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Intuitionistic correspondence theory

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DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for a Masters degree in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.



Abuzer Lookmanjee

09 September 2025

Date

Abstract

This thesis investigates intuitionistic correspondence theory, which aims to identify the frame conditions that ensure the validity of specific intuitionistic formulas. These conditions are typically formulated in first-order logic because of its well-established properties. A formal introduction to the correspondence framework is provided, with particular attention to syntactic classes of intuitionistic formulas that are first-order definable. Additionally, a detailed analysis of methods for proving that certain intuitionistic formulas are not first-order definable. This thesis also examines the ALBA algorithm, which computes first-order correspondents for a specific class of intuitionistic formulas. The principal contribution of this work is an extension of ALBA, enabling its application to a broader class of intuitionistic formulas, and providing a syntactic characterization of these formulas. As a result, this thesis expands the known syntactic class of intuitionistic formulas that are first-order definable.

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Chapter 1

Introduction

Mathematical logic is a branch of mathematics that studies correct arguments and reasoning. With the use of formal languages and rules of inference, mathematicians can analyse statements and manipulate them while preserving validity. Logic has a long history in philosophy and plays an essential role in the foundations of mathematics. Mathematical logic consists of various subfields, including proof theory, which constructs and studies the properties of formal deductive systems, computability theory, which is the study of computable functions, model theory, which studies the relationship between models and their formalized theories, and many more.

There are numerous types of logic, each differing in various aspects. Propositional logic and first-order logic are the most widely used logics due to their applications in the foundations of mathematics and elsewhere, and due to their desirable proof theoretic and model theoretic properties (completeness, compactness, Löwenheim–Skolem theorem, to name a few), and computational properties. There are logics that expand on the language of first-order logic, such as second-order and other higher-order logics. Modal logic is an example of a highly researched field that adds so called "modal operators" (the box $\Box$ and diamond $\Diamond$) to classical propositional logic. Intuitionistic logic, which is the logic this thesis focuses on, is an example of a logic that "restricts" classical logic by rejecting one of its axioms.

Intuitionistic logic is the mathematical formalization of the philosophical school of thought that arose in opposition to certain aspects of Cantor's work on infinities. Founded by Brouwer, intuitionism assumed a finitist and constructivist approach to mathematics. A popular way of classifying intuitionistic logic is as the logic that rejects the law of excluded middle ($p \vee \neg p$), and in consequence rejects double negation elimination ($\neg\neg p \rightarrow p$). The basis for the rejection of $p \vee \neg p$ can be explained by considering a conjecture that hasn't yet been proven. For example, consider the Goldbach conjecture. Denote it as p. A classical logician would argue that it doesn't matter if p or if $\neg p$ has been proven yet, $p \vee \neg p$ will always be a tautology regardless of what p actually is. An intuitionist would argue that since p as of yet hasn't been proven one cannot say there is a proof until said proof is provided, neither can one say there isn't a proof (i.e $\neg p$ is true) unless that proof is provided hence neither p is proven nor $\neg p$ is proven so $p \vee \neg p$ is not true in this

instance. Keeping in mind that there may be a proof provided in the future that would then make $p \vee \neg p$ true, shows an essential dependence on time in intuitionistic logic. Now, if time were considered to not be a factor, for example, if we live in a world with no future, would $p \vee \neg p$ be true intuitionistically? This is the type of question investigated by correspondence theory.

Correspondence theory, originally introduced by van Benthem in his PhD thesis [31] and further developed in [32] and [33], deals with finding equivalent formulas in first-order logic to formulas in other logics, i.e., it considers the first-order definability of properties expressed by the formulas of other logics that can be interpreted over first-order structures. The study of correspondence theory allows us to use the "nice" properties of first-order logic to describe formulas from another logic. In 1974, Henrik Sahlqvist [27] found a way to identify formulas of propositional modal logic that will have first-order equivalents. These formulas are known as Sahlqvist formulas. An extension to this class of formulas was found in 2006 by Valentin Goranko and Dimiter Vakarelov in [20], which introduced a new class of first-order definable modal formulas, called inductive formulas, which properly extends the class of Sahlqvist formulas. Alternative proofs to the Sahlqvist theorem can be found in [18] and [28], which makes use of a canonicity and a topological argument, respectively. The property of being a first-order definable modal formula was proved by Chagrova in 1991 in [4] to be algorithmically undecidable; hence, the study is constantly expanding, and any correspondence found can be of importance.

The problem of finding first-order equivalents can be formulated as second-order quantifier elimination problems. Various algorithms were introduced that managed to find first-order equivalents to second-order formulas. Among these is SCAN which was formulated by Dov Gabbay and Hans Jürgen Ohlbach in [17], DLS introduced by Patrick Doherty, Witold Łukaszewicz and Andrzej Szałas in [13] is another algorithm that found first-order correspondents of second order formulas. SQEMA, introduced by Conradie, Goranko, and Vakarelov in [6], is an algorithm that finds the first-order correspondents of formulas in classical modal logic. Further generalizing the concept of SQEMA to distributive and non-distributive modal logic, in [7] and [8] respectively, is ALBA.

Intuitionistic correspondence theory attempts to find these correspondences from intuitionistic logic to first-order logic. One of the main references on this topic is considered to be Rodenburg's PhD thesis [26], which was supervised by van Benthem. Rodenburg attempted to adapt his supervisor's work to intuitionistic logic and published what one can consider the main starting point for intuitionistic correspondence theory.

In this thesis, we undertake a comprehensive study of intuitionistic correspondence theory. We begin with preliminary material in Chapter 2, which establishes the foundational concepts necessary to understand the remainder of the thesis. In Chapter 3, we formally introduce the framework of correspondence theory. Within Chapter 3.1, we examine specific intuitionistic formulas and rigorously derive their first-order correspondents. Subsequently, we present the known classes of syntactically definable first-order intuitionistic formulas, namely Sahlqvist, (Ω, ε)-inductive, and deterministic formulas, and compute explicit examples for each of them. Finally, we present

a table that lists all the axioms of intuitionistic logic alongside those of the well-known superintuitionistic logics. Each axiom is classified on the basis of whether it is Sahlqvist, deterministic, inductive, or none of these.

Chapter 4 addresses methods for demonstrating the non-first-order definability of intuitionistic formulas. We analyze three distinct intuitionistic formulas and, for one of them, provide three independent arguments that establish its non-first-order definability, leveraging the Löwenheim–Skolem Theorem, the Compactness Theorem, and ultrapower constructions.

Our principal contribution is presented in Chapter 5, where we first specialize ALBA (an algorithm that computes first-order correspondents of (Ω, ε)-inductive formulas) to intuitionistic logic and compute some examples. In Chapter 5.2, we introduce *Int*-ALBA, which is a novel extension of ALBA designed specifically for the case of intuitionistic logic. The class of formulas that succeeds on Int-ALBA is strictly greater than the class of (Ω, ε)-inductive formulas. Finally, we capture the syntactic classification of the extended class of formulas for which *Int*-ALBA succeeds. This leads to the identification of a novel syntactic class of intuitionistic formulas that are first-order definable, thereby contributing to the broader theory of intuitionistic correspondence.

Chapter 2

Preliminaries

2.1 Language

Intuitionistic language

In this thesis, we consider intuitionistic propositional logic. We define the language $\mathcal{L}_{\mathcal{IP}}$ to consist of a countable set *Prop* of propositional variables $p_0, p_1, p_2, \ldots$, connectives $\vee, \wedge$, and $\rightarrow$, and constant $\bot$. The formulas are defined recursively as:

$$\varphi ::= p \mid \bot \mid \psi \vee \chi \mid \psi \wedge \chi \mid \psi \rightarrow \chi$$

where $p \in Prop$. We also use the following defined connectives:

$\top := \bot \rightarrow \bot$,

$\neg\varphi := \varphi \rightarrow \bot$,

$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Modal language

We define the language of modal logic denoted by $\mathcal{L}_{\mathcal{M}}$ with a countable set $Prop^M$ of propositional variables $p_0, p_1, p_2, \ldots$, binary connective $\vee$, unary connective $\neg$, and constant $\bot$, we further have a unary modal operator $\Diamond$. The formulas are defined recursively as:

$$\varphi ::= p \mid \bot \mid \neg\psi \mid \psi \vee \chi \mid \Diamond\psi$$

where $p \in Prop$. We also use the following defined connectives:

$\top := \neg \bot$,

$\varphi \wedge \psi := \neg\varphi \vee \neg\psi$,

$\varphi \rightarrow \psi := \neg\varphi \vee \psi$,

$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$,

$\Box\varphi = \neg\Diamond\neg\varphi$.

2.2 Semantics

Kripke semantics [21], introduced by Saul Kripke in the late 1950s and early 1960s, will be the semantics of choice for this thesis. It is a relational semantics that originated for use in modal logic and was later adapted to intuitionistic logic. Formally, intuitionistic Kripke semantics is defined as follows.

Intuitionistic Kripke semantics

An *intuitionistic Kripke frame* (we will drop "intuitionistic" when clear from the context) is a tuple $\mathfrak{F} = (W, \leq)$ where W is a non-empty set and $\leq$ is a binary relation on W such that $\mathfrak{F}$ is a partially ordered set (reflexivity, anti-symmetry and transitivity hold). In a frame $\mathfrak{F} = (W, \leq)$, a set $U \subseteq W$ is said to be *upwards closed* if for all $u \in U$ and $w \in W$, $u \leq w$ implies $w \in U$. The collection of upwards closed subsets of $\mathfrak{F}$ will be denoted as $\mathfrak{U}(\mathfrak{F})$. A valuation is a function $V : Prop \to \mathfrak{U}(\mathfrak{F})$. Intuitively, $V(p)$ is the set of all nodes in W where p is true. A *Kripke model* is a tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (W, \leq)$ is a Kripke frame and V is a valuation on the frame.

Let $\mathfrak{M} = (W, \leq, V)$ be a Kripke model and $u \in W$. We write $(\mathfrak{M}, u) \Vdash \varphi$ to indicate that φ is true at u in W and this is defined inductively as follows:

$(\mathfrak{M}, u) \Vdash p$	if and only if	$u \in V(p)$,
$(\mathfrak{M}, u) \Vdash \varphi \wedge \psi$	if and only if	$(\mathfrak{M}, u) \Vdash \varphi$ and $(\mathfrak{M}, u) \Vdash \psi$,
$(\mathfrak{M}, u) \Vdash \varphi \vee \psi$	if and only if	$(\mathfrak{M}, u) \Vdash \varphi$ or $(\mathfrak{M}, u) \Vdash \psi$,
$(\mathfrak{M}, u) \Vdash \varphi \to \psi$	if and only if	for all $v \in W$ such that $u \leq v, (\mathfrak{M}, v) \Vdash \varphi$ implies $(\mathfrak{M}, v) \Vdash \psi$,
$(\mathfrak{M}, u) \nVdash \bot$,		
$(\mathfrak{M}, u) \Vdash \neg\varphi$	if and only if	for all $v \in W$ such that $u \leq v, (\mathfrak{M}, v) \nVdash \varphi$.

Remark 2.2.1. The semantic definition of $(\mathfrak{M}, u) \Vdash \neg\varphi$ is not part of the inductive definition of intuitionistic Kripke semantics, we have just added it on to help us in simplifying the semantics but the definition should be clear given how $\neg\varphi$ is defined as $\varphi \to \bot$.

We define **Int** in the language $\mathcal{L}_{\mathcal{IP}}$, where $\mathbf{For}\mathcal{L}$ is the set of all $\mathcal{L}_{\mathcal{IP}}$ formulas, such that:

$$\mathbf{Int} = \{\phi \in \mathbf{For}\mathcal{L} \mid \mathfrak{F} \Vdash \phi \text{ for all frames } \mathfrak{F}\}.$$

A formula φ is *satisfied* in a model $\mathfrak{M} = (\mathfrak{F}, V)$ if $(\mathfrak{M}, u) \Vdash \varphi$ for some point u in $\mathfrak{F}$. We say φ is *true* in $\mathfrak{M}$ if $(\mathfrak{M}, u) \Vdash \varphi$ for every u in $\mathfrak{F}$ (denoted as $\mathfrak{M} \Vdash \varphi$). If φ is not true in $\mathfrak{M}$ then we say that φ is *refuted* in $\mathfrak{M}$ (denoted as $\mathfrak{M} \nVdash \varphi$).

A formula φ is *satisfied* in a frame $\mathfrak{F} = (W, \leq)$ if φ is satisfied in some model based on $\mathfrak{F}$. We say φ is *valid* at a point u in $\mathfrak{F}$ (denoted as $(\mathfrak{F}, u) \Vdash \varphi$) if φ is true at u

in all models based on $\mathfrak{F}$. We say φ is *valid* in a frame $\mathfrak{F}$ (denoted as $\mathfrak{F} \Vdash \varphi$) if φ is true in all models based on $\mathfrak{F}$, otherwise we say that φ is *refuted* in $\mathfrak{F}$ (denoted as $\mathfrak{F} \nVdash \varphi$).

We further define an inequality $\varphi \leq \psi$ as *true* in a model $\mathfrak{M} = (\mathfrak{F}, V)$, denoted by $\mathfrak{M} \Vdash \varphi \leq \psi$, if $V(\varphi) \subseteq V(\psi)$. Similarly, $\varphi \leq \psi$ is *valid* in a frame $\mathfrak{F} = (W, \leq)$ if it is true in every model based on $\mathfrak{F}$.

Due to how intuitionistic Kripke frames are defined on partially ordered sets, an intuitive way to consider the semantics is through a temporal interpretation. If you are at a point u, you are in the present. Any point v such that $u \leq v$ and $u \neq v$ (we will denote this as $u < v$) would be considered the future, and additionally u would be considered the past from v.

Modal Kripke semantics

The difference between a modal and an intuitionistic Kripke frame is the relation defined on the frame. A modal frame is a pair $\mathfrak{X} = (W, R)$ where W is a set and R is a binary relation on the set (unlike intuitionistic frames, the relation is not defined as a partial order). A valuation on $\mathfrak{X}$ is a function $V^M : Prop \to \mathcal{P}(\mathfrak{X})$ where $\mathcal{P}(\mathfrak{X})$ is the power set of W. A modal Kripke model is the tuple $\mathfrak{Y} = (\mathfrak{X}, V^M)$ where $\mathfrak{X} = (W, R)$ is a modal Kripke frame and V^M a valuation on the frame.

Let $\mathfrak{Y} = (W, R, V^M)$ be a modal Kripke model and $u \in W$, the Kripke semantics of modal logic is defined inductively as follows:

$(\mathfrak{M}, u) \Vdash p$	if and only if	$u \in V^M(p)$,
$(\mathfrak{M}, u) \Vdash \varphi \vee \psi$	if and only if	$(\mathfrak{M}, u) \Vdash \varphi$ or $(\mathfrak{M}, u) \Vdash \psi$,
$(\mathfrak{M}, u) \Vdash \Diamond\varphi$	if and only if	there exists $v \in W$ such that uRv, $(\mathfrak{M}, v) \Vdash \varphi$,
$(\mathfrak{M}, u) \nVdash \bot$,		
$(\mathfrak{M}, u) \Vdash \neg\varphi$	if and only if	$(\mathfrak{M}, u) \nVdash \varphi$.

2.3 Translations

Translations are maps from the set of formulas in one language into the set of formulas of another language, all while preserving the meaning of the formulas. We will state two well known translations, the standard translation and the Gödel–McKinsey–Tarski translation.

The standard translation

Define $\mathcal{L}_0$ to be the first-order language with equality, relation symbol $\leq$, and individual variables $x_0, x_1, \ldots$. Let $\mathcal{L}_1$ be an extension of $\mathcal{L}_0$ which has unary predicates $P_0, P_1, \ldots$ corresponding to propositional variables $p_0, p_1, \ldots$ in $\mathcal{L}_{\mathcal{IP}}$.

The standard translation, $ST : \mathcal{L}_{\mathcal{IP}} \to \mathcal{L}_1$, is a recursively defined function that converts intuitionistic formulas to formulas of first-order logic. Let φ and ψ be a

intuitionistic formulas, p an intuitionistic variable, P a unary predicate symbol and u a first-order variable, then the standard translation of φ, $ST_u(\varphi)$, is defined as:

$$ST_u(p) = P(u),$$

$$ST_u(\bot) = \bot,$$

$$ST_u(\varphi \wedge \psi) = ST_u(\varphi) \wedge ST_u(\psi),$$

$$ST_u(\varphi \vee \psi) = ST_u(\varphi) \vee ST_u(\psi),$$

$$ST_u(\varphi \to \psi) = \forall v(u \leq v \to (ST_v(\varphi) \to ST_v(\psi))).$$

This gives us the following propositions:

Proposition 2.3.1. [3] For every formula φ, every model $\mathfrak{M}$ and every point $u \in \mathfrak{M}$ using induction on the construction of φ one can deduce that:

$$(\mathfrak{M}, u) \Vdash \varphi \text{ if and only if } \mathfrak{M} \models ST_v(\varphi)[u],$$
$$\mathfrak{M} \Vdash \varphi \text{ if and only if } \mathfrak{M} \models \forall v ST_v(\varphi).$$

Where $\mathfrak{M} \models ST_v(\varphi)[u]$ means that the first-order formula $ST_v(\varphi)$ is satisfied in the model $\mathfrak{M}$ when u is assigned to the free variable v.

We can further expand the standard translation to convert formulas from intuitionistic logic to formulas of second-order logic. The standard translation of φ is a first-order equivalent of φ when models are considered, but when dealing with frames then all predicates $P_1, P_2, \ldots, P_n$ are considered as arbitrary monadic predicate variables (the predicates only take one argument) on $\mathfrak{U}(\mathfrak{F})$ [3]. Hence, φ corresponds to the second-order formula $\forall P_1, P_2, \ldots, P_n ST(\varphi)$.

Proposition 2.3.2. [3] For every formula φ with propositional variables $p_0, p_1, \ldots, p_n$, every Kripke frame $\mathfrak{F}$ and every point $u \in \mathfrak{F}$ one can deduce that:

$$(\mathfrak{F}, u) \Vdash \varphi \text{ if and only if } \mathfrak{F} \models \forall P_1 \forall P_2 \ldots \forall P_n ST_v(\varphi)[u],$$
$$\mathfrak{F} \Vdash \varphi \text{ if and only if } \mathfrak{F} \models \forall u \forall P_1 \forall P_2 \ldots \forall P_n ST_v(\varphi).$$

Gödel–McKinsey–Tarski translation

The Gödel–McKinsey–Tarski translation, $GT : \mathcal{L}_{\mathcal{IP}} \to \mathcal{L}_{\mathcal{M}}$, is a recursively defined function that translates intuitionistic formulas to formulas of modal logic. Let φ and ψ be intuitionistic formulas and p an intuitionistic variable, then the Gödel translation of φ, $GT(\varphi)$, is defined as:

$$GT(p) = \Box p,$$

$$GT(\bot) = \bot,$$

$$GT(\varphi \wedge \psi) = GT(\varphi) \wedge GT(\psi),$$

$$GT(\varphi \vee \psi) = GT(\varphi) \vee GT(\psi),$$
$$GT(\varphi \to \psi) = \Box(GT(\varphi) \to GT(\psi)).$$

From the Gödel translation, one can see that intuitionistic logic is a subset of modal logic since all intuitionistic formulas can be represented as a modal formula. This is in line with the fact that intuitionistic logic restricts classical logic, and modal logic extends it.

The following theorem systematically connects **Int** to the modal logic **S4**, which is the modal logic of all transitive and reflexive frames.

Proposition 2.3.3. [3] The Gödel–McKinsey–Tarski translation is an embedding of **Int** into **S4**, i.e., for any formula φ:

$$\varphi \in \mathbf{Int} \quad \text{iff} \quad GT(\varphi) \in \mathbf{S4}.$$

2.4 Truth-Preserving Operations

We will discuss three types of truth-preserving operations, each of which will play a role in Chapter 4 when demonstrating the non-first-order definability of certain formulas. These operations involve: generated subframes and submodels, bounded morphism mappings, and disjoint unions.

Generated subframes and submodels

A frame $\mathfrak{F}' = (W', \leq')$ is a *subframe* of another frame $\mathfrak{F} = (W, \leq)$ if $W' \subseteq W$ and $\leq'$ restricts $\leq$ to W', denoted as $\mathfrak{F}' \subseteq \mathfrak{F}$. Further if W' is an upwards closed subset of W then $\mathfrak{F}'$ is considered a *generated subframe* of $\mathfrak{F}$, denoted as $\mathfrak{F}' \underset{\rightarrow}{\subseteq} \mathfrak{F}$.

A model $\mathfrak{M}' = (\mathfrak{F}', V')$ is a submodel of $\mathfrak{M} = (\mathfrak{F}, V)$ if $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$ and V' is V restricted to W' for all $p \in Prop$, i.e., $V'(p) = V(p) \cap W'$, denoted as $\mathfrak{M}' \subseteq \mathfrak{M}$. If $\mathfrak{F}' \underset{\rightarrow}{\subseteq} \mathfrak{F}$ then $\mathfrak{M}'$ is a *generated submodel* of $\mathfrak{M}$, denoted as $\mathfrak{M}' \underset{\rightarrow}{\subseteq} \mathfrak{M}$.

Theorem 2.4.1. (Generation [3]) Let $\mathfrak{M}' = (\mathfrak{F}', V')$, $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' \underset{\rightarrow}{\subseteq} \mathfrak{M}$. Then for every formula φ and every point $u \in \mathfrak{F}'$

$$(\mathfrak{M}', u) \Vdash \varphi \quad \text{iff} \quad (\mathfrak{M}, u) \Vdash \varphi.$$

For generated subframes, we have the following corollary:

Corollary 2.4.2. [3] If $\mathfrak{F}' \underset{\rightarrow}{\subseteq} \mathfrak{F}$ then, for every formula φ,

i. $(\mathfrak{F}', u) \Vdash \varphi$ iff $(\mathfrak{F}, u) \Vdash \varphi$, for all points u in $\mathfrak{F}'$,

ii. $\mathfrak{F} \Vdash \varphi$ implies $\mathfrak{F}' \Vdash \varphi$.

Bounded morphism

Given two frames $\mathfrak{F} = (W, \leq)$ and $\mathfrak{F}' = (W', \leq')$. A function f that maps W onto W' and which satisfies the following conditions:

(R1) if $u \leq v$ then $f(u) \leq' f(v)$,

(R2) if $f(u) \leq' f(v)$ then $\exists w \in W$ where $u \leq w$ and $f(w) = f(v)$,

is called a reduction of $\mathfrak{F}$ to $\mathfrak{F}'$. f is referred to as a bounded morphism.

A bounded morphism f from $\mathfrak{F}$ to $\mathfrak{F}'$ is a reduction of a model $\mathfrak{M} = (\mathfrak{F}, V)$ to another model $\mathfrak{M}' = (\mathfrak{F}', V')$ if for every $p \in Prop$, $V(p) = f^{-1}(V'(p))$.

Theorem 2.4.3. (Reduction [3]) If f is a reduction of a model $\mathfrak{M} = (\mathfrak{F}, V)$ to another model $\mathfrak{M}' = (\mathfrak{F}', V')$ then, for every formula φ and every point $u \in \mathfrak{F}$,

$$(\mathfrak{M}, u) \Vdash \varphi \quad \text{iff} \quad (\mathfrak{M}', f(u)) \Vdash \varphi.$$

For a bounded morphism between frames, we have the following corollary:

Corollary 2.4.4. [3] If $\mathfrak{F}$ is reducible to $\mathfrak{F}'$ then, for every formula φ,

$$\mathfrak{F} \Vdash \varphi \quad \text{implies} \quad \mathfrak{F}' \Vdash \varphi.$$

Disjoint union

Let $\{\mathfrak{F}_i = (W_i, \leq_i) : i \in I\}$ be a family of frames where $W_i \cap W_j = \emptyset$ for every $i \neq j$, then the disjoint union of the family of frames is given as $\sum_{i \in I} \mathfrak{F}_i = (\bigcup_{i \in I} W_i, \bigcup_{i \in I} \leq_i)$.

Similarly the disjoint union of the family of models $\{\mathfrak{M}_i = (\mathfrak{F}_i, V_i) : i \in I\}$ is defined for pairwise disjoint models as $\sum_{i \in I} \mathfrak{M}_i = (\sum_{i \in I} \mathfrak{F}_i, \sum_{i \in I} V_i)$ and $(\sum_{i \in I} V_i)(p) = \bigcup_{i \in I} V_i(p)$ for every $p \in Prop$.

Theorem 2.4.5. (Disjoint union [3]) Let $\sum_{i \in I} \mathfrak{M}_i$ be the disjoint union of a family $\{\mathfrak{M}_j : j \in I\}$, then for every formula φ, every $u \in \mathfrak{M}_{i,j}$ and every $i, j \in I$,

$$(\sum_{i \in I} \mathfrak{M}_i, u) \Vdash \varphi \quad \text{iff} \quad (\mathfrak{M}_j, u) \Vdash \varphi.$$

For a disjoint union between frames, we have the following corollary:

Corollary 2.4.6. [3] Let $\sum_{i \in I} \mathfrak{F}_i$ be the disjoint union of a family $\{\mathfrak{F}_i : i \in I\}$. Then, for every formula φ and for all $i \in I$,

$$\sum_{i \in I} \mathfrak{F}_i \Vdash \varphi \quad \text{iff} \quad \mathfrak{F}_i \Vdash \varphi.$$

2.5 Lattices and Heyting Algebras

Lattices form an integral part of the study of intuitionistic logic. Heyting algebras, which are the algebraic semantics of intuitionistic logic, are a type of distributive lattice. Lattice theory also forms an integral part in the inner workings of the definition of inductive formulas (Chapter 3.2). The main reference for all the basic terminology on lattices is [10].

Definition 2.5.1. (Lattice) Let $\mathfrak{L}$ be a non-empty ordered set, if $\inf\{a, b\} \in \mathfrak{L}$ and $\sup\{a, b\} \in \mathfrak{L}$ for all $a, b \in \mathfrak{L}$ then $\mathfrak{L}$ is called a *lattice*.

The standard operations defined on a lattice $\mathfrak{L}$ are the "meet" operation $\wedge$ such that $a \wedge b := \inf\{a, b\}$ and the "join" operation $\vee$ such that $a \vee b := \sup\{a, b\}$. If we have a subset $\mathfrak{L}' \subseteq \mathfrak{L}$ then $\bigvee \mathfrak{L}'$ is used to denote the supremum of $\mathfrak{L}'$ if it exists, and dually $\bigwedge \mathfrak{L}'$ is used to denote the infimum of $\mathfrak{L}'$ if it exists. We also have that the lattice ordering is defined such that $a \vee b = b$ and $a \wedge b = a$ whenever $a \leq b$.

Definition 2.5.2. (Complete lattice) Let $\mathfrak{L}$ be an ordered set, if $\bigvee \mathfrak{L}'$ and $\bigwedge \mathfrak{L}'$ exists for all $\mathfrak{L}' \subseteq \mathfrak{L}$ then $\mathfrak{L}$ is called a *complete* lattice.

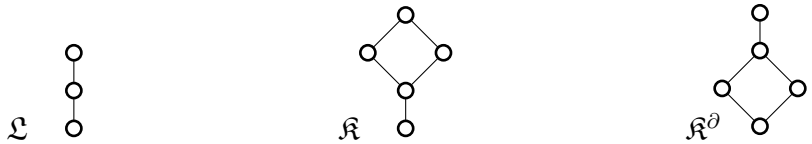


Figure 2.1: Examples of Lattices

Given any lattice $\mathfrak{L}$, we can construct another lattice $\mathfrak{L}^{\partial}$, called the dual of $\mathfrak{L}$, by defining $a \leq b$ to hold in $\mathfrak{L}^{\partial}$ if and only if $b \leq a$ holds in $\mathfrak{L}$. Informally, the dual of a lattice is just the lattice turned upside down. This construction can be defined on any ordered set.

Theorem 2.5.3. Let $\mathfrak{L}$ be a lattice. Then $\wedge$ and $\vee$ satisfy, for all $a, b, c \in \mathfrak{L}$,

(L1) $(a \vee b) \vee c = a \vee (b \vee c)$, (associative law)

(L2) $a \vee b = b \vee a$, (commutative law)

(L3) $a \vee a = a$, (idempotency law)

(L4) $a \vee (a \wedge b) = a$. (absorption law)

Along with (L1)–(L4) $\mathfrak{L}$ also satisfies the dual of the above (L1)–(L4) ((L1)$^{\partial}$–(L4)$^{\partial}$) which are achieved by replacing all occurrences of $\wedge$ with $\vee$ and $\vee$ with $\wedge$.

Definition 2.5.4. Let $\mathfrak{L}$ be a lattice, if $\mathfrak{L}$ satisfies the distributive law then $\mathfrak{L}$ is called a distributive lattice, i.e., for all $a, b, c \in \mathfrak{L}$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Note that a lattice satisfies $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ iff it also satisfies $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

We now adopt the definition of a lattice to an algebra with the binary operations meet ($\wedge$) and join ($\vee$) using the following theorem.

Theorem 2.5.5. [10] Let $(U, \wedge, \vee)$ be a non-empty set equipped with two binary operations which satisfy (L1)–(L4) and (L1)$^{\partial}$–(L4)$^{\partial}$ given in Theorem 2.5.3.

i. For all $a, b \in U$, we have $a \wedge b = b$ iff $a \vee b = a$.

ii. Define $\leq$ on U by $a \leq b$ if $a \vee b = b$. Then $\leq$ is an order relation.

iii. With $\leq$ defined as above, $(U, \leq)$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in U$,

$$a \vee b = \sup\{\text{a,b}\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}.$$

Theorem 2.5.5 allows us to refer to algebras $(U, \wedge, \vee)$ which satisfy (L1)-(L4) and $(\text{L1})^{\partial}$–$(\text{L4})^{\partial}$ as lattices.

Definition 2.5.6. (Heyting Algebra [29]) Let U be a set. $\mathfrak{U} = (U, \wedge, \vee, \rightarrow, \bot)$, with three binary operations $(\wedge, \vee, \rightarrow)$ and a constant $\bot$ ($\top$ is defined as $\bot \rightarrow \bot$), is a *Heyting Algebra* if $\mathfrak{U}$ satisfies:

i. $(U, \wedge, \vee)$ is a distributive lattice,

ii. $a \wedge \bot = \bot; \quad a \vee \top = \top,$

iii. $a \rightarrow a = \top,$

iv. $(a \rightarrow b) \wedge b = b; \quad a \wedge (a \rightarrow b) = a \wedge b,$

v. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c); \quad (a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c).$

Given a Heyting algebra $\mathfrak{U} = (U, \wedge, \vee, \rightarrow, \bot)$, a valuation is a function that sends propositional variables to elements of $\mathfrak{U}$, $V : Prop \rightarrow U$.

Remark 2.5.7. A consequence of the definition of Heyting algebras is that $a \rightarrow b$ is defined as the relative pseudo-complement of $\leq$ ($\leq$ is defined as the normal lattice ordering on U) such that $\forall u \in U$,

$$u \wedge a \leq b \text{ is equivalent to } u \leq a \rightarrow b.$$

Heyting Algebras got their name from Arend Heyting, who introduced them to formalize intuitionistic logic. They can be a framework on which to base a study on algebraic intuitionistic logic. This thesis will not focus on this algebra but instead will only call on it if necessary to define certain operations.

Definition 2.5.8. (Complex algebra of an intuitionistic frame) For every intuitionistic frame $\mathfrak{F}$ the complex algebra of $\mathfrak{F}$ is:

$$\mathfrak{F}^{+} = (\mathfrak{U}(\mathfrak{F}), \cup, \cap, \supset, \bot),$$

where the binary operations $(\cup, \cap)$ are defined with their usual set theoretic definitions of union and intersection, $\bot$ is defined as the empty set $\emptyset$ and $\supset$ is defined such that, for all $U, V \in \mathfrak{U}(\mathfrak{F})$,

$$U \supset V = \bigcup \{W \in \mathfrak{U}(\mathfrak{F}) \mid W \cap U \subseteq V\}.$$

In fact, $\mathfrak{F}^+$ forms a Heyting Algebra. To prove this, we will make use of the following theorem.

Theorem 2.5.9. ([3]) An algebra $\mathfrak{A} = (U, \wedge, \vee, \rightarrow, \bot)$ is a Heyting algebra if and only if the following conditions hold for every $u, v, w \in U$,

i. Commutativity of $\wedge$ and $\vee$, i.e., $u \wedge v = v \wedge u$ and $u \vee v = v \vee u$,

ii. Associativity of $\wedge$ and $\vee$, i.e., $u \wedge (v \wedge w) = (u \wedge v) \wedge w$ and $u \vee (v \vee w) = (u \vee v) \vee w$,

iii. Absorption, i.e., $(u \wedge v) \vee v = v$ and $u \wedge (u \vee v) = u$,

iv. Heyting Axiom, i.e., $w \wedge u \leq v$ iff $w \leq u \rightarrow v$,

v. $\bot$ is the least element, i.e., $\bot \leq u$.

Theorem 2.5.10. The complex algebra $\mathfrak{F}^+$ of any intuitionistic frame $\mathfrak{F}$ is a Heyting algebra.

Proof. Consider our algebra $\mathfrak{F}^+ = (\mathfrak{U}(\mathfrak{F}), \cup, \cap, \supset, \bot)$. Using Theorem 2.5.9 with $U, V, W, Z \in \mathfrak{U}(\mathfrak{F})$.

Clearly $\cup$ and $\cap$ are commutative and associative by definition. Absorption is also relatively easy to show, consider $(U \cap V) \cup V = V$. Now $U \cap V \subseteq V$ by definition of intersection. We also have that if $X \subseteq Y$ then $X \cup Y = Y$ for any set X and Y. Hence, we have $(U \cap V) \cup V = V$. A similar argument suffices for $U \cap (U \cup V) = U$.

