Pricing Interest Rate Derivatives Using The Forward Market Model

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Abstract

The IBOR are due to be discontinued and their replacements have been chosen to be the overnight rates. This change in the risk-free rate comes with challenges of how the new rates will be modelled and how the products will be priced. In this dissertation, we look to explore the classical short-rates and the new generalized Forward Market Model proposed by Andrei Lyanschenko and Fabio Mercurio in 2019. We seek to utilize this model in pricing interest rate derivatives such as caps and swaptions.

Declaration

I, Tshana Tumelo Konaite, hereby declare that the dissertation entitled 'Pricing Interest Rate Derivatives using the Forward Market Model' is my original work. I am submitting it for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. This dissertation has been prepared without any external assistance and has not been previously submitted for any other degree or examination at any other university.

Signature

Date

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Contents

List of Figures

List of Tables

List of abbreviations

ARRC Alternative reference rates commitee

BGM Brace Gatarek and Musiela

BL Backward Looking rate

CBOE Chicago bond options exchange

CIR Cox-Ingersoll-Ross

CCP Central counterparty

CME Chicago Mercantile Exchange

Flr Black floor payoff

FMM Forward market model

FL Forward Looking rate

EMM Equivalent Martingale Measure

FRA Forward rate agreement

GBM Geometric Brownian motion

HJM Heath-Jarrow-Morton

IBOR Interbank offered rates

IRS Interest rate swap

LMM LIBOR market model

OTC Over-the-counter

OIS Overnight index swap

PDE Partial differential equation

PFS European payer swaption payoff

RFR Risk free rates

SARON Swiss average overnight rate

SDE Stochastic differential equation

List of symbols

$\mathbb{Q}$ - Risk neutral measure

$\mathbb{Q}_0$ - Real world measure

$\sim$ - Equivalence

$r(\top)$ - Short rate

$C(\top)$ - Cash account

$S(\top)$ - Underlying asset

$\pi(\top)$ - Portfolio value

$\mathbb{E}^N$ - Expectation under the measure N

$D_f(\top, \top)$ - The discount factor between $\top$ and T.

κ - The strike price

δ - Compounding period

$f(\top)$ - The instantaneous forward rate

$H_\top$ - Time T payoff

$\mu(\top, r(\top))$ - Drift parameter

$\sigma(\top, r(\top))$ -Volatility parameter

$F(\top, T)$ - Forward rate

$P_\top(\top)$ - Zero-coupon bond

ϕ - The standard normal distribution function

ρ - Correlation coefficient

$\mathcal{F}_\top$ - The filtration at time $\top$

$W(\top)$ - The Wiener process

CHAPTER 1

Introduction

1.1 Theory of derivatives

The most commonly traded products in the markets are derivatives. Derivatives are traded either on exchange markets or in the over-the-counter (OTC) markets. Derivatives were initially invented as a risk management tool in the commodities markets. So commodity derivatives were the first types of derivatives to be introduced. By commodity, we are referring to things like wheat, gold, silver, etc. Financial derivatives followed. Examples of financial derivatives are futures, forwards, and options which will be formally defined in the coming chapters. The pricing of interest rate derivatives has been examined and investigated in the last 50 years and resulted in numerous tools and methods, most of which are still being used by practitioners even today.

A derivative exchange defines and standardizes contracts that can then be traded by the market participants. Derivative markets have been in existence for a very long time. As previously mentioned, some derivatives are traded in the OTC markets. These derivatives, however, expose the parties to risks such as counter-party risk, legal risk and operational risk. The OTC market for interest rate derivatives, however, has been expanding swiftly in recent years and it has become one of the biggest and most liquid option markets.

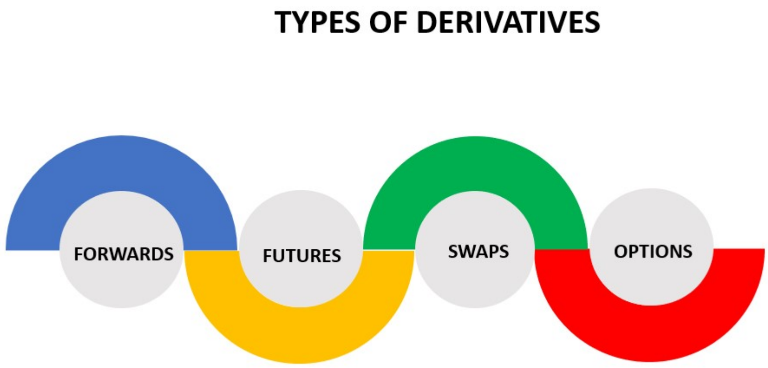


Figure 1.1: Derivatives classification.

Figure (1.1) shows how derivatives evolved through time. Forwards were the first to be introduced in the market. They were then followed by the emergence of futures. Swaps were then introduced in the market and they gained popularity quickly. The first OTC swap is thought to have been between IBM and the World Bank. Options were the last to be introduced in the market.

Ever since the capital markets were integrated globally, there has been an increase in the financial risk due to how frequent the prices of the interest and currency exchange rates and the stock prices. Financial derivatives were invented to manage and overcome the risks that are a result of the fluctuations of the previously mentioned variables. Derivatives can also be used to serve the following purposes: Hedging or reducing the risk one is exposed to; Speculation; and Arbitrage.

1.2 Theory of short-rates

The short-rate models generally have a simple form and they often result in an analytical solution for the bonds and the vanilla derivatives. Since these short-rate models are analytically tractable, it implies that the payoff of a given derivative can be computed in a short time. There are quite a number of different short-rate models that exist in the literature. They generally have the form given by equation

$$dr(\top) = \mu(\top, r(\top))d\top + \sigma(\top, r(\top))dW(\top) \tag{1.1}$$

in Equivalent Martingale Measure $\mathbb{Q}$ but they only differ in how the drift and the volatility parameter are defined. Table (1.1) contains some of the popular equilibrium models as well as their general properties such as their distribution, if they contain mean reversion, and if the bonds are analytically tractable. Equilibrium models were the first to be proposed.

Model	Distribution	Analytic $P_\top(T)$	$r(\top) > 0$?	Mean Reversion
Vasicek	Gaussian	Yes	No	Yes
Dothan	e^{Gaussian}	Yes	Yes	Yes
CIR	Gaussian^2	Yes	Yes	Yes
Exp Vasicek	e^{Gaussian^2}	No	Yes	Yes

Table 1.1: Equilibrium models.

Model	Distribution	Analytic $P_\top(T)$	$r(\top) > 0$?	Mean Reversion
Ho-Lee	Gaussian	Yes	No	No
Hull-White	Gaussian	Yes	No	Yes
Extended CIR	Gaussian^2	Yes	Yes	Yes
Black-Karasinki	e^{Gaussian^2}	No	Yes	Yes

Table 1.2: Arbitrage-free models.

These were then improved upon by the introduction of arbitrage-free models which can be found in Table (1.2). Arbitrage-free models are mostly extensions of the equilibrium models in that they introduce a time-dependent drift function to the equilibrium models. This introduction is done with the aim of trying to match these models to the interest rate term structure that is observed today in the markets. These classes are known as as *affine term structure models.* These models have the special property of being able to relate the zero-coupon bond to the short-rate. Most of these affine models have either a Gaussian or a log-normal distribution. These often have a closed form-solution. Models such as the Cox et al. [1], however, do not fall into this category as they follow the non-central chi-squared distribution.

Despite allowing negative interest rates with positive probability, affine (Gaussian) models remain popular amongst market participants because they are easily implemented and can also provide explicit numerical solutions when used to price derivatives. These models, however, are one-factor models and the user of the models cannot freely choose the volatility structure, which is a major drawback. Multi-factor short-rate models have properties that make them more desirable over single-factor models with the one stand-out property being able to describe the intrinsic volatility structure and the Heath-Jarrow-Morton(HJM) [2] framework being the best multi-factor model on the spectrum. Heath et al. [2] proposed a framework where instead of modelling the short-rate $r(\top)$, they attempted to model the dynamics of the instantaneous forward-rate curve $f(\top, T)$. The main takeaway from this framework is the fact that the drift of $f(\top, T)$ is driven by the volatility. Short-rate models offer the flexibility to select the drift of the short-rate. It is also worth noting that this framework offers more flexibility than the short-rate models in that it perfectly matches the initial term-structure of interest rates. This HJM is very important in the literature because all the arbitrage-free models can also be derived from this framework. However, it proves to be difficult to implement this in practice. Another disadvantage of this model is that the quantity that is being modelled, $f(\top, T)$, is not market-observable, meaning that calibrating the model to the prices of products such as the caplets will be difficult. Market models were then introduced to tackle the issue of the HJM.

Brace et al. [3] proposed an alternative model to deal with the drawbacks of the HJM. We call this model the LIBOR Market Model (LMM) and the driving variable of the model is the forward rates, known as the IBOR rates. This is the cost of borrowing in the interbank market. This model can also be easily calibrated to products that are being actively traded on the markets such as the interest rate cap.

During the 2007-09 financial crisis, there were a number of scandals involving illicit manipulation of the IBOR interest rates. These IBOR rates are set to be discontinued due to the risk they have and they will be replaced by overnight rates. This transition provides major problems because these IBOR rates are used as underlying in many derivatives and another question with this transition is what to do with the legacy LIBOR contracts, some of which have maturities of up to 30 years. The key difference between the IBOR rates and the overnight rates is that the former is reported for different tenors, spanning term lending durations of up to 12 months and the latter cannot go beyond the 24-hour term. The overnight rates are also set-in arrears, meaning that the LMM cannot be used to model these new rates. In 2019, LyashenkO and Mercurio [4] proposed an extension of the LMM to also model these set-in arrears rates. They called this new model, the generalized Forward Market model (FMM).

1.3 Delimitation

A multitude of affine and market models can be found in the literature representing different characteristics and assumptions. We discuss the different classes of interest rate models such as classical short-rates and market models. Interest rate models that have distributions that are normal and lognormal account for several models. Hence, when pricing interest rate derivatives in the empirical study, later on, we restrict ourselves to the selection of the LMM and the generalized forward market models. When pricing the simple exotic options, we will not be addressing the analytical pricing formulas.

Moreover, this dissertation concentrates only on the comparative study of the valuation of the interest models. Therefore, we will only be focusing on implementing the models, and the calibration theory of the aforementioned models is not considered. We also assume a 'smile-less' world. Calibration theory for market models is available in a number of various textbooks that mainly deal with interest rate derivatives valuation. For pricing purposes, the year fractions and volatilities are assumed to be constant. The interest-rate products that are being priced here follow a European style. American-style products are not discussed in the dissertation.

In the dissertation, we first give an introduction to the theory of interest modelling. We then introduce classical short-rate models and then proceed to market models and close off by giving a comparative study of the results. Below is a more detailed breakdown of what the reader can expect from the chapters in the dissertation.

- Chapter 2 *Principles of interest rates.* This section introduces the fundamental theory of interest rates and forward rates, which will be crucial for the subsequent chapters.

- Chapter 3 *Traditional derivatives pricing.* Introduces the theory of term structure modelling and the theory of pricing interest rate caps/floors and swaptions.

- Chapter 4 *Classical short-rate models.* Introduces short-rate models. We look at the equilibrium models, and arbitrage-free models and conclude by discussing the HJM framework.

- Chapter 5 *LIBOR Market Model.* We discuss the theory of the LMM and how it is applied to price interest rate derivatives such as caps and swaptions.

- Chapter 6 *The generalized Forward market model.* We discuss the theory of the generalized forward market model and its characteristics and how it is applied to price interest rate derivatives such as caps and swaptions.

- Chapter 7 *Results.* Discuss the results obtained from the study.

- Chapter 8 *Conclusion.* Concludes the dissertation and also suggests alternative research areas.

CHAPTER 2

Principles of interest rates

This chapter introduces the theory of short-rates, key definitions, forward rates, and forward rate agreements.

2.1 Fundamental concepts

When a deposit is made into a bank account, the expectation is that the money should grow over time. It is common knowledge and wisdom that lending money requires some sort of compensation, making obtaining a particular amount tomorrow not equal to receiving the exact same amount today. For us to express this mathematically, we have to introduce some definitions and concepts. The first definition we look at is that of a cash account.

Definition 2.1. *The cash account is the value of a unitary investment at* $\top \geq 0$ *made* *at* $\top = 0$. *We can denote it by* $C(\top)$ *and has dynamics given by*

$$dC(\top) = r_\top C(\top) d\top, \quad C(0) = 1$$

and has a solution given by

$$C(\top) = \exp\left(\int_0^\top r_u du\right).$$

It is worth noting that the cash account $C(\top)$ has the property of not being affected and retains its properties even when transitioning to a risk-adjusted probability measure and it is the only asset in finance with this property.

Definition 2.2. *A zero-coupon bond for* $\top \in [0, T]$*, denoted by* $P_\top(T)$*, is a contract that guarantees the holder will receive a future payout of* π *on the maturity date,* T*. This can be expressed as*

$$P_\top(T) = \mathbb{E}^{\mathbb{Q}} e^{-\int_\top^T r(u)du}.$$

Zero-coupon bonds play a key role in interest rate modelling. Every interest rate can be expressed using these zero-coupon bonds as a basis. On the subject of interest rates, they can be categorized into government rates, which are related to government bonds, and interbank rates, which are related to the rates that banks charge each other for swap transactions between them. Although we will be mainly interested in the interbank sector, it is worth noting that the mathematical modelling can still be applied to government rates.

Definition 2.3. *The Simply-compounded spot interest rate for* $\top \in [0, T]$*, denoted by* $L(\top, T)$ *is the annualized rate of return from retaining the zero-coupon bond from* $\top$ *until maturity* T*. It is expressed mathematically as*

$$L(\top, T) = \frac{1 - P_\top(T)}{\delta(\top, T)P_\top(T)}.$$

The LIBOR rate is the most important simply-compounded spot rate as it is the borrowing cost in the interbank market and hence we choose $L(\top, T)$ to denote such a rate.

In the market, the preferred notation is the simple rate one. Nevertheless, from a mathematical perspective, working with continuously compounded rates proves to be more straightforward.

Definition 2.4. *The continuously compounded spot rate $R(\top, T)$ is the annualized logarithmic return of investment from holding a zero-coupon bond starting at time $\top$ until maturity T. It is expressed mathematically as*

$$R(\top, T) = -\frac{\ln P_\top(T)}{\delta(\top, T)}.$$

2.2 Forward rates

This section introduces forward rates that are fundamental in interest rate modelling, more especially for market models as we shall see in the coming chapters. We begin with the following definition.

A *forward rate agreement (FRA)* is a contract that is entered into at time $\top$, where the party who is issuing it agrees to pay the holder at time T_j the simply compounded rate $L(T_{j-1}, T_j)$ in exchange for a constant rate κ where the notional is N. The time T_j payoff is given by

$$\delta_j(\kappa - L)N. \tag{2.1}$$

We can assume that the $N = 1$. By definition, it is free to enter into a forward rate agreement. By discounting the payoff to time $\top$ of Equation (2.1) and equating the resulting summation to zero, we can determine the fixed rate κ that makes the contract fair.

Definition 2.5. *A simply-compounded forward LIBOR rate is denoted by $L(\top, T_{j-1}, T_j)$ and is given by:*

$$L(\top, T_{j-1}, T_j) = \frac{1}{\delta_j}\left(\frac{P_\top(T_{j-1})}{P_\top(T_j)} - 1\right).$$

We also consider another compounding method known as annual compounding.

Definition 2.6. *Let $\top \le T \le U$. The continuously-compounded forward interest rate is denoted by $R_\top(\top, U)$ and is given by*

$$R_\top(\top, U) := -\frac{\ln P_\top(U) - \ln P_\top(\top)}{\delta_U}.$$

Where $\delta_U = U - \top$. With some algebra we get

$$e^{R\top(\top,U)} = \frac{P_\top(\top)}{P_\top(U)}.$$

Definition 2.7. *The instantaneous forward rate at time $\top$ with maturity T is given by:*

$$f(\top, T) = -\frac{\partial}{\partial T}\ln P_\top(\top).$$

Definition 2.8. *The instantaneous short interest rate $r(\top)$ is the limit of each of the different spot rates for $\top \in [0, T]$ with $T \to \top^+$. Mathematically, it can be defined by:*

$$r(\top) = f(\top, \top).$$

The instantaneous short-rate and the short-rate are mathematical idealizations but their importance in the literature is not to be understated as we shall see in Chapter 3.

In this chapter, we introduced fundamental concepts pertaining to interest rate modelling such as the cash account, instantaneous short-rates, and forward rates, which will be important in the chapters to follow. A thorough understanding of these concepts will help the reader understand the chapters to follow much better. In the next chapter, we look at how the short-rates can be used to price interest rate derivatives.

CHAPTER 3

Traditional derivatives pricing

The arbitrage-free theory plays a major role when it comes to pricing any financial derivative and it inspired the derivation of the famous Black76 [5] formula for pricing options. In the arbitrage-free framework, we can define the continuous-time dynamics of an interest-rate derivative as a random process and as a Stochastic Differential Equation (SDE). In this chapter, we introduce some of the important definitions and theorems in arbitrage-free theory which are biased towards the ones we will use in the coming chapters.

3.1 Traditional pricing theory

We say that a variable follows a stochastic process if, over time, its dynamics involve some randomness or uncertainty about it. We introduce two major continuous stochastic processes: the Wiener process and geometric Brownian motion.

Definition 3.1. *A stochastic process in discrete time is a sequence* $(S_i)_{i=0}^{\infty}$ *of random* *variables.*

It can also be continuous-time if its index set is some interval of the real line.

Definition 3.2. *A process W is called a Wiener process if it is continuous and stochastic process W and it satisfies the following:*

(i) $W(0) = 0$.

(ii) W has stationary independent increments.

(iii) For each time $\top > 0$, the process $W(\top)$ has a normal distribution.

(iv) The function $\top \to W(\top)$ is continuous in $\top$, with probability 1.

Definition 3.3. *A geometric Brownian motion $B^m(\top)$ is the solution of the SDE*

$$dB^m(\top) = \mu B^m(\top)d\top + \sigma B^m(\top)dW(\top) \tag{3.1}$$

where μ and σ are the drift and the volatility respectively.

Equation (3.3) has the closed-form solution given by

$$B^m(\top) = B^m(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\top + \sigma W(\top)\right), \tag{3.2}$$

which is referred to as the predictor-correct method. Alternatively, we can rewrite the discrete version of (3.1) as follows

$$B^m(\top_{i+1}) = B^m(\top_i) + \mu B^m_{\top_i}\delta\top_i + \sigma B^m(\top_i)\delta W(\top_i). \tag{3.3}$$

Definition 3.4. *An Ito process (stochastic integral) $I(\top)$ is a stochastic process that has the following form*

$$dI(\top) = \mu(\top)I(\top)d\top + \sigma(\top)I(\top)dW(\top). \tag{3.4}$$

Theorem 3.1. *Let $I(\top)$ be an Ito process that satisfies Equation* (3.1) *and let $f(x, \top)$ be a stochastic process on the L^2-space; then $f(I(\top), \top)$ is an Ito process, and*

$$d(f(I(\top), \top)) = \frac{\partial f}{\partial \top}(I(\top), \top)d\top + \frac{\partial f}{\partial I(\top)}dI(\top) + \frac{1}{2}\frac{\partial^2 f}{\partial^2 I(\top)}dI^2(\top), \tag{3.5}$$

where $dI(\top)^2$ is defined by

$$d\top^2 = 0 \tag{3.6}$$

$$d\top dW(\top) = 0 \tag{3.7}$$

$$dW(\top)^2 = d\top \tag{3.8}$$

Definition 3.5. *Let $M(\top)$ be a stochastic process. $M(\top)$ is a martingale under a filtration $\mathcal{F}_\top$ iff*

(i) $\mathbb{E}^{\mathbb{Q}_0}\{|M(\top)|\}$ *is bounded.*

(ii) $\mathbb{E}^{\mathbb{Q}_0}\{M(\top)|\mathcal{F}_u\} = M(u), \ \forall \ u \leq \top.$

Condition *ii* plays a crucial role as it implies that the expected future price M and the price available in the market today are equal. We need this condition to hold when pricing derivatives, however, it is not always observed. We, therefore, have to utilize the change of numeraire technique to move from a probability measure where the dynamics include a non-zero drift to an EMM $\mathbb{Q}$ that is driftless to apply this martingale theory.

Definition 3.6. *Suppose we have two equivalent measures, $\mathbb{Q}_0$ and $\mathbb{Q}$. It is possible to express $\mathbb{Q}_0$ through $\mathbb{Q}$ with the random variable $d\mathbb{Q}_0/d\mathbb{Q}$, which is called the Radon-Nikodym derivative. The expectation of a variable S with respect to $\mathbb{Q}_0$ is given by*

$$\mathbb{E}^{\mathbb{Q}_0}\{S\} = \mathbb{E}^{\mathbb{Q}}\left\{\frac{d\mathbb{Q}_0}{d\mathbb{Q}}S\right\}.$$

We can also define a process on the filtration $\mathcal{F}_\top$ through the Radon-Nikodym derivative as follows

$$\gamma(\top) = \mathbb{E}^{\mathbb{Q}}\left\{\frac{d\mathbb{Q}_0}{d\mathbb{Q}}|\mathcal{F}_\top\right\}.$$

This is another version of the Radon-Nikodym derivative because it also transfers the Radon-Nikodym derivative for random variables to Ito processes. It also satisfies the following property:

$$\mathbb{E}^{\mathbb{Q}}\left\{S\big|\mathcal{F}_u\right\} = \gamma(\top)^{-1}\mathbb{E}^{\mathbb{Q}_0}\left\{\gamma(\top)S\big|\mathcal{F}_u\right\}, \quad u \leq \top \leq T.$$

When using the Radon-Nikodym derivative to move from one measure to another, only the drift parameter is affected, the volatility remains the same and also, this new process is generally not a martingale under the new measure. Girsanov's theorem formalizes the mapping of an SDE from one measure to another to obtain a new martingale process under the EMM. This concept is crucial for the LMM to obtain all the forward rate dynamics martingale under the terminal measure. We now introduce the *Girsanov's theorem.*

Theorem 3.2. *The Cameron-Martin-Girsanov Theorem, Girsanov Theorem states that if* $\zeta = \{\zeta(\top) : \top \in [0, T]\}$ *is a* $\mathcal{F}_\top$ *predictable process such that* $\mathbb{E}^{\mathbb{Q}_0}\left\{e^{\frac{1}{2}\int_0^\top \zeta(\top)^2 d\top}\right\}$ *is bounded. Then there exists a measure* $\mathbb{Q}$ *such that*

(i) $\mathbb{Q}$ *is equivalent to* $\mathbb{Q}_0$.

(ii) $d\mathbb{Q}_0/d\mathbb{Q} = e^{-\int_0^\top \zeta(\top)dW(\top) - \frac{1}{2}\int_0^\top \zeta(\top)^2 d\top}$.

