac{1}{c_2-c_3} = -1,$$

which are again the correct results.

B State/Operator Map

In this appendix we will simply quote the six normalization factors that enter the relation between the restricted Schur polynomials and the normalized Cuntz chain states relevant for the excited two giant graviton bound state[7]. The normalization factors are not exact - we simply quote the leading large N value of these normalizations. These factors are determined completely by the $F_0^{(1)}F_0^{(2)}$ contribution in the language of [4]. The factor f_R is the product of weights of the Young diagram R. The open string word $W^{(1)}$ contains a total number of h_1 Higgs fields; the open string word $W^{(2)}$ contains a total number of h_2 Higgs fields.

State	Normalization
$\|b_0-1,b_1,11,22\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\|b_0-1,b_1,22,11\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\|b_0-1,b_1,12,21\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\|b_0-1,b_1,21,12\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\|b_0-2,b_1+2,22,22\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{b_1+3}{b_1+1}}$
$\|b_0,b_1-2,11,11\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{b_1-1}{b_1+1}}$

[7]See section 8.1 for the restricted Schur polynomials corresponding to these states.

C Boundstate of three Sphere Giants

In this appendix, we will compute the $+1 \to 1$ interaction for two strings attached to a bound state of three sphere giants. This example is interesting because, firstly, it does partially illustrate our claim that the methods we have developed apply to any bound state of giants and secondly, in this situation, we expect an emergent $U(3)$ gauge theory. The three sphere giant boundstate is described by a Young diagram with three columns. When labeling the open string endpoints we will use the labels 'b', 'm' and 'l' for the first column ('b' for big brane), second column ('m' for medium brane) and third column ('l' for little brane) respectively. The relevant Cuntz chain states together with their normalizations are shown in the table below.

State	Normalization
$\lvert b_0, b_1-1, b_2, bb, mm\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+b_2+1)b_1}{(b_1+b_2+2)(b_1+1)}}$
$\lvert b_0-1, b_1+1, b_2-1, bb, ll\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+2)b_2}{(b_2+1)(b_1+1)}}$
$\lvert b_0, b_1-1, b_2, mm, bb\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+b_2+1)b_1}{(b_1+b_2+2)(b_1+1)}}$
$\lvert b_0-1, b_1, b_2+1, mm, ll\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_2+2)(b_1+b_2+3)}{(b_2+1)(b_1+b_2+2)}}$
$\lvert b_0-1, b_1+1, b_2-1, ll, bb\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+2)b_2}{(b_2+1)(b_1+1)}}$
$\lvert b_0-1, b_1, b_2+1, ll, mm\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_2+2)(b_1+b_2+3)}{(b_2+1)(b_1+b_2+2)}}$
$\lvert b_0, b_1, b_2-2, bb, bb\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_2-1)(b_1+b_2)}{(b_2+1)(b_1+b_2+2)}}$
$\lvert b_0, b_1-2, b_2+2, mm, mm\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_2+3)(b_1-1)}{(b_2+1)(b_1+1)}}$
$\lvert b_0-2, b_1+2, b_2, ll, ll\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+3)(b_1+b_2+4)}{(b_1+1)(b_1+b_2+2)}}$
$\lvert b_0, b_1-1, b_2, bm, mb\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+b_2+1)b_1}{(b_1+b_2+2)(b_1+1)}}$
$\lvert b_0, b_1-1, b_2, mb, bm\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+b_2+1)b_1}{(b_1+b_2+2)(b_1+1)}}$
$\lvert b_0-1, b_1+1, b_2-1, bl, lb\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+2)b_2}{(b_2+1)(b_1+1)}}$
$\lvert b_0-1, b_1+1, b_2-1, lb, bl\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_1+2)b_2}{(b_2+1)(b_1+1)}}$
$\lvert b_0-1, b_1, b_2+1, ml, lm\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_2+2)(b_1+b_2+3)}{(b_2+1)(b_1+b_2+2)}}$
$\lvert b_0-1, b_1, b_2+1, lm, ml\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{3b_0+2b_1+b_2+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{(b_2+2)(b_1+b_2+3)}{(b_2+1)(b_1+b_2+2)}}$

The labels b_0, b_1 and b_2 again determine the momenta of the giants. The giant corresponding to the first column has a momentum of $b_0+b_1+b_2$, the giant corresponding to the second column has a momentum of b_0+b_1 and the giant corresponding to the third column has a momentum of b_0. We take b_0 to be $O(N)$ and b_1, b_2 to be $O(1)$.

To determine the boundary interactions, we start by rewriting the identities of section 8.2 for the case that we have a Young diagram with three columns. To obtain the boundary interaction terms in the Hamiltonian, these identities are then inverted and rewritten in terms of normalized Cuntz chain

states.