Now for the Heyting axiom. Assume $W \cap U \subseteq V$, we need to show $W \subseteq U \supset V$. Clearly $W \in \{Z \mid Z \cap U \subseteq V\}$ by assumption, and since we are taking the union of all Z in the set, we have that $W \subseteq \bigcup\{Z \mid Z \cap U \subseteq V\}$ i.e $W \subseteq U \supset V$. Let us now assume $W \subseteq U \supset V$, we need to show $W \cap U \subseteq V$. Now $W \subseteq U \supset V$ implies $W \subseteq \bigcup\{Z \mid Z \cap U \subseteq V\}$ which in turn implies $W \in \{Z \mid Z \cap U \subseteq V\}$. Since $W \in \mathcal{U}(\mathfrak{F})$ we have $W \cap U \subseteq V$.

Finally $\bot := \emptyset$ is the least element since $\emptyset \subseteq U$ for all U.

Hence, we have proved that for any frame $\mathfrak{F}$, its complex algebra $\mathfrak{F}^+$ is a Heyting algebra. □

Another important construction is that of the Cartesian product $\mathfrak{L} \times \mathfrak{K}$ between two lattices $\mathfrak{L}$ and $\mathfrak{K}$. Informally $\mathfrak{L} \times \mathfrak{K}$ produces a lattice whereby you place a copy of the lattice $\mathfrak{K}$ on all points in the lattice $\mathfrak{L}$ and connect all "corresponding" points, this can be visualized in Figure 2.2. Formally, we have that:

Definition 2.5.11. Let $P_1, \ldots, P_n$ be ordered sets. The Cartesian product $P_1 \times \cdots \times P_n$ can be made into an ordered set by imposing the coordinate-wise order defined by declaring that,

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \quad \text{iff } a_i \leq b_i \text{ in } P_i, \text{ for all } 1 \leq i \leq n.$$

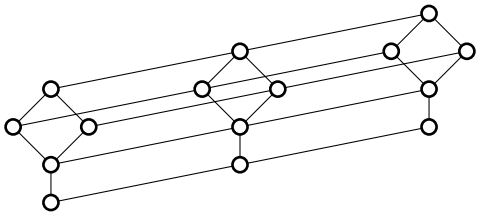


Figure 2.2: Lattice product $\mathfrak{L} \times \mathfrak{K}$

Definition 2.5.12. Let $\mathfrak{L}_i$ be lattices, for all $1 \leq i \leq n$. Define $\wedge$ and $\vee$ coordinate-wise on $\mathfrak{L}_1 \times \mathfrak{L}_2 \times \cdots \times \mathfrak{L}_n$, with $l_1^i, l_2^i \in \mathfrak{L}_i$ as follows:

$$(l_1^1, l_1^2, \ldots, l_1^n) \vee (l_2^1, l_2^2, \ldots, l_2^n) = (l_1^1 \vee l_2^1, l_1^2 \vee l_2^2, \ldots, l_1^n \vee l_2^n),$$

$$(l_1^1, l_1^2, \ldots, l_1^n) \wedge (l_2^1, l_2^2, \ldots, l_2^n) = (l_1^1 \wedge l_2^1, l_1^2 \wedge l_2^2, \ldots, l_1^n \wedge l_2^n).$$

Remark 2.5.13. If both $\mathfrak{L}$ and $\mathfrak{K}$ are lattices then $\mathfrak{L} \times \mathfrak{K}$ is also a lattice.

Lattice Expansions

We make use of another unspecified but fixed language $\mathcal{L}_{\mathcal{LE}}$, to be interpreted over lattices of compatible similarity type. We define an *order-type* over $n \in \mathbb{N}$ as an n-tuple $\varepsilon \in \{1, \partial\}^n$. For every order-type ε its *opposite* is denoted as ε^∂ such that $\varepsilon_i^\partial = 1$ if and only if $\varepsilon_i = \partial$ where $1 \leq i \leq n$. For any lattice $\mathfrak{L}$ and any order-type ε over n, we let $\mathfrak{L}^\varepsilon = \prod_{i=1}^n \mathfrak{L}_i^\varepsilon$.

Our language $\mathcal{L}_{\mathcal{LE}}(F, G)$ (abbreviated as $\mathcal{L}_{\mathcal{LE}}$) consists of a countable set *Prop* of propositional variables $p_0, p_1, p_2, \ldots$ and two sets of connectives F and G that are disjoint. Each $f \in F$ and $g \in G$ has arity $n_* \in \mathbb{N}$, $* \in \{f, g\}$, and is associated with some order-type ε_* over n_*. The formulas are defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \psi \vee \chi \mid \psi \wedge \chi \mid \psi \to \chi \mid f(\overline{\psi}) \mid g(\overline{\chi})$$

where $p \in \mathit{Prop}$, $f \in F$, $g \in G$, $\overline{\psi}$ is an n_f-tuple of formulas, and $\overline{\chi}$ is an n_g-tuple of formulas. Chapter 3.2 will discuss whether the connectives $\vee, \wedge$, and $\to$ will be grouped as either F or G.

Definition 2.5.14. For any tuple (F, G) of disjoint sets of connectives as above, a *lattice expansion* (abbreviated as LE) is a tuple $\mathfrak{L} = (L, F^\mathfrak{L}, G^\mathfrak{L})$ such that L is a bounded lattice, $F^\mathfrak{L} = \{f^\mathfrak{L} | f \in F\}$ and $G^\mathfrak{L} = \{g^\mathfrak{L} | g \in G\}$, such that each $f^\mathfrak{L} \in F^\mathfrak{L}$ is an n_f-ary and $g^\mathfrak{L} \in G^\mathfrak{L}$ is an n_g-ary operation on $\mathfrak{L}$ (we shall drop the $\mathfrak{L}$ when clear and write f for $f^\mathfrak{L}$ etc). An LE is *normal* if every $f \in F$ preserves finite joins in each coordinate with $\varepsilon_f(i) = 1$ and reverses finite meets in each coordinate with $\varepsilon_f(i) = \partial$, and every $g \in G$ preserves finite meets in each coordinate with $\varepsilon_g(i) = 1$ and reverses finite joins in each coordinate with $\varepsilon_g(i) = \partial$, i.e., for any $f \in F$, $g \in G$, $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$:

> if $\varepsilon_f(i) = 1$, then $f(a_0, \ldots, a \vee b, \ldots, a_{n_f}) = f(a_0, \ldots, a, \ldots, a_{n_f}) \vee f(a_0, \ldots, b, \ldots, a_{n_f})$ and $f(a_0, \ldots, \bot, \ldots, a_{n_f}) = \bot$,

if $\varepsilon_f(i) = \partial$, then $f(a_0, \ldots, a \wedge b, \ldots, a_{n_f}) = f(a_0, \ldots, a, \ldots, a_{n_f}) \vee f(a_0, \ldots, b, \ldots, a_{n_f})$ and $f(a_0, \ldots, \top, \ldots, a_{n_f}) = \bot$,

if $\varepsilon_g(i) = 1$, then $g(a_0, \ldots, a \wedge b, \ldots, a_{n_g}) = g(a_0, \ldots, a, \ldots, a_{n_g}) \wedge g(a_0, \ldots, b, \ldots, a_{n_g})$ and $g(a_0, \ldots, \top, \ldots, a_{n_g}) = \top$,

if $\varepsilon_g(i) = \partial$, then $g(a_0, \ldots, a \vee b, \ldots, a_{n_g}) = g(a_0, \ldots, a, \ldots, a_{n_g}) \wedge g(a_0, \ldots, b, \ldots, a_{n_g})$ and $g(a_0, \ldots, \bot, \ldots, a_{n_g}) = \top$.

The above definitions are taken from [8], where they were given for a general lattice expansions setting. Heyting algebras are a variety of lattice expansions, particularly since they are distributive, they fall under *distributive lattice expansions* (DLE). This is the case since $\to$ is a g operation, the details are given in Chapter 3.2.1.

For every complete lattice $\mathfrak{L}$, an element $a \in \mathfrak{L} \setminus \{\bot\}$ is *completely join-irreducible* iff $a = \bigvee S$ implies $a \in S$ for every $S \subseteq \mathfrak{L}$; also, a is *completely join-prime* if, for every $S \subseteq \mathfrak{L}$, if $a \leq \bigvee S$, then $a \leq s$ for some $s \in S$. If a is completely join-prime, then a is completely join-irreducible. The collection of completely join-irreducible elements of $\mathfrak{L}$ is denoted by $J^\infty(\mathfrak{L})$. Dually, the completely meet-irreducible and meet-prime elements are defined, and the collection of completely meet-irreducible elements of $\mathfrak{L}$ is denoted by $M^\infty(\mathfrak{L})$.

Definition 2.5.15. (Perfect Lattice [14]) A complete lattice $\mathfrak{L}$ is *perfect* if $J^\infty(\mathfrak{L})$ is join-dense in $\mathfrak{L}$ and $M^\infty(\mathfrak{L})$ is meet-dense in $\mathfrak{L}$, i.e., if for every $a \in \mathfrak{L}$, $a = \bigvee\{b \in J^\infty(\mathfrak{L}) | b \leq a\}$ and $a = \bigwedge\{c \in M^\infty(\mathfrak{L}) | a \leq c\}$.

Definition 2.5.16. A LE $\mathfrak{L} = (L, F^{\mathfrak{L}}, G^{\mathfrak{L}})$ is perfect if L is a perfect lattice and the following infinitary distribution laws are satisfied for each $f \in F$, $g \in G$, $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$, for every $S \subseteq L$:

$$f(s_1, \ldots, \bigvee S, \ldots, s_{n_f}) = \bigvee\{f(s_1, \ldots, s, \ldots, s_{n_f}) \mid s \in S\} \quad \text{if } \varepsilon_f(i) = 1,$$
$$f(s_1, \ldots, \bigwedge S, \ldots, s_{n_f}) = \bigvee\{f(s_1, \ldots, s, \ldots, s_{n_f}) \mid s \in S\} \quad \text{if } \varepsilon_f(i) = \partial,$$
$$g(s_1, \ldots, \bigwedge S, \ldots, s_{n_g}) = \bigwedge\{g(s_1, \ldots, s, \ldots, s_{n_g}) \mid s \in S\} \quad \text{if } \varepsilon_g(j) = 1,$$
$$g(s_1, \ldots, \bigvee S, \ldots, s_{n_g}) = \bigwedge\{g(s_1, \ldots, s, \ldots, s_{n_g}) \mid s \in S\} \quad \text{if } \varepsilon_g(j) = \partial.$$

Finally, we specialize the definitions to the setting of intuitionistic logic and get the concept of a perfect Heyting algebra.

Definition 2.5.17. (Perfect Heyting algebra [9, 14]) A Heyting algebra $\mathfrak{A} = (U, \wedge, \vee, \to, \bot)$ is perfect if:

i. $(U, \wedge, \vee)$ is a perfect distributive lattice,

ii. $\bigvee X \to u = \bigwedge_{x \in X} x \to u$ and $u \to \bigvee X = \bigwedge_{x \in X} u \to x$, for all $X \subseteq U$.

2.6 Language extensions

This subsection describes the necessary extensions needed for the ALBA algorithm in Chapter 5. There are two extensions needed: one that adds connectives which are intended to be interpreted as residuals of the connectives in F and G, and the next extensions include nominals and co-nominals. For an in-depth understanding, the reader is referred to [8], which is generalised to non-distributive modal algebras.

Any language $\mathcal{L}_{\mathcal{LE}} = \mathcal{L}_{\mathcal{LE}}(F, G)$ can be associated with the language $\mathcal{L}^*_{\mathcal{LE}} = \mathcal{L}_{\mathcal{LE}}(F^*, G^*)$ where $F \subseteq F^*$ and $G \subseteq G^*$ are acquired by expanding $\mathcal{L}_{\mathcal{LE}}$ with the following connectives:

$$
\begin{array}{lll}
f(a_1, \ldots, a_i, \ldots, a_{n_f}) \leq b & \text{iff} & a_i \leq f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f}) \quad \text{if } \varepsilon_f(i) = 1, \\
f(a_1, \ldots, a_i, \ldots, a_{n_f}) \leq b & \text{iff} & f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f}) \leq a_i \quad \text{if } \varepsilon_f(i) = \partial, \\
b \leq g(a_1, \ldots, a_i, \ldots, a_{n_g}) & \text{iff} & g_i^\flat(a_1, \ldots, b, \ldots, a_{n_g}) \leq a_i \quad \text{if } \varepsilon_g(i) = 1, \\
b \leq g(a_1, \ldots, a_i, \ldots, a_{n_g}) & \text{iff} & a_i \leq g_i^\flat(a_1, \ldots, b, \ldots, a_{n_g}) \quad \text{if } \varepsilon_g(i) = \partial.
\end{array}
$$

Where we have that $f_i^\sharp \in G^*$ if $\varepsilon_f(i) = 1$, and $f_i^\sharp \in F^*$ if $\varepsilon_f(i) = \partial$. Dually we have that $g_i^\flat \in F^*$ if $\varepsilon_g(i) = 1$, and $g_i^\flat \in G^*$ if $\varepsilon_g(i) = \partial$. For example if $\varepsilon_f = (1, \partial)$ and $\varepsilon_g = (\partial, 1)$, then $\varepsilon_{f_1^\sharp} = (1, 1)$, $\varepsilon_{f_2^\sharp} = (1, \partial)$, $\varepsilon_{g_1^\flat} = (\partial, 1)$ and $\varepsilon_{g_2^\flat} = (1, 1)$.

For the second extension, we now expand our language $\mathcal{L}_{\mathcal{LE}}$ by adding an denumerably infinite set of sorted variable nominals (denoted as NOM), ranging over the completely join-prime elements of perfect algebras, and an denumerably infinite set of sorted variable co-nominals (denoted as CO-NOM), ranging over the completely meet-prime elements of perfect algebras. The nominals will be denoted with $i, j, \ldots$, and co-nominals will be denoted with $m, n, \ldots$; if needed, they will be indexed. The formulas φ of our extended language $\mathcal{L}_{\mathcal{LE}}{}^+$ are given by the following recursive definition:

$$\varphi ::= \psi \mid i \mid m \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $\psi \in \mathcal{L}_{\mathcal{LE}}$, $i \in$ NOM, $m \in$ CO-NOM, $f \in F^*$ and $g \in G^*$.

If $\mathfrak{L}$ is a perfect LE, then an *assignment* for $\mathcal{L}_{\mathcal{LE}}{}^+$ on $\mathfrak{L}$ is a map $V : \mathit{Prop} \cup \text{NOM} \cup \text{CO-NOM} \to \mathfrak{L}$ sending propositional variables to elements of $\mathfrak{L}$, nominals to $J^\infty(\mathfrak{L})$ and co-nominals to $M^\infty(\mathfrak{L})$.

In the case for intuitionistic logic, nominals and co-nominals become the principal upsets in the intuitionistic Kripke frame. A nominal i, is defined as a propositional variable with the condition that for some u and for every model $\mathfrak{M} = (\mathfrak{F}, V)$, $V(i) = \{v \in W | u \leq v\}$, in other words $V(i) = u{\uparrow}$. A co-nominal, m, similarly is a propositional variable but with the different condition that for every model $V(m) = \{v \in W | \neg(v \leq u)\}$, in other words $V(m) = (u{\downarrow})^c$.

Chapter 3

First-order Correspondence

Correspondence theory attempts to find *links* or *correspondences* between two different logics. Often, it is done to find correspondents from different kinds of logics to first-order logic. First-order logic is the preferred choice because of its centrality in mathematical logic and its "nice" proof theoretic and computational properties. The first systematic study of correspondence theory is by van Benthem in his PhD thesis [31], which focuses on correspondence in modal logic. Correspondence between various other logics have been subsequently studied, in particular, van Bentham's PhD student, Pieter Rodenburg, who examines correspondence in intuitionistic logic [26].

Definition 3.0.1. (Local and Global Correspondents [2, 33]) An $\mathcal{L}_0$ formula $\alpha(x)$ in one free variable is a local correspondent of an intuitionistic formula φ if, for all frames $\mathfrak{F}$ and points $w \in \mathfrak{F}$,

$$(\mathfrak{F}, w) \Vdash \varphi \quad \text{iff} \quad \mathfrak{F} \models \alpha(x)[x := w].$$

Similarly, a first-order sentence α is a (global) frame correspondent of an intuitionistic formula φ if,

$$\mathfrak{F} \Vdash \varphi \quad \text{iff} \quad \mathfrak{F} \models \alpha.$$

Here $\mathfrak{F} \models \alpha(x)[x := w]$ means that the first-order formula $\alpha(x)$ is satisfied in the model $\mathfrak{F}$ when w is assigned to the free variable x.

Correspondence theory seeks to identify conditions on frames that are necessary and sufficient to make certain formulas valid within those frames (these conditions need to be expressed in first-order logic). In the case of intuitionistic logic, we consider intuitionistic formulas and the corresponding intuitionistic Kripke frames, where specific restrictions on the frames are equivalent to the validity of these formulas. For instance, in the example considered in Chapter 1, we discussed how $p \vee \neg p$ is intuitionistically valid at worlds with no future; from the temporal interpretation, the future of a point in a Kripke frame can be considered as follows. If we were to consider ourselves at a point $u \in \mathfrak{F}$ with a point $v \in \mathfrak{F}$ such that $u < v$, then v can be considered as a future point from u. Hence, we would need a world that cannot allow such a v, and this restriction can be described in first-order logic, namely $\forall x, y(x \leq y \Rightarrow x = y)$.

To further justify this claim, assume that our frame has a point, u, with a future, v, then semantically for $p \vee \neg p$ to be true at the present point, u, we need either p to be true at u or $\neg p$ to be true at u. But $\neg p$ will only be true at u if all points in the future of, and including, u have p false. Now, if we assign p to be true at v, then in this valuation $p \vee \neg p$ is false. Hence, we cannot have any point with a future point. A formal argument on how we get the first-order correspondent for the law of excluded middle is produced using ALBA in Chapter 5, the first-order correspondent will end up being:

$$\mathfrak{F} \Vdash p \vee \neg p \quad \text{iff} \quad \mathfrak{F} \models \forall x, y(x \leq y \Rightarrow x = y).$$

We achieve statements like the one above by considering the intuitionistic formula and "breaking it down" using Kripke semantics, then trying to find a frame that, under a specific valuation, refutes the chosen formula. Now, we aim to describe, using first-order logic, a way to restrict the frames such that we cannot find a valuation in the restricted frames that refutes the formula. Finally, we will have to prove the "if and only if" statement to confirm that it holds in both directions.

3.1 Finding first-order correspondents

The axioms of intuitionistic logic listed in Table 3.1 have straightforward first-order correspondents. Since these axioms are valid on all intuitionistic Kripke frames, their first-order correspondents are equivalent to $\top$ on Kripke frames. Hence, for all axioms in Table 3.1 we set their first-order correspondents as $x = x$. First-order correspondents of the added axioms in superintuitionistic logics (sometimes referred to as intermediate logics, a superintuitionistic logic is any logic between intuitionistic propositional logic and classical propositional logic. Table 3.2 gives some examples of axioms of well-known superintuitionistic logics) have been discussed and were found in [3, 26]. A few examples are provided, but Table 3.3 summarizes all the results found in the literature.

Consider the following examples that prove the first-order correspondents of some superintuitionistic axioms. Table 3.3 lists the first-order correspondents of each of the superintuitionistic axioms we considered. The majority of the correspondents are from [3].

Proposition 3.1.1. The axiom added to KC has the following first-order correspondent:

$\mathfrak{F} \Vdash \neg p \vee \neg\neg p$ iff $\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \rightarrow \exists u(y \leq u \wedge z \leq u))$

Proof. The first-order correspondence is taken from [3], and the argument they provide is as below:

($\Leftarrow$) To prove this implication we will prove the contrapositive that is, we will prove if $\mathfrak{F} \nVdash \neg p \vee \neg\neg p$ then $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \forall u(\neg(y \leq u) \vee \neg(z \leq u)))$.

	Axioms
A_1	$p \to (q \to p)$
A_2	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
A_3	$p \wedge q \to p$
A_4	$p \wedge q \to q$
A_5	$p \to (q \to p \wedge q)$
A_6	$p \to p \vee q$
A_7	$q \to p \vee q$
A_8	$(p \to r) \to ((q \to r) \to (p \vee q \to r))$
A_9	$\bot \to p$

Table 3.1: Axioms for intuitionistic logic

Logic	Axioms
For	Int + p
Cl	Int + $p \vee \neg p$
SmL	Int + $(\neg q \to p) \to (((p \to q) \to p) \to p)$
KC	Int + $\neg p \vee \neg\neg p$
LC	Int + $(p \to q) \vee (q \to p)$
SL	Int + $((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$
KP	Int + $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$
WKP	Int + $(\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r)$
ND_k	Int + $(\neg p \to \neg q_1 \vee \cdots \vee \neg q_k) \to (\neg p \to \neg q_1) \vee \cdots \vee (\neg p \to \neg q_k)$
BD_n	Int + bd_n, where $bd_1 = p_1 \vee \neg p_1$ $bd_{n+1} = p_{n+1} \vee (p_{n+1} \to bd_n)$
BW_n	Int + $\bigvee_{i=0}^{n}(p_i \to \bigvee_{i \neq j} p_j)$
BTW_n	Int + $\bigwedge_{0 \leq i < j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{i \neq j} \neg p_j)$
T_n (bb_n)	Int + $\bigwedge_{i=0}^{n}((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$
B_n	Int + $\bigwedge_{i=0}^{n}(\neg p_i \leftrightarrow \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$
NL_n	Int + nf_n, where $nf_0 = \bot, nf_1 = p, nf_2 = \neg p, nf_\omega = \top$ $nf_{2m+3} = nf_{2m+1} \vee nf_{2m+2}$ $nf_{2m+4} = nf_{2m+3} \to nf_{2m+1}$

Table 3.2: List of superintuitionistic logics

Take an arbitrary Kripke frame $\mathfrak{F} = (W, \leq)$ such that $\mathfrak{F} \Vdash \neg p \vee \neg\neg p$. Then there is a point $u \in W$ and a valuation V such that $(\mathfrak{F}, V, u) \Vdash \neg p \vee \neg\neg p$. After applying the definition of Kripke semantics, we get that $(\mathfrak{F}, V, u) \Vdash \neg p \vee \neg\neg p$ iff $(\mathfrak{F}, V, u) \Vdash \neg p$ and $(\mathfrak{F}, V, u) \Vdash \neg\neg p$. The former is the case if there exists a point $v \geq u$ such that $(\mathfrak{F}, V, v) \Vdash p$, the latter is the case if there exists a point $w \geq u$ with $(\mathfrak{F}, V, w) \Vdash \neg p$. Clearly, u, v, and w must be distinct. Furthermore, there cannot be a point, say t, above both v and w. Assume that the point t does exist, then that implies that $(\mathfrak{F}, V, t) \Vdash \neg p$ and $(\mathfrak{F}, V, t) \Vdash p$, which is impossible. Hence, for $\mathfrak{F} \Vdash \neg p \vee \neg\neg p$ it must be that $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \forall u(\neg(y \leq u) \vee \neg(z \leq u)))$.

($\Rightarrow$) To prove this implication, we will use the contrapositive. Let $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \forall u(\neg(y \leq u) \vee \neg(z \leq u)))$, we want to show $\mathfrak{F} \Vdash \neg p \vee \neg\neg p$. Let $\mathfrak{F} = (W, \leq)$ be an arbitrary frame with $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \forall u(\neg(y \leq u) \vee \neg(z \leq u)))$, $W = \{u, v, w\}$, and relations $u \leq v$ and $u \leq w$. Therefore, $\mathfrak{F}$ contains the frame in Figure 3.1 as a subframe.

Define a valuation V on $\mathfrak{F}$ such that $V(p) = v\uparrow$. Note that since $v\uparrow \cap w\uparrow = \emptyset$, $w\uparrow \subseteq W \setminus V(p)$ and hence $(\mathfrak{F}, V, w) \Vdash \neg p$. Also $(\mathfrak{F}, V, v) \Vdash p$. Hence, $(\mathfrak{F}, V, u) \nVdash \neg\neg p$ and $(\mathfrak{F}, V, u) \nVdash \neg p$, i.e., $(\mathfrak{F}, V, u) \nVdash \neg p \vee \neg\neg p$.

□

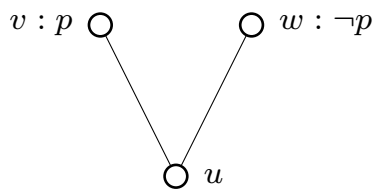


Figure 3.1: Kripke model rejecting KC

Proposition 3.1.2. The axiom added to LC has the following first-order correspondent:

$\mathfrak{F} \Vdash (p \to q) \vee (q \to p)$ iff $\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$

Proof. The first-order correspondence is taken from [3], and the argument they provide is as below:

($\Leftarrow$) To prove this implication we will prove the contrapositive that is, we will prove if $\mathfrak{F} \nVdash (p \to q) \vee (q \to p)$ then $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$.

Take an arbitrary Kripke frame $\mathfrak{F} = (W, \leq)$ such that $\mathfrak{F} \nVdash (p \to q) \vee (q \to p)$. Then there is a point $u \in W$ and a valuation V such that $(\mathfrak{F}, V, u) \nVdash (p \to q) \vee (q \to p)$. After applying the definition of Kripke semantics, we get that $(\mathfrak{F}, V, u) \nVdash (p \to q) \vee (q \to p)$ iff $(\mathfrak{F}, V, u) \nVdash p \to q$ and $(\mathfrak{F}, V, u) \nVdash q \to p$. The former is the case only if there exists a point $v \geq u$ with $(\mathfrak{F}, V, v) \Vdash p$ and $(\mathfrak{F}, V, v) \nVdash q$. The latter, similarly, is the case if there exists a point $w \geq u$ with $(\mathfrak{F}, V, w) \Vdash q$ and $(\mathfrak{F}, V, w) \nVdash p$. Clearly, v and w must be distinct. Moreover, both v and w must be distinct from the point u. If $(\mathfrak{F}, V, u) \nVdash p \to q$ then we cannot have $(\mathfrak{F}, V, u) \nVdash p$ and $(\mathfrak{F}, V, u) \Vdash q$, and

similarly for $(\mathfrak{F}, V, u) \nVdash q \to p$ we cannot have $(\mathfrak{F}, V, u) \Vdash p$ and $(\mathfrak{F}, V, u) \nVdash q$. Also, we can establish that v and w cannot be related to one another, since if $v \leq w$ then p would have to be true in w also, which leads to a contradiction. Hence, if $\mathfrak{F} \nVdash (p \to q) \vee (q \to p)$ then $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$.

($\Rightarrow$) To prove this implication we will make use of its contrapositive. Let $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$ we want to show $\mathfrak{F} \nVdash (p \to q) \vee (q \to p)$. Let $\mathfrak{F}$ be an arbitrary frame with $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$, $W = \{u, v, w\}$, and relations $u \leq v$ and $u \leq w$. Therefore, $\mathfrak{F}$ contains the frame in Figure 3.2 as a subframe.

Now define the valuations V on $\mathfrak{F}$ with $V(p) = v\uparrow$ and $V(q) = w\uparrow$. For sake of contradiction assume $(\mathfrak{F}, V, u) \Vdash (p \to q) \vee (q \to p)$, this is satisfied iff $(\mathfrak{F}, V, u) \Vdash p \to q$ or $(\mathfrak{F}, V, u) \Vdash q \to p$. The former cannot be true since that would imply that for all points above u, p is false or q is true, this is not satisfied at the point v. The latter similarly would imply that all points above u need to have p true and q false, which is not satisfied at w, so we have that $(\mathfrak{F}, V, u) \nVdash (p \to q) \vee (q \to p)$. Hence, if $\mathfrak{F} \Vdash (p \to q) \vee (q \to p)$ then $\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$. □

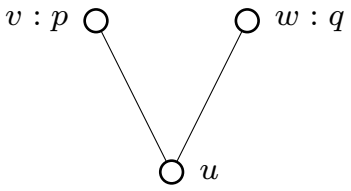


Figure 3.2: Kripke model rejecting LC

Proposition 3.1.3. The axiom added to BD_n has the following first-order correspondent:

$(\mathfrak{F}, u) \Vdash bd_n$ iff $\mathfrak{F} \models \forall x_1 \ldots, x_n((\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \to \bigvee_{i \neq j} x_i = x_j)$

Proof. The first-order correspondence is taken from [3], and they leave the argument as an exercise for the reader.We provide the argument for a local correspondence (which implies global correspondence) below:

($\Leftarrow$) To prove this implication, we will attempt to prove the contrapositive, that is, we will prove if $\mathfrak{F} \nVdash bd_n$ then $\mathfrak{F} \models \exists x_0, \ldots, x_n((\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \wedge \bigwedge_{i \neq j} x_i \neq x_j)$.

The proof will be by induction, so we will start by considering bd_1, $p_1 \vee \neg p_1$. Take an arbitrary Kripke frame $\mathfrak{F} = (W, \leq)$ such that $\mathfrak{F} \nVdash bd_1$. Then there is a point $u \in W$ and a valuation V on $\mathfrak{F}$ such that $(\mathfrak{F}, V, u) \nVdash p_1 \vee \neg p_1$. After applying the definition of Kripke semantics we have $(\mathfrak{F}, V, u_0) \nVdash p_1 \vee \neg p_1$ iff $(\mathfrak{F}, V, u_0) \nVdash p_1$ and $(\mathfrak{F}, V, u_0) \nVdash \neg p_1$. This implies the existence of two distinct points, u_0, u_1 with $u_0 \leq u_1$, since then we can assign p_1 true in u_1.

Consider bd_2, $p_2 \vee (p_2 \to (p_1 \vee \neg p_1))$. $(\mathfrak{F}, V, u_0) \nVdash p_2 \vee (p_2 \to (p_1 \vee \neg p_1))$ iff $(\mathfrak{F}, V, u_0) \nVdash p_2$ and $(\mathfrak{F}, V, u_0) \nVdash p_2 \to (p_1 \vee \neg p_1)$. The latter is the case if there exists a point above u_0, say u_1, with $(\mathfrak{F}, V, u_1) \Vdash p_2$ and $(\mathfrak{F}, V, u_1) \nVdash p_1 \vee \neg p_1$. Again, the

latter is the case iff $(\mathfrak{F}, V, u_1) \nVdash p_1$ and $(\mathfrak{F}, V, u_1) \nVdash \neg p_1$. The latter is the case if there exists a point above u_1 with p_1 true. Hence, we need a frame containing a chain of at least three points to make bd_2 false.

Let the induction hypothesis be that, if $(\mathfrak{F}, u) \nVdash bd_n$ then there is at least an $n+1$ point chain in $\mathfrak{F}$.

Now consider bd_{n+1}, i.e., $p_{n+1} \vee (p_{n+1} \to bd_n)$. Now, $(\mathfrak{F}, V, u_0) \nVdash p_{n+1} \vee (p_{n+1} \to bd_n)$ iff $(\mathfrak{F}, V, u_0) \nVdash p_{n+1}$ and $(\mathfrak{F}, V, u_0) \nVdash p_{n+1} \to bd_n$. The latter is the case if there exists a point above u_0, say u_1, with $(\mathfrak{F}, V, u_1) \Vdash p_{n+1}$ and $(\mathfrak{F}, V, u_1) \nVdash bd_n$. Now, by induction hypothesis, if $\mathfrak{F}, V \nVdash bd_n$ then there is at least an $n+1$ point linear sequence in $\mathfrak{F}$. Adding the point u_0, we have that for $\mathfrak{F} \Vdash bd_{n+1}$ we need a frame with at least an $n+2$ point linear sequence. So $\mathfrak{F} \nVdash bd_n$ if the frame has at least n+1 elements where $u_0 < u_1 < \cdots < u_n$, i.e., $\exists x_0, \ldots, x_n((\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \wedge \bigwedge_{i \neq j} x_i \neq x_j)$.

($\Rightarrow$) To prove this implication, we will make use of the contrapositive. Let $\mathfrak{F} \models \exists x_0, \ldots, x_n((\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \wedge \bigwedge_{i \neq j} x_i \neq x_j)$ we want to show that $\mathfrak{F} \nVdash bd_n$. Let $\mathfrak{F}$ be an arbitrary frame with $\mathfrak{F} \models \exists x_0, \ldots, x_n((\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \wedge \bigwedge_{i \neq j} x_i \neq x_j)$, $W = \{u_0, u_1, \ldots, u_n\}$, and relations $u_i \leq u_{i+1}$ for all $0 \leq i < n$. Therefore, $\mathfrak{F}$ contains the frame for bd_n in Figure 3.3 as a subframe.

Now define the valuation V on $\mathfrak{F}$ with $V(p_n) = u_1{\uparrow}$ and $V(p_{n-1}) = u_2{\uparrow}$ and ... and $V(p_1) = u_n{\uparrow}$. Now for sake of contradiction assume $(\mathfrak{F}, V, u_0) \Vdash bd_n$. This is satisfied iff $(\mathfrak{F}, V, u_0) \Vdash p_n$ or $(\mathfrak{F}, V, u_0) \Vdash p_n \to bd_{n-1}$. The latter is the case if all the points above u_0 have p_n false and bd_{n-1} true, p_n was assigned true at u_1, so we consider bd_{n-1}. For bd_{n-1}, the argument will remain the same; it will only be true if all the points above u_1 have p_{n-1} false, but u_2 had p_{n-1} true. We continue until the last node, and since there are $n+1$ points, we will end off in the same fashion, with needing p_1 false at all points above u_2, but u_1 has p_1 true. Hence, if $\mathfrak{F} \Vdash bd_n$ then $\mathfrak{F} \models \forall x_0, \ldots, x_n((\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \to \bigvee_{i \neq j} x_i = x_j)$. □

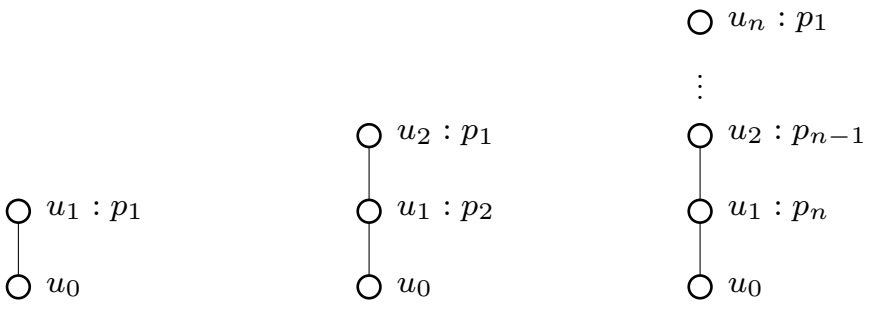


Figure 3.3: Kripke models rejecting bd_1, bd_2 and bd_n respectively

Proposition 3.1.4. The axiom added to BW_n has the following first-order correspondent:

$\mathfrak{F} \Vdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i \neq j} p_j)$ iff $\mathfrak{F} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \to \bigvee_{i \neq j} x_i \leq x_j)$

Proof. The first-order correspondence is taken from [3], where they state it without a proof; we provide the argument below:

($\Leftarrow$) To prove this implication we will prove the contrapositive that is, we will prove if $\mathfrak{F} \Vdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$ then $\mathfrak{F} \models \exists x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \wedge \bigwedge_{i\neq j} \neg(x_i \leq x_j))$.