(iii) $\tilde{W}(\top) = W(\top) = \int_0^\top \zeta(u)du$ *is a* $\mathbb{Q}$*-Brownian motion.*

Theorem 3.3. *Let* V *be the value of a derivative,* S *be an underlying variable at* $\top$ *and* $dC = rCd\top$ *be the evolution of the underlying cash account. If* S *follows Equation (3.3)* *with the constant drift r then the evolution of V is*

$$\frac{\partial V}{\partial \top} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S(\top)} = rV. \tag{3.9}$$

Proof. By equation (3.5),

$$\begin{aligned} dV &= \frac{\partial V}{\partial \top}d\top + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}dS^2 \\ &= \left(\frac{\partial V}{\partial \top}d\top + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S}dS^2\right)d\top + \sigma S\frac{\partial V}{\partial S}dW \end{aligned}$$

since

$$dS = \mu S d\top + \sigma S dW(\top) \tag{3.10}$$

and

$$\begin{aligned} dS^2 &= \mu^2 S^2 d\top^2 + \mu\sigma S^2 d\top dW + \sigma^2 S^2 dW^2 \\ &= \sigma^2 S^2 d\top \end{aligned}$$

by Equations (3.6), (3.7), and (3.8). Now consider a portfolio that consists of the derivative V, say a call option, and β stocks. Then the portfolio has a cost given by $V + \beta S$. Using the same reasoning as previously mentioned, we observe that

$$d(V+\beta S) = \left(\frac{\partial V}{\partial \top} + \mu S\frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S}dS^2\beta\mu S\right)d\top + \sigma S\frac{\partial V}{\partial S}dW + \sigma S\left(\frac{\partial V}{\partial S} + \beta\right)dW(\top). \tag{3.11}$$

Now we let $\beta = -\frac{\partial V}{\partial S}$ to eliminate all risk in the portfolio.

$$d(V+\beta S) = \left(\frac{\partial V}{\partial \top} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S}dS^2\beta\mu S\right)d\top. \tag{3.12}$$

It is clear to see that the portfolio is free of risk or randomness. This result proves to be critical because since it is risk-free, it must grow at a risk-free rate r over time. So, $\frac{d}{d\top}(V + \beta S) = r(V + \beta S) = r(V - S\frac{\partial V}{\partial S})$ and

$$r\left(V - S\frac{\partial V}{\partial S}\right) = \left(\frac{\partial V}{\partial \top} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S}\right).$$

Moving the variables around results in the Black-Scholes equation:

$$\frac{\partial V}{\partial d\top} + rS\frac{\partial V}{\partial dS} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S(\top)} = rV$$

$\square$

In the risk-neutral approach to valuation of derivatives, the value $\pi(\top) = \pi(\top, T, S(\top))$ of an asset $S(\top)$ at time-$\top$ is given by choosing a numeraire $N(\top)$ and taking an expectation with respect to an equivalent martingale measure $\mathbb{Q}^N$ under which the discounted value of the derivative is a martingale. Hence $\pi(\top)$ is defined from the expression

$$\frac{\pi(\top)}{N(\top)} = \mathbb{E}^{\mathbb{Q}^N}\left\{\frac{\pi(\top)}{N(\top)}\middle|\mathcal{F}_\top\right\}. \tag{3.13}$$

If we choose the cash account $C(\top)$ as the numeraire, then (3.13) can be written as

$$\begin{aligned}\pi(\top) &= \mathbb{E}^{\mathbb{Q}^C}\left\{\frac{C(\top)}{N(\top)}\pi(\top)\middle|\mathcal{F}_\top\right\} \\ &= \mathbb{E}^{\mathbb{Q}^C}\left\{e^{-\int_\top^T r_u du}\pi(\top)\middle|\mathcal{F}_\top\right\}\end{aligned} \tag{3.14}$$

where $\mathbb{Q}^C$ is the equivalent martingale measure that makes $\frac{\pi(\top)}{N(\top)}$ a martingale. When the interest rate is deterministic, the exponent can be factored out of the expectation so that Equation (3.14) becomes

$$\pi(\top) = e^{-\int_{\top}^{T} r_u du} \mathbb{E}^{\mathbb{Q}^C}\left\{\pi(\top)\middle|\mathcal{F}_{\top}\right\}. \tag{3.15}$$

Lastly, when interest rates are constant, i.e. $r(u) = r$ just like in the Black and Scholes model, Equation (3.15) further simplifies to

$$\pi(\top) = e^{-r(\top-\top)} \mathbb{E}^{\mathbb{Q}^C}\left\{\pi(\top)\middle|\mathcal{F}_{\top}\right\}. \tag{3.16}$$

If we set $S(\top) = r(\top)$, the payoff of the derivative can be represented as $\pi(\top) = \pi(\top, T, r(\top))$. Furthermore, for term structure modelling, interest rates cannot be assumed to be deterministic as this is too restrictive. Hence the payoff of an interest rate derivative can only be given by Equation (3.13)

$$\pi(\top) = \mathbb{E}^{\mathbb{Q}^C}\left\{e^{-\int_{\top}^{T} r_u du} \pi(\top, T, r(\top))\middle|\mathcal{F}_{\top}\right\} \tag{3.17}$$

when the cash account $C(\top)$ is taken as the numeraire. Unfortunately, Equation (3.17) cannot be simplified to Equation (3.15) or (3.16). It is hard to evaluate the expectation in Equation (3.17) due to the fact that it involves two terms that each depend on the underlying variable.

The realization that the cash account, $C(\top)$ is not the only viable choice as the numeraire was an essential step forward in term-structure modelling.

The zero-coupon bond $P_{\top}(\top)$ can also be considered as a numeraire. In fact, by taking $P_{\top}(\top)$ as the numeraire, Equation (3.17) can now be evaluated. The equivalent martingale measure associated with using $P_{\top}(\top)$ as the numeraire is the *T-forward measure*. We use the fact that $P_{\top}(\top) = 1$. In Equation (3.13), let $N(\top) = P_{\top}(\top)$. Then

$$\frac{\pi(\top)}{P_{\top}(\top)} = \mathbb{E}^{\mathbb{Q}^{\top}}\left\{\frac{\pi(\top)}{P_{\top}(\top)}\middle|\mathcal{F}_{\top}\right\} \tag{3.18}$$

then

$$\begin{aligned}\pi(\top) &= P_{\top}(\top)\mathbb{E}^{\mathbb{Q}^{\top}}\{\pi(\top)\big|\mathcal{F}_{\top}\} \\ &= \mathbb{E}^{\mathbb{Q}^{C}}\{e^{-\int_{\top}^{T} r_u du}\big|\mathcal{F}_{\top}\}\mathbb{E}^{\mathbb{Q}^{\top}}\{\pi(\top)\big|\mathcal{F}_{\top}\}.\end{aligned} \tag{3.19}$$

Equation (3.19) is the key result. We see that by using $P_{\top}(\top)$, we can convert the expectation under $\mathbb{Q}^{C}$ of a product of two terms in Equation (3.17) into the product of two expectations in Equation (3.19), one under $\mathbb{Q}^{C}$ and the other under $\mathbb{Q}^{\top}$, which can be easily evaluated. The key takeaway from this is that $\pi(\top)$ can be expressed as:

$$\pi(\top) = \mathbb{E}^{\mathbb{Q}^{C}}\{e^{-\int_{\top}^{T} r_u du}\pi(\top)\big|\mathcal{F}_{\top}\} \tag{3.20}$$

$$= P_{\top}(\top)\mathbb{E}^{\mathbb{Q}^{\top}}\{\pi(\top)\big|\mathcal{F}_{\top}\}. \tag{3.21}$$

3.2 Pricing interest rate derivatives

In the market, we have a derivative which is composed of FRAs which we call an interest rate swap. This is a bilateral OTC contract to exchange cash flow streams, which in its standard form, a fixed interest rate against a floating rate payment, at a specified future time instant. We refer to these streams as legs of the swap.

Suppose that we have that we have n floating leg payoffs for the tenor structure $\mathbb{T} = T_0, \ldots, T_n$ with the corresponding set of year fractions $\delta = \delta_0, \ldots, \delta_n$. Let $L(\top_{j-1}, T_j)$ be a floating forward rate and n be the number of fixing times of the floating leg. Let κ be the fixed rate and n be the number of fixing times of the fixed rate. Then for every time $T_j \in T_{\eta+1}, \ldots, T_\eta$ one party gives the floating leg payments defined by

$$N\delta_j L(\top_{j-1}, T_j)$$

and receives the fixed leg payments given by

$$N\delta_j\kappa,$$

where N is the notional value. We have two types of interest rate swaps: a *payer*, and a *receiver interest rate swap*. The discounted payoff at a time $\top \leq T_\eta$ of a IRS is given by

$$\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N\delta_j v \left(L(\top_{j-1}, T_j) - \kappa\right),$$

where $v = -1$ is a receiver IRS and $v = 1$ is a payer IRS.

Figure 3.1: A graph showing how swaps work.

In a swap contract, the payer receives the floating interest rate, usually the LIBOR rate, and pays a fixed interest rate. We now denote *payer* and *receiver* interest rate swap as PFS and RFS respectively. The arbitrage-free price at $\top < T_\eta$ of a PFS with $N = 1$ is:

$$\begin{aligned}
PFS(\top, T, \kappa) &= \mathbb{E}^{\mathbb{Q}}\left\{\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N\delta_j (L_\top(\top_{j-1}, T_j) - \kappa) \middle| \mathcal{F}_\top\right\} && (3.22)\\
&= \sum_{j=\eta+1}^{\omega} P_\top(\top_j)\delta_j E^{Q_j}\left\{L_\top(\top_{j-1}, T_j) - \kappa \middle| \mathcal{F}_\top\right\} && (3.23)\\
&= P_\top(\top_\eta) - P_\top(\top_\omega) - \kappa \sum_{j=\eta+1}^{\omega} P_\top(\top_j)\delta_j && (3.24)\\
&= \sum_{j=\eta+1}^{\omega} P_\top(\top_j)\delta_j L_\top(\top_{j-1}, T_j) - \kappa. && (3.25)
\end{aligned}$$

Definition 3.7. *The forward swap rate $S_{\eta,\omega}(\top)$ at time $\top$ is defined by*

$$S_{\eta,\omega}(\top) = \frac{P_\top(\top_\eta) - P_\top(\top_\omega)}{A_{0,\omega}}.$$

where $A_{0,\omega}$ is the annuity and is defined by:

$$A_{O,\omega} := \sum_{j=\eta+1}^{\omega} \delta_j P_\top(\top_j)$$

Definition 3.8. *An interest rate caplet is a call option on a FRA. A cap is a basket of these caplets. The general payoff formula for a cap is*

$$\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N \delta_j \left(L(\top_{j-1}, T_j) - \kappa \right)_+ \tag{3.26}$$

We can define a floor in an analogous way as follows.

Definition 3.9. *A floorlet is a put option on a FRA. A floor is a basket of these floorlets. The general payoff formula for a floor is*

$$\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N \delta_j \left(\kappa - L(\top_{j-1}, T_j) \right)_+ .$$

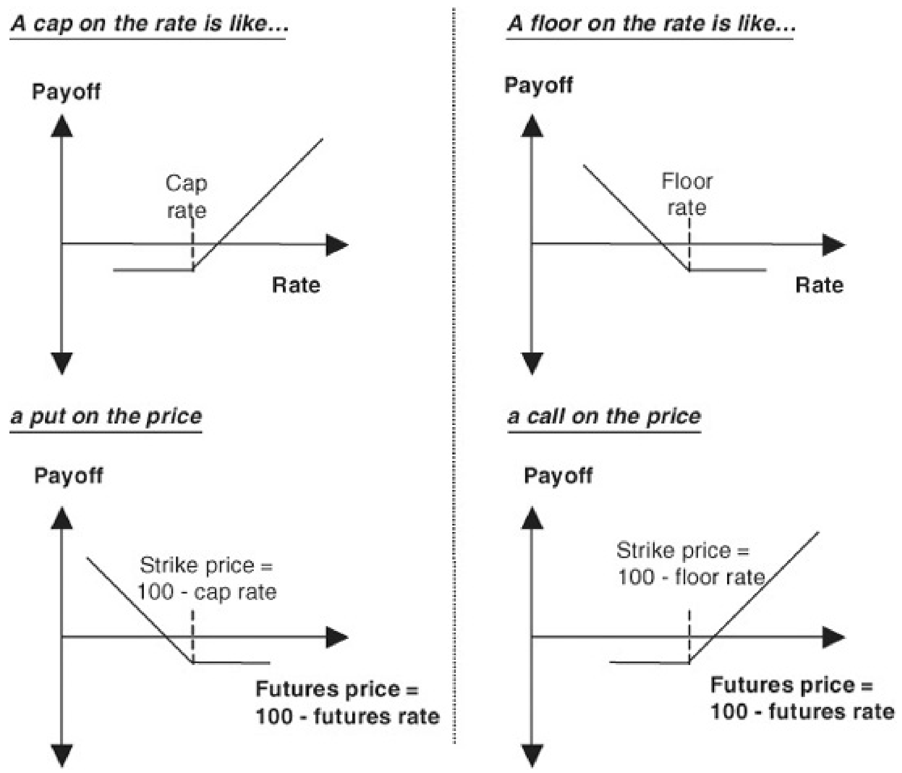


Figure 3.2: Floors. Source:https://www.oreilly.com.

In Figure (3.2), we see a graph of a cap and a floor payoff where both the cap and floor rate strike is $\kappa = 100$. In the case of a cap, the payoff remains constant below zero, which coincides with the premium paid to purchase the contract until the floating rate exceeds the cap rate. The floor case can be explained analogously so.

Black's formula [5] is the most preferable tool by market participants when it comes to the pricing of caps and floors. This formula was proposed in 1976 to price commodity options and it is often referred to as the Black76 model. The time-$\top$ payoff of the cap with an associated tenor structure T and a nominal N is given by

$$\mathbf{Cap}^{Black}(\top, T, \delta, N, \kappa, \sigma) = N \sum_{j=\eta+1}^{\omega} P_{\top}(\top_j)\delta_j \text{Black}\left(\kappa, L_{\top}(\top_{j-1}, T_j), \sigma_j\right), \qquad (3.27)$$

where, ϕ is the CDF function,

$$
\begin{aligned}
\text{Black}(\kappa, L(\top; T_{j-1}, T_j), \sigma_j) &= L(\top; T_{j-1}, T_j)\phi(d_1(\kappa, L(\top; T_{j-1}, T_j), \sigma)) - \kappa\phi(d_1(\kappa, F(\top; T_{j-1}, T_j), \sigma)),\\
d_1(\kappa, L, \sigma) &= \frac{\log L/\kappa + \sigma^2/2}{\sigma},\\
d_2(\kappa, L, \sigma) &= \frac{\log L/\kappa - \sigma^2/2}{\sigma},\\
\sigma_j &= \sigma\sqrt{T_{j-1}},
\end{aligned}
$$

where σ is the volatility that can be implied from the markets. The Black's formula for floorlets can be defined in an analogous way as

$$\mathbf{Flr}^{Black}(\top, T, \delta, N, \kappa, \sigma_j) = N \sum_{j=\eta+1}^{\omega} P_0(\top_j)\delta_j \text{Black}(\kappa, L(0, T_{j-1}, T_j)\sigma_j),$$

We also have nonstandard caps/floors that are available in the market such as Ratchet Caps and Flexi caps. These are often referred to as simple exotic options because their payoff is European style. These caps differ from the standard cap by incorporating conditions to determine how the cap rate κ is set for each caplet. We now define the payoffs for two nonstandard caps known as the ratchet cap and the sticky cap.

Definition 3.10. *A ratchet cap is a contract consisting of a series of caplets paying at* $\{T_{\alpha+1}, \ldots, T_\beta\}$ *where the strike is given by* $R(T_{i-2}, T_{i-1}) + \epsilon$. *In this case,* $\epsilon \in \mathbb{R}$ *and it is a specified spread. The discounted payoff is given by*

$$\sum_{j=\alpha+1}^{\beta} ND_f(\top, T_j)\delta_j(R(T_{j-1}, T_j) - (R(T_{j-2}, T_{j-1}) + \epsilon))_+.$$

Definition 3.11. *A sticky cap is a contract consisting of a series of caplets paying at* $\{T_{\alpha+1}, \ldots, T_\beta\}$ *where the strike is given by* $(\min(R(T_{j-2}, T_{j-1}), \kappa) + \epsilon)$. *In this case,* $\epsilon \in \mathbb{R}$ *and it is a specified spread. The discounted payoff is given by*

$$\sum_{j=\alpha+1}^{\beta} ND_f(\top, T_j)\delta_j(R(T_{j-1}, T_j) - (\min(R(T_{j-2}, T_{j-1}), \kappa) + \epsilon))_+.$$

We cannot use the Black76 [5] formula to price this cap. One has to resort to approximations for the analytical formula for pricing such caps. In this thesis, we will not be addressing the analytical pricing of the ratchet and sticky caps.

We now introduce swaptions. These are options where the underlying assets are interest rate swaps. Because they are dependent on interest rate swaps, swaptions also have a payer and a receiver version.

Definition 3.12. *An RFR swaption, either payer or receiver, can be defined as the option to enter a spot RFR swap on the swaption's maturity date. We denote by T_ϕ the swaption's maturity, by κ the strike price, $T_j,\ j = \phi + 1, \ldots, \eta$ the floating-leg dates, and by $T'_{c+1}, \ldots, T'_d$ the fixed-leg ones. We assume $T'_c = T_\phi$ and $T'_d = T_\eta$. The (payer) swaption payoff at time T_ϕ is thus given by:*

$$[S(T_\phi) - \kappa]_+ \sum_{j=c+1}^{d} \tau'_j P_\top(T'_j)$$

$$S_{\eta,\omega}(\top) = \frac{P_\top(\top_\eta) - P_\top(\top_\omega)}{A_{0,\omega}}.$$

where $A_{0,\omega}$ is the annuity and is defined by:

$$A_{O,\omega} := \sum_{j=\eta+1}^{\omega} \delta_j P_\top(\top_j)$$

This chapter introduces the interest rate derivatives that will be the focus of pricing within this dissertation, namely interest rate swaps, caps/floors, and swaptions. We also discussed the arbitrage-free theory for pricing those derivatives. As the name suggests, the underlying product in these interest rate derivatives is the interest short-rate introduced in Chapter 2, so the modelling of these short-rates is imperative in finance. In the next part of the dissertation, we look at some of the affine term structure models that have been introduced in the literature and conclude with the HJM framework.

CHAPTER 4

Classical short-rate models

In finance, fund managers and other market participants are heavily reliant on the movements of interest rates over some period to make a decision. The first class of stochastic interest rate models described the dynamics of the instantaneous short-rate. The short-rate models generally have a simple form and they often result in an analytical solution for the bonds and the vanilla derivatives. Since these short-rate models are analytically tractable, it implies that the payoff of a given derivative can be computed in a short time. In general, the short-rate models have the form given by Equation (1.1). Short-rate models can be categorized into different types, including equilibrium models, arbitrage-free models, yield curve models such as the Heath-Jarrow-Morton model (HJM), and market models. In this chapter, we discuss some of the famous equilibrium and arbitrage-free models and close off with the HJM model [2]. Market models will be discussed in the coming chapters.

4.1 Equilibrium models

The Vasicek model

Vasicek [6] presented the first one-factor model in the literature. This model is given by the SDE of the form:

$$dr(\top) = \alpha[\rho - r(\top)]d\top + \sigma dW(\top), \tag{4.1}$$

where α, ρ, and σ are constants that are strictly greater than zero. This model has mean reversion. Another special feature of Vasicek's model is that Equation (4.1) has an analytic solution that can be found utilizing the method of variations of constants.

The homogeneous equation

$$dr(\top) = -\lambda r(\top)d\top$$

has the solution given by

$$r(\top) = Ke^{-\lambda\top} \tag{4.2}$$

where K is an arbitrary constant. We require a particular solution to the inhomogeneous equation in the form (4.2) where we replace the constant K by an unknown function $\phi(\top)$,

$$r_1(\top) = \phi(\top)e^{-\lambda\top}.$$

$\phi(\top)$ has to satisfy the ordinary differential equation:

$$d\phi(\top) = \lambda\mu e^{\lambda\top}d\top + \sigma e^{\lambda\top}dW(\top).$$

Consequently

$$\phi(\top) = \mu e^{\lambda\top} + \sigma \int_0^\top e^{\lambda v} dW(v).$$

The solution to our problem is the sum of the solution (4.2) with $K = r_0 - \mu$ and the particular solution $r_1(\top)$:

$$r(\top) = r(v)e^{-\lambda(\top - v)} + \mu(1 - e^{-\lambda(\top)(\top - v)}) + \sigma \int_0^\top e^{-\lambda(\top)} dW(v). \tag{4.3}$$

It is important to note that the Vasicek model permits negative interest rates, which may not always align with real-world expectations. It is also worth noting that these negative interest rates can only be exhibited under extreme conditions.

The Dothan model

Dothan [7] presented a lognormal short-rate model in an attempt to address the shortcomings of the Vasicek model. His model is given by

$$dr(\top) = ar(\top)d\top + \sigma r(\top)dW(\top) \tag{4.4}$$

where $a,\ \sigma \geq 0$ are constants.

The dynamics (4.4) are integrated as follows

$$r(\top) = r(v)e^{(a - \frac{1}{2}\sigma^2)d\top + \sigma dW(\top)}$$

for $v \leq \top$ so that

$$\mathbb{E}^{\mathbb{Q}}\{r(\top)|\mathcal{F}_v\} = r^2(v)e^{2ad\top} \tag{4.5}$$

$$\mathrm{Var}\{r(\top)|\mathcal{F}_v\} = r^2(v)e^{2ad\top}(e^{\sigma^2 d\top} - 1). \tag{4.6}$$

In this model, $r(\top) > 0$ for all time $\top$, while the volatility term $\sigma r(\top)$ is proportional to $r(\top)$. Although this is positive, we observe that the a has to be less than zero in order for the model to have mean-reversion and the mean-reversion level is necessarily zero. The exponential Vasicek model was introduced in an attempt to address this mean-reversion restriction. In this model, $P_\top(\top)$ can be calculated explicitly.

The main drawback of this model is that just like all lognormal models for the interest rate short-rate, $\mathbb{E}^{\mathbb{Q}}\{C(\top)\} = \infty$, in other words, the cash account explodes. This drawback can, however, be overcome by resorting to approximating trees.

The Cox-Ingersoll-Ross model

Cox, Ingersoll, and Ross [1] presented another model as an alternative to the Vasicek model [6]. This model followed the SDE:

$$dr(\top) = \alpha(\rho - r(\top))d\top + \sigma\sqrt{r(\top)}dW(\top) \tag{4.7}$$

where α, ρ and σ are non-negative constants. By imposing the condition $2\alpha\rho > \sigma^2$, we ensure that a zero-valued short-rate is avoided. The CIR model is mean-reverting, but it is not a Gaussian model, meaning that it is rather challenging to analyze.

The mean and variance of $r(\top)$ conditional on $\mathcal{F}_v$ are given by

$$\mathbb{E}^{\mathbb{Q}}\{r(\top)|\mathcal{F}_v\} = r(v)\xi_v^\top + \rho(1 - \xi_v^\top) \tag{4.8}$$

$$\mathrm{Var}\{r(\top)|\mathcal{F}_v\} = \frac{r(v)\sigma^2}{\alpha}(\xi_v^\top - e^{-2\alpha(\top - v)}) + \frac{\rho\sigma^2}{2\rho}(1 - \xi_v^\top)^2 \tag{4.9}$$

where $\xi_v^\top = e^{-\alpha(\top - v)}$. Compared to the Vasicek model, $r(\top)$ cannot be negative if the condition $2\alpha\rho > \sigma^2$ holds.

The exponential Vasicek model

Kolkiewicz and Tan [8] proposed the exponential Vasicek model. In this model, the short-rate $r(\top)$ satisfies the following SDE

$$dr(\top) = r(\top)[\rho + \frac{\sigma^2}{2} - \alpha \ln r(\top)]d\top + \sigma r(\top) dW(\top) \tag{4.10}$$

where ρ, α, and σ are non-negative constants. The explicit solution to (4.10) for all $v \leq \top$, is obtained by

$$r(\top) = \exp\left\{\ln r(v)\xi_v^\top + \frac{\rho}{\alpha}(1 - \xi_v^\top) + \int_s^\top \xi_v^\top + \int_v^\top e^{-\alpha(\top - v)dW(u)}\right\}$$

so that $r(\top)$ conditional on $\mathcal{F}_v$ has a distribution that is log-normal with normal distribution given by

$$\mathbb{E}^{\mathbb{Q}}\{r(\top)\} = \exp\left\{\ln r(v)\xi_v^\top + \frac{\rho}{\alpha}\left(1 - \xi_v^\top\right)\frac{\sigma^2}{4\alpha}\left[1 - e^{-2\alpha(\top - v)}\right]\right\}$$
$$\mathbb{E}^{\mathbb{Q}}\{r(\top)\} = \exp\left\{2\ln r(v)\xi_v^\top + 2\frac{\rho}{\alpha}\left(1 - \xi_v^\top\right)\frac{\sigma^2}{\alpha}\left[1 - e^{-2\alpha(\top - v)}\right]\right\}$$

where $\xi_v^\top = e^{-\alpha(\top - v)}$. Therefore, unlike in the Dothan model, mean-reversion is always observed in this model since

$$\lim_{\top \to \infty} \mathbb{E}^{\mathbb{Q}}\{r(\top)|\mathcal{F}_v\} = \exp\left(\frac{\rho}{\alpha} + \frac{\sigma^2}{4\alpha}\right). \tag{4.11}$$

Another advantage of this model is that it is lognormally distributed, which means that the $r(\top) > 0$ at all times.