The term in the Hamiltonian describing the process in which a Z hops out of the first site of string 1 is given by

$$H_{+1\to 1}\begin{bmatrix} |b_0,b_1-1,b_2,bb,mm\rangle \\ |b_0-1,b_1+1,b_2-1,bb,ll\rangle \\ |b_0,b_1-1,b_2,mm,bb\rangle \\ |b_0-1,b_1,b_2+1,mm,ll\rangle \\ |b_0-1,b_1+1,b_2-1,ll,bb\rangle \\ |b_0-1,b_1,b_2+1,ll,mm\rangle \\ |b_0,b_1,b_2-2,bb,bb\rangle \\ |b_0,b_1-2,b_2+2,mm,mm\rangle \\ |b_0-2,b_1+2,b_2,ll,ll\rangle \\ |b_0,b_1-1,b_2,bm,mb\rangle \\ |b_0,b_1-1,b_2,mb,bm\rangle \\ |b_0-1,b_1+1,b_2-1,bl,lb\rangle \\ |b_0-1,b_1+1,b_2-1,lb,bl\rangle \\ |b_0-1,b_1,b_2+1,ml,lm\rangle \\ |b_0-1,b_1,b_2+1,lm,ml\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M \begin{bmatrix} |b_0,b_1-1,b_2+1,bb,mm\rangle \\ |b_0-1,b_1+1,b_2,bb,ll\rangle \\ |b_0,b_1,b_2-1,mm,bb\rangle \\ |b_0-1,b_1+1,b_2,mm,ll\rangle \\ |b_0,b_1,b_2-1,ll,bb\rangle \\ |b_0,b_1-1,b_2+1,ll,mm\rangle \\ |b_0,b_1,b_2-1,bb,bb\rangle \\ |b_0,b_1-1,b_2+1,mm,mm\rangle \\ |b_0-1,b_1+1,b_2,ll,ll\rangle \\ |b_0,b_1-1,b_2+1,bm,mb\rangle \\ |b_0,b_1,b_2-1,mb,bm\rangle \\ |b_0-1,b_1+1,b_2,bl,lb\rangle \\ |b_0,b_1,b_2-1,lb,bl\rangle \\ |b_0-1,b_1+1,b_2,ml,lm\rangle \\ |b_0,b_1-1,b_2+1,lm,ml\rangle \end{bmatrix},$$

where the non-zero elements of M are presented below. Note that the matrix elements are listed column by column i.e. all non-zero elements of the first column are presented first followed by all non-zero entries of the second column etc.):

$$\begin{aligned}
M_{1\,1} &= -(b_2)_1^2(b_1+b_2)_2, \\
M_{3\,1} &= -\frac{(b_1+b_2)_2}{(b_2+1)^2(b_2+2)}, \\
M_{6\,1} &= -\frac{(b_1+2)\sqrt{b_2+2}\sqrt{b_1}}{\sqrt{b_2+1}\,(b_1+1)^{3/2}\,(b_1+b_2+2)}, \\
M_{4\,1} &= \frac{-b_1-b_2-3}{\sqrt{b_2+1}\,(b_1+1)^{3/2}\,(b_1+b_2+2)\,\sqrt{b_1}\sqrt{b_2+2}}, \\
M_{8\,1} &= -\frac{\sqrt{b_1-1}\sqrt{b_2+3}\sqrt{b_1+b_2+3}}{\sqrt{b_2+1}\sqrt{b_1+b_2+2}\,(b_2+2)\,\sqrt{b_1}},
\end{aligned}$$

$$M_{10\,1} = -\frac{(b_1+b_2)_2(b_2)_1}{(b_2+1)(b_2+2)},$$
$$M_{11\,1} = -\frac{(b_1+b_2)_2(b_2)_1}{(b_2+1)},$$
$$M_{14\,1} = \frac{\sqrt{b_2+2}\sqrt{b_1+2}}{(b_1+1)^{3/2}\sqrt{b_2+1}\,(b_1+b_2+2)},$$
$$M_{15\,1} = \frac{(b_1+b_2+3)\sqrt{b_1+2}}{(b_1+1)^{3/2}\sqrt{b_2+1}\,(b_1+b_2+2)\sqrt{b_2+2}},$$
$$M_{2\,2} = -(b_1+b_2)_2^2(b_2)_1,$$
$$M_{5\,2} = \frac{(b_2)_1}{(b_1+b_2+2)^2(b_1+b_2+3)},$$
$$M_{4\,2} = -\frac{b_1\sqrt{b_1+b_2+3}\sqrt{b_1+2}}{(b_1+1)^{3/2}(b_2+1)\sqrt{b_1+b_2+2}},$$
$$M_{6\,2} = \frac{-b_2-2}{(b_2+1)(b_1+1)^{3/2}\sqrt{b_1+b_2+2}\sqrt{b_1+2}\sqrt{b_1+b_2+3}},$$
$$M_{13\,2} = -\frac{(b_2)_1(b_1+b_2)_2}{(b_1+b_2+2)},$$
$$M_{9\,2} = -\frac{\sqrt{b_1+b_2+4}\sqrt{b_1+3}\sqrt{b_2+2}}{\sqrt{b_2+1}\sqrt{b_1+b_2+2}\sqrt{b_1+2}\,(b_1+b_2+3)},$$
$$M_{12\,2} = -\frac{(b_2)_1(b_1+b_2)_2}{(b_1+b_2+2)(b_1+b_2+3)},$$
$$M_{14\,2} = -\frac{(b_2+2)\sqrt{b_1}}{(b_1+1)^{3/2}(b_2+1)\sqrt{b_1+b_2+2}\sqrt{b_1+b_2+3}},$$
$$M_{1\,3} = \frac{(b_1)_1}{b_2(b_2+1)^2},$$
$$M_{15\,2} = -\frac{\sqrt{b_1+b_2+3}\sqrt{b_1}}{(b_1+1)^{3/2}(b_2+1)\sqrt{b_1+b_2+2}},$$
$$M_{3\,3} = -(b_2)_1^2(b_1)_1,$$
$$M_{2\,3} = \frac{b_1+2}{\sqrt{b_2+1}\,(b_1+1)\,(b_1+b_2+2)^{3/2}\sqrt{b_1+b_2+1}\sqrt{b_2}},$$
$$M_{10\,3} = \frac{(b_1)_1(b_2)_1}{(b_2+1)},$$
$$M_{5\,3} = -\frac{(b_1+b_2+3)\sqrt{b_2}\sqrt{b_1+b_2+1}}{\sqrt{b_2+1}\,(b_1+1)\,(b_1+b_2+2)^{3/2}},$$
$$M_{11\,3} = \frac{(b_1)_1(b_2)_1}{b_2(b_2+1)},$$
$$M_{7\,3} = \frac{\sqrt{b_2-1}\sqrt{b_1+b_2}\sqrt{b_1+2}}{\sqrt{b_2+1}\sqrt{b_1+1}b_2\sqrt{b_1+b_2+1}},$$
$$M_{12\,3} = \frac{\sqrt{b_2}\sqrt{b_1+b_2+3}}{(b_1+b_2+2)^{3/2}(b_1+1)\sqrt{b_2+1}},$$