Take an arbitrary Kripke frame $\mathfrak{F} = (W, \leq)$ such that $\mathfrak{F} \Vdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$. Then there is a point $u \in W$ and a valuation V on $\mathfrak{F}$ such that $(\mathfrak{F}, V, u) \nVdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$. After applying the definition of Kripke semantics we have $(\mathfrak{F}, V, u) \nVdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$ iff $(\mathfrak{F}, V, u) \nVdash p_0 \to (p_1 \vee p_2 \vee \cdots \vee p_n)$ and $(\mathfrak{F}, V, u) \nVdash p_1 \to (p_0 \vee p_2 \vee \cdots \vee p_n)$ and $\ldots$ and $(\mathfrak{F}, V, u) \nVdash p_n \to (p_0 \vee p_1 \vee \cdots \vee p_{n-1})$. Consider the first one, i.e., $(\mathfrak{F}, V, u) \nVdash p_0 \to (p_1 \vee p_2 \vee \cdots \vee p_n)$ iff there exists a point above $u_0 \geq u$ with $(\mathfrak{F}, V, u_0) \Vdash p_0$ and $(\mathfrak{F}, V, u_0) \nVdash p_1$ and $(\mathfrak{F}, V, u_0) \nVdash p_2$ and $\ldots$ and $(\mathfrak{F}, V, u_0) \nVdash p_n$. Similarly for the second one in the set, i.e., $(\mathfrak{F}, V, u) \nVdash p_1 \to (p_0 \vee p_2 \vee \cdots \vee p_n)$ iff there exists a point $u_1 \geq u$ with $(\mathfrak{F}, V, u_1) \Vdash p_1$ and $(\mathfrak{F}, V, u_1) \nVdash p_0$ and $(\mathfrak{F}, V, u_1) \nVdash p_2$ and $\ldots$ and $(\mathfrak{F}, V, u_1) \nVdash p_n$. The same argument continues for all formulas in the set. Clearly, all u_i points need to be distinct from each other since they each have p_i true and p_j false for $i \neq j$. Furthermore, all u_i points are distinct from u. Assume that some $u_i = u$, then u would have p_i true, and therefore all u_j $i \neq j$ would also have to have p_i true, which is a contradiction. Hence, the frame needs to have a width of at least $n + 1$, i.e., $\mathfrak{F} \models \exists x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \wedge \bigwedge_{i\neq j} \neg(x_i \leq x_j))$.

($\Rightarrow$) To prove this implication, we will make use of the contrapositive 12. Let $\mathfrak{F} \models \exists x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \wedge \bigwedge_{i\neq j} \neg(x_i \leq x_j))$ we want to show $\mathfrak{F} \nVdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$. Let $\mathfrak{F}$ be an arbitrary frame with $\mathfrak{F} \models \exists x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \wedge \bigwedge_{i\neq j} \neg(x_i \leq x_j))$, $W = \{u, u_0, u_1, \ldots, u_n\}$, and relations $u \leq u_i$ for all $0 \leq i \leq n$. Therefore, $\mathfrak{F}$ contains the frame in Figure 3.4 as a subframe.

Now define the valuations V on $\mathfrak{F}$ with $V(p_0) = u_0{\uparrow}$, $V(p_1) = u_1{\uparrow}$, $\ldots$, $V(p_n) = u_n{\uparrow}$. Now for sake of contradiction assume $(\mathfrak{F}, V, u) \Vdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$ this is satisfied iff $(\mathfrak{F}, V, u) \Vdash p_0 \to \bigvee_{0\neq j} p_j$ or $(\mathfrak{F}, V, u) \Vdash p_1 \to \bigvee_{1\neq j} p_j$ or $\ldots$ or $(\mathfrak{F}, V, u) \Vdash p_n \to \bigvee_{n\neq j} p_j$. Pick an arbitrary one in the set, $(\mathfrak{F}, V, u) \Vdash p_i \to \bigvee_{i\neq j} p_j$ iff for all points above u we have that p_i is false or some p_j with $i \neq j$ is true, if we consider the point u_i this clearly is a contradiction. Hence, if $\mathfrak{F} \Vdash \bigvee_{i=0}^{n}(p_i \to \bigvee_{i\neq j} p_j)$ then $\mathfrak{F} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \to \bigvee_{i\neq j} x_i \leq x_j)$. □

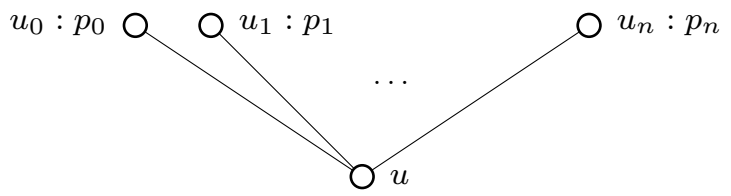


Figure 3.4: Kripke model rejecting BW_n

Proposition 3.1.5. The first-order correspondent for the formula bb_1 has the following first-order correspondent:

$\mathfrak{F} \Vdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$ iff $\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$

Proof. The first-order correspondent was not found in [3] since they incorrectly classified it as non-first-order definable. The argument for its first-order correspondent

is as follows:

($\Leftarrow$) To prove this implication, we will prove the contrapositive, that is, we will prove if $\mathfrak{F} \nVdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$ then $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$.

Take an arbitrary Kripke frame $\mathfrak{F} = (W, \leq)$ such that $\mathfrak{F} \nVdash bb_1$. Then there is a point $u_0 \in W$ and a valuation V on $\mathfrak{F}$ such that $(\mathfrak{F}, V, u_0) \nVdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$. After applying the definition of Kripke semantics we get that $(\mathfrak{F}, V, u_0) \nVdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$ iff there exists a $u_1 \geq u_0$ such that $(\mathfrak{F}, V, u_1) \Vdash ((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)$ and $(\mathfrak{F}, V, u_1) \nVdash p_0 \vee p_1$. The latter is the case if u_1 has both p_0 and p_1 false. The former is the case if $(\mathfrak{F}, V, u_1) \Vdash (p_0 \to p_1) \to p_1$ and $(\mathfrak{F}, V, u_1) \Vdash (p_1 \to p_0) \to p_0$. Again, the former is the case only if all points $u_2 \geq u_1$ have $(\mathfrak{F}, V, u_2) \nVdash p_1 \to p_0$ false or $(\mathfrak{F}, V, u_2) \Vdash p_1$ true. We cannot have $(\mathfrak{F}, V, u_1) \Vdash p_1$ since u_1 had both p_0 and p_1 false. So the only option is $(\mathfrak{F}, V, u_1) \nVdash p_1 \to p_0$, which is the case if there exists a point above u_1 with p_1 true and p_0 false. Similarly, we do the same for $(\mathfrak{F}, V, u_1) \Vdash (p_1 \to p_0) \to p_0$ and achieve a point above that has p_0 true and p_1 false. Now, clearly, all 3 points have to be distinct. Hence, we obtain that there must be three points, i.e, $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$.

($\Rightarrow$) to prove this implication, we will make use of a proof by contradiction. Let $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$ we want to show $\mathfrak{F} \nVdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$. Let $\mathfrak{F}$ be an arbitrary frame with $\mathfrak{F} \models \exists x, y, z(x \leq y \wedge x \leq z \wedge \neg(y \leq z) \wedge \neg(z \leq y))$, $W = \{u, v, w\}$, and relations $u \leq v$ and $u \leq w$. Therefore, $\mathfrak{F}$ contains the frame in Figure 3.5 as a subframe.

Now define the valuations V on $\mathfrak{F}$ with $V(p_0) = v\uparrow$ and $V(p_1) = W \setminus (v\uparrow \cup \{u\})$. Then $(\mathfrak{F}, V, u) \nVdash p_0 \vee p_1$. We are required to show $(\mathfrak{F}, V, u) \Vdash (p_0 \to p_1) \to p_1$ and $(\mathfrak{F}, V, u) \Vdash (p_1 \to p_0) \to p_0$.

Case I: $(\mathfrak{F}, V, u) \Vdash (p_0 \to p_1) \to p_1$: Consider $u \leq u_1$, we want to show $(\mathfrak{F}, V, u_1) \nVdash p_0 \to p_1$ or $(\mathfrak{F}, V, u_1) \Vdash p_1$.

i If $u_1 \in W \setminus (v\uparrow \cup \{u\})$ then $(\mathfrak{F}, V, u_1) \Vdash p_1$.

ii If $u_1 = u$ then $v \geq u_1$ and $(\mathfrak{F}, V, v) \Vdash p_0$ and $(\mathfrak{F}, V, v) \nVdash p_1$, so $(\mathfrak{F}, V, u_1) \nVdash p_0 \to p_1$.

iii If $u_1 \in v\uparrow$ then $(\mathfrak{F}, V, u_1) \nVdash p_1$ and $(\mathfrak{F}, V, u_1) \Vdash p_0$, so $(\mathfrak{F}, V, u_1) \nVdash p_0 \to p_1$.

Case II: $(\mathfrak{F}, V, u) \Vdash (p_1 \to p_0) \to p_0$: Consider $u \leq u_1$, we want to show $(\mathfrak{F}, V, u_1) \nVdash p_1 \to p_0$ or $(\mathfrak{F}, V, u_1) \Vdash p_0$.

i If $u_1 \in W \setminus (v\uparrow \cup \{u\})$ then $(\mathfrak{F}, V, u_1) \Vdash p_1$ and $(\mathfrak{F}, V, u_1) \nVdash p_0$, so $(\mathfrak{F}, V, u_1) \nVdash p_1 \to p_0$.

ii If $u_1 = u$ then $w \geq u_1$ and $(\mathfrak{F}, V, w) \Vdash p_1$ and $(\mathfrak{F}, V, w) \nVdash p_0$, so $(\mathfrak{F}, V, u_1) \nVdash p_0 \to p_1$.

iii If $u_1 \in v\uparrow$ then $(\mathfrak{F}, V, u_1) \Vdash p_1$.

So we have that $\mathfrak{F} \nVdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$. Hence, if $\mathfrak{F} \Vdash (((p_0 \to p_1) \to p_1) \wedge ((p_1 \to p_0) \to p_0)) \to p_0 \vee p_1$ then $\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$ $\square$

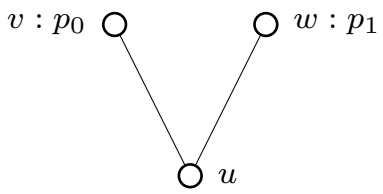


Figure 3.5: Kripke model rejecting bb_1

Logic	Global first-order correspondent
$\mathfrak{F} \Vdash \mathrm{For}$	$\mathfrak{F} \models x \neq x$
$\mathfrak{F} \Vdash \mathrm{Cl}$	$\mathfrak{F} \models \forall x, y(x \leq y \to x = y)$.
$\mathfrak{F} \Vdash \mathrm{SmL}$	$\mathfrak{F} \models \forall x, y(x \leq y \wedge x \neq y \to \neg\exists z(z \neq x \wedge z \neq y \wedge (x \leq z \vee z \leq x)))$ [3]
$\mathfrak{F} \Vdash \mathrm{KC}$	$\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to \exists u(y \leq u \wedge z \leq u))$ (Proposition 3.1.1[3])
$\mathfrak{F} \Vdash \mathrm{LC}$	$\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$ (Proposition 3.1.2[3])
$\mathfrak{F} \Vdash \mathrm{SL}$	No first-order correspondent
$\mathfrak{F} \Vdash \mathrm{KP}$	$\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \wedge \neg y \leq z \wedge \neg z \leq y \to \exists u(x \leq u \wedge u \leq y \wedge u \leq z \wedge \forall v(u \leq v \to \exists w(v \leq w \wedge (y \leq w \vee z \leq w)))))$[3]
$\mathfrak{F} \Vdash \mathrm{WKP}$	$\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to \exists u(y \leq u \wedge z \leq u))$
$\mathfrak{F} \Vdash \mathrm{ND}_k$	$\mathfrak{F} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=1}^{n} x \leq x_i \to \exists y \bigvee_{i \neq j}(x_i \leq y \wedge x_j \leq y))$
$\mathfrak{F} \Vdash \mathrm{BD}_n$	$\mathfrak{F} \models \forall x_0, \ldots, x_n(\bigwedge_{i=0}^{n-1} x_i \leq x_{i+1} \to \bigvee_{i \neq j} x_i = x_j)$ (Proposition 3.1.3 [3])
$\mathfrak{F} \Vdash \mathrm{BW}_n$	$\mathfrak{F} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \to \bigvee_{i \neq j} x_i \leq x_j)$(Proposition 3.1.4 [3])
$\mathfrak{F} \Vdash \mathrm{BTW}_n$	$\mathfrak{F} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=1}^{n} x \leq x_i \to \exists y \bigvee_{i \neq j}(x_i \leq y \wedge x_j \leq y))$ [3]
$\mathfrak{F} \Vdash \mathrm{T}_n$	$\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$ (Proposition 3.1.5) No first-order correspondent for $n \geq 2$
$\mathfrak{F} \Vdash \mathrm{B}_n$	$\mathfrak{F} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=1}^{n} x \leq x_i \to \exists y \bigvee_{i \neq j}(x_i \leq y \wedge x_j \leq y))$ [3]
$\mathfrak{F} \Vdash \mathrm{NL}_n$	$\mathfrak{F} \Vdash nf_{0,1,2,4}$ iff $\mathfrak{F} \models x \neq x$ $\mathfrak{F} \Vdash nf_{\omega}$ iff $\mathfrak{F} \models x = x$ $\mathfrak{F} \Vdash nf_{3,6,7}$ iff $\mathfrak{F} \models \forall x, y(x \leq y \Rightarrow x = y)$ $\mathfrak{F} \Vdash nf_{5,8,9}$ iff $\mathfrak{F} \models \forall x, y, z(x \leq y \wedge x \leq z \to \exists u(y \leq u \wedge z \leq u))$ No first-order correspondent for $n \geq 10$

Table 3.3: First-order correspondences of the superintuitionistic logics

3.2 Inductive Formulas

This chapter introduces Sahlqvist and inductive formulas, both of which provide syntactic sufficient conditions on formulas for them to be elementary (have a first-order correspondent) and canonical (logics built from canonical axioms are complete). These syntactic methods of identifying classes of first-order definable formulas simplify the process vastly and are areas of active research in correspondence theory.

Various definitions will be given that have been adapted from their given generalised versions to the intuitionistic logic considered in this thesis.

3.2.1 Branch classification

Definition 3.2.1 (Signed Generation Tree [8])**.** The positive or negative generation tree of any $\mathcal{L}_{\mathcal{IP}}$-formula φ ($sg(\varphi)$) is defined by labeling the root node of the generation tree of φ with the sign $+$ or $-$, and then propagating the labeling on each remaining node as follows:

- For any node labeled with $\vee$ or $\wedge$, assign the same sign to its children.
- If a node is labeled with $\rightarrow$, assign the opposite sign to its left child and the same sign to its right child.

Nodes in signed generation trees are positive or negative depending on if they are assigned $+$ or $-$.

For simplicity, we shall introduce the following notations: $sg^+(\varphi)$ to denote the positively signed generation tree of φ and similarly $sg^-(\varphi)$ for the negatively signed generation tree of φ. We use $sg(\varphi') \preceq sg^-(\varphi)$ to denote that $sg(\varphi')$ is a subtree of $sg^-(\varphi)$ that inherits its polarities from $sg^-(\varphi)$ (similarly for $sg(\varphi') \preceq sg^+(\varphi)$).

For example, one can use Figure 3.8 to see how to construct the signed generation tree of the axiom in SL.

Connectives are classified as $\triangle$-adjoints, SLR, SRR, or SRA based on how they distribute over lattice expansions (LE) and distributive lattice expansions (DLE). The suggested reading to gain a detailed understanding of how this classification came into being is [8, 7]. For the purpose of this thesis, we shall state a brief overview of how we classify these connectives.

All definitions from [8] can be applied to intuitionistic logic since it is a variety of LE. Table 3.4, which classifies the connectives as Skeleton and PIA, is taken from [8].

To see which connectives are f and g, we have the following. Let $\mathfrak{A}$ be a Heyting algebra, given the connectives $\vee$, $\wedge$, $\rightarrow$ and $\neg$ we have:

$\underline{\wedge : \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}}$

The meet operation is binary, hence it gets classified as either SLR or SRR.

Skeleton	PIA
$\triangle$-adjoints	Syntactically Right Adjoint (SRA)
$+ \quad \vee$	$+ \quad \wedge \quad g \quad$ with $n_g = 1$
$- \quad \wedge$	$- \quad \vee \quad f \quad$ with $n_f = 1$
Syntactically Left Residual (SLR)	Syntactically Right Residual (SRR)
$+ \quad f \quad$ with $n_f \geq 1$	$+ \quad g \quad$ with $n_g \geq 2$
$- \quad g \quad$ with $n_g \geq 1$	$- \quad f \quad$ with $n_f \geq 2$

Table 3.4: Node classification for LEs

$\wedge(a \vee b, c) = (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = \wedge(a, c) \vee \wedge(b, c)$,

$\wedge(a \wedge b, c) = (a \wedge b) \wedge c = (a \wedge c) \wedge (b \wedge c) = \wedge(a, c) \wedge \wedge(b, c)$.

A similar argument can be done to show that, $\wedge(a, b \vee c) = \wedge(a, b) \vee \wedge(a, c)$ and $\wedge(a, b \wedge c) = \wedge(a, b) \wedge \wedge(a, c)$,

$\wedge(\bot, a) = \bot$ and $\wedge(a, \bot) = \bot$.

Hence, the binary meet operation is an f operation, and $+\wedge$ gets classified as SLR, and $-\wedge$ gets classified as SRR.

$\underline{\vee : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}}$

$\vee(a \wedge b, c) = (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) = \vee(a, c) \wedge \vee(b, c)$,

$\vee(a \vee b, c) = (a \vee b) \vee c = (a \vee c) \vee (b \vee c) = \vee(a, c) \vee \vee(b, c)$.

A similar argument can be done to show that, $\vee(a, b \wedge c) = \vee(a, b) \wedge \vee(a, c)$ and $\vee(a, b \vee c) = \vee(a, b) \vee \vee(a, c)$,

$\vee(\top, a) = \top$ and $\vee(a, \top) = \top$.

Hence, the binary join operation is a g operation, and $-\vee$ gets classified as SLR, and $+\vee$ gets classified as SRR.

$\underline{\to : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}}$

$\to (a \vee b, c) = \to (a, c) \wedge \to (b, c)$ by (v) in the definition of a Heyting algebra, Definition 2.5.6.

$\to (a, b \wedge c) = \to (a, b) \wedge \to (a, c)$ by (v) in the definition of a Heyting algebra, Definition 2.5.6.

$a \to \top$ iff $\top \leq a \to \top$ iff $\top \wedge a \leq \top$ iff $a \leq \top$ which is true for all $a \in \mathfrak{A}$.

$\bot \to a$ iff $\top \leq \bot \to a$ iff $\top \wedge \bot \leq a$ iff $\bot \leq a$ which is true for all $a \in \mathfrak{A}$.

Hence, the binary implication operation is a g operation, and $- \to$ gets classified as SLR, and $+ \to$ as SRR.

$\underline{\neg : \mathfrak{A} \to \mathfrak{A}}$

$\neg(a \vee b) = (a \vee b) \to \bot = (a \to \bot) \wedge (b \to \bot)$ by (v) in the definition of a Heyting algebra, Definition 2.5.6, and $(a \to \bot) \wedge (b \to \bot) = \neg a \wedge \neg b$.

$\neg \bot = \bot \to \bot = \top$ by (iii) in the definition of a Heyting algebra, Definition 2.5.6.

Hence, the unary operation negation is a g operation, and $+\neg$ gets classified as SRA, and $-\neg$ as SLR.

Finally, we achieve Table 3.5.

Skeleton	PIA
$\triangle$-adjoints	SRA
$+\vee$ $-\wedge$	$+\wedge$ $-\vee$
	$+\neg$
SLR	SRR
$+\wedge$ $-\vee$	$+\vee$ $-\wedge$
$-\neg$ $-$ $\to$	$+$ $\to$

Table 3.5: Skeleton and PIA nodes for $\mathcal{L}_{\mathcal{IP}}$

Remark 3.2.2. Since $\neg$ is a defined connective, we coloured both $+\neg$ and $-\neg$ in Table 3.5 red; it is equally valid to remove them. We have added them since they simplify the process of classifying $\mathcal{L}_{\mathcal{IP}}$ formulas.

3.2.2 (Ω, ε)-inductive formulas

We are now ready to define our first syntactic class of first-order definable formulas, which is the specialization of the general definition in [7] and [8] to the setting of intuitionistic formulas.

Definition 3.2.3 (Good Branch [8])**.** A branch in a signed generation tree $sg(\varphi)$ is called a good branch if it can be represented as a concatenation of two paths P_1 and P_2, any of which may possibly be of length 0, such that P_1 is a path starting from the leaf of the branch and consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

Definition 3.2.4 (Excellent Branch [8])**.** A good branch is excellent if it is a concatenation of paths P_1 and P_2 as above (each possibly empty), where only SRA-nodes can occur on P_1.

Definition 3.2.5 (Polarity-Type [8])**.** A polarity-type ε maps each propositional variable p to a polarity of either 1 or ∂ ($+$ or $-$ respectively).

In a signed generation tree, an ε-critical node is a leaf node p_i such that if p_i has a polarity of $-$ then $\varepsilon(p_i) = -$, similarly for $+$. An ε-critical branch is the branch that extends to the ε-critical node. With this terminology, we are ready to define one class of syntactic formulas that are first-order definable. The definition has been specialized to our case, where all connectives are considered as binary or unary.

Definition 3.2.6 ((Ω, ε)-inductive generation tree, formulas and inequalities [8])**.** For any order type ε and any irreflexive transitive relation (strict partial order), $<$ on $p_0, p_1, \ldots, p_n$, the signed generation tree $sg(\varphi)$ of a formula φ, with variables $p_0, p_1, \ldots, p_n$, is (Ω, ε)-inductive if,

- for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good;
- every SRR-node, $\star$, occurring in the critical branch is of the form $\star(\gamma, \beta)$, where if the ε-critical variable occurs in γ then:
  - there is no occurrence of a ε-critical variable in $sg(\beta) \preceq sg(\varphi)$,
  - $p_k <_\Omega p_i$ for every p_k occurring in β and for every $1 \leq k \leq n$.

From here onward, when clear from the context, we shall drop the Ω in $<_\Omega$ when listing the orderings of an (Ω, ε)-inductive formula.

An inequality $\psi \leq \chi$ is (Ω, ε)-inductive if the signed generation trees $sg^+(\psi)$ and $sg^-(\chi)$ are (Ω, ε)-inductive. A formula φ can be turned into an inequality by considering $\top \leq \varphi$, then one can check, as in the case of an inequality, if φ is (Ω, ε)-inductive.

Given the definition of (Ω, ε)-inductive formulas, one can easily see that Sahlqvist formulas, the next definition, can be considered as a subset of (Ω, ε)-inductive formulas where there is no $<_\Omega$ ordering.

Definition 3.2.7 (Sahlqvist formula [8])**.** A formula φ is called Sahlqvist if there is a polarity type ε such that in the negative signed generation tree of φ for every variable p_i, if
$\varepsilon(p_i) = 1$ (or $+$) then all branches ending in $+p_i$ are excellent, or
$\varepsilon(p_i) = \partial$ (or $-$) then all branches ending in $-p_i$ are excellent.

The theorems that claim both Sahlqvist and inductive formulas are elementary and canonical are in Chapter 5.1.

We now consider all formulas in both Table 3.1 and 3.2 and classify them as inductive or not. Signed generation trees drawn from now on will have green representing excellent branches and blue representing good branches.

Example 3.2.8. $A_1 = p \to (q \to p)$ (Figure 3.6), for every choice of ε all critical occurrences will be on excellent branches.

Example 3.2.9. $A_8 = (p \to (q \to r)) \to ((p \to q) \to (p \to r))$ (Figure 3.7) cannot be Sahlqvist since both $+q$ and $-q$ occur as a leaf of a branch that is not excellent (branches numbered 5 and 2 respectively). To check that the formula is inductive, we see that it fulfills the first criterion where every ε-critical branch is good when $\varepsilon(p) = \varepsilon(q) = +$ and $\varepsilon(r) = -$ is chosen. Considering $\varepsilon(p) = +$, which is the leaf node of branch 6 (excellent branch), hence has no SRR node in the branch, so it fulfills the second criterion. Considering $\varepsilon(q) = +$ in branch 5 (good branch) the only SRR node in the branch is $+ \to$ and since $+q$ occurs on the right child, we need to look at the propositional variable occurring on the left child which is $-p$

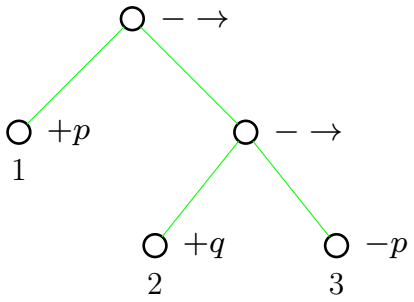


Figure 3.6: Signed generation tree for A_1

which is not ε-critical, hence the second criterion is fulfilled with the ordering $p < q$. Finally, $\varepsilon(r) = -$ occurs in branch 7, which is excellent, and similarly to $\varepsilon(p) = +$ fulfills the second criterion.

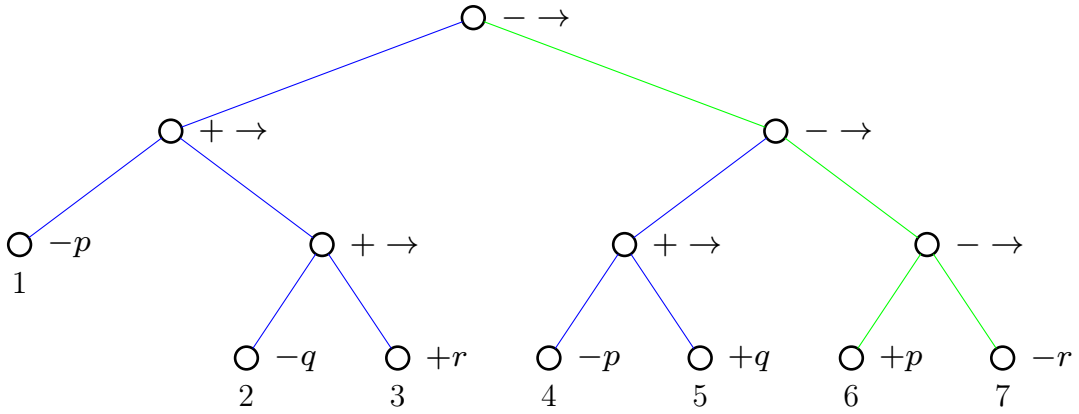


Figure 3.7: Signed generation tree for A_8

Example 3.2.10. The axiom added to SmL, $(\neg q \to p) \to (((p \to q) \to p) \to p)$, (Figure 5.1) is not inductive since both $+q$ and $-q$ occur as a leaf of a branch that is not good (branches 1 and 4 respectively), hence doesn't fulfill the first criterion for ε-inductive formulas.

Example 3.2.11. The axiom added to SL, $((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$, (Figure 3.8) is not inductive since there exists an occurrence of both $+p$ and $-p$ in a branch that is not good (branches 1 and 2 respectively), hence doesn't fulfill the first criterion for inductive formulas.

Example 3.2.12. The axiom added to WKP, $(\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r)$, is inductive for $\varepsilon(p) = -$ and $\varepsilon(q) = \varepsilon(r) = +$ in which all chosen ε-critical nodes occur in good branches, the nodes $+q$ and $+r$ occur in excellent branches and hence have no SRR nodes in the respective ε-critical branches so there is no required ordering. The branch ending in $-p$ has an SRR node, $+ \to$, but the subtree generated by this node does not have another propositional variable, hence the ordering is empty.

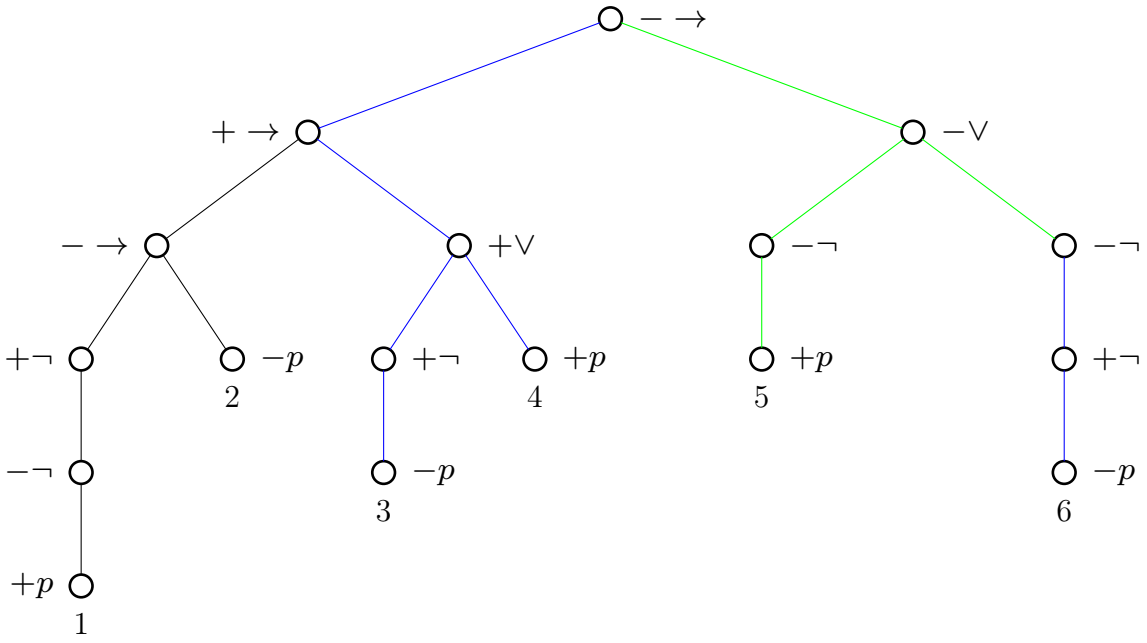


Figure 3.8: Signed generation tree for SL

Example 3.2.13. The axioms added to BW_n, $\bigvee_{i=0}^{n}(p_i \rightarrow \bigvee_{i\neq j} p_j)$, (Figure 3.9) since the only non-leaf nodes are $-V$ and $-\rightarrow$, therefore all branches are excellent. Hence, the formula is Sahlqvist.

Example 3.2.14. The axioms added to T_n, $\bigwedge_{i=0}^{n}((p_i \rightarrow \bigvee_{i\neq j} p_j) \rightarrow \bigvee_{i\neq j} p_j) \rightarrow \bigvee_{i=0}^{n} p_i$, are not inductive since for all polarity types some propositional variable occurs in a branch that is not good.

Example 3.2.15. The axioms added to NL_n, nf_n, are not inductive for $n \geq 6$, since for nf_6 (Figure 5.2) we have that branches 1,2,6, and 7 are excellent, 3 and 5 are good, and 4 is bad. Since $+p$ is on branch 4, we cannot have $\varepsilon(p) = +$. Let $\varepsilon(p) = -$, now there is an instance of a $-p$ (branch 3) such that the SRR node in its critical branch has a $-p$ in its other branch. Hence nf_6 is not inductive. Now, since all nf_n with $n > 6$ have nf_6 as a subtree, we can easily see that $nf_{n>6}$ will also not be inductive.