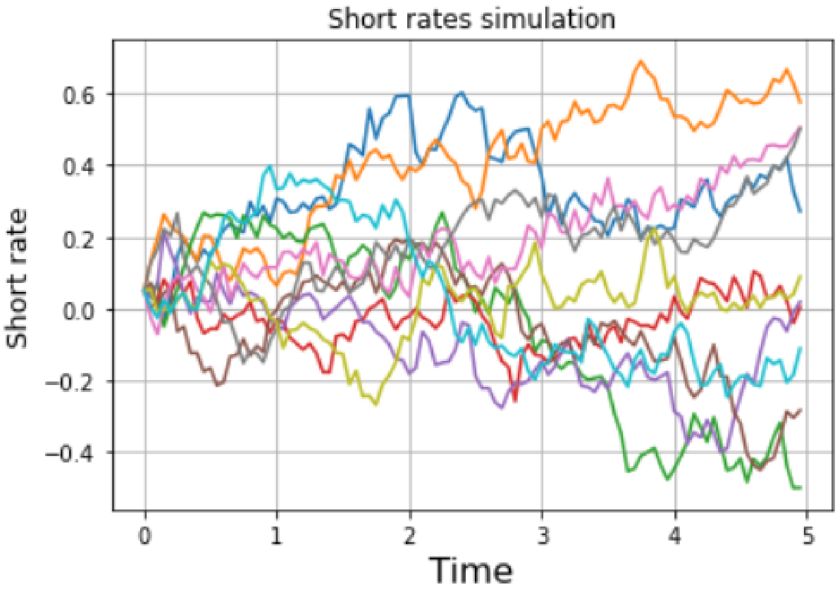


Figure 4.1: The Vasicek model simulation.

Figure (4.1) shows 10 Vasicek simulations where $r(0) = 0.01$, $T = 5$, $\sigma = 0.20$. We can see that five paths always converge to 0.01, which is the mean, as we approach $T = 5$ and the 5 paths suggest that we can expect negative interest rates, which are not usually observed in the market.

Equilibrium models generally assume that the drift and volatilities are constant parameters and this results in the term structure not matching the observable term structure in the market. Arbitrage-free models were developed in an attempt to address these shortcomings.

4.2 Arbitrage-free models

Arbitrage-free models were developed to address the shortcomings of equilibrium models. As already mentioned, the fact that equilibrium models assume constant parameters with the volatility parameter being the most crucial one when deriving the dynamics of the yield curve is not favored by a lot of market participants. In order to ensure that zero-coupon prices are exactly the same as what is observable in the market, we consider non-constant parameters for $\mu(r, \top)$ and $\sigma(r, \top)$. Arbitrage-free models were designed to ensure that the yield curve that is being outputted is consistent with what is observable in the markets. This implies that if we have a yield curve that has a positive slope, the short-rate $r(\top)$ will also have a positive slope, the same applies if the yield curve is decreasing or it is humped. The first arbitrage-free model we look at was proposed by Ho and Lee.

The Ho-Lee model

Ho and Lee [9] presented a model which allows the matching of the initial term structure. It follows the SDE

$$dr(\top) = \theta(\top)d\top + \sigma(r)dW(\top) \tag{4.12}$$

where $\sigma(r)$ is constant. Here, $\theta(\top)$ is a time-dependent function and it is usually chosen by observing market data and calibrating it to the data. This was the first arbitrage-free model in the literature. In choosing the drift term, we have to be very careful or we risk having negative interest rates which is not usually expected in financial applications.

The Hull-White model

Hull and White [10] extended Ho-Lee model [9] as follows:

$$dr(\top) = [\rho(\top) - \alpha r(\top)]d\top + \sigma r(\top)^{\beta}dW(\top) \tag{4.13}$$

where α and σ are constant and ρ is a time-dependent function and it is implied by the market observables. The Hull-White model contains many popular arbitrage-free models as special cases. For example, when $\alpha = 0$ and σ is constant we get the Ho-Lee model. Let $f^M(0,T)$ be the market instantaneous forward rate 0 for the maturity T i.e.

$$f^M(0,T) = -\frac{\partial \ln P_0^M(\top)}{\partial T} \tag{4.14}$$

with $P_0^M(\top)$ the zero-coupon bond in the market for the maturity T, we must have

$$\rho(\top) = f_\top(0,\top) + \alpha f(0,\top) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\top}).$$

We integrate (4.13) and get the following result

$$\begin{aligned} r(\top) &= r(v)\xi_v^\top + \int_v^\top \xi_v^\top \rho(u)du + \sigma \int_v^\top \xi_v^\top dW(u), \\ &= r(v)\xi_v^\top + \beta(\top) - \beta(v)\xi_v^\top + \sigma \int_v^\top \xi_v^\top dW(u) \end{aligned}$$

where

$$\beta(\top) = f^M(0,\top) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha\top})^2.$$

Therefore, $r(\top)$ conditional on $\mathcal{F}_v$ has a normal distribution with mean and variance given respectively by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\{r(\top)|\mathcal{F}_v\} &= r(v)\xi_v^\top + \alpha(\top) - \alpha(v)\xi_v^\top \\ \mathrm{Var}\{r(\top)|\mathcal{F}_v\} &= \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha(\top - v)}\right]. \end{aligned}$$

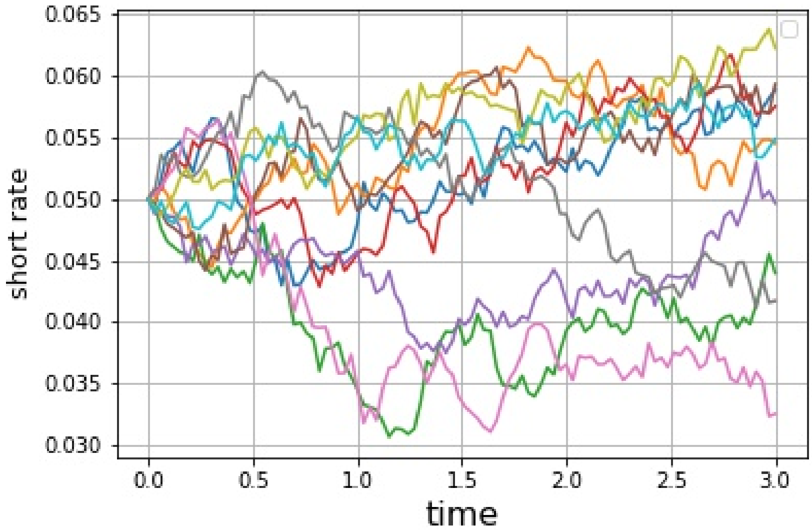


Figure 4.2: The Hull-White model simulations.

Figure (4.2), we computed 10 Hull-White model simulations with $R = 0.05$, $T = 3$, $\sigma = 0.25$, $\rho(\top) = 0.05$. We notice that as we approach $T = 3$, the rate reverts back to $\rho(\top) = 0.05$, except for one path.

Extension of the CIR model

Maghsoodi [11], Jamshidian [12], and Rogers[13] proposed a representation of the extended CIR model. This model follows the SDE

$$dr(\top) = [\rho(\top) - \alpha(\top)r(\top)]d\top + \sigma(\top)\sqrt{r(\top)}dW(\top), \tag{4.15}$$

where α, r, and σ are dependent on time. In this model, $r(\top)$ follows the generalized chi-square distribution and it is known as the squared Bessel process. It proves to be difficult to derive the probability distribution of such a process with time-varying dimensions explicitly. No solution to this problem has been found yet. Just like in the original CIR version, numerical techniques can be used to determine the bond price. In general, the bonds cannot be defined analytically in this framework.

The Black-Karasinski model

Another model that was proposed with the aim of solving the issue of negative interest rates is the Black and Karasinski [14] model. The lognormal dynamics of the short-rate are defined by

$$d\ln(r(\top)) = [\rho(\top) - \alpha(\top)\ln(r(\top))]d\top + \sigma(\top)dW(\top), \tag{4.16}$$

where $\rho(\top)$, $\alpha(\top)$ and $\sigma(\top)$ are time-dependent functions and they are implied from market observables. We can also have a constant parameter version of (4.16) by setting $\alpha(\top) = \alpha > 0$, $\sigma(\top) = \sigma > 0$ which results in

$$d\ln(r(\top)) = [\rho(\top) - \alpha\ln(r(\top))]\top + \sigma dW(\top), \tag{4.17}$$

A major disadvantage is that, in the Black-Karasinski model, a zero-coupon bond $P_\top(\top)$ cannot be calculated explicitly. Therefore, under this model, numerical methods become necessary for pricing interest rate derivatives.

From (4.17), applying Ito's lemma, we derive:

$$d(r(\top)) = r(\top)[\rho(\top) + \frac{\sigma^2}{2} - \alpha(\top)\ln(r(\top))]d\top + r(\top)\sigma(\top)dW(\top)$$

whose explicit solution satisfies, for each $v \leq \top$,

$$r(\top) = \exp\left\{\ln r(v)\xi_v^\top + \int_v^\top \xi_v^\top \rho(u)du + \sigma\int_v^\top \xi_v^\top dW(u)\right\}.$$

Therefore, $r(\top)$ conditional on $\mathcal{F}_v$ has a log-normal distribution with first and second moments given respectively by

$$\mathbb{E}_v^{\mathbb{Q}}\{r(\top)\} = \exp\left\{\ln r(v)\xi_v^\top + \int_v^\top \xi_v^\top \rho(u)du + \frac{\sigma^2}{4\alpha}\left[1 - e^{-2\alpha(\top - v)}\right]\right\}$$
$$\mathbb{E}_v^{\mathbb{Q}}\{r(\top)^2\} = \exp\left\{2\ln r(v)\xi_v^\top + 2\int_v^\top \xi_v^\top \rho(u)du + \frac{\sigma^2}{\alpha}\left[1 - e^{-2\alpha(\top - v)}\right]\right\}.$$

Moreover, setting

$$\theta(\top) = \ln(r_0)e^{-\alpha\top} + \int_0^\top \xi_v^\top \rho(u)du$$

we have that

$$\lim_{\top\to\infty} \mathbb{E}^{\mathbb{Q}}(r(\top)) = \exp\left(\lim_{\top\to\infty} \rho(\top) + \frac{\sigma^2}{4\alpha}\right).$$

The equilibrium and arbitrage-free models discussed above focused on modelling $r(\top)$ in a multifactor framework. After the development of the affine models, practitioners started investigating the possibility of having the forward rate as the driving state variable, which gave birth to the model that will be explored in the coming section.

4.3 The Heath-Jarrow-Morton model

Multifactor short-rate models have properties that make them more desirable over single-factor models with the one stand-out property being that of being able to describe the intrinsic volatility structure and the Heath-Jarrow-Morton framework [2] being the best multifactor model on the spectrum. The HJM model treats the forward rate curve at a point in time as an input and produces the evolution of the curve according to a stochastic process. This process has the general form given by Equation (1.1) but it is more general than arbitrage-free models. Heath, Jarrow, and Morton proposed the first multifactor model in the literature. Their paper described the arbitrage-free conditions that a model of the yield curve must satisfy.

4.3.1 Construction of the Heath-Jarrow-Morton model

The instantaneous rate is given by

$$f(\top, T) = -\frac{\partial \ln P_\top(T)}{\partial T}. \tag{4.18}$$

By differentiating this, we get

$$\implies df(\top, T) = -\frac{\partial}{\partial T} d \ln P_\top(T). \tag{4.19}$$

From Ito's Lemma, we get that

$$\begin{aligned} d \ln P_\top(T) &= \frac{dP_\top(T)}{P_\top(T)} - \frac{\sigma_P(\top, T)' \sigma_P(\top, T)}{2} d\top \qquad (4.20) \\ &= \left(r(\top) - \frac{\sigma_P(\top, T)' \sigma_P(\top, T)}{2} \right) d\top - \sigma_P(\top, T)' dW(\top). \qquad (4.21) \end{aligned}$$

Taking the differential of $d \ln P_\top(T)$ with respect to T results in

$$df(\top, T) = \left(\frac{\partial}{\partial T} \sigma_P(\top, T) \right)' \sigma_P(\top, T) d\top + \left(\frac{\partial}{\partial T} \sigma_P(\top, T) \right)' dW(\top). \tag{4.22}$$

Let

$$\sigma_P(\top, T) = \frac{\partial}{\partial T} \sigma_P(\top, T).$$

We know that at maturity, $\sigma_P(T, T) = 0$, we have that

$$\sigma_P(\top, T) = \int_\top^T \sigma_f(\top, v) dv. \tag{4.23}$$

By substituting back into Equation (4.22), we have that

$$df(\top, T) = \sigma_f(\top, T)' \left(\int_{\top}^{T} \sigma_f(\top, v) dv \right) d\top + \sigma_f(\top, T)' dW(\top) \tag{4.24}$$

The fact that virtually all the arbitrage-free interest models can be derived from the HJM framework makes the theory of the HJM more important in the literature. This model, however, had some drawbacks of its own. The first one is that they have an increase in dimensions compared to previously introduced short-rate models. This increase in dimensions means that the lattice approach to pricing could not be used anymore and one has to resort to Monte Carlo techniques which are time-consuming and are not the preferred option by market participants. This drawback was, however, addressed by the emergence of high dimensional low-discrepancy sequences. These sequences could significantly reduce the time needed for the computation of this model, consequently improving the viability of the HJM approach. The other problem with this framework is that the quantity being modelled is the infinitesimal short-rate which is not observable in the market. This issue can also be circumvented by the HJM models by just interpolating the observed rates The biggest issue with the HJM framework is that it is completely not compatible with the log-normal specifications, implying that $f(\top, T)$ explodes and it is not bounded. This inspired the introduction of the LIBOR market model, also known as the BGM method.

In this chapter, we discussed classical short-rate models such as the affine term structure models (Equilibrium models and Arbitrage-free models). We discussed their drawbacks and how each model inspired the development of the other. We closed off the chapter by discussing the unifying framework known as the HJM framework. As already mentioned, the HJM framework has two major drawbacks. In the next chapter, we will explore a model aimed at resolving the drawbacks of the HJM model.

CHAPTER 5

LIBOR Market Model

So far we only introduced models that had the instantaneous short-rate and the instantaneous forward rate as the state variables. These models have a number of drawbacks that make them unfavorable to practitioners. Firstly, these models only allowed a few parameters to be calibrated and they could not fully reproduce the observed yield curve after calibration. Secondly, the driving state variables in these models, the instantaneous rates, are mathematical idealizations rather than rates that are observed in the markets.

Brace et al. [3] proposed a new model that aimed at solving these problems. In their approach, they modelled forward LIBOR rates that are observed in the market instead of instantaneous spot rates as in the HJM model. Since the LMM is a market model, it is defined by the volatilities imposed on various rates. Furthermore, the LMM prices products such as caps and swaptions that are consistent with the Black76 formulae [5], thus making it more acceptable to market participants. This also makes calibrating the models to be easier.

5.1 Introduction

Consider a term structure $T_0, < T_1 < \ldots, < T_n$ along with an application period $\delta_j = T_i - T_{i-1}$. We defined the LIBOR rate, $L(\top, T_{i-1}, T_i)$, in Chapter 2. This is expressed as

$$L(\top, T_{i-1}, T_j) = \frac{1}{\delta_i}\left(\frac{P_\top(T_{i-1})}{P_\top(T_i)} - 1\right).$$

For ease of notation, we will set $L(\top, T_{i-1}, T_i) = L_i(\top)$. Since we are still modelling interest rates, the dynamics of the LIBOR rate should follow the SDE given by Equation (1.1), i.e. it assumes the dynamics given by

$$dL_i(\top) = \mu_i(L_i(\top), \top)d\top + \sigma_i(\top)L_i(\top)dW_i^i(\top), \quad \top \leq T_{i-1}. \tag{5.1}$$

for $i = 1, \ldots, M$. We showed that the LIBOR rate $L_i(\top)$ is a martingale under the T-forward measure $\mathbb{Q}^{T_i}$. Therefore, under this measure, the dynamics of $L_i(\top)$ should be driftless.

5.2 Construction of the LIBOR Market Model

We now look at how the LMM is constructed. In this LMM, we assume the LIBORs follow the dynamics

$$dL_i(\top) = \sigma_i(\top)L_i(\top)dW_i^i(\top), \quad \top \leq T_{i-1} \tag{5.2}$$

for $i = 1, \ldots, M$. Here σ_i is assumed to be deterministic scalar, and dW_i is the ith component of the $\mathbb{Q}_{T_I}$-Wiener process, hence is a standard Wiener process. This is a *discrete tenor* LIBOR market model.

It is possible to extend this model where the $\sigma_i(\top)$ is determined by an SDE. If σ_i is bounded, then there exists a strong unique solution for the SDE since it describes a geometric Wiener process. From Ito's formula,

$$
\begin{aligned}
d \ln L_i(\top) &= \sigma_i(\top) dW_i^i(\top) - \frac{\sigma_i(\top)^2}{2} d\top \\
\implies \ln L_i(\top) &= \ln L_i(\top) + \int_\top^T \sigma_i(u) dW_i^i(u) - \int_\top^T \frac{\sigma_i(u)^2}{2} du \\
\implies \quad L(\top) &= L_i(\top) e^{\int_\top^T \sigma_i(u) dW_i^i(u) - \frac{1}{2}\int_\top^T \sigma_i(u)^2 du}, \quad 0 \le u \le T \le T_{i-1}.
\end{aligned}
$$

The risk-neutral dynamics are needed to give the valuation of certain products in the market like Eurodollar futures. Let us start by deriving the risk-neutral dynamics of the LMM.

Proposition 5.1. *The LIBOR dynamics where the numeraire is the continuous bank account are given by the following SDE[15]:*

$$
dL_n(\top) = \hat{\mu}_n(\top) d\top + \sigma_n(\top) L_n(\top) d\tilde{W}_n(\top), \tag{5.3}
$$

where

$$
\begin{aligned}
\hat{\mu}_n(\top) &= \sum_{j=\eta(\top)}^{n} \frac{\rho_{n,j}\delta_j \sigma_j(\top)\sigma_n(\top) L_j(\top)}{1+\delta_j L_j(\top)} + \underline{\sigma}_n(\top)\rho \int_\top^{T_{\eta(\top)-1}} \sigma_f(\top, u)' du \\
&= \sum_{j=\eta(\top)}^{n} \frac{\rho_{n,j}\delta_j \sigma_j(\top)\sigma_n(\top) L_j(\top)}{1+\delta_j L_j(\top)} + \sum_{j=\eta(\top)}^{n} \rho_{n,j}\sigma_n(\top) \int_\top^{T_{\eta(\top)-1}} (\sigma_f)_j(\top, u) du.
\end{aligned}
$$

Proof. Let DC be the operator defined in Definition (B.1). We know that the cash account $C(\top)$ has the dynamics given by:

$$dC(\top) = r_\top C(\top) d\top$$

where the volatility is zero. We can derive the risk-neutral dynamics of L_n by using the Radon Nikodym theorem [16] to change the numeraire from $P_\top(\top_n)$ to $C(\top)$. From this, we get the following dynamics:

$$dL_n(\top) = -\underline{\sigma}_{P_\top(\top_n)} \rho \underline{\sigma}_n(\top)' L_n(\top) d\top + \underline{\sigma}_n(\top)' L_n(\top) d\tilde{W}(\top)$$

where $\underline{\sigma}_{P_\top(\top_n)}$ is the bond volatility. Since we can express bond prices as

$$P_\top(\top_n) = P_\top(\top_{\eta(\top)-1}) \prod_{j=\eta(\top)}^{n} \frac{1}{1+\delta_j L_j(\top)}$$

we can compute the above drift as

$$\begin{aligned}
\hat{\mu}_n(\top) :&= -\underline{\sigma}_{P_\top(\top_n)} \rho \underline{\sigma}_n(\top)' \\
&= -\mathrm{DC}\left(\ln P_\top(\top_n)\right) \rho \underline{\sigma}_n(\top)' \\
&= -\mathrm{DC}\left[\ln P_\top(\top_{\eta(\top)-1}) + \sum_{j=\eta(\top)}^{n} \ln\left(\frac{1}{1+\delta_j L_j(\top)}\right)\right] \rho \underline{\sigma}_n(\top)'.
\end{aligned}$$

Now since the following equation holds

$$\ln P_\top(\top) = -\int_\top^T f(\top, u) du$$

we can easily prove that

$$\mathrm{DC}[P_\top(\top)] = -\int_\top^T \underline{\sigma}_f(\top, u) du,$$

where $\underline{\sigma}_f(\top, u)$ is the absolute instantaneous vector volatility of the instantaneous forward rate $f(\top, u)$. We consequently have that

$$dL_n(\top) = \tilde{\mu}_n(\top)d\top + \sigma_n(\top)L_n(\top)d\tilde{W}_n(\top), \tag{5.4}$$

where

$$\begin{aligned}
\hat{\mu}_n(\top) &= \sum_{j=\eta(\top)}^{n} \frac{\rho_{n,j}\delta_j\sigma_j(\top)\sigma_n(\top)L_j(\top)}{1+\delta_j L_j(\top)} + \underline{\sigma}_n(\top)\rho \int_{\top}^{T_{\eta(\top)-1}} \sigma_f(\top, u)' du \\
&= \sum_{j=\eta(\top)}^{n} \frac{\rho_{n,j}\delta_j\sigma_j(\top)\sigma_n(\top)L_j(\top)}{1+\delta_j L_j(\top)} + \sum_{j=\eta(\top)}^{n} \rho_{n,j}\sigma_n(\top) \int_{\top}^{T_{\eta(\top)-1}} (\sigma_f)_j(\top, u) du.
\end{aligned}$$

□

The presentation of the drift in (5.4) does not look good because of the second summation. This summation is a result of the continuous tenor part stemming from the dynamics of $P_{\top}(\top_{\eta(\top)-1})$, which cannot be inferred from forward rates in our family. This equation can only be closed by simulating the dynamics of $\underline{\sigma}_f(\top, u)$ but this is not consistent with the discrete tenor space that we are operating in. The spot measure was introduced in the literature as a discrete-time equivalent of the cash account.

Proposition 5.2. *The LIBOR dynamics where the numeraire is the discrete bank account are given by the following SDE[15]:*

$$dL_n(\top) = \sigma_n(\top)L_n(\top) \sum_{j=\eta(\top)}^{j} \frac{\rho_{n,j}\delta_j\sigma_j(\top)L_j(\top)}{1+\delta_j L_j(\top)} d\top + \sigma_n(\top)L_n(\top)dW_n^i(\top). \tag{5.5}$$

Proof. Given a LIBOR short-rate $L(\top)$, its SDE has the general form given by

$$dL_n(\top) = \mu_n(\top)L_n(\top)d\top + \sigma_n(\top)L_n(\top)dW_n^i(\top). \tag{5.6}$$

When the cash account is taken as the numeraire, we have that

$$C(\top) = P_\top(\top_{\eta(\top)}) \prod_{j=0}^{\eta(\top)-1} [1 + \delta_j L_j(\top)]. \tag{5.7}$$

The deflated bond price, $D_\top(\top_n) = \frac{P_\top(\top_n)}{C(\top)}$ is a martingale with respect to the spot measure, and

$$D_\top(\top_n) = \frac{P_\top(\top_n)}{C(\top)} = \prod_{j=\eta(\top)}^{n-1} \frac{1}{1 + \delta_j L_j(\top)} \prod_{j=0}^{\eta(\top)-1} \frac{1}{1 + \delta_j L_j(\top)}. \tag{5.8}$$

If $D_\top(\top_n)$ is a martingale that is positive, we have that $\frac{dD_\top(\top_{n+1})}{D_\top(\top_{n+1}} = V_{n+1}^i(\top)dW(\top)$ where $n = 1, \ldots, m$.