$$M_{13\,3} = -\frac{(b_1+2)\sqrt{b_1+b_2+3}}{(b_1+b_2+2)^{3/2}(b_1+1)\sqrt{b_2+1}\sqrt{b_2}},$$
$$M_{2\,4} = \frac{\sqrt{b_1+2}(b_1+b_2+1)\sqrt{b_1+b_2+3}}{(b_2+1)\sqrt{b_1+1}(b_1+b_2+2)^{3/2}},$$
$$M_{4\,4} = -(b_1)_1^2(b_2)_1,$$
$$M_{5\,4} = -\frac{b_2}{\sqrt{b_1+1}(b_2+1)(b_1+b_2+2)^{3/2}\sqrt{b_1+b_2+3}\sqrt{b_1+2}},$$
$$M_{6\,4} = -\frac{\sqrt{b_2+2}\sqrt{b_2}}{(b_1+1)^2(b_2+1)(b_1+2)},$$
$$M_{9\,4} = -\frac{\sqrt{b_1+b_2+4}\sqrt{b_1+3}\sqrt{b_2}}{\sqrt{b_2+1}\sqrt{b_1+1}(b_1+2)\sqrt{b_1+b_2+3}},$$
$$M_{12\,4} = -\frac{b_2\sqrt{b_1+b_2+1}}{(b_1+b_2+2)^{3/2}(b_2+1)\sqrt{b_1+1}\sqrt{b_1+2}},$$
$$M_{13\,4} = \frac{\sqrt{b_1+2}\sqrt{b_1+b_2+1}}{(b_1+b_2+2)^{3/2}(b_2+1)\sqrt{b_1+1}},$$
$$M_{14\,4} = -\frac{(b_2)_1(b_1)_1}{(b_1+1)(b_1+2)},$$
$$M_{1\,5} = \frac{b_1}{(b_2+1)^{3/2}(b_1+1)\sqrt{b_1+b_2+2}\sqrt{b_2}\sqrt{b_1+b_2+1}},$$
$$M_{15\,4} = -\frac{(b_2)_1(b_1)_1}{(b_1+1)},$$
$$M_{2\,5} = \frac{(b_1)_1}{(b_1+b_2+2)^2(b_1+b_2+1)},$$
$$M_{3\,5} = \frac{\sqrt{b_1+b_2+1}(b_2+2)\sqrt{b_2}}{(b_2+1)^{3/2}(b_1+1)\sqrt{b_1+b_2+2}},$$
$$M_{5\,5} = -(b_1)_1(b_1+b_2)_2^2,$$
$$M_{7\,5} = \frac{\sqrt{b_2-1}\sqrt{b_1+b_2}\sqrt{b_1}}{\sqrt{b_1+1}\sqrt{b_1+b_2+2}(b_1+b_2+1)\sqrt{b_2}},$$
$$M_{10\,5} = -\frac{\sqrt{b_1+b_2+1}\sqrt{b_2+2}}{(b_2+1)^{3/2}(b_1+1)\sqrt{b_1+b_2+2}},$$
$$M_{11\,5} = -\frac{b_1\sqrt{b_2+2}}{(b_2+1)^{3/2}(b_1+1)\sqrt{b_1+b_2+2}\sqrt{b_1+b_2+1}},$$
$$M_{12\,5} = \frac{(b_1)_1(b_1+b_2)_2}{b_1+b_2+2},$$
$$M_{13\,5} = -\frac{(b_1)_1(b_1+b_2)_2}{(b_1+b_2+2)(b_1+b_2+1)},$$
$$M_{1\,6} = \frac{\sqrt{b_1}b_2\sqrt{b_2+2}}{\sqrt{b_1+1}(b_2+1)^{3/2}(b_1+b_2+2)},$$
$$M_{3\,6} = \frac{b_1+b_2+1}{\sqrt{b_1+1}(b_2+1)^{3/2}(b_1+b_2+2)\sqrt{b_2+2}\sqrt{b_1}},$$