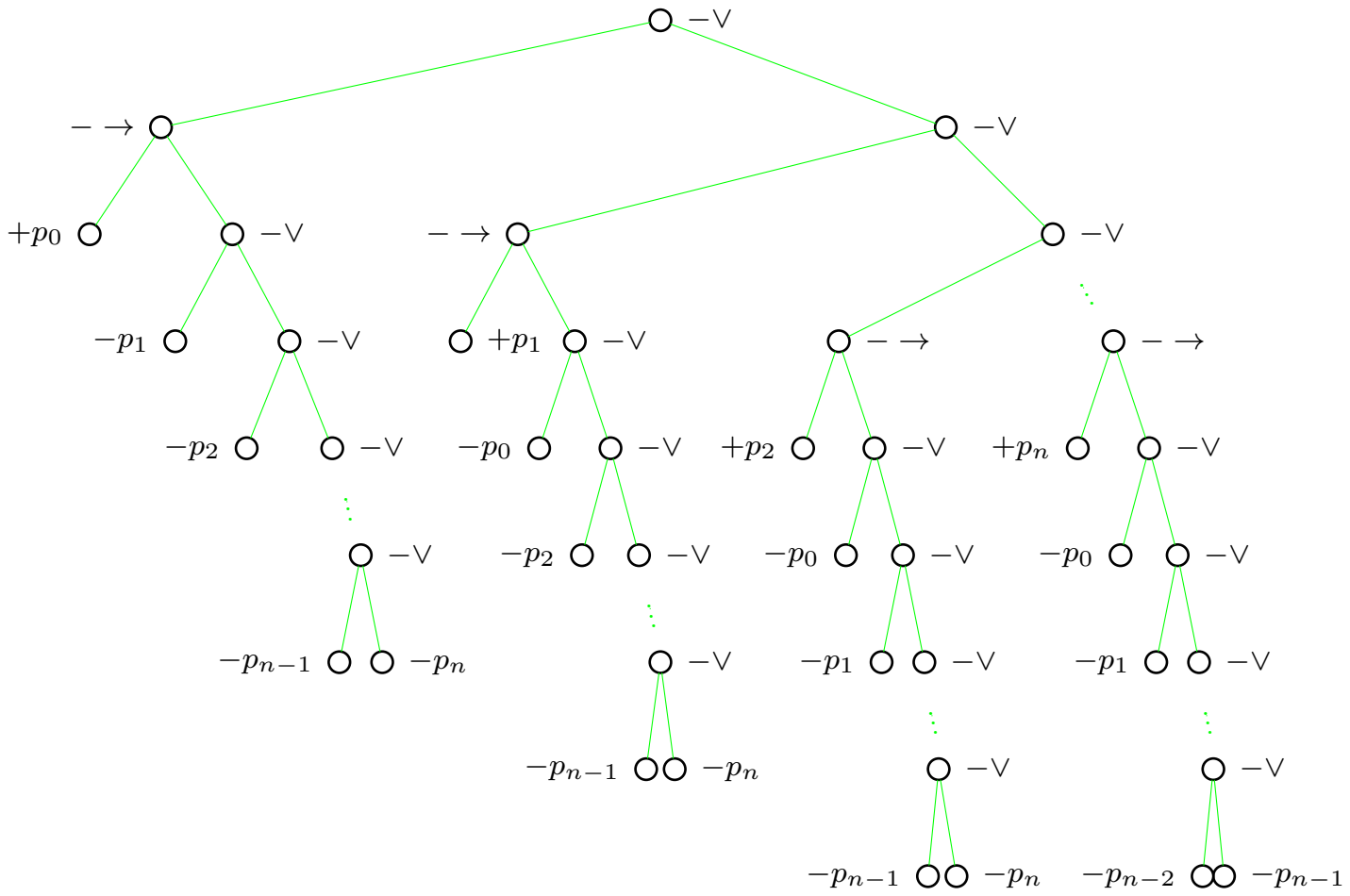


Figure 3.9: Signed generation tree for BW_n

3.3 Deterministic formulas

Deterministic formulas, defined by Rodenburg [26], are another class of elementary formulas that are syntactically classified.

Deterministic formulas form a different class of elementary formulas when compared to ε-inductive formulas. We provide the definition of a deterministic formula without much justification, since the preliminaries needed fall outside of the scope of our thesis. The reasoning for why all deterministic formulas have first-order correspondents can be found in Rodenburg's thesis, Chapter 7.

Definition 3.3.1. Deterministic formulas are defined as follows:

- An intuitionistic implication $\psi \to \chi$ is *determinate* if $+sg(\chi)$ contains no disjunction signed $+$, i.e., contains no $+\vee$.
- An intuitionistic formula φ is *deterministic* if in the negative signed generation tree $-sg(\varphi)$, for all occurrences of a positive implication, $+(\psi \to \chi)$, $\psi \to \chi$ is determinate.

Drawing the negative signed generation tree of a formula φ, it is quite easy to see if it is deterministic or not; we just need to see if there is an occurrence of a $+\vee$ on the consequent of any $+$ signed implication.

The following theorem with its proofs can be found in [26].

Theorem 3.3.2. Deterministic formulas are first-order definable.

Examples of deterministic formulas are as follows:

Example 3.3.3. For the axiom added to SmL, $(\neg q \to p) \to (((p \to q) \to p) \to p)$, since there are no disjunctions, it is deterministic.

Example 3.3.4. For the axiom added to SL, $((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$, the subformula highlighted in red is an instance of a positive disjunction in the consequent of a positive implication, hence SL is not deterministic.

Example 3.3.5. For the axiom added to WKP, $(\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r)$, the subformula highlighted in red indicates that it is not deterministic.

Example 3.3.6. For the axiom added to BD_n, bd_n, where $bd_1 = p_1 \vee \neg p_1$ and $bd_{n+1} = p_{n+1} \vee (p_{n+1} \to bd_n)$ we use induction. We can easily see that bd_1 is deterministic. Assume that bd_n is deterministic. Now for $bd_{n+1} = p_{n+1} \vee (p_{n+1} \to bd_n)$ since bd_n is on the consequent of an implication we know that $sg(bd_n) \preceq sg^-(bd_{n+1})$ is also a negative so the polarities in bd_n remain the same and does not affect the deterministic nature of bd_{n+1}. It can easily be seen that the remainder of the formula also has no occurrences of $+\vee$ in the consequent of a $+\to$. Hence, bd_n is deterministic.

Table 3.6 lists all the formulas considered in this thesis thus far and whether they are deterministic, Sahlqvist, or (Ω, ε)-inductive (with their ε-critical variables and

ordering). There is a column that lists the chosen ε-critical variable and ordering (if there is one) for each formula (there may be different combinations of ε-critical variables and orderings that will work, we just provide one such combination). The formulas that are deterministic and not ε-inductive and vice versa should be made note of since they indicate that neither class is a subclass of the other.

The existence of this syntactic classification allows for interesting results. What classifies a formula as deterministic or not is a particular combination of both the disjunction and implication. The following corollary is a direct consequence:

Corollary 3.3.7. i. An intuitionistic formula φ is first-order definable if φ contains no disjunctions.

ii. An intuitionistic formula φ is first-order definable if φ contains no implications.

Both inductive formulas and deterministic formulas define syntactic classes of formulas that are elementary, but it is to be noted that if a formula is neither inductive nor deterministic, then it does not definitively mean that there does not exist a first-order correspondent. Meaning that the added formulae in the logics SL, T_n (where $n \geq 2$), B_n, and NL_n (where $n \geq 6$) have each to be considered individually to determine the existence of a first-order correspondent or not.

Logic	Deterministic	Sahlqvist	ε-inductive	Ordering
A_1	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
A_2	✓	✓	✓	$\varepsilon(p) = +$ $\varepsilon(q) = \varepsilon(r) = -$
A_3	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
A_4	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
A_5	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
A_6	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
A_7	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
A_8	✓		✓	$\varepsilon(p) = \varepsilon(q) = +$ $\varepsilon(r) = -$ $p < q$
A_9	✓	✓	✓	$\varepsilon(p) = +$
For	✓	✓	✓	$\varepsilon(p) = -$
Cl	✓	✓	✓	$\varepsilon(p) = +$
SmL	✓			Not ε-inductive
KC	✓	✓	✓	$\varepsilon(p) = +$
LC	✓	✓	✓	$\varepsilon(p) = \varepsilon(q) = +$
SL				Not ε-inductive
KP			✓	$\varepsilon(p) = \varepsilon(q) = \varepsilon(r) = -$ empty ordering
WKP			✓	$\varepsilon(p) = -$ $\varepsilon(q) = \varepsilon(r) = +$ empty ordering
ND_k			✓	$\varepsilon(p) = -$ $\varepsilon(q_i) = +$, $1 \leq i \leq k$ empty ordering
BD_n	✓	✓	✓	$\varepsilon(p_i) = +$, $0 \leq i \leq n$
BW_n	✓	✓	✓	$\varepsilon(p_i) = +$, $0 \leq i \leq n$
BTW_n	✓		✓	$\varepsilon(p_i) = -$, $0 \leq i \leq n$ empty ordering
T_n	✓			$n = 1$
				Not ε-inductive for $n \geq 1$
B_n	✓			$n = 1$
				Not ε-inductive for $n \geq 1$
NL_n	✓	✓	✓	$\varepsilon(p) = +$, $1 \leq n \leq 5$
	✓			Not ε-inductive for $n = 6$ and 7
				Not ε-inductive for $n \geq 8$

Table 3.6: Classification of superintuitionistic logics

Chapter 4

Non-elementary intuitionistic formulas

We have discussed formulas that are first-order definable and have stated their frame conditions in Table 3.3, but some of the axioms added to the logics: SL, T_n, and NL_n (for $n \leq 10$) are claimed to be non-first-order definable. The question is, how exactly does one prove non-first-order definability? We make use of the "limits" of first-order logic. There are various methods one can use; the common strategy between each of them is to find two structures that are known to both describe the same set of first-order sentences, then show that our formula φ is validated in one structure while invalidated in the other, this in-turn implies that φ is not first-order definable.

Various first-order preserving operations can be found in [5]. We will discuss and give detailed examples of three, namely using the Löwenheim–Skolem theorem, Compactness Theorem, and ultrapowers.

4.1 Scott Axiom

The Scott axiom, the axiom added to the logic SL, $((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$, we shall denote it with SA from now on, is our first example of a non-first-order definable formula. It will be useful to consider the following frame refuting SA.

Consider the 4 point Kripke model, $\mathfrak{M} = (W, \leq, V)$, given in Figure 4.1, where $W = \{u_0, u_1, u_2, u_3\}$, relations are $u_0 < u_1$, $u_0 < u_3$ and $u_1 < u_2$, and $V(p) = \{u_2\}$ and $V(\neg p) = \{u_3\}$.

We want to show that $\mathfrak{M}, u_0 \nVdash ((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$, in other words:

i. $u_0 \Vdash (\neg\neg p \to p) \to \neg p \vee p$,

ii. $u_0 \nVdash \neg p \vee \neg\neg p$.

Clearly, $u_0 \nVdash \neg p \vee \neg\neg p$. (i) implies that all points above u_0 must have $\neg\neg p \to p$

false or $\neg p \vee p$ true. Clearly, u_2 and u_3 have $\neg p \vee p$ true. For u_0 and u_1, $\neg\neg p \to p$ is false, since p is not true at both points and $\neg\neg p$ is true (since both points have a successor with p true).

Hence, $\mathfrak{M}$ refutes SA. We will call its underlying frame $\mathfrak{SA}$. □

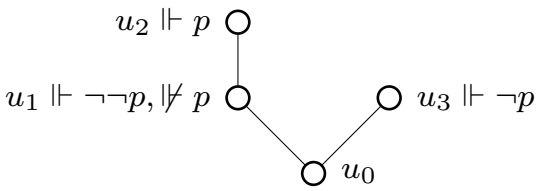


Figure 4.1: Kripke model rejecting SA

We provide three different proofs below of why the Scott axiom is not first-order definable.

4.1.1 Löwenheim–Skolem theorem

First seen in the downwards case, Leopold Löwenheim proved the result known as the downwards Löwenheim–Skolem theorem in [22]. Unfortunately, the proof was considered incorrect [11], since it made use of a lemma that was not a known result at that time. A "correct" proof was provided for the downwards case by Thoralf Skolem in [30], and Anatoly Ivanovich Maltsev in [23] completed the theorems by proving the upwards Löwenheim–Skolem theorem. The theorem needed for this argument is the downwards Löwenheim–Skolem theorem:

Lemma 4.1.1. [12] Let $\mathcal{L}$ be a first-order language and $\mathfrak{M}$ a model for $\mathcal{L}$ with domain W, $U \subseteq W$, μ a cardinal number, such that $\aleph_0, |\mathcal{L}|, |U| \leq \mu \leq |W|$. Then a submodel $\mathfrak{M}'$ with $\mathfrak{M}' \subseteq \mathfrak{M}$ exists such that $U \subseteq W'$ and $|W'| = \mu$.

Theorem 4.1.2. (Downwards Löwenheim–Skolem theorem) [12] Assume the conditions of Lemma 4.1.1 hold, then an elementary submodel $\mathfrak{M}' \prec \mathfrak{M}$ exists such that $U \subseteq W'$ and $|W'| = \mu$.

The theorem, in a nutshell, states that; given any model with some infinite number of points, say κ, we can always find an elementary submodel (two models $\mathfrak{M}'$ and $\mathfrak{M}$ are elementary submodels of one another, denoted as $\mathfrak{M}' \prec \mathfrak{M}$, if $\mathfrak{M}' \subseteq \mathfrak{M}$ and they both satisfy the same set of first-order formulas(with parameters from $\mathfrak{M}'$)) that has an infinite number of points, say κ', with $\kappa' \leq \kappa$.

The following proof is a detailed version of the sketch taken from [33] which makes use of the downwards Löwenheim–Skolem theorem to show that the Scott axiom is not first-order definable.

Definition 4.1.3. (Co-final set) Let $\mathfrak{F} = (W, \leq)$ be an intuitionistic frame. A proper subset of W, say $U \subsetneq W$, is said to be *co-final* in W if for all $w \in W$ there is a point $u \in U$ such that $w \leq u$.

Step 1: Find a frame that validates the SA.

Consider the following Kripke frame $\mathfrak{F} = (W, \leq)$.

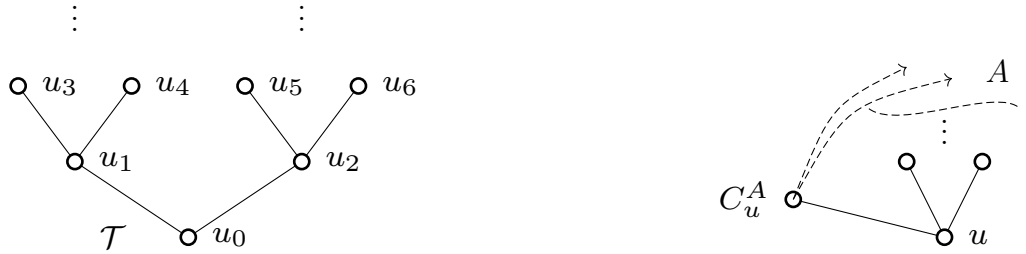


Figure 4.2: Kripke frame validating Scott axiom

Let $\mathcal{T}$ be an infinite binary tree rooted at a point u_0, for all points $u \in \mathcal{T}$, we write the subtree of $\mathcal{T}$ that is rooted at u using capital letters, so U is the subtree rooted at u. We also define for each point $u \in \mathcal{T}$, an extra point C_u^A such that $u \leq C_u^A$ and $C_u^A \leq a$ for all $a \in A$ where A is an upwards closed co-final set of U (C_u^A is an extra point outside of the binary tree). $\mathfrak{F}$ is defined to be both the infinite binary tree together with the points C_u^A for all $u \in \mathcal{T}$ and for all co-final sets A of U. Formally we have that for all $u \in \mathcal{T}$ and upwards closed co-final subset A of U:

i. $u \leq C_u^A \leq a$ for all $a \in A$,

ii. $C_u^A \leq C_u^{A'}$ when $A' \subseteq A$.

It may be beneficial to take some time and properly explain the formulation of $\mathfrak{F}$, in particular, these "extra points" C_u^A. From (ii), we find that if $A' \subseteq A$ then $C_u^A \leq C_u^{A'}$. Now this point $C_u^{A'}$ is related to u by transitivity, but it is not a direct successor of u. Moreover, since U has a least element there is a biggest co-final set, namely the set $U \setminus u{\downarrow}$ (we cannot have the set U alone since then it would imply that $u \leq C_u^U$ and $C_u^U \leq u$ and this contradicts the anti-symmetry property, this is why we defined a co-final set as being a proper subset), Claim 4.1.4 proves that $U \setminus u{\downarrow}$ is upwards closed and co-final. This implies that the only direct successors of u are the two points in the binary tree and the point $C_u^{U \setminus u{\downarrow}}$.

Claim 4.1.4. Let A be an upwards closed co-final set in U, then for any $a \in A$, $A \setminus a{\downarrow}$ is an upwards closed co-final set in U.

Proof. $A \setminus a{\downarrow}$ is upwards closed: Consider any point $a' \in A \setminus a{\downarrow}$ and any point b' with $a' \leq b'$. Both $a' \in A$ and $b' \in A$ since $A \setminus a{\downarrow}$ removes all points below a in A and b' was above a'. Furthermore, $b' \not\leq a$ since then it would imply that $a' \leq a$, which is a contradiction. Hence, we have that $b' \in A \setminus a{\downarrow}$ implying that $A \setminus a{\downarrow}$ is upwards closed.

$A \setminus a{\downarrow}$ is co-final in U: Consider any point $t \in U$ then $t \leq b$, for some $b \in A$, since A is co-final. There exists a point $a' \in A \setminus a{\downarrow}$ with $b \leq a'$ since U is an infinite binary tree, so $t \leq a'$. Thus, $A \setminus a{\downarrow}$ is co-final. □

The existence of the smallest point $C_u^{U \setminus u\downarrow}$ applies to any point $u_i \in \mathcal{T}$. This implies that for any point in the binary tree $\mathcal{T}$, there are only three direct successors. For points of the form $C_{u_i}^A$, we have the following claim.

Claim 4.1.5. All points of the form $C_{u_i}^A$ have more than three direct successors.

Proof. Consider any point u in the binary tree $\mathcal{T}$ and name its points as follows, u^1 and u^2 are the direct successors of u, and u^3, u^4 and u^5, u^6 are the direct successors of u^1 and u^2, respectively.

Consider the point $C_u^{A'}$ that it is a direct successor of u and its direct successor will be the minimal points in the set A' where $A' := U \setminus u\downarrow$, i.e., $C_u^{A'} \leq u^1$ and $C_u^{A'} \leq u^2$ (since there is no other co-final set that is not a subset of A', $C_u^{A'}$ has to be the only direct successor of u that is not in the binary tree). Furthermore, the direct successor of $C_u^{A'}$ are all the points $C_u^{A_i^*}$ where A_i^* are the largest proper upwards closed co-final subsets of A'. There are two, namely $A_1^* := u^1\uparrow \cup u^5\uparrow \cup u^6\uparrow$ and $A_2^* := u^2\uparrow \cup u^3\uparrow \cup u^4\uparrow$. Both A_1^* and A_2^* are upward closed co-final sets (this can be seen if we define A_1^* and A_2^* as $U \setminus u^2\downarrow$ and $U \setminus u^1\downarrow$ respectively, and make use of Claim 4.1.4) and they are the largest such sets that are subsets of A'. Hence, $C_u^{A_1^*}$ and $C_u^{A_2^*}$ are both direct successors of $C_u^{A'}$. There are no other direct successors of $C_u^{A'}$, meaning that it has exactly four direct successors.

For both points $C_u^{A_1^*}$ and $C_u^{A_2^*}$ they already have three direct successors in the binary and there are more points of the form C_u^A such that $C_u^{A_1^*} < C_u^A$, there has to be a least point and therefore making the direct successors of $C_u^{A_1^*}$ at least four. The same applies to $C_u^{A_2^*}$. Now consider any other point that above either $C_u^{A_1^*}$ or $C_u^{A_2^*}$ call it $C_u^{A^*}$. Clearly, $A^* \subsetneqq A_1^*$, the only way to make A' co-final in U and a proper subset of, say, A_1^* would be if we remove at least one minimal point, say v, in A_1^* and replace it with points above v. Since we need to make A^* co-final, we would have to replace v with at least two more points. Hence, A^* has to have more than three direct successors. □

We claim that $\mathfrak{F}$ validates SA.

Proof. For the sake of contradiction, assume $\mathfrak{F} \not\Vdash SA$. This means that there is a point $u \in W$ where $u \not\Vdash ((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$, i.e., there exists a point $u' \geq u$ with:

1. $u' \Vdash (\neg\neg p \to p) \to \neg p \vee p$,
2. $u' \not\Vdash \neg p \vee \neg\neg p$.

u' can be either a point in the binary tree or a point of the form C_u^U. We consider the two points separately.

Case (a): u' is a point in the binary tree.

Consider the point $C_{u'}^A$, where A is the upwards closed co-final set in U,

$$U \cap (V(p) \cup V(\neg p)).$$

At the point $C^A_{u'}$, we have that $\neg\neg p \to p$ is true. If it were false, then we would have a point above where $\neg\neg p$ is true and p is false. Clearly, this point cannot be in $V(p)$. Similarly, the point cannot be in $V(\neg p)$ since $\neg\neg p$ being true requires a successor point with p true. Also, if this point was of the form C^U_u, then $\neg\neg p$ being true requires all points above to have $\neg p$ false, a contradiction due to how A was defined and since $U \subseteq A$. Hence, $\neg\neg p \to p$ is true at $C^A_{u'}$.

Furthermore, $C^A_{u'} \nVdash \neg p \vee p$. Consider if $C^A_{u'} \Vdash p$, then p would be true in the entirety of A, but this contradicts (2) since $u' \nVdash \neg\neg p$ implies that there is a point above u' where $\neg p$ is true. If we consider $C^A_{u'} \Vdash \neg p$, then a similar argument is used and the contradiction is taken from the fact that $u' \nVdash \neg p$. Hence, we have that $C^A_{u'} \Vdash \neg\neg p \to p$ and $C^A_{u'} \nVdash \neg p \vee p$ so $C^A_{u'} \nVdash (\neg\neg p \to p) \to \neg p \vee p$ which is in contradiction with (1).

Case (b): We show that each point C^A_u also validates SA.

Let $a \in A$ and consider the subset $A' \subseteq A$ such that $A' = A \setminus a{\downarrow}$, by Claim 4.1.4, A' is upwards closed and co-final. Furthermore, since any valuation preserved up a frame and A' is A without some downset, there should still be some points in A' with p true and some points with $\neg p$ true.

Now by (ii) we have that $C^A_u \leq C^{A'}_u$ when $A' \subseteq A$, hence we can use the same argument as in case (a) and just substitute the new points where C^A_u behaves as u in case (a) and $C^{A'}_u$ behaves as C^A_u in case (a).

Hence we have that $\mathfrak{F} \Vdash SA$. □

Step 2: Calculate the cardinality of the frame.

We claim that $\mathfrak{F}$ is uncountable.

Proof. We will show that there are an uncountable number of states of the form C^A_u, i.e., there are an uncountable number of unique upwards closed co-final sets that are in U for a particular u. Consider $N \in \mathcal{P}(\mathbb{N})$ assume $N = \{n_0, n_1, n_2, \dots\}$ we create an upwards closed co-final set, call it A, based on N. Illustrated in Figure 4.3, we construct the co-final set as follows:

- From the point u count "rightwards" up to the n_0^{th} point, now everything above that point is in A. For all points in between u and n_0 move once leftwards, then everything above these points are in A.
- Now move "leftwards" once from u to the point u_1 now repeat as before but to the n_1^{th} point.
- Continue for all points in N, if N is finite, let $n = |N|$, then add everything above u_n into A.

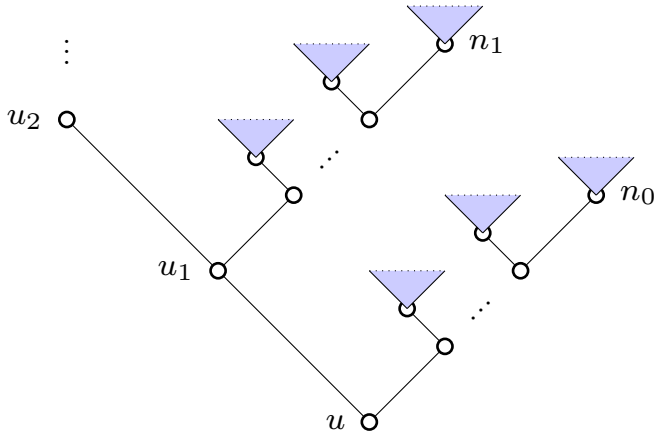


Figure 4.3: Upwards closed co-final set for N

A is upwards closed co-final in $\mathcal{T}$ and, as illustrated in Figure 4.3, it is a union of all the sets in blue. Finally, if we consider a different element of $\mathcal{P}(\mathbb{N})$, then with the above construction, we will produce a different upwards closed co-final set. Hence, we have mapped all upwards closed co-final sets in U to $\mathcal{P}(\mathbb{N})$, concluding that we have an uncountable number of points C_u^A. □

Step 3: Show that a countable elementary sub-structure of $\mathfrak{F}$ refutes SA.

Using Theorem 4.1.2, we consider an elementary subframe of $\mathfrak{F}$, including the binary tree $\mathcal{T}$, let us call it $\mathfrak{F}'$. Since we have shown that there are an uncountable number of points C_u^A in the frame, $\mathfrak{F}'$ is a frame with only a countable number of points C_u^A. To show that SA is not first-order definable, it would be sufficient to show that it is not valid in $\mathfrak{F}$.

The existence of the points C_u^A and that these points are different from the points in the binary tree are definable in first-order logic.

Some shorthand that will be made use of are:

$$x < y := (x \leq y) \wedge \neg(x = y) \text{ [}x \text{ less than } y \text{ and they are not equal]},$$

$$C(x, y) := (x < y) \wedge \neg\exists z(x < z \wedge z < y) \text{ [}y \text{ is a direct cover of } x\text{]},$$

$$Rt(x) := \forall y(x \leq y) \text{ [}x \text{ is the least element, i.e., } x \text{ is the root]}.$$

We know that the points in the binary tree have exactly three successors, and by Claim 4.1.5, the points of the form C_u^U have more than four successors. The following are the first-order formulas that describe the points in the binary tree and the points of the form C_u^U:

$$T(x) := \exists x_1, x_2, x_3(x_1 \neq x_2 \neq x_3 \wedge C(x, x_1) \wedge C(x, x_2) \wedge C(x, x_3) \wedge \neg\exists y(C(x, y) \wedge y \neq x_1 \wedge y \neq x_2 \wedge y \neq x_3)) \text{ [}x \text{ is the point in the binary tree]},$$

$C^A(x) := \exists x_1, x_2, x_3, x_4(x_1 \neq x_2 \neq x_3 \neq x_4 \wedge C(x, x_1) \wedge C(x, x_2) \wedge C(x, x_3) \wedge C(x, x_4))$ [x is a point in the frame described as C^A_u].

These first-order sentences describe our uncountable frame $\mathfrak{F}$ and are preserved in its elementary subframe $\mathfrak{F}'$. We shall consider $\mathfrak{F}'$ to be sectioned into subtrees as shown in Figure 4.4.

- At the root u we move once to the left a point u_1 then the right child subtree of u_1 is indexed $\mathcal{T}_1$,
- then move once to the left from u_1 to u_2 then the right subtree marks the beginning of the tree $\mathcal{T}_2$,
- repeat the process to mark out a countable number of trees $\mathcal{T}_i$.

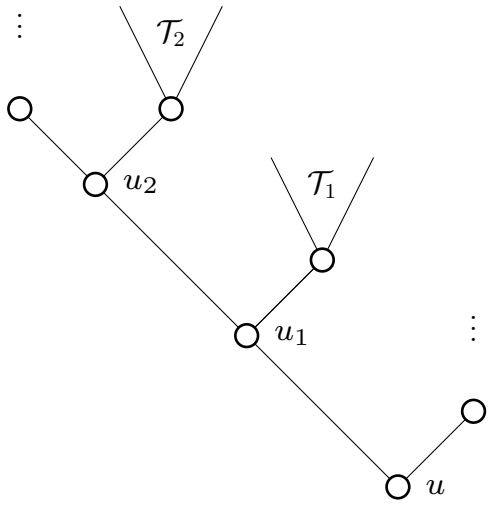


Figure 4.4: Subtrees in $\mathfrak{F}'$

We are now ready to start the argument. Consider all points C^A_u since they are only a countable number of them, let them be enumerated and listed as $C^{A_1}_u, C^{A_2}_u, \ldots$.

Claim 4.1.6. If $A_1, A_2, \ldots, A_n$ are co-final and upwards closed in U, then $A_1 \cap \cdots \cap A_n$ is also upwards closed and co-final in U.

Proof. Since A_1 is co-final in U there exist $u' \in U$ with $u' \leq a_1 \in A_1$ but now $a_1 \in U$ so for $a_2 \in A_2$ we have $a_1 \leq a_2$. Similarly we continue till $a_n \in A_n$ so $a_n \in A_1 \cap \cdots \cap A_n$ hence $A_1 \cap \cdots \cap A_n$ is non-empty. Now let $a \in A_1 \cap \cdots \cap A_n$ then $a \in A_1$ and $a \in A_2$ and $\ldots$ and $a \in A_n$, so for all $u' \in U$, $u' \leq a$ since, for all $1 \leq i \leq n$, A_i is co-final in U. Hence, $A_1 \cap \cdots \cap A_n$ is also co-final. Using contradiction assume $A_1 \cap \cdots \cap A_n$ is not upwards closed, i.e., for some $a \in A_1 \cap \cdots \cap A_n$ there is $a \leq a'$ with $a' \notin A_1 \cap \cdots \cap A_n$, but since $a \in A_1$ all points in $a{\uparrow}$ are also in A_1 because A_1 is upwards closed, similarly for all $A_{2 \leq i \leq n}$ we have $a{\uparrow} \subseteq A_{2 \leq i \leq n}$. Now,

since $a{\uparrow}$ is in all $A_{1\leq i\leq n}$, it is in its intersection. Hence, a contradiction since we cannot have the point $a' \notin A_1 \cap \cdots \cap A_n$.

Hence, we have that $A_1 \cap \cdots \cap A_n$ is also upwards closed and co-final in U. $\square$

Now using Claim 4.1.6 and that $T_i \subseteq U$, we have that:

> Finite intersections $T_i \cap A_1 \cap \cdots \cap A_n$ are upwards closed and co-final in T_i.

Furthermore, we have in $\mathfrak{F}$ that for any point $a \in A$, with A an upwards closed co-final set, there is a point $C_u^{A_i}$ such that $C_u^{A_i} \not\leqslant a$. We can see this from Claim 4.1.4 where we can always create a new upwards closed co-final set which excluded a point (and its downset). This can be written in first-order frame correspondence language, let us call it $\star$:

$$\forall x \forall y \forall z((Rt(x) \wedge C^A(y) \wedge C(x,y) \wedge y < z \wedge T(z)) \to \exists u(C^A(u) \wedge y < u \wedge u \not\leqslant z)).$$

Hence, this implies that:

> The total intersection $T_i \cap \{A_j \mid j = 1, 2, \dots\}$ is empty.

Our final aim is to falsify SA in $\mathfrak{F}'$, and we can achieve this by systematizing a way to always have a point above each point C_u^A that has p false and a point further up that has p true. We define this set as follows:

$$A_0^* = A_1,$$

$$A_{n+1}^* = A_1 \cap \cdots \cap A_k \text{ for smallest } k \text{ such that } T_{n+1} \cap A_1 \cap \cdots \cap A_k \subsetneqq T_{n+1} \cap A_n^*.$$

We know that such a set, A_k, exists by the following argument. Since, by Claim 4.1.6, we have that $A_1 \cap \cdots \cap A_m \neq \emptyset$ and that implies that $T_i \cap A_1 \cap \cdots \cap A_m \neq \emptyset$. We also know that there exists some A_j, $1 \leq j \leq m$ and $v \in T_i \cap A_1 \cap \cdots \cap A_m$ such that $C_u^{A_j} < v$. Then, by $\star$ we know there exists some A_k such that $C_u^{A_j} < C_u^{A_k}$ and $C_u^{A_k} \not\leqslant v$, i.e., $v \notin T_i \cap A_1 \cap \cdots \cap A_m \cap \cdots \cap A_k$. Therefore $T_i \cap A_1 \cap \cdots \cap A_m \cap \cdots \cap A_k \subsetneqq T_i \cap A_1 \cap \cdots \cap A_m$.

We now define the valuation:

$$V(p) = \bigcup\{T_i \cap A_i^* \mid i = 1, 2, \dots\}.$$

The above description constructs a valuation such that for any point $C_u^{A_i}$, there will be a successor in T_i that is not in $V(p)$.

This can easily be seen by the following argument.

Proof. Consider the first set A_1^*, by definition $A_1^* = A_1 \cap \cdots \cap A_j$ for smallest j such that $T_1 \cap A_1 \cap \cdots \cap A_j \subsetneqq T_1 \cap A_1$. Clearly, we need to add some sets to the intersection (at minimum we need to add A_2), i.e., $A_1^* = A_1 \cap A_2 \cap \cdots \cap A_j$. Now consider the next set A_2^*, by definition $A_2^* = A_1 \cap \cdots \cap A_k$ for smallest k such that $T_2 \cap A_1 \cap \cdots \cap A_k \subsetneqq T_2 \cap A_1 \cap A_2 \cap \cdots \cap A_j$. Clearly in the intersection to define A_2^* we need to add A_2 and at minimum A_3, i.e., $A_2^* = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k$. It can easily be seen that this process will repeat for all i. Hence, we have that for all sets A_i there exists a point $a_i \in A_i \cap T_i$ with $a_i \notin A_i^* \cap T_i$. □

Let us consider our frame $\mathfrak{F}'$ together with the above stated valuation. We note that $u \nVdash \neg p$ and $u \nVdash \neg\neg p$. Indeed, for the latter case, it is easy to see since the subtree rooted at the right child of u has no point where p is true. Hence, we can establish that $u \nVdash \neg p \vee \neg\neg p$.

Consider $u \Vdash (\neg\neg p \to p) \to p \vee \neg p$, assume it is false; this would imply that there is a point $u' \in u\uparrow$ where $u' \Vdash \neg\neg p \to p$ and $u' \nVdash p \vee \neg p$. If u' is a point in $V(p)$, then we cannot have $u' \nVdash p \vee \neg p$. If u' is a point of the form C_u^A, then we cannot have $u' \Vdash \neg\neg p \to p$, since this would imply that all points $v \geq u'$ have $v \nVdash \neg\neg p$ or $v \Vdash p$. The latter is clearly not possible. The former would require a successor point with $\neg p$ true, but this cannot be satisfied since it would require all points above to have p false. If u' is in the binary tree, then u' can either be in the subtree rooted in the right child of the root or the left child. If u' is in the right child, then $u' \nVdash p \vee \neg p$ will be false. If u' is in the left child, then $u' \Vdash \neg\neg p \to p$ will be false, since p is not true at u' but is assigned true in a successor point.

Hence, we conclude that SA is not valid in $\mathfrak{F}'$ and that SA is not first-order definable using the Löwenheim–Skolem theorem. □

Comment 4.1.7. There are two points in the argument where we differ from van Benthem [33].

i. Defining the set for our valuation in $\mathfrak{F}'$, van Benthem had his set defined as:

$$A_1^* = A_1$$
$$A_{n+1}^* = A_1 \cap \cdots \cap A_k \text{ for smallest } k \text{ such that } T_{n+1} \cap A_1 \cap \cdots \cap A_k \subsetneqq T_{n+1} \cap A_n^*.$$

We cannot begin by assigning $A_1^* = A_1$ since then the point $C_u^{A_1}$ would not have a successor that is outside $V(p)$, but this can easily be fixed by rather assigning $A_0^* = A_1$ and the rest remains the same.

ii. The valuation from [33] was defined as:

$$A^* = \cup\{T_i \cap A_i^* \mid i = 1, 2, \dots\} \text{ with } V(p) = \{y \mid \exists x \leq y, x \in A^*\}.$$

We consider this particular form for the valuation as being unnecessary since the co-final set A^* is already upwards closed.