Applying Ito's lemma,

$$d\ln D_\top(\top_{n+1}) = -\frac{1}{2}||V_{n+1}(\top)||d\top + V_{n+1}^i(\top)dW(\top). \tag{5.9}$$

For a one factor model, we have that $\frac{dD_\top(\top_{n+1})}{D_\top(\top_{n+1}} = V_{n+1}(\top)dW(\top) \implies d\ln D_\top(\top_{n+1}) = -\frac{1}{2}||V_{n+1}(\top)||d\top + V_{n+1}^i(\top)dW(\top)$

$$d\ln D_\top(\top_{n+1}) = -\sum_{j=\eta(\top)}^{n} d\ln(1 + \delta_j L_j(\top)). \tag{5.10}$$

We apply Ito's lemma and Equation (5.6), and we get

$$V_{n+1}(\top) = -\sum_{j=\eta(\top)}^{n} \frac{\delta_j \sigma_j(\top) L_j(\top)}{1 + \delta_j L_j(\top)}. \tag{5.11}$$

This is the relation between the bond volatility and LIBOR. From this, we have that the drift term μ_n is given by

$$\mu_n = -\sigma_n V_{n+1}(\top) = -\sum_{j=\eta(\top)}^{n} \frac{\delta_j \sigma_j(\top) L_j(\top)}{1 + \delta_j L_j(\top)} \sigma_n \tag{5.12}$$

So from Equation (5.6), we have that the spot LIBOR dynamics are given by

$$dL_n(\top) = \sigma_n(\top) L_n(\top) \sum_{j=\eta(\top)}^{n} \frac{\rho_{n,j} \delta_j \sigma_j(\top) L_j(\top)}{1 + \delta_j L_j(\top)} d\top + \sigma_n(\top) L_n(\top) dW_n^i(\top). \tag{5.13}$$

$\square$

Proposition 5.3. *The LIBOR dynamics where the numeraire is the zero-coupon bond account are given by the following SDE [15]:*

$$n < i : dL_n(\top) = \sigma_n(\top) L_n(\top) \sum_{j=n+1}^{j} \frac{\rho_{n,j} \tau_j \sigma_j(\top) L_j(\top)}{1 + \delta_j L_j(\top)} d\top + \sigma_n(\top) L_n(\top) dW_n^i(\top)$$

$$n = i : dL_i(\top) = \sigma_n(\top) L_n(\top) dW_n^i(\top)$$

$$n > i : dL_n(\top) = -\sigma_n(\top) L_n(\top) \sum_{j=i+1}^{n} \frac{\rho_{n,j} \delta_j \sigma_j(\top) L_j(\top)}{1 + \delta_j L_j(\top)} d\top + \sigma_n(\top) L_n(\top) dW_n^i(\top) \quad \textit{for } \top \leq \min\{T_{n-1}, T_i\}$$

Proof. We assume there exists an LMM that satisfies (5.2). The aim is to find the deterministic functions $\mu_n^i(\top, L_n(\top))$, where:

$$L(\top) = (L_i(\top), \ldots, L_M(\top))'$$

such that the following is satisfied

$$dL_n(\top) = \mu_n^i(\top, L(\top)) L_n(\top) d\top + \sigma_n L_n dW_n^i. \tag{5.14}$$

For the $\mu_n^i(\top, L(\top))$ to be determined, we make use of the Radon-Nikodym theorem to move from $\mathbb{Q}^i$ to $\mathbb{Q}^n$, then impose that the $\mathbb{Q}^n$-resulting drift is null. Applying the Radon-Nikodym of $\mathbb{Q}^{i-1}$ with respect to $\mathbb{Q}^i$ at time $\top$ is

$$\frac{d\mathbb{Q}^{i-1}}{d\mathbb{Q}^i}|\mathcal{L}_t^W = \frac{P_t(\top_{i-1})P_0(\top)}{P_0(T_{i-1})P_t(\top_i)} := \gamma_t^i \tag{5.15}$$

and by the fundamental law of asset pricing,

$$\gamma_t^i = \frac{P_0(T_i)}{P_0(T_{i-1})}(1 + \delta_i L_i(\top)).$$

Therefore, if we assume (5.2), the SDE of γ_i under $\mathbb{Q}^i$ is

$$\begin{aligned} d\gamma_t^i &= \frac{P_0(T_i)}{P_0(T_{i-1})} dL_i(\top)\delta = \frac{P_0(T_i)}{P_0(T_{i-1})}\delta_i\sigma_i(\top)L_i(\top)dW_i(\top) \\ &= \frac{\gamma_t^i}{1 + \delta_i L_i(\top)}\delta_i\sigma_i(\top)L_i(\top)dW_i(\top). \end{aligned}$$

So, the density γ_i is a martingale of exponential form with associated process Λ which is the d-dimensional $n \times n$ vector apart from the i-th component,

$$\Lambda = \left(0, \ldots, -\frac{\delta_i\sigma_i L_i}{1 + \delta_i L_i}, \ldots 0\right)', \tag{5.16}$$

So that we can make use of the change of drift formula with correlation:

$$dW^i(\top) = dW^{i-1}(\top) - \rho d\top,$$

namely with components:

$$dW_j^i(\top) = dW_j^{i-1}(\top) + \rho^{j,i}\frac{\delta_i\sigma_i(\top)L_i(\top)}{1 + \delta_i L_i(\top)}d\top.$$

using induction, we get:

$$n < i : dW_j^i(\top) = dW_j^n(\top) + \sum_{h=n+1}^{i} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + \delta_h L_h(\top)} d\top,$$
$$n > i : dW_j^i(\top) = dW_j^n(\top) - \sum_{h=n+1}^{i} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + \delta_h L_h(\top)} d\top$$

Then, substituting these into (5.14) and imposing the $\mathbb{Q}^n$-drift to be zero we obtain:

$$n < i : L_n(\top)(\mu_n^i(\top, L(\top))) + \sigma_i(\top) \sum_{h=n+1}^{i} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + \delta_h L_h(\top)} d\top = 0,$$
$$\Longrightarrow \mu_n^i(\top, L(\top)) = -\sigma_i(\top) \sum_{h=n+1}^{i} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + \delta_h L_h(\top)} d\top.$$
$$n > i : L_n(\top)(\mu_n^i(\top, L(\top))) - \sigma_i(\top) \sum_{h=n+1}^{i} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + \delta_h L_h(\top)} d\top = 0,$$
$$\Longrightarrow \mu_n^i(\top, L(\top)) = \sigma_i(\top) \sum_{h=n+1}^{i} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + \delta_h L_h(\top)} d\top.$$

□

Proposition 5.4. *The terminal LIBOR dynamics are given by the following SDE:*

$$dL_i(\top) = -\sigma_i(\top) L(\top) \sum_{j=i+1}^{M} \rho^{jh} \frac{\delta_h \sigma_h(\top) L_h(\top)}{1 + L_h(\top)\delta_h} d\top + \sigma_i(\top) L_i(\top) dW_i^M(\top). \tag{5.17}$$

Proof. First, we have to prove that the solution of (5.17) exists. For $i = N$ we easily have

$$dL_i(\top) = \sigma_N(\top) L_N(\top) dW_N^N(\top)$$

which is a martingale of exponential form, where σ_M is bounded. This proves the existence of the solution to (5.17). Now, inductively: assume that (5.17) has a solution for $i+1, \ldots, N$, then we present the i-th dynamics as

$$dL_i(\top) = \mu(\top, L_{i+1}(\top), \ldots, L_M(\top))L_i d\top + \sigma_i(\top)L_i(\top)dW_i^M(\top)$$

where the fact that μ_i is only dependent on L_K for $n = i+1, \ldots, N$ is key. So if we denote $L_{i+1}^M := (L_{i+1}, \ldots, L_M)^T$, we can get the explicit solution of the above SDE by using Ito's formula:

$$\begin{aligned}
d\ln L_i(\top) &= \frac{dL_i(\top)}{L_i(\top)} - \frac{1}{2L_i(\top)^2}\sigma_i(\top)^2 L_i(\top)^2 d\top \\
&= \mu_i(\top, L_{i+1}^N(\top))d\top + \sigma_i(\top)dW_i^M(\top) - \frac{1}{2}\sigma_i(\top)^2 d\top \\
\Longrightarrow \ \ln L(\top) &= \ln L_i(0) + \int_0^\top \left(\mu_i(u, L_{i+1}^M(u)) - \frac{\sigma_i(u)^2}{2}\right) du + \int_0^\top \sigma_i(u)dW_i^i(u) \\
\Longrightarrow \ L_i(\top) &= L_i(\top)\exp\left[\int_0^\top \left(\mu_i(u, L_{i+1}^M(u)) - \frac{\sigma_i(\top)^2}{2}\right) du\right] \exp\left[\int_0^\top \sigma_i(u)dW_i^i(u)\right],
\end{aligned}$$

for $0 \leq \top \leq T_{i-1}$ and this proves existence.

Lastly, by showing that the Novikov condition is satisfied by the process Λ defined in (5.16), in which case the density process γ^i is a $\mathbb{Q}^i$-martingale, which allows us to use the Girsanov Theorem, following the same procedure as in Proposition (5.3). If we have an initial LIBOR term structure that is non-negative, as it is $L(0) = (L_1(0), \ldots, L_M(0))'$, we see that all $L_i(\top)$ will be always non-negative, so the process λ in (5.16) is bounded and consequently satisfies the Novikov condition. $\square$

5.3 Valuation of caps in the LMM

Let us recap a bit on caps. In Chapter 3 we defined caps as a series of European call options, known as caplets that share the same strike price κ. Each caplet has a payoff at $\{T_1, \ldots, T_N\}$. If we take the sum of all the caplets we get the price of the cap. The cap has a discounted payoff $\top$ that is defined by

$$\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N\delta_j \left(L(\top_{j-1}, T_j) - \kappa\right)_+ .$$

The LMM can also be used to price exotic caps such as ratchet caps and sticky caps. The ratchet cap has a discounted payoff that is defined by

$$\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N\delta_j (L(T_{j-1}, T_j) - (L(T_{j-2}, T_{j-1}) + \epsilon))_+.$$

and the discounted payoff of a sticky cap is given by

$$\sum_{j=\eta+1}^{\omega} D_f(\top, T_j) N\delta_j (L(T_{j-1}, T_j) - (\min(L(T_{j-2}, T_{j-1}), \kappa) + \epsilon))_+.$$

Pricing caps in the LMM framework is interesting because it can be used to validate the model and the calibrations by comparing the results from the analytical Black model commonly referred to as the Black76 model. The results obtained from this model can help us in computing the pricing of other models that are not analytically tractable by using the same parameters. In the Black76 formula, the LIBORs are log-normally distributed, hence the LMM has to also account for this property.

Market cap prices are often quoted as implied, flat, or spot volatilities. Let $\top \leq T_0$ and κ be fixed. Assume that for each i there is a traded cap that settles at times $T_0, T_1, \ldots, T_N$.

Let Cp_i^m denote the price of the corresponding market cap. Then

$$Cpl_i^m(\top) = Cap_i^m(\top) - Cap_{i-1}^m(\top), \tag{5.18}$$

where $Cap_i^m(\top) = 0$ and $Cpl_1^m(\top) = Cap_1^m(\top)$. We have that

$$Cap_i^m(\top) = Cpl_i^m(\top) + Cpl_{i-1}^m(\top). \tag{5.19}$$

Thus

$$Cpl_2^m(\top) = Cpl_2^m(\top) + Cl_1^m(\top) \tag{5.20}$$

$$= Cpl_2^m(\top) + Cpl_1^m(\top) \tag{5.21}$$

$$Cpl_3^m(\top) = Cpl_3^m(\top) + Cpl_2^m(\top) + Cpl_1^m(\top) \tag{5.22}$$

In general

$$Cap_i^m(\top) = \sum_{j=1}^{i} Cpl_j^m(\top). \tag{5.23}$$

Definition 5.1. *Consider the market price data above. The implied Black volatilities can be defined as:*

1. *The implied Black volatilities $\hat{\sigma_1}, \ldots, \hat{\sigma_N}$ are the solutions to the below*

$$Cap_i^m(\top) = \sum_{j=1}^{i} Cpl_j^b(\top, \hat{\sigma}_i). \tag{5.24}$$

2. *The implied Black forward volatilities $\hat{\sigma_1}, \ldots, \hat{\sigma_N}$ are the solutions to the below*

$$Cpl_i^m(\top) = \sum_{j=1}^{i} Cpl_j^b(\top, \hat{\sigma}_i). \tag{5.25}$$

Proposition 5.5. *The LMM i-th caplet is equal to the corresponding Black76 caplet formula [15] :*

$$\begin{aligned} cpl_{LMM}(0, T_{i-1}, T_i, \kappa) &= cpl_{Black}(0, T_{i-1}, T_i, \kappa, V_i) \\ &= P_j(\top) Black(R_j(\top), \kappa, \sum_{j}^{F} \sqrt{T_{j-1}}) \\ &= \delta_i P_0(T) Black(\kappa, L(0), T_{i-1}, T_i) v_i \sqrt{T_{i-1}} \end{aligned}$$

where

$$(v_i)^2 T_{i-1} = \int_0^{T_{i-1}} \sigma_i(\top)^2 d\top. \tag{5.26}$$

Proof.

$$\mathrm{capl}_{LMM}(\top, T_{i-1}, T_i, \kappa) = \delta_i P_\top(T_i) \mathbb{E}^{\mathbb{Q}_i}\{(L(T_{i-1}, T_i) - \kappa)_+ | \mathcal{F}_\top\}$$

where $L(T_{i-1}, T_i) = L(T_{i-1}, T_{i-1}, T_i) = L_i(T_{i-1})$ and, for $T < T_{i-1}, L_i(T)$ follows a distribution that is log-normal under $\mathbb{Q}^{T_i}$, indeed

$$L_i(T) = L_i(\top) e^{\int_\top^T \sigma_i(u) dW_i^i(u) - \frac{1}{2}\sigma_i^2(u) du} \qquad 0 \leq \top \leq T \leq T_{i-1}$$

with σ_i assumed to be deterministic. Let

$$X_i(\top, T) := \int_i^T \sigma_i(u) dW_i^i(u) - \frac{1}{2}\sigma_i(u)^2 du,$$

we have

$$L_i(T) = L_i(\top) e^{X_i(\top, T)}, \quad X_i(\top, T) \sim \mathcal{N}(\mu_i(\top, T), \sum_i^2(\top, T)),$$

where

$$\mu_i(\top, T) = -\frac{1}{2} \int_\top^T \sigma_i(u)^2 du, \quad \sum_i^2(\top, T) = \int_\top^T \sigma_i(u)^2 du$$

and $L_i(\top) \in \mathbb{R}$ at time $\top$. Thus the proof follows from $(B.8)$.

$\square$

5.4 Calibration of the LMM to Caplets

It is standard practice to calibrate the LMM to the traded caplets in the market since they are among the most liquid ones. The traders are responsible for converting caplets to caps, then using Proposition (5.5) and market quoted volatilities, calibrating the parameters of the LMM to caps and floors becomes trivial.

$\sigma_1, \ldots, \sigma_N$ such that

$$(\sigma_i)^2 T_{i-1} = \int_0^{T_{i-1}} \sigma_i(\top)^2 d\top, \qquad i = 1, \ldots, N. \tag{5.27}$$

5.4.1 Parameterizations of Volatility of Forward rates

Equation (5.27) is very undetermined, meaning when applying it, it is important to make some assumptions about the structure of the volatility functions. A lot of specifications have been proposed in the literature, with the two most popular ones being the piecewise-constant functions and the functional parameterized forms. In this dissertation, we will only use the former.

- General Piecewise-constant volatilities (GPC):

$$\sigma_i(\top) = \sigma_{i,\eta(\top)}(\top), \qquad 0 < \top \leq T_{i-1}, \tag{5.28}$$

  where

$$\eta(\top) = n \text{ if } T_{n-2} < \top \leq T_{n-1}, \quad n \geq 1. \tag{5.29}$$

The maturity of the first LIBOR rate that is yet to expire at time $\top$ is indicated by $\eta(\top)$. We can organize these σ's in Table (5.1), where the columns are the time intervals and the LIBOR rates are in the rows.

T	$[T_0, T_1)$	$[T_1, T_2)$	$[T_2, T_3)$	$\cdots$	$[T_{N-1}, T_N)$
$L_0(\top)$	$\sigma_{0,0}$	dead	dead	$\cdots$	dead
$L_1(\top)$	$\sigma_{1,0}$	$\sigma_{1,1}$	dead	$\cdots$	dead
$L_2(\top)$	$\sigma_{2,0}$	$\sigma_{2,1}$	$\sigma_{2,2}$	$\cdots$	dead
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$L_N(\top)$	$\sigma_{N,0}$	$\sigma_{N,1}$	$\sigma_{N,2}$	$\cdots$	$\sigma_{N-1,N}$

Table 5.1: GPC volatilities.

The LIBOR rates only exist until the date when they are due to start applying. This implies that the instantaneous volatility that corresponds to the starting period ceases to exist when we are past the period.

To calibrate the model, we insert Equation (5.28) into Equation (5.27), and thus impose:

$$T_{i-1}(v_i)^2 = \int_0^{T_{i-1}} \sigma_{i,\eta}(\top)^2 d\top = \sum_{k=1}^{i} \tau_{k-2,k-1}\sigma_{i,k}, \qquad i = 1, \ldots, N. \tag{5.30}$$

We have a lot of configurations that can perfectly fit caplets. It is a fact that:

$$\begin{aligned} \sqrt{(T_0 - 0)}\sigma_{0,0} \iff \sigma_{0,0} &= v \\ \sqrt{(T_0 - 0)\sigma_{1,0}^2 + (T_1 - T_0)\sigma_{1,1}^2} &= \sqrt{T_1}v \\ \sqrt{(T_0 - 0)\sigma_{3,0}^2 + (T_1 - T_0)\sigma_{3,2}^2 + (T_2 - T_1)\sigma_{3,3}^2} &= \sqrt{T_2}v \\ &\vdots \end{aligned}$$

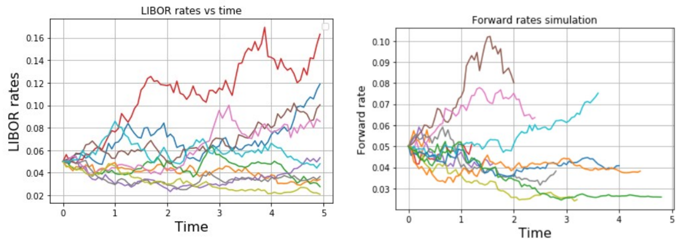
Year, i:	1	2	3	4	5	6	7	8	9	10
v_i	15.50	18.25	17.91	17.74	17.27	16.79	16.30	16.01	15.76	15.54
$\sigma_{10,i-1}$	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

Table 5.2: Volatility data, application period= 1 year.

Lastly, it is important to specify the instantaneous correlation. A lot of interest rate derivatives are dependent on multiple LIBOR rates. There is some correlation between these rates. Because the payoff in caplets relies solely on a single LIBOR rate, they are not suitable for calibrating the instantaneous correlation between LIBOR rates. Instead, we use European swaptions to calibrate the instantaneous correlations. The swaption payoff is dependent on multiple forwards. The instantaneous correlation will be assumed to follow the below structure

$$\rho_{i,j} = \prod_{n}^{j-1} \rho_{n,n+1}$$

$$\rho_{i,j} = \rho_{j,i}.$$



(a) Forward rates under the same application periods.

(b) Forward rates under different application periods.

Figure 5.1: Forward rates simulations.

We simulated forward rates for $T = 5$, $\sigma = 0.25$ with the same application period in Figure (5.1a). In Figure (5.1b), we simulate the forward rates with different application periods. When simulating the forward rates, we start at the initial time, $\top = 0$, and move forward in time. As we move forward in time, more and more forwards reach their maturity dates and then cease to exist for times after their maturity. This means that only a few forward rates will still be alive as we move forward.

We now consider an example of a cap simulation with the following parameters, $L(0) = 0.05$, $\kappa = 0.5$, $\delta = 0.5$, $\sigma = 0.20$. Summing all these caplets results in a cap. Using N number of trials results in an estimate of the Monte Carlo computed value of a cap $\bar{V}^{cap}$ as

$$\bar{V}^{cap} = \frac{1}{N} \sum_{j=1}^{N} V_j^{cap}. \tag{5.31}$$

We can also obtain the standard error of the simulation using Equation (A.3).

Standard cap valuation

	Black76	LMM Sim
	0.00082	0.00069
	0.00103	0.00102
	0.00127	0.00131
	0.00154	0.00133
	0.00164	0.00164
	0.00162	0.00182
	0.00184	0.00186
	0.00212	0.00209
	0.00208	0.00221
Cap	0.01397	0.01397

Table 5.3: Caplet prices.

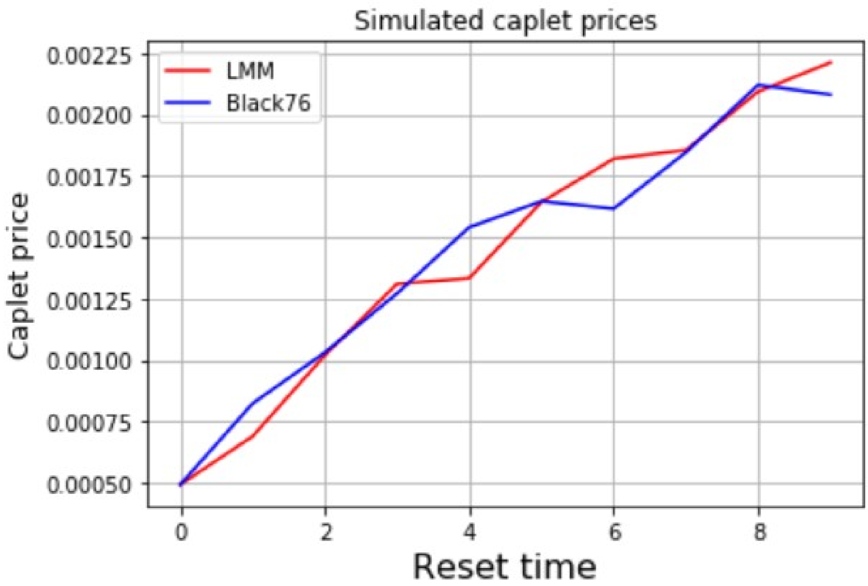


Figure 5.2: Simulated caplet prices.

In Table (5.3) and Figure (5.2), we evaluated the payoff of 9 caplets using the analytical Black76 formula and simulated payoff. The caplets are of different maturities, starting from 1 to 9, where the notional is 1 and $\kappa = 0.05$. We used 2000 Monte Carlo simulations. As time progresses, the caplet payoffs increase. The two methods yielded similar caplet payoffs for different maturities.

Ratchet and sticky cap valuation

	Ratchet	Sticky
	0.00001	0.00001
	0.00004	0.00005
	0.00011	0.00012
	0.00019	0.00022
	0.00028	0.00034
	0.00036	0.00042
	0.00044	0.00053
	0.00053	0.00063
	0.00058	0.00069
Cap	0.00254	0.00301

Table 5.4: Ratchet and sticky Caplet prices.

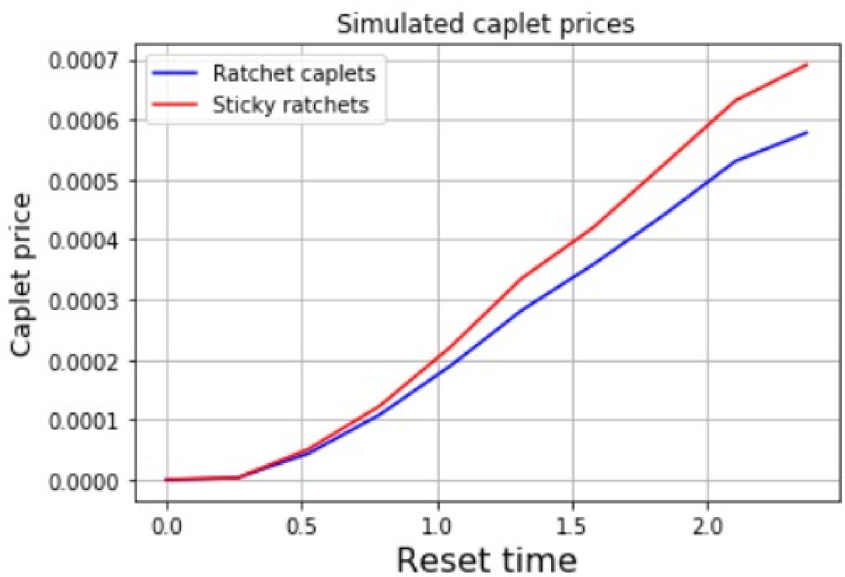


Figure 5.3: Simulated caplet prices.

In Table (5.4) and Figure (5.3) we evaluated 9 Ratchet and sticky caplet prices. The caplets are of different maturities, starting three months from now until 5 years. We set $\kappa = 0.05$ and $\epsilon = 0.01$ and 2000 Monte Carlo simulations. As the maturities increase, the caplet prices also increase, but the Sticky caplets are generally higher than those of ratchet caplets. The standard caplets are always higher than both these nonstandard caplets at each maturity, and hence the cap has a higher payoff.