$$
\begin{aligned}
M_{4\,6} &= \frac{(b_1+b_2)_2}{b_1(b_1+1)^2},\\
M_{6\,6} &= -(b_1)_1^2(b_1+b_2)_2,\\
M_{8\,6} &= \frac{\sqrt{b_1-1}\sqrt{b_2+3}\sqrt{b_1+b_2+1}}{\sqrt{b_1+1}\sqrt{b_1+b_2+2}\sqrt{b_2+2}b_1},\\
M_{10\,6} &= \frac{(b_1+b_2+1)\sqrt{b_2}}{(b_2+1)^{3/2}\sqrt{b_1+1}\,(b_1+b_2+2)\sqrt{b_1}},\\
M_{11\,6} &= \frac{\sqrt{b_1}\sqrt{b_2}}{(b_2+1)^{3/2}\sqrt{b_1+1}\,(b_1+b_2+2)},\\
M_{14\,6} &= \frac{(b_1)_1(b_1+b_2)_2}{b_1+1},\\
M_{1\,7} &= -\frac{\sqrt{b_1}\sqrt{b_2+2}\sqrt{b_1+b_2+3}}{(b_2+1)^2\sqrt{b_1+1}\sqrt{b_1+b_2+2}\sqrt{b_2}},\\
M_{15\,6} &= -\frac{(b_1)_1(b_1+b_2)_2}{b_1(b_1+1)},\\
M_{2\,7} &= -\frac{\sqrt{b_1+2}\sqrt{b_2+2}\sqrt{b_1+b_2+3}}{\sqrt{b_2+1}\sqrt{b_1+1}\,(b_1+b_2+2)^2\sqrt{b_1+b_2+1}},\\
M_{3\,7} &= -\frac{\sqrt{b_1}\sqrt{b_2+2}\sqrt{b_1+b_2+3}\sqrt{b_2}}{\sqrt{b_1+1}\,(b_2+1)^2\sqrt{b_1+b_2+2}},\\
M_{5\,7} &= -\frac{\sqrt{b_1+2}\sqrt{b_2+2}(b_1+b_2)_2}{\sqrt{b_1+1}\sqrt{b_2+1}\,(b_1+b_2+2)},\\
M_{7\,7} &= -\frac{\sqrt{b_1+b_2+3}\sqrt{b_2+2}\sqrt{b_2-1}\sqrt{b_1+b_2}}{\sqrt{b_2}\sqrt{b_1+b_2+1}\sqrt{b_2+1}\sqrt{b_1+b_2+2}},\\
M_{10\,7} &= \frac{\sqrt{b_1}\sqrt{b_1+b_2+3}}{(b_2+1)^2\sqrt{b_1+1}\sqrt{b_1+b_2+2}},\\
M_{11\,7} &= \frac{\sqrt{b_1}\,(b_2+2)\sqrt{b_1+b_2+3}}{(b_2+1)^2\sqrt{b_1+1}\sqrt{b_1+b_2+2}},\\
M_{12\,7} &= \frac{\sqrt{b_1+2}\sqrt{b_2+2}}{(b_1+b_2+2)^2\sqrt{b_2+1}\sqrt{b_1+1}},\\
M_{13\,7} &= \frac{\sqrt{b_1+2}\sqrt{b_2+2}\,(b_1+b_2+3)}{(b_1+b_2+2)^2\sqrt{b_2+1}\sqrt{b_1+1}},\\
M_{1\,8} &= \frac{\sqrt{b_1+2}\sqrt{b_1+b_2+1}(b_2)_1}{(b_2+1)\sqrt{b_1+1}\sqrt{b_1+b_2+2}},\\
M_{3\,8} &= -\frac{\sqrt{b_1+2}\sqrt{b_1+b_2+1}\sqrt{b_2}}{(b_2+1)^2\sqrt{b_1+1}\sqrt{b_1+b_2+2}\sqrt{b_2+2}},\\
M_{4\,8} &= -\frac{\sqrt{b_1+2}\sqrt{b_1+b_2+3}\sqrt{b_2}}{(b_1+1)^2\sqrt{b_2+1}\sqrt{b_1+b_2+2}\sqrt{b_1}},\\
M_{6\,8} &= -\frac{\sqrt{b_1+2}\sqrt{b_1+b_2+3}\sqrt{b_2}\sqrt{b_1}}{\sqrt{b_2+1}\,(b_1+1)^2\sqrt{b_1+b_2+2}},
\end{aligned}
$$

$$M_{8\,8} = -\frac{\sqrt{b_1-1}\sqrt{b_2+3}\sqrt{b_1+2}\sqrt{b_2}}{\sqrt{b_1+1}\sqrt{b_2+1}\sqrt{b_1}\sqrt{b_2+2}},$$

$$M_{10\,8} = -\frac{\sqrt{b_1+2}\sqrt{b_1+b_2+1}b_2}{(b_2+1)^2\sqrt{b_1+1}\sqrt{b_1+b_2+2}},$$

$$M_{11\,8} = \frac{\sqrt{b_1+2}\sqrt{b_1+b_2+1}}{(b_2+1)^2\sqrt{b_1+1}\sqrt{b_1+b_2+2}},$$

$$M_{14\,8} = \frac{\sqrt{b_1+b_2+3}\sqrt{b_2}}{(b_1+1)^2\sqrt{b_2+1}\sqrt{b_1+b_2+2}},$$

$$M_{15\,8} = \frac{(b_1+2)\sqrt{b_1+b_2+3}\sqrt{b_2}}{(b_1+1)^2\sqrt{b_2+1}\sqrt{b_1+b_2+2}},$$

$$M_{2\,9} = \frac{\sqrt{b_1}\sqrt{b_2}(b_1+b_2)_2}{\sqrt{b_1+1}\sqrt{b_2+1}\,(b_1+b_2+2)},$$

$$M_{4\,9} = \frac{\sqrt{b_1}\sqrt{b_1+b_2+1}\sqrt{b_2+2}\sqrt{b_1+2}}{\sqrt{b_1+b_2+2}\sqrt{b_2+1}\,(b_1+1)^2},$$

$$M_{5\,9} = -\frac{\sqrt{b_1}\sqrt{b_1+b_2+1}\sqrt{b_2}}{\sqrt{b_1+1}\sqrt{b_2+1}\,(b_1+b_2+2)^2\sqrt{b_1+b_2+3}},$$