4.1.2 Ultrapowers

An ultrapower is another type of construction whereby a structure and its ultrapower both validate the same set of first-order sentences.

Definition 4.1.8. A *proper filter* $\mathcal{F}$ over a non-empty set W is a set $\mathcal{F} \subseteq \mathcal{P}(W)$ such that:

i. $W \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,

ii. if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$,

iii. if $X \in \mathcal{F}$ and $X \subseteq Y$ then $Y \in \mathcal{F}$.

Definition 4.1.9. An *ultrafilter* $\mathcal{U}$ over W is a filter with the added condition that:

> if $X \in \mathcal{P}(W)$ then either $X \in \mathcal{U}$ or $W \setminus X \in \mathcal{U}$.

Let $\mathcal{U}$ be an ultrafilter over a non-empty set I, and for each $i \in I$ let A_i be a non-empty set. Now let $\mathcal{C}$ be the Cartesian product of all A_i, $\mathcal{C} = \prod_{i \in I} A_i$. For any two functions $f, g \in \mathcal{C}$ if $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$ then we say that f and g are U-equivalent ($f \sim_{\mathcal{U}} g$). We also have that $\sim_{\mathcal{U}}$ is an equivalence relation.

If we define $f_{\mathcal{U}} = \{g \in \mathcal{C} \mid g \sim_{\mathcal{U}} f\}$ then the ultraproduct of A_i modulo $\mathcal{U}$ denoted as $\prod_{\mathcal{U}} A_i$ is given as:

$$\prod_{\mathcal{U}} A_i = \{f_{\mathcal{U}} \mid f \in \prod_{i \in I} A_i\}.$$

An ultraproduct is called an ultrapower of A modulo $\mathcal{U}$ if $A_i = A$ for all $i \in I$, written as $\prod_{\mathcal{U}} A$.

Now, let us apply the same idea, but to models instead of general sets.

Definition 4.1.10. [5, Chapter 4] Let $\mathcal{L}$ be a first-order language and $\mathfrak{U}_i$, where $i \in I$, is an $\mathcal{L}$-model. The ultraproduct $\prod_{\mathcal{U}} \mathfrak{U}_i$ of $\mathfrak{U}_i$ modulo $\mathcal{U}$ is the model described as follows.

> The universe $A_{\mathcal{U}}$ of $\prod_{\mathcal{U}} \mathfrak{U}_i$ is the set $\prod_{\mathcal{U}} A_i$, where A_i is the universe of $\mathfrak{U}_i$.
>
> Let R be an n-place relation symbol, and R_i its interpretation in the model $\mathfrak{U}_i$. The relation $R_{\mathcal{U}}$ in $\prod_{\mathcal{U}} \mathfrak{U}_i$ is given by:
>
> $$R_{\mathcal{U}} f^1_{\mathcal{U}} \dots f^n_{\mathcal{U}} \quad \text{iff} \quad \{i \in I \mid R_i f^1(i), \dots, f^n(i)\} \in \mathcal{U}.$$
>
> Let F be an m-place function symbol, and F_i its interpretation in $\mathfrak{U}_i$. The function $F_{\mathcal{U}}$ in $\prod_{\mathcal{U}} \mathfrak{U}_i$ is given by:
>
> $$F_{\mathcal{U}}(f^1_{\mathcal{U}}, \dots, f^m_{\mathcal{U}}) = \{(i, F_i(f^1(i), \dots, f^m(i))) \mid i \in I\}_{\mathcal{U}}.$$
>
> Let c be a constant, and a_i its interpretation in $\mathfrak{U}_i$. Then c is interpreted by the element $c' \in \prod_{\mathcal{U}} A_i$ where $c' = \{(i, a_i) \mid i \in I\}_{\mathcal{U}}$.

We are now ready to formulate Łoś's Theorem, sometimes referred to as *the fundamental theorem of ultraproducts.*

Theorem 4.1.11. (Łoś's Theorem)[5, Chapter 4] Let $\mathcal{U}$ be an ultrafilter over a non-empty set I. For each $i \in I$, let $\mathfrak{U}_i$ be a model. Given any first-order formula $\alpha(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $f^1_{\mathcal{U}}, \ldots, f^n_{\mathcal{U}}$ in $\prod_{\mathcal{U}} \mathfrak{U}_i$ we have:

$$\prod_{\mathcal{U}} \mathfrak{U}_i \models \alpha[f^1_{\mathcal{U}}, \ldots, f^n_{\mathcal{U}}] \quad \text{iff} \quad \{i \in I \mid \mathfrak{U}_i \models \alpha[f^1(i), \ldots, f^n(i)]\} \in \mathcal{U}.$$

An important corollary to Łoś's Theorem is that truth is preserved in ultrapowers.

Corollary 4.1.12. If φ is any first-order sentence and $\prod_{\mathcal{U}} \mathfrak{U}$ an ultrapower of $\mathfrak{U}$ then,

$$\mathfrak{U} \models \varphi \quad \text{iff} \quad \prod_{\mathcal{U}} \mathfrak{U} \models \varphi.$$

An ultrapower is an ultraproduct where all the models $\mathfrak{U}_i$ considered are equal and an important theorem about ultraproducts is Theorem 4.1.11, Łoś's Theorem, which states that a first-order formula α is true in an ultraproduct if and only if the set of indices $i \in I$ where α is true in $\mathfrak{U}_i$ is in the ultrafilter. This intuitively can be considered as saying that "a first-order formula α is true in the ultraproduct if and only if it is true in *enough* $\mathfrak{U}_i$s".

Based on Łoś's Theorem, we can make a claim that if a first-order sentence is true in a "large enough" set of models, then we can claim it to be true in the ultraproduct.

Now we show how we can prove that SA is not first-order definable using an ultrapower construction.

Step 1: Construct a frame that refutes the SA.

This was done above, and the resulting refutation frame is given in Figure 4.1.

Step 2: Find an infinite frame that validates the SA axiom with its ultrapower refuting SA.

The following proof is a detailed version of the sketch taken from [26]. Consider the frame $\mathfrak{F} = (W, \leq)$ given below in Figure 4.5.

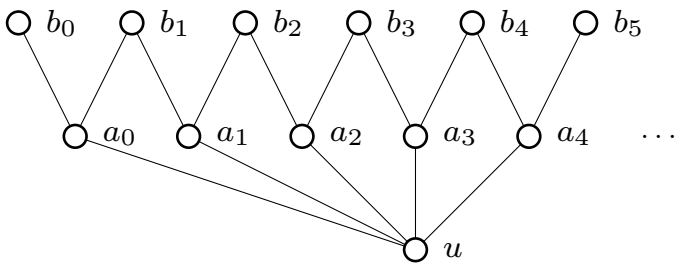


Figure 4.5: Kripke frame, $\mathfrak{F}$, validating SA

We claim that $\mathfrak{F} \Vdash SA$. Assume $\mathfrak{F} \nVdash SA$ we have seen above the conditions needed to refute SA, they are reproduced below, for some $u, u_1, u_2, u_3 \in \mathfrak{F}$:

(i) $u \nVdash ((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$,

(ii) $\exists u_1 \geq u,\ u_1 \Vdash (\neg\neg p \to p) \to \neg p \vee p \quad \text{and} \quad u_1 \nVdash \neg p \vee \neg\neg p$,

(iii) $\forall u_2 \geq u_1,\ (u_2 \nVdash \neg\neg p \to p \quad \text{or} \quad u_2 \Vdash \neg p \vee p) \quad \text{and} \quad (u_1 \nVdash \neg p \quad \text{and} \quad u_1 \nVdash \neg\neg p)$,

(iv) $\exists u_3 \geq u_2,\ (u_3 \Vdash \neg\neg p \quad \text{and} \quad u_3 \nVdash p) \quad \text{or} \quad (u_2 \Vdash \neg p \quad \text{or} \quad u_2 \Vdash p)$.

Now, if $\mathfrak{F}$ does not validate SA, then it is required that there is a point $u_1 \geq u$ where (ii) is true. Let us assume that $u_1 = b_n$ with $n \in \mathbb{N}$, now according to (iii) we have that $u_1 \nVdash \neg p$ and $u_1 \nVdash \neg\neg p$ which requires there to be points say $v_1, v_2 \geq u_1$ such that $v_1 \Vdash p$ and $v_2 \Vdash \neg p$ clearly these points have to be distinct hence require two distinct successors which is not possible since b_n is an end point. Assume $u_1 = a_n$ with $n \in \mathbb{N}$, this allows there to be two distinct successors, let $b_n \Vdash p$ and $b_{n+1} \Vdash \neg p$, now by (iv) we need there to exist a point $u_3 \geq u_2$ where $u_3 \Vdash \neg\neg p$ and $u_3 \nVdash p$ now clearly we cannot assign u_3 to b_{n+1} so if we let $u_3 = b_n$ then we again have to have a distinct successor of b_n since $u_3 \Vdash \neg\neg p$ and $u_3 \nVdash p$ requires a distinct successor $v > u_3$ where $v \Vdash p$ which is again not possible since b_n is an end point. Finally let $u_1 = u$ then we have two distinct successors let them be a_n and a_{n+1} and assign $a_n \Vdash \neg\neg p$ and $a_{n+1} \Vdash \neg p$ furthermore a_n has distinct successors b_n and b_{n+1} let us further assign $b_n \Vdash p$, now the point b_{n+1} is a successor for both a_n and a_{n+1} and we have that $a_n \Vdash \neg\neg p$ and $a_{n+1} \Vdash \neg p$ which implies that we require $b_{n+1} \nVdash \neg p$ and $b_{n+1} \Vdash \neg p$, a contradiction. Hence we have that $\mathfrak{F} \Vdash SA$.

Step 3: Create the ultrapower of $\mathfrak{F}$.

Now we consider the ultrapower of the $\mathfrak{F}$, take a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. Creating the ultrapower of $\mathfrak{F}$ we consider the set of frames $\mathfrak{F}_n$, each identical to $\mathfrak{F}$, where $n \in \mathbb{N}$ and make use of Łoś's Theorem where we can claim that all first-order sentence valid on a *large enough* set of frames hold in the ultrapower, so we describe these sentences as follows.

For simplicity, we shall use the term "height" so that if a point is the cover of the root, then it has height 1; if it is a cover of a point of height 1, then it has a height of 2, and so on. We will make use of some of the short hands as defined in Chapter 4.1.1. The following first-order sentences are valid on $\mathfrak{F}$.

(5.1) $\exists x(Rt(x) \wedge \forall z(z = x \vee C(x,z) \vee \exists u(C(x,u) \wedge C(u,z))))$ [All points in the frame are either the root, a cover of the root or a cover of the cover of the root],

(5.2) $\exists x(Rt(x) \wedge \forall z(C(x,z) \to (\exists u,v(C(z,u) \wedge C(z,v) \wedge \neg(u = v) \wedge \neg\exists w(C(z,w) \wedge \neg(w = u) \wedge \neg(w = v))))))$ [Every cover of the root has exactly 2 covers],

(5.3) $\exists x(Rt(x) \wedge \forall z\forall u(C(x,z) \wedge C(z,u) \to \neg\exists w(C(u,w))))$ [The cover of the cover of the root has no cover, i.e., there is no point with height 3],

(5.4) $\exists x(Rt(x) \wedge \exists r(C(x,r) \wedge \exists s(C(r,s) \wedge \neg\exists t(C(t,s) \wedge \neg(r = t)) \wedge \forall u(\neg(u = r) \wedge C(x,u) \wedge \exists v\exists w(C(u,v) \wedge C(u,w) \wedge \neg(v = w))))))$ [There is one sole height 2 point that is the cover of only one height 1 point].

The first-order sentences can be seen to correctly describe our frame $\mathfrak{F}$, and now we claim that our ultrapower $\mathfrak{F}'$ satisfies the same formulas by Łoś's Theorem. To begin to envision what our ultrapower looks like, we make use of a theorem.

Lemma 4.1.13. There is an isomorphic copy of $\mathfrak{F}$ in $\prod_{\mathcal{U}} \mathfrak{F}$.

Proof. Let I be an index set and $\mathcal{U}$ an ultrafilter on I. Assume we have frames $\mathfrak{F}_i = (W_i, \leq_i)$ and its ultrapower $\mathfrak{F}' = \prod_{i \in I} \mathfrak{F}_{\mathfrak{i}} = (W', \leq')$. Let us describe the points in $\mathfrak{F}_i$ as follows: $w_n^i \in W_i$ where i indicates the frame in the index set we are considering and n indicates which point in $\mathfrak{F}_{\mathfrak{i}}$ we are considering. Now consider a point in any $\mathfrak{F}_i$, let us call this point a_0. Now there exists a point in $\mathfrak{F}'$ that is isomorphic to this point, namely all points that are U-equivalent to $(a_0^0, a_0^1, a_0^2, \dots)$, the same can be done for all points in the chosen frame $\mathfrak{F}_i$. With regards to the relationship between an two points a similar argument works, since if there is any relationship between two points say a_n and a_m then there should also be a relationship that is U-equivalent to all relationships between the points $(a_n^0, a_n^1, a_n^2, \dots)$ and $(a_m^0, a_m^1, a_m^2, \dots)$. Hence showing that an isomorphic copy of $\mathfrak{F}$ exists in $\prod_{\mathcal{U}} \mathfrak{F}$. □

Now by Lemma 4.1.13 we know that $\mathfrak{F}' = (W', \leq')$ has an isomorphic copy of $\mathfrak{F}$ with some extra points, clearly the points in $\mathfrak{F}'$ that are isomorphic to the original frame are U-equivalent to points of the form $(a_n^0, a_n^1, a_n^2, \dots)$. Now consider the series of points $(a_0^0, a_1^1, a_2^2, \dots)$ it is not U-equivalent to any point in the isomorphic copy of $\mathfrak{F}$ in $\mathfrak{F}'$, and is therefore an extra point that should be added to $\mathfrak{F}'$. In fact we can claim there to be an infinite number of extra points just consider the point mentioned above but where the subscript is increased by one and two and so on, so we can conclude that there are an infinite number of extra points (we do not claim that our ultrapower is of a cardinality greater than our original frame it is not a necessary condition for our proof). Hence, we can assume our ultrapower looks like the frame given in Figure 4.6.

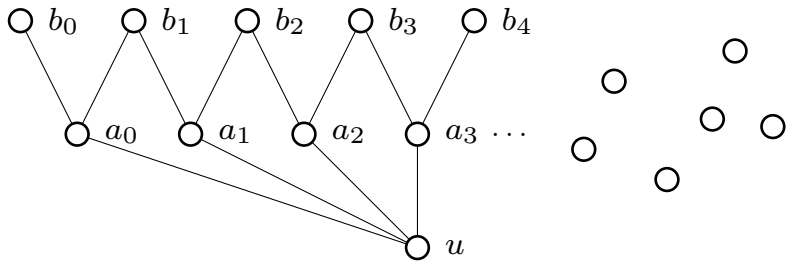


Figure 4.6: Ultrapower of $\mathfrak{F}$ initial assumption

This is our initial assumption on what the ultrapower looks like but now we consider the first-order sentences that described $\mathfrak{F}$ and we know that they must be valid in $\mathfrak{F}'$ so we modify our picture and get the following in Figure 4.7.

We can also claim that the "extra" points in $\mathfrak{F}'$ are incomparable with the points that are part of the isomorphic copy of $\mathfrak{F}$ we consider some case. For simplicity we

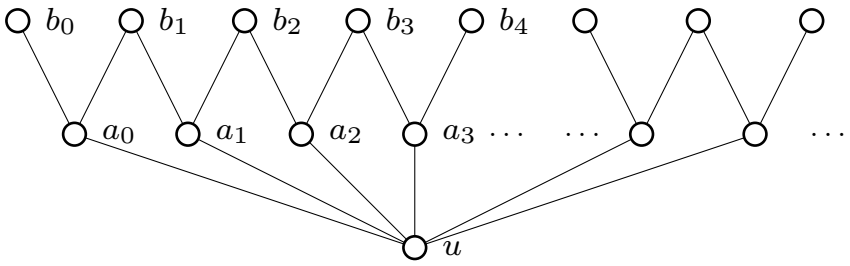


Figure 4.7: Ultrapower of $\mathfrak{F}$ final assumption

shall denote the "extra" points that are covers of the root as a' and the covers of a' as b'.

Assume $a_n \leq a'$ for some $n \in \mathbb{N}$ then this contradicts (5.2) similarly for if $a' \leq a_n$,

Assume $b_n \leq a'$ for some $n \in \mathbb{N}$ then this contradicts (5.3),

Assume $a' \leq b_n$ for some $n \in \mathbb{N}/\{0\}$ then this contradicts (5.2),

Assume $a' \leq b_0$ then this contradicts (5.4),

Assume $a_n \leq b'$ for some $n \in \mathbb{N}$ then this contradicts (5.2),

Assume $b' \leq a_n$ for some $n \in \mathbb{N}$ then this contradicts (5.3),

Assume $b_n \leq b'$ for some $n \in \mathbb{N}$ then this contradicts (5.3) similarly for if $b' \leq b_n$,

Step 4: Show that SA is refutable in the ultrapower.

We show this by using a bounded morphism from the ultrapower in Figure 4.7 to the refutation frame in Figure 4.1. The bounded morphism is $f : \mathfrak{F}' \to \mathfrak{SA}$ defined as $f(u) = u$, $f(a_n) = u_1$, $f(b_n) = u_2$, $f(a') = u_3$ and $f(b') = u_3$. The first condition (R1) of a bounded morphism clearly holds when considering the fact that the frame is reflexive. For the second condition (R2) the points u, a_n and b_n clearly satisfy the conditions with u, u_1 and u_2, for a' and b' it is also easily proved: consider $u \leq u_3$ in $\mathfrak{SA}$ then $\exists z \in W'$ with $u \leq z$ in $\mathfrak{F}'$. Now, if we let $z = a'$ and $z = b'$, then the condition holds.

Hence, we have a bounded morphism between the refutation frame of SA and the ultrapower $\mathfrak{F}'$; this implies that $\mathfrak{F}'$ does not validate SA. Łoś's Theorem states that all first-order sentences should preserve their validity in ultrapowers, and since the frame $\mathfrak{F}$ validated SA and its ultrapower does not validate SA, we have a contradiction and hence imply that SA is not first-order definable. □

4.1.3 Compactness

First proved by Kurt Gödel in 1930 [19], the Compactness Theorem gives us another construction that preserves truth of first-order formulas and thus provides a widely used method for proving non-definability. Formally, the Compactness Theorem is given as:

Theorem 4.1.14 (Compactness Theorem [2])**.** Let Σ be a set of first-order formulas. If each finite subset of Σ has a model, then Σ itself has a model.

The compactness argument is similar to that of ultrapowers. It involves obtaining the refutation frame, finding a construction that produces two structures, one that validates the formula, and another that does not, and then proving that the second structure does not validate the formula. From Chapter 4.1.2, we have the refutation frame for the SA axiom, $\mathfrak{SA}$, in Figure 4.1.

Chagrov and Zakharyaschev, [3], supply us with the frame chosen to validate SA, but they provide a very brief argument. In the following section, we provide the full details of the compactness argument. The frame is produced below in Figure 4.8, we shall call it $\mathfrak{F}$.

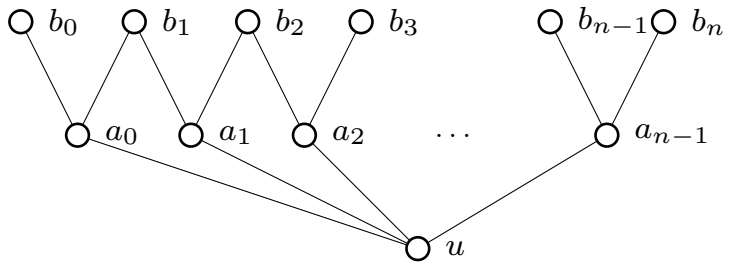


Figure 4.8: Kripke frame, $\mathfrak{F}$, validating SA

The claim is that $\mathfrak{F}$ validates SA, it is easy to see since the reasoning is similar to that used in the case of the infinite frame used in the ultrapower argument.

Compactness claims that if Σ is a set of first-order formulas and every finite subset of Σ is satisfiable, then Σ will also be satisfiable. For the case of SA, we provide the frame $\mathfrak{F}$ that satisfies SA; furthermore, we state some first-order formulas, which are also satisfied in $\mathfrak{F}$, and will all form part of Σ.

We will use the same shorthand as defined in Chapter 4.1.2 with the following added on:

> $A(b) := \exists x(Rt(x) \wedge \exists a(C(x,a) \wedge (C(a,b) \wedge \neg\exists c(C(c,b) \wedge \neg(c = a)))))$ [b is a height 2 point that is a cover of only one height 1 point].

For the first-order formulas, we will make use of the formulas (5.1),(5.2), and (5.3) from Chapter 4.1.2 with the following extra formula:

(5.5) $\exists b \exists b' [A(b) \wedge A(b') \wedge \neg(b = b') \wedge \neg \exists c (A(c) \wedge \neg(b = c) \wedge \neg(c = b'))]$ [There are exactly two height 2 points that are the cover of only one height 1 point],

We let $\delta_n(u, a_0, \ldots, a_{n-1}, b_0, \ldots, b_n)$ be a first-order formula stating that there is a "zig-zag" pattern with length $2n$ from b_0 to b_n,

(5.6) $\delta_n = C(a_0, b_0) \wedge C(a_0, b_1) \wedge C(a_1, b_1) \wedge C(a_1, b_2) \wedge C(a_2, b_2) \cdots \wedge C(a_{n-1}, b_n) \wedge \bigwedge_{0 \leq i < j \leq n-1} \neg(a_i = a_j) \wedge \bigwedge_{0 \leq i < j \leq n} \neg(b_i = b_j)$.

Suppose that λ is a first-order formula equivalent to SA. We define Σ as:

$$\Sigma = \{\lambda\} \cup \{5.1\} \cup \{5.2\} \cup \{5.3\} \cup \{5.5\} \cup \{\delta_n \mid n \in \omega\}.$$

Clearly, any finite subset of Σ is satisfied in $\mathfrak{F}$. When considering Σ as a whole, we make use of the frame below, call it $\mathfrak{F}'$, in Figure 4.9.

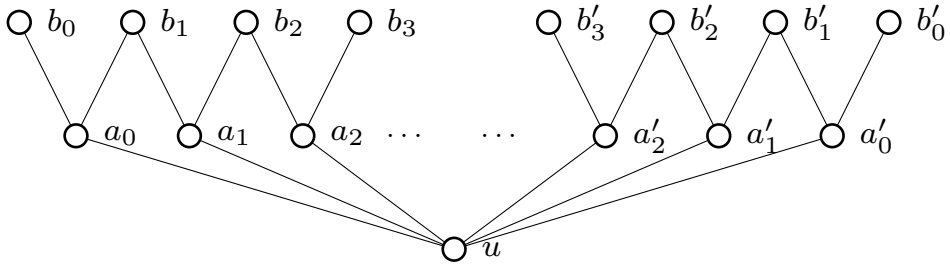


Figure 4.9: Kripke frame, $\mathfrak{F}'$, that should satisfy Σ

In $\mathfrak{F}'$, we have that the points on the "left" of the frame are incomparable with the points on the "right" side of the frame. We establish this considering the cases.

Assume $a_n \leq a'_n$ for some $n \in \mathbb{N}$ then this contradicts (5.2) similarly for if $a'_n \leq a_n$,

Assume $b_n \leq a'_n$ for some $n \in \mathbb{N}$ then this contradicts (5.3),

Assume $a'_n \leq b_n$ for some $n \in \mathbb{N}/\{0\}$ then this contradicts (5.2),

Assume $a'_n \leq b_0$ for some $n \in \mathbb{N}$ then this contradicts (5.5),

Assume $a_n \leq b'_n$ for some $n \in \mathbb{N}$ then this contradicts (5.2),

Assume $b'_n \leq a_n$ for some $n \in \mathbb{N}$ then this contradicts (5.3),

Assume $b_n \leq b'_n$ for some $n \in \mathbb{N}$ then this contradicts (5.3) similarly for if $b'_n \leq b_n$.

It is easy to see that $\Sigma \setminus \{\lambda\}$ is satisfied in $\mathfrak{F}'$. To show that λ is not true in $\mathfrak{F}'$ we can consider the bounded morphism with the frame $\mathfrak{SA}$ in Figure 4.1 defined as $f : \mathfrak{F}' \to \mathfrak{SA}$ by setting $f(u) = u$, $f(a_n) = u_1$, $f(b_n) = u_2$, $f(a'_n) = u_3$ and $f(b'_n) = u_3$. The reasoning is the same as done for the ultrapower.

Hence, by compactness, we have that SA is not first-order definable. $\square$

4.2 Logics of bounded branching

In [16] Dov M Gabbay and D.H.J. de Jongh inductively defined the bb_n axioms and consequently the superintuitionistic logics, $T_n = \mathrm{Int} + bb_n$, which were made to satisfy the following conditions:

$T_n \supsetneqq \mathbf{Int}$,

$\bigcap T_n = \mathbf{Int}$,

$T_n \supsetneqq T_{n+1}$,

T_n is decidable,

T_n has the disjunction property,

T_n is finitely axiomatizable.

bb_n is given as:

$$bb_n = \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i.$$

In this section, we present an ultrapower argument that shows why bb_n is not first-order definable for $n \geq 2$; for $n = 1$, it is deterministic. Additionally, we critically examine a frame proposed by Chagrov and Zakharyaschev in [3], which they claim establishes that SA is not first-order definable. We challenge this claim, arguing that their frame is insufficient for such a conclusion. Moreover, we highlight an interesting observation: the same frame that refutes bb_n can be used to invalidate another formula, yet this formula is first-order definable.

Step 1: Find a refutation frame of bb_n.

Consider the $n+2$ point Kripke model, $\mathfrak{M} = (W, \leq, V)$, given in Figure 4.10, where $W = \{u, u_0, u_1, \ldots, u_n\}$, relations are $u < u_i$ for all $0 \leq i \leq n$ and $V(p_i) = u_i$ for $0 \leq i \leq n$.

We want to show that $\mathfrak{M}, u \nVdash \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i$. This is the case iff there exists a $u' \geq u$ with $\mathfrak{M}, u' \Vdash \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j)$ and $\mathfrak{M}, u' \nVdash \bigvee_{i=0}^{n} p_i$ so for u' we have the following:

$(\mathfrak{M}, u') \Vdash ((p_0 \to \bigvee_{j\neq 0} p_j) \to \bigvee_{j\neq 0} p_j)$,

$(\mathfrak{M}, u') \Vdash ((p_1 \to \bigvee_{j\neq 1} p_j) \to \bigvee_{j\neq 1} p_j)$,

$\vdots$

$(\mathfrak{M}, u') \Vdash ((p_n \to \bigvee_{j\neq n} p_j) \to \bigvee_{j\neq n} p_j)$,

$(\mathfrak{M}, u') \nVdash p_0$,

$(\mathfrak{M}, u') \nVdash p_1$,

$\vdots$

$(\mathfrak{M}, u') \nVdash p_n$.

Let us fix $u = u'$. For any $0 \leq i \leq n$, $(\mathfrak{M}, u) \Vdash ((p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq 1} p_j)$ is the case iff all points above u, say $v \geq u$, we have $(\mathfrak{M}, v) \nVdash p_i \to \bigvee_{j \neq i} p_j$ or $(\mathfrak{M}, v) \Vdash \bigvee_{j \neq i} p_j$. The latter is not satisfied since u does not have any p_i true. The former is only true if there exists a point $v_i \geq v$ with $(\mathfrak{M}, v_i) \Vdash p_i$ and $(\mathfrak{M}, v_i) \nVdash \bigvee_{j \neq i} p_j$. Both are satisfied if we fix $u_i = v_i$ for all $1 \leq i \leq n$.

Hence, $\mathfrak{M}$ refutes bb_n. We will call the refutation frame $\mathfrak{BB}$. □

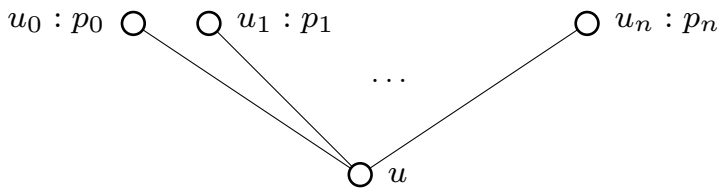


Figure 4.10: Kripke model rejecting bb_n

Step 2: Find a frame that validates bb_n where its ultrapower will invalidate bb_n.

The frame produced is from Rodenburg [26]; he used it to prove non-first-order definability for a different formula, but we noted that it would be sufficient to prove non-first-order definability of bb_n.

Consider the frame given in Figure 4.11, let us call it $\mathfrak{F} = (W, \leq)$, we claim that $\mathfrak{F} \Vdash bb_n$ for $n \geq 2$.

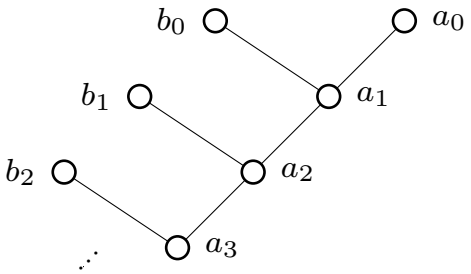


Figure 4.11: Kripke frame, $\mathfrak{F}$, validating bb_n

There are only two types of points to consider for $\mathfrak{F}$, either the point could be an endpoint, b_n, a_0, or a point that has 2 immediate successors, a_n, for $n \in \mathbb{N} \setminus \{0\}$. Considering the endpoints, verifying that bb_n is true in all b_n, it is sufficient to consider if bb_n is true classically, since endpoints validate all classical tautologies. One can easily verify that bb_n is true classically using a truth table.

Given a_{n+1} is a point that has 2 immediate successors, we present the following argument. Consider bb_2 for simplicity and assume that $a_{n+1} \nVdash bb_2$ implying that for some valuation and some point $a_n \geq a_{n+1}$ we have the following:

i. $a_n \Vdash (p_0 \to p_1 \vee p_2) \to p_1 \vee p_2$,

ii. $a_n \Vdash (p_1 \to p_0 \vee p_2) \to p_0 \vee p_2$,

iii. $a_n \Vdash (p_2 \to p_0 \vee p_1) \to p_0 \vee p_1$,

iv. $a_n \nVdash p_0$,

v. $a_n \nVdash p_1$,

vi. $a_n \nVdash p_2$.

Consider (i) $a_n \Vdash (p_0 \to p_1 \vee p_2) \to p_1 \vee p_2$ iff $\forall a_n^1 \in W$ with $a_n \leq a_n^1$ it is the case that $a_n^1 \nVdash p_0 \to p_1 \vee p_2$ or $a_n^1 \Vdash p_1$ or $a_n^1 \Vdash p_2$. We list all three below:

vii. $a_n^1 \nVdash p_0 \to p_1 \vee p_2$ or $a_n^1 \Vdash p_1$ or $a_n^1 \Vdash p_2$,

viii. $a_n^1 \nVdash p_1 \to p_0 \vee p_2$ or $a_n^1 \Vdash p_0$ or $a_n^1 \Vdash p_2$,

ix. $a_n^1 \nVdash p_2 \to p_0 \vee p_1$ or $a_n^1 \Vdash p_0$ or $a_n^1 \Vdash p_1$.

Now consider $a_n^1 = a_n$ in particular. We cannot have $a_n \Vdash p_1$ or $a_n \Vdash p_2$ by (v) and (vi). So we have to have $a_n \nVdash p_0 \to p_1 \vee p_2$ this is satisfied iff $\exists a_n^2$ with $a_n \leq a_n^2$ such that $a_n^2 \Vdash p_0$ and $a_n^2 \nVdash p_1$ and $a_n^2 \nVdash p_2$. Repeating it for (ii) and (iii) we get that, for (ii) $\exists a_n^3$ with $a_n^1 \leq a_n^3$ such that $a_n^3 \Vdash p_1$ and $a_n^3 \nVdash p_2$ and $a_n^3 \nVdash p_0$ and for (iii) $\exists a_n^4$ with $a_n^1 \leq a_n^4$ such that $a_n^4 \Vdash p_2$ and $a_n^4 \nVdash p_0$ and $a_n^4 \nVdash p_1$. Clearly, these points need to be distinct; furthermore, none of the points can be above one another. This implies that either all three points are b points or two are b points, and one is at an a point.

Case A: All points a_n^2,a_n^3 and a_n^4 are b points.

Then there must be some topmost b point with some propositional variable true, without loss of generality, let us say $b_i \Vdash p_0$ with b_i being the topmost point. Then at a_{i+1} we cannot have $a_{i+1} \Vdash p_2$, $a_{i+1} \Vdash p_1$ and $a_{i+1} \Vdash p_0$ (for $a_{i+1} \Vdash p_0$ go to Case B). Similarly we cannot have $a_{i+1} \nVdash p_1 \to p_0 \vee p_2$ and $a_{i+1} \nVdash p_2 \to p_0 \vee p_1$. Hence, we cannot have that $a_n \Vdash (p_0 \to p_1 \vee p_2) \to p_1 \vee p_2$ since we have found a point, a_{i+1}, with (viii) and (ix) false.

Case B: Two points a_n^2,a_n^3 are b points and a_n^4 is an a point.

Clearly, the topmost point needs to be the a point, without loss of generality, let us say $a_i \Vdash p_0$ with a_i being the topmost point. Then similar to the above case the point a_{i+1} cannot have $a_{i+1} \Vdash p_2$, $a_{i+1} \Vdash p_1$ and $a_{i+1} \Vdash p_0$ (since a_i is the top most point with only p_0 true). Similarly we cannot have $a_{i+1} \nVdash p_1 \to p_0 \vee p_2$ and $a_{i+1} \nVdash p_2 \to p_0 \vee p_1$. Hence, we cannot have that $a_n \Vdash (p_0 \to p_1 \vee p_2) \to p_1 \vee p_2$ since we have found a point, a_{i+1}, with (viii) and (ix) false.