5.5 Valuation of swaptions in the LMM

We now look at how swaptions are priced using the LMM. In Chapter 3 we introduced IRS as a financial contract where one party pays the floating leg $L_i(T_{i-1})$ and receives the constant rate κ in exchange. As already mentioned, we have two types of IRS, a payer and a receiver IRS. The time T_ϕ payoff of a payer IRS is given by

$$\sum_{i=\eta+1}^{\omega} D_f(\top, T_i)\delta_i \left(L(\top_{i-1}, T_i) - \kappa\right). \tag{5.32}$$

We can also get the discounted time $\top < T_\phi$ payoff as

$$\sum_{i=\eta+1}^{\omega} D_f(\top, T_i)\delta_i \left(L(\top_{i-1}, T_i) - \kappa\right). \tag{5.33}$$

We compute the payoff of this cap as:

$$\begin{aligned}
PFS(\top, [T_\phi, \ldots, T_\eta], \kappa) &= \mathbb{E}^{\mathbb{Q}} \left\{ \sum_{i=\phi+1}^{\eta} D_f(\top, T_i)\delta_i(L_i(T_{i-1}) - \kappa) \right\} && (5.34)\\
&= \sum_{i=\phi+1}^{\eta} P_\top(T_i)\delta_i \mathbb{E}^{\mathbb{Q}^i} \left\{(L_i(T_{i-1}) - \kappa)\right\} && (5.35)\\
&= \sum_{i=\phi+1}^{\eta} P_\top(T_i)\delta_i(L_i(\top) - \kappa) && (5.36)\\
&= \sum_{i=\phi+1}^{\eta} [P_\top(T_{i-1}) - (1 + \delta_i\kappa)P_\top(T_i)]. && (5.37)
\end{aligned}$$

We see that we can get the same value by defining T_ϕ-payoff

$$\sum_{i=\phi+1}^{\eta} P_{T_\phi}(T_i)\delta_i(L_i(T_\phi) - \kappa)$$

which results in the $\top$-discounted payoff

$$D_f(\top, T_\phi) \sum_{i=\phi+1}^{\eta} P_{T_\phi}(T_i)\delta_i(L_i(T_\phi) - \kappa)$$

and by taking the risk-neutral expectation.

We now define the forward swap rate before moving on to the swaption definition.

Definition 5.2. *The forward swap rate $S_{\eta,\omega}(\top)$ at time $\top$ the fixed interest rate that the receiver requires in return for the uncertainty associated with paying the short-term LIBOR (floating) rate over time*

$$S_{\eta,\omega}(\top) = \frac{P_\top(\top_\eta) - P_\top(\top_\omega)}{A_{0,\omega}}$$

where $A_{0,\omega}$ is the annuity and is defined by:

$$A_{O,\omega} := \sum_{j=\eta+1}^{\omega} \delta_j P_\top(\top_j).$$

Lastly, we observe that the discounted payoff (5.33) for a κ value distinct from the swap rate can also be represented, at time $\top = 0$, also in terms of swap rates as

$$D_f(0, T_\phi)(S_{\phi,\eta}(T_\phi) - \kappa) \sum_{i=\phi+1}^{\eta} \delta_i. \tag{5.38}$$

We now move on to swaptions which were introduced in Chapter 3. Recall that the payer swaption payoff is given by

$$D_f(0, T_\phi)(S_{\phi,\eta}(T_\phi) - \kappa)_+ \sum_{i=\phi+1}^{\eta} \delta_i. \tag{5.39}$$

It was thought that AA-rated banks could not default on their loans. During the financial crisis of 2007-09, there were reports of banks manipulating LIBOR and these reports were later investigated by both the UK and US governments. After the financial crisis, market participants started searching for a more risk-free rate that could replace the IBOR rates. The Overnight Index Swap rate was elected as the best candidate to be the risk-free rate.

In the next chapter, we introduce a model proposed by Andrei Lyashenko and Fabio Mercurio (2018) that aims to address the drawbacks of the LMM.

CHAPTER 6

The generalized Forward Market Model

The financial crisis of 2007-09 prompted the search for an alternative risk-free rate, to transition away from LIBOR. The overnight rates have been chosen as risk-free rates in all major economies. In the LMM framework, two zero rates must be modelled in order to price derivatives that are dependent on LIBOR. This, however, resulted in arbitrage opportunities in the market.

Andrei Lyashenko and Fabio Mercurio [4] proposed a model that aims to address this issue. This LMM extension takes advantage of the idea of extended zero-coupon bonds, which is simple and natural, and is convenient when dealing with the backward-looking setting -in-arrears rates. This concept has been used before by Glasserman and Zhao [17] or Piterbarg [18] to define hybrid numeraires and measures but it has never been used to extend the definition of term rates from Forward-looking to Backward-looking or to provide a generalization of a single-curve LMM. The FMM has some properties that the LMM does not have such as modelling the dynamics of the forward-rate under

$\mathbb{Q}$ so that we do not have to approximate the dynamics under $\mathbb{Q}^d$. We first introduce the concept of backwards rates.

6.1 Introduction

The new FMM is essentially an improvement of the LMM, which we introduced in Chapter 5. The key idea in this model is that of the extended zero coupon bonds, first used by Glasserman and Zhao [17] forms the basis of this LMM extension. The most notable difference between these two models is where the rates are defined. In the case of the LMM, the rates are only defined for times until the beginning of the application period. With the FMM on the other hand, the rate is defined for all times, including the application period and the time after the application period. As we have seen in Chapter 5, the LIBOR rates evolve until T_{j-1}, which is the beginning of the application period. We say that this is a forward-looking rate (FL rate). In the transition from LIBOR, we use backward-looking rates (BL rates). Unlike the FL rates, their fixing is only known at the end of the application period, T_j, and even for times after the end of the application period. Figure (6.1) gives an illustration of the difference between the two rates.

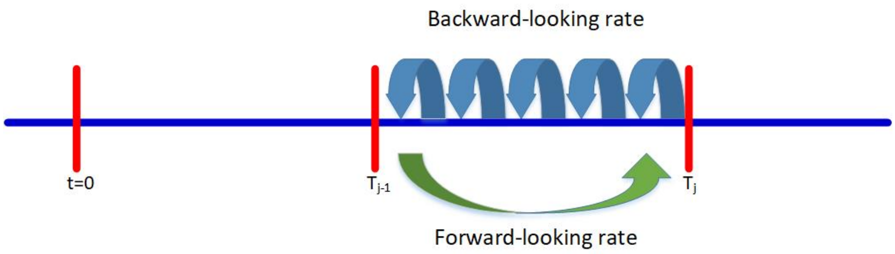


Figure 6.1: Forward-looking rate vs Backward-looking rate.

It is worth noting that financial institutions still have contracts that rely on forward rates with maturities of up to 30 years or even longer, so the new model still has to cater to that. As we will see in the coming chapters, the FMM models both the FL and the

BL rates, and it can still be used to cater for the parties who are still interested in FL rates.

6.2 Fundamentals of the generalized FMM

In the FMM, we will assume the same tenor structure as we did in Chapter 2, $0 := T_0 < T_1.. < T_M$, with an associated year fraction denoted by δ_j for the interval $[T_{j-1}, T_j)$ given as $\delta_j = T_j - T_{j-1}$. For each time $\top$, we define $\eta(\top) = \min j : T_j \geq \top$, which is the next tenor date after time $\top$. Modelling RFRs in the LIBOR transition is of great importance. We also examine a financial market operating in continuous time, with an instantaneous risk-free rate $r(\top)$. When the identical risk-free rate (RFR) is used both for discounting purposes and for deriving term rates, it implies a return to the traditional single-curve modeling framework. This rate $r(\top)$ has an associated cash account $C(\top)$ such that $C(0) = 1$ as shown before, and the time $\top$ price of a zero-coupon bond price with maturity T, is given by:

$$P_\top(T) = \mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_\top^T r(u)du}|\mathcal{F}_t\right\} \tag{6.1}$$

which is defined for $\top \leq T$. We now introduce an important proposition.

Proposition 6.1. *The definition of $P_\top(T)$ can be extended to times $\top > T$.*

Proof. Using (6.1) and the definition of $C(\top)$, we can write:

$$P_\top(T) = \mathbb{E}^{\mathbb{Q}}\left\{e^{\int_T^\top r(v)dv}|\mathcal{F}_\top\right\} = e^{\int_T^\top r(v)dv} = \frac{C(\top)}{C(T)} \tag{6.2}$$

since $\int_T^\top r(v)dv$ is $\mathcal{F}_\top$-measurable. □

Remark 6.1. *If we set the maturity to be $T = 0$, we have that $P_\top(0) = C(\top)$, implying that the cash account can be looked at as a zero-coupon bond expiring immediately.*

Consider a strategy Y_T that entails the following:

1. Buying the T-maturity zero-coupon bond;
2. reinvesting the proceeds of the bond's unit notional received at T at the risk-free rate $r(\top)$ from time T on-wards.

Denoting by $Y(\top)$ the time-$\top$ value of this strategy, we have:

$$Y_T(\top) = \begin{cases} P_\top(T) & \top \leq T \\ e^{\int_T^\top r(u)du} & \top > T \end{cases} \tag{6.3}$$

From here we see that, for $\top > T$, $Y_T(\top)$ corresponds precisely to the extended bond price as defined in Equation (6.1). We can now conclude that, for each given T, $Y_T(\top) = P_\top(T)$ for all times $\top$. The extended zero-coupon bond defined by (6.1) can still be considered as a viable numeraire since it is strictly positive and it is non-dividend paying. Therefore, we can define the extended T-forward measure $\mathbb{Q}^T$ as follows:

Definition 6.1. *The extended T-forward measure $\mathbb{Q}^T$ can be defined the usual way as the EMM associated with the extended bond price $P_\top(T)$ for any time $\top$. Since, for $\top > T$,*

$$P_\top(T) = \frac{C(\top)}{C(T)}.$$

The extended T-forward measure $\mathbb{Q}^T$ is a combination of the T-forward measure up to the maturity time T and the risk-neutral cash account measure $\mathbb{Q}$ after T. Such hybrid T-forward measures have been discussed in the literature. We note that $P_\top(0) = C(\top)$.

6.3 Fundamentals of the forward rates

In this section, we introduce the BL rate and the BL forward rate. The reader must understand the fundamental differences between these two rates as they play a key role in the generalized FMM. Before we introduce the mathematical concepts pertaining to these two rates, we will first explain the difference using the timeline in Figure (6.2) to ease the reader into the mathematical definitions.

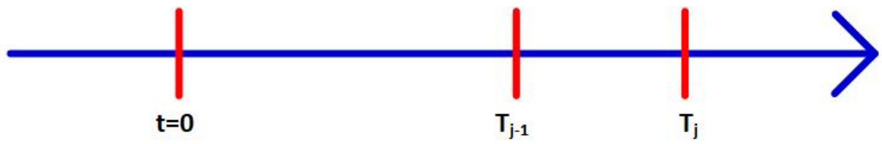


Figure 6.2: Backwards-looking rates vs backward-looking forward rates.

$R(T_{j-1}, T_j)$, which is the BL rate is the rate that will be observed in the market when an application period comes to an end. Assume that $T_j - T_{j-1} = 3$ months in Figure (6.2) and we are standing at T_j. The compounded overnight rates from T_{j-1} to T_j can be defined as the BL rate. The overnight rates are only known at T_j for every period $[T_{j-1}, T_j]$ and are supplied by the market. So for us to get this BL rate, the daily overnight rates can be compounded until the expiry of the application period.

The BL forward rate, on the other hand, is the BL rate observed at $\top = 0$ for a future application period $[T_{j-1}, T_j]$. So, if T_j is yet to be reached for the BL rate, i.e. $\top < T_j$, we must give an expectation of the BL rate over the application period. This BL forward rate will be simulated by the generalized FMM for $\top \in [T_0, T_j)$. It is possible to simulate the expected BL forward rate until T_j. It is only logical that we expect the BL rate to be the same as the BL forward rate at T_j.

In this section, we introduced the BL rate and the BL forward rate and explained how they are different. A thorough understanding of these two rates is important in order to understand the concepts that will be introduced in the coming sections.

6.3.1 Backward-looking rate

We now define the BL rate which we previously denoted by $R(T_{j-1}, T_j)$. To apply a BL rate towards an interest period means applying the overnight interest rate each day.

We first define r_i as the overnight rate on the day i with δ being the corresponding day-count fraction over the period $[T_{j-1}, T_j]$. We assume that it takes n days in the application period. For each $j = 1, ..., M$, we can get an approximation of $R(T_{j-1}, T_j)$ for the interval $[T_{j-1}, T_j)$ as follows:

$$R(T_{j-1}, T_j) = \frac{1}{\delta_j}\left[\prod_{i=1}^{n}(1 + r_i\tau_i) - 1\right] \tag{6.4}$$

$$R(T_{j-1}, T_j) \approx \frac{1}{\delta_j}\left(e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1\right) = \frac{1}{\delta_j}\left[P_{j-1}(T_j) - 1\right]. \tag{6.5}$$

To get the approximation, we take the limit of the day-count fraction to zero. We see that extended zero coupon definition is used in Equation (6.5). The fixing value of in-arrears rates is only determined at the time T_j. This makes them BL in nature. Alternatively one can define FL rates, which are IBOR-like in the sense that they are fixed at the beginning of their application period. Let $F(T_{j-1}, T_j)$ denote such a rate and by arbitrage-free, we have:

$$F(T_{j-1}, T_j) = \mathbb{E}^{\mathbb{Q}_{T_j}}\left\{R(T_{j-1}, T_j)|\mathcal{F}_{T_{j-1}}\right\} \tag{6.6}$$

for each $j = 1, ..., M$. If Equation (6.6) does not hold, it would imply that the forward rate which is fixed at T_{j-1} is not equal to the expectation of the BL rate over the application period $[T_{j-1}, T_j]$.

6.3.2 Backward-looking forward rates

Let us now define the BL forward rate $R(\top, T_{j-1}, T_j)$ mathematically. For ease of reading, we will use the shorthand notation $R_j(\top)$. For instance, the expected BL forward rate in the future over the application period $[T_{j-1}, T_j)$ observed at $\top = 0$ is denoted by $R_j(0)$. Since we are still at $\top = 0$, we still have to simulate this rate until the end of the application period.

Definition 6.2. *The BL forward rate $R_j(\top)$ at time $\top \leq T_{j-1}$, by arbitrage-free theory is given by*

$$R_j(\top) = \mathbb{E}^{\mathbb{Q}_{T_j}} \left\{ R(T_{j-1}, T_j) | \mathcal{F}_{T_{j-1}}] \right\}. \tag{6.7}$$

From (6.4) and (6.6) we see that, for each $j = 1, ..., M$, the FL spot rate $F(T_{j-1}, T_j) = R_j(T_{j-1})$. The dynamics of the BL forward rate under $\mathbb{Q}$ can also be derived. We start from Equation (6.7)

$$R_j(\top) = \mathbb{E}^{\mathbb{Q}_{T_j}} \left\{ R(T_{j-1}, T_j) | \mathcal{F}_{T_{j-1}}] \right\} = \mathbb{E}^{\mathbb{Q}_{T_j}} \left\{ \frac{1}{\delta_j} e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1 \middle| \mathcal{F}_{T_{j-1}} \right\}, \tag{6.8}$$

$$1 + \frac{1}{\delta_j} R_j(\top) = E^{\mathbb{Q}_{T_j}} \left\{ e^{\int_{T_{j-1}}^{T_j} r(u)du} \middle| \mathcal{F}_{T_{j-1}} \right\}. \tag{6.9}$$

We use Definition (3.6) to move from one measure to another:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_{T_j}} = \frac{C(T_j)}{C(\top)} \frac{P_\top(T_j)}{P_{T_j}(T_j)}. \tag{6.10}$$

Using the definition of an expectation, from Equation (6.9), we have that the RHS of the equation can be rewritten as

$$\mathbb{E}^{\mathbb{Q}_{T_j}} \left\{ e^{\int_{T_{j-1}}^{T_j} r(u)du} \middle| \mathcal{F}_{T_{j-1}} \right\} = \int e^{\int_{T_{j-1}}^{T_j} r(u)du} d\mathbb{Q}_{T_j} = \int e^{\int_{T_{j-1}}^{T_j} r(u)du} \frac{C(T_j)}{C(\top)} \frac{P_\top(T_j)}{P_{T_j}(T_j)} d\mathbb{Q} \tag{6.11}$$

$$= \frac{1}{P_j(\top)} \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\int_\top^{T_j} r(u)du} e^{\int_{T_{j-1}}^{T_j} r(u)du} \middle| \mathcal{F}_{T_{j-1}} \right\} = \frac{1}{P_j(\top)} \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\int_\top^{T_{j-1}} r(u)du} \middle| \mathcal{F}_{T_{j-1}} \right\} \tag{6.12}$$

$$= \frac{P_{j-1}(\top)}{P_j(\top)}. \tag{6.13}$$

So we can write

$$R_j(\top) = \frac{1}{\delta_j} \left[\frac{P_{j-1}(\top)}{P_j(\top)} - 1 \right]. \tag{6.14}$$

This represents the classic forward-rate formula, which is simply compounded. Due to our definition of extended bond price, this formula remains valid for every period $\top$, including those occurring after T_j. From Equation (6.14), we observe four key properties of the BL forward rate:

(i) $R_j(\top)$ is a martingale under the T_j-forward measure. To prove this, we can rewrite Equation (6.14) as

$$P_j(\top)R_j(\top) = \frac{1}{\delta_j}\left[P_{j-1}(\top) - P_j(\top)\right].$$

When we examine the movement of $R_j(\top)$ under the $\mathbb{Q}_{T_j}$, it is evident that it is a martingale

$$\begin{aligned}\frac{R_j(\top)P_j(\top)}{P_j(\top)} &= \mathbb{E}^{\mathbb{Q}_{T_j}}\left\{\frac{R_j(U)P_j(U)}{P_j(U)}\middle|\mathcal{F}_{T_{j-1}}\right\},\\ R_j(\top) &= \mathbb{E}^{\mathbb{Q}_{T_j}}\left\{R(U)|\mathcal{F}_{T_{j-1}}\right\}.\end{aligned}$$

Just as in the LIBOR case, only one $R_j(\top)$ is a martingale under the $\mathbb{Q}_{T_j}$, all the other rates are martingales under their respective the $\mathbb{Q}_{T_j}$'s.

(ii) $R_j(\top) = F(T_{j-1}, T_j)$ at time T_{j-1}. To prove this, we look at Equation (6.14).

$$\begin{aligned}R_j(\top) &= \frac{1}{\delta_j}\left[\frac{P_{j-1}(\top)}{P_j(\top)} - 1\right] = \frac{1}{\delta_j}\left[\frac{P_{j-1}(\top)}{P_j(\top)} - \frac{P_j(T_{j-1})}{P_j(T_{j-1})}\right] = \frac{1}{\delta_j}\left[\frac{P_{j-1}(T_{j-1}) - P_j(T_{j-1})}{P_j(T_{j-1})}\right]\\ &= F(T_{j-1}, T_j).\end{aligned}$$

(iii) $R_j(\top)$ is equal to the realized BL rate at time T_j : $R_j(T_j) = R(T_{j-1}, T_j)$. This follows from the definition of BL forward rates. By setting $\top = T_j$ in Equation

(6.14) and compare it with Equation (6.5),

$$
\begin{aligned}
R_j(T_j) &= \frac{1}{\delta_j}\left[\frac{P_{j-1}(T_j)}{P_j(T_j)} - 1\right] = \frac{1}{\delta_j}\left[P_{j-1}(T_j) - 1\right], \\
R(T_{j-1}, T_j) &= \frac{1}{\delta_j}\left[P_{j-1}(T_j) - 1\right].
\end{aligned}
$$

(iv) $R_j(\top)$ stops evolving after T_j : $R_j(\top) = R(T_{j-1}, T_j), \top > T_j$. This is a key property because want the dynamics of $R_j(\top)$ to be fixed after the application period. From Equation (6.14), for $\top > T_j$, where we use the definition of the extended zero-coupon bonds, we get

$$
\begin{aligned}
R_j(\top) &= \frac{1}{\delta_j}\left[\frac{P_{j-1}(\top)}{P_j(\top)} - 1\right] = R_j(\top) = \frac{1}{\delta_j}\left[\frac{P_\top(T_{j-1})}{P_\top(T_j)} - 1\right] = \frac{1}{\delta_j}\left[\frac{C(\top)}{C(T_{j-1})}\frac{C(T_j)}{C(\top)} - 1\right] \\
&= \frac{1}{\delta_j}\left[\frac{C(T_j)}{C(T_{j-1})} - 1\right].
\end{aligned}
$$

6.3.3 Forward-looking forward rates

We already claimed that the generalized FMM models both the FL forward rates and the BL forward rates. In this section, we look to provide a mathematical argument to support this claim. We first recap on FL forward rates.

Definition 6.3. *The FL forward rate $F_j(\top)$ at time $\top \leq T_{j-1}$, by arbitrage-free theory is given by:*

$$
\begin{aligned}
F_j(\top) &= \mathbb{E}^{\mathbb{Q}_{T_j}}\{F(T_{j-1}, T_j)|\mathcal{F}_\top\} = \\
&= \mathbb{E}^{\mathbb{Q}_{T_j}}\{\mathbb{E}^{\mathbb{Q}_{T_j}}\{R(T_{j-1}, T_j)|\mathcal{F}_{T_{j-1}}\}|\mathcal{F}_\top\} \\
&= \mathbb{E}^{\mathbb{Q}_{T_j}}\{F(T_{j-1}, T_j)|\mathcal{F}_\top\} = R_j(\top).
\end{aligned}
$$

From this last equality, we can see that $F_j(\top) = R_j(\top)$ for $\top \leq T_{j-1}$. At time T_{j-1}, $F_j(\top)$ fixes, while $R_j(\top)$ continues its trajectory until it also becomes fixed at time T_j. Finally, $R_j(\top) = R_j(T_j)$, $\top \geq T_j$. So in the new generalized FMM, we can get the dynamics of $R_j(\top)$ over the application period $[0, T_{j-1}]$, which is the same as that of $F_j(\top)$ over the same application period.

6.4 The generalized FMM

In this section we will model the joint evolution of rates $R_j(\top)$ for $j = 1, ..., M$, and introduce a natural extension of the LIBOR Market Model to the case where rates are set in arrears. The advantage of this approach is that both the fixings of the FL and BL rates can be obtained under a single process. Just as in the LMM case, the generalized FMM has the $\mathbb{Q}_{T_j}$ dynamics given by

$$dR_j(\top) = \mu_j(\top)d\top + \hat{\sigma}dW_j(\top), \tag{6.15}$$

where we have the freedom to choose the dynamics of $\hat{\sigma}$. In the previous chapter, we showed that $R_j(\top)$ is a martingale under its associated extended T_j-forward measure (Property i). This implies under that specific measure, the dynamics have a zero drift. We make the assumption that the rates follow a log-normal distribution, just as we did with the LMM. This means that we will be considering the log-normal FMM. We choose the log-normal dynamics because of the possibility of pricing the interest rate derivatives such as caps/floors in the same manner they are priced in the market. We, therefore, choose to specify the volatility parameter in the following manner

$$dR_j(\top) = \sigma_j(\top)R_j(\top)1_{\{\top \leq T_j\}}dW_j(\top) \tag{6.16}$$

where, for each $j = 1, ..., M$, $\sigma_j(\top)$ is an adapted process and $W_j(\top)$ is a standard Wiener such that $dW_i(\top)dW_j(\top) = \rho_{i,j}d\top$. To guarantee that the process is well-defined and remains constant, for times larger than T_j (Property iv), we introduce the indicator function $1_{\{\top \leq T_j\}}$.

Again, $\sigma_j(\top)$ is the instantaneous volatility. It is worth examining the behavior of the instantaneous volatility of the new $R_j(\top)$, with its behavior over the application period $[T_{j-1}, T_j]$ being more interesting. For this examination of the instantaneous volatility, we will consider Figure (6.3). We assume that the application period is 3 months or 90 days. Looking at the upper picture and assuming that we are at T_{j-1}, it means that we are yet to know the overnight rates which we need to compound in order to get the BL rate. Each line represents 5.625 days. In the bottom picture, we already know the overnight rates of 78.75 days and they have been fixed. Since we already know most of the overnight rates, the remaining days will not have an effect much on the realized BL rate at T_j. We see that as we cover more days, the volatility becomes less and less. So we can conclude that the volatility of $R_j(\top)$ should decrease more and more over $[T_{j-1}, T_j]$.

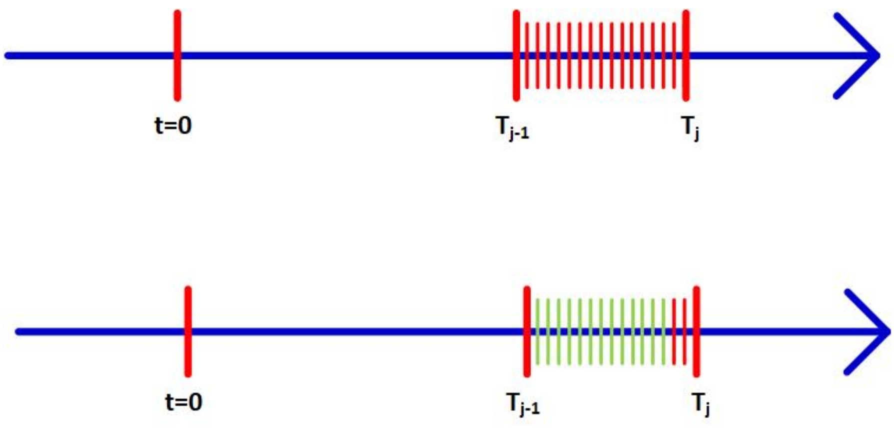


Figure 6.3: Volatility in the application period.