$$M_{6\,9} = -\frac{\sqrt{b_1}\sqrt{b_1+b_2+1}\sqrt{b_2+2}}{\sqrt{b_2+1}\,(b_1+1)^2\sqrt{b_1+b_2+2}\sqrt{b_1+2}},$$

$$M_{9\,9} = -\frac{\sqrt{b_1+b_2+4}\sqrt{b_1+3}\sqrt{b_1}\sqrt{b_1+b_2+1}}{\sqrt{b_1+1}\sqrt{b_1+b_2+2}\sqrt{b_1+b_2+3}\sqrt{b_1+2}},$$

$$M_{12\,9} = -\frac{\sqrt{b_1}\,(b_1+b_2+1)\sqrt{b_2}}{(b_1+b_2+2)^2\sqrt{b_1+1}\sqrt{b_2+1}},$$

$$M_{13\,9} = \frac{\sqrt{b_1}\sqrt{b_2}}{(b_1+b_2+2)^2\sqrt{b_1+1}\sqrt{b_2+1}},$$

$$M_{14\,9} = -\frac{b_1\sqrt{b_1+b_2+1}\sqrt{b_2+2}}{(b_1+1)^2\sqrt{b_2+1}\sqrt{b_1+b_2+2}},$$

$$M_{15\,9} = \frac{\sqrt{b_1+b_2+1}\sqrt{b_2+2}}{(b_1+1)^2\sqrt{b_2+1}\sqrt{b_1+b_2+2}},$$

$$M_{3\,10} = -\frac{\sqrt{b_2+3}(b_1+b_2)_2}{(b_2+2)\sqrt{b_2+1}},$$

$$M_{4\,10} = \frac{\sqrt{b_2+3}}{(b_1+b_2+2)\sqrt{b_2+2}\sqrt{b_1}\sqrt{b_1+1}},$$

$$M_{8\,10} = \frac{\sqrt{b_1-1}\sqrt{b_1+b_2+3}}{\sqrt{b_1+b_2+2}\,(b_2+2)\sqrt{b_1}},$$

$$M_{10\,10} = -\frac{\sqrt{b_2+3}\sqrt{b_1+b_2+1}\sqrt{b_1+b_2+3}\sqrt{b_2}}{\sqrt{b_2+1}\sqrt{b_2+2}\,(b_1+b_2+2)},$$

$$M_{15\,10} = -\frac{\sqrt{b_2+3}\sqrt{b_1+2}}{\sqrt{b_1+1}\,(b_1+b_2+2)\sqrt{b_2+2}},$$

$$M_{1\,11} = \frac{(b_2-1)\sqrt{b_1}\sqrt{b_1+2}}{(b_1+1)\sqrt{{b_2}^2-1}b_2},$$

$$M_{2\,11} = \frac{\sqrt{b_2 - 1}}{(b_1 + 1)\sqrt{b_2}\sqrt{b_1 + b_2 + 1}\sqrt{b_1 + b_2 + 2}},$$
$$M_{7\,11} = -\frac{\sqrt{b_1 + b_2}\sqrt{b_1 + 2}}{\sqrt{b_1 + 1}b_2\sqrt{b_1 + b_2 + 1}},$$
$$M_{11\,11} = -\frac{\sqrt{b_2 - 1}\sqrt{b_1}\sqrt{b_1 + 2}\sqrt{b_2 + 2}}{\sqrt{b_2 + 1}\sqrt{b_2}(b_1 + 1)},$$
$$M_{13\,11} = -\frac{\sqrt{b_2 - 1}\sqrt{b_1 + b_2 + 3}}{\sqrt{b_1 + b_2 + 2}(b_1 + 1)\sqrt{b_2}},$$
$$M_{5\,12} = -\frac{\sqrt{b_1 + b_2 + 4}\sqrt{b_2 + 2}\sqrt{b_2}}{(b_2 + 1)\sqrt{b_1 + b_2 + 2}(b_1 + b_2 + 3)},$$
$$M_{6\,12} = -\frac{\sqrt{b_1 + b_2 + 4}}{(b_2 + 1)\sqrt{b_1 + b_2 + 3}\sqrt{b_1 + 2}\sqrt{b_1 + 1}},$$
$$M_{9\,12} = \frac{\sqrt{b_1 + 3}\sqrt{b_2 + 2}}{\sqrt{b_2 + 1}(b_1 + b_2 + 3)\sqrt{b_1 + 2}},$$
$$M_{12\,12} = -\frac{\sqrt{b_1 + b_2 + 4}\sqrt{b_2 + 2}\sqrt{b_2}\sqrt{b_1 + b_2 + 1}}{\sqrt{b_1 + b_2 + 2}\sqrt{b_1 + b_2 + 3}(b_2 + 1)},$$
$$M_{14\,12} = -\frac{\sqrt{b_1 + b_2 + 4}\sqrt{b_1}}{\sqrt{b_1 + 1}(b_2 + 1)\sqrt{b_1 + b_2 + 3}},$$
$$M_{1\,13} = -\frac{\sqrt{b_1 + b_2}}{(b_1 + 1)\sqrt{b_1 + b_2 + 1}\sqrt{b_2}\sqrt{b_2 + 1}},$$
$$M_{2\,13} = \frac{\sqrt{b_1 + b_2}\sqrt{b_1}\sqrt{b_1 + 2}}{(b_1 + 1)\sqrt{b_1 + b_2 + 2}(b_1 + b_2 + 1)},$$
$$M_{7\,13} = -\frac{\sqrt{b_2 - 1}\sqrt{b_1}}{\sqrt{b_1 + 1}(b_1 + b_2 + 1)\sqrt{b_2}},$$
$$M_{11\,13} = \frac{\sqrt{b_1 + b_2}\sqrt{b_2 + 2}}{\sqrt{b_2 + 1}(b_1 + 1)\sqrt{b_1 + b_2 + 1}},$$
$$M_{13\,13} = -\frac{\sqrt{b_1 + b_2}\sqrt{b_1}\sqrt{b_1 + 2}\sqrt{b_1 + b_2 + 3}}{\sqrt{b_1 + b_2 + 1}\sqrt{b_1 + b_2 + 2}(b_1 + 1)},$$
$$M_{5\,14} = \frac{\sqrt{b_1 + 3}}{(b_2 + 1)\sqrt{b_1 + 2}\sqrt{b_1 + b_2 + 3}\sqrt{b_1 + b_2 + 2}},$$
$$M_{6\,14} = -\frac{(b_1 + 3)\sqrt{b_2 + 2}\sqrt{b_2}}{(b_2 + 1)\sqrt{{b_1}^2 + 4\,b_1 + 3}(b_1 + 2)},$$
$$M_{9\,14} = \frac{\sqrt{b_1 + b_2 + 4}\sqrt{b_2}}{\sqrt{b_2 + 1}(b_1 + 2)\sqrt{b_1 + b_2 + 3}},$$
$$M_{12\,14} = \frac{\sqrt{b_1 + 3}\sqrt{b_1 + b_2 + 1}}{\sqrt{b_1 + b_2 + 2}(b_2 + 1)\sqrt{b_1 + 2}},$$
$$M_{14\,14} = -\frac{\sqrt{b_1 + 3}\sqrt{b_2 + 2}\sqrt{b_2}\sqrt{b_1}}{\sqrt{b_1 + 1}\sqrt{b_1 + 2}(b_2 + 1)},$$
$$M_{3\,15} = \frac{\sqrt{b_1 - 1}}{(b_1 + b_2 + 2)\sqrt{b_1}\sqrt{b_2 + 2}\sqrt{b_2 + 1}},$$