For the rest of the formulas bb_n with $3 \leq n$, one can easily see that an extension of the same argument can be used.

Now we consider an ultrapower of the $\mathfrak{F}$, take a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$.

In addition to the shorthand used in Chapter 4.1.2, we have:

$E(x) := \neg\exists y(x < y)$ [x has no successors, i.e., x is an endpoint],

$S(x) := \exists y\exists z(\neg(y = z)\wedge C(x, y)\wedge C(x, z)\wedge\neg\exists u(\neg(u = y)\wedge\neg(u = z)\wedge C(x, u)))$ [x is a point that has only two direct successors],

$SE(x) := S(x) \wedge \exists y\exists z(\neg(y = z) \wedge C(x, y) \wedge C(x, z) \wedge E(y) \wedge E(z))$ [x is a point that has two direct successors where both of them are endpoints].

The following first-order sentences are valid on $\mathfrak{F}$,

i. $\forall x[E(x) \vee (S(x) \wedge \exists y(C(x, y) \wedge E(y)))]$ [All points are either endpoints or points that have two direct successors],

ii. $\exists x(SE(x) \wedge \neg\exists y(\neg(x = y) \wedge SE(y)))$ [There is only one point that has two endpoints as direct successors].

It is clear that the sentences above are true in $\mathfrak{F}$ and by Łoś's Theorem we have that they will be true in our ultrapower, we can now envision the ultrapower as looking like Figure 4.12, let us call it $\mathfrak{F}'$.

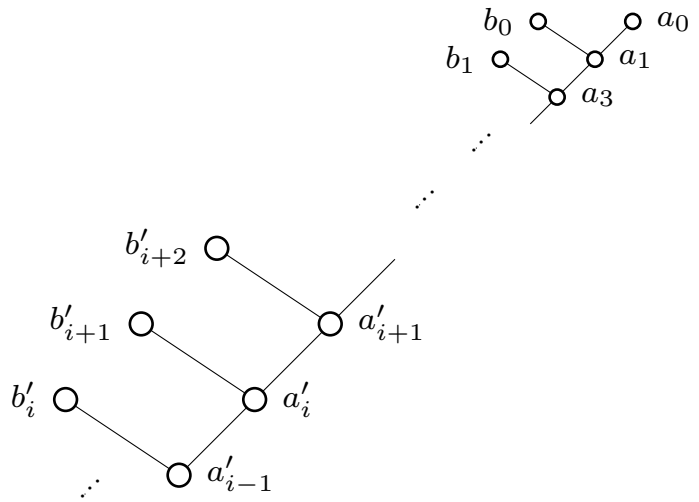


Figure 4.12: Ultrapower of $\mathfrak{F}$

Step 3: Show that bb_n is refutable in the ultrapower

Let us define a bounded morphism from our ultrapower to the refutation frame, $f : \mathfrak{F}' \to \mathfrak{BB}$, as follows:

$f(a'_i) = u$ for all $i \in \mathbb{N}$,

$f(b'_i) = u_0$ for all $i \in \mathbb{N}$ such that $i(mod(n+1)) = 0$,

$f(b'_i) = u_1$ for all $i \in \mathbb{N}$ such that $i(mod(n+1)) = 1$,

$f(b'_i) = u_2$ for all $i \in \mathbb{N}$ such that $i(mod(n+1)) = 2$,

$\vdots$

$f(b'_i) = u_n$ for all $i \in \mathbb{N}$ such that $i(mod(n+1)) = n$,

$f(a_j) = u_n$ if $a_j > a'_i$ for all $i, j \in \mathbb{N}$,

$f(b_j) = u_n$ if $b_j > a'_i$ for all $i, j \in \mathbb{N}$.

The first condition (R1) of a bounded morphism is clearly true for f. For the second condition (R2), take the two points u_k and u, $u \leq u_k$. Let us assume that $f(b'_l) = u_k$, if $l(mod(n+1)) = k$, and $f(a'_{l-1}) = u$ now consider the point b'_{l+n+1} we have the condition that $f(b'_{l+n+1}) = u_k$ also $a'_{l-1} \leq b'_{l+n+1}$, it is clear to see that this works for all points in the frame $\mathfrak{BB}$. Hence, satisfying the (R2).

If we assumed that bb_n for $n \geq 2$ was first-order definable then by Łoś's Theorem it should have been true in $\mathfrak{F}'$ since it was true in $\mathfrak{F}$ but due to the bounded morphism we have a contradiction hence proving that bb_n for $n \geq 2$ is not first-order definable. □

It is possible, even without the use of deterministic formulas, to see that bb_1 would not work in our proof since it is possible to refute it in our frame $\mathfrak{F}$. Consider the bounded morphism that maps the two topmost endpoints in $\mathfrak{F}$, a_0 and b_0, to u_0 and u_1 in $\mathfrak{BB}$ and the rest of the points in $\mathfrak{F}$ to u. This is not in any way a "proof" to show that bb_1 is first-order definable; it is merely showing where the argument would break down for this case, the formal proof was done in Chapter 3.1.

An interesting observation that arises is that both bb_n and BW_n, $\bigvee_{i=0}^{n}(p_i \to \bigvee_{i \neq j} p_j)$, have the same refutation frame while one is first-order definable and the other is not. We have shown both above, but an interesting experiment would be to see where BW_n fails in the proof used to show bb_n was not first-order definable, and vice versa, where bb_n would fail if attempting to use the same first-order condition as BW_n.

<u>BW_n failing.</u>

Consider the frame $\mathfrak{F}$ in Figure 4.11. We will show that $\mathfrak{F} \Vdash\!\!\!\!/\ BW_n$; this will be sufficient to show that the above proof fails for BW_n. Let us consider a point a_n in $\mathfrak{F}$ we want to show that $a_n \Vdash\!\!\!\!/\ BW_n$ which requires the following:

$a_n \Vdash\!\!\!\!/\ p_0 \to p_1 \vee p_2 \vee \cdots \vee p_n$,

$a_n \Vdash\!\!\!\!/\ p_1 \to p_0 \vee p_2 \vee \cdots \vee p_n$,

$\vdots$

$a_n \nVdash p_n \to p_0 \vee p_1 \vee \cdots \vee p_{n-1}$.

The first condition requires the existence of a point, say b_n, above a_n where $b_n \Vdash p_0$ and $b_n \nVdash p_1$ and ... and $b_n \nVdash p_n$. The rest of the conditions are similar. Now if we consider the valuation $V(p_0) = \{a_0\}$, $V(p_1) = \{b_0\}$, $V(p_2) = \{b_1\}$, ..., $V(p_n) = \{b_{n-1}\}$. Then we have all conditions satisfied, hence $a_n \nVdash BW_n$. The fact that $\mathfrak{F}$ does not hold BW_n true while there is a bounded morphism between the refutation frame and the ultrapower is the failing point with this formula. □

$\underline{bb_n \text{ failing.}}$

We want to show that for an arbitrary frame $\mathfrak{G} = (W, \leq)$ we have $\mathfrak{G} \Vdash \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i$ iff $\mathfrak{G} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \to \bigvee_{i\neq j} x_i \leq x_j)$ is not true.

($\Leftarrow$) Using a contrapositive argument we are required to show that if $\mathfrak{G} \nVdash \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i$ then $\mathfrak{G} \models \exists x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \wedge \bigwedge_{i\neq j} \neg(x_i \leq x_j))$. We have shown this when finding the refutation frame of bb_n.

($\Rightarrow$) Let us assume $\mathfrak{G} \Vdash \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i$ we want to show $\mathfrak{G} \models \forall x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \to \bigvee_{i\neq j} x_i \leq x_j)$, but we have a counterexample which is our frame in Figure 4.11, $\mathfrak{F} \Vdash \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i\neq j} p_j) \to \bigvee_{i\neq j} p_j) \to \bigvee_{i=0}^{n} p_i$ but $\mathfrak{F} \not\models \forall x, x_0, \ldots, x_n(\bigwedge_{i=0}^{n} x \leq x_i \to \bigvee_{i\neq j} x_i \leq x_j)$ consider the point a_n which is less than or equal to all the points $a_0, b_0, \ldots, b_{n-1}$ but none of them are comparable with each other. □

Unfortunately, it seems that [3], in the end notes in chapter 6, has mistakenly classified bb_n as being non-first-order definable for all $n \geq 1$ (in actual fact it is first-order definable for $n = 1$) and similarly they claim that, using a compactness argument, an infinitely ascending frame would suffice to show that it is not first-order definable, this is also untrue. Consider Figure 4.13. According to [3], this frame should validate bb_n, but the following argument shows that this is not the case.

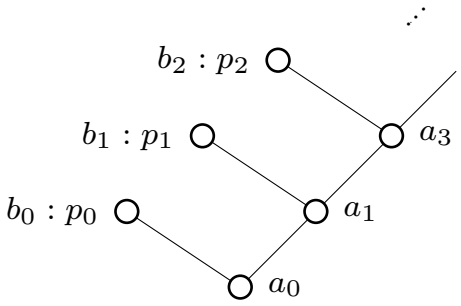


Figure 4.13: Incorrect Kripke frame claimed to validate bb_n

We are required to show that there exists a valuation on the above frame, call it $\mathfrak{F}$, such that $(\mathfrak{F}, V) = \mathfrak{M} \nVdash bb_n$. The necessary conditions to refute bb_n are listed below. We show that $a_0 \nVdash bb_n$.

$$a_0 \Vdash (p_0 \to p_1 \vee p_2 \vee \cdots \vee p_n) \to p_1 \vee p_2 \vee \cdots \vee p_n,$$

$$a_0 \Vdash (p_1 \to p_0 \vee p_2 \vee \cdots \vee p_n) \to p_0 \vee p_2 \vee \cdots \vee p_n,$$

$$\vdots$$

$$a_0 \Vdash (p_n \to p_0 \vee p_1 \vee \cdots \vee p_{n-1}) \to p_0 \vee p_1 \vee \cdots \vee p_{n-1},$$

$$a_0 \nVdash p_0,$$

$$a_0 \nVdash p_1,$$

$$\vdots$$

$$a_0 \nVdash p_n.$$

Now consider the valuation below, where we have the same repeating pattern as the valuation on the ultrapower in Figure 4.12.

$$b_i \Vdash p_0 \text{ for all } i \in \mathbb{N} \text{ such that } i(mod(n+1)) = 0,$$

$$b_i \Vdash p_1 \text{ for all } i \in \mathbb{N} \text{ such that } i(mod(n+1)) = 1,$$

$$\vdots$$

$$b_i \Vdash p_n \text{ for all } i \in \mathbb{N} \text{ such that } i(mod(n+1)) = n.$$

It can be checked that a_0 under the specified valuation does not validate bb_n (a bounded morphism argument could similarly be used). Also, the ultrapower of an ascending frame can easily be seen to have a bounded morphism to the refutation frame; the argument would be similar to the one we provided. Hence, we conclude that an ascending frame is insufficient to prove non-first-order definability for bb_n.

4.3 Rieger–Nishimura formulas

The NL_n logic was defined as Int + nf_n in Table 3.2. nf_n, which are better known as the Rieger–Nishimura formulas, have the interesting property that all one variable intuitionistic formulas are equivalent to one of the recursively defined Rieger–Nishimura formulas. This was proved by Ladislav Rieger in [25] as stated in [26]. Iwao Nishimura in [24] independently proved the same theorem and that no one Rieger–Nishimura formula is equivalent to any other one.

The sequence is recursively defined as:

$nf_0 = \bot, \quad nf_1 = p, \quad nf_2 = \neg p, \quad nf_\omega = \top,$
$nf_{2m+3} = nf_{2m+1} \vee nf_{2m+2},$
$nf_{2m+4} = nf_{2m+3} \to nf_{2m+1}.$

The Rieger–Nishimura formulas form the Heyting algebra shown in Figure 4.14. Known as the Rieger–Nishimura lattice, all nf_n formulas and subsequently all one variable intuitionistic formulas can be ascribed to an element, and only one element, of the lattice.

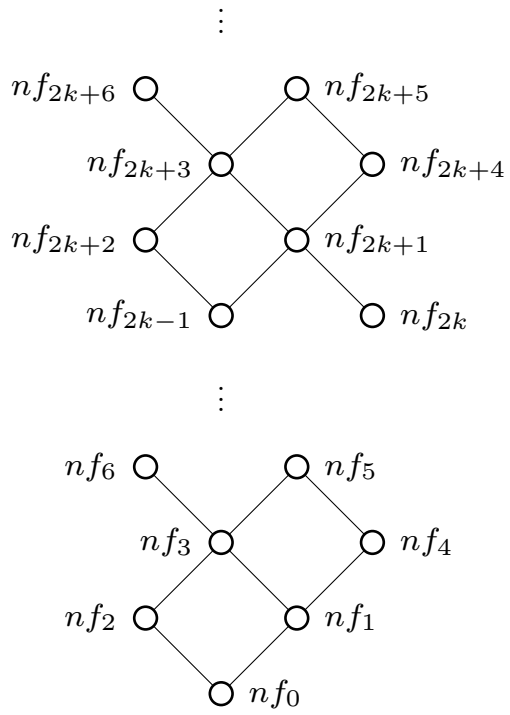


Figure 4.14: Rieger–Nishimura lattice

Rodenburg [26] proved that in the Rieger–Nishimura lattice, the first 10 elements are first-order definable and all other formulas are not first-order definable, but he made use of a different version of the Rieger–Nishimura formulas. His definition is as follows:

$M_0 = \bot, \quad M_1 = p, \quad M_2 = \neg p, \quad M_\omega = \top,$
$M_{2m+3} = M_{2m+1} \vee M_{2m+2},$
$M_{2m+4} = M_{2m+2} \to M_{2m+1}.$

Even though the two versions differ slightly in the indexing of the antecedent for all even formulas, they are equivalent. A proof is given below. We make use of the fact that the Rieger–Nishimura lattice is a Heyting algebra, and from [3] we have the following.

Let $\mathfrak{RN} = (U, \wedge, \vee, \to, \bot)$ be the algebra depicted in Figure 4.14. Define a valuation V on $\mathfrak{RN}$ such that $V(p)$ is the element marked by $nf_1 = p$. Then we have that $V(nf_i)$ is the element nf_i in $\mathfrak{RN}$, for each $i \leq \omega$. It can be checked [3] that, if $* \in \{\wedge, \vee, \to\}$ and $i, j, k \leq \omega$, we have that:

$$V(nf_i * nf_j) = V(nf_k) \quad \text{iff} \quad (nf_i * nf_j) \leftrightarrow nf_k \in \mathbf{Int}.$$

Recall, we also defined $u \to v = \bigvee\{w \in U \mid u \wedge w \leq v\}$ for all $u, v \in \mathfrak{RM}$.

Lemma 4.3.1. nf_i is equivalent to M_i for all $i \in I$.

Proof. The cases in which i is not an even number are all trivially true.

For the case where i is an even number, let us consider both $nf_{2m+3} \to nf_{2m+1}$ and $nf_{2m+2} \to nf_{2m+1}$ independently; it can be easily seen that in the Rieger–Nishimura lattice they are both equal to the point nf_{2m+4} therefore $V(nf_{2m+3} \to nf_{2m+1}) = V(nf_{2m+4})$ hence $(nf_{2m+3} \to nf_{2m+1}) \leftrightarrow nf_{2m+4} \in \mathbf{Int}$. Similarly $V(nf_{2m+2} \to nf_{2m+1}) = V(nf_{2m+4})$ hence $(nf_{2m+2} \to nf_{2m+1}) \leftrightarrow nf_{2m+4} \in \mathbf{Int}$ which would imply that $(nf_{2m+2} \to nf_{2m+1}) \leftrightarrow (nf_{2m+3} \to nf_{2m+1}) \in \mathbf{Int}$ since they are both equivalent to the same formula. □

Now, since we have proved an equivalence between the two formula sequences, we can claim similarly as Rodenburg.

Theorem 4.3.2. nf_i where $i \leq 9$ are first-order definable.

Proof. We make use of some simplifications of the formulas. The following are equivalent,

nf_4 is equivalent to $\neg\neg p$,

nf_5 is equivalent to $\neg\neg p \vee \neg p$,

nf_6 is equivalent to $\neg\neg p \to p$,

nf_7 is equivalent to $(\neg\neg p \to p) \to \neg\neg p$,

nf_8 is equivalent to $(\neg\neg p \to p) \to (p \vee \neg p)$.

A proof of these equivalences can be done by showing $\varphi \leftrightarrow \psi \in \mathbf{Int}$ where φ and ψ are the relevant formulas. These simplifications were taken from [26], and it is easy to check that they are all deterministic and hence first-order definable. □

Theorem 4.3.3. nf_i where $10 \leq i$ are not first-order definable for all i.

Proof. See Rodenburg [26] chapter 11. □

This result is powerful since it gives us a way to characterise first-order definability for all one variable intuitionistic formulas by finding an equivalence to the formulas in the Rieger–Nishimura lattice.

Chapter 5

ALBA

ALBA (Ackermann Lemma Based Algorithm) is an algorithm such that, when given an inductive formula, will produce its first-order correspondent. Introduced by Willem Conradie and Alessandra Palmigiano in [7], ALBA is a generalized version of the SQEMA algorithm [6]. While SQEMA found first-order correspondents of classical modal formulas, ALBA was defined for distributive modal logic. ALBA was further generalized and shown in [8] to compute first-order correspondents for non-distributive logics.

This chapter aims to briefly introduce ALBA, specialize it to intuitionistic logic, and compute some examples. We will further introduce a new rule, specifically for intuitionistic logic, that allows for a greater class of formulas to be computed on this new ALBA (defined as Int-ALBA). Lastly, we shall provide a syntactic classification of the formulas that will compute on Int-ALBA. This new class is a novel extension of (Ω, ε)-inductive intuitionistic formulas.

5.1 Standard ALBA

The Ackermann Lemma based algorithm (ALBA) algorithmically computes first-order correspondents for any intuitionistic formula, given that the formula is inductive. For a more in-depth understanding, the reader is referred to [7] where ALBA is introduced for distributive logics. We provide a specialized version for intuitionistic logic in this thesis.

As the name suggests, the Ackermann lemmas hold importance to the actual algorithm. In [1], Wilhelm Ackermann proved the Ackermann lemmas in the context of second-order logic. We provide a specialized version for intuitionistic logic that is analogous to the lemmas.

Definition 5.1.1. A formula $\varphi \in \mathcal{L}^+_{\mathcal{IP}}$ is positive (respectively negative) in a proposition variable p if in its positive generation tree all p-nodes are signed + (respectively -). An inequality $\varphi \leq \psi \in \mathcal{L}^+_{\mathcal{IP}}$ is positive (respectively negative) in a propositional variable p if φ is negative (respectively positive) in p and φ is positive (respectively negative) in p.

Some terminology needed before we state the lemma. Two valuations V and V' on a frame $\mathfrak{F}$ are called p-variants for $p \in \mathit{Prop}$ (notation: $V \sim_p V'$) if $V(a) = V'(a)$ for all $a \in (\mathit{Prop} - \{p\}) \cup \mathrm{NOM} \cup \mathrm{CO\text{-}NOM}$.

Lemma 5.1.2. (Right Ackermann Lemma) Let $\alpha_0, \ldots, \alpha_n \in \mathcal{L}^+_{\mathcal{IP}}$ with $p \notin \mathit{Prop}(\alpha_i)$, $0 \leq i \leq n$, let $\beta_0(p), \ldots, \beta_m(p) \in \mathcal{L}^+_{\mathcal{IP}}$ be positive in p, and let $\gamma_0(p), \ldots, \gamma_m(p) \in \mathcal{L}^+_{\mathcal{IP}}$ be negative in p. Let V be any valuation on a frame $\mathfrak{F}$. Then,

$$(\mathfrak{F}, V) \Vdash \beta_j((\alpha_0 \vee \cdots \vee \alpha_n) \setminus p) \leq \gamma_j((\alpha_0 \vee \cdots \vee \alpha_n) \setminus p), \quad \text{for all } 0 \leq j \leq m$$

iff there exists some $V' \sim_p V$ such that

$$\mathfrak{F}, V \Vdash \alpha_i \leq p \quad \text{for all } 0 \leq i \leq m, \quad \text{and} \quad \mathfrak{F}, V' \Vdash \beta_i(p) \leq \gamma_i(p) \quad \text{for all } 0 \leq j \leq m.$$

Lemma 5.1.3. (Left Ackermann Lemma) Let $\alpha_0, \ldots, \alpha_n \in \mathcal{L}^+_{\mathcal{IP}}$ with $p \notin \mathit{Prop}(\alpha_i)$, $0 \leq i \leq n$, let $\beta_0(p), \ldots, \beta_m(p) \in \mathcal{L}^+_{\mathcal{IP}}$ be positive in p, and let $\gamma_0(p), \ldots, \gamma_m(p) \in \mathcal{L}^+_{\mathcal{IP}}$ be negative in p. Let V be any valuation on a frame $\mathfrak{F}$. Then,

$$(\mathfrak{F}, V) \Vdash \beta_j((\alpha_0 \vee \cdots \vee \alpha_n) \setminus p) \leq \gamma_j((\alpha_0 \vee \cdots \vee \alpha_n) \setminus p), \quad \text{for all } 0 \leq j \leq m$$

iff there exists some $V' \sim_p V$ such that

$$\mathfrak{F}, V \Vdash p \leq \alpha_i \quad \text{for all } 0 \leq i \leq m, \quad \text{and} \quad \mathfrak{F}, V' \Vdash \beta_i(p) \leq \gamma_i(p) \quad \text{for all } 0 \leq j \leq m.$$

Proof. The proofs for both these lemmas are similar to those that can be found in [7]. □

Given the language $\mathcal{L}_{\mathcal{IP}}$, an $\mathcal{L}_{\mathcal{IP}}$-quasi-inequality (from now on we will drop the $\mathcal{L}_{\mathcal{IP}}$) is an expression of the form $\varphi_0 \leq \psi_0, \varphi_1 \leq \psi_1, \ldots, \varphi_n \leq \psi_n \ \vdash \ \varphi \leq \psi$ where $\varphi_i, \psi_i, \varphi$ and ψ are all $\mathcal{L}_{\mathcal{IP}}$ formulas and the symbols the commas (,) and $\vdash$ are interpreted as conjunction and implication, respectively.

Briefly, what ALBA attempts to do is eliminate all propositional variables in a formula. This is done by utilizing the Ackermann lemmas, which provide a way to eliminate a propositional variable and replace it with a formula that is free from the propositional variable. Now, given any formula, we first have to fashion it into a form that will then allow us to apply the Ackermann lemmas; here, we make use of known lattice properties. Below we have listed the *rules* and will provide justification on some of them; for all rules, the double line means that they are reversible.

First Approximation Rule

$$\frac{\varphi \leq \psi}{i_0 \leq \varphi, \psi \leq m_0 \ \vdash \ i_0 \leq m_0}$$

for each computation, the first approximation rule can only be applied once.

Residuation Rules

$\wedge$-splitting

$$\frac{\varphi \leq \psi \wedge \delta}{\varphi \leq \psi, \varphi \leq \delta}$$

$\vee$-splitting

$$\frac{\varphi \vee \psi \leq \delta}{\varphi \leq \delta, \psi \leq \delta}$$

Heyting rule

$$\frac{\varphi \leq \psi \to \delta}{\varphi \wedge \psi \leq \delta}$$

Swapping rule

$$\frac{\varphi \leq \psi \to \delta}{\psi \leq \varphi \to \delta}$$

Remark 5.1.4. The swapping rule is not atomic; instead, it is derivable from the Heyting rule and application of commutativity of $\wedge$.

Approximation Rules

($\to$) antecedent approximation rule

$$\frac{\varphi \to \psi \leq\ m}{i \leq \varphi, i \to \psi \leq\ m}$$

($\to$) consequent approximation rule

$$\frac{\varphi \to \psi \leq\ m}{\varphi \to n \leq m, \psi \leq n}$$

($\neg$) approximation rule

$$\frac{\neg\varphi \leq\ m}{i \leq \varphi, \neg i \leq\ m}$$

where the introduced nominal and co-nominal are different to any previously introduced nominal or co-nominal.

Ackermann Rules

Left Ackermann rule

$$\frac{p \leq \alpha, \delta_1(p) \leq \beta_1(p), \ldots, \delta_\ell(p) \leq \beta_\ell(p) \vdash i_0 \leq m_0}{\delta(\alpha) \leq \beta(\alpha) \vdash i_0 \leq m_0}$$

Right Ackermann rule

$$\frac{\alpha \leq p, \beta_1(p) \leq \delta_1(p), \ldots, \beta_\ell(p) \leq \delta_\ell(p) \vdash i_0 \leq m_0}{\beta(\alpha) \leq \delta(\alpha) \vdash i_0 \leq m_0}$$

Comment 5.1.5. For the left Ackermann rule, there may be multiple occurrences of $p \leq \alpha$, but we apply the reverse of the $\wedge$-splitting rule to conjoin them into one inequality. Similarly, it is done for the right Ackermann rule by making use of the reverse $\vee$-splitting rule.

For both Ackermann rules, the following conditions must be applied:

i. p doesn't occur in α,

ii. $\beta(p)$ is positive in p, which means that in the positive signed generation tree of β all occurrences of p are positive,

iii. $\delta(p)$ is negative in p, which means that in the positive signed generation tree of δ all occurrences of p are negative.

Comment 5.1.6. Given any quasi-inequality $i_0 \leq \varphi, \psi \leq m_0 \ \vdash \ i_0 \leq m_0$, the antecedent $(i_0 \leq \varphi, \psi \leq m_0)$ is where we apply reisduation or approximation, the Ackermann rules are applied to the quasi-inequality as a whole.

We shall prove soundness for a select few of the above rules. Most of the residuation rules are simply based on standard lattice properties. Let $\mathfrak{A} = (U, \wedge, \vee, \rightarrow, \bot)$ be a perfect Heyting algebra; the other rules have the following derivation:

First approximation rule:

$\varphi \leq \psi$,

iff $\bigvee\{i_0 \in J^\infty(\mathfrak{A}) | i_0 \leq \varphi\} \leq \ \bigwedge\{m_0 \in M^\infty(\mathfrak{A}) | \psi \leq m_0\}$,

iff $i_0 \leq \varphi, \psi \leq \ m_0$ implies $i_0 \leq m_0$, for all $i_0 \in J^\infty(\mathfrak{A})$ and $m_0 \in M^\infty(\mathfrak{A})$.

Swapping rule:

$\varphi \leq \psi \rightarrow \delta$,

iff $\varphi \wedge \psi \leq \delta$,

iff $\psi \wedge \varphi \leq \delta$,

iff $\psi \leq \varphi \rightarrow \delta$.

$(\rightarrow)$ antecedent approximation rule:

$\varphi \rightarrow \psi \leq \ m_0$,

iff $\bigvee\{j \in J^\infty(\mathfrak{A}) | j \leq \varphi\} \rightarrow \psi \leq \ m_0$, since $J^\infty(\mathfrak{A})$ completely $\vee$-generates $\mathfrak{A}$,

iff $\bigwedge\{j \rightarrow \psi | j \in J^\infty(\mathfrak{A}), j \leq \varphi\} \leq \ m_0$, since $\rightarrow$ is completely $\vee$-reversing in the 1^{st} coordinate,

iff $j \rightarrow \psi \leq \ m_0$ for some $j \leq \varphi$, $j \in J^\infty(\mathfrak{A})$, since m_0 is completely $\wedge$-prime in $\mathfrak{A}$.

For the $(\rightarrow)$ consequent approximation rule, the proof is similar but makes use of $M^\infty(\mathfrak{A})$ in place of $J^\infty(\mathfrak{A})$. Similarly, the proof for $(\neg)$ approximation rule can be achieved by considering $\neg\varphi \leq i_0$ as $\varphi \rightarrow \bot \leq i_0$ and treating it the same as the $(\rightarrow)$ antecedent approximation rule.

There are three stages of ALBA.

The preprocessing stage is the stage at the beginning of ALBA, before applying the first approximation rule. In this stage, we can apply any of the residuation rules, but the main aim of this stage is to split the inequality with $\vee$-splitting or $\wedge$-splitting.

The reduction and elimination stage is the stage immediately after applying the first approximation rule. In this stage, we can apply either the residuation rules or the approximation rules, and we finish this stage of with any of the Ackermann rules.

A formula is said to *succeed* on ALBA if all propositional variables are eliminated by the Ackermann lemma. A formula is said to *fail* on ALBA if there is at least one propositional variable that does not get eliminated through the Ackermann lemma.

The Success, failure, and output stage is the stage after the Ackermann rules, where we return either if the formula has failed or succeeded on ALBA. If the formula succeeded on ALBA, then the first-order condition that is output by ALBA needs to be simplified into a more readable form. We use $\mathrm{ALBA}(\varphi \leq \psi)$ to denote the output that ALBA produces after eliminating all propositional variables in $\varphi \leq \psi$, and $\mathrm{ALBA}^{FO}(\varphi \leq \psi)$ to denote the first-order equivalent after introducing the semantic interpretation of the nominals and co-nominals in $\mathrm{ALBA}(\varphi \leq \psi)$.

Below are the necessary theorems that collectively claim that an inductive inequality will compute on ALBA and that ALBA will, in fact, output the first-order correspondent. Taking from [8], we list the theorems without their proofs; the proofs can be found in [8].

Given an intuitionistic Kripke frame $\mathfrak{F} = (W, \leq)$ and σ that is a formula, inequality, or quasi-inequality in our extended language $\mathcal{L}^{+}_{\mathcal{LE}}$. We use the notation $\mathfrak{F}(i := w{\uparrow}) \Vdash \sigma$ to indicate that $(\mathfrak{F}, V) \Vdash \sigma$ for all valuations V that send the nominal variable i to $w{\uparrow}$.

Theorem 5.1.7 (Correctness and correspondence)**.** If ALBA succeeds in reducing an $\mathcal{L}_{LE}$-inequality $\varphi \leq \psi$ and yields $\mathrm{ALBA}(\varphi \leq \psi)$, then $(\mathfrak{F}, w) \Vdash \varphi \leq \psi$ iff $\mathfrak{F}(i := w{\uparrow}) \Vdash \mathrm{ALBA}(\varphi \leq \psi)$ iff $\mathfrak{F} \models \mathrm{ALBA}^{FO}(\varphi \leq \psi)[i := w]$.

Theorem 5.1.8. ALBA succeeds on all inductive inequalities.

Given all the information we now have, we are ready to compute some first-order correspondents using ALBA. What we aim to achieve first, given any formula φ, is to manipulate it such that it morphs into a form where we can apply the first approximation rule. We achieve this by considering φ as the inequality $\top \leq \varphi$. After applying the first approximation, we get a formula of the form $i_0 \leq \varphi, \psi \leq m_0 \vdash i_0 \leq m_0$. We then have to use any of the given rules, apart from the first approximation rule, to separate all the propositional variables so that we can then apply the Ackermann rules. The Ackermann rules will then eliminate the propositional variables from our formula, and we will then be left with a formula of only nominals and co-nominals; any more rules can be applied hereafter to simplify the formula. Now we apply the semantic interpretations in intuitionistic Kripke frame of nominals ($x{\uparrow}$) and co-nominals ($(y{\downarrow})^c$), negation of a nominal or co-nominal adds$\downarrow^c$ to the nominal or co-nominal, this can easily be seen if one was to consider a nominal $\neg(x{\uparrow})$ as $(x{\uparrow}) \to \bot$ and consider it as its set theoretic definition, similarly for co-nominals. This equation we are now left with is the first-order correspondent of our formula φ, just in a possibly unappealing form, hence we then further simplify the equation to arrive at one that is more readable.

Lemma 5.1.9. For any Kripke frame $\mathfrak{F}$ and $U, V \in \mathfrak{U}(\mathfrak{F})$, $U \supset V := \bigcup\{w{\uparrow} \mid w{\uparrow} \cap U \subseteq V\} = (U \cap V^c){\downarrow}^c$.

Proof. Initially, we defined $U \supset V = \bigcup\{W \in \mathfrak{U}(\mathfrak{F}) \mid W \cap U \subseteq V\}$ but we can substitute $w{\uparrow}$ for W since $W \in \mathfrak{U}(\mathfrak{F})$ is also generated by an element of $\mathfrak{F}$.

Show $\bigcup\{w\uparrow | w\uparrow \cap U \subseteq V\} \subseteq (U \cap V^c)\downarrow^c$:

Let $x \in \bigcup\{w\uparrow | w\uparrow \cap U \subseteq V\}$, i.e., there exists a $w\uparrow$ such that $w \leq x$. We want to show that $x \in (U \cap V^c)\downarrow^c$, i.e., $x \notin (U \cap V^c)\downarrow$. For contradiction assume $x \in (U \cap V^c)\downarrow$, which means there exists some $y \in U \cap V^c$ with $x \leq y$, so $y \in U$ and $y \notin V$. By transitivity, $w \leq y$, i.e., $y \in w\uparrow$, so $y\uparrow \cap U \subseteq V$ which is a contradiction.

Show $(U \cap V^c)\downarrow^c \subseteq \bigcup\{w\uparrow | w\uparrow \cap U \subseteq V\}$:

Let $x \in (U \cap V^c)\downarrow^c$, then $x \notin (U \cap V^c)\downarrow$. This implies that there does not exist a $y \geq x$ with $y \in (U \cap V^c)$. Now fix $w = x$, we want to show that $w\uparrow \cap U \subseteq V$. Suppose that $z \in w\uparrow \cap U$ this means that $z \in U$ and $z \in w\uparrow$, i.e., $(x \leq z)$. Since $x \notin (U \cap V^c)\downarrow$ this implies that $z \notin (U \cap V^c)$ implying that $z \notin U$ or $z \in V$ and since it was assumed that $z \in U$ it has to be that $z \in V$. Hence, $w\uparrow \cap U \subseteq V$.