We then choose a piece-wise function g_j such that the dynamics of $R_j(\top)$ are given by

$$dR_j(\top) = \sigma_j(\top)R_j(\top)g_j(\top)dW_j(\top). \tag{6.17}$$

Since we are free to choose this g_j function, we let

$$g_j(\top) = \min\left\{\frac{(T_j - \top)_+}{T_j - T_{j-1}}, 1\right\}. \tag{6.18}$$

The behaviour of this $g_j(\top)$ over $[0, T_j]$ can be described as follows:

(i) For $0 \leq \top < T_{j-1}$, $g_j(\top)$ since $\top$ is much farther away from T_j compared to T_{j-1}. This implies that $R_j(\top) = F_j(\top)$ in this period.

(ii) For $T_{j-1} \leq \top < T_j$, $g_j(\top)$ is monotonically decreasing. This implies that we incorporate the decrease in the volatility of the dynamics already explained above.

(iii) For $\top \geq T_j$, $g_j(\top) = 0$. This means that (Property iv) holds. So we can do away with the indicator function.

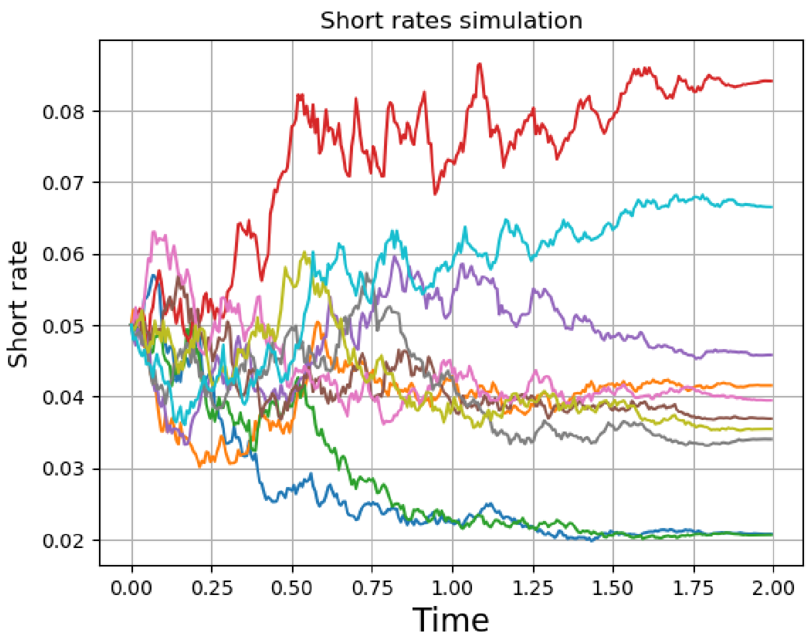


Figure 6.4: 10 paths of the FMM dynamics under the same application period.

In Figure (3.2) we compute 10 paths of $R_j(\top)$. We notice that unlike in the LMM case, these dynamics are defined for any time $\top$. The rate does not stop at $T = 1.75$, but continues to evolve in a stochastic manner until $T = 2$, and proceeds to remain constant thereafter. In this application period, $T = [1.75, 2]$, the effect of the decaying volatility in the rate application period is clearly visible.

6.4.1 Construction of the generalized FMM

The dynamics of $R_j(\top)$ is given by Equation (6.17) under the corresponding T_j-forward measure. A market model where all forward rates, for $j = 1, ..., M$, are modelled jointly can be defined by deriving the dynamics of each forward under a common probability measure. We make use of the formula that relates the drifts of a given process under two measures with known numeraires. In our case, we know the dynamics of $R_j(\top)$ under the T_j-forward measure, and we want to derive its dynamics under a measure $\mathbb{Q}^N$, which is associated with numeraire $N(\top)$. Assuming continuous dynamics, the drift of R_j under $\mathbb{Q}^N$, as a function of time $\top$, is given by:

$$\mu(R_j; \mathbb{Q}^N) = \frac{dR_j(\top) d\ln(N(\top)/P_\top(T_j))}{d\top}. \tag{6.19}$$

Proposition 6.2. *The dynamics of $R_j(\top)$ where the numeraire is the continuous bank account are given by the following SDE:*

$$dR_j(\top) = \sigma_j(\top) g_j(\top) R_j(\top) \sum_{i=1}^{j} \rho_{i,j} \frac{\delta_i \sigma_i(\top) g_i(\top) R_i(\top)}{1 + \delta_i R_i(\top)} d\top + \sigma_j(\top) g_j(\top) R_i(\top) dW_j^{\mathbb{Q}} \tag{6.20}$$

where $W_j^{\mathbb{Q}}$ is a $\mathbb{Q}$-Wiener process.

Proof. We first apply (6.19) to the case where $N(\top) = C(\top)$ and $\mathbb{Q}^N = \mathbb{Q}$ so that the drift of R_j under $\mathbb{Q}$ is given by

$$\mu(R_j;\mathbb{Q})(\top) = \frac{dR_j(\top)d\ln(C(\top)/P_\top(T_j))}{d\top}. \tag{6.21}$$

Thanks to the definition of extended bond prices, we can write:

$$\begin{aligned}
\ln\frac{C(\top)}{P_\top(T_j)} &= \ln\frac{P_\top(0)}{P_\top(T_j)} \\
&= \ln\prod_{i=1}^{j}\frac{P_\top(T_{i-1})}{P_\top(T_i)} \\
&= \ln\prod_{i=1}^{j}[1+\delta_i R_i(\top)] \\
&= \sum_{i=1}^{j}\ln[1+\delta_i R_i(\top)].
\end{aligned}$$

We then have:

$$\begin{aligned}
\mu(R_j;\mathbb{Q}) &= \frac{dR_j(\top)d\sum_{i=1}^{j}\ln[1+\delta_i R_i(\top)]}{d\top} \\
&= \sum_{i=1}^{j}\frac{dR_j(\top)d\ln[1+\delta_i R_i(\top)]}{d\top} \\
&= \sum_{i=1}^{j}\frac{\delta_i}{1+\delta_i R_i(\top)}\frac{dR_j(\top)dR_i(\top)}{d\top}.
\end{aligned}$$

From (6.17), We have that:

$$\mu(R_j;\mathbb{Q}) = \sigma_j(\top)g_j(\top)\sum_{i=1}^{j}\rho_{i,j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_i R_i(\top)}$$

The $\mathbb{Q}$-dynamics of R_j then becomes:

$$dR_j(\top) = \sigma_j(\top)g_j(\top)\sum_{i=1}^{j}\rho_{i,j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}d\top + \sigma_j(\top)g_j(\top)dW_j^{\mathbb{Q}} \tag{6.22}$$

where $W_j^{\mathbb{Q}}$ is a $\mathbb{Q}$-Wiener process.

□

Remark 6.2. *Because of the definition of $g_i(\top)$, which is zero for $\top \geq T_i$, that is for $\eta(\top) > i$, the drift term in* (6.20) *can be defined in terms of the index function $\eta(\top)$. We have*

$$dR_j(\top) = \sigma_j(\top)g_j(\top)\sum_{i=\eta(\top)}^{j}\rho_{i,j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}d\top + \sigma_j(\top)g_j(\top)dW_j^{\mathbb{Q}}. \tag{6.23}$$

Proposition 6.3. *The dynamics of $R_j(\top)$ where the numeraire is the discrete bank account are given by the following SDE:*

$$dR_j(\top) = \sigma_j(\top)g_j(\top)\sum_{i=\eta(\top)+1}^{j}\rho_{i,j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}d\top + \sigma_j(\top)g_j(\top)dW_j^{d} \tag{6.24}$$

where W_j^d is a standard Wiener process under $\mathbb{Q}^d$, and where the drift is zero when $j \leq \eta(\top)$.

Proof. We now apply(6.19) to the case where $N(\top) = C_d(\top)$ and $\mathbb{Q}^N = \mathbb{Q}^d$. The drift of R_j under $\mathbb{Q}^d$ is given by:

$$\begin{aligned}\mu(R_j;\mathbb{Q}^d)(\top) &= \frac{dR_j(\top)d\ln(B_d(\top)/P_\top(T_j))}{d\top} \\ &= \frac{dR_j(\top)d\ln(P(\top,T_{\eta(\top)})/P_\top(T_j))}{d\top}\end{aligned}$$

Using the same steps as we did in the previous section, it can be shown that the $\mathbb{Q}^d$-

dynamics are given by:

$$dR_j(\top) = \sigma_j(\top) g_j(\top) \sum_{i=\eta(\top)+1}^{j} \rho_{i,j} \frac{\delta_i \sigma_i(\top) g_i(\top)}{1+\delta_i R_i(\top)} d\top + \sigma_j(\top) g_j(\top) dW_j^d \tag{6.25}$$

where W_j^d is a standard Wiener process under $\mathbb{Q}^d$, and where the drift is zero when $j \leq \eta(\top)$. □

Remark 6.3. *Comparing the* (6.23) *and* (6.25) *we see that the difference between rate dynamics under* $\mathbb{Q}$ *and* $\mathbb{Q}^d$ *is given by:*

$$\mu(R_j;\mathbb{Q})(\top) - \mu(R_j;\mathbb{Q}^d)(\top) = \frac{\delta_\eta(\top)\sigma_{\eta(\top)}(\top) g_{\eta(\top)}(\top)}{1+\delta_{\eta(\top)} R_{\eta(\top)}(\top)} \sigma_j(\top) g_j(\top). \tag{6.26}$$

In the classic LMM, when approximating $\mathbb{Q}$ *with* $\mathbb{Q}^d$*, one cannot quantify the magnitude of the approximation. The generalized FMM addresses this issue.* (6.26) *gives us the impact of moving from* $\mathbb{Q}$ *to* $\mathbb{Q}^d$*. Note also, since* $g_i(T_i) = 0$*, the* $\mathbb{Q}$*-drift term in* (6.23) *does not experience a jump when time* $\top$ *moves right past* T_i *and index* $\eta(\top)$ *jumps from* i *to* $i+1$*. On the other hand, the* $\mathbb{Q}^d$*-drift in* (6.25) *does jump since it loses the* $(i+1)$*-term.*

Proposition 6.4. *The dynamics of where the numeraire is the zero-coupon bond* $P_\top(T_k)$ *for an arbitrary* k *are given by the following SDE:*

$$dR_j(\top) = \sigma_j(\top) g_j(\top) \sum_{i=K+1}^{j} \rho_{i,j} \frac{\delta_i \sigma_i(\top) g_i(\top)}{1+\delta_i R_i(\top)} d\top + \sigma_j(\top) g_j(\top) dW_j^K \tag{6.27}$$

for $j > k$ *and*

$$dR_j(\top) = -\sigma_j(\top) g_j(\top) \sum_{i=j+1}^{k} \rho_{i,j} \frac{\delta_i \sigma_i(\top) g_i(\top)}{1+\delta_i R_i(\top)} d\top + \sigma_j(\top) g_j(\top) dW_j^K \tag{6.28}$$

when $j < k$ and where W_j^k is a standard Wiener process under $\mathbb{Q}^{T_k}$.

Finally, when $N(\top) = P_\top(T_k)$ and $\mathbb{Q}^N = \mathbb{Q}^{T_k}$ the drift of R_j under $\mathbb{Q}^{T_k}$ is given by:

$$\mu(R_j;\mathbb{Q}^{T_k})(\top) = \frac{dR_j(\top) d\ln(P_\top(T_k)(\top)/P_\top(T_j))}{d\top}. \tag{6.29}$$

Repeating the same procedure as before, we then get

$$dR_j(\top) = \sigma_j(\top)g_j(\top)\sum_{i=K+1}^{j}\rho_{i,j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}d\top + \sigma_j(\top)g_j(\top)dW_j^K \tag{6.30}$$

for $j > k$ and

$$dR_j(\top) = -\sigma_j(\top)g_j(\top)\sum_{i=j+1}^{k}\rho_{i,j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}d\top + \sigma_j(\top)g_j(\top)dW_j^K \tag{6.31}$$

when $j < k$ and where W_j^k is a standard Wiener process under $\mathbb{Q}^{T_k}$.

Remark 6.4. *Since $\mathbb{Q}^{T_k}$ is a hybrid measure that consists of the classic T_k-forward measure until time T_k and of the risk-neutral measure $\mathbb{Q}$ after T_k, we should have:*

$$\mu(R_j;\mathbb{Q}^{T_k}(\top)) = \mu(R_j;\mathbb{Q})(\top) \ for\ \top \geq T_k.$$

In fact, using the drifts formulas from the previous sections, we get for $\top \geq T_k$ and $j > k$

$$\begin{aligned}\mu(R_j,\mathbb{Q})(\top) &= \sigma_j(\top)g_j(\top)\sum_{i=1}^{j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}\rho_{i,j}\\ &= \sigma_j(\top)g_j(\top)\sum_{i=k+1}^{j}\frac{\delta_i\sigma_i(\top)g_i(\top)}{1+\delta_iR_i(\top)}\rho_{i,j} = \mu(R_j,\mathbb{Q}^{T_k})(\top)\end{aligned}$$

where we used the property that $g_i(\top) = 0$ for $\top > T_i$. In particular, when $k = 0$, we *can confirm that*

$$\mu(R_j, \mathbb{Q}^{T_d})(\top) = \mu(R_j, \mathbb{Q})(\top).$$

6.5 Differences between FMM and LMM

The FMM not only retains all the key properties of the LMM that made it favored by market participants in the first place, but it also extends on it by describing the joint dynamics of both FL rates $F_j(\top)$ and BL forward rates $R_j(\top)$, since they are the same for all times $\top$ outside the application period of $F_j(\top)$. This extended model also has additional properties summarized below:

(i) **Model Completeness**

The generalized forward rates $R_j(\top)$ are distinguished from IBORs by their property of being complete in terms of being defined for all times $\top$ on the time grid $[T_0, ..., T_m]$, including the application period. For any $j \in \{1, \ldots, M\}$ and for any time $\top$, the price of zero-coupon bond with maturity T_j can be defined in terms of the cash account $C(\top)$ and forward rates as follows:

$$P_\top(T_j) = C(\top) \prod_{i=1}^{j} \frac{1}{1 + \delta_i R_i(\top)} \tag{6.32}$$

with the equality holding for all $\top$, including $\top > T_j$ which implies

$$\frac{dP_\top(T_j)}{P_\top(T_j)} = r(\top)d\top - \sum_{i=1}^{j} \frac{\delta_i}{1 + \delta_i R_i(\top)} \sigma_i(\top) g_i(\top) dW_i^{\mathbb{Q}}(\top). \tag{6.33}$$

So the volatility of all bonds $P_\top(T_j)$ is known and is a function of rates as well as their instantaneous covariance structure. Analogous representations are not available in the LMM when using FL rates $F_j(\top)$. It is for this reason that we have a closed-form drift term representation under the $\mathbb{Q}$-measure for FMM, but not for LMM. In this respect, FMM is a complete model while LMM is not.

(ii) **Better pricing of futures contracts**

The fact that in this framework, the $\mathbb{Q}$-dynamics of $R_j(\top)$ are available presents a number of advantages in terms of the pricing of contracts. The time-$\top$ future price of a contract that has a payoff of H_T at time $T > \top$ is defined by:

$$f(\top) = \mathbb{E}^{\mathbb{Q}}\{H_T|\mathcal{F}_\top\}.$$

Since no $\mathbb{Q}$-dynamics are directly available in a classic LMM, one typically approximates $\mathbb{Q}$ with $\mathbb{Q}^d$ to explicitly calculate the future price $f(\top)$:

$$f(\top) \approx \mathbb{E}^{\mathbb{Q}^d}\{H_T|\mathcal{F}_\top\}$$

with $\mathbb{E}^{\mathbb{Q}^d}$ denoting the expectation under $\mathbb{Q}^d$. We no longer need this approximation for the FMM, since we know the $\mathbb{Q}$-dynamics of $R_j(\top)$, so the former formula can be used.

6.5.1 More natural hybrid modelling

For a hybrid equity-IR model, the equity risk-neutral drift rate is equal to $r(\top)$, which is not known in the classic LMM, and nor are its integrals. When using an FMM instead, we can express the integral of the drift rate in terms of rates $R_j(\top)$, which are by definition known in the model. To show this, assume that the $\mathbb{Q}$-dynamics of a given stock S is:

$$dS(\top) = r(\top)S(\top)d\top + \sigma S(\top)dW(\top).$$

Then, for each pair of indexes $j < k$, we can write:

$$\begin{aligned} S(T_k) &= S(T_j) e^{\int_{T_j}^{T_k}} e^{-\frac{1}{2}\sigma^2(T_k - T_j) + \sigma[W(T_k) - W(T_j)]} \\ &= S(T_j) \prod_{i=j+1}^{k} e^{\int_{T_j}^{T_k}} e^{-\frac{1}{2}\sigma^2(T_k - T_j) + \sigma[W(T_k) - W(T_j)]} \\ &= S(T_j) \prod_{i=j+1}^{k} e^{\int_{T_j}^{T_k}} e^{-\frac{1}{2}\sigma^2(T_k - T_j) + \sigma[W(T_k) - W(T_j)]}. \end{aligned} \tag{6.34}$$

Therefore, $S(\top)$ can be simulated by simulating the joint evolution of rates $R_j(\top)$ and Wiener process $W(\top)$, assuming a correlation structure among them.

6.6 Pricing in the FMM

6.6.1 The valuation of an RFR fixed-floating swap

We consider a swap that has the floating leg payments at each time $T_j,\ j = \phi+1, \ldots,\ \eta$, a rate that is obtained by compounding the daily fixings of the RFR from T_{j-1} to T_j, and where the fixed leg pays the fixed rate κ on dates $T'_{c+1}, \ldots, T'_d$, with $T'_c = T_\phi$ and $T'_d = T_\eta$. We approximate the floating-leg payment at each T_j by $R(T_{j-1}, T_j)$ times the corresponding year fraction, that is

$$\delta_j R(T_{j-1}, T_j) = e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1.$$

The swap value to the fixed-rate payer at time $\top < T_{\phi+1}$ is thus given by:

$$\sum_{j=\phi+1}^{b} \delta_j P_\top(T_j) R_j(\top) - \kappa \sum_{j=\phi+1}^{d} \delta'_j P_\top(T'_j)$$

where δ'_j denotes the year fraction for the fixed-leg interval $[T'_{j-1}, T'_j)$. Note that for $\top \leq T_\phi$, the price of the swap remains the same when we switch from FL to BL rates. The corresponding forward swap rate is defined as the fixed rate κ such that the swap has a zero value at time $\top$, that is

$$S(\top) = \frac{\sum_{j=\phi+1}^{\eta} \delta_j P_\top(T_j) R_j(\top)}{\sum_{j=c+1}^{d} \delta'_j P_\top(T'_j)} = \frac{P_\top(T_\phi) - P_\top(T_\eta)}{\sum_{j=c+1}^{d} \delta'_j P_\top(T'_j)}$$

where the second equality follows from (6.14). This formula coincides with the classic one in the single-curve framework.

Remark 6.5. *Since the extended zero-coupon prices $P_\top(T_j)$, and forward rates $R_j(\top)$ are defined for all values of time $\top$, the swap price formula above is also defined for all values of $\top$. Similar to the case of a zero-coupon bond, for $\top > T_\phi$ the swap value given by the formula above represents the time-$\top$ value of a self-financing investment strategy where all cash flows are reinvested at the risk-free rate to roll them forward to the present time. This can be considered as a total present value of the strategy, which is inclusive of past cash flows, and can be used to compare the current performance of different investments.*

6.6.2 The valuation of an Risk Free Rate cap

In the generalized FMM framework, we have two types of caplets that we can construct for every application period $[T_{j-1}, T_j]$. The first one is just like the caplet used in the LMM framework, which is called the FL caplet. Given a strike price κ, this caplet has a time T_j payoff given by

$$[R(T_{j-1}; T_{j-1}, T_j) - \kappa]_+. \tag{6.35}$$

We showed that $R_j(\top) = F_j(\top)$ for $\top \leq T_{j-1}$, so we can model this caplet in the same way as modelled the LMM caplet. From property ii of $R_j(\top)$, we get the assurance that this caplet will be the same as the caplet observed in today's market at T_{j-1}.

The second caplet is known as a BL caplet. Given a strike price κ, this caplet has a time

T_j payoff given by

$$[R(T_j; T_{j-1}, T_j) - \kappa]_+. \tag{6.36}$$

In this caplet, the payoff is only known at time T_j, which is the payoff date. There is more uncertainty with the payoff of this type of caplet because the backward rate evolves over a longer period of time.

The difference in the two payoffs lies in that the FL caplet is known at T_{j-1}, whereas the BL caplet is known at T_j. The valuation of the two caplets is dependent on the modelling of the forward rate $R_j(\top)$ in the $\mathbb{Q}^{T_j}$- measure. Since the rate for the BL caplet is known only at T_j, it means that there is more uncertainty with this caplet and therefore, the price of this caplet should be higher than that of the FL caplet. We can also prove this using the tower property of conditional expectations and the Jensen inequality. For $\top \leq T_{j-1}$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_{T_j}}\{(R_j(T_j) - \kappa)_+ | \mathcal{F}_\top\} &= \mathbb{E}^{\mathbb{Q}_{T_j}}\{\mathbb{E}^{\mathbb{Q}_{T_j}}[(R_j(T_j) - \kappa)_+ | \mathcal{F}_{\mathcal{T}_{|-\infty}}]\mathcal{F}_\top\} \\
&\geq \mathbb{E}^{\mathbb{Q}_{T_j}}\{\mathbb{E}^{\mathbb{Q}_{T_j}}\{(R_j(T_j)\mathcal{F}_{T_{j-1}}) - \kappa)\}_+ \mathcal{F}_\top\} \\
&= \mathbb{E}^{\mathbb{Q}_{T_j}}\{(R_j(T_{j-1}) - \kappa))_+ \mathcal{F}_\top\}
\end{aligned}$$

where we applied the martingale property of R_j, that is: $\mathbb{E}^{\mathbb{Q}_{T_j}}[R_j(T_j)|\mathcal{F}_{T_{j-1}}] = R_j(T_{j-1})$. This proves that the payoff of the caplet obtained using the BL rate is higher than that of the caplet that is given by the FL rate. For example, let us choose the dynamics of (6.17) to be lognormal with constant volatility, which with some abuse of notation we denote by σ_j:

$$dR_j(\top) = \sigma_j R_j(\top) g_j(\top) dW_j(\top) \tag{6.37}$$

where we assume the function g_j to be the piece-wise linear function (6.18). These dynamics result in Black-like prices for both caplets. In fact, let us denote the price at time $\top$ of the FL and BL caplets by $C_j^F(\top)$ and $C_j^B(\top)$, respectively. Then, we denote the Black forward price of the caplet by

$$\text{Black}(R, \kappa, \hat{\sigma}) = R\phi\left(\frac{\ln(R/\kappa) + \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}\right) - \kappa\phi\left(\frac{\ln(R/\kappa) - \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}\right)$$

and by ϕ the cdf, we have:

$$C_j^F(\top) = P_j(\top)\text{Black}(R_j(\top), \kappa, \sum_j^F \sqrt{T_{j-1}})$$
$$C_j^B(\top) = P_j(\top)\text{Black}(R_j(\top), \kappa, \sum_j^R \sqrt{T_j})$$

where

$$\sum_j^F = \sigma_j$$
$$\sum_j^R = \sigma_j\sqrt{\frac{1}{3} + \frac{2}{3}\frac{T_{j-1}}{T_j}}.$$

The value of $\sum_j^R$ is obtained by calculating the integrated variance of $R_j(\top)$ up to time T_j:

$$\left(\sum_j^R\right)^2 T_j = \sigma_j^2 T_{j-1} + \int_{T_{j-1}}^{T_j} \sigma_j^2 \left(\frac{T_j - \top}{T_j - T_{j-1}}\right)^2 d\top.$$

Since $(\sum_j^R)^2 T_j \geq \sigma_j^2 T_{j-1}$, we can confirm that $C_j^B(\top) \geq C_j^F(\top)$.

	Backward-looking	Forward-looking
	0.00086	0.00066
	0.00114	0.00010
	0.00137	0.00120
	0.00176	0.00157
	0.00179	0.00167
	0.00185	0.00170
	0.00200	0.00197
	0.00195	0.00187
	0.00204	0.00195
Cap	0.0257	0.0251

Table 6.1: Forward-looking caplets vs Backward-looking caplets.