$$
\begin{aligned}
M_{4\,15} &= \frac{(b_1-1)\,(b_1+b_2)_2}{\sqrt{{b_1}^2-1}b_1},\\
M_{8\,15} &= -\frac{\sqrt{b_2+3}\sqrt{b_1+b_2+1}}{\sqrt{b_1+b_2+2}b_1\,\sqrt{b_2+2}},\\
M_{10\,15} &= \frac{\sqrt{b_1-1}\sqrt{b_2}}{\sqrt{b_2+1}\,(b_1+b_2+2)\,\sqrt{b_1}},\\
M_{15\,15} &= -\frac{\sqrt{b_1-1}\sqrt{b_1+b_2+1}\sqrt{b_1+b_2+3}\sqrt{b_1+2}}{\sqrt{b_1+1}\sqrt{b_1}\,(b_1+b_2+2)}.
\end{aligned}
$$

For large b_1 and b_2, we find that $M=-\mathbf{1}$ with $\mathbf{1}$ the 15×15 identity matrix. We can also identify terms in M that behave as b_1^{-1}

$$
M_1=\frac{1}{b_1}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&1&0&0&-1&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0
\end{bmatrix},
$$

terms that behave as b_2^{-1}

$$M_2 = \frac{1}{b_2}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

and terms that behave as $(b_1 + b_2)^{-1}$

$$M_3 = \frac{1}{b_1 + b_2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By looking at the Cuntz chain states, it is straight forward to see that M_1 is reproduced by ribbon diagrams in which a pair of labels undergoes a $l \leftrightarrow m$ transition, that M_2 is reproduced by ribbon diagrams in which a pair of labels undergoes a $b \leftrightarrow m$ transition and that M_3 is reproduced by ribbon diagrams in which a pair of labels undergoes a $l \leftrightarrow b$ transition. This is exactly the structure expected from an emergent $U(3)$ gauge theory.

References

[1] S. Corley, A. Jevicki and S. Ramgoolam, "Exact correlators of giant gravitons from dual N = 4 SYM theory," Adv. Theor. Math. Phys. **5**, 809 (2002) [arXiv:hep-th/0111222].

[2] V. Balasubramanian, M. Berkooz, A. Naqvi and M. J. Strassler, "Giant gravitons in conformal field theory," JHEP **0204**, 034 (2002) [arXiv:hep-th/0107119].

[3] V. Balasubramanian, D. Berenstein, B. Feng and M. x. Huang, "D-branes in Yang-Mills theory and emergent gauge symmetry," JHEP **0503**, 006 (2005) [arXiv:hep-th/0411205].

[4] R. de Mello Koch, J. Smolic and M. Smolic, "Giant Gravitons - with Strings Attached (I)," JHEP **0706**, 074 (2007), arXiv:hep-th/0701066.

[5] R. de Mello Koch, J. Smolic and M. Smolic, "Giant Gravitons - with Strings Attached (II)," JHEP **0709** 049 (2007), arXiv:hep-th/0701067.

[6] D. Bekker, R. de Mello Koch and M. Stephanou, "Giant Gravitons - with Strings Attached (III)," (2007), arXiv:0710.5372[hep-th].

[7] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[8] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[9] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[10] G. 't Hooft, "Dimensional Reduction in Quantum Gravity," Utrecht THU-93-26 (1993) [gr-qc/9310006].

[11] L. Susskind, "The World as a Hologram," J.Math.Phys. 36 (1995) 6377-6396 [arXiv:hep-th/9409089].