Hence, $\bigcup\{w\uparrow | w\uparrow \cap U \subseteq V\} = (U \cap V^c)\downarrow^c$. □

A few ALBA computations of the axioms of the listed superintuitionistic logics (Table 3.2) will be done in the following examples; we can refer to Table 3.3 to see if our computation arrives at the known first-order correspondents. To make our computations more reader friendly, we shall remove the parenthesis and introduce a priority of ordering from left to right such that $((x\uparrow)\downarrow)^c$ will be written as $x\uparrow\downarrow^c$. We have the following computations:

Example 5.1.10. (CL): Running ALBA on $p \vee \neg p$ produces:

Begin with $\top \leq p \vee \neg p$,

$i \leq \top \ , \ p \vee \neg p \leq m \ \vdash \ i \leq m$ by first approximation,

$p \leq m \ , \ \neg p \leq m \ \vdash \ i \leq m$ by $\vee$ splitting and $i \leq \top$ is trivially true,

$\neg m \leq m \ \vdash \ i \leq m$ by right Ackermann rule,

now introducing the semantic definition of nominal and co-nominal,

$\forall x \forall y \ [y\downarrow^c\downarrow^c \subseteq y\downarrow^c \Rightarrow x\uparrow \subseteq y\downarrow^c]$,

iff $\forall x \forall y \ [y\downarrow \subseteq y\downarrow^c\downarrow \Rightarrow x \in y\downarrow^c]$,

iff $\forall x \forall y \ [y \in y\downarrow^c\downarrow \Rightarrow x \in y\downarrow^c]$,

iff $\forall x \forall y \ [\exists z(y < z) \Rightarrow x \not\leq y]$,

iff $\forall x \forall y \ [x \leq y \Rightarrow \neg\exists z(y < z)]$,

iff $\forall y \ [\exists x(x \leq y) \Rightarrow \neg\exists z(y < z)]$,

iff $\forall y\ [\neg\exists z(y \leq z)]$.

Hence, we achieve our first-order correspondent $\forall x, y(x \leq y \Rightarrow x = y)$.

Example 5.1.11. (KC): Running ALBA on $\neg p \vee \neg\neg p$ produces:

Begin with $\top \leq \neg p \vee \neg\neg p$,

$i \leq \top\ ,\ \neg p \vee \neg\neg p \leq m\ \vdash\ i \leq m$ by first approximation,

$\neg p \leq m\ ,\ \neg\neg p \leq m\ \vdash\ i \leq m$ by $\vee$ splitting and $i \leq \top$ is trivially true,

$j \leq p\ ,\ \neg j \leq m\ ,\ \neg\neg p \leq m\ \vdash\ i \leq m$ by approximation rule,

$\neg\neg j \leq m\ ,\ \neg j \leq m\ \vdash\ i \leq m$ by right Ackermann rule,

$\neg\neg j \vee \neg j \leq m\ \vdash\ i \leq m$ by residual rule,

$i \leq \neg\neg j \vee \neg j$,

now introducing the semantic definition of nominal and co-nominal,

$\forall x \forall z\ [z{\uparrow} \subseteq x{\uparrow}{\downarrow}^c{\downarrow}^c \cup x{\uparrow}{\downarrow}^c]$,

iff $\forall x \forall z\ [z \uparrow \subseteq\ x \uparrow{\downarrow}^c{\downarrow}^c]$ or $\forall x \forall z\ [z \uparrow \subseteq\ x \uparrow{\downarrow}^c]$ since if U and V are up-sets, $x{\uparrow} \subseteq U \cup V$ iff $x{\uparrow} \subseteq U$ or $x{\uparrow} \subseteq V$,

$\forall x \forall z\ [z{\uparrow} \subseteq x{\uparrow}{\downarrow}^c]$ iff x and z are incomparable,

$\forall x \forall z\ [z{\uparrow} \subseteq x{\uparrow}{\downarrow}^c{\downarrow}^c]$, now looking at $x{\uparrow}{\downarrow}^c{\downarrow}^c$. We know $x{\uparrow}{\downarrow}^c{\downarrow}$ includes all points in $x{\uparrow}{\downarrow}^c$ with the points in its downset, and if we consider its complement, $x{\uparrow}{\downarrow}^c{\downarrow}^c$, this includes $x{\uparrow}$ but also may include some points other that are in $x{\uparrow}{\downarrow}$, we shall denote the set of these *extra points* as X. What is important to note about X is that for all points in X there cannot be a point with a successor in $x{\uparrow}{\downarrow}^c$, because then that point would be included in $x{\uparrow}{\downarrow}^c{\downarrow}$. This implies that all points in X need to have a successor in $x{\uparrow}$. So $z{\uparrow}$ can either be in $x{\uparrow}$ or X, either of which requires there to be a common successor between x and z.

Hence, we achieve our first-order correspondent, $\forall x, y, z(x \leq y \wedge x \leq z \rightarrow \exists u(y \leq u \wedge z \leq u))$.

Example 5.1.12. (LC): Running ALBA on $(p \rightarrow q) \vee (q \rightarrow p)$ produces:

Begin with $\top \leq (p \rightarrow q) \vee (q \rightarrow p)$,

$i \leq \top\ ,\ (p \rightarrow q) \vee (q \rightarrow p) \leq m\ \vdash\ i \leq m$ by first approximation,

$p \rightarrow q \leq m\ ,\ q \rightarrow p \leq m\ \vdash\ i \leq m$ by $\vee$ splitting and $i \leq \top$ is trivially true,

$j \leq p\ ,\ j \rightarrow q \leq m\ ,\ k \leq q\ ,\ k \rightarrow p \leq m\ \vdash\ i \leq m$ by approximation rule,

$j \leq p\ ,\ k \rightarrow p \leq m\ ,\ k \leq q\ ,\ j \rightarrow q \leq m\ \vdash\ i \leq m$,

$k \to j \leq m \ , \ j \to k \leq m \ \vdash \ i \leq m$ by right Ackermann rule,

$(k \to j) \vee (j \to k) \leq m \ \vdash \ i \leq m$,

$i \leq (k \to j) \vee (j \to k)$,

now introducing the semantic definition of nominal and co-nominal,

$\forall x \forall y \forall z[z{\uparrow} \subseteq ((x{\uparrow} \to y{\uparrow}) \cup (y{\uparrow} \to x{\uparrow}))]$,

iff $\forall x \forall y \forall z[z{\uparrow} \subseteq (x{\uparrow} \cap y{\uparrow}^c){\downarrow}^c \cup (y{\uparrow} \cap x{\uparrow}^c){\downarrow}^c]$.

There are 3 cases to consider that exhaust all possibilities: x and y are above one another, x and y are not above one another and there is no common successor between them, and the last case is if x and y are not above one another and there is a common successor between them.

For the first case:

> If $x \leq y$ or $y \leq x$ then the above becomes equivalent to $z{\uparrow} \subseteq W$.

For the second case:

> If $x{\uparrow} \cap y{\uparrow} = \emptyset$ then $(x{\uparrow} \cap y{\uparrow}^c){\downarrow}^c \cup (y{\uparrow} \cap x{\uparrow}^c){\downarrow}^c = x{\uparrow}{\downarrow}^c \cup y{\uparrow}{\downarrow}^c$ i.e. $z{\uparrow} \subseteq x{\uparrow}{\downarrow}^c \cup y{\uparrow}{\downarrow}^c$, this implies that z can be anywhere except $x{\uparrow}{\downarrow} \cap y{\uparrow}{\downarrow}$ i.e. z cannot be below both x and y.

For the third case:

> if $x{\uparrow} \cap y{\uparrow} \neq \emptyset$ then $(x{\uparrow} \cap y{\uparrow}^c){\downarrow}^c \cup (y{\uparrow} \cap x{\uparrow}^c){\downarrow}^c = (x{\uparrow} \setminus y{\uparrow}){\downarrow}^c \cup (y{\uparrow} \setminus x{\uparrow}){\downarrow}^c$ i.e. $z{\uparrow} \subseteq (x{\uparrow} \setminus y{\uparrow}){\downarrow}^c \cup (y{\uparrow} \setminus x{\uparrow}){\downarrow}^c$.
>
> Let $X = x{\uparrow} \setminus y{\uparrow}$ and $Y = y{\uparrow} \setminus x{\uparrow}$ then $z{\uparrow} \subseteq X{\downarrow}^c \cup Y{\downarrow}^c$ which implies $z \in X{\downarrow}^c \cup Y{\downarrow}^c$ which implies $z \notin X{\downarrow} \cap Y{\downarrow}$.
>
> This implies that z can be anywhere except z cannot be a point that has two successors, one in X and one in Y, i.e, there we cannot have a point with 2 successors.

Hence, we achieve our first-order correspondent, $\forall x, y, z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$.

Example 5.1.13. (BW_n): Running ALBA on $\bigvee_{i=0}^{n}(p_i \to \bigvee_{i \neq j} p_j)$ produces:

Begin with $\top \leq \bigvee_{i=0}^{n}(p_i \to \bigvee_{i \neq j} p_j)$,

$i \leq \top \ , \ (p_0 \to p_1 \vee p_2 \vee \cdots \vee p_n) \vee (p_1 \to p_0 \vee p_2 \vee \cdots \vee p_n) \vee \cdots \vee (p_n \to p_0 \vee p_1 \vee \cdots \vee p_{n-1}) \leq m \ \vdash \ i \leq m$ by first approximation,

$(p_0 \to p_1 \vee p_2 \vee \cdots \vee p_n) \leq m \ , \ (p_1 \to p_0 \vee p_2 \vee \cdots \vee p_n) \leq m \ , \ \ldots \ , \ (p_n \to p_0 \vee p_1 \vee \cdots \vee p_{n-1}) \leq m \ \vdash \ i \leq m$ by $\vee$ splitting and $i \leq \top$ is trivially true,

$i_0 \leq p_0$, $(i_0 \to p_1 \vee p_2 \vee \cdots \vee p_n) \leq m$, $i_1 \leq p_1$, $(i_1 \to p_0 \vee p_2 \vee \cdots \vee p_n) \leq m$, $\ldots$, $i_n \leq p_n$, $(i_n \to p_0 \vee p_1 \vee \cdots \vee p_{n-1}) \leq m \vdash i \leq m$ by approximation rule,

$(i_0 \to i_1 \vee i_2 \vee \cdots \vee i_n) \leq m$, $(i_1 \to i_0 \vee i_2 \vee \cdots \vee i_n) \leq m$, $\ldots$, $(i_n \to i_0 \vee i_1 \vee \cdots \vee i_{n-1}) \leq m \vdash i \leq m$ by Right-Ackermann rule,

$(i_0 \to i_1 \vee i_2 \vee \cdots \vee i_n) \vee (i_1 \to i_0 \vee i_2 \vee \cdots \vee i_n) \vee \cdots \vee (i_n \to i_0 \vee i_1 \vee \cdots \vee i_{n-1}) \leq m \vdash i \leq m$ by $\vee$ splitting,

$i \leq (i_0 \to i_1 \vee i_2 \vee \cdots \vee i_n) \vee (i_1 \to i_0 \vee i_2 \vee \cdots \vee i_n) \vee \cdots \vee (i_n \to i_0 \vee i_1 \vee \cdots \vee i_{n-1})$,

now introducing the semantic definition of nominal and co-nominal,

$\forall x \forall y_0, \ldots \forall y_n [x\uparrow \subseteq ((y_0\uparrow \to y_1\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow) \cup (y_1\uparrow \to y_0\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow) \cup \cdots \cup (y_n\uparrow \to y_0\uparrow \cup y_1\uparrow \cup \cdots \cup y_{n-1}\uparrow))]$,

iff $\forall x \forall y_0, \ldots \forall y_n [x\uparrow \subseteq (y_0\uparrow \to y_1\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow)$ or $x\uparrow \subseteq (y_1\uparrow \to y_0\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow)$ or $\ldots$ or $x\uparrow \subseteq (y_n\uparrow \to y_0\uparrow \cup y_1\uparrow \cup \cdots \cup y_{n-1}\uparrow)]$,

iff $\forall x \forall y_0, \ldots \forall y_n [x\uparrow \cap y_0\uparrow \subseteq (y_1\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow)$ or $x\uparrow \cap y_1\uparrow \subseteq (y_0\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow)$ or $\ldots$ or $x\uparrow \cap y_n\uparrow \subseteq (y_0\uparrow \cup y_1\uparrow \cup \cdots \cup y_{n-1}\uparrow)]$.

We can assume by Theorem 5.1.7, without loss of generality, that $W = x\uparrow$.

iff $\forall y_0, \ldots \forall y_n [\bigwedge_{i=0}^{n} x \leq y_i \to (y_0\uparrow \subseteq (y_1\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow)) \vee (y_1\uparrow \subseteq (y_0\uparrow \cup y_2\uparrow \cup \cdots \cup y_n\uparrow)) \vee \cdots \vee (y_n\uparrow \subseteq (y_0\uparrow \cup y_1\uparrow \cup \cdots \cup y_{n-1}\uparrow))]$,

iff $\forall y_0, \ldots \forall y_n [\bigwedge_{i=0}^{n} x \leq y_i \to \bigvee_{i \neq 0} y_i \leq y_0 \vee \bigvee_{i \neq 1} y_i \leq y_1 \vee \cdots \vee \bigvee_{i \neq n} y_i \leq y_n]$,

iff $\forall y_0, \ldots \forall y_n [\bigwedge_{i=0}^{n} x \leq y_i \to \bigvee_{0 \leq i \neq j \leq n} y_i \leq y_i]$.

Hence, we have arrived at our first-order correspondent.

The above are examples that show the effectiveness of ALBA in computing first-order correspondents. The semantic interpretation, as seen above, does require some prowess to achieve a more appealing form; an area of current interest in algorithmic first-order definability is to produce a more systematic way of simplifying these formulas. Let us now attempt some non-inductive formulas and see where they fail.

Example 5.1.14. (SL): Running ALBA on $(\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$ produces:

Begin with $\top \leq (\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$,

$(\neg\neg p \to p) \to p \vee \neg p \leq \neg p \vee \neg\neg p$,

$i \leq (\neg\neg p \to p) \to p \vee \neg p$, $\neg p \vee \neg\neg p \leq m \vdash i \leq m$ by first approximation,

$i \wedge (\neg\neg p \to p) \leq p \vee \neg p$, $\neg p \leq m$, $\neg\neg p \leq m \vdash i \leq m$ by $\vee$ splitting and Heyting rule,

$i \wedge (\neg\neg p \to p) \leq p \vee \neg p$, $i \leq p$, $\neg i \leq m$, $\neg\neg p \leq m \vdash i \leq m$ by $(\neg)$ approximation rule,

Looking at $i \wedge (\neg\neg p \to p) \leq p \vee \neg p$, no rule can be used to simplify this inequality further. Hence, we are at a point where we need to use the Ackermann rules.

Since, $i \leq p$ we use the right Ackermann rule but $\beta(p) = i \wedge (\neg\neg p \to p)$ is not positive in p. Hence, we cannot eliminate all occurrences of p and ALBA fails.

We do not get to a point where we can apply the Ackermann lemmas for non-inductive formulas, and this is expected, especially with SL, since it is not-first-order definable. What about deterministic formulas that are not inductive? They do have first-order correspondents, and a natural question would be if it is possible to somehow compute them with ALBA?

Example 5.1.15. (SmL): Running ALBA on $(\neg q \to p) \to (((p \to q) \to p) \to p)$ produces:

Begin with $\top \leq (\neg q \to p) \to (((p \to q) \to p) \to p)$,

$\neg q \to p \leq ((p \to q) \to p) \to p$,

$i \leq \neg q \to p$, $(((p \to q) \to p) \to p) \leq m \vdash i \leq m$ by first approximation,

$i \wedge \neg q \leq p$, $(((p \to q) \to p) \to p) \leq m \vdash i \leq m$ by Heyting rule,

$i \wedge \neg q \leq p$, $(((p \to q) \to p) \to n) \leq m$, $p \leq n \vdash i \leq m$ by Approximation rule,

$i \wedge \neg q \leq n$, $(((n \to q) \to n) \to n) \leq m \vdash i \leq m$ by left Ackermann rule,

$\neg q \leq i \to n$, $(((n \to q) \to n) \to n) \leq m \vdash i \leq m$.

We are now stuck as we cannot solve for q in $\neg q \leq i \to n$, the $(\neg)$ approximation rule only works when the right-hand side of the inequality is a co-nominal, but $i \to n$ has an interesting property.

Lemma 5.1.16. In a Kripke model $\mathfrak{M} = (W, \leq, V)$, for any nominal i and co-nominal m, $V(i \to m) = x{\uparrow} \to y{\downarrow^c}$ for some $x, y \in W$. The following cases are produced:

i. if $x \leq y$ then $x{\uparrow} \supset y{\downarrow^c} = y{\downarrow^c} = m$,

ii. if $x \not\leq y$ then $x{\uparrow} \supset y{\downarrow^c} = W$.

Proof. We have that $x{\uparrow} \supset y{\downarrow^c} = (x{\uparrow} \cap y{\downarrow}){\downarrow^c}$.

Let $x \leq y$, consider an element $u \in x{\uparrow} \cap y{\downarrow}$ then $u \in x{\uparrow}$ and $u \in y{\downarrow}$. Clearly $y \in x{\uparrow} \cap y{\downarrow}$. Furthermore, for all u, we have $u \leq y$. Hence, $(x{\uparrow} \cap y{\downarrow}){\downarrow^c} = y{\downarrow^c}$ since we have $u \in y{\downarrow}$.

If $x \leq y$ then for all $u \in x{\uparrow} \cap y{\downarrow}$ we have that $u \leq y$. Furthermore, all points that are included consist of y and maybe some points in $y{\downarrow}$ (depending on what points are between x and y), these other points are irrelevant since we then take the complement of their downset. Hence, we are left with $y{\downarrow}^c$.

If $x \not\leq y$ then $x{\uparrow} \cap y{\downarrow}$ is empty and $\emptyset{\downarrow}^c$ the whole set W.

Hence, proving the lemma for any nominal and co-nominal. □

The cases simplify the complex implication into either a co-nominal or the entire set. Both these cases allow for us to apply any of the approximation rules. So it seems we can introduce a new rule that will allow us to tackle the inequalities where the right side has the form $i \to m$.

5.2 Intuitionistic ALBA

Lemma 5.1.16 allows us to handle an inequality of the form $\varphi \leq i \to m$ by splitting it into two cases. If we consider both these cases separately, we can further split φ into smaller segments where, in the normal ALBA, we could not. In this section, we define new rules to add to the existing rules for ALBA in order to tackle these problem inequalities. We define *Int*-ALBA as follows: the symbol $\parr$ will be interpreted as a disjunction, and in addition to all the rules for ALBA, we add:

Approximation Rule

$(\to)$ splitting approximation rule

$$\frac{\varphi \leq i \to m}{\varphi \leq m \parr i \leq m}$$

Primality Rules

Alternative $\vee$-splitting

$$\frac{i \leq \varphi \vee \psi}{i \leq \varphi \parr i \leq \psi}$$

Alternative $\wedge$-splitting

$$\frac{\varphi \wedge \psi \leq m}{\varphi \leq m \parr \psi \leq m}$$

Quasi-inequality rule

$\parr$-splitting rule

$$\frac{\varphi \leq \psi \parr \varphi' \leq \psi' \ , \ \varphi_0 \leq \psi_0 \ , \ \dots \ , \ \varphi_n \leq \psi_n \ \vdash \ \chi \leq \kappa}{[\varphi \leq \psi \ , \ \varphi_0 \leq \psi_0 \ , \ \dots \ , \ \varphi_n \leq \psi_n \ \vdash \ \chi \leq \kappa] \ , \ [\varphi' \leq \psi' \ , \ \varphi_0 \leq \psi_0 \ , \ \dots \ , \ \varphi_n \leq \psi_n \ \vdash \ \chi \leq \kappa]}$$

Remark 5.2.1. The $(\to)$ splitting approximation rule is the only new rule introduced into ALBA; it is the rule derived from what we saw at the end of the last section. The other could also be rules of the standard ALBA, but are normally excluded, since they are redundant if the preprocessing stage is completed correctly. The proof of the soundness of the new rule is given below.

Soundness of the rules is shown below:

($\rightarrow$) splitting approximation rule:

If $\varphi \leq i \rightarrow m$ then $\varphi \leq m$ or $i \leq m$. Suppose $(W, \leq, V) \Vdash \varphi \leq i \rightarrow m$, that is $V(\varphi \leq i \rightarrow m) = W$. By definition of satisfaction of an inequality we have $V(\varphi) \subseteq V(i \rightarrow m)$ and $V(i) = x{\uparrow}$, $V(m) = y{\downarrow}^c$ for some $x, y \in W$. Now, by Lemma 5.1.16, two cases can happen: either $x \leq y$ or $x \not\leq y$.

If $x \leq y$ then $V(i \rightarrow m) = V(m)$ as seen in the Lemma i.e. $(W, \leq, V) \Vdash \varphi \leq m$.

If $x \not\leq y$ then $V(i \rightarrow m) = W$ as seen in the Lemma, this implies $V(\varphi) \subseteq W$ which is always true, but this is true only when $x \not\leq y$, i.e., $x{\uparrow} \subseteq y{\downarrow}^c = V(i) \subseteq V(m)$.

If $\varphi \leq m$ or $i \leq m$, then $\varphi \leq i \rightarrow m$.

Assume $(W, \leq, V) \Vdash \varphi \leq m$, we have trivially that $(W, \leq, V) \Vdash \varphi \leq \varphi$ which implies $(W, \leq, V) \Vdash \varphi \wedge i \leq \varphi$ which implies $(W, \leq, V) \Vdash \varphi \wedge i \leq m$ (by assumption) which implies $(W, \leq, V) \Vdash \varphi \leq i \rightarrow m$ (by Heyting rule).

Assume $(W, \leq, V) \Vdash i \leq m$, trivially $(W, \leq, V) \Vdash \top \wedge i \leq i$ which implies $(W, \leq, V) \Vdash \top \wedge i \leq m$ (by assumption) which implies $(W, \leq, V) \Vdash \top \leq i \rightarrow m$ (by Heyting rule) which implies $(W, \leq, V) \Vdash \varphi \leq i \rightarrow m$ (since $\varphi \leq \top$).

Alternative $\vee$-splitting

$i \leq \varphi \vee \psi$,

iff $i \leq \varphi$ or $i \leq \psi$ (due to i being interpreted as join-prime),

iff $i \leq \varphi \parr i \leq \psi$.

$\parr$-splitting rule

Given $\varphi \leq \psi \parr \varphi' \leq \psi'$, $\varphi_0 \leq \psi_0$, $\ldots$, $\varphi_n \leq \psi_n \vdash \chi \leq \kappa$ we can distribute the disjunctions (represented by $\parr$) over the conjunctions (represented by a comma ",") and will be left with a disjunction between two conjunctions of finite inequalities in the antecedent. Since this disjunction is in the antecedent and $\vdash$ has the semantics of classical implication, we can form a conjunction of two quasi-inequalities, $[\varphi \leq \psi$, $\varphi_0 \leq \psi_0$, $\ldots$, $\varphi_n \leq \psi_n \vdash \chi \leq \kappa]$, $[\varphi' \leq \psi'$, $\varphi_0 \leq \psi_0$, $\ldots$, $\varphi_n \leq \psi_n \vdash \chi \leq \kappa]$.

Remark 5.2.2. The ($\rightarrow$) splitting approximation rule can be generalized by a composition of the Heyting rule and the alternative $\wedge$-splitting rules, respectively. $\varphi \leq \psi \rightarrow m$ iff $\varphi \wedge \psi \leq m$ iff $\varphi \leq m \parr \psi \leq m$.

Let us see the new rule in action.

Example 5.2.3. Continuing SmL on Int-ALBA we have:

$(i \leq n \parr \neg q \leq n)\ ,\ (((n \to q) \to n) \to n) \leq m\ \vdash\ i \leq m,$

$[i \leq n\ ,\ (((n \to q) \to n) \to n) \leq m\ \vdash\ i \leq m]\ ,\ [\neg q \leq n\ ,\ (((n \to q) \to n) \to n) \leq m\ \vdash\ i \leq m]$, by $\parr$-splitting,

$[i \leq n\ ,\ \bot \leq q\ ,\ (((n \to q) \to n) \to n) \leq m\ \vdash\ i \leq m]\ ,\ [k \leq q\ ,\ \neg k \leq n\ ,\ (((n \to q) \to n) \to n) \leq m\ \vdash\ i \leq m]$, by $(\neg)$ approximation rule and since $\bot \leq q$ is trivially true,

$[i \leq n\ ,\ (((n \to \bot) \to n) \to n) \leq m\ \vdash\ i \leq m]\ ,\ [\neg k \leq n\ ,\ (((n \to k) \to n) \to n) \leq m\ \vdash\ i \leq m]$, by right Ackermann rule,

now introducing the semantic definition of nominal and co-nominal,

$\forall w \forall x \forall y \forall z([([x\uparrow \subseteq y\downarrow^c] \wedge [(((y\downarrow^c \to \emptyset) \to y\downarrow^c) \to y\downarrow^c) \subseteq z\downarrow^c]) \to (x\uparrow \subseteq z\downarrow^c)] \wedge [([w\uparrow\downarrow^c \subseteq y\downarrow^c] \wedge [(((y\downarrow^c \to w\uparrow) \to y\downarrow^c) \to y\downarrow^c) \subseteq z\downarrow^c]) \to (x\uparrow \subseteq z\downarrow^c)]).$

5.2.1 Extending the (Ω, ε)-inductive formulas

As seen above, Int-ALBA can compute the SmL axiom, which is not (Ω, ε)-inductive, and there must be more intuitionistic formulas that will arrive at a point $\varphi \leq i \to m$ which will cause problems in the ALBA computation. These extra formulas that work on Int-ALBA can be syntactically classified; this class, as shown below, is a strict extension of the (Ω, ε)-inductive formulas.

To start, we show that Int-ALBA allows one to skip the preprocessing stage and proceed directly to the first approximation, as considered in the following lemma.

Lemma 5.2.4. Let φ be an (Ω, ε)-inductive intuitionistic formula. We, can skip the preprocessing stage in the computation of ALBA(φ) if we add both the alternative $\vee$-splitting and alternative $\wedge$-splitting rules.

Proof. Consider any intuitionistic formula φ and the computation ALBA(φ). The $\vee$-splitting and $\wedge$-splitting are the only rules that are necessary to apply in the preprocessing stage, if they can be done. For each of them, we tackle them as follows in Int-ALBA:

Case I: Assume we have $\varphi \leq \psi \wedge \delta$. Then in preprocessing, we apply the $\wedge$-splitting and arrive at $\varphi \leq \psi, \varphi \leq \delta$. Applying first approximation on that gets us to

$$[i_0 \leq \varphi, \psi \leq m_0\ \vdash\ i_0 \leq m_0], [i_0 \leq \varphi, \delta \leq m_0\ \vdash\ i_0 \leq m_0].$$

Similarly, if we skip the preprocessing stage, after first approximation, we will arrive at,

$i_0 \leq \varphi, \psi \wedge \delta \leq m_0 \ \vdash \ i_0 \leq m_0$,
$i_0 \leq \varphi, \psi \leq m_0 \parr \delta \leq m_0 \ \vdash \ i_0 \leq m_0$ by alternative $\wedge$-splitting,
$[i_0 \leq \varphi, \psi \leq m_0 \ \vdash \ i_0 \leq m_0], [i_0 \leq \varphi, \delta \leq m_0 \ \vdash \ i_0 \leq m_0]$ by $\parr$-splitting.

Since they both arrive at the same quasi-inequalities, the proceeding computation would be the same for both. Hence, if two rules are added, we can skip the preprocessing stage and arrive at the same first-order correspondent.

Case II: Assume we have $\varphi \vee \psi \leq \delta$. The proof that both computations converge is similar to Case I. Only here we use the $\vee$-splitting and alternative $\vee$-splitting rules.

For any combination of $\vee$ and/or $\wedge$, the argument can be easily seen to be similar. For any $\bigvee_{1 \leq i \leq n}$ on the right side of the inequality, it will split into n inequalities. Similarly, for $\bigwedge_{1 \leq i \leq m}$ on the left side of the inequality. So we would have $n+m$ different quasi-inequalities after first approximation. If we are skipping preprocessing, we will apply first approximation and then apply the alternative $\vee$-splitting n times and the alternative $\wedge$-splitting m times and end up with $n+m$ quasi-inequalities. $\square$

Comment 5.2.5. The preprocessing stage essentially provides a method of handling any top-level $\vee$ that appears on the left side of the inequality. If we jumped to first approximation without applying $\vee$-splitting, we would not be able to further split out inequalities in the antecedent of the quasi-inequality since the formulas would then be on the right side of the inequality, and we would not be able to further proceed. Similarly, for the $\wedge$ on the right side of the inequality. The new rules, alternative $\vee$-splitting and alternative $\wedge$-splitting, allow us to tackle these situations after the first approximation stage. Note that the alternative $\vee$-splitting and alternative $\wedge$-splitting rules require a nominal and a co-nominal on the left and right side of the inequality, respectively, whereas the normal splitting rules have no such restriction. Yet, this condition will always be met as far as occurrences of $\vee$ and $\wedge$ amenable to pre-processing are concerned, as we saw in the proof above.

Definition 5.2.6 (Super excellent variable)**.** A variable p in an intuitionistic formula φ is called a *super excellent variable in* φ if, in $sg^-(\varphi)$, for all occurrences of $-p$ the following are true:

i. all nodes on the branch from the root to $-p$ have polarity $-$,

ii. the branch from the root to $-p$ does not have an occurrence of $\wedge$.

The following proposition follows immediately by noting that all possible nodes in the branch from $sg^-(\varphi)$ to $-p$, if p is super excellent, are skeleton nodes.

Proposition 5.2.7. If p is a super excellent variable in an intuitionistic formula φ, then the branch extending from the root of $sg^-(\varphi)$ to $-p$ is excellent.

When extending our class of formulas we will see that we require this particular variable entirely due to the fact that we will be able to solve for $-p$ in ALBA and then all occurrences of $+p$ will eliminate to $\bigwedge_{1 \leq i \leq n} m_i$ where m_i is a co-nominal and $n \in \mathbb{N}$. The following lemma proves this claim.

Lemma 5.2.8. If p is a super excellent variable, in an intuitionistic formula φ, then in the ALBA computation on φ (and by extension in the Int-ALBA computation) we can eliminate all occurrences of $+p$ by substituting with $\bigwedge_{1\leq i\leq n} m_i$ where m_i is a co-nominal and $n \in \mathbb{N}$.

Proof. Let p be super excellent in φ. Now when computing ALBA(φ) we begin with $\top \leq \varphi$. Assume that we have already completed the preprocessing stage, we now have a collection of inequalities, each of which will be computed on their own. Consider any one of the inequalities. If it is of the form $\top \leq \psi' \to \psi$, we can apply the Heyting rule and get $\psi' \leq \psi$. Applying the first approximation, we get an instance of the following,

$$i \leq \psi', \psi \leq m \ \vdash \ i \leq m.$$

If the Heyting rule was not used in the preprocessing stage, we have $\psi' = \top$. This will not affect the proof.

All occurrences of p we are going to solve for are in ψ as the root of $sg(\psi') \preceq sg^-(\psi' \to \psi)$ is signed $+$ and so there can be no occurrences of $-p$ in $sg(\psi') \preceq sg^-(\psi' \to \psi)$.

Given the definition of a super excellent variable, we have that on each branch extending from the root to $-p$, only $-\vee$ and $- \to$ will occur. We know $-\neg$ cannot occur in the branch as its child will then have polarity $+$.

Using induction, we have the following:

> Base case: Assume ψ in $\psi \leq m$ is atomic i.e. $p \leq m$ this agrees with our claim.
>
> Inductive hypothesis: Assume $\psi \leq m$, where ψ has r connectives, and we are able to solve for all occurrences of $-p$ in $sg^-(\psi)$ and combine them into a single inequality $p \leq \bigwedge_{1\leq i\leq n} m_i$, for all $1 \leq r \leq t$ with $t \in \mathbb{N}$.
>
> Inductive step: Let us assume ψ and χ have u and v connectives respectively such that $u + v = t$. We will add an extra connective such that the total connectives will be $t + 1$.
>
> Assume the connective added is $-\vee$ then $\psi \vee \chi \leq m$ expands to $\psi \leq m, \chi \leq m$ by $\vee$-splitting, and by inductive hypothesis we can solve for $-p$ in both of them and after combining them we get $p \leq \bigwedge_{1\leq i\leq a} m_i$ in ψ and $p \leq \bigwedge_{a\leq j\leq b} m_j$ in χ. We now apply the reverse $\wedge$-splitting to both and get $p \leq \bigwedge_{1\leq k\leq b} m_k$. Hence $\psi \vee \chi \leq m$ satisfies the claim.
>
> Assume the connective added is $- \to$ then $\psi \to \chi \leq m$ expands to $\psi \to m_0 \leq m, \chi \leq m_0$ by $(\to)$ consequent approximation rule, and by the definition, $-p$ can only occur in χ since the main connective of ψ is signed $+$ in $sg^-(\psi \to \chi)$. Furthermore, by the inductive hypothesis, we can solve for all occurrences of $-p$ in χ and combine them into $p \leq \bigwedge_{1\leq i\leq v} m_i$. Hence $\psi \to \chi \leq m$ satisfies the claim. □

Definition 5.2.9 (Adequate path)**.** A path from one node to another, in the signed generation tree of a formula φ, is called an *adequate path* if the subtree rooted at the right child of every $+ \rightarrow$ is of the form $sg^{+}(\bigwedge_{1\leq i\leq n} p_i)$, where p_i is super excellent in φ for each $1 \leq i \leq n$.

Since $\bigwedge_{1\leq i\leq n} p_i$ occurs in the right child of $+ \rightarrow$, the polarity will be preserved throughout the right subtree of the $+ \rightarrow$ and all occurrences of p_i, in that subtree, will have polarity $+$.

Remark 5.2.10. since $\neg\varphi$ is intuitionistically defined as $\varphi \rightarrow \bot$ no occurrence of $+\neg$ can occur on an adequate path.

Definition 5.2.11 (Adequately (Ω, ε)-inductive subtree)**.** A subtree, $sg(\gamma) \preceq sg^{-}(\varphi)$, where φ is an intuitionistic formula, is called an *adequately* (Ω, ε)*-inductive subtree* if $sg(\gamma)$ is the negatively signed generation tree of an (Ω, ε)-inductive formula and furthermore the path (possibly empty) from the root of $sg(\gamma)$ to the root of $sg^{-}(\varphi)$ is an adequate path.