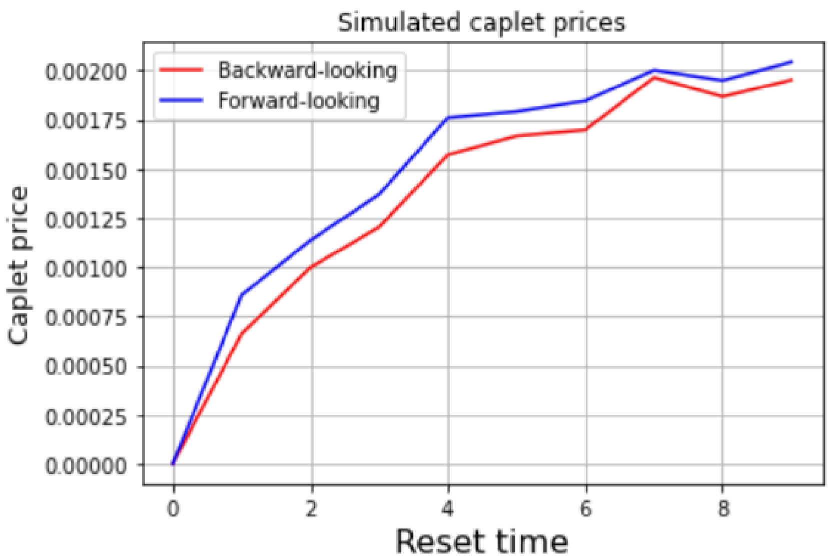


Figure 6.5: Forward-looking caplets vs Backward-looking caplets.

In Table (6.1) and Figure (6.5), we evaluated the payoff of 9 caplets using the analytical Black76 formula and simulated payoff as we did in the LMM case. The caplets are of different maturities, starting from 1 to 9, where the notional is 1 and $\kappa = 0.05$. We used 2000 Monte Carlo simulations. As we move forward in time, the caplet payoffs increase. The BL rates yielded payoffs that were higher than the FL caplets. This is consistent with the theory discussed above.

6.6.3 The valuation of a Risk-Free rate ratchet and sticky cap

In Chapter 5, we introduced ratchet and sticky caps. Since the FMM is just an improvement of the LMM, it can therefore be used to price these exotic options. Since in this framework we have two types of caps, it implies that we also have two types of ratchet and sticky caps. The first ratchet cap is a FL ratchet-cap, which is analogous to the ratchet cap defined in the LMM framework. The second type is a BL ratchet cap.

Definition 6.4. *An interest rate ratchet cap is a contract consisting of a series of caplets paying at* $\{T_{\alpha+1}, \dots, T_\beta\}$ *where the strike is given by* $R(T_{i-2}, T_{i-1})+\epsilon$. *In this case,* $\epsilon \in \mathbb{R}$ *and it is a specified spread. The discounted payoff is*

$$\sum_{i=\alpha+1}^{\beta} ND_f(\top, T_i)\delta_i(R(T_{i-1}, T_i) - (R(T_{i-2}, T_{i-1}) + \epsilon))_+.$$

Definition 6.5. *An interest rate sticky cap is a contract consisting of a series of caplets paying at* $\{T_{\alpha+1}, \dots, T_\beta\}$ *where the strike is given by* $(\min(R(T_{i-2}, T_{i-1}), \kappa) + \epsilon)$. *In this case,* $\epsilon \in \mathbb{R}$ *and it is a specified spread. The discounted payoff is*

$$\sum_{i=\alpha+1}^{\beta} ND_f(\top, T_i)\delta_i(R(T_{i-1}, T_i) - (\min(R(T_{i-2}, T_{i-1}), \kappa) + \epsilon))_+.$$

	Ratchet	Sticky
	0.00025	0.00031
	0.00062	0.00069
	0.00088	0.00101
	0.00101	0.00118
	0.00135	0.00159
	0.00143	0.00165
	0.00160	0.00185
	0.00177	0.00205
	0.00200	0.00229
Cap	0.0251	0.0257

Table 6.2: Pricing of the caps using the FMM.

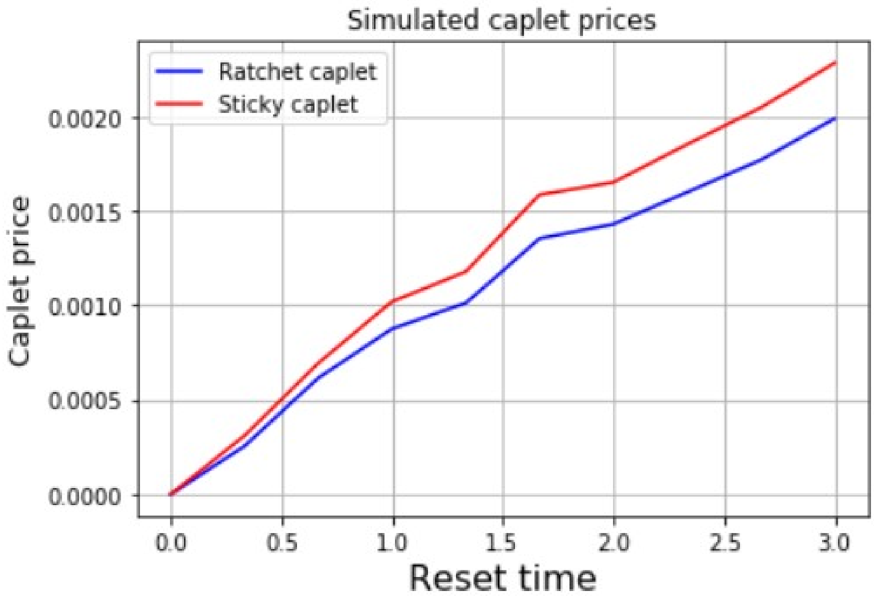


Figure 6.6: Forward-looking caplets vs Backward-looking caplets.

In Table (6.2) and Figure (6.6) we simulated both ratchet and sticky caplet prices for different maturities. The caplets are of different maturities, starting three months from now until 5 years. We set $\kappa = 0.05$ and $\epsilon = 0.01$ and used 2000 Monte Carlo simulations. Just like the standard caplets, as the maturities increase, both the caplet payoff increase. The sticky caplets are generally higher than the ratchet caplets. We notice that both these caplets have payoffs that are less than that given by the standard caplet payoffs.

6.6.4 The valuation of a RFR swaption

In this section, we address the RFR swaptions. We first define the swaption in the FMM framework.

Definition 6.6. *An RFR swaption, either payer or receiver, can be defined as the option to enter a spot RFR swap on the swaption's maturity date. We denote by T_ϕ the swaption's maturity, by κ the strike price, $T_j,\ j = \phi + 1, \ldots, \eta$ the floating-leg dates,* *and by $T'_{c+1}, \ldots, T'_d$ the fixed-leg ones. We assume $T'_c = T_\phi$ and $T'_d = T_\eta$. The (payer) swaption payoff at time T_ϕ is thus given by:*

$$[S(T_\phi) - \kappa]_+ \sum_{j=c+1}^{d} \delta'_j P_t(T'_j)$$

where

$$S(\top) = \frac{\sum_{j=\phi+1}^{\eta} \delta_j P_t(T_j) R_j(\top)}{\sum_{j=c+1}^{d} \delta'_j P_t(T'_j)} = \frac{P_t(T_\phi) - P_t(T_\eta)}{\sum_{j=c+1}^{d} \delta'_j P_t(T'_j)}$$

Similar to the LIBOR-based swap rate, an RFR is a martingale under the forward swap measure associated with annuity numeraire, $\sum_{j=c+1}^{d} \delta'_j P_\top(T'_j)$. When valuing swaptions in the FMM, where each forward rate evolves according to (6.17), is equivalent to the valuation of LIBOR-based swaptions in the single-curve LMM.

In this chapter, we gave an introduction to the generalized FMM. In the LIBOR transition, we noticed that the rates do not stop evolving at the beginning of their application period, but continue to evolve until the end of the application period. The FMM models these new rates, called the Backward-looking rates. The concept of the extended zero coupon bond is the main foundation of this new model. In this chapter, we first gave a definition of the BL rate and then gave a mathematical expression of it. We discussed the properties of this new BL rate. A key takeaway from the theory we discussed in this chapter is that the FL forward rates and the BL forward rates were equal for all times $\top < T_{j-1}$ for a given application period $[T_{j-1}, T_j]$ and the FL rate froze at T_{j-1}, which is the start of the application period while the BL rates continued to evolve until the end of the application period.

We also derived the dynamics of the forward rates under three different measures, one of the measures being the risk-neutral measure. In the LIBOR framework, we could not derive the dynamics of the rates under the risk-neutral measure. We also saw that in the application period, the volatility decreases as we move closer to T_j. The FMM incorporates this with the g function. We concluded the chapter by pricing caplets using 2000 Monte Carlo simulations. In the FMM framework, two types of caps exist, the FL cap and the BL cap. The latter can be shown that it is more expensive than the FL cap. We also introduced swaptions in the new FMM framework. In the next chapter, we will compare all caps using more Monte Carlo simulations and also under different times. We will also compare swaptions using both the FL rates and the BL rates.

CHAPTER 7

Numerical Results

This chapter focuses on determining the prices of the various derivatives introduced in the preceding chapters. We first consider the standard caplets and simulate the payoff using the FL rates and the BL rates. We then look at the nonstandard caplets and simulate their payoffs using the FL and the BL rates. We conclude by comparing the payoffs of swaptions given by the FL rates and the BL rates.

7.1 Valuation of the standard caps

In this section, we compare the caplet payoff given by the FL and the BL rates. We compare the results implied by the FMM and the analytical Black76 formula. We compare the results for two tenors, $\mathbb{T} = 5$ and $\mathbb{T} = 10$, with the rates resetting every three months.

Caplet	Maturity	Forward caplets	Backward Caplets	CB	CF
	0.5	0.00072	0.00094	0.00106	0.00088
	0.75	0.00104	0.00118	0.00128	0.00116
	1	0.00131	0.00145	0.00154	0.00141
	1.25	0.00151	0.00163	0.00170	0.00160
	1.5	0.00165	0.00178	0.00185	0.00173
	1.75	0.00183	0.00194	0.00200	0.00189
	2	0.00196	0.00203	0.00209	0.00202
	2.25	0.00219	0.00228	0.00233	0.00225
	2.5	0.00217	0.00224	0.00229	0.00223
	2.75	0.00239	0.00248	0.00252	0.00244
	3	0.00241	0.00245	0.00250	0.00246
	3.25	0.00254	0.00260	0.00265	0.00259
	3.5	0.00256	0.00261	0.00266	0.00260
	3.75	0.00259	0.00266	0.00270	0.00264
	4	0.00278	0.00285	0.00289	0.00282
	4.25	0.00277	0.00283	0.00287	0.00281
	4.5	0.00291	0.00297	0.00301	0.00295
	4.75	0.00299	0.00306	0.00310	0.00302
	5	0.00291	0.00293	0.00296	0.00295
Cap		0.041248	0.042886	0.043997	0.042462

Table 7.1: FL vs BL caplets where $\mathbb{T} = 5$.

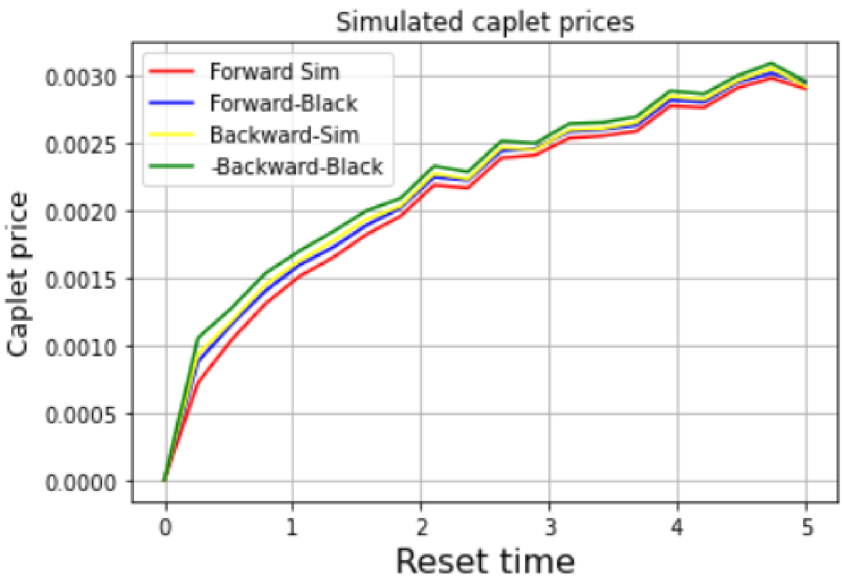


Figure 7.1: FL vs BL caplets where $\mathbb{T} = 5$.

In Table (7.1 and Figure (7.1, we compared the FL caplets with the BL caplets using the Black76 formula and the option payoff definition. After 10,000 trials, we see that the BL caplets are higher than the FL caplets. As the maturities increase, so do the caplet prices. This can be attributed to the fact that the rates evolve over the accrual period as well.

Caplet	Maturity	Forward Caplets	Backward Caplets	CF	CB
	0.5	0.00100	0.00112	0.00127	0.00136
	0.75	0.00150	0.00159	0.00173	0.00180
	1	0.00201	0.00208	0.00216	0.00222
	1.25	0.00252	0.00257	0.00265	0.00271
	1.5	0.00270	0.00276	0.00283	0.00288
	1.75	0.00306	0.00310	0.00318	0.00322
	2	0.00374	0.00379	0.00376	0.00381
	2.25	0.00387	0.00391	0.00404	0.00408
	2.5	0.00409	0.00413	0.00425	0.00429
	2.75	0.00399	0.00403	0.00421	0.00425
	3	0.00433	0.00436	0.00453	0.00456
	3.25	0.00512	0.00516	0.00519	0.00522
	3.5	0.00525	0.00528	0.00537	0.00540
	3.75	0.00542	0.00545	0.00549	0.00552
	4	0.00459	0.00462	0.00462	0.00465
	4.25	0.00487	0.00490	0.00498	0.00501
	4.5	0.00540	0.00544	0.00554	0.00558
	4.75	0.00567	0.00569	0.00591	0.00594
	5	0.00679	0.00683	0.00696	0.00698
	5.25	0.00594	0.00596	0.00600	0.00603
	5.5	0.00596	0.00598	0.00603	0.00605
	5.75	0.00617	0.00619	0.00633	0.00636
	6	0.00669	0.00672	0.00646	0.00648
	6.25	0.00595	0.00598	0.00610	0.00612
	6.5	0.00589	0.00591	0.00602	0.00604
	6.75	0.00648	0.00651	0.00643	0.00646
	7	0.00658	0.00660	0.00670	0.00672
	7.25	0.00622	0.00625	0.00659	0.00661
	7.5	0.00602	0.00605	0.00600	0.00602
	7.75	0.00656	0.00659	0.00660	0.00662
	8	0.00579	0.00582	0.00593	0.00595
	8.25	0.00560	0.00562	0.00565	0.00567
	8.5	0.00559	0.00562	0.00561	0.00563
	8.75	0.00572	0.00574	0.00563	0.00565
	9	0.00558	0.00560	0.00565	0.00567
	9.25	0.00683	0.00685	0.00680	0.00682
	9.5	0.00559	0.00562	0.00547	0.00549
	9.75	0.00515	0.00517	0.00525	0.00527
	10	0.00524	0.00523	0.00533	0.00535
Cap		0.19547	0.19682	0.199253	0.20049

Table 7.2: Forward-looking vs Backward-looking standard caplets where $T = 10$.

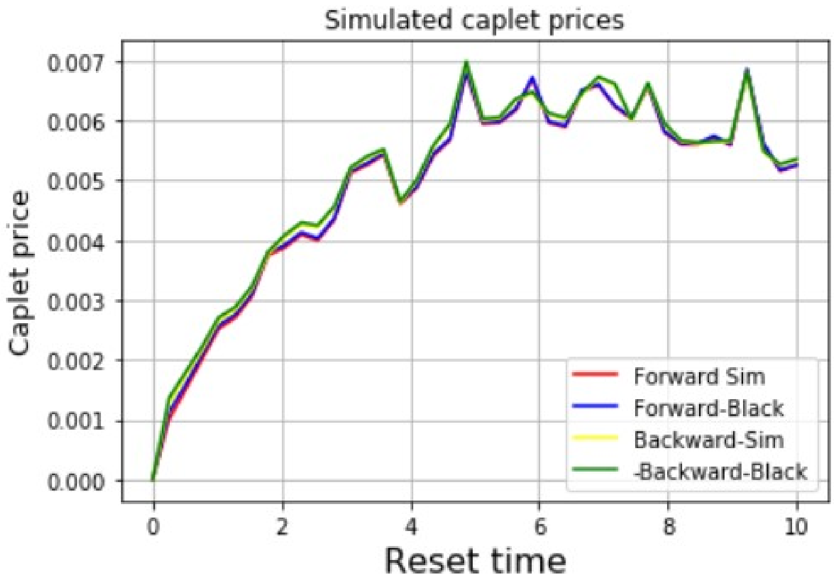


Figure 7.2: FL vs BL caplets where $\mathbb{T} = 10$.

In Table (7.2) and Figure (7.2), we compared the FL caplets with the BL caplets using the Black76 formula and the option payoff definition. After 10,000 trials, we see that the BL caplets are higher than the FL caplets. As the maturities increase, so do the caplet prices. This can also be attributed to the fact that the rates evolve over the accrual period as well.

7.2 Valuation of the nonstandard caps

In this section, we compare the nonstandard caplet payoff given by the FL and the BL rates. We compare the results implied by the FMM and the analytical Black76 formula. We compare the results for two tenors, $\mathbb{T} = 5$ and $\mathbb{T} = 10$, with the rates resetting every three months.

Caplet	Maturity	Forward Ratchet	Backward Ratchet	Forward Sticky	Backward Sticky
	0.5	0.00028	0.00028	0.00068	0.00079
	0.75	0.00088	0.00103	0.00119	0.00139
	1	0.00134	0.00156	0.00159	0.00185
	1.25	0.00164	0.00192	0.00186	0.00218
	1.5	0.00203	0.00239	0.00221	0.00259
	1.75	0.00227	0.00265	0.00241	0.00281
	2	0.00259	0.00300	0.00276	0.00319
	2.25	0.00284	0.00328	0.00300	0.00345
	2.5	0.00306	0.00354	0.00322	0.00368
	2.75	0.00316	0.00365	0.00332	0.00381
	3	0.00342	0.00389	0.00354	0.00403
	3.25	0.00351	0.00397	0.00364	0.00412
	3.5	0.00381	0.00430	0.00397	0.00448
	3.75	0.00402	0.00456	0.00418	0.00472
	4	0.00397	0.00449	0.00407	0.00458
	4.25	0.00426	0.00478	0.00437	0.00488
	4.5	0.00427	0.00479	0.00440	0.00494
	4.75	0.00431	0.00483	0.00444	0.00497
	5	0.00440	0.00491	0.00449	0.00500
Cap		0.041248	0.042886	0.043997	0.042462

Table 7.3: FL vs BL caplets where $\mathbb{T} = 5$.

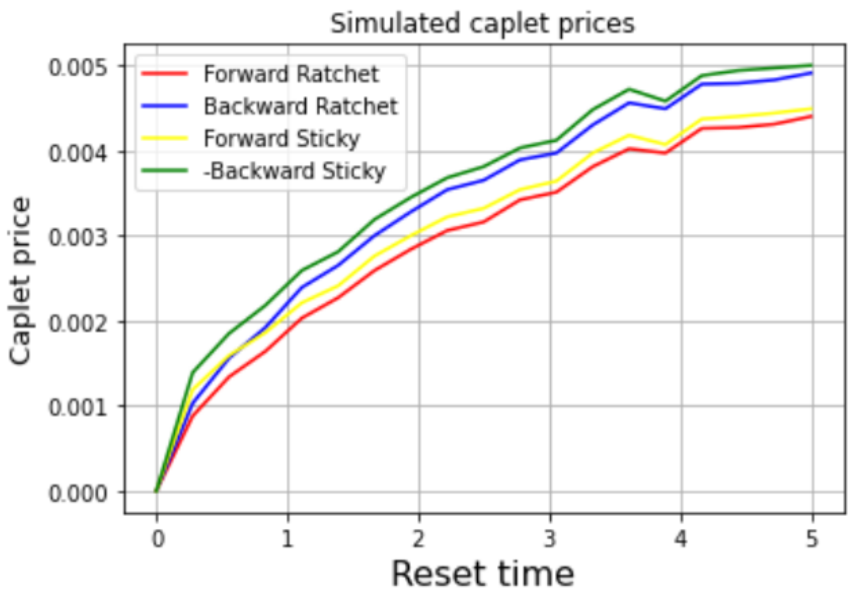


Figure 7.3: FL vs BL caplets where $\mathbb{T} = 5$.

In Table (7.3) and Figure (7.3) we evaluated Ratchet and sticky caplet prices where the tenor is $\mathbb{T} = 5$. We set $\kappa = 0.05$ and $\epsilon = 0.01$ and 10 000 Monte Carlo simulations. We notice that the BL sticky caplets yield prices that are more than those of the FL sticky caplets and the ratchet caplets. Just as in the case of the standard caplets, as we move through the tenor points, the payoffs increase. We notice that these nonstandard caplets yield payoffs that are less than those of the standard caplets at the corresponding tenor points.

Caplet	Maturity	Forward Ratchet	Backward Ratchet	Forward Sticky	Backward Sticky
	0.5	0.000296	0.00110	0.00071	0.00147
	0.75	0.00081	0.00162	0.00111	0.00192
	1	0.00133	0.00217	0.00160	0.00246
	1.25	0.00167	0.00260	0.00194	0.00283
	1.5	0.00230	0.00327	0.00246	0.00343
	1.75	0.00263	0.00378	0.00288	0.00402
	2	0.00264	0.00379	0.00295	0.00410
	2.25	0.00314	0.00425	0.00338	0.00442
	2.5	0.00326	0.00453	0.00337	0.00462
	2.75	0.00364	0.00499	0.00372	0.00503
	3	0.00430	0.00567	0.00443	0.00582
	3.25	0.00414	0.00551	0.00428	0.00571
	3.5	0.00406	0.00526	0.00405	0.00526
	3.75	0.00416	0.00526	0.00445	0.00566
	4	0.00427	0.00562	0.00451	0.00580
	4.25	0.00469	0.00596	0.00480	0.00607
	4.5	0.00527	0.00676	0.00538	0.00685
	4.75	0.00515	0.00643	0.00531	0.00659
	5	0.00549	0.00674	0.00574	0.00696
	5.25	0.00556	0.00670	0.00566	0.00678
	5.5	0.00566	0.00707	0.00583	0.00723
	5.75	0.00566	0.00708	0.00575	0.00718
	6	0.00590	0.00729	0.00593	0.00729
	6.25	0.00578	0.00706	0.00586	0.00715
	6.5	0.00545	0.00657	0.00560	0.00668
	6.75	0.00620	0.00728	0.00622	0.00732
	7	0.00603	0.00742	0.00616	0.00757
	7.25	0.00596	0.00711	0.00603	0.00715
	7.5	0.00574	0.00686	0.00596	0.00707
	7.75	0.00663	0.00777	0.00676	0.00793
	8	0.00645	0.00751	0.00646	0.00750
	8.25	0.00665	0.00766	0.00675	0.00772
	8.5	0.00632	0.00725	0.00646	0.00734
	8.75	0.00661	0.00743	0.00672	0.00756
	9	0.00611	0.00697	0.00601	0.00684
	9.25	0.00641	0.00723	0.00654	0.00734
	9.5	0.00618	0.00694	0.00624	0.00703
	9.75	0.00637	0.00716	0.00652	0.00703
	10	0.00579	0.00645	0.00589	0.00703
Cap		0.18051	0.22288	0.18587	0.22792

Table 7.4: FL vs BL nonstandard caplets where $\mathbb{T} = 10$.