[12] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B72 (1974) 46.

[13] J. McGreevy, L. Susskind and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," JHEP **0006**, 008 (2000) [arXiv:hep-th/0003075].

[14] D. Berenstein, "A toy model for the AdS/CFT correspondence," JHEP 07 (2004) 018 [arXiv:hep-th/0403110].

[15] T. W. Brown, P.J. Heslop and S. Ramgoolam, "Diagonal multi-matrix correlators and BPS operators in N = 4 SYM," (2007) arXiv:0711.0176[hep-th].

[16] H. Lin, O. Lunin and J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," JHEP **0410**, 025 (2004) [arXiv:hep-th/0409174],
H. Y. Chen, D. H. Correa and G. A. Silva, "Geometry and topology of bubble solutions from gauge theory," Phys. Rev. D **76**, 026003 (2007) [arXiv:hep-th/0703068].

[17] R. de Mello Koch and M. Stephanou, work in progress.

[18] D. Berenstein, D. H. Correa and S. E. Vazquez, "A study of open strings ending on giant gravitons, spin chains and integrability," [arXiv:hep-th/0604123].

[19] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].

[20] N. Beisert and M. Staudacher, "The N = 4 SYM integrable super spin chain," Nucl. Phys. B **670** (2003) 439 [arXiv:hep-th/0307042].

[21] D. Berenstein, D. H. Correa and S. E. Vazquez, "Quantizing open spin chains with variable length: An example from giant gravitons," Phys.

Rev. Lett. **95**, 191601 (2005) [arXiv:hep-th/0502172].

[22] D. H. Correa and G. A. Silva, "Dilatation operator and the Super Yang-Mills duals of open strings on AdS Giant Gravitons," [arXiv:hep-th/0608128].

[23] R. de Mello Koch, N. Ives, J. Smolic and M. Smolic, "Unstable giants," Phys. Rev. D **73**, 064007 (2006) [arXiv:hep-th/0509007].

[24] M. Kruczenski, "Spin Chains and String Theory," Phys. Rev. Lett. **93**, 161602 (2004) [arXiv:hep-th/0311203].

[25] E. D'Hoker, D. Z. Freedman and W. Skiba, "Field theory tests for correlators in the AdS/CFT correspondence," Phys. Rev. D **59**, 045008 (1999) [arXiv:hep-th/9807098],
W. Skiba, "Correlators of short multi-trace operators in N = 4 supersymmetric Yang-Mills," Phys. Rev. D **60**, 105038 (1999) [arXiv:hep-th/9907088],
N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, "PP-wave string interactions from perturbative Yang-Mills theory," JHEP **0207**, 017 (2002) [arXiv:hep-th/0205089].

[26] J.-Q. Chen, *Group Representation Theory for Physicists*, Singapore, World Scientific (1989).

[27] D. Berenstein and S. E. Vazquez, "Integrable open spin chains from giant gravitons," JHEP **0506**, 059 (2005) [arXiv:hep-th/0501078].

[28] V. Balasubramanian and A. Naqvi, "Giant gravitons and a correspondence principle," Phys. Lett. B **528**, 111 (2002) [arXiv:hep-th/0111163].

[29] S. Hirano, "Fat magnon," JHEP **0704**, 010 (2007) [arXiv:hep-th/0610027].

[30] M. Ali-Akbari, M. M. Sheikh-Jabbari and M. Torabian, "Tiny graviton matrix theory / SYM correspondence: Analysis of BPS states," Phys. Rev. D **74**, 066005 (2006) [arXiv:hep-th/0606117],
M. M. Sheikh-Jabbari and M. Torabian, "Classification of all 1/2 BPS solutions of the tiny graviton matrix theory," JHEP **0504**, 001 (2005) [arXiv:hep-th/0501001],
M. M. Sheikh-Jabbari, "Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture," JHEP **0409**, 017 (2004) [arXiv:hep-th/0406214].

[31] R. C. Myers, "Dielectric-branes," JHEP **9912** (1999) 022 [arXiv:hep-th/9910053].

[32] Y. Kimura and S. Ramgoolam, "Branes, Anti-Branes and Brauer Algebras in Gauge-Gravity duality," arXiv:0709.2158 [hep-th].

[33] J. P. Rodrigues, "Large N spectrum of two matrices in a harmonic potential and BMN energies," JHEP **0512**, 043 (2005) [arXiv:hep-th/0510244],
A. Donos, A. Jevicki and J. P. Rodrigues, "Matrix model maps in AdS/CFT," Phys. Rev. D **72**, 125009 (2005) [arXiv:hep-th/0507124],
R. de Mello Koch, A. Jevicki and J. P. Rodrigues, "Instantons in c = 0 CSFT," JHEP **0504**, 011 (2005) [arXiv:hep-th/0412319],
R. de Mello Koch, A. Donos, A. Jevicki and J. P. Rodrigues, "Derivation of string field theory from the large N BMN limit," Phys. Rev. D **68**, 065012 (2003) [arXiv:hep-th/0305042],
R. de Mello Koch, A. Jevicki and J. P. Rodrigues, "Collective string field theory of matrix models in the BMN limit," Int. J. Mod. Phys. A **19**, 1747 (2004) [arXiv:hep-th/0209155].