Remark 5.2.12. We will, for the rest of the thesis, assume, without loss of generality, that the last node of an adequate path is the last $+ \rightarrow$ with a conjunction of super excellent variable as its right child. We can take any node between the last $+ \rightarrow$ in the adequate path and the first $+ \rightarrow$ in the (Ω, ε)-inductive subtree to be the end of the adequate path and the beginning of the (Ω, ε)-inductive subtree, as it can easily be seen that any node taken in between will still satisfy the conditions for both an adequate path and an (Ω, ε)-inductive subtree.

Assume we have a formula φ that is not (Ω, ε)-inductive, this means that there is a variable, say q, that cannot be eliminated using the Ackermann rules when doing the ALBA computations of φ. The reason why we have this definition of an adequately (Ω, ε)-inductive subtree is because, as we will see in the next lemma, we will be able to eliminate q using the Ackermann rules if all occurrences of q are in adequate (Ω, ε)-inductive subtrees.

Lemma 5.2.13. Let φ be an intuitionistic formula. If there exists $* \in \{+, -\}$ and (Ω, ε) such that $\varepsilon(q) = *$ and all occurrences of $*q$ in $sg^{-}(\varphi)$ are in adequately (Ω, ε)-inductive subtrees, then q can be eliminated in Int-ALBA.

Proof. Let $sg^{-}(\varphi)$ be the negatively signed generation tree of φ. Further, let $* \in \{+, -\}$ and assume we want to eliminate q. By definition of an adequately (Ω, ε)-inductive subtree we know that $*q$ must occur in subtrees that are (Ω, ε)-inductive with $\varepsilon(q) = *$, call these subtrees γ_i. We take an arbitrary adequately (Ω, ε)-inductive subtree, call it γ.

If the adequate branch extending to γ is empty, then we can solve for $*q$, since then $*q$ will satisfy all the conditions for a (Ω, ε)-inductive formulas.

Now let us assume that the adequate branch from the root of $sg^{-}(\varphi)$ to the root of γ is non-empty, in fact let us further assume without loss of generality that the addition of this adequate branch explicitly places $*q$ in the scope of a bad branch.

Clearly φ is not (Ω, ε)-inductive, and we should not be able to solve for $*q$ in ALBA, but this is not the case given the new $(\rightarrow)$ splitting approximation rule.

To briefly sketch the proof: we will show that given the conditions that we have, at the end of the adequate path we will, in the Int-ALBA computation, find ourselves at a point that resembles the same form as when we apply first approximation (we will have extra inequalities in the entire equation but they will only consist of nominals and co-nominals, therefore will not cause any difficulty while applying the Ackermann rules). Now, since the ends of the adequate paths must be followed by (Ω, ε)-inductive subtrees, we know that we should succeed in eliminating all variables. Hence, the entire formula φ should succeed on Int-ALBA. The formal details are given below.

Using Lemma 5.2.4 we skip the preprocessing stage and now arrive at,

$$i \leq \varphi_0, \varphi_1 \leq m \ \vdash \ i \leq m.$$

We get to $i \leq \varphi_0, \varphi_1 \leq m \ \vdash \ i \leq m$ only if our inequality at the beginning was $\top \leq \varphi_0 \rightarrow \varphi_1$, then we can only apply the Heyting rule. This shows us that $i \leq \varphi_0$ will happen only when $sg(\varphi_0) \preceq sg^-(\varphi)$ has the polarity $+$ at its root. Similarly $sg(\varphi_1) \preceq sg^-(\varphi)$ has the polarity $-$ at its root. Let us first look at $\varphi_1 \leq m$, which we will call $\dagger$.

If needed, we can either use $\vee$-splitting or alternative $\wedge$-splitting to break up $\varphi_1 \leq m$ into various smaller segments, but due the the nature of the rules, all the smaller segments will be of the form $\chi \leq m$. Only when we get an implication $\varphi_1^1 \rightarrow \varphi_1^2 \leq m$ do we use both the $(\rightarrow)$ consequent and $(\rightarrow)$ antecedent approximation rules and get $i_0 \leq \varphi_1^1, \varphi_1^2 \leq m_0, i_0 \rightarrow m_0 \leq m$.

The inequality $i_0 \rightarrow m_0 \leq m$ is independent of any propositional variable, so it can remain as is until the success, failure, and output stage.

The inequality $\varphi_1^2 \leq m_0$ is back to the form of $\dagger$, and we can again repeat the same steps as above.

The inequality $i_0 \leq \varphi_1^1$ is of the same form as $i \leq \varphi_0$ from just after the preprocessing stage. For these types of inequalities, we have the following.

Claim 5.2.14. There is no occurrence of $-p$, where p is a super excellent variable in φ, in $sg(\varphi_0) \preceq sg^-(\varphi)$.

Proof. Since p is super excellent and φ_0 has polarity $+$ at its root, $-p$ cannot occur in $sg(\varphi)$. □

For these inequalities of the form $i \leq \varphi_0$ is where our new rule comes into play. We know $i \leq \varphi_0$ happens only when $sg(\varphi_0) \preceq sg^-(\varphi)$ has the polarity $+$. Now we can further split this using either $\wedge$-splitting or alternative $\vee$-splitting to get various smaller segments, but again, these smaller segments will be of the form $i \leq \chi$. We then progress down the tree until we arrive at the first implication $i \leq \varphi_0^0 \rightarrow \varphi_0^1$, this implication has to be of polarity $+$ (since only

$\vee$ and $\wedge$ could have been between the root and this implication) and since the adequate path was non-empty φ_0^1, the right child of the implication, will be the conjunction of super excellent variables, i.e. $i \leq \varphi_0^0 \to \bigwedge_{1\leq a\leq n} p_a$. We can solve for all the negative super excellent variables $-p$ and Claim 5.2.14 and Lemma 5.2.8, each p_a can be substituted with $\bigwedge_{1\leq b\leq r} m_b$ where m_b are co-nominals, for all $1 \leq b \leq r$. Hence, we have $i \leq \varphi_0^0 \to \bigwedge_{1\leq c\leq s} m_c$ applying the swapping rule gets us to $\varphi_0^0 \leq i \to \bigwedge_{1\leq c\leq s} m_c$ which we can further split into $\varphi_0^0 \leq i \to m_1, \varphi_0^0 \leq i \to m_2, \ldots, \varphi_0^0 \leq i \to m_s$. For each of these inequalities, we can apply the new rule, $(\to)$ splitting approximation rule, and we are left with $\varphi_0^0 \leq m_1 \parr i \leq m_1, \varphi_0^0 \leq m_2 \parr i \leq m_2, \ldots, \varphi_0^0 \leq m_s \parr i \leq m_s$. Consider an arbitrary $\varphi_0^0 \leq m_i \parr i \leq m_i$ in the full equation it will be:

$$\varphi_0^0 \leq m_i \ \parr \ i \leq m_i, \ldots \ \vdash \ i \leq m,$$

by $\parr$-splitting, we are left with

$$[\varphi_0^0 \leq m_i, \ldots \ \vdash \ i \leq m] \ , \ [i \leq m_i, \ldots \ \vdash \ i \leq m].$$

So from $i \leq \varphi_0$ we achieved the above that again has changed its form to that of † plus some inequality with only nominals and co-nominals.

The above process goes on for the entirety of the adequate path until we reach the last node of the path. By Remark 5.2.12, we let the last node be a $+ \to$, i.e., if $i \leq \psi_0 \to \psi_1$ then ψ_1 will eliminate into some conjunction of co-nominals, and as seen above, after applying the $(\to)$ splitting approximation rule we will end up with $\psi_0 \leq m$ plus some other inequalities built from nominals or co-nominals. Since this is the last node in the adequate path, the signed subtree reached now needs to be of an adequately (Ω, ε)-inductive formula, so ψ_0 will be a (Ω, ε)-inductive formula.

Consider any arbitrary (Ω, ε)-inductive formula ψ. If we skip the preprocessing stage in ALBA (Lemma 5.2.4 allows us to do so), we get the following:

$$i \leq \top, \psi \leq m \ \vdash \ i \leq m.$$

$i \leq \top$ is trivially true and we are left with $\psi \leq m \ \vdash \ i \leq m$. Consider $\psi \leq m$, we can solve for all (Ω, ε)-critical variables in ψ since it is (Ω, ε)-inductive. Similarly, we can apply the same argument to $\psi_0 \leq m$ and we should be able to solve for all (Ω, ε)-critical variables in ψ_0.

The above process applies to all (Ω, ε)-inductive subtrees, and we solve for all occurrences of $*q$ in all γ_i. Applying either the reverse of the $\vee$-splitting or $\wedge$-splitting, we can conjoin all inequalities to one, either $q \leq \lambda$ or $\lambda \leq q$. Ackermann will then replace all the occurrences of $*^\delta q$, where $*^\delta$ represents the polarity opposite to $*$, with λ. □

Similarly to ALBA, for any intuitionistic inequality $\varphi \leq \psi$ we define Int-ALBA$(\varphi \leq \psi)$ to denote the output that Int-ALBA produces after eliminating all propositional variables in $\varphi \leq \psi$, and Int-ALBA$^{FO}(\varphi \leq \psi)$ to denote the first-order equivalent after introducing the semantic interpretation of the nominals and co-nominals in ALBA$(\varphi \leq \psi)$. Finally, for correctness of Int-ALBA, we have the following.

Theorem 5.2.15 (Correctness and correspondence)**.** If Int-ALBA succeeds in reducing a $\mathcal{L}_{IP}$-inequality $\varphi \leq \psi$ and yields Int-ALBA$(\varphi \leq \psi)$, then $(\mathfrak{F}, w) \Vdash \varphi \leq \psi$ iff $\mathfrak{F}(i := w\uparrow) \Vdash Int$-ALBA$(\varphi \leq \psi)$ iff $\mathfrak{F} \models Int$-ALBA$^{FO}(\varphi \leq \psi)[i := w]$.

Proof. Beginning with $\varphi \leq \psi$ on Int-ALBA by Lemma 5.2.4, we can skip the preprocessing stage, apply the first approximation, and arrive at $i \leq \varphi, \psi \leq m \vdash i \leq m$. Due to soundness of all the rules given, we will be able to reduce the inequality and arrive at Int-ALBA$(\varphi \leq \psi)$. The rest of the proof follows the same as in standard ALBA. □

We are now ready to complete our definition of the extended (Ω, ε)-inductive formulas for the intuitionistic setting. What we have proved so far is that we can eliminate specific propositional variables that occur in certain bad branches that would originally fail to eliminate if we did not have our new $(\rightarrow)$ splitting approximation rule.

Definition 5.2.16. (Adequately (Ω, ε)-inductive formula) We call an intuitionistic formula φ an *adequately* (Ω, ε)*-inductive formula* if, all ε-critical occurrences of p_i are in adequately (Ω, ε)-inductive subtrees of $sg^{-}(\varphi)$.

Theorem 5.2.17. If φ is an adequately (Ω, ε)-inductive formula, then it is first-order definable.

Proof. This proof makes use of Theorem 5.2.15 that states that a formula that succeeds on Int-ALBA is first-order definable. By the definition of an adequately (Ω, ε)-inductive formula, all propositional variables are in adequately (Ω, ε)-inductive subtrees, and by Lemma 5.2.13 we can eliminate all these propositional variables. Hence, Int-ALBA succeeds on adequately (Ω, ε)-inductive formula proving that they are first-order definable. □

We conclude with four examples, each of the four are not (Ω, ε)-inductive (this can be independently checked). We will check whether they satisfy the conditions for being adequately (Ω, ε)-inductive or not.

Example 5.2.18. SmL: $(\neg q \rightarrow p) \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p)$, its negatively signed generation tree is given in Figure 5.1. To show that it is adequately (Ω, ε)-inductive we proceed as follows:

Step 1 Find all propositional variables that are super excellent (branches extending from the root to super excellent variables are colored orange). For SmL, p is super excellent, q is not, since there is a $+ \rightarrow$ in the branch from the root to $-q$. Furthermore, since p is super excellent, we know that $\varepsilon(p) = -$, so we do not need to check for any adequate subtrees with respect to p.

Step 2 List all adequate paths. For SmL, there are two; they are colored light blue in Figure 5.1.

Step 3 Check if all the subtrees at the end of the adequate paths are (Ω, ε)-inductive. For SmL, since there are two adequate paths, there are two subtrees to consider, namely, the negatively signed generation trees of $\neg q$ and $p \to q$. These are both (Ω, ε)-inductive with $\varepsilon(q) = \pm$. If we choose $\varepsilon(q) = +$, the ordering will be empty. If $\varepsilon(q) = -$, the ordering will be empty. Hence, all subtrees of adequate branches are (Ω, ε)-inductive, for the same Ω and ε, implying that SmL is adequately (Ω, ε)-inductive.

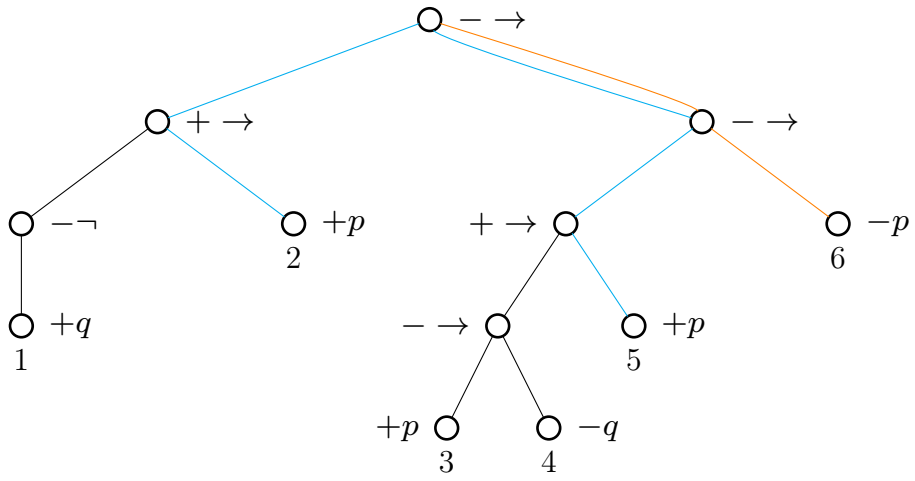


Figure 5.1: Signed generation tree for SmL

Example 5.2.19. nf_6: $[(p \vee \neg p) \vee ((p \vee \neg p) \to p)] \to [p \vee \neg p]$, its negatively signed generation tree is given in Figure 5.2. We have the following:

Step 1 For nf_6, there are no super excellent variables.

Step 2 Since there are no (non-empty) super excellent variables, there are no adequate paths.

Step 3 Since there are no adequate paths, we would need to show that the entire formula is (Ω, ε)-inductive, which we know is not the case from Example 3.2.15. Hence, nf_6 is not adequately (Ω, ε)-inductive.

Interestingly enough, nf_6 is deterministic, showing that the adequately (Ω, ε)-inductive formulas are not a subset of the deterministic formulas. Finally, we show two more examples of engineered non-adequately (Ω, ε)-inductive formulas that are a bit more complex.

Example 5.2.20. Consider this constructed formula $\varphi_1 = [\neg\neg(q \vee \neg q) \to p] \to [(\neg(\neg q \vee \neg\neg q) \to p) \to p]$. Its negatively signed generation tree is given in Figure 5.3. We have the following:

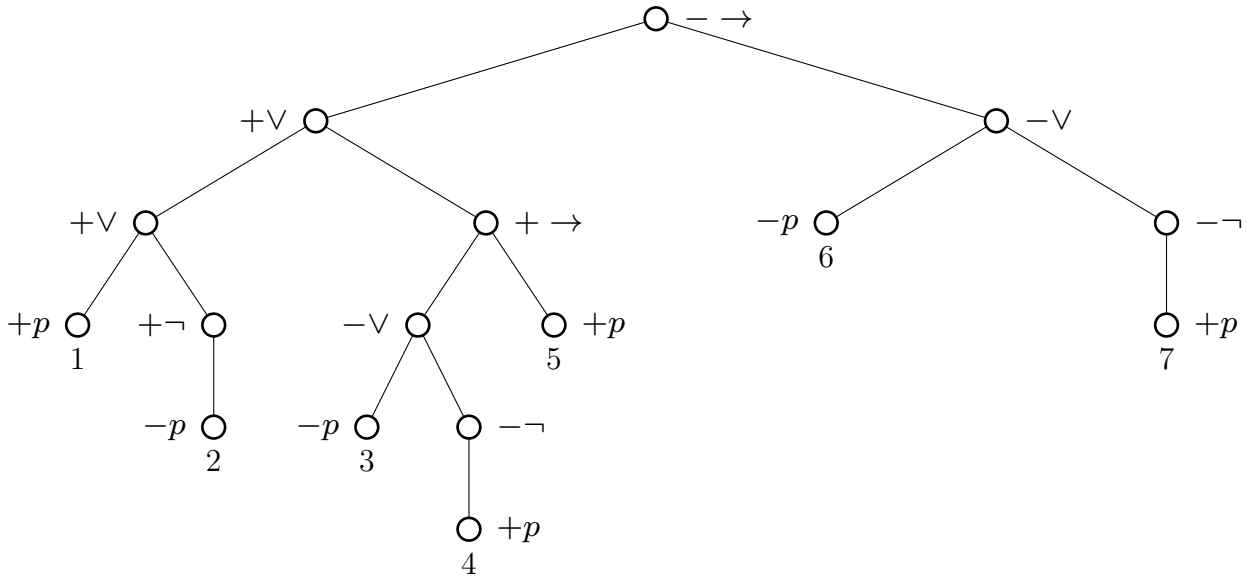


Figure 5.2: Signed generation tree for nf_6

Step 1 Find all propositional variables that are super excellent (branches extending from the root to super excellent variables are colored orange). p is super excellent, q is not since there is a $+\neg$ in the branch from the root to $-q$. Furthermore, we fix $\varepsilon(p) = -$.

Step 2 List all adequate paths. There are two; they are colored light blue in Figure 5.3. By Remark 5.2.12 we can, without loss of generality, let the last node of the adequate path be the $+ \rightarrow$ with $+p$ as its right child.

Step 3 Check if all the subtrees at the end of the adequate paths are (Ω, ε)-inductive. Since there are two adequate paths, there are two subtrees to consider, namely, the negatively signed generation trees of $\neg\neg(q \vee \neg q)$ and $\neg(\neg q \vee \neg\neg q)$. These are both (Ω, ε)-inductive with $\varepsilon(q) = -$ and empty ordering for $\neg\neg(q \vee \neg q)$ and $\varepsilon(q) = -$ and empty ordering for $\neg(\neg q \vee \neg\neg q)$. Hence, since they both satisfy the same Ω and ε, the formula is adequately (Ω, ε)-inductive.

Example 5.2.21. Consider this constructed formula $\varphi_2 = [\neg\neg((p \rightarrow q) \vee q) \rightarrow p] \rightarrow [(\neg(\neg q \vee \neg\neg q) \rightarrow p) \rightarrow p]$. Its negatively signed generation tree is given in Figure 5.4. We have the following:

Step 1 Find all propositional variables that are super excellent (branches extending from the root to super excellent variables are colored orange). p is super excellent, q is not since there is a $+\neg$ in the branch from the root to $-q$. Furthermore, we fix $\varepsilon(p) = -$.

Step 2 List all adequate paths. There are two; they are colored light blue in Figure 5.4. By Remark 5.2.12 we can, without loss of generality, let the last node of the adequate path be the $+ \rightarrow$ with $+p$ as its right child.

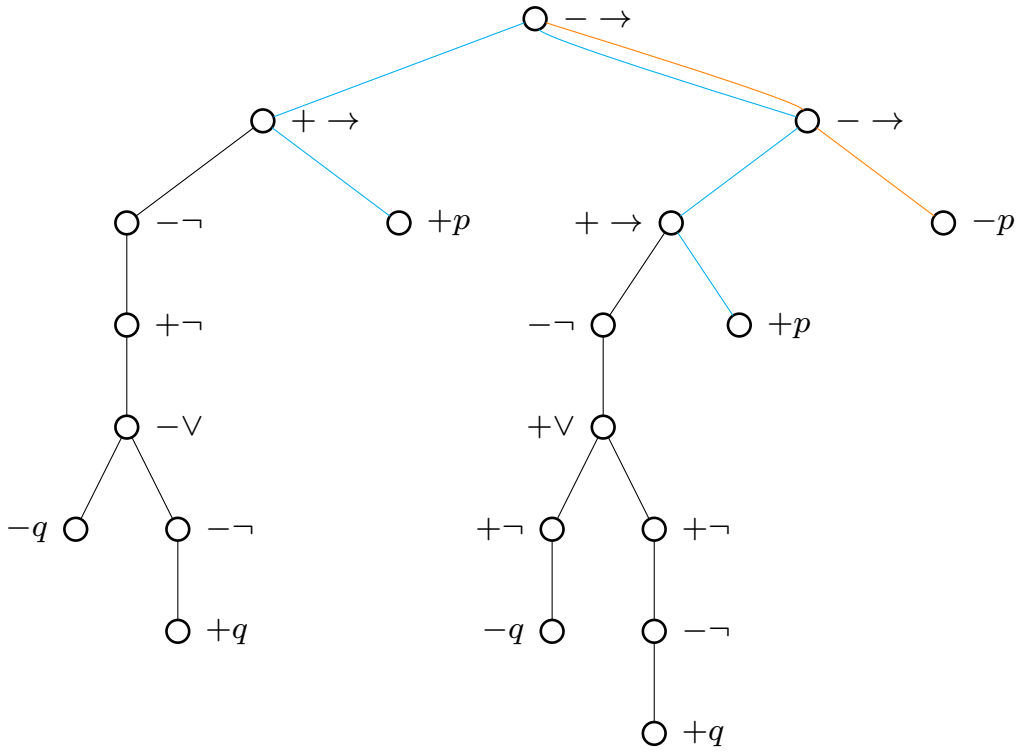


Figure 5.3: Signed generation tree for φ_1

Step 3 Check if all the subtrees at the end of the adequate paths are (Ω, ε)-inductive. Since there are two adequate paths, there are two subtrees to consider, namely, the negatively signed generation trees of $\neg\neg((p \rightarrow q) \vee q)$ and $\neg(\neg q \vee \neg\neg q)$. The former is (Ω, ε)-inductive with $\varepsilon(q) = +$ ($-q$ occurs in a bad branch so we cannot have $\varepsilon(q) = -$) and empty ordering for the latter we have $\varepsilon(q) = -$ (again $+q$ occurs in a bad branch so we cannot have $\varepsilon(q) = +$) and empty ordering. Hence, since both adequately (Ω, ε)-inductive subtrees are (Ω, ε)-inductive but for different polarities of q, the formula is not adequately (Ω, ε)-inductive.

Remark 5.2.22. In a standard (Ω, ε)-inductive formula, if there was a branch that had a $+ \rightarrow$ followed by $- \rightarrow$ (there could be other connectives between them), then there could not be an ε-critical variable after the $- \rightarrow$. This could be understood as the fact that an ε-critical variable occurrence can only happen in a branch with at most two alternating (in polarity) implications. For the adequately (Ω, ε)-inductive formulas, we allow for any finite number of alternating implications as long as all right children of every $+ \rightarrow$ satisfy the particular property of them being a conjunction of super excellent variables.

5.2.2 Possible further extensions?

SmL is an example of a non-(Ω, ε)-inductive formula that surprisingly Int-ALBA succeeded on. This worked due to the fact that we simplified the “problem" variable,

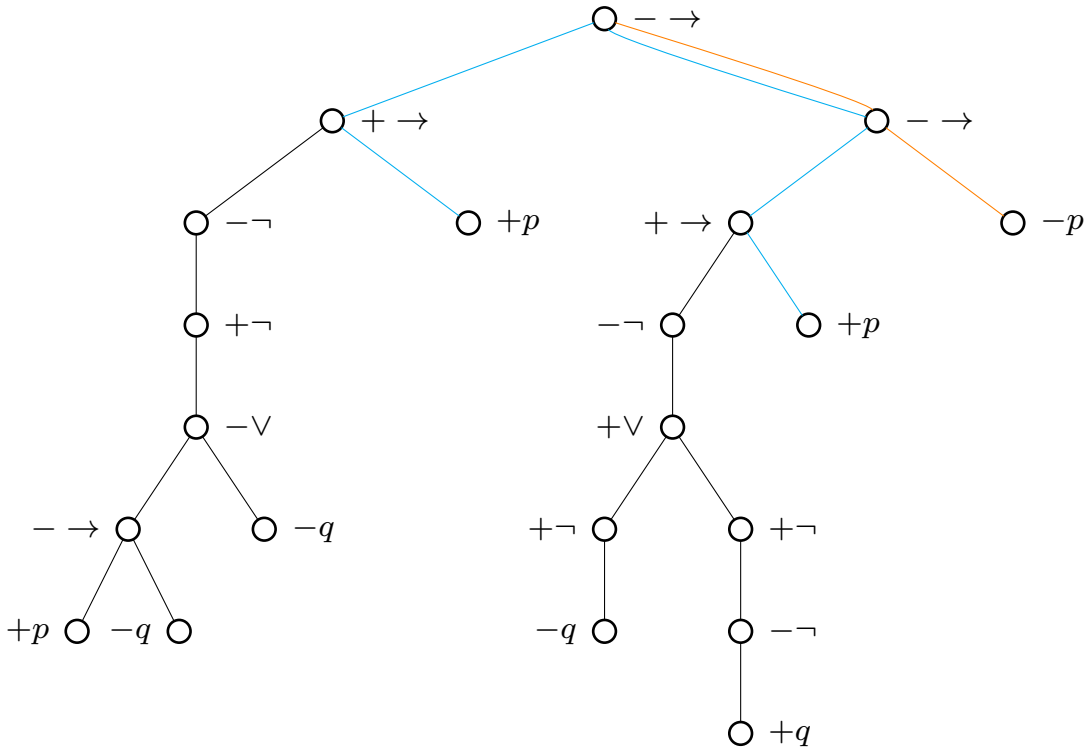


Figure 5.4: Signed generation tree for φ_2

q, to a point where we had $\neg q \leq i \to n$ normally we would not be able to apply $(\neg)$ approximation rule since $\neg q$ is not less than a co-nominal but due to the nature of our nominals and co-nominals being principal upsets and complements of principal down-sets $i \to n$ simplified nicely. Working with other combinations of implications, we get the following.

Remark 5.2.2 showed us a generalized version of the $(\to)$ splitting approximation rule where, in the inequality $\varphi \leq i \to m$, we can replace the i with any formula ψ. So we end up with the rule.

$$\frac{\varphi \leq \psi \to m}{\varphi \leq m \between \psi \leq m}$$

Letting $\psi = i$, where i is a nominal, allowed for our extension of the class of formulas that succeed on Int-ALBA. A natural question would be what if we fixed $\psi = n$, where n is a co-nominal.

Lemma 5.2.23. An inequality of the form $\varphi \leq n \to m$ will never occur in an ALBA computation.

Proof. For the sake of contradiction, assume that we have $\varphi \leq n \to m$,

$\varphi \leq n \to m$,

iff $\varphi \wedge n \leq m$ by Heyting rule,

iff $\varphi \wedge p \leq m$ where p is a variable that gets eliminated with n by an Ackermann rule.

According to our defined rules, these are the only steps that can be taken to get to $\varphi \leq n \to m$. For $\varphi \wedge p \leq m$, let us consider $n \leq p$, it is important to take notice that with any of the given rules, we will never be able to get the form where a co-nominal is less than a variable. Hence, we are only left with $p \leq n$ now, according to the left Ackermann rule, $\varphi \wedge p$ has to have p negative in the positive signed generation tree, which is a contradiction. □

5.2.3 The scope of adequately (Ω, ε)-inductive formulas

Interestingly enough, SmL is deterministic, so it was initially known that it does, in fact, have a first-order correspondence. The natural question that then arises is, do all deterministic formulas succeed on Int-ALBA, and is it possible to compute their first-order correspondents in this manner? The answer to that is a disappointing no (at least in terms of the extension discovered in this thesis), the nf_6 axiom is deterministic but when performing Int-ALBA computations we do not arrive at a point where we can perform the $(\to)$ splitting approximation rule as done for SmL, there maybe a different rule that is yet to be discovered that, when added to Int-ALBA, will make it succeed on a bigger set or the entire class of deterministic formulas. All is not lost, though, since we have arrived at a more interesting result. The computation was not just a fluke that only worked on this particular deterministic formula or a subset of deterministic formulas; we have stumbled upon a different class of formulas that can be computed on ALBA. Take the following formula:

$$(\neg q \to p) \to (((p \to q) \to (p \vee \neg q)) \to p).$$

This formula is an adaptation of SmL that we engineered such that it is neither (Ω, ε)-inductive nor deterministic, but it is an adequately (Ω, ε)-inductive formula:

Begin with $\top \leq (\neg q \to p) \to (((p \to q) \to (p \vee \neg q)) \to p)$,

$\neg q \to p \leq ((p \to q) \to (p \vee \neg q)) \to p$,

$i \leq \neg q \to p, (((p \to q) \to (p \vee \neg q)) \to p) \leq m \ \vdash \ i \leq m$ by first approximation,

$i \wedge \neg q \leq p, (((p \to q) \to (p \vee \neg q)) \to p) \leq m \ \vdash \ i \leq m$ by Heyting rule,

$i \wedge \neg q \leq p, (((p \to q) \to (p \vee \neg q)) \to n) \leq m, p \leq n \ \vdash \ i \leq m$ by Approximation rule,

$i \wedge \neg q \leq n, (((n \to q) \to (n \vee \neg q)) \to n) \leq m \ \vdash \ i \leq m$ by left Ackermann rule,

$\neg q \leq i \to n, (((n \to q) \to (n \vee \neg q)) \to n) \leq m \ \vdash \ i \leq m$,

$(\neg q \leq \top$ or $\neg q \leq n), (((n \to q) \to (n \vee \neg q)) \to n) \leq m \ \vdash \ i \leq m$,

$[(\neg q \leq \top, (((n \to q) \to (n \vee \neg q)) \to n) \leq m)$ or $(\neg q \leq n, (((n \to q) \to (n \vee \neg q)) \to n) \leq m)] \vdash i \leq m$, by distributive property,

$[\neg q \leq \top, (((n \to q) \to (n \vee \neg q)) \to n) \leq m \vdash i \leq m], [\neg q \leq n, (((n \to q) \to (n \vee \neg q)) \to n) \leq m \vdash i \leq m]$, since $\to$ is a g operation,

$[j \leq q, \neg j \leq \top, (((n \to q) \to (n \vee \neg q)) \to n) \leq m \vdash i \leq m], [k \leq q, \neg k \leq n, (((n \to q) \to (n \vee \neg q)) \to n) \leq m \vdash i \leq m]$, by $(\neg)$ approximation rule,

$[\neg j \leq \top, (((n \to j) \to (n \vee \neg j)) \to n) \leq m \vdash i \leq m], [\neg k \leq n, (((n \to k) \to (n \vee \neg k)) \to n) \leq m \vdash i \leq m]$, by right Ackermann rule,

$[(((n \to j) \to (n \vee \neg j)) \to n) \leq m \vdash i \leq m], [\neg k \leq n, (((n \to k) \to (n \vee \neg k)) \to n) \leq m \vdash i \leq m]$ since $\neg j \leq \top$ is trivially true,

$[i \leq (((n \to j) \to (n \vee \neg j)) \to n)], [\neg k \leq n, (((n \to k) \to (n \vee \neg k)) \to n) \leq m \vdash i \leq m]$.

We have now eliminated all propositional variables, and when substituting all nominals and co-nominals for their semantic definition, we will arrive at its first-order correspondent. We will not provide the correspondent of this formula; it was just an example to show that, in the case of intuitionistic logic, adequately (Ω, ε)-inductive formulas extends the known class of syntactic first-order definable intuitionistic formulas.

Chapter 6

Conclusion and further work

This thesis has explored various aspects of intuitionistic correspondence theory. After introducing the concept, we derived the first-order correspondents of a wide array of intuitionistic and superintuitionistic axioms and classified them with respect to known syntactically defined classes of elementary intuitionistic formulas. We then explored the limits of this concept in non-first-order definability. We finally arrived at the topic of algorithmically computing first-order correspondences, which was ripe for further research and, as such, yielded an extension to the known class of syntactically first-order definable intuitionistic formulas. In doing so, we have unified various approaches and results found in the literature and found some inconsistencies.

We now turn to open questions and further research directions. One could focus on the topic of non-first-order definability explicitly (there does not seem to be any reference for non-first-order definability on its own right; it seems to just present itself when proving that a specific formula is not first-order definable). We did not delve into the topic fundamentally; we just used some of the concepts to arrive at the needed results. Some of the questions that arose when researching the topic are:

i. It is generally challenging to find the frames used in the arguments for non-first-order definability. Can this process be systematized to a point where one can systematically find the frame needed for any, or at least some class of, formulas?

ii. Does a syntactic class of non-first-order definable formulas exist? Either in the intuitionistic case or in any other logic. For the intuitionistic case, there seems to be one for one propositional variable intuitionistic formulas; they just need to be equivalent to the Rieger–Nishimura formulas for $n \geq 10$. Can this be extended to a class of formulas with more than one propositional variable, including the non-first-order definable formulas studied in this thesis?

The most significant novel result in this thesis is an extension of ALBA in the form of Int-ALBA and the syntactic class of first-order definable intuitionistic formulas in the form of adequately (Ω, ε)-inductive formulas. These results lay the basis for much further research, including the exploration of the following questions:

iii. Are the adequately (Ω, ε)-inductive formulas canonical?

iv. Int-ALBA forms the basis for an extended correspondence theory for other logics with a Heyting implication, including any logics on an intuitionistic propositional base, like intuitionistic modal logics [15].

v. If we were to replace the $(\rightarrow)$ splitting approximation rule in Int-ALBA with the more general rule described in Remark 5.2.2, would the resulting algorithm succeed on any formulas on which Int-ALBA does not succeed and hence would we be able to obtain a proper superclass of the adequately (Ω, ε)-inductive formulas?

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