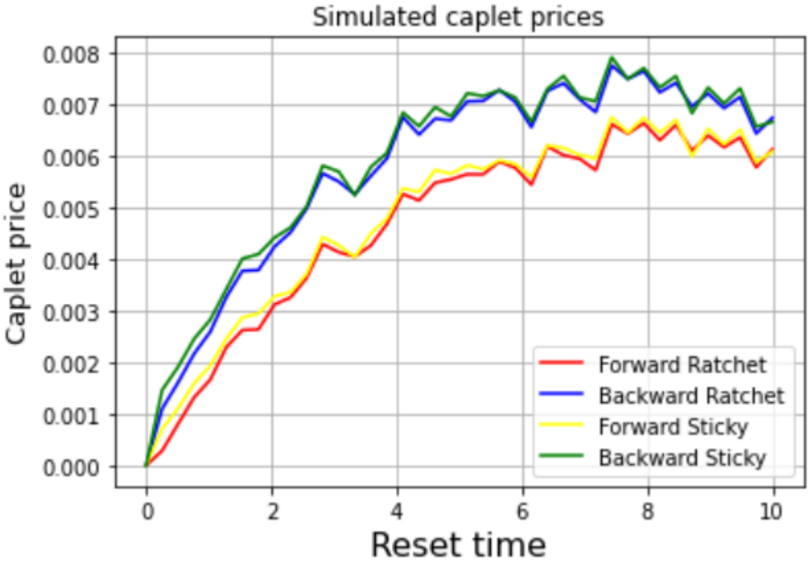


Figure 7.4: FL vs BL nonstandard caplets where $\mathbb{T} = 10$.

In Table (7.4) and Figure (7.4) we evaluated Ratchet and sticky caplet prices where the tenor is $\mathbb{T} = 10$. We set $\kappa = 0.05$ and $\epsilon = 0.01$ and 10 000 Monte Carlo simulations. We notice that the BL sticky caplets yield prices that are more than those of the FL sticky caplets and the ratchet caplets. Just as in the case of the standard caplets, as we move through the tenor points, the payoffs increase. We notice that these nonstandard caplets yield payoffs that are less than those of the standard caplets at the corresponding tenor points.

7.3 Valuation of swaptions

In this section, we compare the swaption payoffs given by the FL rates and the BL rates with tenors from 1 to 10 years and swaption maturities from 1 to 5 years.

Length (Years)	Swaption maturity (Years)				
	1	2	3	4	5
1	0.33877	0.28893	0.23752	0.19252	0.16537
2	0.62753	0.54543	0.49127	0.36536	0.31860
3	0.88480	0.78203	0.71550	0.52729	0.46591
4	1.12042	1.00427	1.00027	0.68316	0.61411
5	1.34470	1.22486	1.15817	0.84090	0.76696
6	1.56379	1.44671	1.39136	1.00041	1.0001
7	1.78649	1.68069	1.64532	1.17106	1.11099
8	2.02093	1.93380	1.92843	1.36121	1.31995
9	2.27243	2.21902	1.83419	1.57726	1.56915
10	2.55804	2.31424	2.10183	1.83520	1.67802

Table 7.5: A table of swaption payoffs given by the forward-looking rates.

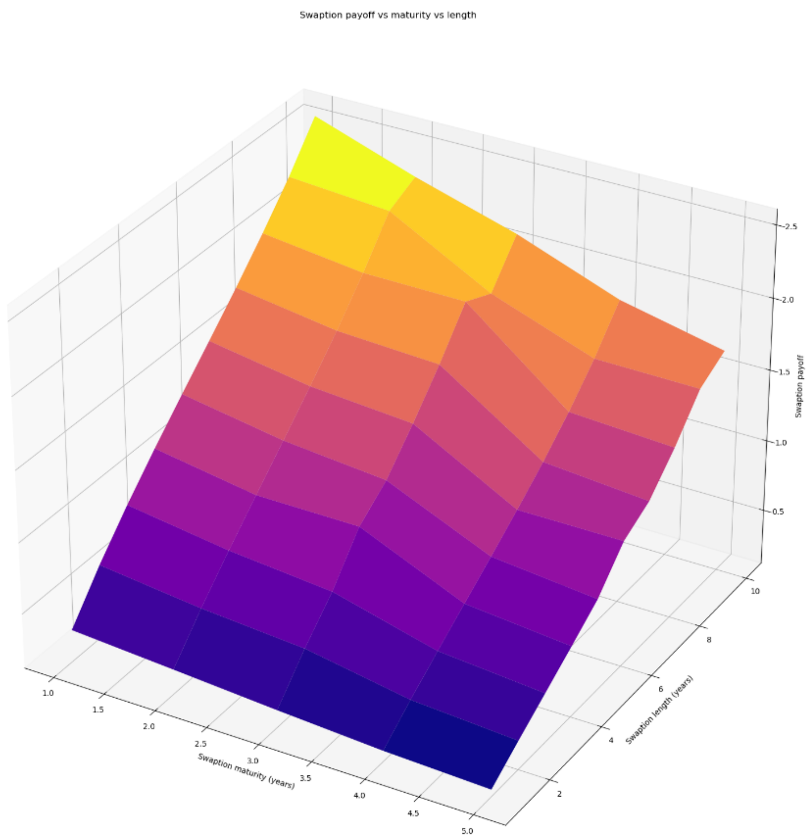


Figure 7.5: A graph of swaption payoffs given by the forward-looking rates.

In Table (7.5) and Figure (7.5), we set the $\kappa = 0.05$ and $\sigma = 0.20$ and conducted

10 000 Monte Carlo trials. We notice that the swaption payoff decreases as maturity increases. For a given maturity, the swaption payoff increases with the tenor.

	Swaption maturity (Years)				
Length (Years)	1	2	3	4	5
1	0.28739	0.24629	0.21099	0.18577	0.16099
2	0.53827	0.46397	0.40065	0.35756	0.31147
3	0.76464	0.66062	0.57967	0.52027	0.45507
4	0.96968	0.84752	0.75465	0.67883	0.59693
5	1.17136	1.02728	0.92272	0.83359	0.74086
6	1.36859	1.20817	1.10142	0.99701	0.88871
7	1.56431	1.39322	1.29028	1.16611	1.05054
8	1.76654	1.58401	1.42663	1.34903	1.22460
9	2.00247	1.79920	1.63372	1.55559	1.42469
10	2.23389	2 .04676	1.87311	1.78794	1.65019

Table 7.6: A table of swaption payoffs given by the backward-looking rates.

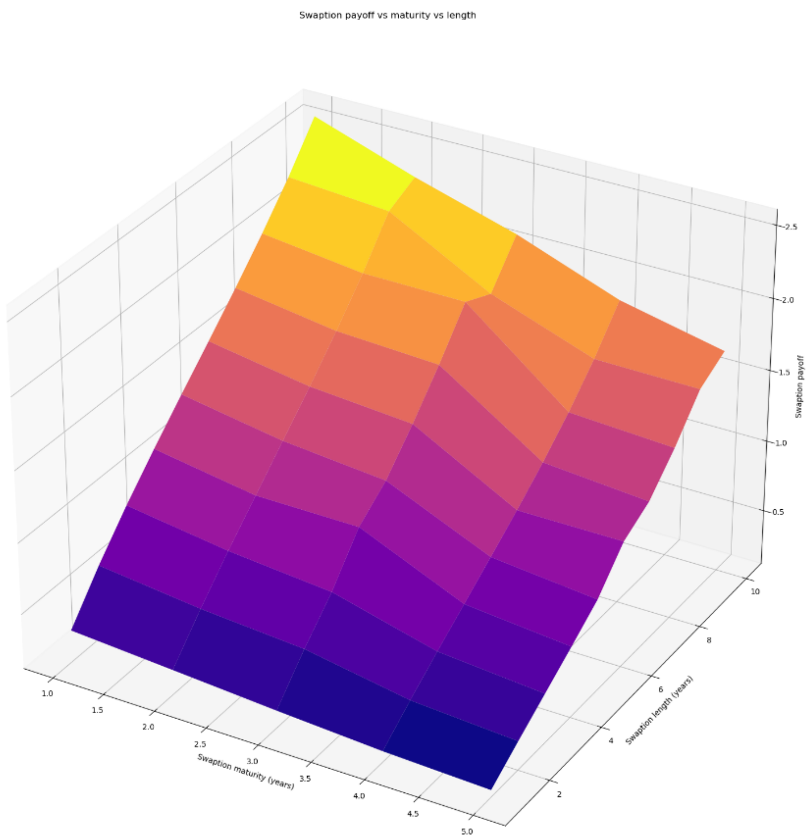


Figure 7.6: A graph of swaption payoffs given by the backward-looking rates.

In Table (7.6) and Figure (7.6), we set the $\kappa = 0.05$ and $\sigma = 0.20$ and conducted 10 000 Monte Carlo trials. We notice that the swaption payoff decreases as maturity increases. For a given maturity, the swaption payoff increases with the tenor. We notice that the swaptions given by the BL rates are generally lower than those given by FL rates. Both these swaptions, however, depict a similar trend as shown by the two graphs above.

CHAPTER 8

Conclusion and further research

The financial crisis of 2007-09 exposed the notion of too big to fall idea that was prevalent in the market and ever since then, the interbank borrowing activity has declined rapidly. This had an impact on the LIBOR rates setting since this rate relied on a few AA-rated banks and above borrowings submissions. The market participants have since settled on alternate risk-free rates that will be replacing the LIBOR rates at the beginning of 2022. Most countries have already settled on their LIBOR replacements. These new risk-free rates should first be converted into term rates, and the best approach to doing this is by using the setting-in-arrears method. The LMM, also known as the BGM model, which has been the most favored model for pricing products such as caps, swaptions, and even exotic products does not have the capability of modelling the new risk-free rates, and thus a new model is needed. The reason behind this is that the LMM only models the forward rates until the beginning of their application periods and then stops evolving afterward. These new rates, however, only cease to exist at the end of their application period. A new model which is essentially an extension of the LMM was proposed in the literature in 2018 which has the capability of modelling the new risk-free rates.

In this dissertation, we first gave a background of the interest rate models and how they progressed from equilibrium models to the market models. In the LMM framework, we derived the LIBOR dynamics under the spot measure and the terminal measure. We noticed that the risk-neutral dynamics were hard to derive under this framework. When implementing the LMM, we made use of Ito's lemma and made the assumption of constant drift per step (pricer predictor method). When using this method, we can move from one tenor point to another. We then proceeded to simulate the payoffs of the standard caplet using the analytical Black76 formula and the option payoff formula. We concluded that the payoffs were very close to each other after using 2000 Monte Carlo simulations. We also simulated two nonstandard caplets, namely, the ratchet cap and the sticky cap. We noticed that the sticky caps had a higher payoff compared to the ratchet caps. We concluded by discussing swaptions in the LMM framework.

We then introduced the generalized Forward market model. A notable difference between the LMM and the FMM is that the former is defined in terms of the normal zero-coupon bonds and the latter makes use of the extended zero-coupon bonds. The two zero-coupon bonds differ in that one is only defined for certain times $\top$ while the latter is defined for all times $\top$, including for times $\top > T$. The generalized Forward market model can therefore have dynamics that are defined for all times $\top$. This model also preserves the properties of the LMM that made it very attractive and desirable in the first place and also provides some extra properties that cannot be found in the LMM framework. We also gave the mathematical expression of these new BL rates and introduced their properties. We noticed that just as in the LMM framework, this new BL rate is a martingale under the corresponding T_j-measure. For times $\top \leq T$, the FL rate and the BL rate are the same under the new FMM. We also introduced the dynamics of the BL dynamics under the martingale measure and realized that the volatility had special characteristics. We noticed that the volatility of the new BL rates decreased as more and more overnight rates were set. We then proceeded to derive the dynamics of the new rates under different measures just like we did in the LIBOR market model framework using the change-of-measure technique. We derived the dynamics of the forward rates under the risk-neutral measure, spot measure, and terminal measure. It is worth noting that the risk-neutral dynamics could not be obtained in the LMM framework. When implementing the FMM, we used a combination of the pricer predictor method and Euler's method. The former was used for times $\top \leq T_j$ whereas the latter was used for the application period. Just as we did in the LMM framework, we proceeded to simulate the payoffs of the standard caplet using the analytical Black76 formula and the option payoff formula. We concluded that the payoffs were very close to each other after using 2000 Monte Carlo simulations. We also simulated two nonstandard caplets, namely, the ratchet cap and the sticky cap. We noticed that the sticky caps had a higher payoff compared to the ratchet caps.

We concluded the chapter by discussing swaptions in the FMM framework and discovered that it is the same as in the LMM framework.

In the last chapter, we proceeded to test the generalized Forward market model caplets payoffs under different tenors and a larger number of Monte Carlo simulations, 10,000 simulations to validate the model. We compared the FL caplets with the BL caplets and saw that the BL payoff was always higher than the FL payoff. We also compared the swaption payoffs given by FL rates with the BL rates. We noticed that the FL rates yielded a higher payoff than the BL swaption payoff.

Further research

In this dissertation, we priced European caplets, and two exotic caplets, namely the ratchet and the sticky caplets and swaptions. Another area that can be explored to expand upon the research conducted in this dissertation is to consider pricing non-simple exotic options such as Barrier options and American options. One might also be interested in calibrating different parameters of this model to different model parameters. Since swaptions are dependent on multiple forward rates, it would be interesting to observe the behavior of the model to the calibration of swaption volatilities. When pricing caplets, the favored analytical formula is the Black76 formula. Baaquie [19] has proposed an alternative to this theory, called the quantum field theory. In the Black76 formula, each caplet has its volatility function, whereas, in the quantum field, a single volatility function can be used to price a bond option. This field looks very attractive and it would be very interesting to see how the new BL caplets would behave under this theory.

APPENDIX A

Pricing caps using Monte Carlo

Boyle [20] provided an alternative way of obtaining the value of derivatives known as the Monte Carlo method. This method is generally used to evaluate integrals of the form

$$M(f) = \int_R f(Y)dY \tag{A.1}$$

where f is an L^2 arbitrary function, R is the range of integration, and $X \in R$.

We can use this method to find an estimate of the integral as follows

$$\hat{M}(f) = \frac{1}{n}\sum_{i=1}^{n} f(Y_i). \tag{A.2}$$

We can obtain the standard deviation sd of the estimate as follows

$$sd = \frac{1}{n-1}\sum_{i=1}^{n}(I(Y_i - \hat{I})^2.$$

Monte Carlo is used to find an approximation of Equation (A.1) the more preferred method as we increase the dimension of the function increases. The standard error, denoted by se of the estimate as follows

$$\frac{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(I(Y_i - \hat{I})^2.}}{\sqrt{n}} \tag{A.3}$$

APPENDIX B

Stochastic Calculus

Definition B.1. *A diffusion coefficient operator $DC(\cdot)$ can be defined as an operator that extracts the vector diffusion coefficient from the dynamics of a given stochastic process. Consider indeed $DC_\top(S)$ as the row vector $v_\top$ in*

$$dS(\top) = \mu(\top)S(\top)d\top + \sigma_\top S(\top)dW_\top$$

for diffusion processes. Consider the following SDE $dS_1(\top) = \sigma_1 S_1(\top)(CdW)_1(\top)$, then

$$DC_\top(S_1) = [\sigma_1 S_1(\top),\ 0,\ 0, \dots,\ 0] = \sigma_1 S_1(\top)e_1 \tag{B.1}$$

where $e_1 = [1,\ 0\dots\ ,0]$ is the first vector in the canonical basis.

Definition B.2. *A stochastic process $V = (V(\top);\ \top \geq 0)$ is called a generalized*

Ornstein-Uhlenbeck type process if it is given by

$$V(\top) = V_0\rho(\top) + \int_0^{\top} \rho(\top - u)dL(u), \quad \forall \top \geq 0, \tag{B.2}$$

where

$$\frac{d\rho(\top)}{d\top} = -\int_0^{\top} \rho(u)d\mu_{\top}(u), \quad with \rho(0) = 1. \tag{B.3}$$

In (B.3), μ_t is a signed measure for each $\top \geq 0$, $V(0) = V_0$ is the initial condition that is either a stochastic or a real deterministic constant. $\rho(\cdot)$ is a deterministic function that is referred to as the *memory kernel function.*

Lemma B.1. *If S follows a log-normal distribution with parameters characterized by a mean of μ and variance σ^2, denoted as $S = Ce^X$ where X is distributed as $\mathcal{N}(\mu, \sigma^2)$ and C is a real constant and K is a positive real constant, then: Then:*

$$\mathbb{E}\{(S-K)^+\} = Ce^{\frac{\sigma^2}{2}+\mu}\Phi\left(\frac{\ln\frac{C}{K} + \mu + \sigma^2}{\sigma}\right) - K\Phi\left(\frac{\ln\frac{C}{K} + \mu}{\sigma}\right) \tag{B.4}$$

where Φ is the CDF for the $\mathcal{N}(0,1)$ distribution.

Proof.

$$
\begin{aligned}
\mathbb{E}\{(S-K)^+\} &= \int_{\mathbb{R}} \max(Ce^x - K, 0) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{(Ce^x > K)} (Ce^x - K) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{(x > \ln\frac{K}{C})} Ce^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx - K \int_{(x > \ln\frac{K}{C})} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{(\ln\frac{K}{C})}^{+\infty} Ce^{-\frac{(x^2+(\mu)^2-2x\mu-2x\sigma^2)}{2\sigma^2}} dx - K \int_{(\ln\frac{K}{C})}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{(\ln\frac{K}{C})-\mu-\sigma^2}{\sigma}}^{+\infty} Ce^{-\frac{z^2}{2}} e^{\frac{\sigma^2}{2}+\mu} \sigma dz - K \int_{\frac{(\ln\frac{K}{C})-\mu}{\sigma}}^{+\infty} e^{-\frac{u^2}{2}} \sigma dx \qquad \text{(B.5)} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(Ce^{\frac{\sigma^2}{2}+\mu} \int_{\frac{(\ln\frac{K}{C})-\mu-\sigma^2}{\sigma}}^{+\infty} e^{-\frac{z^2}{2}} \sigma dz - K \int_{\frac{(\ln\frac{K}{C})-\mu}{\sigma}}^{+\infty} e^{-\frac{u^2}{2}} \sigma dx \right) \qquad \text{(B.6)} \\
&= Ce^{\frac{\sigma^2}{2}+\mu} \Phi\left(\frac{\ln\frac{C}{K}+\mu+\sigma^2}{\sigma}\right) - K\Phi\left(\frac{\ln\frac{C}{K}+\mu}{\sigma}\right) \qquad \text{(B.7)}
\end{aligned}
$$

where the two summands in (B.5)-(B.6) are obtained respectively with the following change of variable:

$$z = \frac{x-\mu-\sigma^2}{\sigma}, u = \frac{x-\mu}{\sigma}.$$

□

Corollary 1. *If S follows a log-normal distribution with parameters characterized by a mean of $-\frac{1}{2}\sigma^2$ and variance σ^2, denoted as $S = Ce^X$ where X is distributed as $\mathcal{N}(-\frac{1}{2}\sigma^2, \sigma^2)$ and C is a real constant and K is a positive real constant, then :*

$$\mathbb{E}\{(S-K)^+\} = C\Phi(d1(K, C, \sigma^2)) - K\Phi(d2(K, C, \sigma^2)), \qquad \text{(B.8)}$$

where

$$d1(K, S, \sigma^2) := \frac{\ln(\frac{S}{K}) + \frac{\sigma^2}{2}}{\sigma}, \tag{B.9}$$

$$d2(K, S, \sigma^2) := d1(K, S, \sigma^2) - \sigma. \tag{B.10}$$

Proof. It follows immediately from (B.4) by substituting μ with $-\frac{1}{2}\sigma^2$. $\square$

APPENDIX C

Code

```
def vasicek_model(r0, theta, sigma, k, n, m, T):
    r = np.zeros((n, m))
    r[:, 0] = r0
    dt = T/m
    for i in range(0, n):
        for j in range(0, m-1):
            r[i, j+1] = r[i, j] + k*(theta-r[i, j])*dt + sigma*dw(dt)
    return r
```

```
def ho_lee_model(r0, sigma, n, m, T):
    r = np.zeros((n, m))
    r[:, 0] = r0
    dt = T/m
    thetat =np.linspace(0.02*standard_normal(), 0.5*standard_normal(), m)
    for i in range(0, n):
        for j in range(0, m-1):
            r[i, j+1] = r[i, j] + thetat[j]*r[i, j]*dt + sigma*r[i, j]*dw
```

```
(dt)
    return r

def gFMM2(r0, delt, sigma, m, T,K):
    Time = np.linspace(0, T, m) #Time
    n=len(Time)                 #Length of time
    r = np.zeros((n, n))        #matrix of the LIBORs
    dt = T/n                    #Steps
    r[:, 0] = r0                #Initial Values
    RFR=np.zeros((n,32))        #Matrix of backwardlooking rates
    RFR[0,0]=r0                 #First backward-looking rate
    D=np.zeros(n)

    #Simulating the forward-looking rates
    for j in range(0, n):
        for i in range(j+1, n):
            summation = 0
            for k in range(i+1, n):
                        summation+=(sigma*delt*r[k, j])/(1+delt*r[k, j])
            r[i, j+1] = r[i, j]*np.exp((-summation*sigma-0.5*sigma**2)*dt
+sigma*dw(dt)) #Simulating the LIBOR forward rates
            if j+1==i:
                RFR[i,0]=r[i,j+1]   #Storing the LIBORs

    #Simulating the application period
    X=np.linspace(0,0.25,32)  #This spans the application period
    g=[]
    for j in range(len(X)):
        g.append(min( max(0.25-X[j],0)/0.25,1))

    for i in range(n):
        for j in range(1,len(X)):
            summation=0
```

```
            for k in range(0,j):
                summation+=((sigma/4)*sigma*RFR[i,k]*g[k])/(1+sigma*RFR[i,k])
            RFR[i,j]=RFR[i,j-1]+(-summation*RFR[i,j-1]*g[j-1]*(sigma/4)*0.025+RFR[i,j-1]*(sigma/4)*g[j-1]*dw(dt))


#Calculating the bonds
    P  = np.zeros((n, n))
for j in range(n):
        for row in range(j, n):
            Prd = 1
            for k in range(j, i+1):
                Prd = Prd * (1 + delt * r[k][j])**-1
            P[i][j] = Prd
            if i==j:
                D[j]=P[i][j]
##Cap payoff
C=np.zeros(n)
C2=np.zeros(n)
for i in range(n):
    C[i]=D[i]*0.25*max(RFR[i,0]-K,0) #Forward looking cap
    C2[i]=D[i]*0.25*max(RFR[i,-1]-K,0) #Backward looking cap\


  #Black76
Caps=np.zeros(n) #This matrix stores the black caplets
d1=np.zeros(n)
d2=np.zeros(n)
T=np.arange(0,n,0.25)
for i in range(n):
    d1[i]=(log(RFR[i,0]/K)+0.5*0.25*((sigma*sqrt(T[0]))**2))/(sigma*sqrt(T[0]))
```

```
        d2[i]=d1[i]-sigma*sqrt(0.25)
    for i in range(n):
        Caps[i]=max(D[i]*0.25*((RFR[i,0]*scipy.stats.norm.cdf(d1[i])-K*scipy.stats.norm.cdf(d2[i]))),0)
    #Backward-looking Black 76
    Caps2=np.zeros(n) #This matrix stores the black caplets
    d1=np.zeros(n)
    d2=np.zeros(n)
    T=np.arange(0,n,0.25)
    for i in range(n):
        d1[i]=(log(RFR[i,-1]/K)+0.5*0.25*((sigma*sqrt(1/3+(2/3)*(T[i])/(T[i]+0.25))*sqrt(0.5))**2))/(sigma*sqrt(1/3+(2/3)*(T[i])/(T[i]+0.25))*sqrt(0.5))
        d2[i]=d1[i]-sigma*sqrt(0.25)
    for i in range(n):
        Caps2[i]=max(D[i]*0.25*((RFR[i,-1]*scipy.stats.norm.cdf(d1[i])-K*scipy.stats.norm.cdf(d2[i]))),0)
    BCap2=sum(Caps2)
     ##Ratchet Cap payoff
    Ratchet=np.zeros(n)
    Ratchet2=np.zeros(n)
    e=0.01
    for i in range(1, n):
        Ratchet[i]=D[i]*0.25*max(RFR[i,0]-(RFR[i-1,0]+e),0) #Forward looking cap
        Ratchet2[i]=D[i]*0.25*max(RFR[i,-1]-(RFR[i-1,-1]+e),0) #Backward looking cap

    ##Sticky Cap payoff
    Sticky=np.zeros(n)
    Sticky2=np.zeros(n)
    e=0.01
    for i in range(1, n):
```

```
        Sticky[i]=D[i]*0.25*max(RFR[i,0]-(min(RFR[i-1,0],K)+e),0) # Forward looking cap
        Sticky2[i]=D[i]*0.25*max(RFR[i,-1]-(min(RFR[i-1,-1],K)+e),0) # Backward looking cap


     #Calculating the swaption payoff
    S=np.zeros(n)
    for i in range(16,56):
        S[i]=D[i]*0.25*(F[i,-1]-K)
    Swap=D[-1]*max(sum(S),0)
    return RFR, Caps, Caps2, Ratchet,Ratchet2, Sticky, Sticky2

```

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