[34] D. Berenstein and R. Cotta, "A Monte-Carlo study of the AdS/CFT correspondence: An exploration of quantum gravity effects," JHEP **0704**, 071 (2007) [arXiv:hep-th/0702090],
D. Berenstein and R. Cotta, "Aspects of emergent geometry in the

AdS/CFT context," Phys. Rev. D **74**, 026006 (2006) [arXiv:hep-th/0605220],
D. Berenstein and D. H. Correa, "Emergent geometry from q-deformations of N = 4 super Yang-Mills," JHEP **0608**, 006 (2006) [arXiv:hep-th/0511104],
D. Berenstein, "Large N BPS states and emergent quantum gravity," JHEP **0601**, 125 (2006) [arXiv:hep-th/0507203],
D. Berenstein, "Strings on conifolds from strong coupling dynamics, part I," arXiv:0710.2086 [hep-th].

[35] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[36] J. de Boer, "Introduction to the AdS/CFT correspondence," Prepared for 10th International Conference on Supersymmetry and Unification of Fundamental Interactions (SUSY02), Hamburg, Germany, 17-23 Jun 2002

[37] R. de Mello Koch, "Lectures on the gauge theory / gravity correspondence," Prepared for 4th International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH4), Contonou, Republic of Benin, 5-11 Nov 2005

[38] D. Berenstein, " Shape and Holography: Studies of dual operators to giant gravitons," Nucl. Phys. **B675** 179, (2003) [arXiv:hep-th/0306090].

[39] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[40] R. Russo and A. Tanzini, "The duality between IIB string theory on pp-wave and N = 4 SYM: A status report," Class. Quant. Grav. **21**, S1265 (2004) [arXiv:hep-th/0401155],

D. Sadri and M. M. Sheikh-Jabbari, "The plane-wave / super Yang-Mills duality," Rev. Mod. Phys. **76**, 853 (2004) [arXiv:hep-th/0310119].

[41] M. T. Grisaru, R. C. Myers and O. Tafjord, "SUSY and Goliath," JHEP **0008**, 040 (2000) [arXiv:hep-th/0008015].

[42] A. Hashimoto, S. Hirano and N. Itzhaki, "Large branes in AdS and their field theory dual," JHEP **0008**, 051 (2000) [arXiv:hep-th/0008016].

[43] S. R. Das, A. Jevicki and S. D. Mathur, "Giant gravitons, BPS bounds and noncommutativity," Phys. Rev. D **63**, 044001 (2001) [arXiv:hep-th/0008088],
S. R. Das, A. Jevicki and S. D. Mathur, "Vibration Modes of Giant Gravitons," Phys. Rev. D **63** 024013 (2001) [arXiv:hep-th/0009019].

[44] V. Balasubramanian, M. x. Huang, T. S. Levi and A. Naqvi, "Open strings from N = 4 super Yang-Mills," JHEP **0208**, 037 (2002) [arXiv:hep-th/0204196],
O. Aharony, Y.E. Antebi, M. Berkooz and R. Fishman, "Holey sheets: Pfaffians and subdeterminants as D-brane operators in large N gauge theories," JHEP **0212**, 096 (2002) [arXiv:hep-th/0211152],
D. Berenstein, " Shape and Holography: Studies of dual operators to giant gravitons," Nucl. Phys. **B675** 179, (2003) [arXiv:hep-th/0306090].

[45] D. Berenstein and S. E. Vazquez, "Integrable open spin chains from giant gravitons," JHEP **0506**, 059 (2005) [arXiv:hep-th/0501078],
D. Berenstein, D. H. Correa and S. E. Vazquez, "Quantizing open spin chains with variable length: An example from giant gravitons," Phys. Rev. Lett. **95**, 191601 (2005) [arXiv:hep-th/0502172],
R. de Mello Koch, N. Ives, J. Smolic and M. Smolic, "Unstable giants," Phys. Rev. D **73**, 064007 (2006) [arXiv:hep-th/0509007],
A. Agarwal, "Open spin chains in super Yang-Mills at higher loops: Some potential problems with integrability," [arXiv:hep-th/0603067],

K. Okamura and K. Yoshida, "Higher loop Bethe ansatz for open spin-chains in AdS/CFT," [arXiv:hep-th/0604100],
D. H. Correa and G. A. Silva, "Dilatation operator and the Super Yang-Mills duals of open strings on AdS Giant Gravitons," [arXiv:hep-th/0608128],
N. Mann and S. E. Vazquez, "Classical Open String Integrability," [arXiv:hep-th/0612038].

[46] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, "Entropy of near-extremal black holes in AdS_5," arXiv:0707.3601 [hep-th].

[47] D. M. Hofman and J. M. Maldacena, "Reflecting magnons," arXiv:0708.2272 [hep-th].

[48] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in N = 4 SYM theory," Nucl. Phys. B **641**, 131 (2002) [arXiv:hep-th/0205221],
R. de Mello Koch and R. Gwyn, "Giant graviton correlators from dual SU(N) super Yang-Mills theory," JHEP **0411**, 081 (2004) [arXiv:hep-th/0410236],
T. W. Brown, "Half-BPS SU(N) correlators in N = 4 SYM," arXiv:[hep-th/0703202],
S. Ramgoolam, "Wilson loops in 2-D Yang-Mills: Euler characters and loop equations," Int. J. Mod. Phys. A **11**, 3885 (1996) [arXiv:hep-th/9412110].

[49] D. Sadri and M. M. Sheikh-Jabbari, "Giant hedge-hogs: Spikes on giant gravitons," Nucl. Phys. B **687**, 161 (2004) [arXiv:hep-th/0312155],
M. Ali-Akbari and M. M. Sheikh-Jabbari, "Electrified BPS Giants: BPS configurations on Giant Gravitons with Static Electric Field," arXiv:0708.2058 [hep-th].